Real Analysis Homework
Chapter 1. Measure theory
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Recall Theorem 1.3:

Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be writen uniquely as a countable union of disjoint open intervals.

Exercise of 9/11:
1. Please prove the uniqueness of Theorem 1.3.

Proof.
Suppose that open subset $\mathcal{O}$ of $\mathbb{R}$ can be writen in two different countable union of disjoint open intervals $\mathcal{I}$ and $\widetilde{\mathcal{I}}$.

Fixed $x \in \mathcal{O}$, and

$$\mathcal{I}_x = \underset{y\in Y}{\cup}(\mathcal{I}_x \cap \widetilde{\mathcal{I}}_y).$$

Then $\underset{y\in Y}{\cup}(\mathcal{I}_x \cap \widetilde{\mathcal{I}}_y)$ are disjoint open subsets of $\mathcal{I}_x$.
But $\mathcal{I}_x$ is connected. So $\mathcal{I}_x \cap \widetilde{\mathcal{I}}_y \neq \emptyset$, and thus $\mathcal{I}_x \cap \widetilde{\mathcal{I}}_y = \mathcal{I}_x$ for some $\widetilde{\mathcal{I}}_y$, that is $\mathcal{I}_x \subseteq \widetilde{\mathcal{I}}_y$.

Then prove similarly that for each $y \in Y$, there must be exactly one x so that $\widetilde{\mathcal{I}}_y \subseteq \mathcal{I}_x$ and $\widetilde{\mathcal{I}}_y \cap \mathcal{I}_{x'} = \emptyset$ if $x' \neq x$.

Recall Definition:

Let $E \in \mathbb{R}^n$ and $\{Q_k\}$ be the covering of E and Q_k be the closed cubes, then we define that

$$\text{Jordan outer meaure: } \sigma(E) = \inf_{N\in\mathbb{N}} \Sigma_{k=1}^{N} |Q_k|$$

and

$$\text{Lebesgue outer meaure: } m_*(E) = \inf_{\{Q_k\}:\ \text{countable set}} \Sigma_{k=1}^{\infty} |Q_k|$$

Exercise of 9/16:
2. Let R be a rectangle in $\mathbb{R}^n$. Prove $\sigma(R) = |R| = m_*(R)$.

Proof.
(1) Prove $\sigma(R) = |R|$.

By Example 2 in the text book (p.11), we know that $|R| \leq \sigma(R)$.

To obtain the reverse inequality, consider a grid in $\mathbb{R}^d$ formed by cubes of side length $1/k$.
Then, if $\mathcal{Q}$ consists of the (finite) collection of all cubes entirely contained in R, and $\mathcal{Q}'$ is the (finite) collection of all cubes that intersect R and the complement of R.

Note that $R \subset \bigcup\limits_{i=1,\, Q\in(\mathcal{Q}\cup\mathcal{Q}')}^{N} Q_i$. Also, a simple argument yields

$$\sum_{i=1}^{N} |Q_i| \leq |R|.$$

Moreover, there are $O(k^{d-1})$ cubes in $\mathcal{Q}'$ and these cubes have volume k^{-d}, so that $\sum\limits_{i=1}^{N}|Q| = O(1/k)$. Hence

$$\sum_{i=1}^{N}|Q| \leq |R| + O(1/k),$$

and letting k tend to infinity yields $\sigma(R) \leq |R|$.
Therefore, $\sigma(R) = |R|$.

(2) Prove $m_*(R) = |R|$.

By Example 2 in the text book (p.11), we know that $|R| \leq m_*(R)$.

To obtain the reverse inequality, consider a grid in $\mathbb{R}^d$ formed by cubes of side length $1/k$.
Then, if $\mathcal{Q}$ consists of the (finite) collection of all cubes entirely contained in R, and $\mathcal{Q}'$ is the (finite) collection of all cubes that intersect R and the complement of R.

Note that $R \subset \cup_{Q\in(\mathcal{Q}\cup\mathcal{Q}')} Q$. Also, a simple argument yields

$$\sum_{q\in\mathcal{Q}} |Q| \leq |R|.$$

Moreover, there are $O(k^{d-1})$ cubes in $\mathcal{Q}'$ and these cubes have volume k^{-d}, so that $\sum\limits_{Q\in\mathcal{Q}'} |Q| = O(1/k)$. Hence

$$\sum_{Q\in(\mathcal{Q}\cup\mathcal{Q}')} |Q| \leq |R| + O(1/k),$$

and letting k tend to infinity yields $m_*(R) \leq |R|$.
Therefore, $m_*(R) = |R|$.

By (1) and (2), then $\sigma(R) = |R| = m_*(R)$.

Exercise of 9/18:
3. *Theorem 1.3* states that every open set in $\mathbb{R}$ is the disjoint union of open intervals. The analogue in $\mathbb{R}^d$, $d \geq 2$, is generally false. Prove the following:

(a) An open disc in $\mathbb{R}^2$ is not the disjoint union of open rectangles.
[Hint: What happens to the boundary of any of these rectangles?]

(b) An open connected set Ω is the disjoint union of open rectangles if and only if Ω is itself an open rectangle.

Proof.

(a) Suppose for a contradiction that an open disc $\mathcal{O} \subset \mathbb{R}^2$ and $\mathcal{O} = \underset{n\in\mathbb{N}}{\cup} R_n$ where R_n are the open rectangles and $R_i \cap R_j = \emptyset$ for $i \neq j$.

Choose some open rectangle $R_1 \in \mathcal{O}$ and let $x \in \partial R_1$. Then, for all $\epsilon > 0$, $B(x,\epsilon) \cap R_1 \neq \emptyset$ and $B(x,\epsilon) \cap R_1^c \neq \emptyset$. Hence, $x \notin R_1$, and so there must be an open rectangle $R_2 \in \mathcal{O}$ with $x \in R_2$. This implies that there is $\epsilon_0 > 0$ such that $B(x,\epsilon_0) \subset R_2$.

By our previous observation, $B(x,\epsilon_0) \cap R_1 \neq \emptyset$. Taken together, then $R_1 \cap R_2 \neq \emptyset$, which is a contradiction with the fact that $\mathcal{O}$ is the disjoint union of open rectangles.

(b) ($\Rightarrow$) Let Ω be the disjoint union of open rectangles. Suppose, to the contrary, that Ω is not itself an open rectangle.

Then, Ω contains at least two open rectangles. By the argument in part (a), so these rectangles cannot be disjoint, which is a contradiction. Hence, it must be that Ω is itself an open rectangle.

($\Leftarrow$) Let Ω be an open rectangle. Then Ω is the disjoint union of a single open rectangle.

4. At the start of the theory, one might define the outer measure by taking coverings by rectangles instead of cubes. More precisely, we define

$$m_*^{\mathcal{R}}(E) = \inf \sum_{j=1}^{\infty} |R_j|,$$

where the inf is now taken over all countable coverings $E \subset \cup_{j=1}^{\infty} \mathcal{R}_j$ by (closed) rectangles.

Show that this approach gives rise to the same theory of measure developed in the text, by proving that $m_*(E) = m_*^{\mathcal{R}}(E)$ for every subset E of $\mathbb{R}^d$.
[Hint: Use *Lemma 1.1.*]

Proof.

(1) Let E be some subset of $\mathbb{R}^d$.

First, we begin by considering the outer measure of E, given by $m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$. Consider any covering of E by closed cubes $E \subset \cup_{j=1}^{\infty} Q_j$.

For each Q_j with side length l_j, we can pick the rectangle R_j, whose sides have length $r_1 = l + \frac{\epsilon}{j}\, l^{d-1}$ and $r_k = l$ for every other side. Then $R_j \supset E_j$ and

$$|R_j| = |Q_j| + \frac{\epsilon}{2^j}.$$

So

$$\sum_{j=1}^{\infty} |Q_j| < \sum_{j=1}^{\infty} |R_j| = \sum_{j=1}^{\infty} |Q_j| + \epsilon.$$

Passing to the infimum gives

$$m_*^R(E) \leq \sum_{j=1}^{\infty} |R_j| \leq m_*(E) + \epsilon.$$

Then letting $\epsilon \to 0$ gives $m_*^R(E) \leq m_*(E)$.

(2) Now begin with a covering of E by rectangles R_j. Then we divide $\mathbb{R}^n$ into a grid of cubes with side length $\frac{1}{k}$. Then we can define $Q_{j,k}$ to be the set of rectangles that have non-empty intersection with R_j.

We need to be precise about how much extra measure these cubes add. We know that there are Ck^{d-1} that could intersect partially intersect R_j for some suitably large C. The total volume these add is $\frac{C}{k}$ and so if $k \geq 2^j C$, then the total volume is less than $\frac{\epsilon}{2^j}$.

That is

$$\sum_k |Q_{j,k}| \leq |R_j| + \frac{\epsilon}{2^j}.$$

Then

$$\sum_j |R_j| \leq \sum_j \sum_k |Q_{j,k}| \leq \sum_j |R_j| + \epsilon.$$

Letting $\epsilon \to 0$ again gives $m_*(E) \leq m_*^R(E)$.

By (1) and (2), then $m_*(E) = m_*^{\mathcal{R}}(E)$.