1081 Calculus 模組 07 Homework 4

Due Date: Oct 31, 2019

Please hand in the following exercise of textbook in Part I and all problems in Part II. The rigorous and clear explanation is needed. An answer without process will get no point.

Part I:

1. (Ch.4.3, Ex.11, 17)
   Please find the intervals on which f is increasing, decreasing, concave upward, or concave downward. Also, find the local and absolute maximum and minimum of f on assigned domain.

   (a) $f(x) = x^4 - 2x^2 + 3$ on $\mathbb{R}$

   (b) $f(x) = x^2 - x - \ln x$ on $\mathbb{R}^+ := \{x \big| x > 0\}$

2. (Ch.4.3, Ex.73, 74)
   Determine the constants a, b, c, d such that the following statements hold.

   (a) $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum value of 3 at $x = -2$, and a local minimum value of 0 at $x = -1$.

   (b) $f(x) = axe^{bx^2}$ has the maximum value $f(2) = 1$.

3. (Ch.4.4, Ex.30, 37, 40, 44, 47, 53, 57, 59, 68)
   Evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x - \tan x}$ (b) $\displaystyle\lim_{x\to 0^+} \frac{\tan^{-1}(2x)}{\ln x}$ (c) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ (d) $\displaystyle\lim_{x\to 0^+} \sin x \ln x$ (e) $\displaystyle\lim_{x\to\infty} x^3 e^{-x^2}$

   (f) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$ (g) $\displaystyle\lim_{x\to 0^+} x^{\sqrt{x}}$ (h) $\displaystyle\lim_{x\to 0} (1-2x)^{1/x}$ (i) $\displaystyle\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$

4. (Ch.4.3, Ex.51, 52, 55; Ch.4.5, Ex.61, 62)
   Find the equations of all (horizontal, vertical, slant) asymptotes.

   (a) $y = \sqrt{x^2+1} - x$

   (b) $y = \dfrac{e^x}{1 - e^x}$

   (c) $y = \ln(1 - \ln x)$

   (d) $y = \dfrac{x^2+1}{x+1}$

   (e) $y = \dfrac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x}$

5. (Ch.4.5, Ex.11, 24, 54)
   Sketch the following curves.

   (a) $y = \dfrac{x - x^2}{2 - 3x + x^2}$

   (b) $y = \sqrt{x^2 + x} - x$

   (c) $y = \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$

Part II:

1. (Ch.4.3, Ex.85)
   Let $p(x) = ax^3 + bx^2 + cx + d$ be a polynomial with degree 3, i.e. $a \neq 0$.
   (a) Show that p has exactly one point of inflection.
   (b) Determine how to choose suitable constants a, b, c, d such that p has a local maximum and a local minimum. Is it possible that p attains the absolute maximum or absolute minimum? Explain why.
   (c) Show that if p has real roots x_1, x_2, x_3 (including multiple roots), then the coordinate of the inflection point is $\dfrac{x_1 + x_2 + x_3}{3}$.

2. (Ch.4.4, Ex.89)
   (a) If f' is continuous, use l'Hospital's Rule to show that
   $$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x). \tag{1}$$
   (b) If f is differentiable, prove that (1) is still true.
   (c) Suppose that
   $$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} \text{ exists for all } x.$$
   Is f differentiable?

(The below part is not contained in the homework.)

Remark 1. (i) Compare the condition of 2.(b) with 2.(a). Notice that even f is differentiable, its derivative f' may be discontinuous. To see this, we consider the following function (Ch.2.7, Ex.60)

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

You can show that f is differentiable everywhere. Moreover, write down the expansion of f' explicitly and show that f' is discontinuous at $x = 0$.

(ii) Recall that the definition of derivative is given by

$$f'(x) = \lim_{h \to \infty} \frac{f(x+h) - f(x)}{h}. \tag{2}$$

In general, (2) is called forward difference and (1) is called central difference. As the result in 2.(b), we see that there are two ways to calculate f' if f is differentiable. Moreover, in numerical region, the central difference (1) has higher accuracy (2nd order) than the forward difference (2) (1st order). However, the conclusion of 2.(c) indicates that the definition of derivative (2) can not be replaced by (1).