Real Analysis
Homework 9

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1. (Exercise 7.4)
If E_1 and E_2 are measurable subsets of $\mathbb{R}^1$ with $|E_1| > 0$ and $|E_2| > 0$, prove that the set $\{x : x = x_1 - x_2, x_1 \in E_1, x_2 \in E_2\}$ contains an interval. (cf. Lemma 3.37.)

Proof.
Consider the set $-E_1$ and E_2, with positive finte measure, then both χ_{-E_1} and χ_{E_2} are integrable, and

$$\begin{aligned}\int_x \chi_{-E_1} * \chi_{E_2} dx &= \int_x \int_y \chi_{-E_1}(x-y)\chi_{E_2}(y)dydx \\ &\overset{\text{Fubini}}{=} \int_y \int_x \chi_{-E_1}(x-y)\chi_{E_2}(y)dxdy \\ &= \int_y \chi_{E_2}(y) \int_x \chi_{-E_1}(x-y)dxdy \\ &= \int_y \chi_{E_2}(y)|-E_1|dy \\ &= |E_2||-E_1| > 0\end{aligned}$$

So there must exist some point where $\chi_{-E_1} * \chi_{E_2} > 0$. Convolution is continuous, so $\chi_{-E_1} * \chi_{E_2} > 0$ on the interval, (x_1, x_2), then for all $t \in (x_1, x_2)$,

$$\chi_{-E_1} * \chi_{E_2}(t) = \int_x \chi_{-E_1}(t-x)\chi_{E_2}(x)dx > 0$$

there must be some $x \in \mathbb{R}$ for which $\chi_{-E_1}(t-x)\chi_{E_2}(x) > 0$, then we have $x \in E_2$ and $t - x \in -E_1$,

$$t = (t-x) + (x) \in -E_1 + E_2 \Rightarrow t \in E_2 - E_1$$

Hence the set $\{x : x = x_1 - x_2, x_1 \in E_1, x_2 \in E_2\}$ contains an interval.

2. (Exercise 7.5)
Let f be of bounded variation on $[a, b]$. If $f = g + h$, where g is absolutely continuous and h is singular, show that

$$\int_a^b \phi \, df = \int_a^b \phi f' \, dx + \int_a^b \phi \, dh$$

for any continuous ϕ.

Proof.
$\int_a^b \phi dg$ exists because g is absolutely continuous and therefore continuous.
$\int_a^b \phi df$ exists because f is of bounded variation.

Since $\int_a^b \phi dg$, $\int_a^b \phi df$ exist and

$$\int_a^b \phi df - \int_a^b \phi dg = \int_a^b \phi d(f-g) = \int_a^b \phi dh$$

then $\int_a^b \phi dh$ also exists.

By Theorem 7.32, so

$$\begin{aligned}\int_a^b \phi df &= \int_a^b \phi d(g+h) \\ &= \int_a^b \phi dg + \int_a^b \phi dh \\ &= \int_a^b \phi g' dx + \int_a^b \phi dh \\ &= \int_a^b \phi f' dx + \int_a^b \phi dh\end{aligned}$$

where $\int_a^b \phi f' dx = \int_a^b \phi g' dx$ since h is singular ($h' = 0$), then $f' = g'$ a.e.

3. (Exercise 7.8)
Prove the following converse of Theorem 7.31: If f is of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Proof.
Since f is of bounded variation on $[a, b]$ and the function $V(x) = V[a, x]$,

$$V(x) = V[a, x] \leq V[a, b] < \infty$$

for all $x \in [a, b]$.

Since $V(x)$ is absoultely continuous on $[a, b]$, for given $\epsilon > 0$, there exists $\delta > 0$ such that for any collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$,

$$\sum |V(b_i) - V(a_i)| < \epsilon, \quad \text{if } \sum (b_i - a_i) < \delta$$

$V(b_i) - V(a_i)$ is well-defined since $V(x) < \infty$ for all $x \in [a, b]$, then if $\sum (b_i - a_i) < \delta$,

$$\begin{aligned}\sum |f(b_i) - f(a_i)| &\leq \sum V[a_i, b_i] \\ &= \sum |V[a, b_i] - V[a, a_i]| \\ &= \sum |V(b_i) - V(a_i)| \\ &< \epsilon\end{aligned}$$

Hence f is absolutely continuous on $[a, b]$.

4. (Exercise 7.9)
If f is of bounded variation on $[a, b]$, show that

$$\int_a^b |f'| \leq V[a, b]$$

Show that if equality holds in this inequality, then f is absolutely continuous on $[a, b]$. (For the second part, use Theorems 2.2(ii) and 7.24 to show that $V(x)$ is absolutely continuous and then use the result of Exercise 8.)

Proof.

(i) .

By Theorem 7.24, since f is of bounded variation on $[a, b]$, then

$$V'(x) = |f'(x)| \text{ for a.e. } x \in [a, b]$$

The integral

$$\int_a^b |f'| = \int_a^b V' \leq V(b^-) - V(a^+) \leq V[a, b]$$

(ii) .

If the equality holds in this inequality.

By Theorem 7.24, we have

$$\begin{aligned}
\int_a^x V' &= \int_a^x |f'| \\
&= \int_a^b |f'| - \int_x^b |f'| \\
&= V[a, b] - \int_x^b |f'| \\
&= V[a, x] + V[x, b] - \int_x^b |f'| \\
&\geq V[a, x]
\end{aligned}$$

for all $x \in [a, b]$.

This completes the prove by Theorem 7.29 and Exercise 7.8.

5. (Exercise 7.10)

(a) Show that if f is absolutely continuous on $[a, b]$ and Z is a subset of $[a, b]$ of measure zero, then the image set defined by $f(Z) = \{w : w = f(z), z \in Z\}$ also has measure zero. Deduce that the image under f of any measurable subset of $[a, b]$ is measurable. (Compare Theorem 3.33.) (Hint: use the fact that the image of an interval $[a_i, b_i]$ is an interval of length at most $V(b_i) - V(a_i)$.)

(b) Give an example of a strictly increasing Lipschitz continuous function f and a set Z with measure 0 such that $f^{-1}(Z)$ does not have measure 0 (and consequently, f^{-1} is not absolutely continuous). (Let $f^{-1}(x) = x + C(x)$ on $[0, 1]$, where $C(x)$ is the Cantor–Lebesgue function.)

Proof.

(a) .

Let $\epsilon > 0$, since f is absolutely continuous on $[a, b]$, so is the variation V of f over $[a, b]$, then there exists $\delta > 0$ such that

$$\sum_i |V(b_i) - V(a_i)| < \epsilon$$

for any nonoverlapping subintervals $[a_i, b_i]$ of $[a, b]$ the sum of whose length $\sum_i (b_i - a_i)$ is less than δ.

Let Z be any subset of $[a, b]$ with measure zero, there exists an open set G contains Z such that $|G| < \delta$.

The open set G can be written as the countable union of nonoverlapping subintervals $[a_i', b_i']$ of $[a, b]$.
Thus
$$\sum_i (b_i' - a_i') < \delta$$
This implies that
$$\begin{aligned}
|f(Z)|_e &\leq |f(\cup_i [a_i', b_i'])|_e \\
&\leq \sum_i |f([a_i', b_i'])|_e \\
&\leq \sum_i [\sup_{x \in [a_i', b_i']} f(x) - \inf_{x \in [a_i', b_i']} f(x)] \\
&\leq \sum_i [V(b_i') - V(a_i')] \\
&< \epsilon
\end{aligned}$$
So $f(Z)$ is measure zero.
For E be any measurable subset of $[a, b]$, written as $E = F \cup Z$ where F is of type F_σ, Z is a set with measure zero and $F \cap Z = \phi$.
Note that F is union of compact subsets of $[a, b]$, then $f(F)$ is measurable since f is continuous on $[a, b]$.
Hence $f(E)$ is measurable.

(b) .
Let $f^{-1}(x) = x + C(x)$ on $[0, 1]$.
Since $f^{-1}(x)$ is strictly increasing, its inverse $f(x)$ exists and f is strictly increasing.
Let $x, y \in f^{-1}([0, 1]) = [0, 2]$.
Suppose $x < y$ and write $x = p + C(p)$, $y = q + C(q)$ where $p < q$.
Since $f^{-1}(x) = x + C(x) \Rightarrow x = f(x + C(x))$ and the Cantor function $C(x)$ is increasing, then
$$\begin{aligned}
f(y) - f(x) &= f(q + C(q)) - f(p + C(p)) \\
&= q - p \\
&\leq q + C(q) - p - C(p) \\
&= y - x
\end{aligned}$$
So f is Lipschitz continuous.
Let Z be the Cantor set, which has measure zero.
Since $C(x)$ is constant on each disjoint interval in $[0, 1] \setminus Z$, $f^{-1}(x)$ maps each interval to an interval of the same length. ✔
Thus
$$|f^{-1}([0, 1] \setminus Z)| = |[0, 1] \setminus Z| = 1$$
Since $f^{-1}([0, 1]) = [0, 2]$, thus $|f^{-1}(Z)| = 1 > 0$.