1. (13pts)

   (a) Sketch the curve $r = 3 - 4\sin^2\dfrac{\theta}{2}$

   (b) Compute the area of the region that is inside the larger loop of the curve $r = 1 + 2\cos\theta$ and outside the smaller loop of the curve $r = 1 + 2\cos\theta$

   Solution

   (a) Since $3 - 4\sin^2\dfrac{\theta}{2} = 1 + 2(1 - 2\sin^2\dfrac{\theta}{2}) = 1 + 2\cos\theta$, $0 = 1 + 2\cos\theta \Rightarrow \cos\theta = \dfrac{-1}{2} \Rightarrow \theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

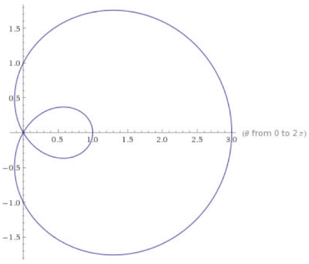

Figure 1: Problem 1(a)

   (b)

$$\begin{aligned}
A &= A_{total} - A_{inner\ circle} \\
&= \int_0^{\frac{2\pi}{3}} (1 + 2\cos\theta)^2 \, d\theta - \int_{\frac{2\pi}{3}}^{\pi} (1 + 2\cos\theta)^2 \, d\theta \\
&= \int_0^{\frac{2\pi}{3}} (1 + 4\cos\theta + 4\cos^2\theta) \, d\theta - \int_{\frac{2\pi}{3}}^{\pi} (1 + 4\cos\theta + 4\cos^2\theta) \, d\theta \\
&= (3\theta + 4\sin\theta + \sin 2\theta)\Big|_0^{\frac{2\pi}{3}} - (3\theta + 4\sin\theta + \sin 2\theta)\Big|_{\frac{2\pi}{3}}^{\pi} = \pi + 3\sqrt{3}
\end{aligned}$$

   (5 pts for (a), 2 pts for finding r=0, 3 pts for correct sketch. 8 pts for (b), 5 pts for the equation and 3 pts for correct answer.)

2. (20pts) Find the area of the region that lies inside the curve $r = 1 + \cos\theta$ but outside the curves $r = 2\cos\theta$ and $r = -\cos\theta$

   Solution

   Two of the intersection points for $r = 1 + \cos\theta$ and $r = -\cos\theta$ can be found by considering the equation $1 + \cos\theta = -\cos\theta$, which yields $\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$. Therefore, $(\dfrac{1}{2}, \dfrac{2\pi}{3})$ and $(\dfrac{1}{2}, \dfrac{4\pi}{3})$ (in polar coordinates) are two points of intersection of the two curves (and the remaining intersection point is the pole).

   By symmetry, we only need to compute the area $A_2 + A_3$.

   The area A_2 is given by

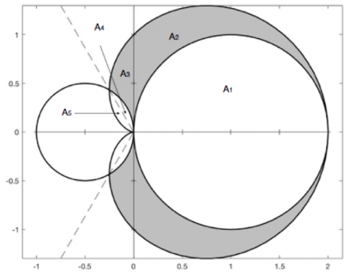

Figure 2: Problem 2

$$A_2 = (A_2 + A_1) - A_1 = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos\theta)^2\,\mathrm{d}\theta - \int_0^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2\,\mathrm{d}\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + 2\cos\theta - 3\cos^2\theta\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + 2\cos\theta - 3\frac{(1+\cos 2\theta)}{2}\,\mathrm{d}\theta$$
$$= \frac{1}{2}\left(-\frac{\theta}{2} + 2\sin\theta - \frac{3\sin 2\theta}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = 1 - \frac{\pi}{8}$$

The area A_3 is given by

$$A_3 = (A_3 + A_4) - A_4 = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2}(1+\cos\theta)^2\,\mathrm{d}\theta - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{1}{2}(-\cos\theta)^2\,\mathrm{d}\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 1 + 2\cos\theta\,\mathrm{d}\theta$$
$$= \frac{1}{2}(\theta + 2\sin\theta)\Bigg|_{\frac{\pi}{2}}^{\frac{2\pi}{3}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

Answer $= 2(A_2 + A_3) = 2(1 - \frac{\pi}{8} + \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1) = \sqrt{3} - \frac{\pi}{12}$

(4 pts for calculating intersection of the two curves, 6 pts for the equation for calculating A_2, 6 pts for the equation for calculating A_3, and 4 pts for the correct final answer.)

3. (12pts) Find the arc length of the curve. $x = \cos t + \ln\tan\frac{t}{2}, y = \sin t, \frac{\pi}{4} \le t \le \frac{3\pi}{4}$

Solution

$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t + \frac{\sec^2\frac{t}{2}}{2\tan\frac{t}{2}} = -\sin t + \csc t$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \cos t$. Thus,

$$\textit{Arc length} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$
$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{\csc^2 t - 1}\,\mathrm{d}t = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} |\cot t|\,\mathrm{d}t = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot t\,\mathrm{d}t - \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \cot t\,\mathrm{d}t$$
$$= \ln|\sin t|\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \ln|\sin t|\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \ln 2$$

(3 pts for correct calculating $\frac{\mathrm{d}x}{\mathrm{d}t}$, 3 pts for correct calculating $\frac{\mathrm{d}y}{\mathrm{d}t}$, 4 pts for calculating arc length, and 2 pts for the correct answer.)

4. (25 pts) A curve called the **folium of Descartes** is defined by the parametric equations

$$x = \frac{3t}{1+t^3} \qquad y = \frac{3t^2}{1+t^3}$$

Cont.

(a) Show that if (a,b) lies on the curve, then so does (b,a); that is, the curve is symmetric with respect to the line $y=x$. Where does the curve intersect this line?

(b) Find the points on the curve where the tangenet lines are horizontal or vertical.

(c) Show that the line $y=-x-1$ is a slant asymptote.

(d) Sketch the curve.

(e) Show that a Cartesian equation of this curve is $x^3+y^3=3xy$.

(f) Show that the polar equation can be written in the form

$$r=\frac{3\sec\theta\tan\theta}{1+\tan^3\theta}$$

(g) Find the area enclosed by the loop of this curve.

(h) (Optional) Show that the area of the loop is the same as the area that lies between the asymptote and the infinite branches of the curve.

Solution

(a) If (a,b) lies on the curve, $\exists t_1\in\mathrm{R}$ such that $\dfrac{3t_1}{1+t_1^3}=a$ and $\dfrac{3t_1^2}{1+t_1^3}=b$.
If $t_1=0$, the point is $(0,0)$, which lies on the line $y=x$.
If $t_1\neq 0$, then the point corresponding to $t=\dfrac{1}{t_1}$ is given by $x=\dfrac{3(1/t_1)}{1+(1/t_1)^3}=\dfrac{3t_1^2}{t_1^3+1}=b$, $y=\dfrac{3(1/t_1)^2}{1+(1/t_1)^3}=\dfrac{3t_1}{t_1^3+1}=a$.
So (b,a) also lies on the curve. The curve intersects the line $y=x$ when $\dfrac{3t}{1+t^3}=\dfrac{3t^2}{1+t^3}\Rightarrow t=t^2\Rightarrow$ $t=0$ or 1, so the points are $(0,0)$ and $(\frac{3}{2},\frac{3}{2})$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{(1+t^3)(6t)-3t^2(3t^2)}{(1+t^3)^2}=\dfrac{6t-3t^4}{(1+t^3)^2}=0$ when $6t-3t^4=3t(2-t^3)=0\Rightarrow t=0$ or $\sqrt[3]{2}$, so there are horizontal tangents at $(0,0)$ and $\sqrt[3]{2},\sqrt[3]{4}$. Using the symmetry from part (a), we see that there are vertical tangents at $(0,0)$ and $(\sqrt[3]{4},\sqrt[3]{2})$.

(c) Notice that as $t\to -1^+$, we have $x\to-\infty$ and $y\to\infty$. As $t\to -1^-$, we have $x\to\infty$ and $y\to-\infty$. Also, $y-(-x-1)=y+x+1=\dfrac{3t+3t^2+(1+t^3)}{1+t^3}=\dfrac{(t+1)^3}{1+t^3}=\dfrac{(t+1)^2}{t^2-t+1}\to 0$ as $t\to -1$. So $y=-x-1$ is a slant asymptote.

(d) $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{3(1+t^3)-3t(3t^2)}{(1+t^3)^2}=\dfrac{3-6t^3}{(1+t^3)^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{6t-3t^4}{(1+t^3)^2}$.
So $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\dfrac{6t-3t^4}{3-6t^3}=\dfrac{t(2-t^3)}{1-2t^3}$. Also, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{2(1+t^3)^4}{3(1-2t^3)^3}>0\Leftrightarrow t<\dfrac{1}{\sqrt[3]{2}}$.
Thus, the curve is concave upward there and has a minimum point at $(0,0)$ and a maximum point at $(\sqrt[3]{2},\sqrt[3]{4})$. Sketch the curve as following.

(e) $x^3+y^3=\left(\dfrac{3t}{1+t^3}\right)^3+\left(\dfrac{3t^2}{1+t^3}\right)^3=\dfrac{27t^3(1+t^3)}{(1+t^3)^3}=\dfrac{27t^3}{(1+t^3)^2}=3\left(\dfrac{3t}{1+t^3}\right)\left(\dfrac{3t^2}{1+t^3}\right)=3xy$, so $x^3+y^3=3xy$.

(f) Substitute $x=r\cos\theta$, $y=r\sin\theta$. Then $x^3+y^3=3xy\Rightarrow r^3\cos^3\theta+r^3\sin^3\theta=3r^2\cos\theta\sin\theta$. For $r\neq 0$, $r=\dfrac{3\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}=\dfrac{3\sec\theta\tan\theta}{1+\tan^3\theta}$.

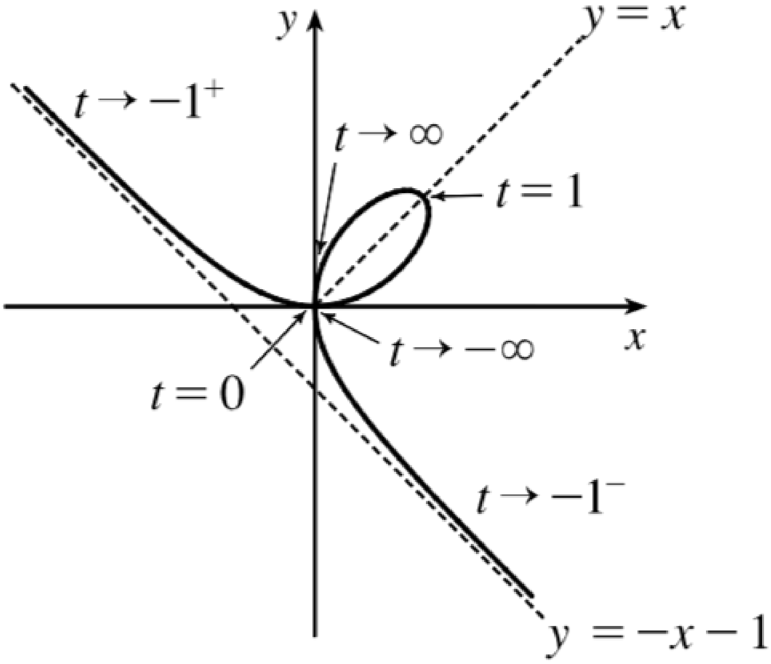

Figure 3: Problem 4(d)

(g) The loop corresponds to $\theta \in (0, \frac{\pi}{2})$, so its area is

$$A = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{3\sec\theta\tan\theta}{1+\tan^3\theta}\right)^2 \mathrm{d}\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta}{(1+\tan^3\theta)^2}\,\mathrm{d}\tan\theta$$
$$= \lim_{b\to\frac{\pi}{2}} \frac{9}{2}\left[\frac{-1}{3}(1+\tan^3\theta)^{-1}\right]\Big|_0^b = \frac{3}{2}$$

(h) By symmetry, the area between the folium and the line $y = -x - 1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the forth quadrant.
The area in the third quadrant is $\frac{1}{2}$ and since $y = -x-1 \Rightarrow r\sin\theta = -r\cos\theta - 1 \Leftrightarrow r = -\frac{1}{\sin\theta + \cos\theta}$, the area in the forth quadrant is

$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \left[\left(-\frac{1}{\sin\theta+\cos\theta}\right)^2 - \left(\frac{3\sec\theta\tan\theta}{1+\tan^3\theta}\right)^2\right] \mathrm{d}\theta$$
$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \left[\left(\frac{1}{\tan\theta+1}\right)^2 - \left(\frac{3\tan\theta}{1+\tan^3\theta}\right)^2\right] \mathrm{d}\tan\theta = \frac{1}{2}$$

Therefore, the total area is $\frac{1}{2} + 2\left(\frac{1}{2}\right) = \frac{3}{2}$.

(3 pts for (a) for finding the intersect points, 4 pts for (b), 2 pts for finding horizontal and 2 pts for finding vertical tangent lines, 3 pts for (c) for correct showing slant asymptote, 3 pts for (d), 3 pts for sketch the curve, 4 pts for (e) for correct computing Cartesian equation, 4 pts for (f) for correct showing the polar equation, 4 pts for (g) for correct calculating enclosed area.)

5. (16pts) Solve the differential equation.

(a) $\dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{1+t^4}{ut^2+u^4t^2}$. (b) $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dfrac{t\sec\theta}{\theta e^{t^2}}$.

Solution

(a) $\dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{1+t^4}{t^2(u+u^4)} \Rightarrow (u+u^4)\,\mathrm{d}u = \dfrac{1+t^4}{t^2}\,\mathrm{d}t$
$\Rightarrow \displaystyle\int (u+u^4)\,\mathrm{d}u = \int \frac{1+t^4}{t^2}\,\mathrm{d}t \Rightarrow \frac{u^2}{2}+\frac{u^5}{5} = \frac{-1}{t}+\frac{t^3}{3}+C$

(b) $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dfrac{t\sec\theta}{\theta e^{t^2}} \Rightarrow \theta\cos\theta\,\mathrm{d}\theta = te^{-t^2}\,\mathrm{d}t$
$\Rightarrow \displaystyle\int \theta\cos\theta\,\mathrm{d}\theta = \int te^{-t^2}\,\mathrm{d}t \Rightarrow \theta\sin\theta+\cos\theta = \frac{-1}{2}e^{-t^2}+C$

(8 pts for each problem, 4 pts for separate variables and 4 pts for correct procedure)

6. (24pts) Solve the initial-value problem.

(a) $xy' = y + x^2\sin x,\ y(\pi) = 0.$ (b) $(x^2+1)\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3x(y-1) = 0,\ y(0) = 2.$

Solution

(a) $xy' = y + x^2\sin x \Rightarrow y' - \dfrac{1}{x}y = x\sin x$.
Thus, $I(x) = e^{\int \frac{-1}{x}\,\mathrm{d}x} = e^{-\ln x} = \dfrac{1}{x}$.
Multiplying by $\dfrac{1}{x}$ gets $\dfrac{1}{x}y' - \dfrac{1}{x^2}y = \sin x \Rightarrow \left(\dfrac{y}{x}\right)' = \sin x \Rightarrow \dfrac{y}{x} = -\cos x + c \Rightarrow y = -x\cos x + Cx$.
Since $y(\pi) = 0$, $\pi + C\pi = 0 \Rightarrow C = -1$. We get $y = -x\cos x - x$

(b) $(x^2+1)\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3x(y-1) = 0 \Rightarrow (x^2+1)y' + 3xy = 3x \Rightarrow y' + \dfrac{3x}{x^2+1}y = \dfrac{3x}{x^2+1}$.
Thus, $I(x) = e^{\int \frac{3x}{x^2+1}\,\mathrm{d}x} = e^{\frac{3}{2}\ln|x^2+1|} = (x^2+1)^{\frac{3}{2}}$.
Multiplying by $(x^2+1)^{\frac{3}{2}}$ gets $(x^2+1)^{\frac{3}{2}}y' + 3x(x^2+1)^{\frac{1}{2}}y = 3x(x^2+1)^{\frac{1}{2}} \Rightarrow \left[(x^2+1)^{\frac{3}{2}}y\right]' = 3x(x^2+1)^{\frac{1}{2}} \Rightarrow (x^2+1)^{\frac{3}{2}}y = \displaystyle\int 3x(x^2+1)^{\frac{1}{2}}\,\mathrm{d}x = (x^2+1)^{\frac{3}{2}} + C \Rightarrow y = 1 + C(x^2+1)^{\frac{-3}{2}}$.
Since $y(0) = 2$, $2 = 1 + C \Rightarrow C = 1$. We get $y = 1 + (x^2+1)^{\frac{-3}{2}}$

(12 pts for each problem, 4 pts for calculating $I(x)$, 3 pts for computing integral, 2 pts for solving C, and 3 pts for correct a answer.)

The End.