Real Analysis Extra Homework
Chapter 2. Integration Theory

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4. Suppose f is integrable on $[0, b]$, and
$$g(x) = \int_x^b \frac{f(t)}{t} dt \qquad \text{for } 0 \le x \le b.$$
Prove that g is integrable on $[0, b]$ and
$$\int_0^b g(x)dx = \int_0^b f(t)dt\,.$$

We begin by noting that because we can always write $f = f^+ - f^-$ we can assume without loss of generality that f is non-negative (otherwise we would just look at each non-negative part separately). We then consider the function

$$g(x) = \int_x^b \frac{f(t)}{t} dt$$

which is defined on the interval $I = (0, b]$. More generally, let $I_x = (x, b]$. We want to integrate g, so we observe that

$$\int_I g(x)dx = \int_I \int_{I_x} \frac{f(t)}{t} dtdx$$

This would lead us to consider the function

$$h(x,t) = \frac{f(t)}{t} \chi_{I_x}$$

Note that h is measurable because it is the quotient of $f(t)$ and t, which are both measurable on I_x, multiplied by χ_{I_x}, which is clearly measurable. Furthermore, because we took f to be non-negative, h is also non-negative. We then rewrite the above to see

$$\int_I g(x)dx = \int_I \int_{I_x} \frac{f(t)}{t} dtdx = \int_{\mathbb{R}} \int_{\mathbb{R}} h(x,t)\, \chi_I \, dtdx$$

We then note that we satisfy the hypotheses for Fubini's theorem, so we can exchange the order of integration to get

$$\int_{\mathbb{R}} \int_{\mathbb{R}} h(x,t)\, \chi_I \, dtdx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x,t)\, \chi_I \, dx \right) dt$$

Simplifying we get

$$\begin{aligned}
\int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x,t)\, \chi_I \, dx \right) dt &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{f(t)}{t}\, \chi_I \, dx \right) \chi_{I_x} \, dt \\
&= \int_0^b \left(\int_0^t \frac{f(t)}{t} dx \right) dt \\
&= \int_0^b t \frac{f(t)}{t} \\
&= \int_0^b f(t) dt
\end{aligned}$$

Because f is integrable on $(0,b]$, we have that g must also be integrable and that

$$\int_0^b g(x)dx = \int_0^b f(t)dt$$

6. Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

(a) There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but yet $\limsup_{x\to\infty} f(x) = \infty$.

(b) However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

[Hint: For (a), construct a continuous version of the function equal to n on the segment $[n, n+1/n^3)$, $n \geq 1$.]

(a) Consider the function f such that for each $n \geq 1$

$$f(x) = \begin{cases} n & x \in [n, n+1/n^3) \\ 0 & \text{otherwise} \end{cases}$$

Pictorially, f consists of rectangles with width $1/n^3$ and height n on each interval $[n, n+1)$. We can see that f is integrable because if we consider the interval $I_n = [n, n+1]$, then

$$\int_{I_n} f = n/n^3 = 1/n^2$$

Then note that

$$\int_{\mathbb{R}} f = \sum_{n=1}^{\infty} \inf_{I_n} f = \sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6}$$

However, we can see that

$$\limsup_{x\to\infty} f(x) = \lim_{x\to\infty}(\sup_{y\geq x} f(y)) = \infty$$

because the "rectangles" get arbitrarily high and so for every x we can find $y > x$ such that $f(y) > M$ for any M.

(b) We will prove the contrapositive. Namely, if f is uniformly continuous and $f \not\to 0$ as $|x| \to \infty$ then f is not Lebesgue integrable. Let $\epsilon > 0$ be given and find $\delta_0 > 0$ such that $|x - y| < \delta_0$ implies $|f(x) - f(y)| < \epsilon/2$, then set $\delta = \min\{\delta_0, 1/2\}$. Because $f \not\to 0$ we can find some x_0 such that $f(x_0) > \epsilon$. But then in some δ-neighborhood of x_0, $|f| > \epsilon/2$. We iterate this process, because we know we can always find an $x_{n+1} > x_n + 1$ such that $f(x_{n+1}) > \epsilon$ and so in some δ-neighborhood of x_{n+1} we have that $|f| > \epsilon/2$. Furthermore, each of these neighborhoods, $\{N_n\}$ are disjoint because $|x_{n+1} - x_n| > 1$ and $\delta > 1/2$. This means that

$$\int_{\mathbb{R}} |f| \geq \int_{\cup N_n} |f| \geq \sum_{n=1}^{\infty} \frac{\epsilon}{2}(2\delta) = \sum_{n=1}^{\infty} \epsilon\delta$$

The sum on the right diverges and so f is not integrable. This verifies the contrapositive, and completes the proof.

7. Let $\Gamma \subseteq \mathbb{R}^d \times \mathbb{R}$, $\Gamma = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\}$, and assume f is measurable on $\mathbb{R}^d$. Show that Γ is a measurable subset of $\mathbb{R}^{d+1}$, and $m(\Gamma) = 0$.

Consider the function

$$\begin{aligned} F : \mathbb{R}^d \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto f(x)\,. \end{aligned}$$

Given $\alpha \in \mathbb{R}$, the set

$$F^{-1}((\alpha, \infty)) = f^{-1}((\alpha, \infty)) \times \mathbb{R}$$

is measurable in $\mathbb{R}^{d+1}$; hence G is a measurable function. Also the projection

$$\begin{aligned} \mathbb{R}^d \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto y \end{aligned}$$

is measurable (in fact continuous). Thus their difference

$$\begin{aligned} G : \mathbb{R}^d \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto f(x) - y \end{aligned}$$

is a measurable function, thus $G^{-1}(\{0\}) = \Gamma$ is a measurable subset of $\mathbb{R}^{d+1}$. And by Corollary 2.3.3,

$$m(\Gamma) = \int_{\mathbb{R}^d} m(\Gamma^x)dx = \int_{\mathbb{R}^d} m(\{f(x)\})dx = \int 0dx = 0.$$

8. If f is integrable on $\mathbb{R}$, show that $F(x) = \int_{-\infty}^{x} f(t)dt$ is uniformly continuous.

Given $x, y \in \mathbb{R}$ with $x \leq y$, by additivity of the Lebesgue integral we have

$$\begin{aligned} \int_{(-\infty,x]} f + \int_{[x,y]} f &= \int_{(-\infty,y]} f \\ \int_{-\infty}^{x} f(t)dt + \int_{x}^{y} f(t)dt &= \int_{-\infty}^{y} f(t)dt\,. \end{aligned}$$

Since f is integrable on $\mathbb{R}$, the above integrals are all finite. Therefore we can perform usual algebra to get

$$\begin{aligned} \int_{x}^{y} f(t)dt &= \int_{-\infty}^{y} f(t)dt - \int_{-\infty}^{x} f(t)dt \\ &= F(y) - F(x)\,. \end{aligned}$$

Given $\varepsilon > 0$, by Proposition 1.12 part (ii) in Stein & Shakarchi's text, there exists $\delta > 0$ such that

$$|F(y) - F(x)| = \left| \int_{x}^{y} f(t)dt \right| \leq \int_{x}^{y} |f(t)|dt < \varepsilon$$

whenever $|x - y| < \delta$ (taking $E = [x, y]$ in the statement of the proposition). This is precisely uniform continuity for F.

9. Tchebychev inequality. Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, prove that

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f.$$

To see this inequality observe that if fix $\alpha > 0$ and define

$$E_\alpha = \{x \,|\, f(x) > \alpha\}$$

Then we have that

$$\int_{E_\alpha} f \geq \int_{\mathbb{R}^n} f\chi(E_\alpha) \geq \alpha m(E_\alpha)$$

Re-ordering the terms gives

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f$$

10. Suppose $f \geq 0$, and let $E_{2^k} = \{x : f(x) > 2^k\}$ and $F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}$. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\},$$

and the sets F_k are disjoint.

Prove that f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \quad \text{if and only if} \quad \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

Use this result to verify the following assertions. Let

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is integrable on $\mathbb{R}^d$ if and only if $a < d$; also g is integrable on $\mathbb{R}^d$ if and only if $b > d$.

Note that F_k's are disjoint so letting $E = \{x : f(x) > 0\}$, as $f = 0$ outside E by non-negativity we have

$$\int f = \int_E f = \sum_{k=-\infty}^{\infty} \int_{F_k} f$$

and by the definition of F_k, we have

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) \leq \sum_{-\infty}^{\infty} \int_{F_k} f \leq \sum_{k=-\infty}^{\infty} 2^{k+1} m(F_k) = 2 \cdot \sum_{k=-\infty}^{\infty} 2^k m(F_k)$$

so $\int f$ is finite if and only if $\sum_{k=-\infty}^{\infty} 2^k m(F_k)$ is finite.

Observe that for every k, we have

$$E_{2^k} = E_{2^{k+1}} \sqcup F_k$$

hence

$$\begin{aligned}\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) &= \sum_{k=-\infty}^{\infty} 2^k m(E_{2^{k+1}}) + \sum_{k=-\infty}^{\infty} 2^k m(F_k) \\ &= \frac{1}{2} \sum_{k=-\infty}^{\infty} 2^{k+1} m(E_{2^{k+1}}) + \sum_{k=-\infty}^{\infty} 2^k m(F_k)\,.\end{aligned}$$

Now since $\sum_{k=-\infty}^{\infty} 2^{k+1} m(E_{2^{k+1}}) = \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k})$, we obtain

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) = \frac{1}{2} \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k})$$

so we get the second if and only if.

Consider the given f and E_{2^k}'s defined by it. If $a < 0$ then f is clearly integrable. So we may assume $a > 0$. Then we have

$$\begin{aligned}E_{2^k} &= \{x \in \mathbb{R}^d : |x|^{-a} > 2^k, |x| \leq 1\} \\ &= \{x \in \mathbb{R}^d : |x|^a < 2^{-k}, |x| \leq 1\} \\ &= \{x \in \mathbb{R}^d : |x| < \min\{1, 2^{-k/a}\}\}\end{aligned}$$

and since $2^{-k/a} \leq 1$ if and only if $-k/a \leq 0$ if and only if $k \geq 0$, we get

$$E_{2^k} = \begin{cases} B_{\mathbb{R}^d}(1) & \text{if } k < 0 \\ B_{\mathbb{R}^d}(2^{-k/a}) & \text{if } k \geq 0 \end{cases}$$

where $B_{\mathbb{R}^d}(r)$ denotes the ball centered at the origin of radius r in $\mathbb{R}^d$. Thus

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = \sum_{k=-\infty}^{-1} 2^k m(B_{\mathbb{R}^d}(1)) + \sum_{k=0}^{\infty} 2^k m(B_{\mathbb{R}^d}(2^{-k/a}))$$

Observe that a ball of radius r in $\mathbb{R}^d$ contains a cube of side-length r and be contained in a cube of side-length $2r$ in $\mathbb{R}^d$. Thus

$$r^d \le m(B_{\mathbb{R}^d}(r)) \le (2r)^d = 2^d r^d$$

therefore

$$\sum_{k=-\infty}^{-1} 2^k + \sum_{k=0}^{\infty} 2^k (2^{-k/a})^d \le \sum_{k=-\infty}^{k=\infty} 2^k m(E_k) \le 2^d \left(\sum_{k=-\infty}^{-1} 2^k + \sum_{k=0}^{\infty} 2^k (2^{-k/a})^d \right)$$

so as $\sum_{k=-\infty}^{-1} 2^k = 1$, it is enough to determine when $\sum_{k=0}^{\infty} 2^k (2^{-k/a})^d$ converges for the integrability of f. And

$$\sum_{k=0}^{\infty} 2^k (2^{-k/a})^d = \sum_{k=0}^{\infty} 2^{k-kd/a} = \sum_{k=0}^{\infty} 2^{k(1-d/a)} = \sum_{k=0}^{\infty} (2^{(1-d/a)})^k$$

converges iff $2^{1-d/a} < 1$ iff $1 - d/a < 0$ iff $d/a > 1$ iff $a < d$.

Now consider the given g and E_{2^k}'s defined by it. We may define g to be 1 when $|x| \le 1$ which does not affect the integrability of g. If $b < 0$ then g is clearly *not* integrable. So we may assume $b > 0$. In this case we have

$$\begin{aligned} E_{2^k} &= \{x \in \mathbb{R}^d : |x|^{-b} > 2^k, |x| > 1\} \cup \{x \in \mathbb{R}^d : 1 > 2^k, |x| \le 1\} \\ &= \{x \in \mathbb{R}^d : |x|^b < 2^{-k}, |x| > 1\} \cup \{x \in \mathbb{R}^d : 1 > 2^k, |x| \le 1\} \\ &= \{x \in \mathbb{R}^d : 1 < |x| < 2^{-k/b}\} \cup \{x \in \mathbb{R}^d : 1 > 2^k, |x| \le 1\}. \end{aligned}$$

Note that since $2^{-k/b} > 1$ iff $-k/b > 0$ iff $k < 0$; so $E_{2^k} = \varnothing$ if $k \ge 0$. And for $k < 0$ we have

$$\begin{aligned} E_{2^k} &= \{x \in \mathbb{R}^d : 1 < |x| < 2^{-k/b}\} \cup \{x \in \mathbb{R} : |x| \le 1\} \\ &= \{x \in \mathbb{R}^d : |x| < 2^{-k/b}\}. \end{aligned}$$

Because $E_{2^k} = B_{\mathbb{R}^d}(2^{-k/b})$, we have

$$\sum_{k=-\infty}^{-1} 2^k (2^{-k/b})^d \le \sum_{k=-\infty}^{-1} 2^k m(E_k) \le 2^d \sum_{k=-\infty}^{-1} 2^k (2^{-k/b})^d.$$

Since $\sum_{k=-\infty}^{-1} 2^k (2^{-k/b})^d = \sum_{k=-\infty}^{-1} 2^{k(1-d/b)} = \sum_{k=1}^{\infty} 2^{(d/b-1)k}$ converges iff $2^{(d/b-1)} < 1$ iff $b > d$, the series

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = \sum_{k=-\infty}^{-1} 2^k m(E_{2^k})$$

also converges iff $b > d$ as desired.

11. Prove that if f is integrable on $\mathbb{R}^d$, real-valued, and $\int_E f(x)dx \geq 0$ for every measurable E, then $f(x) \geq 0$ a.e. x. As a result, if $\int_E f(x)dx = 0$ for every measurable E, then $f(x) = 0$ a.e.

Write $E_n = \{x \in \mathbb{R}^d : f(x) < -1/n\}$. E_n's are measurable. Note that

$$\{x \in \mathbb{R}^d : f(x) < 0\} = \bigcup_{n \in \mathbb{N}} E_n \,,$$

So it is enough to show that every E_n has measure zero. Suppose not, so $m(E_n) > 0$ for some n. So using the assumption on E_n, we have

$$0 \leq \int_{E_n} f \leq \int_{E_n} \frac{-1}{n} = \frac{-1}{n} m(E_n) < 0 \,,$$

a contradiction.

Now let's do the second part. By the first part, we have $f \geq 0$ a.e. Writing $g = -f$, since $\int_E g = \int_E (-f) = 0$ for every measurable E, again by the first part we deduce that $-f = g \geq 0$ a.e. Thus $f \leq 0$ a.e. and hence $f = 0$ a.e.

15. Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. However, observe that the series is unbounded on every interval, and in fact, any function $\widetilde{F}$ that agrees with F a.e. is unbounded in any interval.

Note that

$$\int f \chi_{(1/n,1)} = \int_{1/n}^{1} x^{-1/2} dx = 2\sqrt{x}\Big|_{1/n}^{1} = 2 - 2\sqrt{1/n}$$

so by monotone convergence theorem

$$\int f = \int f \chi_{(0,1)} = \lim_{n \to \infty} (2 - 2\sqrt{1/n}) = 2 \,.$$

Note that by the translation invariance of the Lebesgue integral, we have

$$\int f(x)dx = \int f(x - r)dx$$

for any $r \in \mathbb{R}$. Thus

$$\int \sum_{n=1}^{N} 2^{-n} f(x - r_n) dx = \sum_{n=1}^{N} 2^{-n} \int f(x) dx = \sum_{n=1}^{N} 2^{-n+1} = \sum_{n=0}^{N-1} 2^{-n}$$

and again by the monotone convergence theorem

$$\int F = \sum_{n=0}^{\infty} 2^{-n} = 2\,.$$

Let I be a interval. Let r_N be a rational number in I. Then for every $M > 1$, whenever $x \in (r_N, r_N + 2^{-2N}/M^2)$ we have $0 < x - r_N < 2^{-2N}/M^2 < 1$ so since f is decreasing

$$f(x - r_N) > f(2^{-2N}/M^2) = 2^N M\,.$$

But $(r_N, r_N + 2^{-2N}/M^2)$ intersects I in a nonempty interval which has positive measure. So on a set of positive measure contained in I, we have

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n) \geq 2^{-N} f(x - r_N) > M\,.$$

So if $\widetilde{F}$ is a.e. equal to F then $\widetilde{F}$ also has to be larger than M in a set of positive measure zero contained in I. Since M was arbitrary, we get the desired conclusion.