

Real Analysis

Homework 10

Yueh-Chou Lee

December 28, 2018

1. (Exercise 7.11)

Prove the following result concerning *changes of variable*. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f(g(t))g'(t) dt$$

(Consider the cases when f is the characteristic function of an interval, an open set, etc.)

Proof.

Since $g(t)$ is monotone increasing and absolutely continuous on $[\alpha, \beta]$, then g is of bounded variation on $[\alpha, \beta]$, g' also exists a.e. and $g' \in L[\alpha, \beta]$.

Let f and g' be the non-negative function.

Since $f \in L[a, b]$ and $g' \in L[\alpha, \beta]$, then f is measurable on $[a, b]$ and g is measurable on $[\alpha, \beta]$.

Thus, $f(g(t))$ is also measurable on $[\alpha, \beta]$ since $g(t)$ is continuous on $[\alpha, \beta]$.

Then

$$f(g(t))g'(t) = \frac{1}{2} \left\{ [f(g(t)) + g'(t)]^2 - [f(g(t))]^2 - g'(t)^2 \right\}$$

is measurable.

f and g' are finite on $[a, b]$ and $[\alpha, \beta]$ since $f \in L[a, b]$ and $g' \in L[\alpha, \beta]$, so

$$\int_{\alpha}^{\beta} f(g(t))g'(t) dt \leq |[\alpha, \beta]| \sup_{[\alpha, \beta]} \{f(g(t))g'(t)\} < \infty$$

thus, $f(g(t))g'(t)$ is integrable on $[\alpha, \beta]$.

Let $\Gamma = \{t_i\}$ be a partition of $[\alpha, \beta]$ with norm $|\Gamma|$.

Then

$$\begin{aligned} \int_{\alpha}^{\beta} f(g(t))g'(t) dt &= \sum \int_{t_{i-1}}^{t_i} f(g(t))g'(t) dt \\ &= \sum f(g(t_{i-1})) \int_{t_{i-1}}^{t_i} g'(t) dt + \sum \int_{t_{i-1}}^{t_i} [f(g(t)) - f(g(t_{i-1}))]g'(t) dt \end{aligned}$$

The first term on the right equals

$$\sum f(g(t_{i-1}))[g(t_i) - g(t_{i-1})]$$

which converges to

$$\int_a^b f dg \quad \text{as } |\Gamma| \rightarrow 0$$

The second term on the right is majorized in absolute value by

$$\left[\sup_{|x-y| \leq |\Gamma|} |f(x) - f(y)| \right] \sum \int_{t_{i-1}}^{t_i} |g'| dt = \left[\sup_{|x-y| \leq |\Gamma|} \right] \int_{\alpha}^{\beta} |g'| dt$$

Since $f(g(t))$ is uniformly continuous on $[\alpha, \beta]$, the last expression tends to 0 as $|\Gamma| \rightarrow 0$.
Thus

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f(g(t)) g'(t) dt$$

2. (Exercise 7.12)

Use Jensen's inequality to prove that if $a, b \geq 0$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

More generally, show that

$$a_1 \dots a_N \leq \sum_{j=1}^N \frac{a_j^{p_j}}{p_j}$$

where $a_j \geq 0$, $p_j > 1$, $\sum_{j=1}^N \frac{1}{p_j} = 1$.

(Write $a_j = e^{x_j/p_j}$ and use the convexity of e^x .)

Proof.

Let $a_j = e^{x_j/p_j}$ and $\phi(x) = e^x$, then $x_j = \ln(a_j^{p_j})$ and ϕ is a convex function.

By using Jensen's inequality and $\sum_{j=1}^N \frac{1}{p_j} = 1$,

$$\begin{aligned} a_1 \dots a_N &= \phi \left(\sum_{j=1}^N \frac{x_j}{p_j} \right) = \phi \left(\frac{\sum_{j=1}^N \frac{1}{p_j} \cdot x_j}{\sum_{j=1}^N \frac{1}{p_j}} \right) \\ &\leq \frac{\sum_{j=1}^N \frac{1}{p_j} \phi(x_j)}{\sum_{j=1}^N \frac{1}{p_j}} = \sum_{j=1}^N \frac{e^{x_j}}{p_j} \\ &= \sum_{j=1}^N \frac{a_j^{p_j}}{p_j} \end{aligned}$$

If $N = 2$, then the inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ holds.

3. (Exercise 7.14)

Prove that ϕ is convex on (a, b) if and only if it is continuous and

$$\phi \left(\frac{x_1 + x_2}{2} \right) \leq \frac{\phi(x_1) + \phi(x_2)}{2}$$

for $x_1, x_2 \in (a, b)$.

Proof.

(\Rightarrow)

By Theorem 7.40, since ϕ is convex on (a, b) , then ϕ is continuous in (a, b) .

By the formula 7.34, if ϕ is convex in (a, b) , then

$$\phi \left(\frac{p_1 x_1 + p_2 x_2}{p_1 + p_2} \right) \leq \frac{p_1 \phi(x_1) + p_2 \phi(x_2)}{p_1 + p_2} \quad \text{holds.}$$

Set $p_1 = p_2 = \frac{1}{2}$, we have

$$\phi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{\phi(x_1) + \phi(x_2)}{2}$$

(\Leftarrow)

Suppose that ϕ is continuous and satisfies

$$\phi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{\phi(x_1) + \phi(x_2)}{2}$$

for $x_1, x_2 \in (a, b)$. Given $x_1, x_2 \in (a, b)$, then for any $t \in [x_1, x_2]$ can be written as

$$t = \left(1 - \sum_{k=1}^{\infty} \frac{a_k}{2^k}\right) x_1 + \left(\sum_{k=1}^{\infty} \frac{a_k}{2^k}\right) x_2$$

where $a_i \in [0, 1]$ for all i .

Let t_n be the n th partial sum of the series t .

We claim that

$$\phi\left(\left(1 - \sum_{k=1}^n \frac{a_k}{2^k}\right) x_1 + \left(\sum_{k=1}^n \frac{a_k}{2^k}\right) x_2\right) \leq \left(1 - \sum_{k=1}^n \frac{a_k}{2^k}\right) \phi(x_1) + \left(\sum_{k=1}^n \frac{a_k}{2^k}\right) \phi(x_2)$$

for all n is any positive integer.

For $n = 1$, then

$$\phi\left(\left(1 - \frac{a_1}{2}\right) x_1 + \frac{a_1}{2} x_2\right) \leq \left(1 - \frac{a_1}{2}\right) \phi(x_1) + \frac{a_1}{2} \phi(x_2)$$

Suppose that this inequality holds for $n = r$.

For $n = r + 1$, we have

$$\begin{aligned} & \phi\left(\left(1 - \sum_{k=1}^{r+1} \frac{a_k}{2^k}\right) x_1 + \left(\sum_{k=1}^{r+1} \frac{a_k}{2^k}\right) x_2\right) \\ & \leq \frac{1}{2} \left[\phi\left(\left(1 - a_1\right) x_1 + a_1 x_2\right) + \phi\left(\left(1 - \sum_{k=2}^{r+1} \frac{a_k}{2^k}\right) x_1 + \left(\sum_{k=2}^n \frac{a_k}{2^k}\right) x_2\right) \right] \\ & \leq \frac{1}{2} \left[\phi\left(\left(1 - a_1\right) x_1 + a_1 x_2\right) + \left(1 - \sum_{k=2}^{r+1} \frac{a_k}{2^k}\right) \phi(x_1) + \left(\sum_{k=2}^n \frac{a_k}{2^k}\right) \phi(x_2) \right] \\ & = \left(1 - \sum_{k=1}^{r+1} \frac{a_k}{2^k}\right) \phi(x_1) + \left(\sum_{k=1}^{r+1} \frac{a_k}{2^k}\right) \phi(x_2) \end{aligned}$$

By the induction, then we have

$$\phi(t) = \lim_{n \rightarrow \infty} \phi(t_n) \leq \left(1 - \sum_{k=1}^{\infty} \frac{a_k}{2^k}\right) \phi(x_1) + \left(\sum_{k=1}^{\infty} \frac{a_k}{2^k}\right) \phi(x_2)$$

since ϕ is continuous and the above inequality as $n \rightarrow \infty$, therefore, ϕ is convex.

4. (Exercise 7.15)

Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\phi(x) = \int_a^x f(t)dt + \phi(a)$ in (a, b) and f is monotone increasing, then ϕ is convex in (a, b) . (Use Exercise 14.)

Proof.

Given any interval $[x_1, x_2] \in (a, b)$, since f is monotone increasing, then we have

$$\begin{aligned}
 \frac{\phi(x_1) + \phi(x_2)}{2} - \phi\left(\frac{x_1 + x_2}{2}\right) &= \frac{[\phi(x_2) - \phi(\frac{x_1+x_2}{2})] - [\phi(\frac{x_1+x_2}{2}) - \phi(x_1)]}{2} \\
 &= \frac{1}{2} \int_{\frac{x_1+x_2}{2}}^{x_2} f(x) dx - \frac{1}{2} \int_{x_1}^{\frac{x_1+x_2}{2}} f(x) dx \\
 &\geq \frac{1}{2} \left[\left(x_2 - \frac{x_1+x_2}{2} \right) f\left(\frac{x_1+x_2}{2}\right) - \left(\frac{x_1+x_2}{2} - x_1 \right) f\left(\frac{x_1+x_2}{2}\right) \right] \\
 &= \left(\frac{x_2 - x_1}{4} \right) \left(f\left(\frac{x_1+x_2}{2}\right) - f\left(\frac{x_1+x_2}{2}\right) \right) \\
 &= 0
 \end{aligned}$$

By Exercise 7.14, ϕ is convex since $\int_a^x f(t) dt$ is continuous.

5. (Exercise 7.16)

Show that the formula

$$\int_{-\infty}^{+\infty} f g' = - \int_{-\infty}^{+\infty} f' g$$

for integration by parts may not hold if f is of bounded variation on $(-\infty, +\infty)$ and g is infinitely differentiable with compact support. (Let f be the Cantor–Lebesgue function on $[0, 1]$, and let $f = 0$ elsewhere.)

Proof.

Follow the hint, let f be the Cantor–Lebesgue function on $[0, 1]$ and $f = 0$ elsewhere. Since $f \in [0, 1]$ on $[0, 1]$ and $f = 0$ elsewhere, then f is of bounded variation on $(-\infty, +\infty)$.

Let

$$g(x) = \begin{cases} e^{\frac{1}{(x-1)^2-1}} & \text{if } x \in (0, 2) \\ 0 & \text{otherwise} \end{cases}$$

Then g vanishes outside the bounded set $[0 + \epsilon, 2 - \epsilon]$ where $\epsilon > 0$ and $g^{(n)} = f_n(x) e^{\frac{1}{(x-1)^2-1}}$, hence, g is infinitely differentiable with compact support.

Since g is also increasing on $(0, 1)$, then

$$\int_{-\infty}^{+\infty} f \cdot g' = \int_0^1 f \cdot g' \geq \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{1}{2} \cdot g' > 0$$

But $f' = 0$ since f is the Cantor–Lebesgue function on $[0, 1]$ and $f = 0$ elsewhere, then $\int_{-\infty}^{+\infty} f' g = 0$.

Thus,

$$\int_{-\infty}^{+\infty} f \cdot g' > 0 = \int_{-\infty}^{+\infty} f' g$$

and then the exercise follows.

6. (Exercise 7.17)

A sequence $\{\phi_k\}$ of set functions is said to be *uniformly absolutely continuous* if given $\epsilon > 0$, there exists $\delta > 0$ such that if E satisfies $|E| < \delta$, then $|\phi_k(E)| < \epsilon$ for all k . If $\{f_k\}$ is a sequence of integrable functions on $(0, 1)$ which converges pointwise a.e. to an integrable f , show that $\int_0^1 |f - f_k| \rightarrow 0$ if and only if the indefinite integrals of the f_k are uniformly absolutely continuous. (cf. Exercise 23 of Chapter 10.)

Proof.

Given $\epsilon > 0$.

If the indefinite integrals of the f_k are uniformly absolutely continuous and $f \in L(0, 1)$, there exists $\delta > 0$ such that if $E \subseteq (0, 1)$ satisfies $|E| < \delta$, then

$$\left| \int_E f_k \right| < \epsilon$$

for all k and

$$\int_E |f| < \epsilon$$

By Egorov's theorem, there is a closed subset F of $(0, 1)$ such that

$$|(0, 1) - F| < \delta$$

and $\{f_k\}$ converges uniformly to f on F .

Then choose $M > 0$ such that for all $k \geq M$, we have

$$\begin{aligned} \int_0^1 |f - f_k| &= \int_F |f - f_k| + \int_{(0,1) \setminus F} |f - f_k| \\ &< \epsilon |F| + \int_{(0,1) \setminus F} |f| + \int_{(0,1) \setminus F} |f_k| \\ &< \epsilon + \epsilon + \int_{(0,1) \setminus F} f_k^+ + \int_{(0,1) \setminus F} f_k^- \\ &< 4\epsilon \end{aligned}$$

So

$$\int_0^1 |f - f_k| \rightarrow 0$$

Conversely, given $\epsilon > 0$.

For all k , since the indefinite integral of f_k is absolutely continuous, there exists $\delta_k > 0$ such that for any $E \subseteq (0, 1)$ with $|E| < \delta_k$, then we have

$$\left| \int_E f_k \right| < \epsilon$$

Since the indefinite integral of f is absolutely continuous, choose $M > 0$ and $\delta > 0$ such that for any $|E| < \delta$ and $k \geq M + 1$, then we have

$$\left| \int_E f_k \right| \leq \int_E |f_k| \leq \int_E |f_k - f| + \int_E |f| < \epsilon$$

Let $\delta' = \min\{\delta, \delta_1, \delta_2, \dots, \delta_M\}$, then

$$\left| \int_E f_k \right| < \epsilon$$

for all k .