# Real Analysis
# Homework 10

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**EXERCISE 1**

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Show that if $f \in C^0(\mathbb{R}^n)$, then its support is identical with the support of the distribution

$$\langle f, \phi \rangle = \int f \phi \, dx, \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

Is this true when $f \in L^1_{\text{loc}}(\mathbb{R}^n)$?

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***Proof.***
If $\phi$ is such that $\operatorname{supp} \phi \cap \operatorname{supp} f = \emptyset$, then $\int_{\mathbb{R}^n} f(x)\phi(x)\, dx = 0$, so $\operatorname{supp} \mathcal{D}_f \subseteq \operatorname{supp} f$.
Now fixed an $x_0$ such that $f(x_0) \neq 0$, $f(x_0) > 0$, then $f(x) > \frac{f(x_0)}{2}$ for some ball $B_\eta(x_0)$. Assume that $x_0 \notin \operatorname{supp} \mathcal{D}_f$, then we can find some $\delta > 0$ such that $B_\delta(x_0) \cap \operatorname{supp}(\mathcal{D}_f) = \emptyset$. We can assume that $\delta < \eta$.
Now find a $\phi$ such that supp $\phi \subseteq B_\delta(x_0)$ and that $\phi(x) = 1$ on $B_{\frac{\delta}{2}}(x_0)$. Then

$$\int_{\mathbb{R}^n} f(x)\phi(x)\, dx \geq \int_{B_{\frac{\delta}{2}}(x_0)} f(x)\phi(x)\, dx \geq \frac{(f(x_0))\, \delta^n}{2^{n+1}} > 0,$$

so $\phi$ is such that supp $\phi \cap \operatorname{supp} \mathcal{D}_f = \emptyset$ but $\int_{\mathbb{R}^n} f(x)\phi(x)\, dx \neq 0$, this is a contradiction.
We conclude that $x_0 \in \operatorname{supp} \mathcal{D}_f$ and hence $\operatorname{supp} f \subseteq \operatorname{supp} \mathcal{D}_f$.
Thus

$$\operatorname{supp} f = \operatorname{supp} \mathcal{D}_f.$$

This will not be true when $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, consider the function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

The supp $f = \mathbb{R}$, but supp $\mathcal{D}_f = \emptyset$.

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**EXERCISE 2**

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Show that the principal value integral

$$\text{p.v.} \int \frac{\phi(x)}{x}\, dx = \lim_{\varepsilon \to 0^+} \left( \int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x}\, dx \right)$$

exists for all $\phi \in C_c^\infty(\mathbb{R}^n)$, and is a distribution. What is its order?

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***Proof.***

$$\text{p.v.} \int \frac{\phi(x)}{x}\,dx = \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x}\,dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x}\,dx\right)$$
$$= \lim_{\varepsilon\to 0^+} \int_{\varepsilon\le|x|<1} \frac{\phi(x)-\phi(0)}{x}\,dx + \int_{1\le|x|} \frac{\phi(x)}{x}\,dx$$

Since $\phi \in C_c^\infty(\mathbb{R}^n)$, $\phi$ has compact support. Then

$$\int_{1\le|x|} \frac{\phi(x)}{x}\,dx = \int_{1\le|x|} \frac{|x\,\phi(x)|}{x^2}\,dx \le \sup_{x\in\mathbb{R}}\{|x\,\phi(x)|\} \int_{1\le|x|} \frac{1}{x^2}\,dx = 2\sup_{x\in\mathbb{R}}\{|x\,\phi(x)|\} < \infty.$$

Also, we see that

$$\chi_{\varepsilon\le|x|<1}\left|\frac{\phi(x)-\phi(0)}{x}\right| \le \chi_{|x|<1}\,||\phi||_\infty \quad \text{and} \quad \chi_{|x|<1}\,||\phi||_\infty \in L^1(\mathbb{R}),$$

so by Lebesgue Dominated Convergence Theorem, we know that

$$\lim_{\varepsilon\to 0^+} \int_{\varepsilon\le|x|<1} \frac{\phi(x)-\phi(0)}{x}\,dx \le \chi_{|x|<1}\,||\phi||_\infty < \infty.$$

Hence

$$\text{p.v.} \int \frac{\phi(x)}{x}\,dx = \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x}\,dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x}\,dx\right) \quad \text{exists.}$$

Moreover, if $\operatorname{supp}\phi \subset [-a,a]$, then

$$\left|\text{p.v.} \int \frac{\phi(x)}{x}\,dx\right| \le 2a\,\sup\{|\phi'|\}.$$

This implies that the p.v. of $\dfrac{1}{x}$ is a distribution of order at most 1.

Finally, the order cannot be 0. Indeed, if $0 \le \phi_\varepsilon \le 1$ such that $\operatorname{supp}\phi_\varepsilon \subset [\varepsilon, 4\varepsilon]$ and $\phi_\varepsilon = 1$ on $[2\varepsilon, 3\varepsilon]$ then

$$\text{p.v.} \int \frac{\phi(x)}{x}\,dx \ge \frac{1}{4\varepsilon}\,\sup\{|\phi_\varepsilon|\}.$$

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**EXERCISE 3**

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Find a distribution $u \in \mathcal{D}'(\mathbb{R})$ such that $u = \frac{1}{x}$ on $(0,\infty)$ and $u = 0$ on $(-\infty, 0)$.

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***Proof.***
Consider the function $g(x) = \ln x$ for $x > 0$ and $g(x) = 0$ for $x \le 0$. Then $g \in L^1_{\text{loc}}$, so defines a distribution, and $g'(x) = \frac{1}{x}$ for $x > 0$. So $g'$ is an admissible $u$. Therefore

$$\langle u, \phi\rangle = -\int_0^\infty \phi'(x)\,\ln x\,dx$$

works.

## EXERCISE 4

Show that
$$\langle u, \phi \rangle = \sum_{k=1}^{\infty} \partial^k \phi(\frac{1}{k})$$
is a distribution in $(0, \infty)$? What is its order?

***Proof.***
Let $\phi$ be such that supp $\phi \subset [\frac{1}{N}, N]$. Then
$$\langle u, \phi \rangle = \sum_{k=1}^{N} \partial^k \phi(\frac{1}{k}) \leq \sum_{k=1}^{N} \sup_{x \in [1/N, N]} |\partial^k \phi|.$$
Since the compacts $[1/N, N]$ exhaust $(0, \infty)$, it follows that $u$ is a distribution on $(0, \infty)$.

Suppose that $u = v|_{(0,\infty)}$ for $v \in \mathcal{D}'(\mathbb{R})$. Then there must exist $N_0$ and $C_0$ such that
$$|\langle v, \phi \rangle| \leq C_0 \sum_{k=1}^{N_0} \sup |\partial^k \phi|, \quad \text{supp } \phi \subset [-1, 1].$$
So, if we take $N > N_0$, we will have
$$\left| \partial^N \phi(\frac{1}{N}) \right| \leq |\langle u, \phi \rangle| \leq |\langle v, \phi \rangle| \leq C_0 \sum_{k=1}^{N_0} \sup |\partial^k \phi|,$$
if supp $\phi \subset \left( \frac{1}{N+1}, \frac{1}{N-1} \right)$. This would imply that $\partial^N \delta_{\frac{1}{N}}$ is order at most $N_0 < N$ and consequently that $\partial^N \delta$ is of order at most $N_0$ on a small interval $(-\varepsilon, \varepsilon)$.

We claim that this is impossible.
Indeed, let $\psi \in C_c^\infty((-\varepsilon, \varepsilon))$ be such that $\partial^N \psi(0) \neq 0$. Consider then the test functions
$$\psi_\lambda(x) = \lambda^N \psi(\frac{x}{\lambda}) \quad \text{for small } \lambda > 0.$$
We have supp $\psi_\lambda \subset (-\varepsilon\lambda, \varepsilon\lambda)$. Moreover,
$$\partial^N \psi_\lambda(0) = \partial^N \psi(0) \quad \text{and} \quad \partial^k \psi_\lambda = \lambda^{N-k} \partial^k \psi.$$
Thus, we would have an estimate
$$|\partial^N \psi(0)| \leq C_0 \sum_{k=1}^{N_0} \lambda^{N-N_0} \sup |\partial^k \psi| \quad \text{for any } \lambda > 0.$$
This is clearly a contradiction for small $\lambda$.

## EXERCISE 5

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ have the property that $\langle u, \phi \rangle \geq 0$ for all real valued nonnegative $\phi \in C_c^\infty(\mathbb{R}^n)$. Show that $u$ is of order 0.

***Proof.***
Let $K \subset\subset \mathbb{R}^n$ and $\psi_K \in C_c^\infty(\mathbb{R}^n)$ non-negative cut-off function such that $\psi_K = 1$ on $K$. Then for real-valued test functions $\phi$ with supp $\phi \subset K$, we would have

$$\left(\sup_K |\phi|\right) \psi_K(x) - \phi(x) \geq 0.$$

Hence

$$\left\langle u, \left(\sup_K |\phi|\right) \psi_K(x) - \phi(x) \right\rangle \geq 0.$$

This implies

$$\langle u, \phi(x) \rangle \leq \langle u, \psi_K \rangle \left(\sup_K |\phi|\right).$$

For complex valu $\phi$ we obtain

$$| \langle u, \phi(x) \rangle | \leq 2 \langle u, \psi_K \rangle \left(\sup_K |\phi|\right)$$

by considering the real and imaginary parts of $\phi$.

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**EXERCISE 6**

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Let $\{f_k\}_{k=1}^\infty \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a sequence of real valued functions such that

$$\operatorname{supp} f_k \subset \{|x| \leq k^{-1}\}, \qquad \int f_k(x)\, dx = 1, \quad k = 1, 2, \cdots .$$

Show that the sequence $\{f_k^2\}_{k=1}^\infty$ does not converge in $\mathcal{D}'(\mathbb{R}^n)$ as $k \to \infty$.

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***Proof.***
Let $\phi \in C_c^\infty(\mathbb{R}^n)$ and

$$f_k(x) = \begin{cases} \frac{k}{2}, & |x| \leq \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases},$$

then $\{f_k\}_{k=1}^\infty \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\int f_k(x)\, dx = 1$. So

$$f_k^2(x) = \begin{cases} \frac{k^2}{4}, & |x| \leq \frac{1}{k} \\ 0, & |x| > \frac{1}{k} \end{cases},$$

hence, if $\phi \in (\mathbb{R}^n)$ and $\phi = 1$ in $|x| \leq 1$, then

$$\left\langle f_k^2, \phi \right\rangle = \int f_k^2(x)\phi(x)\, dx \geq \inf_{|x| \leq \frac{1}{k}} \phi \int_{|x| \leq \frac{1}{k}} f_k^2(x)\, dx = \frac{k}{2},$$

which is divergent as $k \to \infty$. Therefore, the sequence $\{f_k^2\}_{k=1}^\infty$ does not converge in $\mathcal{D}'(\mathbb{R}^n)$ as $k \to \infty$.