Calculus 1 10/3 Note
Module Class 07

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Section 3.1: Derivatives of Polynomials and Exponential Functions

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The Power Rule

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example:

(1) Given any constant c, then $\dfrac{d}{dx}\ c = \dfrac{d}{dx}\ cx^0 = c\ (0 \cdot x^{-1}) = 0.$

(2) $\dfrac{d}{dx}\ \left(\sqrt[3]{x^2}\right) = \dfrac{d}{dx}\ x^{2/3} = \dfrac{2}{3}\ x^{-1/3}.$

Some Rules for Derivatives

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1. **Constant Multiple:**
   If c is a constant and f is a differentiable function, then
   $$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x).$$
2. **Sum and Difference:**
   If f and g are both differentiable, then
   $$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$$

Definition of the Number e

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e is the number such that $\displaystyle\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$ Moreover, the derivative of the natural exponential function is

$$\frac{d}{dx}(e^x) = e^x.$$

Example:
Suppose that the position of an object is given by

$$s(t) = te^t$$

Does the object ever stop moving?

Sol.
First, we will need the derivative. We need this to determine if the object ever stops moving since at that point (provided there is one) the velocity will be zero and recall that the derivative of the position function is the velocity of the object.

The derivative is

$$s'(t) = e^t + te^t = (1+t)e^t$$

So we need to determine if the derivative is ever zero.
To do this we will need to solve

$$(1+t)e^t = 0$$

Now, we know that exponential functions are never zero and so this will only be zero at $t = -1$.
So, if we are going to allow negative values of t then the object will stop moving once at $t = -1$. If we aren't going to allow negative values of t then the object will never stop moving.

Exercise:
Differentiate the following functions.

1. $y = \dfrac{\sqrt{x} + x}{x^2}$
2. $G(x) = (1 + x^{-1})^2$
3. $y = e^{x+1} + 1$

Sol.
1. $-\frac{3}{2}x^{-5/2} - x^{-2}$ 2. $-2x^{-2} - 2x^{-3}$ 3. e^{x+1}

Section 3.2: The Product and Quotient Rules

The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

The Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Example:

1. Differentiate each of the following functions.

   (a) $f(x) = \sqrt[3]{x^2}\,(2x - x^2)$

   Sol.

   $$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(2x - x^2) + x^{\frac{2}{3}}(2 - 2x) = \frac{10}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{\frac{5}{3}}$$

   (b) $h(x) = \dfrac{4\sqrt{x}}{x^2 - 2}$

   Sol.

   $$h'(x) = \frac{4(\frac{1}{2})x^{-\frac{1}{2}}(x^2 - 2) - 4x^{\frac{1}{2}}(2x)}{(x^2 - 2)^2} = \frac{-6x^{\frac{3}{2}} - 4x^{-\frac{1}{2}}}{(x^2 - 2)^2}$$

2. Suppose that the amount of air in a balloon at any time t is given by

   $$V(t) = \frac{6\sqrt[3]{t}}{4t + 1}$$

   Determine if the balloon is being filled with air or being drained of air at $t = 8$.

   Sol.

   If the balloon is being filled with air then the volume is increasing and if it's being drained of air then the volume will be decreasing.

   In other words, we need to get the derivative so that we can determine the rate of change of the volume at $t = 8$.

   This will require the quotient rule.

   $$V'(t) = \frac{2t^{-\frac{2}{3}}(4t + 1) - 6t^{\frac{1}{3}}(4)}{(4t + 1)^2} = \frac{-16t^{\frac{1}{3}} + 2t^{-\frac{2}{3}}}{(4t + 1)^2}$$

   The rate of change of the volume at $t = 8$ is then

   $$V'(8) = -\frac{7}{242}$$

   So the rate of change of the volume at $t = 8$ is negative and so the volume must be decreasing.

   Therefore, air is being drained out of the balloon at $t = 8$.

Exercise:
Compute $Q'(0)$, where

$$Q(x) = \frac{1 + x + x^2 + xe^x}{1 - x + x^2 - xe^x}$$

Sol.
$Q'(0) = 4$.

Differentiation Formulas

$$\frac{d}{dx}\, c = 0 \qquad \frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x \qquad (cf)' = cf'$$

$$(f+g)' = f' + g' \qquad (f-g)' = f' - g' \qquad (fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

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Section 3.3: Derivatives of Trigonometric Functions

Note:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x\ \cot x \qquad \frac{d}{dx}(\sec x) = \sec x\ \tan x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Example:

1. $\lim\limits_{\theta \to 0} \frac{\sin\theta}{6\theta}$

   Sol.

$$\lim_{\theta \to 0} \frac{\sin\theta}{6\theta} = \frac{1}{6}\left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right) = \frac{1}{6}(1) = \frac{1}{6}$$

2. $\lim\limits_{x \to 4} \frac{\sin(x-4)}{x-4}$

   Sol.

$$\lim_{x \to 4} \frac{\sin(x-4)}{x-4} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

3. Let $g(x) = 3\sec x - 10\cot x$. Find $g'(x)$.

   Sol.

$$g'(x) = 3\sec x\ \tan x - 10(-\csc^2 x) = 3\sec x\ \tan x + 10\csc^2 x$$

Exercise:

1. Differentiate the following functions.
   (a) $f(x) = e^x(\cos x + cx)$ where c is the constant.
   (b) $h(x) = xe^x \cot x$

2. Find the following limit.
   (a) $\lim_{x\to 0} \frac{\sin x}{\sin(\pi x)}$
   (b) $\lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2}$

Sol.
1.(a) $f'(x) = e^x(\cos x - \sin x + cx + c)$ (b) $h'(x) = e^x(\cot x + x\cot x - x\csc^2 x)$
2.(a) $\frac{1}{\pi}$ (b) $\frac{1}{3}$

Section 3.4: The Chain Rule

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x))\dot{g}'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The Power Rule Combined with the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x).$$

Example:
Find $\frac{d}{dx}(b^x)$.

Sol.
Recall that $b = e^{\ln b}$. So

$$b^x = \left(e^{\ln b}\right)^x = e^{(\ln b)x}$$

and Chain Rule gives

$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{(\ln b)x}\right) = e^{(\ln b)x}\frac{d}{dx}(\ln b)x = e^{(\ln b)x} \cdot \ln b = b^x \ln b.$$

Hence,

$$\frac{d}{dx}(b^x) = b^x \ln b.$$

Exercise:

1. Find the derivative of the following function.

   (a) $f(x) = e^{\sin(2x)} + \sin(e^{2x})$

   (b) $h(x) = 2^{3^{4^x}}$

2. Find y' and y''.

   (a) $y = \dfrac{1}{(1+\tan x)^2}$

   (b) $y = e^{e^x}$

3. If $F(x) = f(xf(xf(x)))$, where $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$, $f'(2) = 5$, and $f'(3) = 6$, find $F'(1)$.

Sol.
1.(a) $f'(x) = 2\cos(2x)e^{\sin(2x)} + 2e^{2x}\cos(e^{2x})$ (b) $h'(x) = (\ln 2)(\ln 3)(\ln 4)4^x 3^{4^x} 2^{3^{4^x}}$

2.(a) $y' = [-2(1+\tan x)^{-3}](\sec x)^2$, $y'' = \dfrac{2\sec^2 x(\tan^2 x - 2\tan x + 3)}{(1+\tan x)^4}$

(b) $y' = e^{e^x + x}$, $y'' = e^{e^x + x}(1 + e^x)$

3. $F'(1) = 198$