Real Analysis
Homework 3

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Yueh-Chou Lee

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1. (Exercise 4.2) Let f be a simple function, taking its distinct values on disjoint sets $E_1, ..., E_N$. Show taht f is measurable if and only if $E_1, ..., E_N$ are measurable.

Proof.
By the statement in Exercise 4.2, we may assume that f is a simple function and takes its distinct values $a_i \in \mathbb{R}$ on disjoint sets E_i for all $i \in \{1, 2, ..., N\}$, then

$$E_i = \{x \in \cup_{i=1}^N E_i \mid f(x) = a_i\}, \quad for\ all\ i \in \{1, 2, ..., N\}$$

$(\Rightarrow)$
Since f is measurable, then for all $a_i \in \mathbb{R}$ and any $\epsilon > 0$ such that $\{x \in \cup_{i=1}^N E_i \mid f(x) > a_i + \epsilon\}$ and $\{x \in \cup_{i=1}^N E_i \mid f(x) > a_i - \epsilon\}$ are measurable for all $i \in \{1, 2, ..., N\}$.
Futhermore, for all $i \in \{1, 2, ..., N\}$, we know that

$$\begin{aligned} E_i &= \{x \in \cup_{i=1}^N E_i \mid f(x) = a_i\} \\ &= \{x \in \cup_{i=1}^N E_i \mid f(x) > a_i - \epsilon\} - \{x \in \cup_{i=1}^N E_i \mid f(x) > a_i + \epsilon\} \end{aligned}$$

Hence, E_i is also measurable for all $i \in \{1, 2, ..., N\}$.

$(\Leftarrow)$
To prove f is measurable function, that is to prove for any $a \in \mathbb{R}$, the set $\{x \in \cup_{i=1}^N E_i \mid f(x) > a\}$ is measurable.
Take $a \in \mathbb{R}$.

(a) If $a > a_1, ..., a_N$, then there is NO $x \in \cup_{i=1}^N E_i$ such that $f(x) > a$, so the set $\{x \in \cup_{i=1}^N E_i \mid f(x) > a\}$ is measure zero and also measurable.

(b) If $a < a_1, ..., a_N$, then for all $x \in \cup_{i=1}^N E_i$ such that $f(x) > a$, so the set

$$\{x \in \cup_{i=1}^N E_i \mid f(x) > a\} = \{x \in \cup_{i=1}^N E_i \mid f(x) = a_1, a_2, ..., a_N\} = \cup_{i=1}^N E_i$$

is measurable.

(c) If $a_{i_1} < a_{i_2} < ... < a_{i_k} \leq a < a_{j_1} < a_{j_2} < ... < a_{j_l}$ where $k, l \in \mathbb{N}$ and $k + l = N$, then

$$\{x \in \cup_{i=1}^N E_i \mid f(x) > a\} = \{x \in \cup_{i=1}^N E_i \mid f(x) = a_{j_1}, ..., a_{j_l}\} = \cup_{i=j_1,...,j_l} E_i$$

is measurable. ✓

By above (a), (b) and (c), we know that for any $a \in$ R,the set $\{x \in \cup_{i=1}^N E_i \mid f(x) > a\}$ is measurable, therefore, f is measurable.

2. (Exercise 4.3) Theorem 4.3 can be used to define measurability for vector-valued (e.g., complex-valued) functions. Suppose, for example, that f and g are realvalued and finite in $\mathbb{R}^n$, and let $F(x) = (f(x), g(x))$. Then F is said to be measurable if $F^{-1}(G)$ is measurable for every open $G \in \mathbb{R}^2$. Prove that F is measurable if and only if both f and g are measurable in $\mathbb{R}^n$.

Proof.
($\Rightarrow$)
Suppose that F is measurable. Then $F^{-1}((a,\infty)\times \mathrm{R}) = \{f > a\}$ and $F^{-1}(\mathrm{R}\times(b,\infty)) = \{g > b\}$ are measurable for $a, b \in \mathrm{R}$, hence, f and g are measurable. ✓

($\Leftarrow$)
Suppose that f and g are measurable. Then

$$\{a \le f \le b\} \quad and \quad \{c \le g \le d\}$$

are measurable for all real a, b, c and d.

Recall:
. All open sets in R^2 can be written as a union of nonoverlapping closed rectangles.
Then, if G is an open set in R^2, we have

$$\begin{aligned} F^{-1}(G) &= F^{-1}\left(\cup_{k=1}^{\infty}[a_k, b_k] \times [c_k, d_k]\right) \\ &= \cup_{k=1}^{\infty} F^{-1}([a_k, b_k] \times [c_k, d_k]) \\ &= \cup_{k=1}^{\infty}(\{a_k \le f \le b_k\}) \cap (\{c_k \le g \le d_k\}) \end{aligned}$$

This is a countable union of measurable sets, hence F is measurable. ✓

3. (Exercise 4.4) Let f be defined and measurable in R^n. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(Tx)$ is measurable. (If $E_1 = \{x : f(x) > a\}$ and $E_2 = \{x : f(Tx) > a\}$, show that $E_2 = T^{-1}E_1$.)

Proof.
Since f is defined and measurable in R^n, then E_1 is measurable.
Follow the hint, let $E_1 = \{x : f(x) > a\}$ and $E_2 = \{x : f(Tx) > a\}$, we continue to show that $E_2 = T^{-1}E_1$.

(a) For every $x \in E_2$, there will exist y such that $Tx = y$, then $y \in E_1$ and $x = T^{-1}y$. Hence, $x \in T^{-1}E_1$.

(b) Futhermore, for every $x \in T^{-1}E_1$, there will exist $y \in E_1$ such that $x = T^{-1}y$, then $Tx = y$. Hence, $x \in E_2$.

By above (a) and (b), we know that $E_2 = T^{-1}E_1$. Since T is a nonsingular linear transformation, T^{-1} will also be a linear transformation.
By Theorem 3.33 in the textbook, since E_1 is measurable and $E_2 = T^{-1}E_1$, then T^{-1} will map the measurable set E_1 into the measurable set E_2. ✓
Hence, E_2 is measurable, and so is $f(Tx)$.

4. (Exercise 4.5) Give an example to show that $\phi(f(x))$ may not be measurable if ϕ and f are measurable and finite. (Let F be the Cantor–Lebesgue function and let f be its inverse, suitably defined. Let ϕ be the characteristic function of a set of measure zero whose image under F is not measurable.) Show that the same may be true even if f is continuous. (Let $g(x) = x + F(x)$, where F is the Cantor–Lebesgue function, and consider $f = g^{-1}$.)

Proof.

(a) Follow the hint, let F be the Cantor–Lebesgue function and let f be its inverse, suitably defined, where f be defined as

$$f(x) = \inf\{a \in [0,1] : F(a) = x\}$$

for $x \in [0,1]$, then we will have $f(F(x)) = F(f(x))$ for all $x \in C'$, where C' is the Cantor-Lebesgue set removed all right end-points of every subinterval. Hence, f is the inverse of F restricted to C'.
See the proof in Exercise 3.17 (in Hw2), the above statement implies $F(C') = [0,1]$. Since $|[0,1]| = 1 > 0$, there exists $B \subseteq F(C')$ such that B is a non-measurable set.
Let

$$A = \{x \in C' | F(x) \in B\}$$

However, C' is measure zero and $A \subseteq C'$, therefore, A is also measurable zero.
Define characteristic function $\phi(x)$ as the same as the function in the textbook,

$$\phi(x) = \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Then

$$\begin{aligned}\{x \in C' : \phi(f(x)) = 1\} &= f^{-1}(\phi^{-1}(1)) \\ &= F(\phi^{-1}(1)) \\ &= F(A)\end{aligned}$$

By above, we know that $F(x \in A) \in B$ and B is non-measurable, therefore, $\{x \in C' : \phi(f(x)) = 1\}$ is non-measurable, which implies $\phi(f(x))$ is also non-measurable.

(b) Follow the hint, let $g(x) = x + F(x)$, where F is the Cantor–Lebesgue function, $x \in C$ where C is the Cantor set and consider $f = g^{-1}$. Then $g : [0,1] \to [0,2]$ is strictly monotone and continuous, thus it has a continuous inverse.
We claim that $|g(C)| = 1$, since F is constant on every interval in $[0,1] \setminus C$, so g maps such an interval to an interval of the same length, therefore $|g([0,1]) \setminus C| = 1$. Since $|g([0,1])| = |[0,2]| = 2$, this proves the claim that $|g(C)| = 1$.
Similarly in (a), there exists $B \subseteq g(C)$ such that B is a non-measurable set.
Let

$$A = \{x \in C' | g(x) \in B\},$$

then A is measure zero.
Define $\phi(x) = \chi_A(x)$, then

$$f^{-1}(\phi^{-1}(0,2)) = f^{-1}(A) = g(A)$$

By above, we know that $g(x \in A) \in B$ and B is non-measurable, therefore, $\phi(f(x))$ is also non-measurable.

5. (Exercise 4.7) Let f be usc and less than $+\infty$ on a compact set E. Show that f is bounded above on E. Show also that f assumes its maximum on E, that is, that there exists $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.

Proof.

(a) Suppose that $x_1, x_2, ..., x_N$ are the limit points of E.
f is usc and less than $+\infty$ on the set E, so for all x_i where $i \in \{1, 2, ..., N\}$, then $f(x_i)$ will also be finite and usc at x_i.
Therefore, for all $M \in$ R such that $f(x_i) < M$, then there must exist $\delta_{x_i} > 0$ such that $f(x) < M$ where $x \in B(x_i, \delta_{x_i}) \cap E$.
Pick $M = max\{f(x_1), f(x_2), ..., f(x_N)\} + 1$.
Futhermore, E is compact so we will have $\cup_{i=1}^{N} B(x_i, \delta_{x_i}) \supset E$.
Hence, f is bounded above by M on E.

(b) By above, since f is bounded above on the set E, so there must exist the sequence $f(x_k)$ such that $f(x_k) \to \sup f(E)$.
Hence, it has convergent subsequence $\{x_{k_i}\}$ in $\{x_k\}$. Let $x_{k_i} \to x_0$, then for every $\epsilon > 0$, there exists an integer $n > 0$ such that for $i \geq n$, we have

$$f(x_{k_i}) < f(x_0) + \epsilon$$

Thus that

$$\sup f(E) \leq f(x_0) + \epsilon$$

for any $\epsilon > 0$.
Since ϵ is arbitrary chosen then $\sup f(E) \leq f(x_0)$.
Hence, f has its maximum on E. ✓

6. (Exercise 4.8)

(a) Let f and g be two functions that are usc at x_0. Show that $f + g$ is usc at x_0. Is $f - g$ usc at x0? When is fg usc at x_0?

(b) If $\{f_k\}$ is a sequence of functions that are usc at x_0, show that $\inf_k f_k(x)$ is usc at x_0.

(c) If $\{f_k\}$ is a sequence of functions that are usc at x_0 and converge uniformly near x_0, show that $\lim f_k$ is usc at x_0.

Proof.

(a) i. Since f and g are two functions that are usc at x_0, we will have

$$\begin{aligned} \limsup_{x \to x_0,\, x \in E}(f(x) + g(x)) &\leq \limsup_{x \to x_0,\, x \in E} f(x) + \limsup_{x \to x_0,\, x \in E} g(x) \\ &\leq f(x_0) + g(x_0) \end{aligned}$$

Hence, $f + g$ is usc at x_0. ✓

ii. Since f is usc at x_0, then

$$\limsup_{x \to x_0,\, x \in E} f(x) \leq f(x_0).$$

Since g is usc at x_0, then

$$\limsup_{x \to x_0,\, x \in E} g(x) \leq g(x_0) \Rightarrow \liminf_{x \to x_0,\, x \in E}(-g(x)) \geq (-g(x_0))$$

There is "NOT" sufficient to say that

$$\limsup_{x \to x_0,\, x \in E}[f(x) + (-g(x))] \leq f(x) + (-g(x)) = f(x) - g(x)$$

Hence, $f - g$ is "NOT" usc at x_0.

iii. Since f and g are usc at x_0, then

$$\begin{aligned}\limsup_{x\to x_0,\ x\in E}(fg)(x) &\leq (\limsup_{x\to x_0,\ x\in E} f(x))(\limsup_{x\to x_0,\ x\in E} g(x)) \\ &\leq f(x_0)g(x_0)\end{aligned}$$

Hence, fg is usc at x_0.

(b) Let $f(x) = \inf_{k\in \mathrm{N}} f_k(x)$, then

$$\limsup_{x\to x_0} f(x) = \limsup_{x\to x_0}(\inf_{k\in \mathrm{N}} f_k(x)) \leq \limsup_{x\to x_0} f_k(x) \leq f_k(x_0)$$

for all $k \in \mathrm{N}$.
Then

$$\limsup_{x\to x_0} f(x) \leq \inf_{k\in \mathrm{N}} f_k(x_0) = f(x_0)$$

Hence, $f(x) = \inf_k f_k(x)$ is usc at x_0.

(c) By definition of uniformly convergence, let $f(x) = \lim_{k\to\infty} f_k(x)$, then for $\epsilon > 0$, there exists a $k \in \mathrm{N}$ such that $\sup_{x\in E}\{|f(x) - f_k(x)|\} < \epsilon$.
Since $\{f_k\}_{k=1}^{\infty}$ converge uniformly and are usc at x_0, we have

$$\limsup_{x\to x_0} f(x) < \limsup_{x\to x_0} f_k(x) + \epsilon \leq f_k(x_0) + \epsilon < f(x_0) + 2\epsilon$$

for any $\epsilon > 0$.
However, ϵ is arbitrary chosen, hence, $f(x) = \lim_{k\to\infty} f_k(x)$ is usc at x_0.

7. (Exercise 4.9)

(a) Show that the limit of a decreasing (increasing) sequence of functions usc (lsc) at x_0 is usc (lsc) at x_0. In particular, the limit of a decreasing (increasing) sequence of functions continuous at x_0 is usc (lsc) at x_0.

(b) Let f be usc and less than $+\infty$ on $[a, b]$. Show that there exist continuous f_k on $[a, b]$ such that $f_k \searrow f$. (First show that there are usc step functions $f_k \searrow f$.)

Proof.

(a) i. Let $\{f_k\}_{k=1}^{\infty}$ be the decreasing sequence such that $f_k \searrow f$, then

$$\lim_{x\to x_0} \sup_{x\in E} f(x) \leq \lim_{x\to x_0} \sup_{x\in E} f_k(x) \leq f_k(x_0).$$

Since $f_k(x_0) \searrow f(x_0)$, we will have

$$\lim_{x\to x_0} \sup_{x\in E} f(x) \leq f(x_0)$$

Hence, f is usc at x_0.

ii. Let $\{f_k\}_{k=1}^{\infty}$ be the increasing sequence such that $f_k \nearrow f$, then $-f_k \searrow -f$, therefore,

$$\lim_{x\to x_0} \sup_{x\in E} -f(x) \leq -f(x_0) \Rightarrow \lim_{x\to x_0} \inf_{x\in E} f(x) \geq f(x_0).$$

Hence, f is lsc at x_0.

iii. In particular, if every f_k is continuous at x_0, it follows that $|f(x_0)| < +\infty$ and f is both usc and lsc at x_0.
If $f_k \searrow f$, then f is usc at x_0.
If $f_k \nearrow f$, then f is lsc at x_0.

(b) First, we assume $f \le 0$ and finite-valued, then let $f_n : [a,b] \to \mathrm{R}$ with

$$f_n(x) = \sup\{f(t) - n|x-t| \,|\, t \in [a,b]\}.$$

Then for all n,

$$\begin{aligned} f_n(x) &= \sup\{f(t) - n|x-t| \,|\, t \in [a,b]\} \\ &\le f(x) - n|x-x| \\ &= f(x) \end{aligned}$$

and for every $\epsilon > 0$ and $x, y \in [a,b]$ with $|x-y| < \frac{\epsilon}{n}$,

$$\begin{aligned} |f_n(x) - f_n(y)| &\le |\sup\{f(t) - n|x-t| - f(t) + n|y-t| \,|\, t \in [a,b]\}| \\ &\le n|x-y| \\ &< \epsilon \end{aligned}$$

For every $\epsilon > 0$, for each n we can choose $t_n \in [a,b]$ such that

$$f(x) \le f_n(x) < f(t_n) - n|x-t_n| + \frac{\epsilon}{2} \le -n|x-t_n| + \frac{\epsilon}{2}$$

Then $|x-t_n| \to 0$ as $n \to \infty$. Since f is usc, then $\lim_{n\to\infty}\sup f(t_n) \le f(x)$. There is $M > 0$ such that for all $n \ge M$, we have $f(t_n) < f(x) + \epsilon$.
For $n \ge M$, then

$$\begin{aligned} f_n(x) - f(x) &< f(t_n) - n|x-t_n| + \frac{\epsilon}{2} - f(x) \\ &\le f(t_n) + \frac{\epsilon}{2} - f(x) \\ &< f(x) + \frac{\epsilon}{2} + \frac{\epsilon}{2} - f(x) \\ &= \epsilon \end{aligned}$$

Thus $\{f_n\}$ is decreasing sequence of continuous functions with $f_n \searrow f$. In general f, let $h(x) = -\frac{1}{2} + \arctan x$, $x \in \bar{\mathrm{R}}$, then hf is finite-valued, usc and $f \le 0$. So we can find contiuous function $g_n \searrow hf$.
Let $f_n = h^{-1}g_n$, then $f_n \searrow f$.