Calculus 1 10/31 Note
Module Class 07

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Section 4.7: Optimization Problems

The First Derivative Test for Absolute Extrema

Suppose that c us a critical number of a continuous function f defined on the interval I.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval I.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval I.

The Second Derivative Test for Absolute Extrema

Suppose that c us a critical number of a continuous function f defined on the interval I.

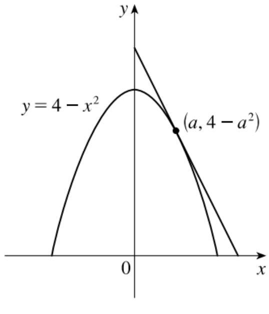
(a) If $f''(x) > 0$ for all $x \in I$, then $f(c)$ will be the absolute maximum value of $f(x)$ on the interval I.

(b) If $f''(x) < 0$ for all $x \in I$, then $f(c)$ will be the absolute minimum value of $f(x)$ on the interval I.

Example:

What is the smallest possible area of the triangle that is cut off by the first quadrant and whose hypotenuse is tangent to the parabola $y = 4 - x^2$ at some point?

Sol.



Since

$$y = 4 - x^2 \;\Rightarrow\; y' = -2x,$$

so an equation of the tangent line at $(a,\, 4 - a^2)$ is $y - (4 - a^2) = -2a(x - a)$ or $y = -2ax + a^2 + 4$.

The y-intercept $(x = 0)$ is $a^2 + 4$. The x-intercept $(y = 0)$ is $\frac{a^2+4}{2a}$.

The area A of the triangle is

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2} \cdot \frac{a^2 + 4}{2a}(a^2 + 4) = \frac{1}{4}\,\frac{a^4 + 8a^2 + 16}{a} = \frac{1}{4}\left(a^3 + 8a + \frac{16}{a}\right).$$

Moreover,

$$\begin{aligned} A' = 0 &\Rightarrow \frac{1}{4}\left(3a^2 + 8 - \frac{16}{a^2}\right) = 0 \\ &\Rightarrow 3a^4 + 8a^2 - 16 = 0 \\ &\Rightarrow (3a^2 - 4)(a^2 + 4) = 0 \\ &\Rightarrow a^2 = \frac{4}{3} \\ &\Rightarrow a = \frac{2}{\sqrt{3}} \qquad (a > 0). \end{aligned}$$

Also,

$$A'' = \frac{1}{4}\left(6a + \frac{32}{a^3}\right) > 0.$$

By **The Second Derivative Test for Absolute Extrema**, there is an absolute minimum at

$$a = \frac{2}{\sqrt{3}}.$$

Thus,

$$A = \frac{1}{2} \cdot \frac{\frac{4}{3} + 4}{2 \cdot \frac{2}{\sqrt{3}}}\left(\frac{4}{3} + 4\right) = \frac{1}{2} \cdot \frac{4\sqrt{3}}{3} \cdot \frac{16}{3} = \frac{32}{9}\sqrt{3}.$$

Exercise:

Find an equation of the line through the point (3, 5) that cuts off the least area from the first quadrant.

Sol.

Absolute minimum area occurs when $m = -\frac{5}{3}$, then the line is $y - 5 = -\frac{5}{3}(x - 3)$ or $y = -\frac{5}{3}x + 10$.