Introduction to Computational Mathematics
Quiz 3

December 12, 2018

1. (3 pts) Find a sequence $\{x_k\}$ such that $\lim_{k\to\infty}(x_k - x_{k-1}) = 0$ but $\{x_k\}$ diverges.

2. (3 pts) Consider the equations

$$\begin{aligned} &u^2 \log u + v \log v = -0.2 \\ &u^4 + v^2 u = 1. \end{aligned}$$

   Write the intersection of these curves in the form $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ for two-dimensional $\mathbf{f}$ and $\mathbf{x}$, and compute the Jacobian matrix of $\mathbf{f}$.

3. (4 pts) The iteration formula of the secant method is

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

   Let $\epsilon_k = r - x_k$ be the errors in successive root approximations. By using Taylor's expansion, show that

$$\epsilon_{k+1} \approx -\frac{1}{2}\frac{f''(r)}{f'(r)}\epsilon_k \epsilon_{k-1}$$

Solution:

1. Take $x_k = \sum_{n=1}^{k} \frac{1}{n}$.

2.
$$\begin{aligned}
f_1(u,v) &= u^2 \log u + v \log v + 0.2 \\
f_2(u,v) &= u^4 + v^2 u - 1 \\
\mathbf{J}(\mathbf{x}) &= \begin{bmatrix} 2u \log u + u & \log v + 1 \\ 4u^3 + v^2 & 2vu \end{bmatrix}
\end{aligned}$$

3. Note that $x_k = r - \epsilon_k$ and $x_{k-1} = r - \epsilon_{k-1}$. By Taylor's expansion,

$$\begin{aligned}
f(r - \epsilon_k) &= f(r) - f'(r)\epsilon_k + \frac{1}{2} f''(r)\epsilon_k^2 + \cdots \\
f(r - \epsilon_{k-1}) &= f(r) - f'(r)\epsilon_{k-1} + \frac{1}{2} f''(r)\epsilon_{k-1}^2 + \cdots ,
\end{aligned}$$

and,

$$f(x_k) - f(x_{k-1}) = -f'(r)(\epsilon_k - \epsilon_{k-1}) + \frac{1}{2} f''(r)(\epsilon_k^2 - \epsilon_{k-1}^2) + \cdots$$

Since r is a root of f, $f(r) = 0$, and

$$\begin{aligned}
\epsilon_{k+1} &= r - x_{k+1} \\
&= r - \left\{ x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} \right\} \\
&= \epsilon_k - \frac{f(x_k)(\epsilon_k - \epsilon_{k-1})}{f(x_k) - f(x_{k-1})} \\
&= \epsilon_k - \frac{\left(-f'(r)\epsilon_k + \frac{1}{2} f''(r)\epsilon_k^2 + \cdots\right)(\epsilon_k - \epsilon_{k-1})}{(\epsilon_k - \epsilon_{k-1})\left(-f'(r) + \frac{1}{2} f''(r)(\epsilon_k + \epsilon_{k-1}) + \cdots\right)} \\
&= \frac{\left[f'(r)\epsilon_k - \frac{1}{2} f''(r)\epsilon_k^2 - \frac{1}{2} f''(r)\epsilon_k\epsilon_{k-1} + \cdots\right] - \left[f'(r)\epsilon_k + \frac{1}{2} f''(r)\epsilon_k^2 + \cdots\right]}{f'(r) - \frac{1}{2} f''(r)(\epsilon_k + \epsilon_{k-1}) + \cdots} \\
&\approx -\frac{1}{2} \frac{f''(r)}{f'(r)} \epsilon_k \epsilon_{k-1}
\end{aligned}$$