Calculus 2 12/12 Note
Module Class 07

Yueh-Chou Lee

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Section 7.8: Improper Integrals

Definition of an Improper Integral of Type 1

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1. If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

2. If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

3. If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Example:
Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx$.

   Sol.

$$\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to\infty} \int_0^t \frac{x^2}{\sqrt{1+x^3}}\,dx = \lim_{t\to\infty} \left[\frac{2}{3}\sqrt{1+x^3}\right]_{x=0}^t = \lim_{t\to\infty} \left(\frac{2}{3}\sqrt{1+t^3} - \frac{2}{3}\right) = \infty$$

   Thus,

$$\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx \quad \text{is divergent.}$$

2. $\int_{-\infty}^{0} \frac{x}{x^4+4}\,dx$.

Sol.

$$\int_{-\infty}^{0} \frac{x}{x^4+4}\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{x}{x^4+4}\,dx = \lim_{t\to-\infty}\frac{1}{4}\int_{t}^{0} \frac{x}{(x^2/2)^2+1}\,dx$$
$$= \lim_{t\to-\infty}\frac{1}{4}\left[\tan^{-1}\left(\frac{x^2}{2}\right)\right]_{x=t}^{0} = \lim_{t\to-\infty}\frac{1}{4}\left[0-\tan^{-1}\left(\frac{t^2}{2}\right)\right]$$
$$= -\frac{\pi}{8}$$

Thus,

$$\int_{-\infty}^{0} \frac{x}{x^4+4}\,dx \quad \text{is convergent.}$$

Property

$$\int_{1}^{\infty} \frac{1}{x^p}\,dx \quad \text{is convergent if } p>1 \text{ and divergent if } p\le 1.$$

Example:
Find the values of p for which the integral converges and evaluate the integral for those values of p.

1. $\int_{0}^{1} \frac{1}{x^p}\,dx$.

Sol.
If $p=1$, then

$$\int_{0}^{1} \frac{dx}{x^p} = \lim_{t\to0^+}\int_{t}^{1} \frac{dx}{x} = \lim_{t\to0^+}[\ln x]_{x=t}^{1} = \infty$$

Thus,

$$\int_{0}^{1} \frac{dx}{x^p} \quad \text{is divergent when } p=1.$$

If $p\neq 1$, then

$$\int_{0}^{1} \frac{dx}{x^p} = \lim_{t\to0^+}\int_{t}^{1} \frac{dx}{x^p} = \lim_{t\to0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_{x=t}^{1} = \lim_{t\to0^+}\frac{1}{1-p}\left[1-\frac{1}{t^{p-1}}\right].$$

If $p>1$, then $p-1>0$, so

$$\frac{1}{t^{p-1}} \to \infty \quad \text{as } t\to 0^+ \text{ and the integral diverges.}$$

If $p<1$, then $p-1<0$, so

$$\frac{1}{t^{p-1}} \to 0 \quad \text{as } t\to 0^+ \text{ and } \int_{0}^{1} \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to0^+}(1-t^{1-p})\right] = \frac{1}{1-p}.$$

So the integral converges if and only if $p<1$, and in that case its value is $\frac{1}{1-p}$.

Definition of an Improper Integral of Type 2

1. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limits exists (as a finite number).

2. If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limits exists (as a finite number).

The improper integrals $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

3. If f has a discontinuous at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example:
Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

1. $\int_0^1 \frac{3}{x^5}\,dx$.

Sol.

$$\int_0^1 \frac{3}{x^5}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{3}{x^5}\,dx = \lim_{t\to 0^+} \left[-\frac{3}{4x^4}\right]_{x=t}^1 = \lim_{t\to 0^+} \left(-\frac{3}{4} + \frac{3}{4t^4}\right) = \infty$$

Thus,

$$\int_0^1 \frac{3}{x^5}\,dx \quad \text{is divergent.}$$

2. $\int_0^3 \frac{dx}{x^2-6x+5}$.

Sol.

$$\int_0^3 \frac{dx}{x^2-6x+5} = \int_0^3 \frac{dx}{(x-1)(x-5)} = \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}$$

So

$$\frac{1}{(x-1)(x-5)} = \frac{A}{x-1} + \frac{B}{x-5} \;\Rightarrow\; 1 = A(x-5) + B(x-1)$$

Set $x = 5$ to get $1 = 4B$, so $B = \dfrac{1}{4}$. Set $x = 1$ to get $1 = -4A$, so $A = -\dfrac{1}{4}$.

Thus

$$\begin{aligned}
\int_0^1 \frac{dx}{(x-1)(x-5)} &= \lim_{t\to 1^-} \int_0^t \left(\frac{-1/4}{x-1} + \frac{1/4}{x-5} \right) dx \\
&= \lim_{t\to 1^-} \left[-\frac{1}{4} \ln|x-1| + \frac{1}{4} \ln|x-5| \right]_{x=0}^{t} \\
&= \lim_{t\to 1^-} \left[\left(-\frac{1}{4} \ln|t-1| + \frac{1}{4} \ln|t-5| \right) - \left(-\frac{1}{4} \ln|-1| + \frac{1}{4} \ln|-5| \right) \right] \\
&= \infty \qquad \left[\text{since } \lim_{t\to 1^-} \left(-\frac{1}{4} \ln|t-1| \right) = \infty \right]
\end{aligned}$$

Since $\int_0^1 \frac{dx}{(x-1)(x-5)}$ is divergent, $\int_0^3 \frac{dx}{x^2-6x+5}$ is divergent.

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Example:
Use the Comparison Theorem to determine whether the integral is convergent or divergent.

1. $\int_0^\infty \frac{x}{x^3+1}\,dx$.

Sol.
For $x > 1$, $\dfrac{x}{x^3+1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$.

$$\int_1^\infty \frac{x}{x^3+1}\,dx < \int_1^\infty \frac{1}{x^2}\,dx \quad \text{is convergent.} \qquad (\text{by property } p = 2 > 1)$$

For $0 < x < 1$, $\dfrac{x}{x^3+1} < 1$.

$$\int_0^1 \frac{x}{x^3+1}\,dx < \int_0^1 dx = 1 \quad \text{is convergent.}$$

By comparison theorem, $\int_0^1 \frac{x}{x^3+1}\,dx$ and $\int_1^\infty \frac{x}{x^3+1}\,dx$ are convergent, so

$$\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^\infty \frac{x}{x^3+1}\,dx \quad \text{is also convergent.}$$

2. $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$.

Sol.

For $0 < x \le 1$, $\dfrac{\sec^2 x}{x\sqrt{x}} > \dfrac{1}{x^{3/2}}$.

$$\int_0^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\left[-2x^{-1/2}\right]_{x=t}^1 = \lim_{t\to 0^+}\left(-2+\frac{2}{\sqrt{t}}\right) = \infty,$$

so $\int_0^1 x^{-3/2}\,dx$ is divergent, and by comparison theorem, then

$$\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx \quad \text{is divergent.}$$

Exercise:

1. Evaluate the integral or show that it is divergent.
$$\int_1^\infty \frac{\ln x}{x^4}\,dx$$
2. Evaluate the integral or show that it is divergent.
$$\int_2^6 \frac{x}{\sqrt{x-2}}\,dx$$
3. Evaluate the integral or show that it is divergent.
$$\int_0^\infty \frac{x\ \arctan x}{(1+x^2)^2}\,dx$$
4. Use the Comparison Theorem to determine whether the integral is convergent or divergent.
$$\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$$

Sol.

1. $\dfrac{1}{9}$ 2. $\dfrac{40}{3}$ 3. $\dfrac{\pi}{8}$ 4. convergent.