Calculus 2 1/2 Note
Module Class 07

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Section 10.1: Curves Defined by Parametric Equations

Parametric Curves

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which w can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**.

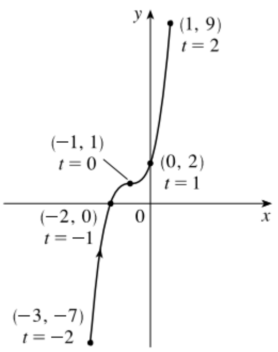
Example:

1. Given $x = t - 1,\ y = t^3 + 1,\ -2 \le t \le 2$

   (a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

   Sol.

t	-2	-1	0	1	2
x	-3	-2	-1	0	1
y	-7	0	1	2	9



   (b) Eliminate the parameter to find a Cartesian equation of the curve.

   Sol.

$$x = t - 1 \ \Rightarrow\ t = x + 1,$$

so

$$y = t^3 + 1 \ \Rightarrow\ y = (x+1)^3 + 1 \quad \text{with } -3 \le x \le 1.$$

Section 10.2: Calculus with Parametric Curves

Tangents

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t), y = g(t)$, where y is also a differentiable function of x. Then the **Chain Rule** gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we can solve for dy/dx:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if } \frac{dx}{dt} \neq 0.$$

Example:

1. Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

$$x = e^t \sin \pi t,\ y = e^{2t};\ t = 0$$

Sol.

$$\frac{dy}{dt} = 2e^{2t}, \quad \frac{dx}{dt} = e^t(\pi \cos \pi t + \sin \pi t),$$

and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2e^{2t}}{e^t(\pi \cos \pi t + \sin \pi t)} = \frac{2e^t}{\pi \cos \pi t + \sin \pi t}.$$

When $t = 0$, then $(x, y) = (0, 1)$ and $dy/dx = 2/\pi$, so equation of the tangent to the curve at the point corresponding to $t = 0$ is

$$y - 1 = \frac{2}{\pi}(x - 0) \quad \text{or} \quad y = \frac{2}{\pi}x + 1.$$

2. Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

$$x = 2\sin t,\ y = 3\cos t,\ 0 < t < 2\pi$$

Sol.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3\sin t}{2\cos t} = -\frac{3}{2}\tan t,$$

so

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{3}{2}\sec^2 t}{2\cos t} = -\frac{3}{4}\sec^3 t.$$

The curve is concave upward when

$$\sec^3 t < 0 \ \Rightarrow\ \sec t < 0 \ \Rightarrow\ \cos t < 0 \ \Rightarrow\ \frac{\pi}{2} < t < \frac{3\pi}{2}.$$

Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geq 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the **Substitution Rule for Definite Integrals** as follow:

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt \qquad \left[\text{or } \int_\beta^\alpha g(t)f'(t)\,dt\right]$$

Example:

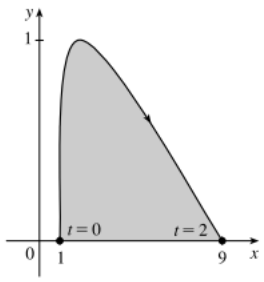
1. Find the area enclosed by the x-axis and the curve $x = t^3 + 1$, $y = 2t - t^2$.

   Sol.

   The curve $x = t^3 + 1,\ y = 2t - t^2 = t(2-t)$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 2$. (The corresponding values of x are 1 and 9.)

   The shaded area is given by

$$\int_0^2 [y(t) - 0]\,x'(t)\,dt = \int_0^2 (2t - t^2)(3t^2)\,dt = \frac{24}{5}.$$



Arc Length

If a curve C is decribed by the parametric equations $x = f(x), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases form α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example:

1. Find the exact length of the curve.

$$x = t\sin t,\ y = t\cos t,\ 0 \leq t \leq 1$$

   Sol.

$$\frac{dx}{dt} = t\cos t + \sin t \quad \text{and} \quad \frac{dy}{dt} = -t\sin t = \cos t,$$

so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (t\cos t + \sin t)^2 + (-t\sin t = \cos t)^2 = t^2 + 1.$$

Thus,

$$\text{arc length} = \int_0^1 \sqrt{t^2+1}\,dt = \left[\frac{1}{2}t\sqrt{t^2+1} + \frac{1}{2}\ln(t + \sqrt{t^2+1})\right]_{t=0}^1 = \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1+\sqrt{2}).$$

Surface Areas

Suppose the curve c gives by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f', g' are continuous, $g(t) \geq 0$, is rotated about the $x-$axis. If C is traversed exactly once as t increases from α to β, then the area of the resulting surface is given by

$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

The general symbolic formula $S = \int 2\pi y\,ds$ and $S = \int 2\pi x\,ds$ are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Example:

1. Find the exact area of the surface obtain by rotating the given curve about the x-axis.

$$x = 3t - t^3,\ y = 3t^2,\ 0 \leq t \leq 1$$

Sol.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = [3(1+t^2)]^2.$$

Thus,

$$\text{surface area} = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 2\pi \cdot 3t^2 \cdot 3(1+t^2)\,dt = 18\pi \int_0^1 (t^2+t^4)\,dt = \frac{48}{5}\pi.$$

Exercise:

1. Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

$$x = \sqrt{t},\ y = t^2 - 2t;\ t = 4$$

2. Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

$$x = \cos t,\ y = \sin(2t),\ 0 < t < \pi$$

3. Find the area enclosed by the curve $x = t^2 - 2t, y = \sqrt{t}$ and the y-axis.

4. Find the exact length of the curve.

$$x = e^t + e^{-t},\ y = 5 - 2t,\ 0 \le t \le 3$$

5. Find the exact area of the surface obtain by rotating the given curve about the x-axis.

$$x = 2t^2 + \frac{1}{t},\ y = 8\sqrt{t},\ 1 \le t \le 3$$

Sol.

1. $y = 24x - 40$

2. $\dfrac{dy}{dx} = \dfrac{2\cos(2t)}{-\sin t},\ \dfrac{d^2y}{dx^2} = -\dfrac{\cos t(4\sin^2 t + 2)}{\sin^3 t}$ and the curve is concave upward when $\frac{\pi}{2}, t < \pi$.

3. $\dfrac{8\sqrt{2}}{15}$

4. $e^3 - e^{-3}$

5. $\frac{32\pi}{15}(103\sqrt{3} + 3)$

Section 10.3: Polar Coordinates

Polar Coordinates

If the point P has Cartesian coordinates (x, y) and the polar coordinates (x, y) and polar coordinates (r, θ), then we have

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

and so

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Moreover, we have

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

Example:

1. Find a polar equation for the curve represented by the given Cartesian equation.

$$xy = 4$$

Sol.

$$xy = 4 \;\Leftrightarrow\; (r\cos\theta)(r\sin\theta) = 4 \;\Leftrightarrow\; r^2\left(\frac{1}{2}\cdot 2\sin\theta\cos\theta\right) = 4 \;\Leftrightarrow\; r^2\sin(2\theta) = 8 \;\Leftrightarrow\; r^2 8\csc(2\theta)$$

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Tangent to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equation as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then, using the method for finding slopes of parametric curves and the **Product Rule**, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Example:

1. Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

$$r = \frac{1}{\theta},\ \theta = \pi$$

Sol.

Since $r = \frac{1}{\theta}$, then

$$x = r\cos\theta = \frac{\cos\theta}{\theta}, \quad y = r\sin\theta = \frac{\sin\theta}{\theta}$$

such that

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta(-1/\theta^2) + (1/\theta)\cos\theta}{\cos\theta(-1/\theta^2) + (-1/\theta)\cos\theta} = \frac{-\sin\theta + \theta\cos\theta}{-\cos\theta - \theta\sin\theta}.$$

When $\theta = \pi$, then

$$\frac{dy}{dx} = -\pi.$$

Exercise:

1. Find a polar equation for the curve represented by the given Cartesian equation.

$$x^2 - y^2 = 4$$

2. Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

$$r = 1 + 2\cos\theta,\ \theta = \frac{\pi}{3}$$

Sol.

1. $r^2\cos(2\theta) = 4$ 2. $\dfrac{\sqrt{3}}{9}$