Work out **ALL** questions below. Provide sufficient justification to every step of your arguments.
Write your solutions as well as your ID number clearly on A4-sized paper and submit them to *Instructor's office* **before 6pm (GMT +8)** on **2nd January, 2019**.

Recommended time limit: 150 minutes.

1. Let

$$f(x) = \int_0^{\frac{1}{x}} \frac{t^2}{t^4+1}\,dt + \int_0^x \frac{1}{t^4+1}\,dt, \qquad x \neq 0$$

   (a) (4 pts) Find $f'(x)$.

   (b) (5 pts) Find $f(1)+f(-1)$.

   (c) (4 pts) Using the above results, find $f(3)+f(-2)$.

(a) By the Fundamental Theorem of Calculus, we have

$$f'(x) = \frac{(1/x)^2}{(1/x)^4+1}\cdot\frac{-1}{x^2} + \frac{1}{x^4+1} = 0$$

(Using the Fundamental Theorem of Calculus + Answer : 2 + 2 points)

(b)

$$f(1)+f(-1) = \int_0^1 \frac{t^2}{t^4+1}dt + \int_0^1 \frac{1}{t^4+1}dt + \int_0^{-1} \frac{t^2}{t^4+1}dt + \int_0^{-1} \frac{1}{t^4+1}dt$$

. (Plug $x=1$ and $x=-1$ into the function $f(x)$: 1 point)

$$= \int_0^1 \frac{t^2}{t^4+1}dt + \int_0^1 \frac{1}{t^4+1}dt - \int_{-1}^0 \frac{t^2}{t^4+1}dt - \int_{-1}^0 \frac{1}{t^4+1}dt$$

$$= \left(\int_0^1 \frac{t^2}{t^4+1}dt - \int_{-1}^0 \frac{t^2}{t^4+1}dt\right) + \left(\int_0^1 \frac{1}{t^4+1}dt - \int_{-1}^0 \frac{1}{t^4+1}dt\right)$$

Let $u=-t$, then $du=-dt$. When $t=-1 \Rightarrow u=1$, when $t=0 \Rightarrow u=0$.

. (Substitution rule : 2 points)

$$= \left(\int_0^1 \frac{t^2}{t^4+1}dt - \int_1^0 \frac{u^2}{u^4+1}(-du)\right) + \left(\int_0^1 \frac{1}{t^4+1}dt - \int_1^0 \frac{1}{u^4+1}(-du)\right)$$

$$= \left(\int_0^1 \frac{t^2}{t^4+1}dt - \int_0^1 \frac{u^2}{u^4+1}du\right) + \left(\int_0^1 \frac{1}{t^4+1}dt - \int_0^1 \frac{1}{u^4+1}du\right)$$

$$= 0$$

(Answer : 2 points)

(Note: $\frac{t^2}{t^4+1}$ and $\frac{1}{t^4+1}$ are even function.)

(c) Since $f'(x) = 0$ for all $x \in \mathbb{R}$ and $x \neq 0$, f is constant on the intervals $(0, \infty)$ and $(-\infty, 0)$ (but not necessarily constant on $(-\infty, 0) \cup (0, \infty)$).
Therefore, $f(3) = f(1)$ and $f(-2) = f(-1)$. (f is locally constant : 2 points)
So

$$f(3) + f(-2) = f(1) + f(-1) = 0$$

(Answer : 2 points)

2. (a) (8 pts) Evaluate the integral $\int_0^{\frac{\pi}{2}} \left|\cos^2 x - 3\sin^2 x\right| dx$.

(b) (8 pts) Compute $\int \frac{1}{e^{2x}+e^x+1} dx$.

(a) We know that

$$\cos^2 x = \frac{1+\cos 2x}{2}, \ \sin^2 x = \frac{1-\cos 2x}{2}$$

The indefinite integral can then be computed to obtain

$$\begin{aligned}\int \cos^2 x - 3\sin^2 x dx &= \int \frac{1+\cos 2x}{2} - \frac{3-3\cos 2x}{2} dx \\ &= \int -1 + 2\cos 2x \, dx \\ &= -x + \sin 2x + C, \quad C \in \mathbb{R}\end{aligned}$$

For $x \in [0, \frac{\pi}{2}]$, we have

$$\cos^2 x - 3\sin^2 x \geq 0 \iff 0 \leq x \leq \frac{\pi}{6}$$

(Find the postive or negative inteval : 3 points)

Then

$$\int_0^{\frac{\pi}{2}} \left|\cos^2 x - 3\sin^2 x\right| dx$$

$$= \int_0^{\frac{\pi}{6}} \cos^2 x - 3\sin^2 x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 x \ - \cos^2 x \, dx$$

. (Divided the definite integral into two parts : 2 points)

$$= (-x + \sin 2x)\Big|_0^{\frac{\pi}{6}} + (x - \sin 2x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

. (Correcet indefinite integral : 2 points)

$$= \frac{\pi}{6} + \sqrt{3}$$

(Answer : 1 point)

(b) Let $t = e^x \Rightarrow x = \ln t \Rightarrow dx = \frac{dt}{t}$. (Substitution rule : 2 points)

The integral becomes

$$\int \frac{1}{e^{2x}+e^x+1} dx = \int \frac{1}{t^2+t+1} \cdot \frac{dt}{t}$$

Cont.

Suppose that

$$\frac{1}{t(t^2+t+1)} = \frac{A}{t} + \frac{P(t)}{t^2+t+1} \Rightarrow A(t^2+t+1) + P(t)\cdot t = 1$$

Let $t = 0 \Rightarrow A = 1$, and then

$$P(t)\cdot t = -t^2 - t \Rightarrow P(t) = -t-1$$

(Compute A and $P(t)$: 1 + 1 points)

Consider the integral

$$\begin{aligned}\int \frac{1}{t^2+t+1}\cdot\frac{dt}{t} &= \int \frac{1}{t}dt + \int \frac{-t-1}{t^2+t+1}dt \\ &= \int \frac{1}{t}dt + \left(\frac{-1}{2}\right)\int \frac{2t+1}{t^2+t+1}dt + \int \frac{-1/2}{t^2+t+1}dt \\ &= \ln|t| - \frac{1}{2}\ln|t^2+t+1| + \int \frac{-1/2}{t^2+t+1}dt\end{aligned}$$

The thrid term can be computed as

$$\int \frac{-1/2}{t^2+t+1}dt = \left(\frac{-1}{2}\right)\int \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}} = \frac{-1}{\sqrt{3}}\arctan(\frac{2t+1}{\sqrt{3}})$$

So, the integral in question is given by

$$\begin{aligned}&\ln|t| - \frac{1}{2}\ln|t^2+t+1| + \int \frac{-1/2}{t^2+t+1}dt \\ &= \ln|t| - \frac{1}{2}\ln|t^2+t+1| - \frac{1}{\sqrt{3}}\arctan(\frac{2t+1}{\sqrt{3}}) + C \\ &= x - \frac{1}{2}\ln|e^{2x}+e^x+1| - \frac{1}{\sqrt{3}}\arctan(\frac{2e^x+1}{\sqrt{3}}) + C,\end{aligned}$$

where C is a constant

(Compute the indefinite integral + Answer : 3 + 1 points)

3. (a) (5 pts) Determine whether the improper integral

$$\int_0^1 \frac{\cos t}{t^{4/3}}\,dt$$

is convergent or divergent?

(b) (5 pts) Evaluate the following limit

$$\lim_{x\to 0^+} x^{1/6}\int_{\sqrt{x}}^1 \frac{\cos t}{t^{4/3}}\,dt$$

(a) Since $0 = \cos\frac{\pi}{2} < \cos 1 \leq \cos x \leq \cos 0 = 1$ when $x \in [0, 1]$, then

$$\int_0^1 \frac{\cos t}{t^{4/3}}\, dt > \int_0^1 \frac{\cos 1}{t^{4/3}}\, dt = \cos 1 \int_0^1 t^{\frac{-4}{3}}\, dt$$

However $\int_0^1 t^{\frac{-4}{3}}\, dt$ is divergent, thus

$$\int_0^1 \frac{\cos t}{t^{4/3}}\, dt \text{ is also divergent.}$$

(Comparison Test + Answer : 3 + 2 points)

(b)

$$\begin{aligned}
\lim_{x\to 0^+} x^{1/6} \int_{\sqrt{x}}^1 \frac{\cos t}{t^{4/3}}\, dt &= \lim_{x\to 0^+} \frac{\int_{\sqrt{x}}^1 \frac{\cos t}{t^{4/3}}\, dt}{x^{-1/6}} \\
&\underset{(\frac{\infty}{\infty})}{\overset{L'H}{=}} \lim_{x\to 0^+} \frac{\frac{-\cos\sqrt{x}}{x^{2/3}} \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}}}{\frac{-1}{6} x^{-\frac{7}{6}}} \\
&= \lim_{x\to 0^+} 3\cos\sqrt{x} \\
&= 3
\end{aligned}$$

(Using L'Hospital rule + Answer : 3 + 2 points)

4. Figure 1 shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, a vertical line through P bounded a region A between the curves $y = 2x^2$ and $y = x^2$ while a horizontal line through P bounded a region B between the curves $y = 2x^2$ and C, and the area of B is twice the area of A.

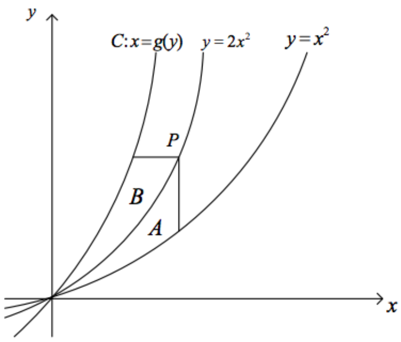

Figure 1: The three curves $C : x = g(y)$, $y = 2x^2$ and $y = x^2$.

(a) (9 pts) Find an equation $x = g(y)$ for C.
(Hint: Compute the areas of A and B.)

(b) (6 pts) Let R be the region bounded by the curve C, $y = x^2$, $x = 2$ and $y = 8$. Find the volume of the solid obtained by rotating R about the x-axis.

(a) Assume $P(t, 2t^2)$ is a point on the curve $y = 2x^2$.
The area A is

$$A = \int_0^t (2x^2 - x^2)dx = \int_0^t x^2 dx = \frac{1}{3}t^3$$

(Compute the area of A : 2 points)

The area B is

$$B = \int_0^{2t^2} (\sqrt{\frac{y}{2}} - g(y))dy = \frac{4}{3}t^3 - \int_0^{2t^2} g(y)dy$$

(Compute the area of B : 2 points)

Since $B = 2A$, then

$$\int_0^{2t^2} g(y)dy = \frac{2}{3}t^3 \Rightarrow \frac{d}{dt}\int_0^{2t^2} g(y)dy = \frac{d}{dt}(\frac{2}{3}t^3)$$

By the Fundamental Theorem of Calculus,

$$g(2t^2) \cdot 4t = 2t^2 \Rightarrow g(2t^2) = \frac{t}{2}$$

(Using the Fundamental Theorem of Calculus : 3 points)

Let $y = 2t^2 \Rightarrow t = \sqrt{\frac{y}{2}} \Rightarrow g(y) = \frac{1}{2}\sqrt{\frac{y}{2}}$ (Answer : 2 points)

(b) Note that the point $(2, 8)$ is on the curve $y = 2x^2$.

(Method 1.)

$$\begin{aligned} V &= \int_0^8 2\pi y(\sqrt{\frac{y}{2}} - \frac{1}{2}\sqrt{\frac{y}{2}})dy + \int_0^2 \pi((2x^2)^2 - (x^2)^2)dx \\ &= 2\pi \int_0^8 \frac{1}{2\sqrt{2}} y^{\frac{3}{2}} dy + \pi \int_0^2 3x^4 dx \\ &= \frac{256}{5}\pi + \frac{96}{5}\pi \\ &= \frac{352}{5}\pi \end{aligned}$$

(Compute the volumes of B and A : 3 + 3 points)

(Method 2.) On the curve C, when $y = 8 \Rightarrow x = \frac{1}{2}\sqrt{\frac{8}{2}} = 1$

$$\begin{aligned} V &= \int_0^1 \pi((8x^2)^2 - (x^2)^2)dx + \int_1^2 \pi((8)^2 - (x^2)^2)dx \\ &= \pi \int_0^1 63x^4 dx + \pi \int_1^2 (64 - x^4)dx \\ &= \frac{63}{5}\pi + (128 - \frac{32}{5} - 64 + \frac{1}{5})\pi \\ &= \frac{352}{5}\pi \end{aligned}$$

(Compute the volumes on the intervals $(0, 1)$ and $(1, 2)$: 3 + 3 points)

5. Consider the plane curve $3ay^2 = x(a - x)^2$ where $a > 0$ is a constant.

(a) (6 pts) Find the arc length of the loop defined by the curve.

(b) (4 pts) Find the surface area of the surface obtained by rotating the loop around x-axis.

(a) The graph of the curve is shown in Fig2.

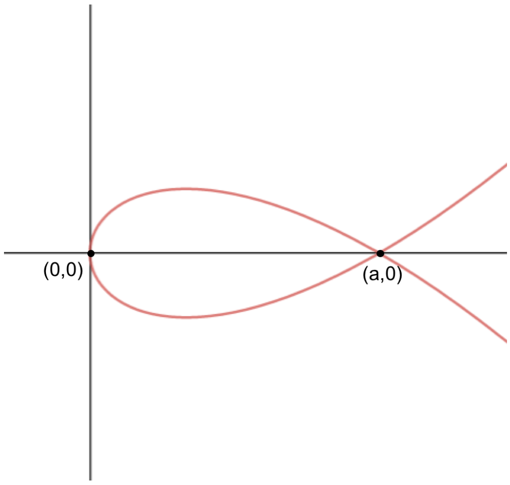

Figure 2: The plane curve of $3ay^2 = x(a-x)^2$

$$
\begin{aligned}
&3ay^2 = x(a-x)^2 \\
&\Rightarrow 3a(2y)\frac{dy}{dx} = 2x(a-x)(-1) + (a-x)^2 \\
&\Rightarrow \frac{dy}{dx} = \frac{(a-x)(a-3x)}{6ay}
\end{aligned}
$$

(Compute $\frac{dy}{dx}$: 2 points)

Hence the length L is

$$
\begin{aligned}
L &= 2 \cdot \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= 2 \cdot \int_0^a \sqrt{1 + \frac{(a-x)^2(a-3x)^2}{36a^2y^2}}\, dx \\
&= 2 \cdot \int_0^a \sqrt{\frac{(a+3x)^2}{12ax}}\, dx \\
&= 2\sqrt{\frac{1}{12a}} \cdot \int_0^a \frac{a+3x}{\sqrt{x}}\, dx \\
&= \frac{4\sqrt{3}}{3}a
\end{aligned}
$$

(Compute the length L + Answer : 3 + 1 points)

(b) The surface area A is

$$\begin{aligned}
A &= \int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= 2\pi\int_0^a \left[\frac{\sqrt{x}(a-x)}{\sqrt{3a}}\right]\left[\frac{1}{\sqrt{12a}}\frac{(a+3x)}{\sqrt{x}}\right]dx \\
&= 2\pi\int_0^a \frac{1}{6a}(a+3x)(a-x)dx \\
&= \frac{\pi}{3a}\int_0^a (a^2-ax+3ax-3x^2)dx \\
&= \frac{\pi}{3}a^2
\end{aligned}$$

(Compute the area A + Answer : 3 + 1 points)

6. (a) (7 pts) Find all points of intersection of the two polar curves $r=\sqrt{2}\sin\theta$ and $r^2=\cos 2\theta$.
   (b) (6 pts) Find the area of the shaded region in Figure 3.

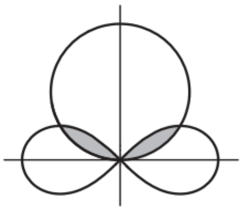

Figure 3: The two curves $r=\sqrt{2}\sin\theta$ and $r^2=\cos 2\theta$.

(a) If we solve the equations $r=\sqrt{2}\sin\theta$ and $r^2=\cos 2\theta$, we get

$$\begin{cases} r=\sqrt{2}\sin\theta \\ r^2=\cos 2\theta \end{cases}$$

$$\Rightarrow 2\sin^2\theta=\cos 2\theta = 1-2\sin^2\theta$$

$$\Rightarrow (r,\theta)=\left(\frac{1}{\sqrt{2}},\frac{\pi}{6}\right),\ \left(\frac{1}{\sqrt{2}},\frac{5\pi}{6}\right),\ \left(-\frac{1}{\sqrt{2}},\frac{7\pi}{6}\right),\ \left(-\frac{1}{\sqrt{2}},\frac{11\pi}{6}\right).$$

(The values of (r,θ) : 4 points)

Hence we have found two points of intersection given in polar coordinates as $(\frac{1}{\sqrt{2}},\frac{\pi}{6})=(-\frac{1}{\sqrt{2}},\frac{7\pi}{6})$ and $(-\frac{1}{\sqrt{2}},\frac{11\pi}{6})=(\frac{1}{\sqrt{2}},\frac{5\pi}{6})$. (Find two intersection points : 2 points)

However, if we plug $\theta=0$ into $r=\sqrt{2}\sin\theta$ and $\theta=\pi/4$ into $r^2=\cos 2\theta$, we can find one more point of intersection at the pole (origin).
. (Find the third intersection point (origin) : 1 point)

(b) Since the area in $x > 0$ is the same as the area in $x < 0$, then the area A is

$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}} (\sqrt{2}\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2\theta \, d\theta\right) \\ &= \int_0^{\frac{\pi}{6}} 1 - \cos 2\theta \, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2\theta \, d\theta \\ &= \frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{2} \end{aligned}$$

(Compute the areas when θ in $(0, \frac{\pi}{6})$ and $(\frac{\pi}{6}, \frac{\pi}{4})$: 3 + 3 points)

7. (a) (7 pts) Solve the differential equation $x\frac{dy}{dx} - 2y = x^3 \cdot \tan x \cdot \sec x$, $x > 0$ and $y(\pi/3) = 0$.

(b) (5 pts) Find the orthogonal trajectories of the family of curves $y = \frac{k}{x+1}$, where k is an arbitrary constant.

(a)

$$\begin{aligned} x\frac{dy}{dx} - 2y = x^3 \cdot \tan x \cdot \sec x, &\quad \text{where } x > 0 \text{ and } y(\frac{\pi}{3}) = 0 \\ \Rightarrow \frac{dy}{dx} - \frac{2}{x}y = x^2 \cdot \tan x \cdot \sec x \end{aligned}$$

Let

$$I = \exp\left\{\int \frac{-2}{x} dx\right\} = \exp^{-2\ln|x| + C} = \frac{1}{x^2}$$

by choosing $C = 0$. (Compute the integrating factor I : 3 points)

$$\begin{aligned} Iy &= \int \frac{1}{x^2}\left(x^2 \cdot \tan x \cdot \sec x\right) dx = \sec x + C' \\ &\Rightarrow y = \frac{\sec x + C'}{I} = x^2(\sec x + C') \\ &\Rightarrow y(\frac{\pi}{3}) = (\frac{\pi}{3})^2 \cdot (\sec(\frac{\pi}{3}) + C') = 0 \\ &\Rightarrow C' = -2 \\ &\Rightarrow y = x^2(\sec x - 2) \end{aligned}$$

(Compute y + Answer : 3 + 1 points)

(b)

$$y = \frac{k}{x+1} \Rightarrow k = y(x+1)$$

$$\frac{dy}{dx} = -\frac{k}{(x+1)^2} = -\frac{y(x+1)}{(x+1)^2} = -\frac{y}{x+1}$$

(Find the slope field of the curve : 2 points)

Slope field of the orthogonal trajectories

$$\frac{dy}{dx} = \frac{x+1}{y}$$
$$\Rightarrow \int y dy = \int (x+1) dx$$
$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + x + C, \quad \text{where } C \text{ is a constant}$$

(Compute the orthogonal trajectories + Answer : 2 + 1 points)

8. (a) (7 pts) Solve the initial value problem:

$$\begin{cases} 2x(x+3)y' + (4x+3)y = 2x^{\frac{1}{2}}(x+3)^{\frac{1}{2}} \\ y(1) = \frac{1}{2}, \quad x > 0 \end{cases}$$

(b) (4 pts) Find $\lim_{x\to\infty} y(x)$ and $\lim_{x\to 0^+} y(x)$.

(a) Divide the equation by $2x(x+3)$, then we get

$$y' + \frac{4x+3}{2x(x+3)}y = \frac{1}{\sqrt{x(x+3)}}$$

Let $I(x) = \exp\{\int \frac{4x+3}{2x(x+3)}\, dx\}$, where

$$\int \frac{4x+3}{2x(x+3)}\, dx = \frac{1}{2}\int \left(\frac{1}{x} + \frac{3}{x+3}\right) dx = \frac{1}{2}\ln x + \frac{3}{2}\ln(x+3) + C \quad \text{when } x > 0$$

Hence

$$I(x) = e^C\sqrt{x(x+3)^3}$$

Let $C = 0$, then

$$I(x) = \sqrt{x(x+3)^3}$$

(Compute the integrating factor I : 3 points)

$$Iy' + I\frac{4x+3}{2x(x+3)}y = Iy' + I'y = (Iy)' = I\frac{1}{\sqrt{x(x+3)}} = x+3$$
$$Iy = \int (x+3)dx = \frac{1}{2}x^2 + 3x + D, \quad \text{where } D \text{ is a constant}$$
$$\Rightarrow \sqrt{x(x+3)^3}\; y = \frac{1}{2}x^2 + 3x + D$$

Bringing $y(1) = \frac{1}{2}$ into the equation, we find $D = \frac{1}{2}$.
Hence

$$y = \frac{x^2 + 6x + 1}{2\sqrt{x(x+3)^3}}$$

(Compute y + Answer : 3 + 1 points)

(b)

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{x^2+6x+1}{2\sqrt{x(x+3)^3}} = \lim_{x\to\infty} \frac{1+\frac{6}{x}+\frac{1}{x^2}}{2\sqrt{\left(1+\frac{3}{x}\right)^3}} = \frac{1}{2}$$

(Compute the limit when $x \to \infty$: 2 points)

Since $\lim\limits_{x\to 0^+} x^2+6x+1 = 1$ and $\lim\limits_{x\to 0^+} 2\sqrt{x(x+3)^3} = 0$, then

$$\lim_{x\to 0^+} y = +\infty$$

(Compute the limit when $x \to 0^+$: 2 points)

The End.