Real Analysis
Homework 2

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1. (Exercise 3.15) If E is measurable and A is any subset of E, show that $|E| = |A|_i + |E - A|_e$.

Proof.
Let F be a closed set and $F \subseteq A$, then $E - A \subseteq E - F$. So we have

$$|F| + |E - A|_e \leq |F| + |E - F|_e = |E|$$

Taking the supremum of the both sides, then

$$\sup |F| + |E - A|_e = |A|_i + |E - A|_e \leq |E|$$

However

$$|A|_i + |E - F|_e \geq |F| + |E - F|_e = |E|$$

Since $E - A \subseteq E - F \Rightarrow \inf(E - A) \subseteq E - F$, taking the infimum of the both sides, then

$$|A|_i + \inf |E - F|_e \geq |E| \Rightarrow |A|_i + |E - A|_e \geq |E|$$

By above two inequalities, we will have

$$|E| \leq |A|_i + |E - A|_e \leq |E|$$

Hence

$$|E| = |A|_i + |E - A|_e$$

2. (Exercise 3.17) Give an example which shows that the image of a measurable set under a continuous transformation may not be measurable. (Consider the Cantor-Lebesgue function and the pre-image of an appropriate nonmeasurable subset of its range.) See also Exercise 10 of Chapter 7.

Proof.
Let f be Cantor-Lebesgue function and C be the Cantor set, then $f(C) = [0, 1]$, therefore for all $x \in C$, we have

$$x = \sum_{k=1}^{\infty} c_k 3^{-k}, \ c_k = 0 \text{ or } 2 \text{ for all } k$$

Then

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2} c_k 2^{-k}$$

Hence, $f(C) \subseteq [0, 1]$.
However, for every $y \in [0, 1]$, let

$$y = \sum_{k=1}^{\infty} a_k 2^{-k}$$

where $a_k = 0$ or 1.
Let

$$x = \sum_{k=1}^{\infty} 2a_k 3^{-k} \in C$$

Then

$$f(x) = f(\sum_{k=1}^{\infty} 2a_k 3^{-k}) = \sum_{k=1}^{\infty} a_k 2^{-k} = y$$

This implies $f(C) = [0,1]$. Since $|[0,1]| = 1 > 0$, there exists $B \subseteq [0,1]$ such that B is non-measurable set.
Let

$$A = \{x \in C | f(x) \in B\}$$

However, C is measure zero and $A \subseteq C$, therefore A is also measure zero.
Hence, f is continuous and $f(A = B$ where A is measurable set and B is non-measurable set.

3. (Exercise 3.18) Prove that outer measure is *translation invariant*; that is, if $E_h = \{x+h : x \in E\}$ is the translate of E by h, $h \in \mathbb{R}^n$, show that $|E_h|_e = |E|_e$. If E is meaurable, show that E_h is also measurable. (This fact was used in proving Lemma 3.37.)

   Proof.
   Let $\{I_k\}_{k=1}^{\infty}$ be a family of intervals such that $E \subseteq \cup_{k=1}^{\infty} I_k$.
   Since $E_h = x + h : x \in E$, then $\{I_k + h\}_{k=1}^{\infty}$ is also a family such that $E_h \subseteq \cup_{k=1}^{\infty}(I_k + h)$.
   Hence,

$$|E_h|_e \leq \sum_{k=1}^{\infty} v(I_k + h) = \sum_{k=1}^{\infty} v(I_k)$$

   Taking the infremum of the both sides in $|E_h|_e$‘ $\sum_{k=1}^{\infty} v(I_k)$, we will have

$$\inf |E_h|_e = |E_h|_e \leq \inf \sum_{k=1}^{\infty} v(I_k) = |E|_e$$

   However,

$$|E|_e \leq \sum_{k=1}^{\infty} v(I_k) = \sum_{k=1}^{\infty} v(I_k + h)$$

   Similarily, taking the infremum of the both sides in $|E|_e \leq \sum_{k=1}^{\infty} v(I_k + h)$ Hence,

$$\inf |E|_e = |E|_e \leq \inf \sum_{k=1}^{\infty} v(I_k + h) = |E_h|_e$$

   From $|E_h|_e \leq |E|_e$ and $|E_h|_e \geq |E|_e$, we can conclude that

$$|E_h|_e = |E|_e$$

4. (Exercise 3.20) Show that there exist disjoint $E_1, E_2, ..., E_k, ...$ such that $|\cup E_k|_e < \sum |E_k|_e$ with strict inequality. (Let E be a nonmeasurable subset of $[0,1]$ whose rational translates are disjoint. Consider the translates of E by all rational numbers r, $0 < r < 1$, and use Exercise 18.)

   Proof.
   Follow the hint, let E be a nonmeasurable subset of $[0,1]$ whose rational translates are disjoint.

Consider the translates of E by the sequence of rational numbers $\{r_k\}$ where $0 < r_k < 1$ for all k.
Define the $E_{r_k} = x + r_k : x \in E$.
Then we have $\cup_{r_k} E_{r_k} \subseteq [0, 2]$, that is $|\cup_{r_k} E_{r_k}| \le 2$.
By Exercise 3.18, we know that $|E_{r_k}|_e = |E|_e$ for all k.
Hence,

$$|\cup E|_e \le |\cup_{r_k} E_{r_k}|_e \le 2 < \sum_{r_k} |E_{r_k}|_e = \sum |E|_e$$

5. (Exercise 3.21) Show that there exist sets $E_1, E_2, ..., E_k, ...$ such that $E_k \searrow E, |E_k|_e < +\infty$, and $\lim_{k\to\infty} |E_k|_e > |E|_e$ with strict inequality.

Proof.
Let a non-measurable set $A \subset [0, 1]$ and satisfy $|A|_e > 0$.
Let $\{r_i\}_{i=1}^{\infty} = \mathbb{Q} \cap [0, 1]$, $A_{r_i} = \{a + r_i | a \in A\}$ and

$$E_k = \cup_{i=k}^{\infty} A_{r_i}$$

Since $A_{r_i} \cap A_{r_j} = \phi$ for $i \neq j$, then $E_k \searrow \phi$, and $E_k \subset [0, 2]$ implies

$$|E_k|_e \le 2 < +\infty$$

Hence

$$|E_k|_e = |\cup_{i=k}^{\infty} A_{r_i}|_e \ge |A_{r_k}|_e = |A|_e > 0$$

for all k. But $|\phi|_e = 0$, therefore $\lim_{k\to\infty} |E_k|_e$ must be larger than $|E|_e$.

6. (Exercise 3.24) Let $0.\alpha_1\alpha_2...$ be the dyadic development of any x in $[0, 1]$, that is, $x = \alpha_1 2^{-1} + \alpha_2 2^{-2} + ...$ with $\alpha_i = 0, 1$. Let $k_1, k_2, ...$ be a fixed permutation of the positive integers $1, 2, ...,$ and consider the transformation T that sends $x = 0.\alpha_1\alpha_2...$ to $Tx = 0.\alpha_{k_1}\alpha_{k_2}....$ If E is a measurable subset of $[0, 1]$, show that its image TE is also measurable and that $|TE| = |E|$. (Consider first the special cases of E a dyadic interval $[s2^{-k}, (s + 1)2^{-k}]$, $s = 0, 1, ..., 2^k - 1$, and then of E an open set [which is a countable union of nonoverlapping dyadic intervals]. Also show that if E has small measure, then so has TE.)

Proof.
Let $I_{k,s}$ be the s-th dynamic interval where for any element $x = 0.\alpha_1\alpha_2... \in I_{k,s}$, only the first k dynamic intergers $\alpha_1, \alpha_2, ..., \alpha_k$ are the same (which means $\alpha_{k+1}, \alpha_{k+2}, ...$ are different).
So

$$I_{k,s} = [s2^{-k}, (s + 1)2^{-k}],\ s = 0, 1, ..., 2^{k-1}$$

Since $I_{k,s}$ is closed interval, $I_{k,s}$ is measurable.
Suppose the tranformation T_i only permutes the first i dynamic intergers $\alpha_1, \alpha_2, ..., \alpha_i$ in any x where x is in any dynamic interval $I_{k,s}$, therefore, $T_i \nearrow T$.
Since for any interger k, we will have $\cup_s I_{k,s} = [0, 1]$.
Hence, for any open set $E \subset [0, 1]$, pick k and $k > i$, then we can find the finite j dynamic intervals such that

$$E = \cup_{s=\{s_1,s_2,...,s_j\}} I_{k,s}$$

For any dynamic interval $I_{k,s}$ where $\cup_{s=\{s_1,s_2,...,s_j\}} I_{k,s}$, since the first k dynamic intergers in every $x \in I_{k,s}$ are the same, and for any tranformation T_i only permutes the first i dynamic intergers where $k > i$.
Hence, the first k dynamic intergers of all elemnets $x' \in T_i(I_{k,s})$ are still the same, this means $T_i(I_{k,s})$ just be moved to another dynamic interval $I_{k,s'}$ where $s' = 0, 1, ..., 2^{k-1}$.

$T_i(I_{k,s}) = I_{k,s'}$, so $I_{k,s'}$ is also a closed interval and also measurable.
However $T_i(E) = T_i(\cup_{s=\{s_1,s_2,\ldots,s_j\}} I_{k,s}) = \cup_{s'=\{s'_1,s'_2,\ldots,s'_j\}} I_{k,s'}$ and all $I_{k,s'}$ are measurable, then $T_i(E)$ is measurable.
Since $T_i \nearrow T$, then $T_i(E) \nearrow T(E)$ is also measurable.

Let $I'_{k,s}$ be the interval which is removed two boundary points $s2^{-k}$ and $(s+1)2^{-k}$ in $I_{k,s}$, therefore all intervals $I'_{k,s}$ are open and measurable.
Also, fixed k and for all s, $I'_{k,s}$ are disjoint.
Since those two boundary points measure zero, then we have

$$|I_{k,s}| - |I'_{k,s}| \leq |I_{k,s} - I'_{k,s}| < \epsilon$$

Since

$$T(E) = T(\cup_s I_{k,s}) = \cup_{s'} I_{k,s'}$$

we will have

$$|T(E)| - |E| = |\cup_{s'} I_{k,s'}| - |\cup_s I_{k,s}| < \sum_{s'} |I_{k,s'}| - |\cup_s I'_{k,s}|$$

Since all $I'_{k,s}$ are disjoint, then

$$|T(E)| - |E| < \sum_{s'} |I_{k,s'}| - |\cup_s I'_{k,s}| < \sum_{s'} |I_{k,s'}| - \sum_s |I'_{k,s}|$$

Fixed k, if $\sum_s |I'_{k,s}|$ is the sum of the n disjoint open intervals $I'_{k,s}$ and $\sum_{s'} |I'_{k,s'}|$ is the sum of the another n disjoint open intervals $I'_{k,s'}$, since $|I'_{k,s}| = |I'_{k,s'}| = 2^{-k}$, therefore, $\sum_s |I'_{k,s}| = \sum_{s'} |I'_{k,s'}|$.
Hence,

$$|T(E)| - |E| < \sum_{s'} |I_{k,s'}| - \sum_s |I'_{k,s}| = \sum_{s'} |I_{k,s'}| - \sum_{s'} |I'_{k,s'}| < n \cdot \epsilon$$

Since ϵ is arbitary chosen small, $|T(E)| = |E|$.