Calculus 2 12/26 Note
Module Class 07

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Section 9.4: Models for Population Growth

Law of Natural Growth

If $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

$$\frac{dP}{dt} = kP$$

where k is the constant.

Solve the Problem of Natural Growth

The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

If the rate of emigration is a constant m, then the rate of change of the population is modeled by the differential equation

$$\frac{dP}{dt} = kP - m.$$

Moreover, we want to reflect the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** M, the maximum population that the enviroment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incorporates these assumption is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right).$$

Multiplying by P, we obtain the model for population growth known as the **logistic differential equation**:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

So we have

$$\begin{aligned}
&\int \frac{dP}{P(1-P/M)} = \int k\,dt = \int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP \\
\Rightarrow\ & \ln|P| - \ln|M-P| = \ln\left|\frac{P}{M-P}\right| = kt + C \\
\Rightarrow\ & -\ln\left|\frac{P}{M-P}\right| = \ln\left|\frac{M-P}{P}\right| = -kt - C \\
\Rightarrow\ & \left|\frac{M-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt} \\
& \frac{M-P}{P} = Ae^{-kt}
\end{aligned}$$

where $A = \pm e^{-C}$. So

$$P = \frac{M}{1+Ae^{-kt}}.$$

Therefore, if $t = 0$, then $P = P_0$ and

$$\frac{M-P_0}{P_0} = Ae^0 = A.$$

Thus the solution to the logistic equation is

$$P(t) = \frac{M}{1+Ae^{-kt}} \qquad \text{where } A = \frac{M-P_0}{P_0}.$$

Example:

1. A population grows accroding to the given logistic equation, where t is measured in weeks.

$$\frac{dP}{dt} = 0.02P - 0.0004P^2, \quad P(0) = 40.$$

   (a) What is the carrying capacity M? What is the value of k?

   (b) Write the solution of the equation.

   (c) What is the population after 10 weeks?

   Sol.

   (a)

$$\frac{dP}{dt} = 0.02P - 0.0004P^2 = 0.02P\left(1 - \frac{P}{500}\right),$$

   so the carrying capacity is $M = 500$ and the value of k is 0.02.

   (b) The solution of the equation is

$$P(t) = \frac{M}{1+Ae^{-kt}},$$

   where

$$A = \frac{M-P_0}{P_0}.$$

   Since $P(0) = P_0 = 40$, we have

$$A = \frac{500-40}{40} = 11.5,$$

and hence,

$$P(t) = \frac{500}{1 + 11.5e^{-0.02t}}.$$

(c) The population after 10 weeks is

$$P(10) = \frac{500}{1 + 11.5e^{-0.02(10)}} \approx 48.$$

2. Suppose a population $P(t)$ satisfies

$$\frac{dP}{dt} = 0.4P - 0.001P^2 \qquad P(0) = 50$$

where t is measured in years.

(a) What is the carrying capacity?
(b) What is $P'(0)$?
(c) When will the population reach 50% of the carrying capacity?

Sol.

(a)

$$\frac{dP}{dt} = 0.4P - 0.001P^2 = 0.4P\left(1 - \frac{P}{400}\right),$$

so the carrying capacity is $M = 400$ and the value of k is 0.4.

(b) Using the fact that $P(0) = 50$ and the formula for dP/dt, we get

$$P'(0) = \left.\frac{dP}{dt}\right|_{t=0} = 0.4(50) - 0.001(50)^2 = 20 - 2.5 = 17.5.$$

(c)

$$A = \frac{M - P_0}{P_0} = \frac{400 - 50}{50} = 7,$$

so

$$P = \frac{400}{1 + 7e^{-0.4t}}.$$

The population reaches 50% of the carrying capacity, $400 * 50\% = 200$, when

$$200 = \frac{400}{1 + 7e^{-0.4t}} \Rightarrow 1+7e^{-0.4t} = 2 \Rightarrow -0.4t = \ln\frac{1}{7} = -\ln 7 \Rightarrow t = \frac{\ln 7}{0.4} \approx 4.86 \text{ (years.)}$$

Exercise:

1. (a) Assume that the carrying capacity for the US population is 800 million. Use it and the fact that the population was 282 million in 2000 to formulate a logistic model for the US population.

(b) Determine the value of k in your model by using the fact that the population in 2010 was 309 million.

(c) Use your model to predict the US population in the years 2100 and 2200.

(d) Use your model to predict the year in which the US population will exceed 500 million.

Sol.

1. (a) $P(t) = \dfrac{800}{1 + \frac{259}{141}\, e^{-kt}}$.

   (b) $k = -\dfrac{1}{10} \ln \dfrac{23077}{26677} \approx 0.0145$.

   (c) $P(100) = \dfrac{800}{1 + \frac{259}{141}\, e^{-100k}} \approx 559$ and $P(200) = \dfrac{800}{1 + \frac{259}{141}\, e^{-200k}} \approx 727$.

   (d) $t = 10\,\dfrac{\ln(423/1295)}{\ln(23077/26677)} \approx 77.18$, that is the US population will exceed 500 million in the year 2077.

Section 9.5: Linear Equations

A first-order **linear** differential equation is one that can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval.

Solve the Linear Equations

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor**

$$I(x) = e^{\int P(x)\,dx}$$

and the integrate both sides.

Example:
Solve the initial-value problem.

1. $2xy' + y = 6x, \quad x > 0, \quad y(4) = 20.$

   Sol.

   $$2xy' + y = 6x, \quad x > 0 \;\Rightarrow\; y' + \frac{1}{2x}y = 3,$$

   then

   $$I(x) = e^{\int 1/(2x)\,dx} = e^{(1/2)\ln x} = \sqrt{x}.$$

   Multiplying by $\sqrt{x}$ gives

   $$\sqrt{x}y' + \frac{1}{2\sqrt{x}}y = 3\sqrt{x} = (\sqrt{x}y)'$$
   $$\Rightarrow \sqrt{x}y = \int 3\sqrt{x}\,dx = 2x^{3/2} + C$$
   $$\Rightarrow y = 2x + \frac{C}{\sqrt{x}}.$$

   So

   $$y(4) = 20 \;\Rightarrow\; 8 + \frac{C}{2} = 20 \;\Rightarrow\; C = 24,$$

   and hence,

   $$y = 2x + \frac{24}{\sqrt{x}}.$$

Exercise:
Solve the initial-value problem.

1. $xy' + y = x\ln x, \quad y(1) = 0.$

2. $(x^2+1)\dfrac{dy}{dx} + 3x(y-1) = 0, \quad y(0) = 2.$

Sol.
1. $y = \frac{1}{2}x\ln x - \frac{1}{4}x + \frac{1}{4x}$ 2. $y = 1 + (x^2+1)^{-3/2}.$