Calculus A-05

NTU 107-1 MATH1201 Exercise set 4 Solution 23th November, 2018

> Please write down your solutions on a separate sheet of paper and submit it to your TA or instructor.
> Submit your solutions to Problems (1) $\sim$ (2) on 30**th** **November, 2018**.
> Submit your solutions to Problems (3) $\sim$ (6) on 12**th** **December, 2018**.

1. Evaluate the integral.

   (a) (4 pts) $\int (\cos x + \sin x)^2 \cos 2x \, dx$

   (b) (5 pts) $\int \frac{x^2}{\sqrt{9-x^2}} \, dx$, $|x| < 3$

   (c) (5 pts) $\int \frac{dx}{\sqrt{x^2+a^2}}$

   (d) (5 pts) $\int \frac{x^2+8x-3}{x^3+3x^2} \, dx$

   (e) (6 pts) $\int_0^a \frac{dx}{(x^2+a^2)^{3/2}}$, $a > 0$

   (f) (6 pts) $\int_0^x \sqrt{a^2-t^2} \, dt$, $0 \le x \le a$

   (g) (7 pts) $\int e^{\sqrt[3]{x}} \, dx$

   (a) (i)

$$\int (\cos x + \sin x)^2 \cos 2x \, dx = \int (\cos x + \sin x)^2 (\cos^2 x - \sin^2 x) \, dx$$
$$= \int (\cos x + \sin x)^3 (\cos x - \sin x) \, dx$$

Let $u = \cos x + \sin x$, then $du = (-\sin x + \cos x) \, dx$, so
.
(Substitution rule : 2 points)

$$\begin{aligned} \int (\cos x + \sin x)^2 \cos 2x \, dx &= \int (\cos x + \sin x)^3 (\cos x - \sin x) \, dx \\ &= \int u^3 \, du \\ &= \frac{1}{4} u^4 + C \\ &= \frac{1}{4} (\cos x + \sin x)^4 + C, \text{ where } C \text{ is a constant} \end{aligned}$$

(Answer : 2 points)

(ii) (alternative solution)

$$\begin{aligned}
\int (\cos x + \sin x)^2 \cos 2x \, dx &= \int (\cos^2 x + 2\cos x \sin x + \sin^2 x) \cos 2x \, dx \\
&= \int (1 + \sin 2x) \cos 2x \, dx \\
&= \int \cos 2x \, dx + \int \sin 2x \cos 2x \, dx \\
&= \frac{\sin 2x}{2} + C_1 + \int \frac{1}{2} \sin 4x \, dx \\
&= \frac{\sin 2x}{2} + C_1 + \frac{1}{2} \cdot \frac{-1}{4} \cos 4x + C_2 \\
&= \frac{\sin 2x}{2} - \frac{1}{8} \cos 4x + C
\end{aligned}$$

where C_1 and C_2 are constants, $C = C_1 + C_2$.

. (Computing the integral + Answer : 2 + 2 points)

(b) Let $x = 3\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$, then $dx = 3\cos\theta d\theta$ and $\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$.

. (Trigonometric substitution : 2 points)

Thus,

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9 - x^2}} \, dx &= \int \frac{9\sin^2\theta}{3\cos\theta} 3\cos\theta \, d\theta \\
&= 9 \int \sin^2\theta \, d\theta \\
&= 9 \int \frac{1}{2} (1 - \cos 2\theta) \, d\theta \\
&= \frac{9}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C \\
&= \frac{9}{2}\theta - \frac{9}{4} (2\sin\theta\cos\theta) + C \\
&= \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2} \cdot \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} + C \\
&= \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2} x\sqrt{9 - x^2} + C, \text{ where } C \text{ is a constant}
\end{aligned}$$

(Computing the integral + Answer : 2 + 1 points)

(c) (i) Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\sqrt{x^2 + a^2} = a\sec\theta$.

. (Trigonometric substitution : 2 points)

Thus,

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+a^2}} &= \int \frac{a\sec^2\theta}{a\sec\theta}\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta+\tan\theta|+C' \\ &= \ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|+C' \\ &= \ln(x+\sqrt{x^2+a^2})+C,\text{ where } C=C'-\ln|a| \text{ and } C' \text{ is a constant}\end{aligned}$$

(Computing the integral + Answer : 2 + 1 points)

(ii) Let $x = a\sinh t$, so that $dx = a\cosh t\,dt$ and $\sqrt{x^2+a^2} = a\cosh t$.
. (Trigonometric substitution : 2 points)

Thus,

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t}{a\cosh t}\,dt = t + C = \sinh^{-1}\left(\frac{x}{a}\right)+C,\text{ where } C \text{ is a constant}$$

(Computing the integral + Answer : 2 + 1 points)

Note: The hyperbolic trigonometric functions are given by $\sinh x = \frac{e^x-e^{-x}}{2}$, $\cosh x = \frac{e^x+e^{-x}}{2}$.

(d)

$$\frac{x^2+8x-3}{x^3+3x^2} = \frac{x^2+8x-3}{x^2(x+3)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+3}$$

$$\Rightarrow x^2+8x-3 = Ax(x+3)+B(x+3)+Cx^2$$

Taking $x=0$, we get $-3=3B$, so $B=-1$.
Taking $x=-3$, we get $-18=9C$, so $C=-2$.
Taking $x=1$, we get $6=4A+4B+C=4A-4-2$, so $4A=12$ and $A=3$.
. (Partial fraction : 3 points)

Now

$$\begin{aligned}\int \frac{x^2+8x-3}{x^3+3x^2}dx &= \int\left(\frac{3}{x}-\frac{1}{x^2}-\frac{2}{x+3}\right)dx \\ &= 3\ln|x|+\frac{1}{x}-2\ln|x+3|+C,\ C \text{ is a constant}\end{aligned}$$

(Answer : 2 points)

(e) Let $x = a\tan\theta$, where $a>0$ and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, then $dx = a\sec^2\theta\,d\theta$.
. (Trigonometric substitution : 2 points)

When $x = 0 \Rightarrow \theta = 0$, when $x = a \Rightarrow \theta = \frac{\pi}{4}$.

. (Change of limits of integration : 1 point)

Thus,

$$\begin{aligned}
\int_0^a \frac{dx}{(x^2+a^2)^{3/2}} &= \int_0^{\pi/4} \frac{a\sec^2\theta}{[a^2(1+\tan^2\theta)]^{3/2}}\,d\theta \\
&= \int_0^{\pi/4} \frac{a\sec^2\theta}{a^3\sec^3\theta}\,d\theta \\
&= \frac{1}{a^2}\int_0^{\pi/4} \cos\theta\,d\theta \\
&= \frac{1}{a^2}\left[\sin\theta\right]_0^{\pi/4} \\
&= \frac{1}{\sqrt{2}a^2}
\end{aligned}$$

(Computing the integral + Answer : 2 + 1 points)

(f) Let $t = a\sin\theta$, $dt = a\cos\theta\,d\theta$.

. (Trigonometric substitution : 2 points)

When $t = 0 \Rightarrow \theta = 0$, when $t = x \Rightarrow \theta = \sin^{-1}(x/a)$.

. (Changed of limits of integration : 1 point)

Thus,

$$\begin{aligned}
\int_0^x \sqrt{a^2-t^2}\,dt &= \int_0^{\sin^{-1}(x/a)} a\cos\theta(a\cos\theta\,d\theta) \\
&= a^2\int_0^{\sin^{-1}(x/a)} \cos^2\theta\,d\theta \\
&= \frac{a^2}{2}\int_0^{\sin^{-1}(x/a)} (1+\cos 2\theta)d\theta \\
&= \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\sin^{-1}(x/a)} \\
&= \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{\sin^{-1}(x/a)} \\
&= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right) - 0\right] \\
&= \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2}
\end{aligned}$$

(Computing the integral + Answer : 2 + 1 points)

(g) Let $w = \sqrt[3]{x}$, then $w^3 = x$ and $3w^2\,dw = dx$, so

$$\int e^{\sqrt[3]{x}}dx = \int e^w \cdot 3w^2\,dw = 3I$$

(Substitution rule : 1 point)

Cont.

To evaluate I, let $u = w^2$ and $dv = e^w\, dw$, then $du = 2w\, dw$ and $v = e^w$, so

$$I = \int w^2 e^w\, dw = w^2 e^w - \int 2we^w\, dw$$

(Integration by parts : 2 points)

Now let $U = w$, $dV = e^w\, dw$, then $dU = dw$, $V = e^w$.
Thus,

$$I = w^2 e^w - 2\left[we^w - \int e^w\, dw\right] = w^2 e^w - 2we^w + 2e^w + C', \text{ where } C' \text{ is a constant.}$$

(Integration by parts : 2 points)

Hence

$$\begin{aligned}\int e^{\sqrt[3]{x}}\, dx &= 3I \\ &= 3e^w(w^2 - 2w + 2) + 3C' \\ &= 3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C, \text{ where } C = 3C'\end{aligned}$$

(Answer : 2 points)

2. (7 pts) If $0 < a < b$, find

$$\lim_{t\to 0}\left\{\int_0^1 [bx + a(1-x)]^t\, dx\right\}^{1/t}$$

Let $u = bx + a(1-x)$, then $du = (b-a)dx$, since $0 < a < b$, we have

$$\int_0^1 [bx + a(1-x)]^t\, dx = \int_a^b \frac{u^t}{(b-a)}\, du = \left[\frac{u^{t+1}}{(t+1)(b-a)}\right]_a^b = \frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}$$

(Substitution rule : 2 points)

Now let $y = \lim\limits_{t\to 0}\left[\frac{b^{t+1}-a^{t+1}}{(t+1)(b-a)}\right]^{1/t}$, then

$$\ln y = \lim_{t\to 0}\frac{\ln \frac{b^{t+1}-a^{t+1}}{(t+1)(b-a)}}{t}$$

This limit is of the form $0/0$, so we can apply l'Hospital's Rule to get

$$\begin{aligned}\ln y &\overset{(\frac{0}{0})}{\underset{l'H}{=}} \lim_{t\to 0}\left[\frac{b^{t+1}\ln b - a^{t+1}\ln a}{b^{t+1} - a^{t+1}} - \frac{1}{t+1}\right] \\ &= \frac{b\ln b - a\ln a}{b-a} - 1 \\ &= \frac{b\ln b}{b-a} - \frac{a\ln a}{b-a} - \ln e \\ &= \ln\frac{b^{b/(b-a)}}{ea^{a/(b-a)}}\end{aligned}$$

Cont.

(L'Hospital's rule : 3 points)

Therefore, $y = e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}$. (Answer : 2 points)

3. Evaluate the integral or show that it is divergent.
   (a) (4 pts) $\int_{-3}^{3} \frac{x}{1+|x|}\,dx$
   (b) (5 pts) $\int_{1}^{\infty} \frac{\tan^{-1} x}{x^2}\,dx$
   (c) (7 pts) $\int_{-1}^{1} \frac{dx}{x^2-2x}$
   (d) (7 pts) $\int_{-1}^{\infty} \left(\frac{x^4}{1+x^6}\right)^2 dx$
   (Hint: $\left(\frac{x^4}{1+x^6}\right)^2$ can be writen as $x^3 \cdot f(x)$)

   (a) The integrand is an odd function, so
   . (Odd function : 2 points)

$$\int_{-3}^{3} \frac{x}{1+|x|}\,dx = 0$$

(Answer : 2 points)

   (b)

$$\int_{1}^{\infty} \frac{\tan^{-1} x}{x^2}\,dx = \lim_{t\to\infty} \int_{1}^{t} \frac{\tan^{-1} x}{x^2}\,dx$$

(Improper integral : 1 point)

Let $u = \tan^{-1} x$ and $dv = \frac{1}{x^2}dx$, then $du = \frac{1}{1+x^2}dx$ and $v = -\frac{1}{x}$.
Using integration by parts,

$$\int_1^\infty \frac{\tan^{-1} x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1} x}{x^2}\,dx$$
$$= \lim_{t\to\infty}\left\{\left[-\frac{\tan^{-1} x}{x}\right]_1^t + \int_1^t \frac{1}{x}\cdot\frac{1}{1+x^2}\,dx\right\}$$

.

(Integration by parts : 2 points)

.

$$= \lim_{t\to\infty}\left\{-\frac{\tan^{-1} t}{t} + \frac{\tan^{-1} 1}{1} + \int_1^t \frac{1}{x} - \frac{x}{x^2+1}\,dx\right\}$$
$$= 0 + \frac{\pi}{4} + \lim_{t\to\infty}\left[\ln|x| - \frac{1}{2}\ln|x^2+1|\right]_1^t$$
$$= \frac{\pi}{4} + \lim_{t\to\infty}\left[\ln|t| - \frac{1}{2}\ln|t^2+1| - \ln|1| + \frac{1}{2}\ln|2|\right]$$
$$= \frac{\pi}{4} + \frac{1}{2}\ln 2 + \frac{1}{2}\lim_{t\to\infty}\ln\frac{t^2}{t^2+1}$$
$$= \frac{\pi}{4} + \frac{1}{2}\ln 2$$

(Answer : 2 points)

(c)

$$I = \int_{-1}^1 \frac{dx}{x^2-2x} = \int_{-1}^1 \frac{dx}{x(x-2)} = \int_{-1}^0 \frac{dx}{x(x-2)} + \int_0^1 \frac{dx}{x(x-2)} = I_1 + I_2$$

(Improper integral : 2 points)

Now

$$\frac{dx}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} \Rightarrow 1 + A(x-2) + Bx$$

Set $x = 2$ to get $1 = 2B$, so $B = \frac{1}{2}$.
Set $x = 0$ to get $1 = -2A$, so $A = -\frac{1}{2}$.

.

(Partial fraction : 2 points)

Thus,

$$I_1 = \lim_{t\to 0-}\int_{-1}^t \left(\frac{-\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x-2}\right)dx$$

and

$$\begin{aligned} I_2 &= \lim_{t\to 0+}\int_t^1 \left(\frac{-\frac{1}{2}}{x}+\frac{\frac{1}{2}}{x-2}\right)dx \\ &= \lim_{t\to 0+}[-\frac{1}{2}\ln|x|+\frac{1}{2}\ln|x-2|]_t^1 \\ &= \lim_{t\to 0+}[(0+0)-(-\frac{1}{2}\ln t+\frac{1}{2}\ln|t-2|)] \\ &= -\frac{1}{2}\ln 2+\lim_{t\to 0+}\frac{1}{2}\ln t \\ &= -\infty \end{aligned}$$

(Computing the limit : 2 points)

Since I_2 diverges, I is divergent.
.

(Answer : 1 point)

(d) Let

$$I=\int\frac{x^8}{(1+x^6)^2}\,dx=\int x^3\cdot\frac{x^5}{(1+x^6)^2}\,dx$$

(Correct $f(x)$: 1 point)

Let $u=x^3$ and $dv=\frac{x^5}{(1+x^6)^2}\,dx$, then $du=3x^2dx$ and $v=-\frac{1}{6(1+x^6)}$, so

$$\begin{aligned} I &= -\frac{x^3}{6(1+x^6)}+\frac{1}{2}\int\frac{x^2}{1+x^6}\,dx \\ &= -\frac{x^3}{6(1+x^6)}+\frac{1}{6}\tan^{-1}(x^3)+C,\text{ where } C \text{ is a constant} \end{aligned}$$

(Integration by parts : 4 points)

Hence

$$\begin{aligned} \int_{-1}^{\infty}\left(\frac{x^4}{1+x^6}\right)^2dx &= \lim_{t\to\infty}\left[-\frac{x^3}{6(1+x^6)}+\frac{1}{6}\tan^{-1}(x^3)\right]_{-1}^{t} \\ &= \lim_{t\to\infty}\left(-\frac{t^3}{6(1+t^6)}+\frac{1}{6}\tan^{-1}(t^3)+\frac{(-1)^3}{6(1+(-1)^6)}-\frac{1}{6}\tan^{-1}((-1)^3)\right) \\ &= 0+\frac{1}{6}\cdot\frac{\pi}{2}-\frac{1}{12}-\frac{1}{6}\left(-\frac{\pi}{4}\right) \\ &= \frac{\pi}{8}-\frac{1}{12} \end{aligned}$$

(Answer : 2 points)

4. We can extend our definition of average value of a continuous function to an infinite interval by defining the average value of f on the interval $[a,\infty)$ to be

$$\lim_{t\to\infty}\frac{1}{t-a}\int_a^t f(x)dx$$

(a) (6 pts) Find the average value of $f(x) = \tan^{-1} x$ on the interval $[0, \infty)$.

(b) (5 pts) If $f(x) \geq 0$ and $\int_a^\infty f(x)dx$ is divergent, show that the average value of f on the interval $[a, \infty)$ is $\lim\limits_{x\to\infty} f(x)$, if this limit exists.

(c) (5 pts) If $\int_a^\infty f(x)dx$ is convergent, what is the average value of f on the interval $[a, \infty)$?

(d) (3 pts) Find the average value of $f(x) = \sin x$ on the interval $[0, \infty)$.

(a)

$$f_{ave} = \lim_{t\to\infty} \frac{1}{t-0} \int_0^t \tan^{-1} x \, dx$$

(Definition of average function : 1 point)

Let $u = \tan^{-1} x$ and $dv = dx$, then $du = \frac{1}{1+x^2}\, dx$ and $v = x$, so

. (Substitution rule : 2 points)

$$\begin{aligned}
f_{ave} &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \tan^{-1} x \, dx \\
&= \lim_{t\to\infty} \left\{ \frac{1}{t} \left[x \tan^{-1} x \right]_0^t - \frac{1}{t} \int_0^t \frac{x}{1+x^2}\, dx \right\} \\
&= \lim_{t\to\infty} \left[\tan^{-1} t - \frac{\ln(1+t^2)}{2t} \right] \\
&= \frac{\pi}{2} - \lim_{t\to\infty} \frac{\ln(1+t^2)}{2t} \\
&\overset{(\frac{\infty}{\infty})}{\underset{l'H}{=}} \frac{\pi}{2} - \lim_{t\to\infty} \frac{2t/(1+t^2)}{2} \\
&= \frac{\pi}{2} - 0 \\
&= \frac{\pi}{2}
\end{aligned}$$

(L'Hospital's rule + Answer : 1 + 2 points)

(b) $f(x) \geq 0$ and $\int_a^\infty f(x)\, dx$ is divergent, then $\lim\limits_{t\to\infty} \int_a^t f(x)dx = \infty$.

. (Computing the limit : 2 points)

$$f_{ave} = \lim_{t\to\infty} \frac{\int_a^t f(x)\, dx}{t-a} \overset{(\frac{\infty}{\infty})}{\underset{l'H}{=}} \lim_{t\to\infty} \frac{f(t)}{1} = \lim_{t\to\infty} f(t), \text{ if this limit exists.}$$

(L'Hospital's rule + Answer : 1 + 2 points)

(c) Suppose $\int_a^\infty f(x)\, dx$ converges; that is

$$\lim_{t\to\infty} \int_a^t f(x)\, dx = L \neq \infty$$

(Improper intergal : 1 point)

Then

$$f_{ave} = \lim_{t\to\infty}\left[\frac{1}{t-a}\int_a^t f(x)\,dx\right] = \lim_{t\to\infty}\frac{1}{t-a}\cdot\lim_{t\to\infty}\int_a^t f(x)\,dx = 0\cdot L = 0$$

(Limit laws + Answer : 2 + 2 points)

(d)

$$f_{ave} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sin x\,dx = \lim_{t\to\infty}\left(\frac{1}{t}[-\cos x]_0^t\right) = \lim_{t\to\infty}\left(-\frac{\cos t}{t}+\frac{1}{t}\right) = \lim_{t\to\infty}\frac{1-\cos t}{t} = 0$$

(Improper integral + Answer : 1 + 2 points)

5. Find the exact length of the curve.
   (a) (5 pts) $y = 1 + 6x^{3/2}$, $0 \le x \le 1$.
   (b) (5 pts) $y = \ln(\sec x)$, $0 \le x \le \pi/4$.
   (c) (7 pts) $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$ on the whole domain.
   (d) (7 pts) $y = \int_1^x \sqrt{t^3 - 1}\,dt$, $1 \le x \le 4$.

(a)

$$y = 1 + 6x^{3/2} \Rightarrow dy/dx = 9x^{1/2} \Rightarrow 1 + (\frac{dy}{dx})^2 = 1 + 81x$$

(Computing the integrand : 2 points)

So

$$L = \int_0^1 \sqrt{1+81x}\,dx = \left[\frac{2}{3}\cdot\frac{1}{81}\cdot(1+81x)^{3/2}\right]_0^1 = \frac{2}{243}\cdot(82\sqrt{82}-1)$$

(Computing the length L + Answer : 2 + 1 points)

(b)

$$y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{\sec x\cdot\tan x}{\sec x} = \tan x \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$$

(Computing the integrand : 2 points)

So

$$\begin{aligned} L &= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx \\ &= \int_0^{\pi/4} |\sec x|\, dx \\ &= \int_0^{\pi/4} \sec x\, dx \\ &= [\ln(\sec x + \tan x)]_0^{\pi/4} \\ &= \ln(\sqrt{2}+1) - \ln(1+0) \\ &= \ln(\sqrt{2}+1) \end{aligned}$$

(Computing the length L + Answer : 2 + 1 points)

(c) The curve has endpoints $(0,0)$ and $(1, \frac{\pi}{2})$.

.

(Endpoints : 2 points)

$$y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x}) \Rightarrow \frac{dy}{dx} = \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{2-2x}{2\sqrt{x}\sqrt{1-x}} = \sqrt{\frac{1-x}{x}}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1-x}{x} = \frac{1}{x}$$

(Computing the integrand : 2 points)

Thus

$$L = \int_0^1 \sqrt{1/x}\, dx = \lim_{t \to 0+} \int_t^1 \sqrt{1/x}\, dx = \lim_{t \to 0+} \left[2\sqrt{x}\right]_t^1 = \lim_{t \to 0+} \left[2\sqrt{1} - 2\sqrt{t}\right] = 2 - 0 = 2$$

(Computing the length L + Answer : 2 + 1 points)

(d)

$$y = \int_1^x \sqrt{t^3 - 1}\, dt \Rightarrow dy/dx = \sqrt{x^3 - 1}$$

$$\Rightarrow 1 + (dy/dx)^2 = 1 + \left(\sqrt{x^3-1}\right)^2 = x^3$$

(The Fundamental Theorem of Calculus + Computing the integrand : 2 + 2 points)

Hence,

$$L = \int_1^4 \sqrt{x^3}\, dx = \frac{2}{5}\left[x^{5/2}\right]_1^4 = \frac{2}{5}(32 - 1) = \frac{62}{5}$$

(Computing the length L + Answer : 2 + 1 points)

6. Find the exact area of the surface of revolution.

   (a) (5 pts) The curve $y = x^3$, $0 \le x \le 2$, rotated about the x-axis.

   (b) (9 pts) The curve $y = e^{-x}$, $x \ge 0$, rotated about the x-axis.

(c) (5 pts) The curve $x = \sqrt{a^2 - y^2}$, $0 \le y \le a/2$, rotated about the y-axis.

(d) (9 pts) The curve $x^2 + y^2 = r^2$, rotated about the line $y = r$.

(a)

$$y = x^3 \Rightarrow y' = 3x^2$$

So

$$\begin{aligned} S &= \int_0^2 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^2 x^3\sqrt{1+9x^4}\,dx \\ &= 2\pi\left[\frac{1}{36}\cdot\frac{2}{3}(1+9x^4)^{3/2}\right]_0^2 = \frac{\pi}{27}\cdot(145\sqrt{145}-1) \end{aligned}$$

(Computing the surface S + Answer : 3 + 2 points)

(b)

$$S = \int_0^\infty 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^\infty e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$$

(The function of the surface S : 2 points)

Evaluate $I = \int e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$ by using the substitution $u = -e^{-x}$, $du = e^{-x}dx$.
. (Substitution rule : 1 point)

Then let $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, so

$$\begin{aligned} I &= \int \sqrt{1+u^2}\,du \\ &= \int \sqrt{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta \\ &= \int \sec^3\theta\,d\theta \\ &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C \\ &= \frac{1}{2}\tan\theta\sqrt{1+\tan^2\theta} + \frac{1}{2}\ln\left|\tan\theta+\sqrt{1+\tan^2\theta}\right| + C \\ &= \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln|u+\sqrt{1+u^2}| + C \\ &= \frac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}} + \frac{1}{2}\ln|-e^{-x}+\sqrt{1+e^{-2x}}| + C, \text{ where } C \text{ is a constant} \end{aligned}$$

(Computing I : 2 points)

Returning to the surface area integral, we have

$$
\begin{aligned}
S &= 2\pi \lim_{t\to\infty} \int_0^t e^{-x}\sqrt{1+(-e^{-x})^2}\,dx \\
&= 2\pi \lim_{t\to\infty} \left[\frac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}} + \frac{1}{2}\ln\left|(-e^{-x}) + \sqrt{1+e^{-2x}}\right|\right]_0^t \\
&= 2\pi \lim_{t\to\infty} \left\{\left[\frac{1}{2}(-e^{-t})\sqrt{1+e^{-2t}} + \frac{1}{2}\ln\left|-e^{-t} + \sqrt{1+e^{-2t}}\right|\right] - \left[\frac{1}{2}(-1)\sqrt{1+1} + \frac{1}{2}\ln(-1+\sqrt{1+1})\right]\right\} \\
&= 2\pi \left\{\left[\frac{1}{2}(0)\sqrt{1} + \frac{1}{2}\ln(0+\sqrt{1})\right] - \left[-\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(-1+\sqrt{2})\right]\right\} \\
&= \pi\left[\sqrt{2} - \ln(\sqrt{2}-1)\right]
\end{aligned}
$$

(Computing the surface S + Answer : 3 + 1 points)

(c)

$$
\begin{aligned}
x = \sqrt{a^2-y^2} &\Rightarrow \frac{dx}{dy} = \frac{1}{2}(a^2-y^2)^{-1/2}\cdot(-2y) = \frac{-y}{\sqrt{a^2-y^2}} \\
&\Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{a^2-y^2} = \frac{a^2}{a^2-y^2}
\end{aligned}
$$

(Computing the integrand : 2 points)

Hence,

$$
S = \int_0^{a/2} 2\pi\cdot\sqrt{a^2-y^2}\cdot\frac{a}{\sqrt{a^2-y^2}}\,dy = 2\pi\int_0^{a/2} a\,dy = \pi a^2
$$

(**Note:** this is $\frac{1}{4}$ the surface area of a sphere of radius a, and the length of the interval $y=0$ to $y=a/2$ is $\frac{1}{4}$ the length of the interval $y=-a$ to $y=a$.)

. (Computing the surface S + Answer : 2 + 1 points)

(d) For the upper semicircle, $f(x)=\sqrt{r^2-x^2}$, $f'(x) = -\frac{x}{\sqrt{r^2-x^2}}$.
The surface area generated is

$$
\begin{aligned}
S_1 &= \int_{-r}^{r} 2\pi\cdot\left(r-\sqrt{r^2-x^2}\right)\cdot\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx \\
&= 4\pi\int_0^r \left(r-\sqrt{r^2-x^2}\right)\cdot\frac{r}{\sqrt{r^2-x^2}}\,dx \\
&= 4\pi\int_0^r \left(\frac{r^2}{\sqrt{r^2-x^2}} - r\right)\,dx \\
&= 4\pi\cdot\lim_{t\to r}\int_0^t \left(\frac{r^2}{\sqrt{r^2-x^2}}\right)\,dx - 4\pi\int_0^r r\,dx
\end{aligned}
$$

(Upper semicircle : 3 points)

For the lower semicircle, $f(x) = -\sqrt{r^2-x^2}$ and $f'(x) = \frac{x}{\sqrt{r^2-x^2}}$, so

$$
S_2 = 4\pi\cdot\lim_{t\to r}\int_0^t \left(\frac{r^2}{\sqrt{r^2-x^2}}\right)\,dx + 4\pi\int_0^r r\,dx
$$

Cont.

(Lower semicircle : 3 points)

Thus, the total area is

$$S = S_1 + S_2 = 8\pi \cdot \lim_{t \to r} \int_0^t \left(\frac{r^2}{\sqrt{r^2 - x^2}} \right) dx = 8\pi \cdot \lim_{t \to r} \left[r^2 \sin^{-1}\left(\frac{x}{r}\right) \right]_0^t = 4\pi^2 r^2$$

(Computing the surface S + Answer : 2 + 1 points)

Note: No matter whether the curve is rotated about the line $y = r > 0$ or the line $y = r < 0$, the surface generated will be of the same size. If $r = 0$, then $x^2 + y^2 = 0$ is just a point (the origin) and thus no surface is generated.

The End.