Calculus 2 11/21 Note
Module Class 07

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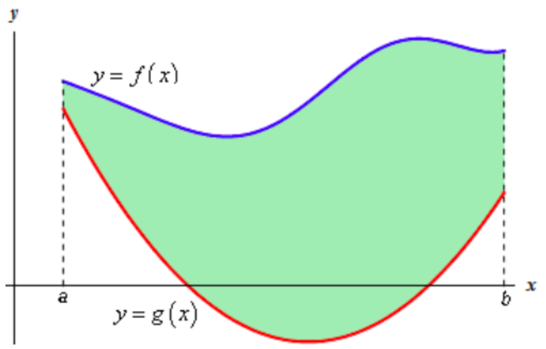
Section 6.1: Areas and Distances

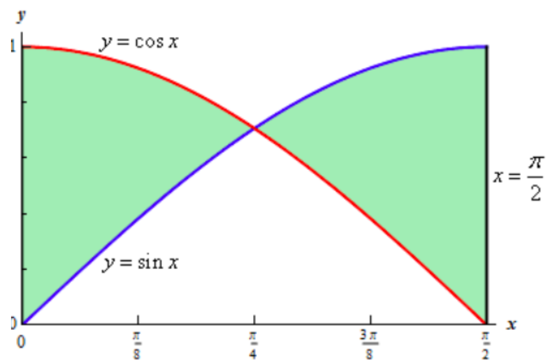
Area Formula

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous for all x in $[a, b]$, is

$$A = \int_a^b |f(x) - g(x)|\, dx$$

Graph:





Example:

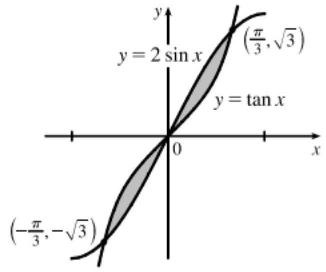
Sketch the region enclosed by the gievn curves and find its area.

1. $y = \tan x,\ y = 2\sin x,\ -\pi/3 \le x \le \pi/3$

   Sol.

   The curves intersect when $\tan x = 2\sin x$, and

$$\begin{aligned} \tan x = 2\sin x \ &\Leftrightarrow\ \sin x = 2\sin x\cos x \ \Leftrightarrow\ \sin x(2\cos x - 1) = 0 \\ &\Leftrightarrow\ \sin x = 0 \text{ or } \cos x = \frac{1}{2} \ \Leftrightarrow\ x = 0,\ \pm\frac{\pi}{3} \end{aligned}$$

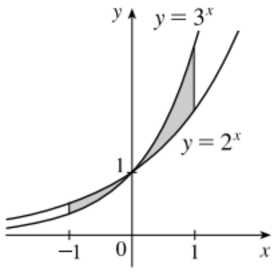


Since $\tan x$ and $\sin x$ are odd functions, if $f(x) = 2\sin x - \tan x$, then f is odd $[f(x) = f(-x)]$.

$$A = 2\int_0^{\pi/3} (2\sin x - \tan x)\, dx = 2\left[-2\cos x - \ln|\sec x|\right]_{x=0}^{\pi/3} = 2 - 2\ln 2$$

2. $y = 3^x,\ y = 2^x,\ -1 \le x \le 1$

Sol.



$$\begin{aligned} A &= \int_{-1}^{1} |3^x - 2^x|\, dx = \int_{-1}^{0} (2^x - 3^x)\, dx + \int_0^1 (3^x - 2^x)\, dx \\ &= \left[\frac{2^x}{\ln 2} - \frac{3^x}{\ln 3}\right]_{x=-1}^{0} + \left[\frac{3^x}{\ln 3} - \frac{2^x}{\ln 2}\right]_{x=0}^{1} \\ &= \frac{4}{3\ln 3} - \frac{1}{2\ln 2} \end{aligned}$$

Exercise:

Find the area enclosed by the gievn curves.

1. $y = x^3$, $y = x$

2. $y = \sinh x,\ y = e^{-x},\ x = 0,\ x = 2$

Sol.
1. $\frac{1}{2}$ 2. $2 - 2\sqrt{3} + \frac{1}{2}e^2 + \frac{3}{2}e^{-2}$

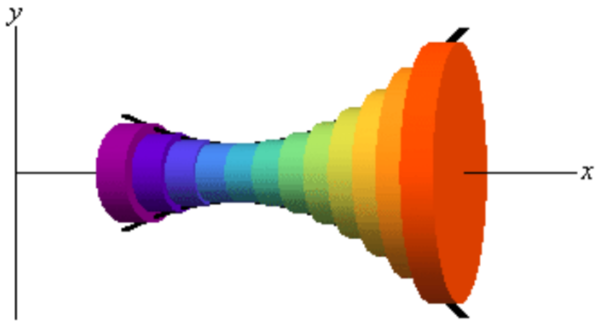
Section 6.2: Volumes

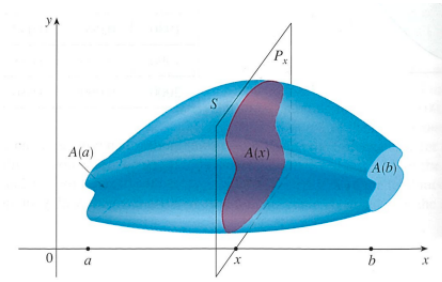
Definition of Volume

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \int_a^b A(x)\,dx$$

Graph:

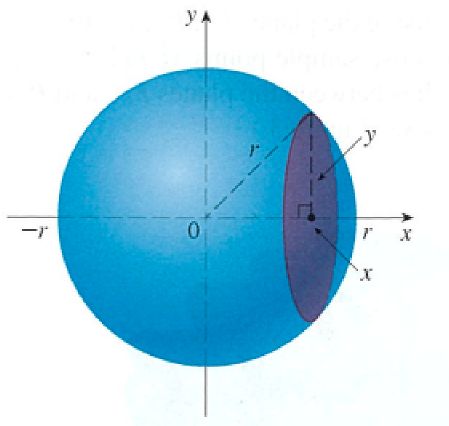




Example:

1. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

   Sol.



See the above figure, then $y = \sqrt{r^2 - x^2}$. So the cross-sectional area is

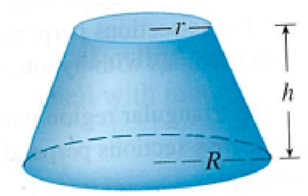
$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Thus,

$$\begin{aligned}
V &= \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = 2\pi \int_{0}^{r} (r^2 - x^2)\,dx \\
&= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_{x=0}^{r} = 2\pi \left(r^3 - \frac{r^3}{3} \right) \\
&= \frac{4}{3}\pi r^3
\end{aligned}$$

Exercise:

1. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.
   (a) $y = x^3$, $y = 1$, $x = 2$; about $y = -3$.
   (b) $y = \sin x$, $y = \cos x$, $0 \le x \le \pi/4$; about $y = -1$.
2. A frustum of a right circular cone with height h, lower base radius R, and top radius r. Find the volume of a frustum.



Sol.
1.(a) $\frac{471\pi}{14}$ (b) $\left(2\sqrt{2} - \frac{3}{2}\right)\pi$
2. $V = \frac{1}{3}\pi h(R^2 + Rr + r^2)$ or $V = \frac{h}{3}\left[\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)}\right]$

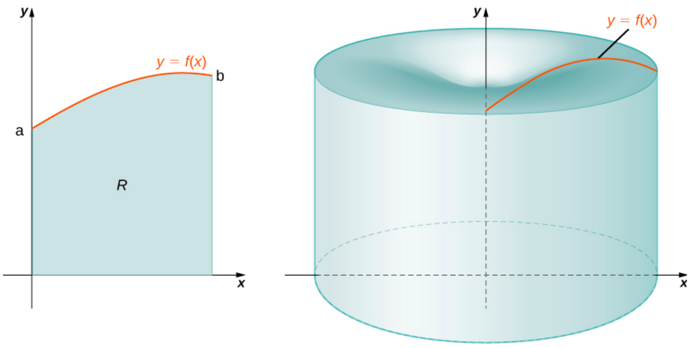
Section 6.3: Volumes by Cylindrical Shells

Volume Formula

The volume of the solid in the following graph, obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b, is

$$V = \int_a^b 2\pi x\, f(x)\, dx \qquad \text{where } 0 \le a < b$$

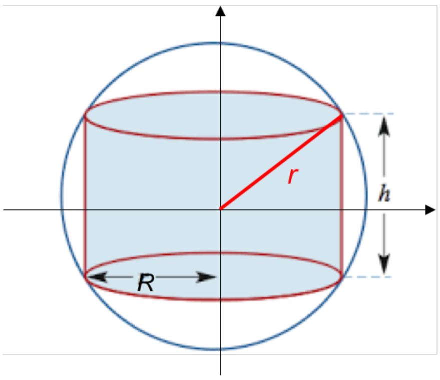
Graph:



Example:

1. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Sol.



See the above figure, then $h(x) = 2\sqrt{r^2 - x^2}$. Thus,

$$\begin{aligned} V &= \int_0^r 2\pi x\, h(x)\, dx = \int_0^r 2\pi x\, (2\sqrt{r^2 - x^2})\, dx \\ &= 4\pi \int_0^r x(r^2 - x^2)^{1/2}\, dx \\ &= 4\pi \left[-\frac{1}{3}(r^2 - x^2)^{3/2} \right]_{x=0}^{r} \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

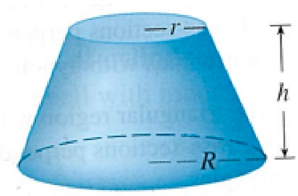
Exercise:

1. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

   (a) $y = 4 - 2x$, $y = 0$, $x = 0$; about $x = -1$.

   (b) $y = \sqrt{x}$, $x = 2y$; about $x = 5$.

2. A frustum of a right circular cone with height h, lower base radius R, and top radius r. Use the method of cylindrical shells to find the volume of a frustum.



Sol.

1.(a) $\frac{40\pi}{3}$ (b) $\frac{136\pi}{15}$

2. $V = \frac{1}{3}\pi h(R^2 + Rr + r^2)$ or $V = \frac{h}{3}\left[\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)}\right]$

Section 6.5: Average Value of a Function

The Mean Value Theorem for Integrals

If f is continuous on $[a,\, b]$, then there exists a number c in $[a,\, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is,

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Example:

1. (a) Find the average value of on the given interval.
   (b) Find c such that $f_{\text{ave}} = f(c)$.
   (c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

$$f(x) = 2\sin x - \sin(2x),\ \ [0,\, \pi]$$

Sol.

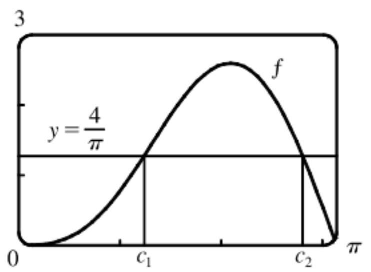
(a)

$$f_{\text{ave}} = \frac{1}{\pi - 0}\int_0^{\pi} [2\sin x - \sin(2x)]\ dx = \frac{1}{\pi}\left[-2\cos x + \frac{1}{2}\cos(2x)\right]_{x=0}^{\pi} = \frac{4}{\pi}$$

(b)

$$f(c) = f_{\text{ave}} \ \Leftrightarrow\ 2\sin c - \sin(2c) = \frac{4}{\pi}$$

(c) .



2. Find the numbers b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0,\, b]$ is equal to 3.

Sol.

$$\begin{aligned}
&\frac{1}{b-0}\int_0^b f(x)\,dx = 3 \\
\Leftrightarrow\ &\frac{1}{b}\int_0^b (2 + 6x - 3x^2)\,dx = 3 \\
\Leftrightarrow\ &2 + 3b - b^2 = 3 \\
\Leftrightarrow\ &b^2 - 3b + 1 = 0 \\
\Leftrightarrow\ &b = \frac{3 \pm \sqrt{5}}{2}
\end{aligned}$$

Both roots are valid since they are positive.

Exercise:

1. Find the average value of the function on the given interval.

   (a) $f(x) = x^2\sqrt{1+x^3}$, $[0, 2]$

   (b) $f(x) = (\ln x)/x$, $[1, 5]$

2. (a) Find the average value of on the given interval.

   (b) Find c such that $f_{\text{ave}} = f(c)$.

   (c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

$$f(x) = (x-3)^2,\ [2, 5]$$

Sol.

1.(a) $\frac{26}{9}$ (b) $\frac{1}{8}(\ln 5)^2$

2.(a) $f_{\text{ave}} = 1$ (b) $c = 2, 4$ (c)