Calculus 2 12/19 Note
Module Class 07

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Section 9.3: Separable Equations

Separable Equation

A **separable equation** is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y).$$

If $f(y) \neq 0$, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$. To solve this equation we rewrite it in the differential form

$$h(y)\,dy = g(x)\,dx,$$

then we integrate both sides of the equation:

$$\int h(y)\,dy = \int g(x)\,dx.$$

Example:
Solve the following initial value problems (IVP).

1. $\dfrac{dy}{dx} = 6y^2x, \quad y(1) = \dfrac{1}{25}.$

   Sol.

$$\begin{aligned} \frac{dy}{dx} = 6y^2x \ &\Rightarrow\ y^{-2}\,dy = 6x\,dx \\ &\Rightarrow\ \int y^{-2}\,dy = \int 6x\,dx \\ &\Rightarrow\ -\frac{1}{y} = 3x^2 + c \qquad \text{where } c \text{ is the consatnt.} \end{aligned}$$

   So apply the initial condition and find the value of c, then

$$-\frac{1}{1/25} = 3(1)^2 + c \ \Rightarrow\ -25 = 3 + c \ \Rightarrow\ c = -28.$$

   Plug this into the general solution and then solve to get an explicit solution.

$$-\frac{1}{y} = 3x^2 + c = 3x^2 - 28 \ \Rightarrow\ y = \frac{1}{28 - 3x^2}.$$

2. $\dfrac{dy}{dx} = e^{y-x} \sec y \, (1 + x^2), \quad y(0) = 0.$

Sol.

$$\begin{aligned}
\frac{dy}{dx} = e^{y-x} \sec y \, (1+x^2) \;&\Rightarrow\; e^{-y} \cos y \, dy = e^{-x}(1+x^2) \, dx \\
&\Rightarrow\; \int e^{-y} \cos y \, dy = \int e^{-x}(1+x^2) \, dx \\
&\Rightarrow\; \frac{e^{-y}}{2}(\sin y - \cos y) = -e^{-x}(x^2+2x+3) + c \qquad \text{where } c \text{ is the consatnt.}
\end{aligned}$$

Applying the initial condition gives

$$\frac{1}{2}(-1) = -(3) + c \;\Rightarrow\; c = \frac{5}{2}.$$

Therefore, the implicit solution is

$$\frac{e^{-y}}{2}(\sin y - \cos y) = -e^{-x}(x^2+2x+3) + \frac{5}{2}.$$

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally, that is, at right angles (see Figure 1).

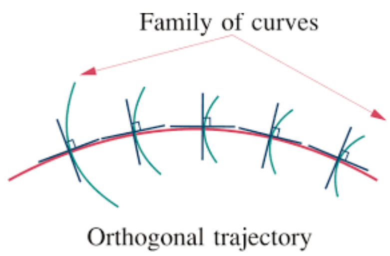
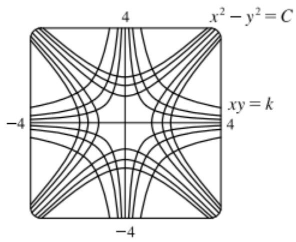


Figure 1: Orthogonal trajectory

Example:
Find the orthogonal trajectories of the family of the curves.

1. $y = \dfrac{k}{x}$.

Sol.



The curves $y = k/x$ form a family of hyperbolas with asymptotes $x = 0$ and $y = 0$. Differentiating gives

$$y = \frac{k}{x} \;\Rightarrow\; \frac{dy}{dx} = \frac{k/x}{dx} \;\Rightarrow\; y' = -\frac{k}{x^2} = -\frac{xy}{x^2} = -\frac{y}{x},$$

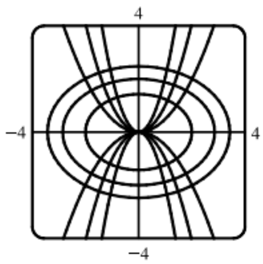
since $y = k/x \;\Rightarrow\; xy = k$.

Thus, the slope of the tangent line at any point (x, y) on one of the hyperbolas is $y' = -y/x$, so the orthogonal trajectories must satisfy

$$\begin{aligned} y' = \frac{dy}{dx} = \frac{x}{y} &\Leftrightarrow y\,dy = x\,dx \\ &\Leftrightarrow \int y\,dy = \int x\,dx \\ &\Leftrightarrow \frac{y^2}{2} = \frac{x^2}{2} + C_1 \\ &\Leftrightarrow x^2 - y^2 = C \qquad \text{where } C_1 \text{ is any constant and } C = -2C_1. \end{aligned}$$

2. $x^2 + 2y^2 = k^2$.

Sol.



The curves $x^2 + 2y^2 = k^2$ form a family of ellipses with major axis on the x-axis. Differentiating gives

$$\begin{aligned} \frac{d}{dx}(x^2 + 2y^2) = \frac{d}{dx}(k^2) &\Rightarrow 2x + 4yy' = 0 \\ &\Rightarrow 4yy' = -2x \\ &\Rightarrow y' = \frac{-x}{2y}. \end{aligned}$$

Thus, the slope of the tangent line at any point (x, y) on one of the ellipses is $y' = \dfrac{-x}{2y}$, so the orthogonal trajectories must satisfy

$$\begin{aligned} y' = \frac{dy}{dx} = \frac{2y}{x} &\Leftrightarrow \frac{dy}{y} = 2\,\frac{dx}{x} \\ &\Leftrightarrow \int \frac{dy}{y} = 2\int \frac{dx}{x} \\ &\Leftrightarrow \ln|y| = 2\ln|x| + C_1 = \ln x^2 + C_1 \\ &\Leftrightarrow |y| = e^{\ln x^2 + C_1} \\ &\Leftrightarrow y = \pm x^2 \cdot e^{C_1} = Cx^2 \qquad \text{where } C_1 \text{ is any constant and } C = \pm e^{C_1}. \end{aligned}$$

Exercise:

1. Solve the following initial value problems (IVP).

   (a) $x + 3y^2\sqrt{x^2+1}\,\dfrac{dy}{dx} = 0, \quad y(0) = 1.$

   (b) $\dfrac{dy}{dx} = ky^2 \ln x, \quad y(1) = -1.$

2. Find the orthogonal trajectories of the family of the curves.

   (a) $y = \dfrac{x}{1+kx}.$

   (b) $y = \dfrac{1}{x+k}.$

Sol.

1.(a) $y = (2 - \sqrt{x^2+1})^{1/3}.$ (b) $y = \dfrac{1}{kx - kx\ln x - k - 1}.$

2.(a) $y = \sqrt[3]{C - x^3}$, where C is any constant. (b) $y = (3x + C)^{1/3}$, where C is any constant.