Calculus 2 11/28 Note
Module Class 07

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Section 7.1: Integration by Parts

Integration by Parts

If f and g are differentiable functions, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Let $u = f(x)$ and $v = g(x)$. By the Substitution Rule, the formula for integration by parts becomes

$$\int u\,dv = uv - \int v\,du$$

Example:
Evaluate the following integrals.

1. $\int \arctan(4x)\,dx$

   Sol.

   Consider the formula of integration by parts $\int u\,dv = uv - \int v\,du$. Let

   $$u = \arctan(4x),\ dv = dx \ \Rightarrow\ du = \frac{4}{1+(4x)^2}\,dx = \frac{4}{1+16x^2}\,dx,\ v = x$$

   Then

   $$\begin{aligned}\int \arctan(4x)\,dx &= x\arctan(4x) - \int \frac{4x}{1+16x^2}\,dx \\ &= x\arctan(4x) - \frac{1}{8}\ln(1+16x^2) + C, \quad \text{where } C \text{ is the constant}\end{aligned}$$

2. $\int_0^\pi x\,\sin x\,\cos x\,dx$

   Sol.

   $$\int_0^\pi x\,\sin x\,\cos x\,dx = \frac{1}{2}\int_0^\pi x\,\sin(2x)\,dx$$

Let

$$u = x,\ dv = \sin(2x)\,dx \ \Rightarrow\ du = dx,\ v = -\frac{1}{2}\cos(2x)$$

Then

$$\begin{aligned}\frac{1}{2}\int_0^{\pi} x\,\sin(2x)\,dx &= \frac{1}{2}\left[-\frac{x}{2}\cos(2x)\right]_{x=0}^{\pi} - \frac{1}{2}\int_0^{\pi} -\frac{1}{2}\cos(2x)\,dx \\ &= -\frac{1}{4} + \frac{1}{4}\left[\frac{1}{2}\sin(2x)\right]_{x=0}^{\pi} \\ &= -\frac{1}{4}\end{aligned}$$

3. $\int_0^t e^x\,\sin(t-x)\,dx$

Sol.

Let

$$u = \sin(t-x),\ dv = e^x\,dx \ \Rightarrow\ du = -\cos(t-x)\,dx,\ v = e^x$$

Then

$$\begin{aligned}I &= \int_0^t e^x\,\sin(t-x)\,dx \\ &= \left[e^x\,\sin(t-x)\right]_{x=0}^{t} - \int_0^t -e^x\cos(t-x)\,dx \\ &= -\sin t + \int_0^t e^x\cos(t-x)\,dx\end{aligned}$$

For $\int_0^t e^x\cos(t-x)\,dx$, let

$$U = \cos(t-x),\ dV = e^x\,dx \ \Rightarrow\ dU = \sin(t-x)dx,\ V = e^x$$

Then

$$\begin{aligned}\int_0^t e^x\cos(t-x)\,dx &= \left[e^x\cos(t-x)\right]_{x=0}^{t} - \int_0^t e^x\sin(t-x)\,dx \\ &= e^t - \cos t - I\end{aligned}$$

So

$$\begin{aligned}&I = -\sin t + \int_0^t e^x\cos(t-x)\,dx = -\sin t + e^t - \cos t - I \\ &\Rightarrow 2I = e^t - \sin t - \cos t \\ &\Rightarrow I = \frac{1}{2}(e^t - \sin t - \cos t)\end{aligned}$$

Exercise:
Evaluate the following integrals.

1. $\int \ln\sqrt[3]{x}\,dx$ 2. $\int (\arcsin x)^2\,dx$ 3. $\int_0^{\pi} e^{\cos x}\sin(2x)\,dx$

Sol.
1. $x\ln\sqrt[3]{x} - \frac{1}{3}x + C$ 2. $x(\arcsin x)^2 + 2\sqrt{1-x^2}\ \arcsin x - 2x + C$ 3. $\frac{4}{e}$

Section 7.2: Trigonometric Integrals

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

1. The power of cosine is odd, then using $\cos^2 x = 1 - \sin^2 x$.

$$\begin{aligned}\int \sin^m x \, \cos^{2k+1} x \, dx &= \int \sin^m x \, (\cos^2 x)^k \, \cos x \, dx \\ &= \int \sin^m x \, (1 - \sin^2 x)^k \, \cos x \, dx\end{aligned}$$

2. The power of sine is odd, then using $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned}\int \sin^{2k+1} x \, \cos^n x \, dx &= \int (\sin^2 x)^k \, \cos^n x \, \sin x \, dx \\ &= \int (1 - \cos^2 x)^k \, \cos^n x \, \sin x \, dx\end{aligned}$$

3. The powers of both sine and cosine are even, then using half-angle identities.

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \sin x \cos x = \frac{1}{2} \sin 2x$$

Example:

1. $\int_0^{\pi/2} \sin^7 x \, \cos^5 x \, dx$

Sol.

$$\begin{aligned}\int_0^{\pi/2} \sin^7 x \, \cos^5 x \, dx &= \int_0^{\pi/2} \sin^7 x \, \cos^4 x \, \cos x \, dx \\ &= \int_0^{\pi/2} \sin^7 x \, (1 - \sin^2 x)^2 \, \cos x \, dx\end{aligned}$$

Let $u = \sin x$, then $du = \cos x \, dx$. $u = 0$ when $x = 0$; $u = 1$ when $x = \frac{\pi}{2}$.

$$\begin{aligned}\int_0^{\pi/2} \sin^7 x \, (1 - \sin^2 x)^2 \, \cos x \, dx &= \int_0^1 u^7 (1 - u^2)^2 \, du \\ &= \int_0^1 u^{11} - 2u^9 + u^7 \, du \\ &= \frac{1}{120}\end{aligned}$$

Another solution.

$$\begin{aligned}\int_0^{\pi/2} \sin^7 x \, \cos^5 x \, dx &= \int_0^{\pi/2} \sin^6 x \, \cos^5 x \, \sin x \, dx \\ &= \int_0^{\pi/2} (1 - \cos^2 x)^3 \, \cos^5 x \, \sin x \, dx\end{aligned}$$

Let $u = \cos x$, then $du = \sin x \, dx$. $u = 1$ when $x = 0$; $u = 0$ when $x = \frac{\pi}{2}$.

$$
\begin{aligned}
\int_0^{\pi/2} (1-\cos^2 x)^3 \, \cos^5 x \, \sin x \, dx &= \int_1^0 (1-u^2)^3 \, u^5 \, du \\
&= \int_1^0 (-u^6+3u^4-3u^2+1) \, u^5 \, du \\
&= \int_1^0 -u^{11}+3u^9-3u^7+u^5 \, du \\
&= \frac{1}{120}
\end{aligned}
$$

2. $\int_0^{\pi/2} \sin^2 x \, \cos^2 x \, dx$.

Sol.

$$
\begin{aligned}
\int_0^{\pi/2} \sin^2 x \, \cos^2 x \, dx &= \int_0^{\pi/2} \frac{1}{4}(4\sin^2 x \, \cos^2 x) \, dx \\
&= \frac{1}{4}\int_0^{\pi/2} (2\sin x \, \cos x)^2 \, dx \\
&= \frac{1}{4}\int_0^{\pi/2} \sin^2(2x) \, dx \\
&= \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}(1-\cos(4x)) \, dx \\
&= \frac{\pi}{16}
\end{aligned}
$$

Another solution.

$$
\begin{aligned}
\int_0^{\pi/2} \sin^2 x \, \cos^2 x \, dx &= \int_0^{\pi/2} \left[\frac{1}{2}(1-\cos(2x))\right]\left[\frac{1}{2}(1+\cos(2x))\right] \, dx \\
&= \frac{1}{4}\int_0^{\pi/2} 1-\cos^2(2x) \, dx \\
&= \frac{1}{4}\int_0^{\pi/2} \sin^2(2x) \, dx \\
&= \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}(1-\cos(4x)) \, dx \\
&= \frac{\pi}{16}
\end{aligned}
$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

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1. The power of secant is even ($n=2k$, $k \geq 2$), then using $\sec^2 x = 1+\tan^2 x$.

$$
\begin{aligned}
\int \tan^m x \, \sec^{2k} x \, dx &= \int \tan^m x \, (\sec^2 x)^{k-1} \, dx \\
&= \int \tan^m x \, (1+\tan^2 x)^{k-1} \, \sec^2 x \, dx
\end{aligned}
$$

2. The power of tangent is odd, then using $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x \, dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx\end{aligned}$$

Recall

$$\int \tan x \, dx = \ln|\sec x| + C \qquad \text{and} \qquad \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Example:

1. $\int \tan^3 x \sec^6 x$.

Sol.

$$\begin{aligned}\int \tan^3 x \sec^6 x \, dx &= \int \tan^3 x \sec^4 x \sec^2 x \, dx \\ &= \int \tan^3 x \, (1 + \tan^2 x)^2 \sec^2 x \, dx\end{aligned}$$

Let $u = \tan x$, then $du = \sec^2 x \, dx$. Then

$$\begin{aligned}\int \tan^3 x \, (1 + \tan^2 x)^2 \sec^2 x \, dx &= \int u^3 (1 + u^2)^2 \, du \\ &= \int u^7 + 2u^5 + u^3 \, du \\ &= \frac{1}{8} u^8 + \frac{1}{3} u^6 + \frac{1}{4} u^4 + C \\ &= \frac{1}{8} \tan^8 x + \frac{1}{3} \tan^6 x + \frac{1}{4} \tan^4 x + C\end{aligned}$$

Another solution.

$$\begin{aligned}\int \tan^3 x \sec^6 x \, dx &= \int \tan^2 x \sec^6 x \tan x \, dx \\ &= \int (\sec^2 x - 1) \sec^6 x \tan x \, dx\end{aligned}$$

Let $u = \sec x$, then $du = \sec x \tan x \, dx$. Then

$$\begin{aligned}\int (\sec^2 x - 1) \sec^5 x \sec x \tan x \, dx &= \int (u^2 - 1) \, u^5 \, du \\ &= \int u^7 - u^5 \, du \\ &= \frac{1}{8} u^8 - \frac{1}{6} u^6 + C \\ &= \frac{1}{8} \sec^8 x - \frac{1}{6} \sec^6 x + C \\ &= \frac{1}{8} \tan^8 x + \frac{1}{3} \tan^6 x + \frac{1}{4} \tan^4 x + C'\end{aligned}$$

Formula

1. $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
2. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Example:

1. $\int \sin(8x)\ \cos(5x)\, dx$

Sol.

$$\begin{aligned}\int \sin(8x)\ \cos(5x)\, dx &= \int \frac{1}{2}[\sin(8x-5x) + \sin(8x+5x)]\, dx \\ &= \frac{1}{2}\int (\sin(3x) + \sin(13x))\, dx \\ &= -\frac{1}{6}\cos(3x) - \frac{1}{26}\cos(13x) + C\end{aligned}$$

Exercise:
Evaluate the following integrals.

1. $\int \tan^2 x\ \cos^3 x\, dx$ 2. $\int \tan^2 x\ \sec^4 x\, dx$ 3. $\int \sin(3x)\ \sin(6x)\, dx$ 4. $\int_{\pi/4}^{\pi/2} \csc^4 x\ \cot^4 x\, dx$

Sol.

1. $\frac{1}{3}\sin^3 x + C$ 2. $\frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$ 3. $\frac{1}{6}\sin(3x) - \frac{1}{18}\sin(9x) + C$ 4. $\frac{12}{35}$

Section 7.3: Trigonometric Substitution

Recall: Integration by Substitution for Single Variable

Let $I \subseteq \mathbb{R}$ be an interval and $\varphi : [a, b] \to I$ be a differentiable function with integrable derivative. Suppose that $f : I \to \mathbb{R}$ is a continuous function. Then

$$\int_{\varphi(a)}^{\varphi(b)} f(u)\, du = \int_a^b f(\varphi(x))\varphi'(x)\, dx.$$

Example:

Consider the definite integral $\int_0^{\frac{5\pi}{6}} \sin\theta\, d\theta$.

If $u(\theta) = \sin\theta$, then $u(0) = 0$ when $\theta = 0$ and $u(\frac{5\pi}{6}) = \frac{1}{2}$ when $\theta = \frac{5\pi}{6}$. But

$$u^{-1} : [0, \frac{1}{2}] \not\to [0, \frac{5\pi}{6}].$$

Hence substitution fails.

Table

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example:

1. $\int x^3\sqrt{9 - x^2}\, dx$

Sol.

Let $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta\, d\theta$. So

$$\begin{aligned}\int x^3\sqrt{9 - x^2}\, dx &= \int 3^3\sin^3\theta\sqrt{9 - 9\sin^2\theta}\,(3\cos\theta)\, d\theta \\ &= 3^4\int \sin^3\theta(3\cos\theta)\, \cos\theta\, d\theta \\ &= 3^5\int \sin^3\theta\, \cos^2\theta\, d\theta \\ &= 3^5\int \sin^2\theta\, \cos^2\theta\, \sin\theta\, d\theta \\ &= 3^5\int (1 - \cos^2\theta)\, \cos^2\theta\, \sin\theta\, d\theta\end{aligned}$$

Let $u = \cos\theta$. Then $du = -\sin\theta \, d\theta$. So

$$\begin{aligned}
3^5 \int (1-\cos^2\theta)\cos^2\theta \sin\theta \, d\theta &= 3^5 \int (u^2-1)u^2 \, du \\
&= 3^5 \int u^4 - u^2 \, du \\
&= 3^5 \left(\frac{1}{5}u^5 - \frac{1}{3}u^3 + C \right) \\
&= 3^5 \left(\frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C \right) \\
&= 3^5 \left[\frac{1}{5}\left(\frac{9-x^2}{9}\right)^{5/2} - \frac{1}{3}\left(\frac{9-x^2}{9}\right)^{3/2} + C \right] \\
&= \frac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C' \\
&= -\frac{1}{5}(x^2+6)(9-x^2)^{3/2} + C'
\end{aligned}$$

Exercise:
Evaluate the following integrals.

1. $\int \frac{x^2}{\sqrt{9-x^2}} \, dx$ 2. $\int \frac{x}{\sqrt{1+x^2}} \, dx$ 3. $\int_0^1 \sqrt{x^2+1} \, dx$

Sol.

1. $\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$ 2. $\sqrt{1+x^2} + C$ 3. $\frac{1}{2}[\sqrt{2} + \ln(1+\sqrt{2})]$