1081 Calculus 模組 07 Quiz 1 Solution

Time: 17:50$\sim$ 18:20 Date: Oct 03, 2019

DEPARTMENT: ______________ ID NUMBER: ______________ NAME: ______________

It's necessary to explain all the reasons in detail and show all of your work on the answer sheet. Or you will NOT get any credits. If you used any theorems in textbook or proved in class, state it carefully and explicitly.

1. (25%) **True/False** (Write down **T** or **F** only. Any explanations are not required.)

   (a) $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$.

   (b) $\sin^{-1}(\sin x) = x$ for all $x \in \mathbb{R}$.

   (c) If $f(x) > 0$ for all $x \in (c-1, c+1)$ and $\lim\limits_{x\to c} f(x) = L$ exists, then $L > 0$.

   (d) If $\lim\limits_{x\to c}(2f(x) + 3g(x))$ and $\lim\limits_{x\to c}(3f(x) - 2g(x))$ both exist, then $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ both exist.

   (e) If f is continuous at $x = c$, then f is differentiable at $x = c$ ($f'(c)$ exists).

   Answer: (a)__T__ (b)__F__ (c)__F__ (d)__T__ (e)__F__

   Solution.

   (a) Note that $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$.

   (b) Note that $\sin^{-1}(\sin x) = x$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

   (c) Define

   $$f(x) = \begin{cases} |x| & , \text{if } x \neq 0 \\ 1 & , \text{if } x = 0 \end{cases}.$$

   Then $\lim\limits_{x\to 0} f(x) = 0$.

   (d) Let $2f(x) + 3g(x) = h(x)$ and $3f(x) - 2g(x) = k(x)$. Then

   $$f(x) = \frac{2}{13}h(x) + \frac{3}{13}k(x),$$
   $$g(x) = \frac{3}{13}h(x) - \frac{2}{13}k(x).$$

   Due to the assumption, we conclude that $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ both exist since $\lim\limits_{x\to c} h(x)$ and $\lim\limits_{x\to c} k(x)$ both exist.

   (e) Consider $f(x) = |x|$. Note that f is continuous everywhere but not differentiable at 0. We remark that the converse is true. That is, if f is differentiable at $x = c$, then f must be continuous at $x = c$.

   $\square$

2. (15%) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2 \\ ax^2 - bx + 3, & 2 \le x < 3 \\ 2x - a + b, & x \ge 3 \end{cases}.$$

Find the values of a and b such that f is continuous everywhere.

Proof. Since f is a rational function on $(-\infty, 2)$ and a polynomial on $(2,3) \cup (3,\infty)$, f is continuous on $\mathbb{R} \setminus \{2,3\}$. In order to ensure that f is continuous at $x = 2, 3$, we need to choose suitable a, b such that

$$\lim_{x\to 2} f(x) = f(2) \quad \text{and} \quad \lim_{x\to 3} f(x) = f(3).$$

We notice that

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= 4, \\ \lim_{x\to 2^+} f(x) &= 4a - 2b + 3, \\ f(2) &= 4a - 2b + 3, \\ \lim_{x\to 3^-} f(x) &= 9a - 3b + 3, \\ \lim_{x\to 3^+} f(x) &= 6 - a + b, \\ f(3) &= 6 - a + b. \end{aligned}$$

Hence, we have to solve

$$\begin{aligned} 4a - 2b + 3 &= 4, \\ 9a - 3b + 3 &= 6 - a + b. \end{aligned}$$

Therefore, we obtain $a = b = \dfrac{1}{2}$. $\square$

3. Evaluate the following limits.

(a) (15%) $\lim_{x\to 0^-} \dfrac{|x|-x}{|x|-x^3}$

Solution.

$$\begin{aligned}\lim_{x\to 0^-} \frac{|x|-x}{|x|-x^3} &= \lim_{x\to 0^-} \frac{-x-x}{-x-x^3} \\ &= \lim_{x\to 0^-} \frac{-2}{-1-x^2} = 2.\end{aligned}$$

□

(b) (15%) $\lim_{x\to 1} \dfrac{1}{x-1}\left(\dfrac{1}{x+3} - \dfrac{2}{3x+5}\right)$

Solution.

$$\begin{aligned}\lim_{x\to 1} \frac{1}{x-1}\left(\frac{1}{x+3} - \frac{2}{3x+5}\right) &= \lim_{x\to 1} \frac{x-1}{(x-1)(x+3)(3x+5)} \\ &= \lim_{x\to 1} \frac{1}{(x+3)(3x+5)} = \frac{1}{32}.\end{aligned}$$

□

(c) (15%) $\lim_{x\to\infty} \dfrac{\sqrt{x+3}-\sqrt{x}}{\sqrt{x+2}-\sqrt{x+1}}$

Solution.

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x+3}-\sqrt{x}}{\sqrt{x+2}-\sqrt{x+1}} &= \lim_{x\to\infty} \left(\frac{\sqrt{x+3}-\sqrt{x}}{\sqrt{x+2}-\sqrt{x+1}} \cdot \frac{\sqrt{x+3}+\sqrt{x}}{\sqrt{x+3}+\sqrt{x}} \cdot \frac{\sqrt{x+2}+\sqrt{x+1}}{\sqrt{x+2}+\sqrt{x+1}}\right) \\ &= \lim_{x\to\infty} \left(3 \cdot \frac{\sqrt{x+2}+\sqrt{x+1}}{\sqrt{x+3}+\sqrt{x}}\right) \\ &= 3 \lim_{x\to\infty} \frac{\sqrt{1+\frac{2}{x}}+\sqrt{1+\frac{1}{x}}}{\sqrt{1+\frac{3}{x}}+1} = 3.\end{aligned}$$

□

(d) (15%) $\lim_{x\to 0} \sqrt[3]{x}\cos(\dfrac{1}{x^2})e^{\sin(\frac{1}{x^4})}$

Solution. Note that

$$-e|\sqrt[3]{x}| \le \sqrt[3]{x}\cos(\frac{1}{x^2})e^{\sin(\frac{1}{x^4})} \le e|\sqrt[3]{x}|$$

for all $x \in \mathbb{R}$, and

$$\lim_{x\to 0} \sqrt[3]{x} = 0.$$

By squeeze theorem, we conclude that

$$\lim_{x\to 0} \sqrt[3]{x}\cos(\frac{1}{x^2})e^{\sin(\frac{1}{x^4})} = 0.$$

□