Calculus 1 $\quad$ 9/12 Note
Module Class 07

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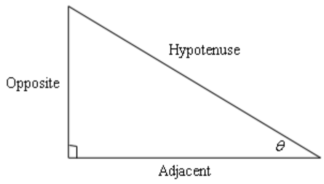
Review Triangular Functions:



$$\cos(x) = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \ \sin(x) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

Then

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \ \cot(x) = \frac{\sin(x)}{\cos(x)}, \ \sec(x) = \frac{1}{\cos(x)}, \ \csc(x) = \frac{1}{\sin(x)}$$

Angle:
We will present $180^o = \pi$, then

$$2\pi = 360^o, \ \frac{\pi}{2} = 90^o, \ \frac{\pi}{3} = 60^o, \ \frac{\pi}{4} = 45^o, \ \frac{\pi}{6} = 30^o$$

Section 1.4: Exponential Functions

Review (Exponential Functions)

Let $a > 0$ and $a \neq 1$, then the exponential function is a function in this form,

$$f(x) = a^x.$$

Example:
Consider an exponential function $g(x) = 4^x$, then

$$g(2) = 4^2 = 16, \ \ g(-3) = 4^{-3} = \left(4^3\right)^{-1} = 64^{-1} = \frac{1}{64}.$$

How about the function $h(x) = (-4)^x$?
We will have

$$h(2) = (-4)^2 = 16, \;\; h\left(\frac{1}{2}\right) = (-4)^{\frac{1}{2}} = \sqrt{-4} = 2i$$

To avoid this problem, we need to require $a > 0$.

Properties of $f(x) = a^x$

1. $f(0) = 1$. The function will always take the value of 1 at $x = 0$.
2. $f(x) \neq 0$. An exponential function will never be zero.
3. $f(x) > 0$. An exponential function is always positive.
4. The previous two properties can be summarized by saying that the range of an exponential function is $(0, \infty)$.
5. The domain of an exponential function is $(-\infty, \infty)$.
6. If $0 < a < 1$, then
$$f(x) \to 0 \text{ as } x \to \infty \text{ and } f(x) \to \infty \text{ as } x \to -\infty.$$
7. If $a > 1$, then
$$f(x) \to \infty \text{ as } x \to \infty \text{ and } f(x) \to 0 \text{ as } x \to -\infty.$$

Laws of Exponents

If a and b are positive numbers and x and y are any real numbers, then

1. $b^{x+y} = b^x b^y$
2. $b^{x-y} = \dfrac{b^x}{b^y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^y$

Definition

The ***natural exponential function*** is $f(x) = e^x$, where $e = 2.71828182845905\ldots$

By above properties, since $e > 1$ we also know that $e^x \to \infty$ as $x \to \infty$ and $e^x \to 0$ as $x \to -\infty$.

Graph:

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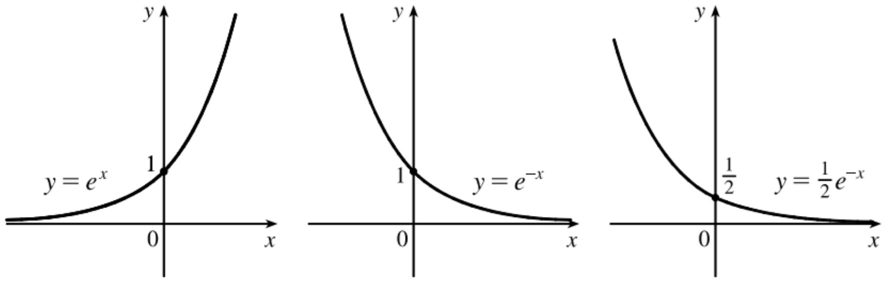

Exercise:
Rewrite and simplify the following expression.
1. $\dfrac{4^{-3}}{2^{-8}}$ 2. $8^{4/3}$ 3. $\dfrac{1}{\sqrt[3]{x}^4}$ 4. $\dfrac{x^{2n} \cdot x^{3n-1}}{x^{n+2}}$ 5. $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}}$

Sol: 1. 4 2. 16 3. $x^{-\frac{4}{3}}$ 4. x^{4n-3} 5. $a^{\frac{1}{6}}b^{-\frac{1}{12}}$

Section 1.5: Inverse Functions and Logarithms

Inverse Functions:

Example: Let the function $f(x) = 3x - 2$, then what's the inverse function of $f(x)$?

Sol:
Let $f(x) = y$ and $g(x)$ be the inverse function of $f(x)$, then

$$y = 3x - 2 \Rightarrow y + 2 = 3x \Rightarrow \frac{y+2}{3} = x$$

So

$$g(x) = \frac{x+2}{3} = f^{-1}(x).$$

Definition (one-to-one)

A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

Finding the Inverse of a Function

Given **the one-to-one function** $f(x)$, we want to find the inverse function $f^{-1}(x)$.

1. First, replace $f(x)$ with y. This is done to make the rest of the process easier.
2. Solve this equation for x in the terms of y. That is to represent the equation in this form: $x = \cdots$.
3. Interchange x and y.
4. Replace y with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that

$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = x$$

and

$$\left(f^{-1} \circ f\right)(x) = f^{-1}\left(f(x)\right) = x$$

are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

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Logarithms Functions:

Review (Logarithms Functions)

Let $b > 0$ and $b \neq 1$, then the logarithms function is a function in this form,

$$f(x) = \log_b x \quad \text{is equivalent to} \quad x = b^y.$$

So

$$\log_b\left(b^x\right) = x \text{ for every } x \in \mathbb{R} \quad \text{and} \quad b^{\log_b x} = x \text{ for every } x > 0$$

Laws of Logarithms

If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b\left(x^r\right) = r\log_b x \quad$ (where r is any real number)
4. $\log_b x = \dfrac{\ln x}{\ln b}$

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Natural Logarithms:

1. $\log_e x = \ln x$
2. $\ln x = y \ \Rightarrow\ e^y = x$
3. $\ln\left(e^x\right) = x \ x \in \mathbb{R}$ and $e^{\ln x} = x \ x > 0$
4. $\ln e = 1$

Exercise:

1. If $f(x) = x^5 + x^3 + x$, find $f^{-1}(3)$ and $f\left(f^{-1}(2)\right)$.
2. Find a formula for the inverse of the function.
   (a) $f(x) = 1 + \sqrt{2+3x}$ (b) $f(x) = \dfrac{4x-1}{2x+3}$ (c) $f(x) = \ln(x+3)$
3. Solve each equation fo x.
   (a) $e^{7-4x} = 6$ (b) $\ln\left(x^2-1\right) = 3$ (c) $e^{2x} - 3e^x + 2 = 0$

Sol:

1. $f^{-1}(3) = 1, \ f\left(f^{-1}(2)\right) = 2.$
2. (a) $f^{-1}(x) = \dfrac{1}{3}(x-1)^2 - \frac{2}{3}, \ x \geq 1$ (b) $f^{-1}(x) = \dfrac{3x+1}{4-2x}$ (c) $f^{-1}(x) = e^x - 3$
3. (a) $x = \dfrac{1}{4}(7 - \ln 6)$ (b) $x = \pm\sqrt{1+e^3}$ (c) $x = 0$ or $\ln 2$

Note:

1. $\sin^{-1} x = y \ \Leftrightarrow\ \sin y = x$ and $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$
2. $\cos^{-1} x = y \ \Leftrightarrow\ \cos y = x$ and $0 \leq y \leq \pi$
3. $\tan^{-1} x = y \Leftrightarrow \tan y = x$ and $-\dfrac{\pi}{2} < y\dfrac{\pi}{2}$