Calculus 1 9/19 Note
Module Class 07

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Section 2.1: The Tangent and Velocity Problem

In this section, we learn that

1. What's the secant line and the tangent line?
2. How to estimate the slope?

Section 2.2: The Limit of a Function

Finite Limits:

Intuitive Definition of Limit

Suppose that $f(x)$ is defined when x is near the number a. Then we write

$$\lim_{x \to a} f(x) = L$$

and also we say

"the limit of $f(x)$, as x approaches a, equals L".

However, $x \to a$ only means that x is very closed to a but **NOT** equals to a. That's

$$\lim_{x \to a} f(x) \text{ may not equal to } f(a).$$

Example:

Find the value of the following limit.

1.
$$\lim_{x \to 1} \frac{2 - 2x^2}{x - 1}$$

Sol.
If $x \neq 1$, then

$$\frac{2 - 2x^2}{x - 1} = \frac{-2(x-1)(x+1)}{x-1} = -2x - 2.$$

So, we have

$$\lim_{x \to 1} \frac{2 - 2x^2}{x - 1} = \lim_{x \to 1} -2x - 2 = -4.$$

2.

$$\lim_{x \to 0} \cos\left(\frac{\pi}{x}\right)$$

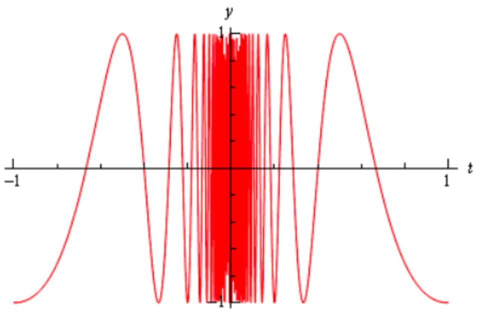
Sol.
Let $f(x) = \cos\left(\frac{\pi}{x}\right)$, then

$$f\left(\frac{1}{2001}\right) = -1, \quad f\left(\frac{2}{2001}\right) = 0, \quad f\left(\frac{4}{4001}\right) = \frac{\sqrt{2}}{2}.$$

Since this function $f(x)$ is concussive, we cannot find out the value of its limit.

So

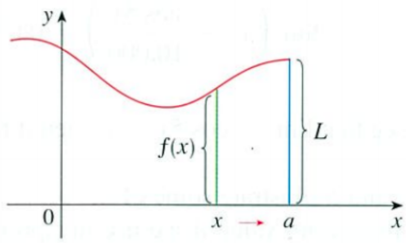
$$\lim_{x \to 0} \cos\left(\frac{\pi}{x}\right) = \text{DNE}$$



One-Sided Limits

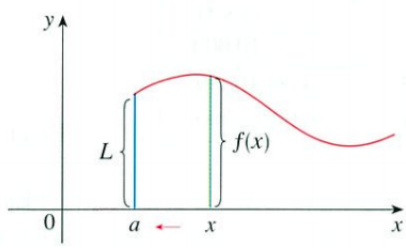
Left-hand limit of $f(x)$:

$$\lim_{x \to a^-} f(x) = L$$



Right-hand limit of $f(x)$:

$$\lim_{x \to a^+} f(x) = L$$



Property

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

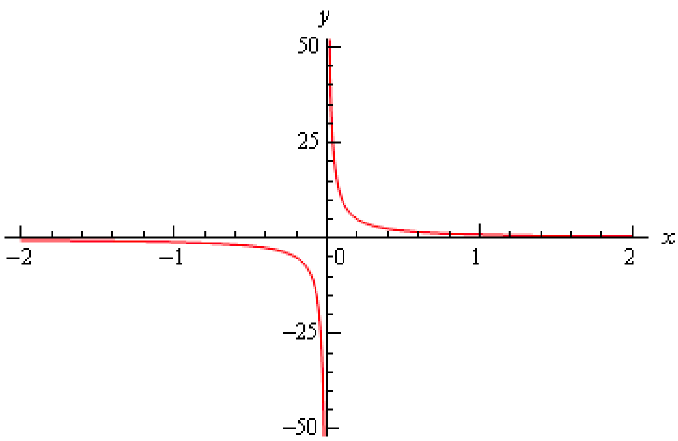
Infinite Limits:
Let's take some examples to introduce the concept of infinite limits.

Example:

Consider the following limits.

$$(a) \lim_{x \to 0^+} \frac{1}{x} \qquad (b) \lim_{x \to 0^-} \frac{1}{x} \qquad (c) \lim_{x \to 0} \frac{1}{x}$$

Sol.



Consider

$$\frac{1}{1} > 0, \quad \frac{1}{\frac{1}{2}} > 0, \quad \frac{1}{\frac{1}{3}} > 0, \quad \frac{1}{\frac{1}{4}} > 0, \ldots$$

and

$$\frac{1}{-1} < 0, \quad \frac{1}{-\frac{1}{2}} < 0, \quad \frac{1}{-\frac{1}{3}} < 0, \quad \frac{1}{-\frac{1}{4}} < 0, \ldots$$

So

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Therefore, by above property, we know that

$$\lim_{x \to 0} \frac{1}{x} = \text{DNE}.$$

Note:
In the above examples, we also know that

$$x = 0 \text{ is a vaertical asymptote.}$$

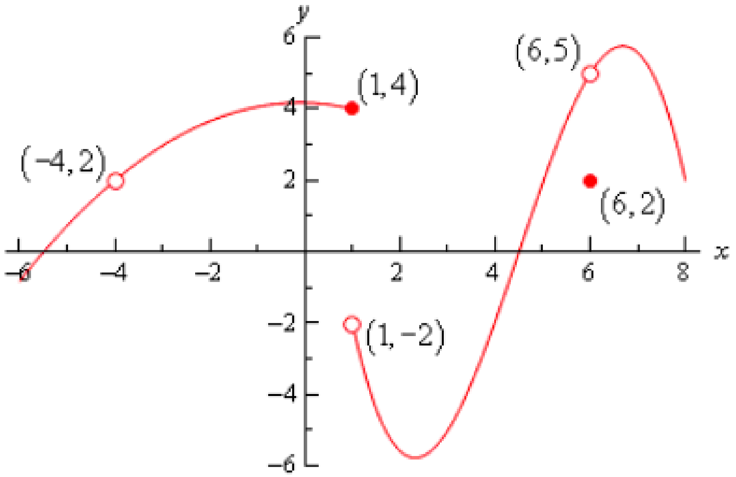
Definition

The vertical line $x = a$ is called aa **vertical asymptote** of the curve $y = f(x)$ if

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \quad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \quad \lim_{x\to a^-} f(x) = -\infty \quad \lim_{x\to a^+} f(x) = -\infty$$

Exercise:

1. Given the graph of $f(x)$.



Compute each of the following limits.

$$(a) f(-4) \quad (b) \lim_{x\to -4^-} f(x) \quad (c) \lim_{x\to -4^+} f(x) \quad (d) \lim_{x\to -4} f(x)$$
$$(e) f(1) \quad (f) \lim_{x\to 1^-} f(x) \quad (g) \lim_{x\to 1^+} f(x) \quad (h) \lim_{x\to 1} f(x)$$
$$(i) f(6) \quad (j) \lim_{x\to 6^-} f(x) \quad (k) \lim_{x\to 6^+} f(x) \quad (l) \lim_{x\to 6} f(x)$$

2. Find the following limits.

$$(a) \lim_{x\to 2} \frac{x^2+4x-12}{x^2-2x} \qquad (b) \lim_{h\to 0} \frac{2(-3+h)^2-18}{h} \qquad (c) \lim_{t\to 4} \frac{t-\sqrt{3t+4}}{4-t}$$

3. Given the function,

$$g(y) = \begin{cases} y^2+5 & \text{if } y < -2 \\ 1-3y & \text{if } y \geq -2 \end{cases}$$

Compute the following limits.

$$(a) \lim_{y\to 6} g(y) \quad (b) \lim_{x\to -2} g(y)$$

4. Evaluate each of the following limits.

$$(a) \lim_{x\to 4^+} \frac{3}{(4-x)^3} \quad (b) \lim_{x\to 4^-} \frac{3}{(4-x)^3} \quad (c) \lim_{x\to 4} \frac{3}{(4-x)^3}$$

Sol.

1. (a)DNE $(b)2$ $(c)2$ $(d)2$ $(e)4$ $(f)4$ $(g)-2$ (h)DNE $(i)2$ $(j)5$ $(k)5$ $(l)5$
2. $(a)\,4$ $(b)\,-12$ $(c)\,-\frac{5}{8}$
3. $(a)\,-17$ (b) DNE
4. $(a)\,-\infty$ $(a)\,\infty$ (a) DNE

Section 2.3: Calculating Limits Using the Limit Laws

Limit Laws

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $$\lim_{x\to a}\,[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$
2. $$\lim_{x\to a}\,[c\,f(x)] = c\lim_{x\to a} f(x)$$
3. $$\lim_{x\to a}\,[f(x)\,g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$
4. $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$$
5. $$\lim_{x\to a}\,[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n \quad \text{where } n \text{ is a positive integer}$$
6. $$\lim_{x\to a} c = c \quad \text{and} \quad \lim_{x\to a} x = a$$
7. $$\lim_{x\to a} x^n = a^n \quad \text{and} \quad \lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},\ \ (a > 0) \quad \text{where } n \text{ is a positive.}$$
8. $$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)},\ \ (\lim_{x\to a} f(x) > 0) \quad \text{where } n \text{ is a positive.}$$

Example:

1.
$$\lim_{x\to\infty} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}}$$

2.
$$\lim_{x\to-\infty} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}}$$

3.
$$\lim_{t\to-\infty} \frac{e^{6t} - 4e^{-6t}}{2e^{3t} - 5e^{-9t} + e^{-3t}}$$

Sol.

1.
$$\begin{aligned}\lim_{x\to\infty} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}} &= \lim_{x\to\infty} \frac{e^{4x}(6 - e^{-6x})}{e^{4x}(8 - e^{-2x} + 3e^{-5x})} = \lim_{x\to\infty} \frac{6 - e^{-6x}}{8 - e^{-2x} + 3e^{-5x}} \\ &= \frac{6-0}{8-0+0} = \frac{3}{4}\end{aligned}$$

2.
$$\begin{aligned}\lim_{x\to-\infty} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}} &= \lim_{x\to-\infty} \frac{e^{-x}(6e^{5x} - e^{-x})}{e^{-x}(8e^{5x} - e^{3x} + 3)} = \lim_{x\to-\infty} \frac{6e^{5x} - e^{-x}}{8e^{5x} - e^{3x} + 3} \\ &= \frac{0-\infty}{0-0+3} = -\infty\end{aligned}$$

3.
$$\begin{aligned}\lim_{t\to-\infty} \frac{e^{6t} - 4e^{-6t}}{2e^{3t} - 5e^{-9t} + e^{-3t}} &= \lim_{t\to-\infty} \frac{e^{-9t}\left(e^{15t} - 4e^{3t}\right)}{e^{-9t}\left(2e^{12t} - 5 + e^{6t}\right)} = \lim_{t\to-\infty} \frac{e^{15t} - 4e^{3t}}{2e^{12t} - 5 + e^{6t}} \\ &= \frac{0-0}{0-5+0} = 0\end{aligned}$$

Theroem

If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then
$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

The Squeeze Theroem

If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$
then
$$\lim_{x\to a} g(x) = L$$

Example:
Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Proof.
Since $-1 \le \sin\frac{1}{x} \le 1$, then

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2.$$

Also,

$$\lim_{x\to 0} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0} -x^2 = 0.$$

By squeeze theorem, we know that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

Exercise:

1. Evaluate the following limits, if it exists.

$$(a)\lim_{x\to 2}\frac{x^2+x-6}{x-2} \qquad (b)\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right) \qquad (c)\lim_{h\to 0}\frac{(x+h)^3-x^3}{h} \qquad (d)\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)$$

2. Prove that $\lim_{x\to 0} x^4 \cos\frac{2}{x} = 0$.

Sol.

1. $(a)\,5 \quad (b)\,-\frac{1}{2} \quad (c)\,3x^2 \quad (d)\,$DNE

2. Consider squeeze theorem to prove.

Section 2.4: The Precise Definition of a Limit

We will skip this section, because this section is optional in our schedule.

Definition

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of** $f(x)$ **as** x **approaches** a **is** L, and we write

$$\lim_{x\to a} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x-a| < \delta \quad \text{then} \quad |f(x)-L| < \epsilon$$
