Distributed Adaptive Resource Allocation over Digraphs: an Uncertain Saddle-point Dynamics Viewpoint

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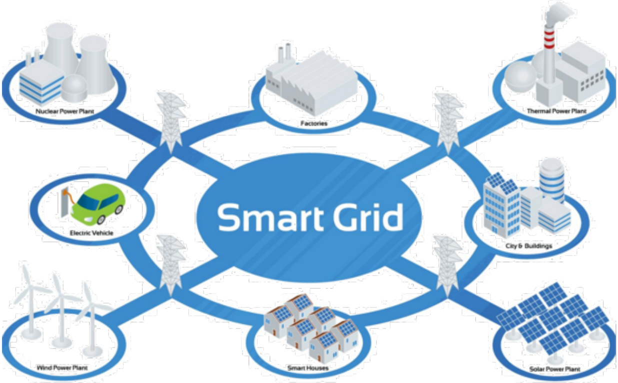
BACKGROUND: DISTRIBUTED RESOURCE ALLOCATION



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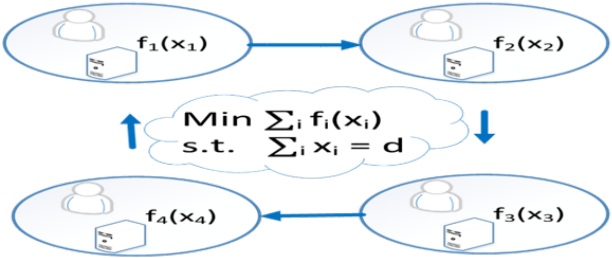
PROBLEM STATEMENT

$$\min_{x \triangleq \text{col}(x_1, \cdots, x_N)} f(x) \triangleq \sum_{i=1}^{N} f_i(x_i), \tag{1}$$

$$\text{subject to } \sum_{i=1}^{N} x_i = d, \text{ where } d = \sum_{i=1}^{N} d_i.$$



PROBLEM STATEMENT

Assumption 1

The communication graph $\mathcal{G}$ is strongly connected and weight-balanced.

Assumption 2

Each local cost function $f_i(\cdot)$ is continuously differentiable and strictly convex.

Problem Statement

Assumption 1

The communication graph $\mathcal{G}$ is strongly connected and weight-balanced.

Assumption 2

Each local cost function $f_i(\cdot)$ is continuously differentiable and strictly convex.

Goal:

Solve the distributed resource allocation problem **WITHOUT** the knowledge of the underlying Laplacian eigenvalues (which has been widely used in related literature).

Uncertain Saddle-point Dynamics

Centralized Saddle-point dynamics:

$$\begin{aligned} \dot{x} &= -\nabla f(x) - \mathbf{1}_N \otimes y; \\ \dot{y} &= (\mathbf{1}_N^T \otimes \mathbf{I}_n)(x - D). \end{aligned} \tag{2}$$

Note: y cannot be undated in a distributed way!

UNCERTAIN SADDLE-POINT DYNAMICS

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(Distributed) Uncertain Saddle-point dynamics:

$$\mathcal{O}: \quad \dot{x} = -\kappa_1(\nabla f(x) + y) \tag{3a}$$

$$\dot{y} = x - D - (\Upsilon \otimes \mathbf{I}_n)y - (\mathcal{L} \otimes \mathbf{I}_n)z \tag{3b}$$

$$\dot{z} = (\Upsilon \otimes \mathbf{I}_n)y \tag{3c}$$

$\mathcal{L}$: the Laplacian of $\mathcal{G}$;
$\Upsilon \in \mathcal{M}_{\mathrm{r}}^N$ (square with zero row sums): unknown a priori.

UNCERTAIN SADDLE-POINT DYNAMICS

Definition 1 (GEP)

The triple $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn}$ is called a generalized equilibrium point of (3), if for any $\Upsilon \in \mathcal{M}_{\mathrm{r}}^{N}$, there holds $\mathcal{O}|_{(\tilde{x},\tilde{y},\tilde{z})} = 0$.

Lemma 1 (GEPs of (3))

Under Assumptions 1 and 2:

- *The uncertain system (3) has infinitely many GEPs.*
- *If $(\tilde{x}, \tilde{y}, \tilde{z})$ is a GEP of (3), then $(\tilde{x}, \tilde{y}) = (x^*, \mathbf{1}_N \otimes y^*)$, i.e., $\tilde{x}$ is the optimizer of problem (1).*
- *$(\tilde{x}, \tilde{y})$ is unique.*

ALGORITHMS

DST-based:

$$
\begin{aligned}
\dot{x} &= -\kappa_1(\nabla f(x) + y) \\
\dot{y} &= x - D - (\mathcal{L}^a \otimes \mathbf{I}_n)y \\
&\quad - (\mathcal{L} \otimes \mathbf{I}_n)z \\
\dot{z} &= (\mathcal{L}^a \otimes \mathbf{I}_n)y \\
\dot{a}_{ij} &= \begin{cases} g(y_{i_k}, y_{k+1}, \sum_{j \in \bar{\mathcal{N}}_{\text{out}}(k+1)} y_j), \\ \qquad\qquad \text{if } e_{ji} \in \bar{\mathcal{E}} \\ 0, \qquad \text{if } e_{ji} \in \mathcal{E} \setminus \bar{\mathcal{E}} \end{cases}
\end{aligned}
$$

DST: directed spanning tree.
Note: the specific form of g is omitted here.

ALGORITHMS

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\end{aligned}
$$

Node-based:

$$
\begin{aligned}
\dot{x} &= -\kappa_1(\nabla f(x) + y) \\
\dot{y} &= x - D - (\mathcal{A}\mathcal{L} \otimes \mathbf{I}_n)y \\
&\quad - (\mathcal{L} \otimes \mathbf{I}_n)z \\
\dot{z} &= (\mathcal{A}\mathcal{L} \otimes \mathbf{I}_n)y \\
\dot{\alpha}_i &= \kappa_2 \xi_i^T \xi_i \\
\xi_i &= \sum_{j\in\mathcal{N}_{\mathrm{in}}(i)} w_{ij}(y_i - y_j)
\end{aligned}
$$

DST: directed spanning tree.
Note: the specific form of g is omitted here.

Main Results

Theorem 1

Under Assumptions 1-2, the DST-based algorithm drives (x, y) to $(x^*, \boldsymbol{1}_N \otimes y^*)$ asymptotically for any initial condition $(x(0), y(0), z(0) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn}$ and any $a_{ij}(0) \in \mathbb{R}$ provided there exists a scalar $m \in \mathbb{R}^+$, such that the following condition (referred to as spanning-tree-based m-strongly convex) holds $\forall x, y \in \mathbb{R}^{Nn}$:

$$(x-y)^T(\bar{\mathcal{L}}^U \otimes \boldsymbol{I}_n)(\nabla f(x) - \nabla f(y)) \geq m(x-y)^T(\bar{\mathcal{L}}^U \otimes \boldsymbol{I}_n)(x-y) \tag{4}$$

where $\bar{\mathcal{L}}^U = \Xi^T \Xi$ is the un-weighted Laplacian matrix of the undirected spanning tree $\bar{\mathcal{G}}^U$ based on $\bar{\mathcal{G}}$. Moreover, the adaptive gains $\bar{a}_{k+1,i_k}$, $k \in \mathcal{I}_{N-1}$, converge to some finite constant values.

MAIN RESULTS

Consider the special case of quadratic local costs

$$f_i(x) \triangleq x^T \Theta x + x^T \varphi_i, \qquad \Theta \succ 0,\ \varphi_i \in \mathbb{R}^n. \tag{5}$$

In this case, the spanning-tree-based m-strongly convex condition (4) holds with any $m \leq \underline{\lambda}(\Theta)$ and for any DST. Immediately, we have the following corollary:

Corollary 1

Under Assumptions 1-2, the resource allocation problem (1) with local costs (5) can be solved with the DST-based algorithm for any initial conditions $(x(0), y(0), z(0) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn}$ *and any* $a_{ij}(0) \in \mathbb{R}$, *i.e.,* $(x, y) \to (x^*, \mathbf{1}_N \otimes y^*)$. *Moreover, the adaptive gains* $\bar{a}_{k+1,i_k}$, $k \in \mathcal{I}_{N-1}$, *converge to some finite constant values.*

Main Results

Theorem 2

Under Assumptions 1-2, the node-based algorithm drives (x, y) *to* $(x^*, \mathbf{1}_N \otimes y^*)$ *asymptotically for any initial conditions* $(x(0), y(0), z(0) \in \mathbb{R}^{Nn} \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn}$ *and any* $\alpha_i(0) \in \mathbb{R}^+$ *provided there exists a scalar* $m \in \mathbb{R}^+$*, such that the following condition (referred to as jointly m-strongly convex) holds* $\forall x, y \in \mathbb{R}^{Nn}$*:*

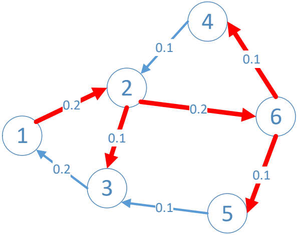
$$(x-y)^T(\mathcal{L}^T\mathcal{L} \otimes \boldsymbol{I}_n)(\nabla f(x) - \nabla f(y)) \geq m(x-y)^T(\mathcal{L}^T\mathcal{L} \otimes \boldsymbol{I}_n)(x-y). \tag{6}$$

Moreover, the adaptive gains α_i*,* $i \in \mathcal{I}_N$*, converge to some finite constant values.*

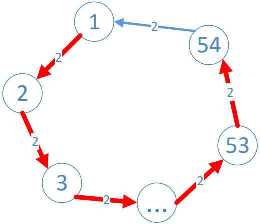
Corollary 2

Under Assumptions 1-2, the resource allocation problem (1) with local costs (5) can be solved with the node-based algorithm $\cdots$

TEST ON IEEE 30/118-BUS POWER GRIDS



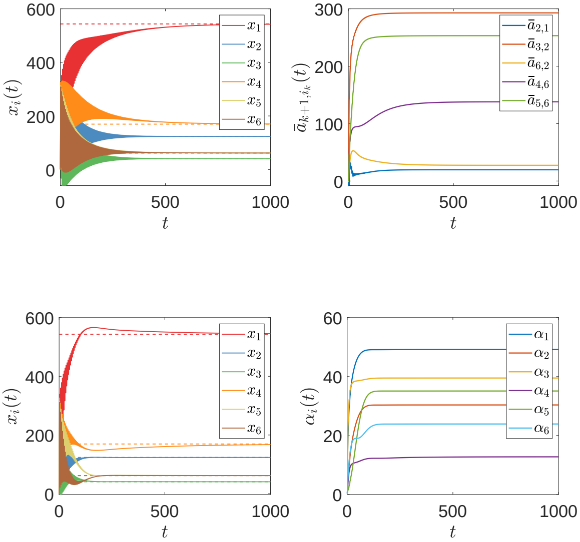
(a) $\mathcal{G}_1$



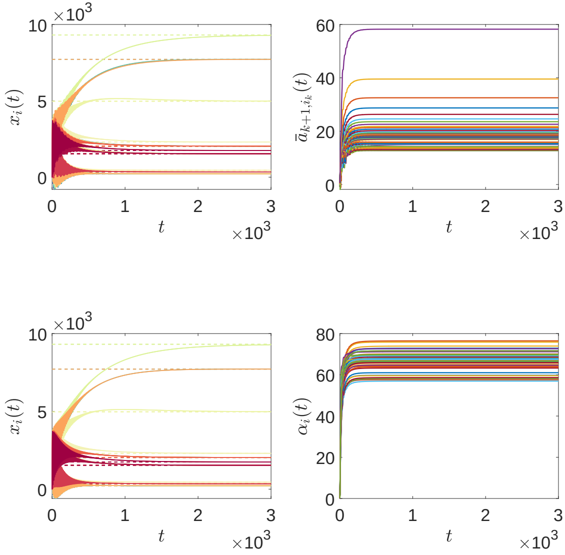
(b) $\mathcal{G}_2$

Figure: Two communication topology between power generators.

IEEE 30-BUS: DST/NODE-BASED EVOLUTION



IEEE 118-BUS: DST/NODE-BASED EVOLUTION



CONCLUSION

List of Contributions:

- A distributed uncertain saddle-point dynamics is proposed for resource allocation problem.
- Two adaptive saddle-point algorithms named DST-based and node-based have been proposed.
- The proposed algorithms successfully remove the knowledge of the underlying Laplacian eigenvalues.

Future works?

Thank you for listening!

Question?