The Vibration of Simple Pendulum

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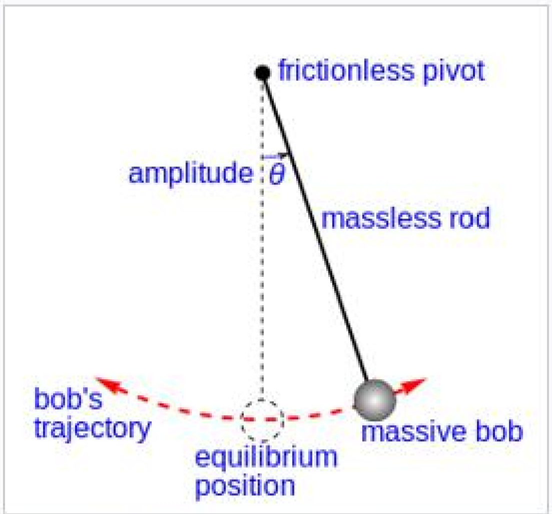
Abstract: The article discusses four kinds of idealized models to find the differences between the different ways to calculate the relationship between simple pendulum amplitude and time. Then, I will increase the complexity of the model by means of increasing the damping coefficient,increasing the pressure and adding non-linear solution. I will choose one of these ways to have a discussion of simple pendulum movement.

Keywords: vibration; pendulum; Euler Method; Euler-Cromer Method; Second-order Runge-Kutta Method; Verlet Method

Introduction

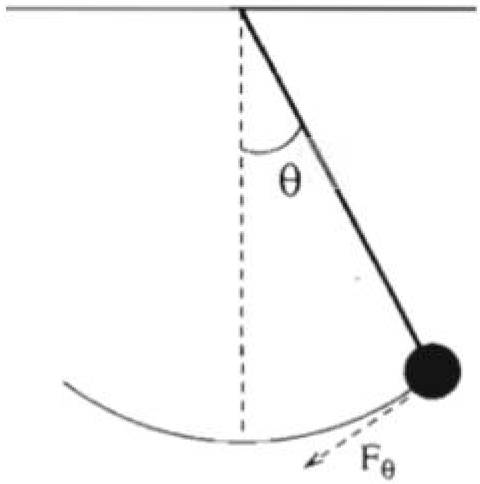
A pendulum is a weight suspended from a pivot so that it can swing freely. When a pendulum is displaced sideways from its resting, equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position. When released, the restoring force combined with the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the period. The period depends on the length of the pendulum and also to a slight degree on the amplitude, the width of the pendulum's swing.



"Simple gravity pendulum" model assumes no friction or air resistance.

From the first scientific investigations of the pendulum around 1602 by Galileo Galilei, the regular motion of pendulums was used for timekeeping, and was the world's most accurate timekeeping technology until the 1930s. The pendulum clock invented by Christian Huygens in 1658 became the world's standard timekeeper, used in homes and offices for 270 years, and achieved accuracy of about one second per year before it was superseded as a time standard by quartz clocks in the 1930s. Pendulums are also used in scientific instruments such as accelerometers and seismometers. Historically they were used as gravimeters to measure the acceleration of gravity in geophysical surveys, and even as a standard of length. Today, I will introduce the different solutions to simple pendulum movement. and the chaos phenomenon it could lead to.

Ⅰ The algorithm analysis under the simplified model



When we consider the most ideal system of a simple pendulum, we will ignore any other outside force including the friction. We will see the small ball as a particle hung by an inextensible rope and the angle of suspension line and the vertical direction (θ) is small. As the picture above, the small ball is in the reciprocating motion under the action of gravity.

So, we can find the relation(1):

$$F_\theta = -mg\sin\theta$$

Using Newton's second law: $F_\theta = \frac{d^2 s}{dt^2}$

in the equation, $s = l\theta$, l is the length of the suspension line. When θ is small, we may find $\sin\theta = \theta$, then we will find the equation(2):

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

It's easy for us to find one set of general solution(3):

$$\theta = \theta_0 \sin(\Omega t + \phi)$$

in the equation, $\Omega = \sqrt{g/l}$, θ_0 和 ϕ is the constant.

Then, we want to get the θ (t).So we have to solve the equation(2). We may translate it into the following equations(4):

$$\frac{d\omega}{dt} = -\frac{g}{l}\theta$$

$$\frac{d\theta}{dt} = \omega$$

in which, the ω is the angular velocity of the simple pendulum.

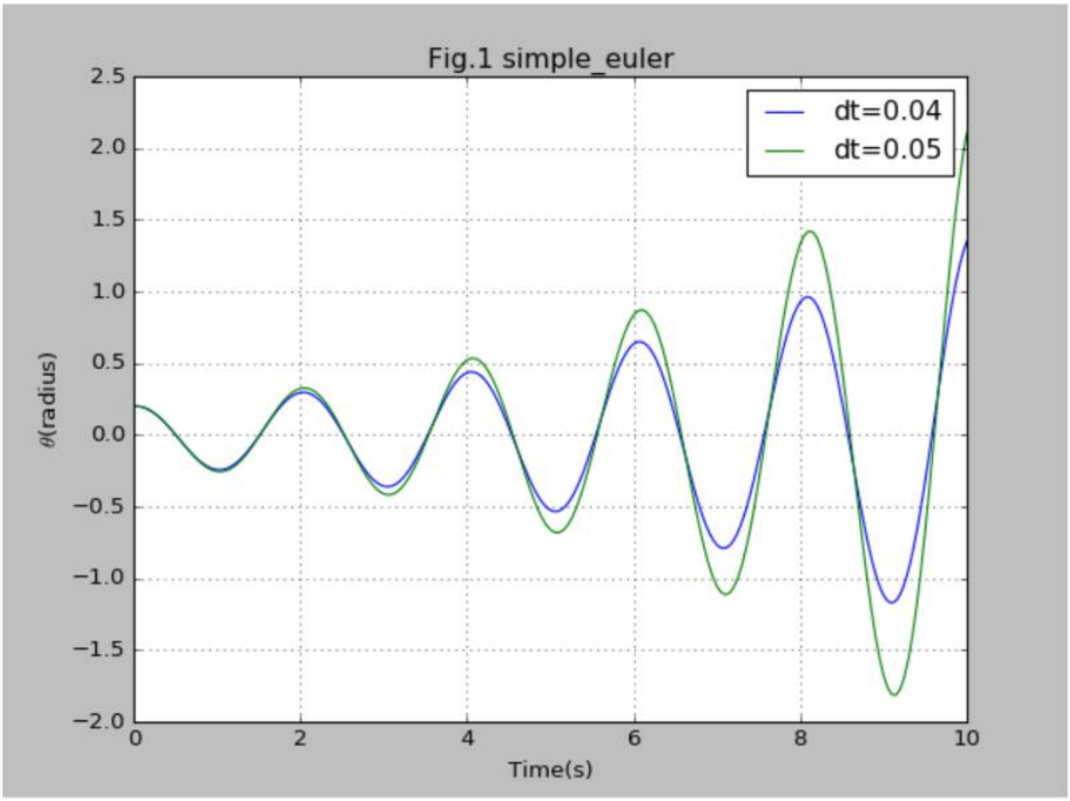
A Euler Method and it's limitations

The basic principle of the Euler Method is to translate the differential equation into the difference equation and to apply the first order approximate. When the t=n△t, the equations(4) are changed into the following equations(5) :

$$\omega_{i+1} = \omega_i - \frac{g}{l}\theta_i \Delta t$$

$$\theta_{i+1} = \theta_i + \omega_i \Delta t$$

when the l=1m,dt=0.05s and dt=0.04s, we get the following figure:



From the figure, it is obvious that, although we can reduce error by reduce the time interval, the energy is always increasing as time goes by in the condition that any time interval cannot be zero, which disagree with the facts.

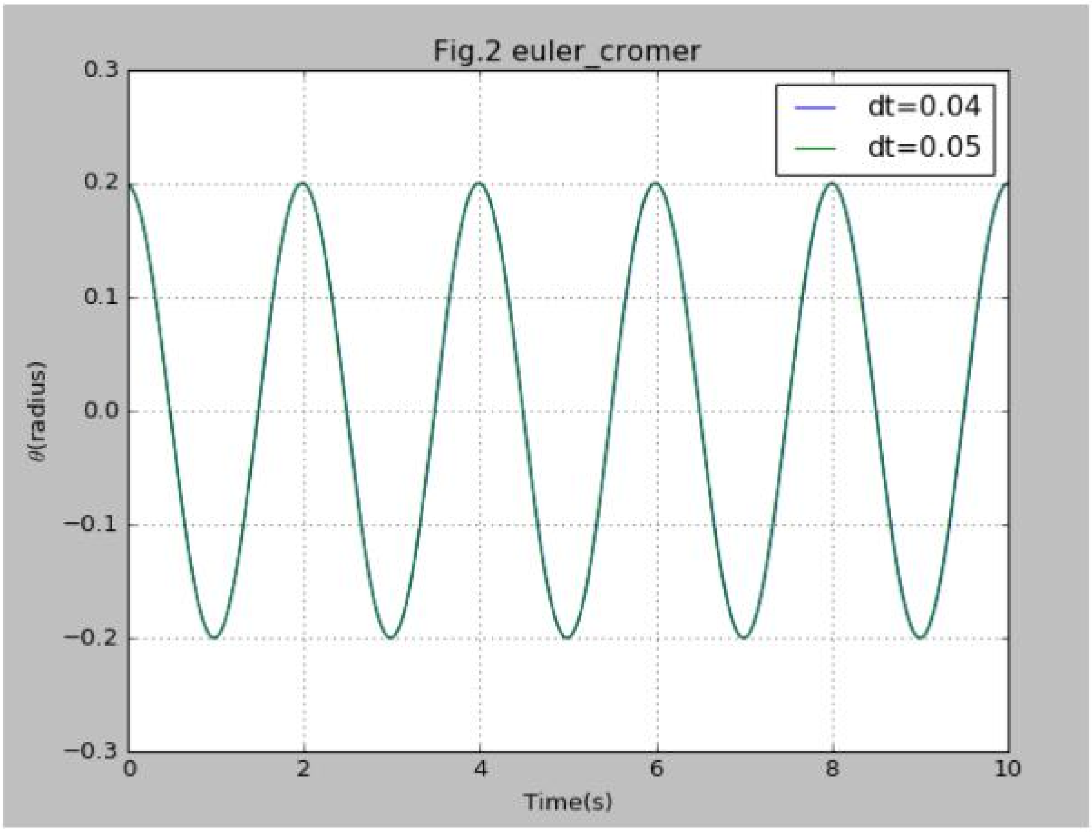
B Euler-Cromer Method

Euler-Cromer Method only change a little. We take the left end point as the mid-value when we calculate the ω_{i+1}, and We take the right end point as the mid-value when we calculate the θ_{i+1}.

. In such cases, the equations(4) will be changed into the equation(6):

$$\omega_{i+1} = \omega_i - \frac{g}{l}\theta_i \Delta t$$
$$\theta_{i+1} = \theta_i + \omega_{i+1}\Delta t$$

when the l=1m,dt=0.05s and dt=0.04s, we get the following figure:



We can see that computed result is so stable. In such ideal condition the system will be moving all the time.

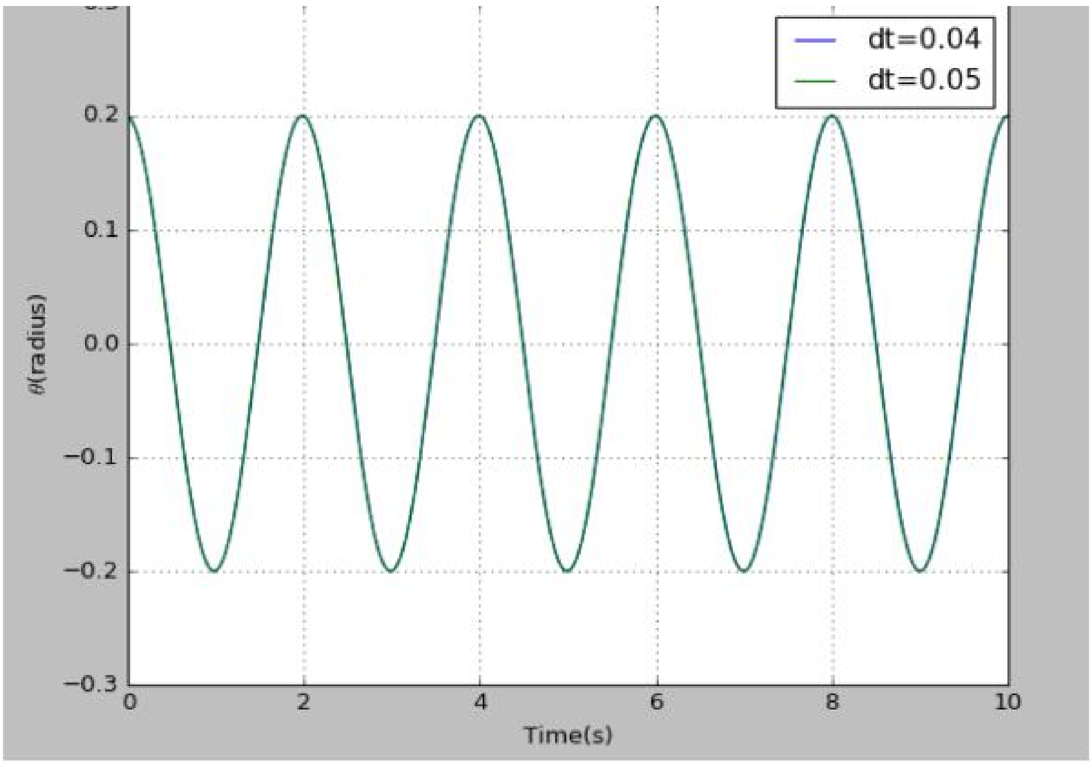
C Second-order Runge-Kutta Method

All order Runge-Kutta Method give the estimated value of the mid-value. The equations(4) will be changed into the equations(7):

$$\omega_{i+1} = 2\omega_i - \frac{g}{l}\theta_{t_m}\Delta t$$
$$\theta_{i+1} = \theta_i + \omega_{t_m}\Delta t$$

when the l=1m,dt=0.02s, we get the following figure:

We can find the figure is the same as the Euler-Cromer Method's one.

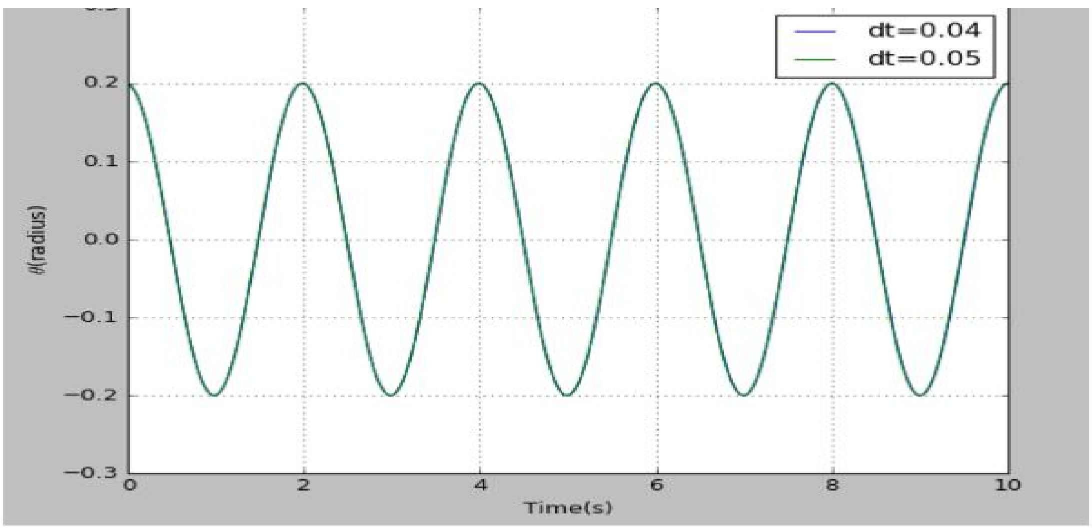


D Verlet Method

Verlet Method use the central difference to change the equations(4) into the equations(8):

$$\theta_{i+1} = 2\theta_i - \theta_{i-1} + \frac{d^2\theta}{dt^2} \times \Delta t^2$$
$$\omega_i = \frac{\theta_{i+1} - \theta_{i-1}}{2\Delta t}$$

when the l=1m,dt=0.02s, we get the following figure:



We can find the figure is the same as the Euler-Cromer Method's one.

Ⅱ Linear harmonic vibration(solved by Euler-Cromer Method)

1. Taking the loss into consideration, the equation(2) is changed into the equation(9):

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}$$

then solve the equation and we can get three defferent models:

Underdamping $\theta_{(t)} = \theta_0 e^{-\frac{qt}{2}} \sin(\sqrt{\Omega^2 - \frac{q^2}{4}}t + \phi)$

Overdamping $\theta_{(t)} = \theta_0 e^{-(q/2 \pm \sqrt{q^2/4 - \Omega^2})t}$

critical damping $\theta_{(t)} = (\theta_0 + Ct)e^{-qt/2}$

2. Taking the damping coefficient q and impressed pressure F_D into consideration,the equation (2) is changed into the equation(10):

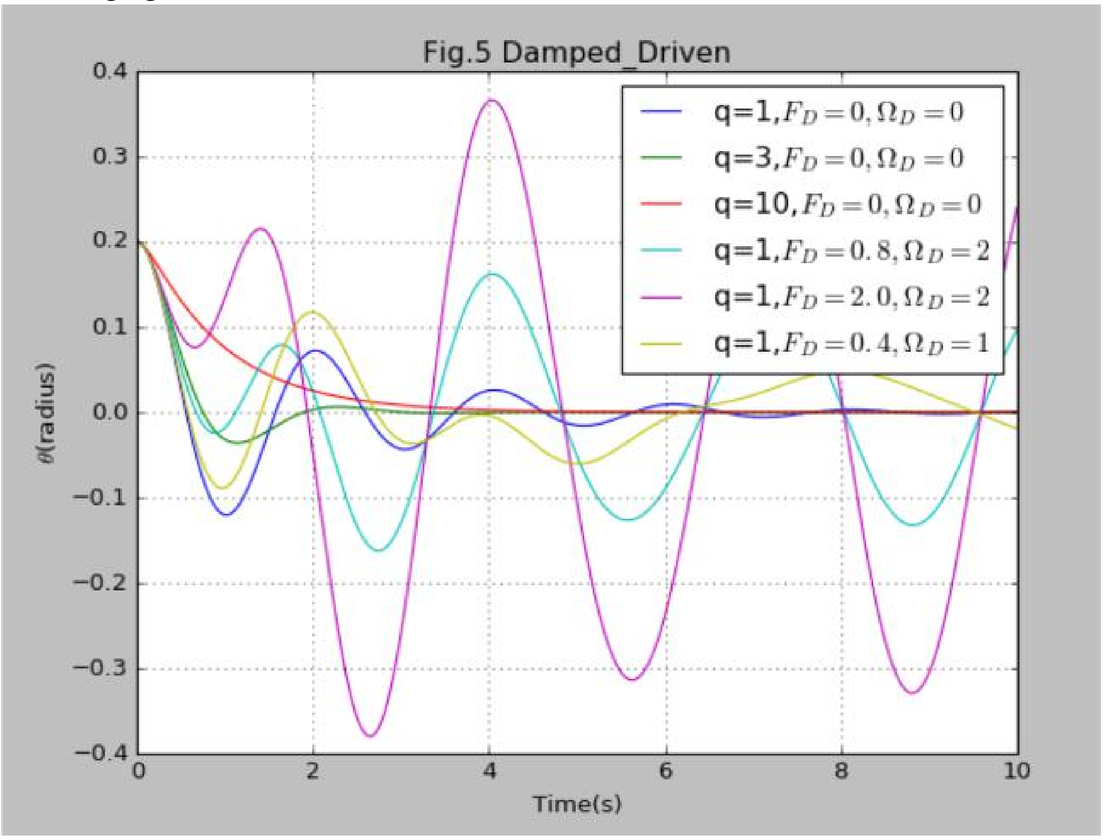
$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + F_D(\sin\Omega_D t)$$

If we consider the F_D as a sine function, Ω_D is the angular velocity. We can get the general solution:

$$\theta = \theta_0 \sin(\Omega_D t + \phi)$$

in which: $$\theta_0 = \frac{F_D}{\sqrt{(\Omega^2 - {\Omega_D}^2)^2 + (q\Omega_D)^2}}$$

change the the damping coefficient q and impressed pressure F_D, we can get the following figure:



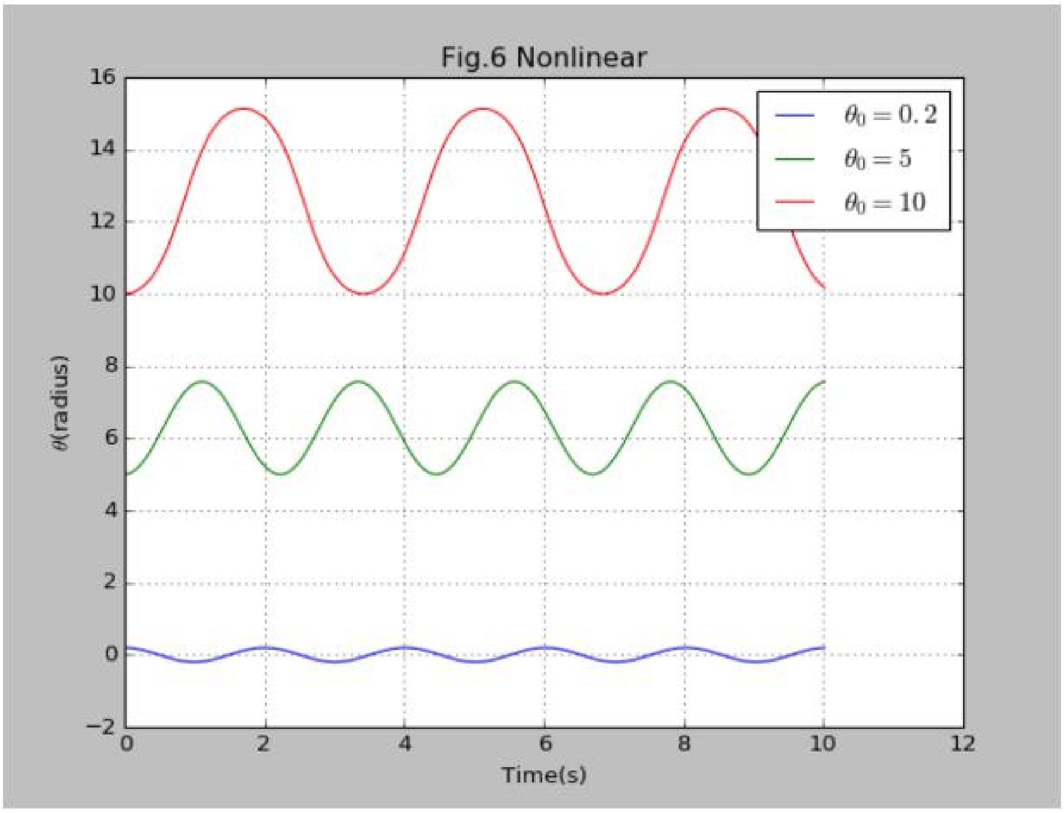
when the the damping coefficient q is the same, the value and frequency of the impressed pressure F_D will affect the frequency、amplitude of the simple pendulum and the value of the energy of the system.

III Nonlinear simple pendulum

Now, we have changed nearly all the conditions in the initial ideal model, except that "the angle of suspension line and the vertical direction (θ) is small". When the angle of suspension line and the vertical direction (θ) is not small, we cannot have the relation sin θ = θ , so the equation(2) will be changed into the equation (13):

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

Assuming that there is no loss and the impressed pressure, we know the conversation of energy so that the simple pendulum must be periodic. calculate with Runge-Kutta Method and get the following figure:



From the figure, we know that the frequency and amplitude of the simple pendulum relate to the initial angle.

Assuming that there exists loss and the impressed pressure,the equation(13) will be changed into the equation (14):

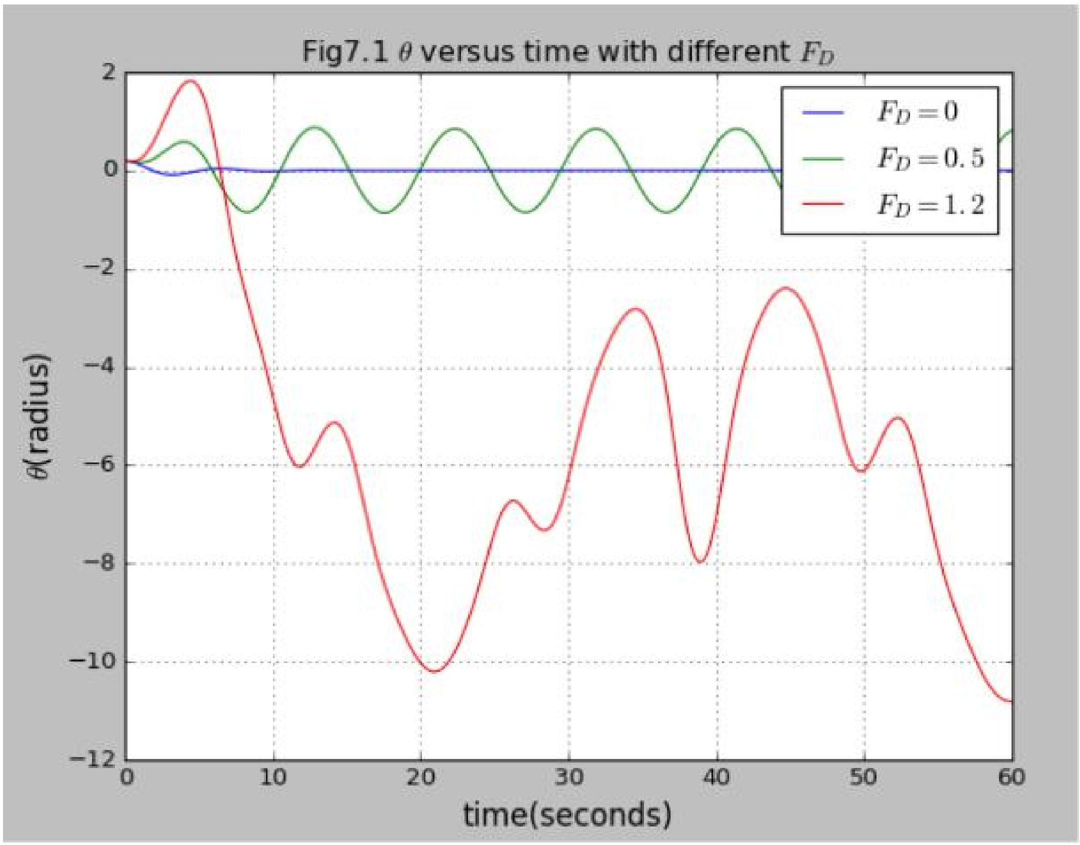
$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - q\frac{d\theta}{dt} + F_D\sin(\Omega_D t)$$

deal with the equation(14) in the same way:

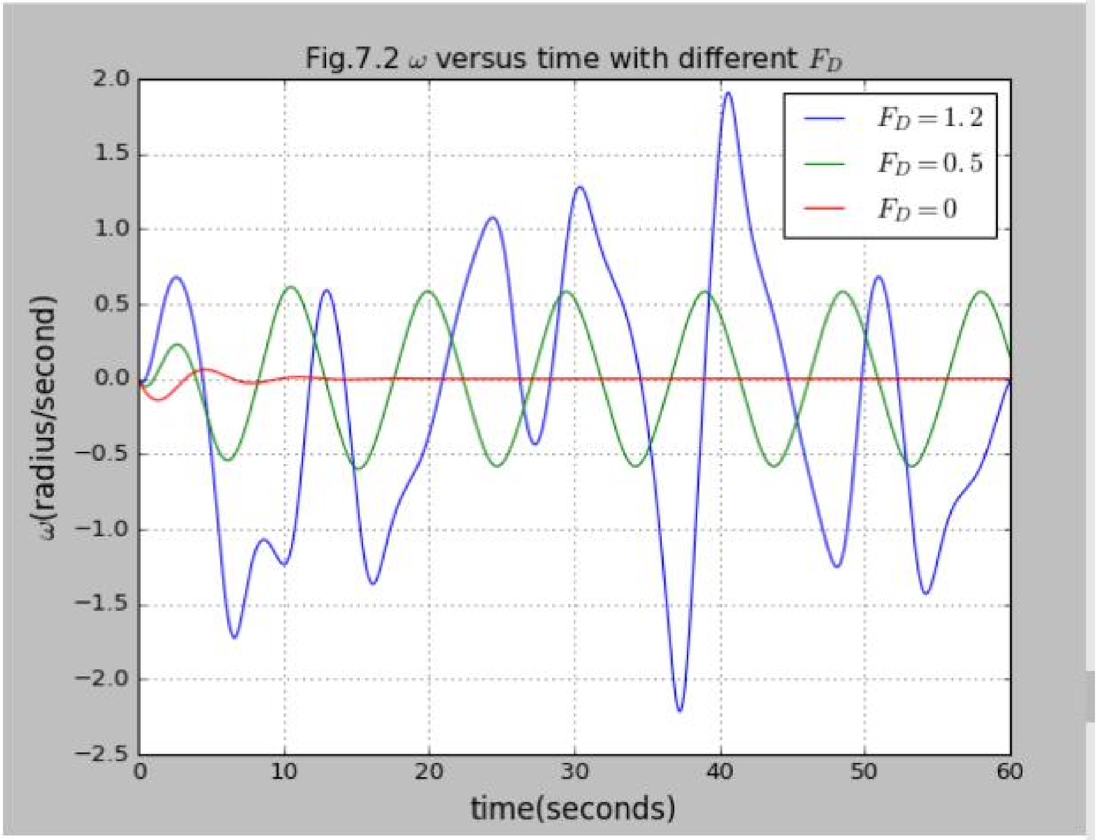
$$\frac{d\omega}{dt} = -\frac{g}{l}\sin(\theta) - q\frac{d\theta}{dt} + F_D \sin(\Omega_D t)$$
$$\frac{d\theta}{dt} = \omega$$

When we solve it with the Euler-Cromer Method, we need pay attention to the the range of 0. Considering the meaning and convenient, we assume the relation |0|⩽ π .

Under the circumstances that the 0 and ω are time dependent and there exist the impressed pressure of defferent values. The vibration of the simple pendulum will have different patterns. We can get the following figures:



the figure about 0(t) and the impressed pressure of defferent values F_D:



the figure about ω (t) and the impressed pressure of defferent values F_D.

From the two figures above, we know:
a. when there is no the impressed pressure, the system will gradually stop moving because of the loss.
b. when the impressed pressure is small, the system will move in the same frequency as the impressed pressure.
c. when the impressed pressure is large enough, there seems no rules in the movement.

Ⅳ Conclusion

During the discussion above, we have a better idea of the movement of the simple pendulum so that we could understand the vibration which is a universal phenomenon but ignored by us. In terms of computing, different computing methods provides different ways of thinking for us. We could draw inferences about other cases from one instance to get the best solution when encountering the similar problems.

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