Real Analysi
Homework 5

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Yueh-Chou Lee

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EXERCISE 10.12

Give an example of a pair of measures ν and μ such that ν is absolutely continuous with respect to μ, but given $\varepsilon > 0$, there is no $\delta > 0$ such that $\nu(A) < \varepsilon$ for every A with $\nu(A) < \delta$. (Thus, the analogue for measures of Theorem 10.34 may fail.)
Prove the analogue of Theorem 10.35 for mutually singular measures ν and μ.

Proof.
Let μ be Lebesgue measure and

$$\mu(A) = \int_A \frac{1}{t} dt$$

for any measurable set A. So for any $\varepsilon, \delta > 0$ and let $A = [0, \delta]$, then

$$\mu(A) = \infty > \varepsilon$$

✓

But ν is absolutely continuous with respect to μ.

Next, we will prove the analogue of Theorem 10.35 for mutually singular measures ν and μ.

If ν is singular on E with respect to μ, there exists $A \subset E$ with $\mu(A) = 0$ such that $\nu(E - A) = 0$. Taking $E_0 = A$, then we will obtain the neccessity of the condition. To prove its sufficiency, choose for each $k = 1, 2, \cdots$ a measurable $E_k \subset E$ with $\mu(E_k) < 2^{-k}$ and $\nu(E - E_k) < 2^{-k}$.

Let $A = \limsup E_k$. Since $A = \cap_{k=m}^{\infty} E_k$ for every m, it follows as usual that $\mu(A) = 0$. Moreover,

$$\begin{aligned} \nu(E - A) = \nu(E - \limsup E_k) &= \nu(\liminf(E - E_k)) \\ &\leq \liminf(E - E_k) = 0 \end{aligned}$$

✓

Hence, ν is singular with respect to μ, which completes the proof.

EXERCISE 10.22

Let μ be a measure and A be a set with $0 < \mu(A) < \infty$. Let f be measurable and bounded on A, and let ϕ be convex in an interval containing the range of f. Prove that

$$\phi\left(\frac{\int_A f d\mu}{\int_A d\mu}\right) \leq \frac{\int_A \phi(f) d\mu}{\int_A d\mu}$$

(This is Jensen's inequality for measures. See Theorem 7.44.)

Proof.
By hypothesis, f is finite a.e. withe respect to μ in A. Choose (a,b), $-\infty \le a < b \le -\infty$, so that ϕ is convex in (a,b), and so that $a < f(\mu) < b$ for every μ at which $f(\mu)$ is finite. The number γ defined by

$$\gamma = \frac{\int_A f d\mu}{\int_A d\mu}$$

is finite and satisfies $a < \gamma < b$. If m is the slope of a supprting line at γ and $a < t < b$, then $\phi(\gamma) + m(t-\gamma) \le \phi(t)$. Hence, for almost every μ,

$$\phi(\gamma) + m[f(\mu) - \gamma] \le \phi(f(\mu)).$$

Multiplying both sides of this inequality by μ and integrating the result with respect to μ, we obatian

$$\phi(\gamma)\int_A d\mu + m\left(\int_A f d\mu - \gamma \int_A d\mu\right) \le \int_A \phi(f) d\mu$$

Here the existence of $\int_A \phi(f)d\mu$ follows from the integrability of μ and $f\mu$. [The continuity of ϕ implies that $\phi(f)$ is measurable.] Since $\int_A f d\mu - \gamma \int_A d\mu = 0$, the last inequality reduces to

$$\phi(\gamma)\int_A d\mu \le \int_A \phi(f) d\mu$$

which is the desired result.

EXERCISE 10.23

A sequence $\{\phi_k\}$ of set functions is said to be uniformly absolutely continuous with respect to a measure μ if given $\varepsilon > 0$, there exists $\delta > 0$ such that if E satisfies $\mu(E) < \delta$, then $|\phi_k(E)| < \varepsilon$ for all k. If $\{f_k\}$ is a sequence of integrable functions on a finite measure space $(\mathscr{S}, \Sigma, \mu)$ that converges pointwise a.e.(μ) to an integrable f, show that $f_k \to f$ in $L(d\mu)$ norm if and only if the indefinite integrals of the f_k are uniformly absolutely continuous with respect to μ.
(Cf. Exercise 17 of Chapter 7.)

Proof.
$(\Rightarrow)$
Let $\phi_k(A) = \int_A f_k d\mu$. Suppose $f_k \to f$ in $L(d\mu)$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $k \ge N$, then

$$\int |f_k - f| \to 0 \quad \text{as} \quad k \to \infty$$

So $\exists \delta > 0$ such that if $A \in \Sigma$ with $\mu(A) < \delta$, then

$$|\phi_k(A)| = |\int_A f_k| \le \int_A |f_k| < \varepsilon, \quad \forall k = 1, \cdots, N-1$$

$$|\phi_k(A)| \le \int_A |f_k| \le \int_A |f_k - f| + \int_A |f| < \varepsilon, \quad k \ge N$$

$(\Leftarrow)$
Given $\varepsilon > 0$. For all k, since the indefinite integral of f_k is absolutely continuous, there exists $\delta_k > 0$ such that for any $A \subseteq \Sigma$ with $|A| < \delta_k$, we have $|\int_E f_k| < \varepsilon$.

Since the indefinite integral of f is absolutely continuous, choose $N \in \mathbb{N}$ and $\delta > 0$ such that for any $\mu(A) < \delta$ and $k \ge N+1$, we have

$$|\int_A f_k| \le \int_A |f_k| \le \int_A |f_k - f| + \int_A |f| < \varepsilon$$

Let $\delta' = \min\{\delta, \delta_1, \delta_2, \cdots, \delta_N\}$, then $|\int_A f_k| < \varepsilon$ for all k.

EXERCISE 10.24

Let $(\mathscr{S}, \Sigma, \mu)$ be a σ-finite measure space, and let f be Σ-measurable and integrable over $\mathscr{S}$. Let Σ_0 be a σ-algebra satisfying $\Sigma_0 \subset \Sigma$. Of course, f may not be Σ_0-measurable. Show that there is a unique function f_0 that is Σ_0-measurable such that $\int f g d\mu = \int f_0 g d\mu$ for every Σ_0-measurable g for which the integrals are finite. The function f_0 is called the *conditional expectation* of f with respect to Σ_0, denoted $f_0 = E(f|\Sigma_0)$.

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(Apply the Radon–Nikodym theorem to the set function $\phi(E) = \int_E f d\mu$, $E \in \Sigma_0$.)

Proof.

Let ϕ be an additive set function on the measurable subsets of a measurable $E \in \Sigma$ and f be σ-measurable and integrable over $\mathscr{S}$, then μ is a σ-finite measurable on E, by Radon–Nikodym theorem, we will have that there exists a unique $f \in L(E; d\mu)$ such that

$$\phi(A) = \int_A f d\mu$$

for every measurable $A \subset E$.

For every Σ_0-measurable g where $\Sigma_0 \subset \Sigma$, so

$$\int_E f g d\mu \leq (\sup_E g) \int_E f d\mu < \infty$$

since $\sup_E g$ and $\int_E f d\mu$ are finte.

To prove the uniqueness, let f_0 and f_1 are Σ_0-measurable such that

$$\int f g d\mu = \int f_0 g d\mu \quad \text{and} \quad \int f g d\mu = \int f_1 g d\mu$$

then

$$\int f_0 g d\mu - \int f_1 g d\mu = \int (f_0 - f_1) g d\mu = \int f g d\mu - \int f g d\mu = 0$$

so $f_0 - f_1 = 0$, hence f_0 is unique.

EXERCISE 10.25

Using the notation of the preceding exercise, prove the following:

(a) $E(af + bg|\Sigma_0) = aE(f|\Sigma_0) + bE(g|\Sigma_0)$, a, b constants.

(b) $E(f|\Sigma_0) \geq 0$ if $f \geq 0$.

(c) $E(fg|\Sigma_0) = gE(f|\Sigma_0)$ if g is Σ_0-measurable.

(d) If $\Sigma_1 \subset \Sigma_0 \subset \Sigma$, then $E(f|\Sigma_1) = E(E(f|\Sigma_0)|\Sigma_1)$.

Proof.

(a) For every $A \in \Sigma_0$ and h is Σ_0-measurable, we have

$$\begin{aligned}\int E(af + bg|\Sigma_0)hd\mu &= \int_A (af + bg)hd\mu = a\int_A fhd\mu + b\int_A ghd\mu \\ &= a\int E(f|\Sigma_0)hd\mu + b\int E(g|\Sigma_0)hd\mu \\ &= \int aE(f|\Sigma_0)hd\mu + \int bE(g|\Sigma_0)hd\mu \\ &= \int [aE(f|\Sigma_0) + bE(g|\Sigma_0)]hd\mu\end{aligned}$$

Hence $E(af + bg|\Sigma_0) = aE(f|\Sigma_0) + bE(g|\Sigma_0)$.

(b) By Exercise 10.24, we know that f is Σ-measurable and $\Sigma_0 \subset \Sigma$, so if $f \geq 0$ in Σ, then $f \geq 0$ in Σ_0, hence $E(f|\Sigma_0) \geq 0$.

(c) For every $A \in \Sigma_0$ and h is Σ_0-measurable, we have

$$\int E(fg|\Sigma_0)hd\mu = \int_A fghd\mu = \int_A f \cdot (gh)d\mu = \int E(f|\Sigma_0)ghd\mu = \int [gE(f|\Sigma_0)]hd\mu$$

Hence $E(fg|\Sigma_0) = gE(f|\Sigma_0)$.

(d) For every $A \in \Sigma_1 \subset \Sigma_0 \subset \Sigma$ and g is Σ_0-measurable, we have

$$\int E(E(f|\Sigma_0)|\Sigma_1)gd\mu = \int_A E(f|\Sigma_0)gd\mu = \int_A fgd\mu = \int E(f|\Sigma_0)gd\mu$$

Hence $E(f|\Sigma_1) = E(E(f|\Sigma_0)|\Sigma_1)$.