NTU 107-1 MATH1201 Calculus A-05
Quiz 1 Solution

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Work ONLY on the problems indicated by your teaching assistants or instructor for this quiz. The rest are left for your self-revision. Pocket calculator is not very helpful. Internet is not allowed. Time limit: 40 minutes

1. (a) Using the precise definition of limits, show that, for any function f, if $\lim_{x\to x_0} |f(x)| = 0$, then $\lim_{x\to x_0} f(x) = 0$. (6 points)

   (b) Give a counter-example to the statement "if a is a non-zero number and $\lim_{x\to x_0} |f(x)| = |a|$, then $\lim_{x\to x_0} f(x) = a$ or $\lim_{x\to x_0} f(x) = -a$". Justify your answer. (6 points)

   Solution.

   (a) By the definition of limit, $\lim_{x\to x_0} |f(x)| = 0$ means that, *for every number* $\varepsilon > 0$, *there is a number* $\delta > 0$ such that

   $$\text{whenever } \ 0 <| x - x_0 |< \delta \ , \quad \text{we have } \ ||f(x)| - 0| < \varepsilon \ .$$

   (definition of $\lim_{x\to x_0} |f(x)|$: 3 points)

   But $||f(x)| - 0| = |f(x)| = |f(x) - 0|$, so the above statement means precisely that $\lim_{x\to x_0} f(x) = 0$. (valid argument: 3 points)

   (b) Any valid counter-examples with explanations are acceptable.

   Here we claim that $f(x) := \frac{x}{|x|}$ with $x_0 := 0$ is a counter-example.

   (correct example: 2 points)

   Indeed, we have

   $$|f(x)| = \left|\frac{x}{|x|}\right| = 1$$

   and thus $\lim_{x\to 0} |f(x)| = 1$. (limit of $|f(x)|$: 2 points)

   However, we notice that

   $$\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} = -1 \ .$$

   As the two sided limits do not match, $\lim_{x\to 0} f(x)$ does not even exist. It can never be equal to 1 or -1. (limit of $f(x)$: 2 points)

2. Evaluate the following limits or explain why they don't exist.

   (a) $\displaystyle\lim_{x\to 0} \frac{|\ 3x-1\ | - |\ 3x+1\ |}{x}$. (5 points)

   (b) $\displaystyle\lim_{x\to a^+} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2-a^2}}$, where $a > 0$. (5 points)

   (c) $\displaystyle\lim_{x\to\infty} (\sqrt{324x^2 + 73x + \pi} - \sqrt{324x^2 + 17\pi})$. (5 points)

   (d) $\displaystyle\lim_{x\to 0} x^3 \cos\left(\frac{1}{x^5}\right)$. (6 points)

(e) $\displaystyle\lim_{x\to 0^+} \frac{x^{1+x}}{(1+x)^x}$. (6 points)

Solution.

(a) **[Method 1]**

If x is close to 0 (in this case, when $|x| < \frac{1}{3}$), then one has

$$\begin{aligned} |\ 3x-1\ | &= -(3x-1) \quad &&(\text{as } 3x-1<0), \text{ and} \\ |\ 3x+1\ | &= 3x+1 &&(\text{as } 3x+1>0). \end{aligned}$$

Therefore, the given limit is equal to

$$\lim_{x\to 0} \frac{-(3x-1)-(3x+1)}{x} = \lim_{x\to 0} \frac{-6x}{x} = -6\ . \qquad \text{(5 points)}$$

[Method 2]

The given limit is equal to

$$\begin{aligned} &\lim_{x\to 0} \frac{(|\ 3x-1\ | - |\ 3x+1\ |)(|\ 3x-1\ | + |\ 3x+1\ |)}{x(|\ 3x-1\ | + |\ 3x+1\ |)} \\ &= \lim_{x\to 0} \frac{(3x-1)^2-(3x+1)^2}{x(|\ 3x-1\ | + |\ 3x+1\ |)} \\ &= \lim_{x\to 0} \frac{-12x}{x(|\ 3x-1\ | + |\ 3x+1\ |)} \\ &= \frac{-12}{1+1} = -6\ . \end{aligned} \qquad \text{(5 points)}$$

(b) Given that $a > 0$, the given limit is equal to

$$\begin{aligned} &\lim_{x\to a^+} \frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}} + \lim_{x\to a^+} \frac{\sqrt{x-a}}{\sqrt{x^2-a^2}} \\ &= \lim_{x\to a^+} \frac{x-a}{\sqrt{x+a}\sqrt{x-a}(\sqrt{x}+\sqrt{a})} + \lim_{x\to a^+} \frac{1}{\sqrt{x+a}} \\ &= \lim_{x\to a^+} \frac{\sqrt{x-a}}{\sqrt{x+a}(\sqrt{x}+\sqrt{a})} + \lim_{x\to a^+} \frac{1}{\sqrt{x+a}} \\ &= \frac{1}{\sqrt{2a}}\ . \end{aligned} \qquad \text{(5 points)}$$

Note that $\frac{x-a}{\sqrt{x-a}} = \sqrt{x-a}$ since we have $x > a$ when we are considering the right-hand limit as $x \to a^+$.

(c) Rationalising the expression in the limit, the given limit is then equal to

$$\begin{aligned} &\lim_{x\to\infty} (\sqrt{324x^2+73x+\pi} - \sqrt{324x^2+17\pi}) \frac{\sqrt{324x^2+73x+\pi} + \sqrt{324x^2+17\pi}}{\sqrt{324x^2+73x+\pi} + \sqrt{324x^2+17\pi}} \\ &= \lim_{x\to\infty} \frac{(324x^2+73x+\pi)-(324x^2+17\pi)}{\sqrt{324x^2+73x+\pi} + \sqrt{324x^2+17}} \\ &= \lim_{x\to\infty} \frac{73x-16\pi}{\sqrt{324x^2+73x+\pi} + \sqrt{324x^2+17}} \\ &= \lim_{x\to\infty} \frac{73-\frac{16\pi}{x^2}}{\sqrt{324+\frac{73}{x}+\frac{\pi}{x^2}} + \sqrt{324+\frac{17\pi}{x^2}}} \\ &= \frac{73}{\sqrt{324}+\sqrt{324}} = \frac{73}{36}\ . \end{aligned} \qquad \text{(5 points)}$$

(d) By the fact that $-1 \le \cos x \le 1$ (which implies that $|\cos x| \le 1$) for all real numbers x, we have

$$0 \le \left| x^3 \cos\left(\frac{1}{x^5}\right) \right| \le |x|^3 \ . \qquad \text{(inequality: 2 points)}$$

By the Squeeze Theorem, since $\lim_{x\to 0} |x|^3 = 0$, we obtain $\lim_{x\to 0} \left|x^3 \cos(\frac{1}{x^5})\right| = 0$.

(use of the Squeeze Thm.: 2 points)

By the statement in Problem (1a), it follows that $\lim_{x\to 0} x^3 \cos(\frac{1}{x^5}) = 0$.

(use of the statement in Problem (1a): 1 point)

(answer: 1 point)

(e) Taking logarithm of the given function in the limit yields

$$\ln\left(\frac{x^{1+x}}{(1+x)^x}\right) = (1+x)\ln x - x\ln(1+x) =: g(x) \ .$$

(transformation that makes Limit Laws applicable: 1 point)

By the Limit Laws and using the fact that ln is a *continuous* function, it follows that

$$\lim_{x\to 0^+} x\ln(1+x) = \lim_{x\to 0^+} x \cdot \lim_{x\to 0^+} \ln(1+x) = 0 \cdot \ln\left(1 + \lim_{x\to 0^+} x\right) = 0 \cdot \ln 1 = 0 \ .$$

Since $\lim_{x\to 0^+} \ln x = -\infty$ while $\lim_{x\to 0^+}(1+x)$ is finite, we have $\lim_{x\to 0^+}(1+x)\ln x = -\infty$. As a result, we have

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} ((1+x)\ln x - x\ln(1+x)) = -\infty \ .$$

Composite the exponential function $f(x) := e^x$ with g, we obtain

$$\lim_{x\to 0^+} \frac{x^{1+x}}{(1+x)^x} = \lim_{x\to 0^+} e^{g(x)} = \lim_{y\to -\infty} e^y = 0 \ .$$

(correct use of Limit Laws and continuity: 4 points)

(answer: 1 point)

3. Let

$$f(x) = \begin{cases} x^2 - m & \text{if } x < 3 \ , \\ 1 - mx & \text{if } x \ge 3 \ . \end{cases}$$

If $f(x)$ is continuous for all x on the real line, find m. (10 points)

Solution. Notice that, by a simple use of Limit Laws, we have

$$\lim_{x\to 3^-} f(x) = 9 - m \qquad \text{and} \qquad \lim_{x\to 3^+} f(x) = 1 - 3m = f(3) \ .$$

(both sided limits: 3+3 points)

((sided) limit equals $f(3)$: 1 point)

Since f is continuous at 3, we must have $9 - m = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 1 - 3m$, which then yields $m = -4$. (equality of both sided limits: 2 points)

(answer: 1 point)

4. Find all asymptotes of the graph of $f(x) = \dfrac{5^x + 4^{-x}}{5^x - 4^{-x}}$. (20 points)

Solution. We look for the horizontal and vertical asymptotes separately.

(a) We first look for the horizontal asymptotes of the graph of f.
To consider the limit $\lim_{x\to\infty} f(x)$, write $f(x)$ as

$$f(x) = \frac{1 + 5^{-x}4^{-x}}{1 - 5^{-x}4^{-x}} = \frac{1 + \frac{1}{20^x}}{1 - \frac{1}{20^x}} \ .$$

Notice that $\lim_{x\to\infty} \frac{1}{20^x} = 0$. We then obtain, via Limit Laws, that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1 + \frac{1}{20^x}}{1 - \frac{1}{20^x}} = \frac{1 + \lim_{x\to\infty} \frac{1}{20^x}}{1 - \lim_{x\to\infty} \frac{1}{20^x}} = \frac{1+0}{1-0} = 1 \ .$$

(correct computation of the limit: 5 points)

Therefore, *the line* $y = 1$ is a horizontal asymptote of the graph of f (note that the expression "$\lim_{x\to\infty} f(x) = 1$" does not represent any line, asymptote or geometric object at all). (conclusion: 1 point)

To consider the limit $\lim_{x\to-\infty} f(x)$, write $f(x)$ as

$$f(x) = \frac{5^x4^x + 1}{5^x4^x - 1} = \frac{20^x + 1}{20^x - 1} \ .$$

Now, since $\lim_{x\to-\infty} 20^x = \lim_{y\to\infty} \frac{1}{20^y} = 0$, we obtain, again via Limit Laws, that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{20^x + 1}{20^x - 1} = \frac{0+1}{0-1} = -1 \ .$$

(computation: 5 points)

Therefore, the line $y = -1$ is also a horizontal asymptote of the graph of f.

(conclusion: 1 point)

(b) Next we search for the vertical asymptotes of the graph of f.
We have to search for the points $x_0 \in \mathbb{R}$ such that $\lim_{x\to x_0^+} f(x) = \pm\infty$ or $\lim_{x\to x_0^-} f(x) = \pm\infty$ (we use $\lim_{x\to x_0^\pm}$ to denote either $\lim_{x\to x_0^+}$ or $\lim_{x\to x_0^-}$ below for convenience). Since, by the continuity of exponential functions, the limit of the numerator

$$\lim_{x\to x_0^\pm} \left(5^x + 4^{-x}\right) = 5^{x_0} + 4^{-x_0}$$

is finite and non-zero for any $x_0 \in \mathbb{R}$, the limit $\lim_{x\to x_0^\pm} f(x)$ diverges to ∞ or $-\infty$ only when we have $\lim_{x\to x_0^\pm}(5^x - 4^{-x}) = 0$. The continuity of exponential functions leads us to the equation

$$0 = \lim_{x\to x_0^\pm} \left(5^x - 4^{-x}\right) = 5^{x_0} - 4^{-x_0} \ ,$$

which has the unique solution $x_0 = 0$.

(argument for assuring only one possible x_0: 4 points)

We note that, since $\lim_{x\to 0}(5^x + 4^{-x}) = 2 > 0$ (and thus $\lim_{x\to 0^\pm}(5^x + 4^{-x}) = 2 > 0$), we have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{5^x + 4^{-x}}{5^x - 4^{-x}} = \infty \qquad \left(\text{or} \quad \lim_{x\to 0^-} f(x) = -\infty\right) \ .$$

(Either one of the above limits suffices to make the following conclusion.)

(computation of the limit as $x \to 0^{\pm}$: 3 points)

As a result, the line $x = 0$ is *the* vertical asymptote of the graph of f.

(conclusion: 1 point)

See Figure 1 for the graph of f and its asymptotes for easy reference.

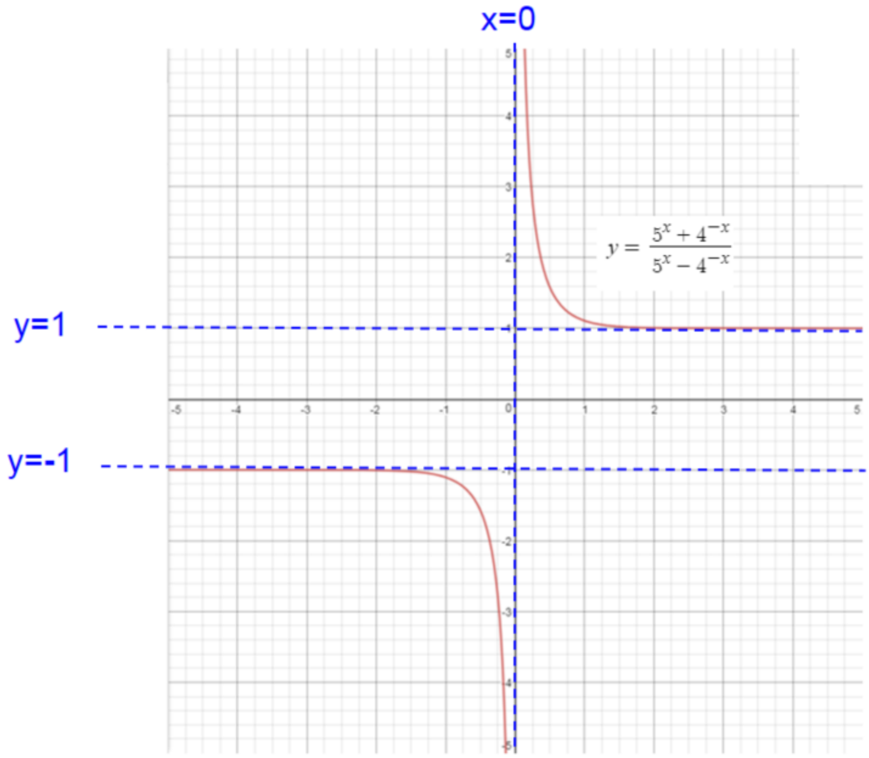

Figure 1: Graph of f and its asymptotes in Problem (4)

Remark. (This is not part of the solution!)

A *slant asymptote* of the graph of f is a line given by the equation $y = mx+b$, where both m and b are constants and, most importantly, $m \neq 0$ (so the line is neither horizontal nor vertical), such that

$$\lim_{x\to\infty} (f(x) - mx - b) = 0 \quad \text{or} \quad \lim_{x\to-\infty} (f(x) - mx - b) = 0 \,.$$

In the case of Problem (4), there is no need of searching for slant asymptotes because we have found two horizontal asymptotes already (thus both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist and finite). There cannot be any more slant asymptotes, as can be easily perceived from the graph of the function or deduced from the definition of slant asymptotes (How?).

Here we present a general technique of finding a slant asymptote which in itself is a good exercise to test your understanding of limits.

Let's write the limit as $\lim_{x\to\pm\infty}$ to refer to either $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$ for simplicity. Suppose the line $y = mx + b$ with $m \neq 0$ is a slant asymptote of the graph of f. As the limit

$$\lim_{x\to\pm\infty} x\left(\frac{f(x)}{x} - m\right) = \lim_{x\to\pm\infty} (f(x) - mx) = b \tag{1}$$

exists by assumption, we must have

$$\lim_{x\to\pm\infty}\left(\frac{f(x)}{x}-m\right)=0 \quad \text{(Why?)}\ .$$

This implies that

$$m=\lim_{x\to\pm\infty}\frac{f(x)}{x}\ . \tag{2}$$

Therefore, we can find the slant asymptote by first finding the slope m of the slant asymptote using equation (2) (in which the limit exists and non-zero if the slant asymptote of the graph of f exists). Then we can find the constant b using (1).