Real Analysis Homework
Chapter 2. Integration Theory
Due Date: 12/2

National Taiwan University, Department of Mathematics
R06221012 Yueh-Chou Lee

December 2, 2019

Exercise 2.17(a)

Suppose f is defined on $\mathbb{R}^2$ as follows:

$$f(x,y)=\begin{cases} a_n & \text{if } n\le x<n+1 \text{ and } n\le y<n+1,\ (n\ge 0)\\ -a_n & \text{if } n\le x<n+1 \text{ and } n+1\le y<n+2,\ (n\ge 0)\\ 0 & \text{o.w.}\end{cases}$$

Here $a_n=\sum_{k\le n} b_k$, with $\{b_k\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s<\infty$.
Verify that each slice f^y and f_x is integrable. Also for all x, $\int f_x(y)\,dy=0$, and hence $\int(\int f(x,y)\,dy)\,dx=0$.

Proof.
By the definition of $f(x,y)$, we have

$$f^y(x)=\begin{cases} -a_{\lfloor y\rfloor-1} & y\ge 1 \text{ and } (\lfloor y\rfloor-1)\le x<\lfloor y\rfloor\\ a_{\lfloor y\rfloor} & y\ge 0 \text{ and } \lfloor y\rfloor\le x<(\lfloor y\rfloor+1)\\ 0 & \text{o.w.}\end{cases}$$

Similarly,

$$f_x(y)=\begin{cases} a_{\lfloor x\rfloor} & x\ge 0 \text{ and } \lfloor x\rfloor\le y<(\lfloor x\rfloor+1)\\ -a_{\lfloor x\rfloor} & x\ge 0 \text{ and } (\lfloor x\rfloor+1)\le y<(\lfloor x\rfloor+2)\\ 0 & \text{o.w.}\end{cases}$$

To show that each slice $f^y(x)$ is integrable, we compute, for $y\ge 1$, then

$$\begin{aligned}\int f^y(x)\,dx &= \int_{\lfloor y\rfloor-1}^{\lfloor y\rfloor} -a_{\lfloor y\rfloor-1}\,dx+\int_{\lfloor y\rfloor}^{\lfloor y\rfloor+1} a_{\lfloor y\rfloor}\,dx\\ &= a_{\lfloor y\rfloor}-a_{\lfloor y\rfloor-1}\\ &= b_{\lfloor y\rfloor} \qquad \text{is bounded,}\end{aligned}$$

since $\{b_n\}$ is convergent ($\sum_{k=0}^{\infty} b_k=s<\infty$). For $0\le y<1$, then

$$\int f^y(x)\,dx=a_0.$$

Similarly, to show that each slice $f_x(y)$ is integrable, we compute

$$\int f_x(y)\,dy=\int_{\lfloor x\rfloor}^{\lfloor x\rfloor+1} a_{\lfloor x\rfloor}\,dx+\int_{\lfloor x\rfloor+1}^{\lfloor x\rfloor+2} -a_{\lfloor x\rfloor}\,dx = a_{\lfloor x\rfloor}-a_{\lfloor x\rfloor}=0$$

So

$$\int \left(\int f(x,y)\,dy \right)\,dx = \int \left(\int f_x(y)\,dy \right)\,dx = \int 0\,dx = 0$$

Exercise 2.17(b)

However,

$$\int f^y(x)\,dx = \begin{cases} a_0 & \text{if } 0 \le y < 1 \\ a_n - a_{n-1} & \text{if } n \le y < n+1 \text{ with } n \ge 1 \end{cases}$$

Hence $y \mapsto \int f^y(x)\,dx$ is integrable on $(0, \infty)$ and

$$\int \left(\int f(x,\,y)\,dx \right)\,dy = s.$$

Proof.

$$\begin{aligned} \int \left(\int f(x,\,y)\,dx \right)\,dy &= \sum_{n=0}^{\infty} \int_n^{n+1} \left(\int_{\mathbb{R}} f^y(x)\,dx \right)\,dy \\ &= a_0 + \sum_{n=1}^{\infty} (a_n - a_{n-1}) \end{aligned}$$

Exercise 2.17(c)

Note that $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|\,dxdy = \infty$.

Proof.

Since $0 \le |f(x,y)| < \infty$, by Tonelli's theorem, we then have

$$\begin{aligned} \int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|\,dxdy &= \int_{\mathbb{R}} \int_{\mathbb{R}} |f_x(y)|\,dydx \\ &= \sum_{n=0}^{\infty} \int_n^{n+1} \int_{\mathbb{R}} |f_x(y)|\,dydx \\ &= \sum_{n=0}^{\infty} 2a_n \end{aligned}$$

However, each of the $a_n \ge a_0 > 0$, then $\sum_{n=0}^{\infty} 2a_n$ is divergent, so

$$\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|\,dxdy = \infty$$

Exercise 2.18

Let f be a measurable finite-valued function on $[0,\,1]$, and suppose that $|f(x)-f(y)|$ is integrable on $[0,\,1]\times[0,\,1]$. Show that $f(x)$ is integrable on $[0,\,1]$.

Proof.

Let the function $g(x,y)=|f(x)-f(y)|$. By Fubini's theorem, since g is integrable on $[0,\,1]\times[0,\,1]$, the slice $g^y(x)$ is integrable for a.e. $y\in[0,\,1]$. Fixed $y\in[0,\,1]$, then

$$f(x)-f(y)\le|f(x)-f(y)|$$

Also, we can use the monotonicity of the integral to see that

$$\int_{[0,1]} f(x)-f(y)\,dx\le\int_{[0,1]}|f(x)-f(y)|\,dx\le\infty$$

That is

$$\begin{aligned}\int_{[0,1]} f(x)\,dx&\le\int_{[0,1]} f(y)\,dx+\int_{[0,1]}|f(x)-f(y)|\,dx\\&=f(y)+\int_{[0,1]}|f(x)-f(y)|\,dx\end{aligned}$$

Since $f(y)$ and $\int_{[0,1]}|f(x)-f(y)|\,dx$ are finite, then $f(x)$ is integrable on $[0,\,1]$.

Exercise 2.19

Suppose f is integrable on $\mathbb{R}^d$. For each $\alpha>0$, let $E_\alpha=\{x\,:\,|f(x)|>\alpha\}$. Prove that

$$\int_{\mathbb{R}^d}|f(x)|\,dx=\int_0^\infty m(E_\alpha)\,d\alpha$$

Proof.

Observe that

$$m(E_\alpha)=\int_{\mathbb{R}^d}\chi_{E_\alpha}(x)\,dx.$$

Let $f(\alpha,\,x)=\chi_{E_\alpha}(x)$. By Tonelli's theorem, since f is non-negative measurable function on $(0,\,\infty)\times\mathbb{R}^d$, then

$$\begin{aligned}\int_0^\infty m(E_\alpha)\,d\alpha&=\int_0^\infty\int_{\mathbb{R}^d} f(\alpha,\,x)\,dxd\alpha\\&=\int_{\mathbb{R}^d}\left(\int_0^\infty\chi_{E_\alpha}(x)\,d\alpha\right)dx\\&=\int_{\mathbb{R}^d}|f(x)|\,dx\end{aligned}$$

Exercise 2.20

The problem (highlighted in the discussion preceding Fubini's theorem) that certain slices of measurable sets can be non-measurable may be avoided by restricting attention to Borel measurable functions and Borel sets. In fact, prove the following:

Suppose E is a Borel set in $\mathbb{R}^2$. Then for every y, the slice E^y is a Borel set in $\mathbb{R}$.

[Hint: Consider the collection $\mathcal{C}$ of subsets E of $\mathbb{R}^2$ with the property that each slice E^y is a Borel set in $\mathbb{R}$. Verify that $\mathcal{C}$ is a σ-algebra that contains the open sets.]

Proof.
Define

$$\mathcal{C} = \{E \subset \mathbb{R}^2 \,|\, \forall y,\, E^y \text{ is Borel.}\}$$

We first prove the following.

1. Prove that $\mathcal{C}$ is σ-algebra.

   Proof.

   It's clear that $\mathcal{C}$ is non-empty.

   Next, we will show that $E \in \mathcal{C}$, then $E^c \in \mathcal{C}$.

   Let $E \in \mathcal{C}$. Since for any y the slice E^y is Borel and $(E^c)^y = (E^y)^c$ is Borel (the Borel sets are a σ-algebra), then $E^c \in \mathcal{C}$.

   Finally, suppose that $\{E_k\}_{k=1}^{\infty}$ is a countable collection of sets in $\mathcal{C}$. Since

   $$\left(\bigcup_{k=1}^{\infty} E_k\right)^y = \bigcup_{k=1}^{\infty} E_k^y,$$

   and the fact that each of the E_k^y is Borel, then $\bigcup_{k=1}^{\infty} E_k^y$ must be Borel so as $(\bigcup_{k=1}^{\infty} E_k)^y$.

   By the above, we then know that $\mathcal{C}$ is σ-algebra.

2. Prove that if $E \subset \mathbb{R}^2$ is open then $E \in \mathcal{C}$.

   Proof.

   If E is an open cube, then we must have $E = (a,\, b) \times (a + h,\, b + h)$ for some $h > 0$.

   Take the slice E^y which is the same as $(a,\, b) + y$ and $y \in \mathbb{R}$. In geometric, we simply have the line segment $(a,\, b)$ translated in the y-direction a distance y.

   So E^y is an open interval and hence is also a Borel set.

   If E is a closed cube, then the slice E^y is a closed interval and can be written as the countable intersection of intervals

   $$\bigcap_n \left(a - \frac{1}{n},\, b + \frac{1}{n}\right)$$

   Thus, E^y is a Borel set.

Then, we use the fact that every open set in $\mathbb{R}^2$ can be written as the countable union of almost disjoint closed cubes such as

$$E = \bigcup_{j=1}^{\infty} Q_k$$

where each of the Q_k is closed cube. Since $Q_k \in \mathcal{C}$ for all k, then $E \in \mathcal{C}$.

In the end, recall that the Borel sets is the smallest σ-algebra containing the open sets. Also, we have shown that $\mathcal{C}$ contains the open sets. Therefore, $\mathcal{C}$ contains the Borels sets.

Exercise 2.21(a)

Suppose that f and g are measurable functions on $\mathbb{R}^d$. Prove that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$.

Proof.
Since f is a measurable function on $\mathbb{R}^d$, then by **Proposition 3.9** in the textbook, we have $\tilde{f}(x,y) = f(x-y)$ is measurable on $\mathbb{R}^d \times \mathbb{R}^d$.

Also, since g is a measurable function on $\mathbb{R}^d$, then by **Corollary 3.7** in the textbook, we have $\tilde{g}(x,y) = g(y)$ is measurable on $\mathbb{R}^d \times \mathbb{R}^d$.

By above two, we know that $f(x-y)g(y)$ is measurable on $\mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$.

Exercise 2.21(b)

Suppose that f and g are measurable functions on $\mathbb{R}^d$. Show that if f and g are integrable on $\mathbb{R}^d$, then $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$.

Proof.
Since f and g are measurable functions on $\mathbb{R}^d$, then by **Tonelli's theorem**, we know that

$$\int_{\mathbb{R}^{2d}} |f(x-y)g(y)|\, d(x,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)|\, dxdy$$

Since f and g are integrable on $\mathbb{R}^d$, then $\int_{\mathbb{R}^d} f = M < \infty$ and $\int_{\mathbb{R}^d} g = N < \infty$. Thus,

$$\begin{aligned}
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)|\, dxdy &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)||g(y)|\, dxdy \\
&= \int_{\mathbb{R}^d} |g(y)| \int_{\mathbb{R}^d} |f(x-y)|\, dxdy \\
&= \int_{\mathbb{R}^d} |g(y)|\, M\, dy \\
&= MN < \infty
\end{aligned}$$

So $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$.

Exercise 2.21(c)

Recall the definition of convolution of f and g given by

$$(f * g) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$$

Suppose that f and g are measurable functions on $\mathbb{R}^d$. Show that $f * g$ is well defined for a.e. x (that is, $f(x-y)g(y)$ is integrable on $\mathbb{R}^d$ for a.e. x).

Proof.
Since we know that $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2d}$ by **Exercise 2.21(b)**, then by **Fubini's theorem**, so for a.e. $x \in \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} f(x-y)g(y)\,dy = (f * g)(x) < \infty$$

Hence $f * g$ is well defined for a.e. x.

Exercise 2.21(d)

Suppose that f and g are measurable functions on $\mathbb{R}^d$. Show that $f * g$ is integrable whenever f and g are integrable, and that

$$||f * g||_{L^1(\mathbb{R}^d)} \leq ||f||_{L^1(\mathbb{R}^d)}||g||_{L^1(\mathbb{R}^d)},$$

with equality if f and g are non-negative.

Proof.
From **Exercise 2.21(b)** and **Exercise 2.21(c)**, we have

$$\begin{aligned} ||f * g||_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |(f * g)(x)|\,dx \\ &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)g(y)\,dy \right| dx \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-y)g(y)|\,dydx \\ &= MN = ||f||_{L^1(\mathbb{R}^d)}||g||_{L^1(\mathbb{R}^d)} \end{aligned}$$

Moreover, if f and g are positive functions, then $|f(x-y)g(y)| = f(x-y)g(y)$, so the equality holds.

Exercise 2.24(a)

Consider the convolution

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy.$$

Show that $f * g$ is uniformly continuous when f is integrable and g bounded.

Proof.

If f is integrable and $\exists M \geq 0$ such that $|g| \leq M$. So $\forall x,\ z \in \mathbb{R}^d$, we have

$$\begin{aligned}|(f * g)(x) - (f * g)(z)| &= \left| \int_{\mathbb{R}^d} (f(x-y) - f(z-y))f(y)\,dy \right| \\ &\leq M \int_{\mathbb{R}^d} |f(x-y) - f(z-y)|\,dy \\ &= M \int_{\mathbb{R}^d} |f(-y) - f(z-y)|\,dy.\end{aligned}$$

Since $f \in L^1(\mathbb{R}^d)$, by **Proposition 2.5** in the textbook, $\forall \varepsilon > 0$, $\exists \delta$ such that $||z-x|| < \delta \Rightarrow ||f(y) - f(y-(z-x))||_{L^1} < \varepsilon$. Thus,

$$|(f * g)(x) - (f * g)(z)| \leq M||f(y) - f(y-(z-x))||_{L^1} < M\varepsilon.$$

Hence, the convolution $(f * g)(x)$ is uniformly continuous.

Exercise 2.24(b)

If in addition g is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

Proof.

By **Exercise 2.21(d)**, if $f,\ g \in L^1(\mathbb{R}^d)$, then $(f * g)(x)$ will be $L^1(\mathbb{R}^d)$. Moreover, by **Exercise 2.24(a)**, we have $(f * g)(x)$ is uniformly continuous, and integrable. By **Exercise 2.6(b)**, then

$$\lim_{|x| \to \infty} (f * g)(x) = 0.$$