Calculus 1   10/10 Note
Module Class 07

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Section 3.5: Implicit Differentiation

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Example:

1. Differentiate e^{x^2-9x} and $e^{y(x)}$.

   Sol.

   $$\frac{d}{dx}\left(e^{x^2-9x}\right) = e^{x^2-9x}\frac{d}{dx}(x^2-9x) = (2x-9)e^{x^2-9x} \qquad \text{and} \qquad \frac{d}{dx}\left(e^{y(x)}\right) = y'(x)e^{y(x)}$$

2. Given $x^2\tan(y) + y^{10}\sec x = 2x$. Find y'.

   Sol.

   We've got two product rules to deal with this time. Here is the derivative of this function.

   $$2x\tan y + x^2\sec(y)\, y' + 10y^9\, y' \sec x + y^10 \sec x \tan x = 2$$

   So

   $$y' = \frac{2 - y^{10}\sec x\tan x - 2x\tan y}{x^2\sec^2 y + 10y^9\sec x}$$

3. Assume that $x = x(t)$ and $y = y(t)$ and differentiate the following equation with respect to t.

   $$x^3y^6 + e^{1-x} - \cos(5y) = y^2$$

   Sol.

Note that the first term will be a product rule since both x and y are functions of t.

$$3x^2\, x'\, y^6 + 6x^3\, y^5\, y' - x'\, e^{1-x} + 5y'\, \sin(5y) = 2y\, y'$$

Excerise:

1. Find dy/dx by implicit differentiation.
   (a) $x^4(x+y) = y^2(3x-y)$
   (b) $e^y \sin x = x + xy$
2. Find the derivative of the function. Simplify where possible.
   (a) $g(x) = arccos\sqrt{x}$
   (b) $f(x) = \cos^{-1}(\sin^{-1} x)$

Sol.

1.(a) $y' = \frac{3y^2-5x^4-4x^3y}{x^4+3y^2-6xy}$ (b) $y' = \frac{1+y-e^y\cos x}{e^y \sin x - x}$

2.(a) $g'(x) = -\dfrac{1}{2\sqrt{x}\sqrt{1-x}}$ (b) $f'(x) = -\dfrac{1}{\sqrt{1-(\sin^{-1}x)^2}} \cdot \dfrac{1}{\sqrt{1-x^2}}$

Section 3.6: Derivatives of Logarithmic

Derivatives of Logarithmic

$$\frac{d}{dx}(\log_b x) = \frac{1}{x\ln b} \qquad \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

The Power Rule

If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

The Number e as a Limit

$$e = \lim_{x\to 0}(1+x)^{1/x} \qquad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Example:
Differentiate the following functions.

1. $f(x) = 3e^x + 10x^3 \ln x$

   Sol.
   $$f'(x) = 3e^x + 30x^2 \ln x + 10x^3\left(\frac{1}{x}\right) = 3e^x + 30x^2 \ln x + 10x^2$$

2. $g(x) = \dfrac{5e^x}{3e^x + 1}$

   Sol.
   $$g'(x) = \frac{5e^x(3e^x+1) - (5e^x)(3e^x)}{(3e^x+1)^2} = \frac{5e^x}{(3e^x+1)^2}$$

Excerise:

1. Differentiate the following functions.
   (a) $p(x) = \dfrac{\ln x}{1-x}$
   (b) $h(x) = \ln\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ where a is the constant.
2. Use logarithmic differentiation to find the derivative of the following functions.
   (a) $y = \sqrt{x}e^{x^2-x}(x+1)^{2/3}$
   (b) $y = (\sin x)^{\ln x}$
3. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$

Sol.
1.(a) $p'(x) = \dfrac{1-x+x\ln x}{x(1-x)^2}$ (b) $h'(x) = \frac{2a^2x}{x^4-a^4}$
2.(a) $y' = \sqrt{x}e^{x^2-x}(x+1)^{2/3}\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x+3}\right)$ (b) $y' = (\sin x)^{\ln x}\left(\ln x \cot x + \dfrac{\ln(\sin x)}{x}\right)$
3. $\dfrac{d^9}{dx^9}(x^8 \ln x) = \dfrac{8!}{x}$

Section 3.8: Exponential Growth and Decay

Theorem

The only solutions of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{ky}$$

Example:

1. If \$1000 is borrowed at 8% interest, find the amounts due at the end of 3 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) weekly, (v) daily, (vi) hoursly, and (vii) continuously.

2. Suppose \$1000 is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years, where $0 \le t \le 3$, graph $A(t)$ for each of the interest rates 6%, 8%, and 10% on a common screen.

Sol.
1. Using $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$ with $A_0 = 1000$, $r = 0.08$, and $t = 3$, we have:
(i) Annually: $n = 1$; \$1259.71
(ii) Quarterly: $n = 4$; \$1268.24
(iii) Monthly: $n = 12$; \$1270.24
(iv) Weekly: $n = 52$; \$1271.01
(v) Daily: $n = 365$; \$1271.22
(vi) Hoursly: $n = 365 \cdot 24$; \$1271.25
(vii) Continuously: $n = 1000e^{(0.08)3}$; \$1271.25

2. $A_{0.10}(3) = \$1349.86$, $A_{0.08}(3) = \$1271.25$, $A_{0.06}(3) = \$1197.22$