1. (15 pts) The area of a triangle with sides of lengths a and b and contained angle θ is $A = \frac{1}{2}ab\sin\theta$.
   (a) If $a = 2$ cm, $b = 3$ cm, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $\theta = \pi/3$?
   (b) If $a = 2$ cm, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $b = 3$ cm and $\theta = \pi/3$?
   (c) If a increases at a rate of 2.5 cm/min, b increases at a rate of 1.5 cm/min, and θ increases at a rate of 0.2 rad/min, how fast is the area increasing when $a = 2$ cm, $b = 3$ cm and $\theta = \pi/3$?

   Solution

   (a) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}ab\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}\cdot 2\cdot 3\cdot\cos\left(\frac{\pi}{3}\right)\cdot 0.2 = 0.3\ \mathrm{cm}^2/\mathrm{min}.$

   (b) Since $A = \frac{1}{2}ab\sin\theta$

   $$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}a\left(b\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} + \sin\theta\frac{\mathrm{d}b}{\mathrm{d}t}\right) = \frac{1}{2}\cdot 2\cdot\left(3\cdot\cos\left(\frac{\pi}{3}\right)\cdot 0.2 + \sin\left(\frac{\pi}{3}\right)\cdot 1.5\right) = \frac{3}{4}\sqrt{3} + 0.3\ \mathrm{cm}^2/\mathrm{min}$$

   (c) Since $A = \frac{1}{2}ab\sin\theta$

   $$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}\left(\frac{\mathrm{d}a}{\mathrm{d}t}b\sin\theta + a\frac{\mathrm{d}b}{\mathrm{d}t}\sin\theta + ab\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\\ &= \frac{1}{2}\cdot\left(2.5\cdot 3\cdot\sin\left(\frac{\pi}{3}\right) + 2\cdot 1.5\cdot\sin\left(\frac{\pi}{3}\right) + 2\cdot 3\cdot\cos\left(\frac{\pi}{3}\right)\cdot 0.2\right) = \frac{21}{8}\sqrt{3} + 0.3\ \mathrm{cm}^2/\mathrm{min}\end{aligned}$$

   (5 pts each, 4 pts for computing product rule correctly and 1 pts for correct anwser.)

2. (10pts) Let $f(x) = \dfrac{1+\cos x}{1+\sin x}$. Use a differential to estimate $f(44^\circ)$.

   Solution

   $f(x+h) \simeq f(x) + f'(x)\cdot h$ and $f'(x) = \dfrac{-\sin x(1+\sin x) - \cos x(1+\cos x)}{(1+\sin x)^2} = \dfrac{-\sin x - \cos x - 1}{(1+\sin x)^2}$.

   Thus, $f(44^\circ) = f(45^\circ + (-1)^\circ) \simeq f(45^\circ) + f'(45^\circ)\cdot(-1)^\circ = f(\frac{\pi}{4}) + f'(\frac{\pi}{4})\cdot\frac{-\pi}{180}$

   Compute $f(\frac{\pi}{4}) = \dfrac{1+\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}} = 1$ and $f'(\frac{\pi}{4}) = \dfrac{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - 1}{(1+\frac{\sqrt{2}}{2})^2} = \dfrac{-\sqrt{2}-1}{\frac{3}{2}+\sqrt{2}} = 2 - 2\sqrt{2}$.

   Finally, we get $f(44^\circ) \simeq 1 + (2-2\sqrt{2})\cdot\dfrac{-\pi}{180} = 1 + \dfrac{(\sqrt{2}-1)\pi}{90}$.

   (5 pts for correct derivative, 3 pts for correct linear approximation formula and 2 points for correct answer.)

3. (15pts) Find the linear approximation of the function $g(x) = \sin^{-1}\left(\dfrac{x-1}{x+1}\right) - \tan^{-1}\left(\sqrt{x}\right)$ at the point $x = 3$.

   Solution

   Linear approximation at $x = 3$ is $g(x) \approx g(3) + g'(3)(x-3)$. Since

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}\left(\frac{x-1}{x+1}\right)\right) = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-1}{x+1}\right) = \frac{1}{\sqrt{x}(x+1)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\sqrt{x}\right) = \frac{1}{1+(\sqrt{x})^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}(x+1)}$$

Compute $g'(x)$ directly by $g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}\left(\frac{x-1}{x+1}\right)\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\sqrt{x}\right) = \frac{1}{2\sqrt{x}(x+1)}$. Thus, we get $g(3) = -\frac{\pi}{6}$ and $g'(3) = \frac{1}{8\sqrt{3}}$. The required linear approximation is given by $g(x) \approx -\frac{\pi}{6} + \frac{1}{8\sqrt{3}}(x-3)$.

(5 pts for correct derivative of $\sin^{-1}$, 5 pts for correct derivative of $\tan^{-1}$, 3 pts for correct linear approximation formula and 2 points for correct answer.)

4. (16pts) Find the absolute maximum and absolute minimum values of f on the given interval.

   (a) $f(x) = \frac{\sqrt{x}}{1+x^2}$, $[0, 2]$.

   (b) $f(x) = x^{-2}\ln x$, $[\frac{1}{2}, 4]$.

Solution

(a) $f'(x) = \frac{(1+x^2)(\frac{1}{2\sqrt{x}}) - \sqrt{x}(2x)}{(1+x^2)^2} = \frac{1-3x^2}{2\sqrt{x}(1+x^2)^2}$.

$f'(x) = 0 \Leftrightarrow 1-3x^2 = 0 \Leftrightarrow x = \pm\frac{1}{\sqrt{3}}$, but $x = -\frac{1}{\sqrt{3}}$ is not in the given interval. $f'(x)$ does not exist when $x = 0$, which is an endpoint. Direct computation yield $f(0) = 0, f(\frac{1}{\sqrt{3}}) = \frac{3^{3/4}}{4}$ and $f(2) = \frac{\sqrt{2}}{5}$. So $f(\frac{1}{\sqrt{3}}) = \frac{3^{3/4}}{4}$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

(b) $f'(x) = x^{-2}\cdot\frac{1}{x} + \ln x\cdot(-2x^{-3}) = \frac{1-2\ln x}{x^3}$.

$f'(x) = 0 \Leftrightarrow 1-2\ln x = 0 \Leftrightarrow x = \sqrt{e}$. $f'(x)$ does not exist when $x = 0$ which is not in the given interval. Direct computation yield $f(\frac{1}{2}) = -4\ln 2, f(\sqrt{e}) = \frac{1}{2e}$ and $f(4) = \frac{\ln 4}{16}$. So $f(\sqrt{e}) = \frac{1}{2e}$ is the absolute maximum value and $f(\frac{1}{2}) = -4\ln 2$ is the absolute minimum value.

(8 pts for each problem, 3 pts for correct derivative, 4 pts for finding critical values and 1 points for correct answer.)

5. (4pts) If a current I passes through a resistor with resistance R, Ohm's Law states that the voltage drop is $V = RI$. If V is constant and R is measured with a certain error, use differentials to show that the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

Solution

$I = \frac{V}{R} \Rightarrow \mathrm{d}I = -\frac{V}{R^2}\,\mathrm{d}R$. The relative error in calculating I is $\frac{\Delta I}{I} \approx \frac{\mathrm{d}I}{I} = \frac{-\left(V/R^2\right)\mathrm{d}R}{V/R} = -\frac{\mathrm{d}R}{R}$. Hence, the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

(3 pts for calculating relative error and 1 pts for the conclusion.)

6. (8pts) Show that $\left|\tan\frac{x}{2} - \tan\frac{y}{2}\right| \geq \frac{|x-y|}{2}$ for $x, y \in (-\pi, \pi)$.

Solution

Let $x, y \in (-\pi, \pi)$, W.L.O.G, set $x < y$. Let $f(t) = \tan\frac{t}{2}$, then f is continuous on $[x, y]$ and differentiable on (x, y). By the Mean Value Theorem, there is a number c between x and y such that $\frac{f(x)-f(y)}{x-y} = f'(c)$.

Since $|\sec\theta| \geq 1,\ \forall\theta \in (-\pi, \pi) \Rightarrow \frac{\left|\tan\frac{x}{2} - \tan\frac{y}{2}\right|}{|x-y|} = |f'(c)| = \left|\frac{1}{2}\sec^2\frac{c}{2}\right| \geq \frac{1}{2}$.

We get $\left|\tan\frac{x}{2} - \tan\frac{y}{2}\right| \geq \frac{|x-y|}{2}$.

(5 pts for correct use of mean value theorem and 3 pts for using inequality $\sec^2 \geq 1$.)

Cont.

7. (12pts) Find the following limits if they exists.

   (a) $\displaystyle\lim_{x\to\infty}\left[x+x^2\ln\left(1-\frac{2}{x}\right)\right]$ (b) $\displaystyle\lim_{x\to 0^+}(1-\cos x)^{\frac{1}{\ln x}}$

   Solution

   (a)

$$\lim_{x\to\infty}\left[x+x^2\ln\left(1-\frac{2}{x}\right)\right]=\lim_{x\to\infty}x^2\left[\ln\left(1-\frac{2}{x}\right)+\frac{1}{x}\right]=\lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}\right)+\frac{1}{x}}{\frac{1}{x^2}}$$

$$\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to\infty}\frac{\frac{1}{1-\frac{2}{x}}\cdot\frac{2}{x^2}-\frac{1}{x^2}}{-2\cdot\frac{1}{x^3}}=\lim_{x\to\infty}\frac{\frac{1}{x^2}\left(\frac{2x}{x-2}-1\right)}{\frac{-2}{x^3}}=\lim_{x\to\infty}\frac{x^2+2x}{-2x+4}=\infty$$

   (b) Let $y=(1-\cos x)^{\frac{1}{\ln x}}$, then $\ln y=\dfrac{\ln(1-\cos x)}{\ln x}$. We use l'Hospital's Rule to get

$$\lim_{x\to 0^+}\frac{\ln(1-\cos x)}{\ln x}\overset{\left(\frac{-\infty}{-\infty}\right)}{=}\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\frac{x^2}{1-\cos x}=\lim_{x\to 0^+}\frac{\sin x}{x}\cdot\lim_{x\to 0^+}\frac{x^2}{1-\cos x}=\lim_{x\to 0^+}\frac{x^2}{1-\cos x}$$

$$=\lim_{x\to 0^+}\frac{x^2(1+\cos x)}{\sin^2 x}=\lim_{x\to 0^+}\frac{1+\cos x}{\frac{\sin^2 x}{x^2}}=2$$

$$\ln\left(\lim_{x\to 0^+}y\right)=\lim_{x\to 0^+}\ln y=2.\ \text{Thus, }\lim_{x\to 0^+}(1-\cos x)^{\frac{1}{\ln x}}=e^2$$

   (6 pts each.)

8. (10pts) Sketch the curve $y=1+\dfrac{1}{x}+\dfrac{1}{x^2}$. (Hint: You should first find all the intercepts, asymptotes, local extema and inflection points of the given curve. Determine also the intervals on which the given function is increasing, decreasing, concave upward or concave downward.)

   Solution

   $y=f(x)=1+\dfrac{1}{x}+\dfrac{1}{x^2}=\dfrac{x^2+x+1}{x^2}$. Domain of $f(x)$ is $(-\infty,0)\cup(0,\infty)$.

   Since $x\neq 0$, there is no y-intercept. $f(x)=0\Leftrightarrow x^2+x+1=0$, there is no real solution, hence, no x-intercept. $\displaystyle\lim_{x\to\infty}\frac{x^2+x+1}{x^2}=1$, $\displaystyle\lim_{x\to-\infty}\frac{x^2+x+1}{x^2}=1\Rightarrow y=1$ is the horizontal asymptote. $\displaystyle\lim_{x\to 0}f(x)=\infty\Rightarrow x=0$ is the vertical asymptote. $f'(x)=\dfrac{-x-2}{x^3}$. $f'(x)>0\Leftrightarrow -2<x<0$ and $f'(x)<0\Leftrightarrow x<-2$ or $x>0$. Thus, f is increasing on $(-2,0)$ and decreasing on $(-\infty,-2)$ and $(0,\infty)$. Local minimum value $(-2)=\dfrac{3}{4}$ and no local maximum.

   $f''(x)=\dfrac{2x+6}{x^4}$. $f''(x)<0\Leftrightarrow x<-3$ and $f''(x)>0\Leftrightarrow -3<x<0$ and $x>0$. Thus, f is concave downward on $(-\infty,-3)$ and concave upward on $(-3,0)$ and $(0,\infty)$. Points of inflection is $(-3,\dfrac{7}{9})$. The graph of f is sketched at below.

   (2 pts for finding x and y intercepts, 2 pts for finding vertical and horizontal asymptotes, 2 pts for finding local maximum and minimum, 1 pts for finding inflection points and 3 pts for sketch the curve.)

9. (10pts) Let p and q be two functions given by

$$\begin{aligned}p\colon(0,+\infty)&\to\mathbb{R}\\ x&\mapsto e^x\end{aligned}\quad\text{and}\quad\begin{aligned}q\colon(0,+\infty)&\to\mathbb{R}\\ x&\mapsto x^e\end{aligned}.$$

   Consider the function defined by $f(x):=\frac{p(x)}{q(x)}$ for all $x\in(0,+\infty)$.

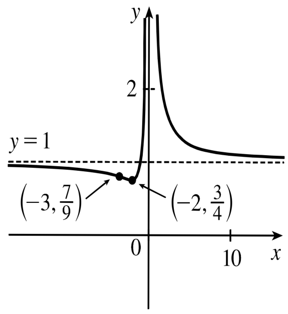

Figure 1: Problem 8

(a) Find all the (either global or local) extremal points of f on $(0, +\infty)$ and determine whether f attains its maximum or minimum there.

(b) Which number is larger, e^π or π^e?

Solution

(a) Set $g(x) := \ln f(x) = \ln e^{x - e \ln x} = x - e \ln x$. The first derivative of g is given by $g'(x) = 1 - \frac{e}{x}$, which follows that

$$g'(x) \begin{cases} < 0 & \text{when } 0 < x < e \ , \\ = 0 & \text{when } x = e \ , \\ > 0 & \text{when } x > e \ . \end{cases}$$

This means that g is strictly decreasing on $(0, e)$, reaches a local minimum at e and is strictly increasing on $(e, +\infty)$. This also implies that g has one and only one extremal point at e on the domain $(0, +\infty)$, at which g attains its minimum. Since the function $x \mapsto e^x$ is one-to-one, the above conclusion for g also holds for $f = e^g$.

(b) Since $f(x)$ attains its minimum at $x = e$, one has

$$f(x) \geq f(e) = 1 \ \Rightarrow \ e^x \geq x^e$$

for all $x \in (0, +\infty)$, and equality holds only when $x = e$.
Putting $x = \pi$ yields $e^\pi > \pi^e$, i.e. e^π is larger.

(6 pts for (a), 2 pts for finding extremal points, 1 pt for declaring it being a minimum and 3 pts for computation and test for minimality. 4 pts for (b), 2 pts for finding whether the equality holds and 2 pts for the correct answer.)

The End.