Please write down your solutions on a separate sheet of paper and submit it to your TA or instructor on 26th **December, 2019**.

Recommended time limit: 150 minutes.

1. Evaluate the following integrals.

   (a) $\int \sin(3x)\cos(5x)\,dx$.

Let $u_1 = \sin(3x)$ and $dv_1 = \cos(5x)\,dx$, then

$$du_1 = 3\cos(3x)\,dx \quad \text{and} \quad v_1 = \frac{\sin(5x)}{5}.$$

Use integration by parts, therefore,

$$\begin{aligned}\int \sin(3x)\cos(5x)\,dx &= \int u_1\,dv_1 = u_1v_1 - \int v_1\,du_1\\ &= \frac{1}{5}\sin(3x)\sin(5x) - \frac{3}{5}\int \cos(3x)\sin(5x)\,dx.\end{aligned}$$

Use integration by parts again. Let $u_2 = \cos(3x)$ and $dv_2 = \sin(5x)\,dx$, then

$$du_2 = -3\sin(3x)\,dx \quad \text{and} \quad v_2 = -\frac{\cos(5x)}{5}.$$

Therefore,

$$\begin{aligned}\int \sin(3x)\cos(5x)\,dx &= \frac{1}{5}\sin(3x)\sin(5x) - \frac{3}{5}\int \cos(3x)\sin(5x)\,dx.\\ &= \frac{1}{5}\sin(3x)\sin(5x) - \frac{3}{5}\left[\int u_2\,dv_2\right]\\ &= \frac{1}{5}\sin(3x)\sin(5x) - \frac{3}{5}\left[u_2v_2 - \int v_2\,du_2\right]\\ &= \frac{1}{5}\sin(3x)\sin(5x) - \frac{3}{5}\left[\cos(3x)\left(-\frac{\cos(5x)}{5}\right) - \int\left(-\frac{\cos(5x)}{5}\right)(-3\sin(3x))\,dx\right]\\ &= \frac{1}{5}\sin(3x)\sin(5x) + \frac{3}{25}\cos(3x)\cos(5x) + \frac{9}{25}\int \sin(3x)\cos(5x)\,dx\end{aligned}$$

So

$$\begin{aligned}\frac{16}{25}\int \sin(3x)\cos(5x)\,dx &= \frac{1}{5}\sin(3x)\sin(5x) + \frac{3}{25}\cos(3x)\cos(5x) + C_1\\ \Rightarrow \int \sin(3x)\cos(5x)\,dx &= \frac{5}{16}\sin(3x)\sin(5x) + \frac{3}{16}\cos(3x)\cos(5x) + C_2\end{aligned}$$

where C_1 is the constant and $C_2 = \frac{25}{16}C_1$.

(b) $\int \frac{3x+1}{x^2(x^2+25)}\,dx$.

Decompose $\frac{3x+1}{x^2(x^2+25)}$ into partial fractions, getting

$$\frac{3x+1}{x^2(x^2+25)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+25},$$

then

$$Ax(x^2+25) + B(x^2+25) + (Cx+D)x^2 = (A+C)x^3 + (B+D)x^2 + 25Ax + 25B = 3x+1.$$

So $A = \frac{3}{25}$, $B = \frac{1}{25}$, $C = -\frac{3}{25}$ and $D = -\frac{1}{25}$.
Therefore,

$$\begin{aligned}\int \frac{3x+1}{x^2(x^2+25)}\,dx &= \int \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+25}\,dx \\ &= \int \frac{3/25}{x} + \frac{1/25}{x^2} + \frac{(-3x)/25 - 1/25}{x^2+25}\,dx \\ &= \frac{3}{25}\ln|x| - \frac{1}{25x} - \frac{3}{50}\ln|x^2+25| - \frac{1}{125}\int \frac{1/5}{(x/5)^2+1}\,dx \\ &= \frac{3}{25}\ln|x| - \frac{1}{25x} - \frac{3}{50}\ln|x^2+25| - \frac{1}{125}\arctan\left(\frac{x}{5}\right)\end{aligned}$$

2. Suppose that f is a continuous and positive function on $[0,5]$, and the area between the graph of $y = f(x)$ and the x-axis for $0 \le x \le 5$ is 8. Let $A(c)$ denote the area between the graph of $y = f(x)$ and the x-axis for $0 \le x \le c$, and let $B(c)$ denote the area between the graph of $y = f(x)$ and the x-axis for $c \le x \le 5$. Let $R(c) = A(c)/B(c)$. If $R(3) = 1$ and $\left.\frac{dR}{dc}\right|_{c=3} = 7$, find $f(3)$.

We have $A(3) + B(3) = 8$ and $R(3) = 1$, then

$$A(3) = B(3) = 4.$$

Moreover,

$$A(c) = \int_0^c f(x)\,dx \quad \text{and} \quad B(c) = \int_c^5 f(x)\,dx.$$

By the Fundamental Theorem of Calculus, then we have

$$A'(c) = f(c) \quad \text{and} \quad B'(c) = -f(c).$$

Therefore, $A'(3) = f(3)$ and $B'(3) = -f(3)$.

On the other hand,

$$\frac{d}{dc}R(c) = \frac{d}{dc}\frac{A(c)}{B(c)} = \frac{A'(c)B(c) - A(c)B'(c)}{B^2(c)}$$

and letting $c = 3$ gives

$$7 = \left.\frac{dR}{dc}\right|_{c=3} = \frac{A'(3)B(3) - A(3)B'(3)}{B^2(3)} = \frac{f(3)B(3) + A(3)f(3)}{B^2(3)} = \frac{f(3)}{2}.$$

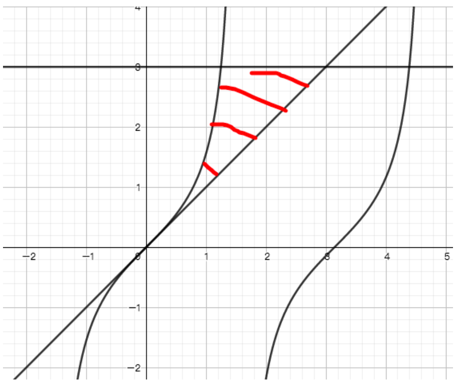
Hence $f(3) = 14$.

3. Compute the area of the region enclosed by the graphs of the equations $y = \tan x$, $y = x$ and below $y = 3$.

Begin by finding the points of intersection of the two graphs. From $y = \tan x$ and $y = x$, then

$$\tan x = x \Rightarrow x = 0.$$

Now see the given graph of the enclosed region.



Using horizontal cross-sections to describe this region, we get that

$$0 \le y \le 3 \quad \text{and} \quad \arctan y \le x \le y.$$

So the area of this region is

$$\int_0^3 (y - \arctan y)\, dy = \int_0^3 y\, dy - \int_0^3 \arctan y\, dy = \frac{9}{2} - \int_0^3 \arctan y\, dy.$$

Let $u = \arctan y$ and $dv = dy$, then

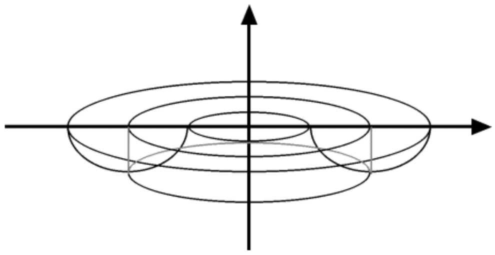
$$du = \frac{1}{y^2+1}\, dy \quad \text{and} \quad v = y.$$

Use integrating by parts, then

$$\begin{aligned}
\int_0^3 (y - \arctan y)\, dy &= \frac{9}{2} - \int_0^3 \arctan y\, dy \\
&= \frac{9}{2} - \left([y \arctan y]_{y=0}^3 - \int_0^3 \frac{y}{y^2+1}\, dy \right) \\
&= \frac{9}{2} - 3 \arctan 3 + \frac{1}{2} \left[\ln |y^2+1| \right]_{y=0}^3 \\
&= \frac{9}{2} - 3 \arctan 3 + \frac{\ln 10}{2}
\end{aligned}$$

4. Sketch the solid obtained by rotating the region bounded by $y = 0$ and $y = \cos x$ for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$ about the y-axis and find its volume.

Observe that $\cos(x) \leq 0$ for $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$. The solid is



We will find the volume of this solid using the method of cylindrical shells.

Since we rotated about a vertical line, we will use x as the variable. Note that the cylinder whose edge passes through x has height $h = 0 - \cos x = -\cos x$ and radius $r = x - 0 = x$.

The volume of the solid is

$$\int_{\pi/2}^{3\pi/2} 2\pi r h \, dx = \int_{\pi/2}^{3\pi/2} 2\pi x(-\cos x) \, dx = -2\pi \int_{\pi/2}^{3\pi/2} x \cos x \, dx.$$

Let $u = x$ and $dv = \cos x \, dx$, then $du = dx$ and $v = \sin x$.

Use integrating by parts, then

$$\begin{aligned} -2\pi \int_{\pi/2}^{3\pi/2} x \cos x \, dx &= -2\pi \left[[x \sin x]_{x=\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x \, dx \right] \\ &= -2\pi \left[-2\pi - [-\cos x]_{x=\pi/2}^{3\pi/2} \right] \\ &= 4\pi^2. \end{aligned}$$

So the volume is $4\pi^2$.

5. (a) Determine whether $\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x}} \, dx$ converges or diverges. Evaluate the value if it converges.

This is an improper integral since $\dfrac{x+1}{\sqrt[3]{x}}$ and $\dfrac{1}{\sqrt[3]{x}}$ have an asymptote at $x = 0$. Then

$$\begin{aligned}
\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x}}\,dx &= \int_{-1}^{1} \frac{x}{\sqrt[3]{x}} + \frac{1}{\sqrt[3]{x}}\,dx \\
&= \int_{-1}^{1} x^{2/3}\,dx + \int_{-1}^{1} x^{-1/3}\,dx \\
&= \frac{6}{5} + \int_{-1}^{0} x^{-1/3}\,dx + \int_{0}^{1} x^{-1/3}\,dx \\
&= \frac{6}{5} + \lim_{t\to 0^-} \int_{-1}^{t} x^{-1/3}\,dx + \lim_{t\to 0^+} \int_{t}^{1} x^{-1/3}\,dx \\
&= \frac{6}{5} + \lim_{t\to 0^-} \left(\frac{3}{2}t^{2/3} - \frac{3}{2}\right) + \lim_{t\to 0^+} \left(\frac{3}{2} - \frac{3}{2}t^{2/3}\right) \\
&= \frac{6}{5} - \frac{3}{2} + \frac{3}{2} \\
&= \frac{6}{5}.
\end{aligned}$$

(b) Determine whether $\int_2^\infty \frac{1+\cos^2 x}{\sqrt{x}[2-\sin^4 x]}\,dx$ converges or diverges. Evaluate the value if it converges.

Since $0 \le \cos^2 x \le 1$, then

$$\frac{1+\cos^2 x}{\sqrt{x}[2-\sin^4 x]} \ge \frac{1}{\sqrt{x}[2-\sin^4 x]}.$$

Also, since $0 \le \sin^4 x \le 1$, then

$$\frac{1+\cos^2 x}{\sqrt{x}[2-\sin^4 x]} \ge \frac{1}{\sqrt{x}[2-\sin^4 x]} \ge \frac{1}{2\sqrt{x}}.$$

Finally, we know that

$$\int_2^\infty \frac{1}{2\sqrt{x}}\,dx \quad \text{is divergent.}$$

So by the Comparison Test, then

$$\int_2^\infty \frac{1+\cos^2 x}{\sqrt{x}[2-\sin^4 x]}\,dx \quad \text{is also divergent.}$$

6. Water is run at a constant rate of 1 ft^3/min to fill a cylindrical tank of radius 3 ft and height 5 ft. Assuming that the tank is initially empty, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and check your conjecture by integrating. [Take the weight density of water to be 62.4 lb/ft^3].

The total volume of the tank if

$$V = \pi r^2 h = \pi \cdot 3^2 \cdot 5 = 45\pi.$$

The time to fill the tank is

$$t = \frac{\text{Volume}}{\text{Rate}} = \frac{45\pi\,\text{ft}^3}{1\,\text{ft}^3/\text{min}} = 45\,\text{min}.$$

Cont.

Thus the average weight should occur when the tank is half way full, which is at time $t = \dfrac{45\pi}{2}$, with the average weight being

$$62.4\left(\frac{45\pi}{2}\right) = 1404\pi.$$

We can check this by integratin $62.4\,t$ from $t = 0$ to $t = 45\pi$.

$$\text{Weight}_{\text{ave}} = \frac{1}{45\pi - 0}\int_0^{45\pi} 62.4\,t\,dt = 1404\pi.$$

7. Solve the following differential equations.

(a) $x\ln x = y(1+\sqrt{3+y^2})y',\quad y(1) = 1.$

$$x\ln x = y(1+\sqrt{3+y^2})y' = y(1+\sqrt{3+y^2})\frac{dy}{dx}$$

$$\Rightarrow \int x\ln x\,dx = \int y + y\sqrt{3+y^2}\,dy$$

Let $u = \ln x$ and $dv = x\,dx$, then $du = \dfrac{dx}{x}$ and $v = \dfrac{x^2}{2}$.

$$\Rightarrow \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \frac{y^2}{2} + \frac{(3+y^2)^{3/2}}{3}$$

$$\Rightarrow \frac{x^2}{2}\ln x - \frac{x^2}{4} + C = \frac{y^2}{2} + \frac{(3+y^2)^{3/2}}{3}.$$

Now

$$y(1) = 1 \;\Rightarrow\; 0 - \frac{1}{4} + C = \frac{1}{2} + \frac{4^{3/2}}{3} \;\Rightarrow\; C = \frac{41}{12},$$

so

$$\frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{41}{12} = \frac{y^2}{2} + \frac{(3+y^2)^{3/2}}{3}.$$

(b) $y'\tan x = a + y,\quad y(\pi/3) = a,\quad 0 < x < \pi/2.$

$$\begin{aligned}
y'\tan x = a + y \;&\Rightarrow\; \frac{dy}{dx} = \frac{a+y}{\tan x}\\
&\Rightarrow\; \frac{dy}{a+y} = \cot x\,dx, \qquad \text{where } a + y \neq 0\\
&\Rightarrow\; \int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x}\,dx\\
&\Rightarrow\; \ln|a+y| = \ln|\sin x| + C\\
&\Rightarrow\; |a+y| = e^{\ln|\sin x| + C} = e^{\ln|\sin x|}\cdot e^C = e^C\,|\sin x|\\
&\Rightarrow\; a + y = K\sin x, \qquad \text{where } K = \pm e^C.
\end{aligned}$$

In our derivation, K was nonzero, but we can restore the excluded case $y = -a$ by allowing K to be zero.

$$y(\pi/3) = a \;\Rightarrow\; a + a = K\sin\left(\frac{\pi}{3}\right) \;\Rightarrow\; 2a = \frac{\sqrt{3}}{2}K \;\Rightarrow\; K = \frac{4a}{\sqrt{3}}.$$

Thus,

$$a + y = \frac{4a}{\sqrt{3}}\sin x \;\Rightarrow\; y = \frac{4a}{\sqrt{3}}\sin x - a.$$

The End.