NTU 107-1 MATH1201 Calculus A-05
Exercise set 1 Solution

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> Write your solutions to the following problems on a separate sheet of paper and submit it to your TA or instructor.
> Submit your solutions to Problems *(4), (5), (6) and (7)* on **October 12**.
> Submit your solutions to Problems *(8), (9) and (10)* on **October 17**.
> The rest are left for your self-revision.

1. Given $g(2) = 4, f(2) = 2$ and $g'(x) = \sqrt{x^2+5}, f'(x) = \sqrt{x^3+1}$ for all $x > 0$, find the derivative of $g(f(x))$ at $x = 2$.

   Solution. We know

   $$f'(2) = 3 \quad \text{and} \quad g'(f(2)) = g'(2) = 3\ .$$

   Therefore, the derivative of $g(f(x))$ at $x = 2$ can be written as

   $$\left.\frac{d}{dx}g(f(x))\right|_{x=2} = g'(f(x))f'(x)\ |_{x=2} = g'(f(2))f'(2) = 3 \cdot 3 = 9\ .$$

2. Let $f(x) = e^x \cdot \ln(2+\sin x)$. Find $f'(x)$ and $f'(0)$.

   Solution. By the chain rule, it follows that

   $$\begin{aligned} f'(x) &= (e^x)' \ln(2+\sin x) + e^x(\ln(2+\sin x))' \\ &= e^x \ln(2+\sin x) + e^x \frac{(2+\sin x)'}{2+\sin x} \\ &= e^x \ln(2+\sin x) + e^x \frac{(2)' + (\sin x)'}{2+\sin x} \\ &= e^x \ln(2+\sin x) + e^x \frac{\cos x}{2+\sin x}\ . \end{aligned}$$

   Then, substituting $x = 0$ yields

   $$f'(0) = e^0 \ln(2+\sin 0) + e^0 \frac{\cos 0}{2+\sin 0} = 1 \cdot \ln 2 + 1 \cdot \frac{1}{2} = \ln 2 + \frac{1}{2}\ .$$

3. If $y^4 + xy^2 - 2 = 0$, find y'.

   Solution.

   $$\begin{aligned} & \frac{d}{dx}(y^4 + xy^2 - 2) = \frac{d}{dx}0 \\ \Rightarrow\quad & \frac{d}{dx}(y^4) + \frac{d}{dx}(xy^2) = 0 \\ \Rightarrow\quad & 4y^3\frac{dy}{dx} + (y^2 + x \cdot 2y\frac{dy}{dx}) = 0 \\ \Rightarrow\quad & (4y^3 + 2xy)\frac{dy}{dx} + y^2 = 0 \\ \Rightarrow\quad & \frac{dy}{dx} = \frac{-y^2}{4y^3+2xy}\ . \end{aligned}$$

4. Find the following limits or explain why they do not exist.

   (a) (5 points) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x\sin x}}{x}$

   (b) (5 points) $\displaystyle\lim_{x\to \frac{\pi}{4}} \frac{1-\tan x}{\sin x - \cos x}$

   (c) (5 points) $\displaystyle\lim_{x\to 0} \csc x \sin(\sin x)$

   (d) (5 points) $\displaystyle\lim_{x\to 0} \frac{e^{\sin x}-1}{x}$

   Solution.

   (a) Notice that

$$\frac{\sqrt{x\sin x}}{x} = \frac{\sqrt{x^2}}{x}\sqrt{\frac{\sin x}{x}} = \frac{|x|}{x}\sqrt{\frac{\sin x}{x}} \quad \text{and} \quad \lim_{x\to 0}\sqrt{\frac{\sin x}{x}} = \sqrt{1} = 1\,.$$

   Since $\lim_{x\to 0}\frac{|x|}{x}$ does not exist, the required limit cannot exist (otherwise, the limit of $\frac{|x|}{x} = \frac{\sqrt{x\sin x}}{x}\sqrt{\frac{x}{\sin x}}$ as x tends to 0 would have existed by the Limit Law for products).

   (b) By the Limit Law for quotients and the continuity of cos function, it follows that

$$\begin{aligned}\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\sin x-\cos x} &= \lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\frac{\sin x}{\cos x}-1}\cdot\frac{1}{\cos x} = \lim_{x\to\frac{\pi}{4}} \cancelto{-1}{\frac{1-\tan x}{\tan x-1}}\cdot\frac{1}{\cos x} \\ &= -\frac{1}{\cos\frac{\pi}{4}} = -\sqrt{2}\,.\end{aligned}$$

   (c) Using $\lim_{x\to 0}\sin x = 0$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$, it follows that

$$\lim_{x\to 0}\csc x\sin(\sin x) = \lim_{x\to 0}\frac{\sin(\sin x)}{\sin x} = \lim_{y\to 0}\frac{\sin y}{y} = 1\,.$$

   (d) Using $\lim_{h\to 0}\frac{e^h-1}{h} = 1$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$, it follows that

$$\begin{aligned}\lim_{x\to 0}\frac{e^{\sin x}-1}{x} &= \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\cdot\frac{\sin x}{x} = \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\cdot\lim_{x\to 0}\frac{\sin x}{x} \\ &= \lim_{h\to 0}\frac{e^h-1}{h}\cdot 1 = 1\,.\end{aligned}$$

5. (12 points) Find the n-th derivative of the function $f(x) = \dfrac{x^n}{1-x}$.

   Solution. Rewrite $f(x)$ as

$$\begin{aligned}f(x) = \frac{x^n}{1-x} &= \frac{x^n-1}{1-x} + \frac{1}{1-x} \\ &= -(x^{n-1}+x^{n-2}+\cdots+1) + \frac{1}{1-x}\,.\end{aligned}$$

   Note that the terms in parentheses become zero after differentiating n times.

(reducing $f(x)$ into a more computable form: 4 points)

(n-th derivatives of x^r: 2 points)

Therefore, it remains to see that, by induction,

$$
\begin{aligned}
(\frac{1}{1-x})' &= (-1)\cdot\frac{1}{(1-x)^2}\cdot(-1) = \frac{1}{(1-x)^2} \\
(\frac{1}{1-x})'' &= \frac{2}{(1-x)^3} \\
&\vdots \\
(\frac{1}{1-x})^{(k)} &= \frac{k!}{(1-x)^{k+1}} .
\end{aligned}
$$

As a result, we have $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$. (inductive argument: 4 points)

(answer: 2 points)

6. (18 points) Let $f(x) = x^r|x|$, where $r > 0$ is a positive number such that x^r is a well-defined function on $\mathbb{R}$ (e.g. r is a rational number $\frac{p}{q}$ with q being odd). Determine whether f is differentiable at 0 and find $f'(0)$ if it does.

Solution.

We want to evaluate

$$
f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^r\cdot|h|-0}{h} .
$$

(definition of limits: 3 points)

To evaluate the last limit, irrespective of whether $r < 1$ or not, note that

$$
\begin{aligned}
\lim_{h\to 0^+}\frac{h^r\cdot|h|}{h} &= \lim_{h\to 0^+}\frac{h^r\cdot h}{h} = \lim_{h\to 0^+} h^r = 0 , \\
\lim_{h\to 0^-}\frac{h^r\cdot|h|}{h} &= \lim_{h\to 0^-}\frac{h^r\cdot(-h)}{h} = -\lim_{h\to 0^-} h^r = 0 .
\end{aligned}
$$

(both sided limits: 5+5 points)

Therefore, we see that

$$
\lim_{h\to 0}\frac{h^r\cdot|h|}{h} = 0 \qquad \text{(3 points)}
$$

i.e. f is differentiable at 0 and $f'(0) = 0$. (conclusion: 2 points)

7. (10 points) The figure shows a circular arc of length s and a chord of length d, both subtended by a central angle θ. Find $\lim_{\theta\to 0^+}\dfrac{s}{d}$.

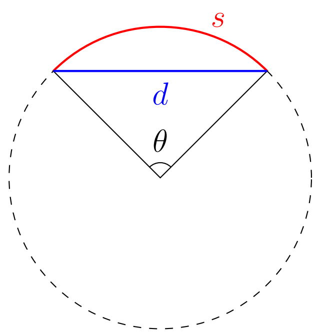

Figure of Problem 7

Solution. Let r be the radius of the circle. Then, we have

$$s = r\theta \quad \text{and} \quad d = 2r\sin\frac{\theta}{2} \ .$$

(formulae for s and d in terms of radius and θ: 3+3 points)

Therefore,

$$\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{\cancel{r}\theta}{2\cancel{r}\sin\frac{\theta}{2}} = \lim_{h\to 0^+}\frac{h}{\sin h} = \frac{1}{\displaystyle\lim_{h\to 0^+}\frac{\sin h}{h}} = 1 \ . \qquad \text{(4 points)}$$

8. Find the derivatives of the following functions. (106-1 Midterm 2)

(a) (5 points) $f(x) = \dfrac{\sin x}{1+\cos x}$

(b) (5 points) $f(x) = \log_2\sqrt{x} + \tan^{-1}(x^3)$

(c) (5 points) $f(x) = x^{\cos x}$

(d) (5 points) $y = \dfrac{(2x+1)^5(x^2+1)^3}{(3x-2)^6(x^3+1)^4}$, find $y'(0)$.

Solution.

(a) [**Method 1**] By the quotient rule,

$$f'(x) = \frac{\cos x(\cos x+1) - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x} \ .$$

[**Method 2**] By the product rule,

$$\begin{aligned} f'(x) &= \frac{\cos x}{1+\cos x} + (\sin x)\cdot\left(-\frac{(1+\cos x)'}{(1+\cos x)^2}\right) = \frac{\cos x}{1+\cos x} + \frac{\sin^2 x}{(1+\cos x)^2} \\ &= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x} \ . \end{aligned}$$

(b) Rewrite $f(x)$ as

$$f(x) = \frac{\ln x}{2\ln 2} + \tan^{-1}(x^3) \ .$$

Then, since $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$, the chain rule yields

$$f'(x) = \frac{1}{(2\ln 2)x} + 3x^2\cdot\frac{1}{1+x^6} \ .$$

(c) [**Method 1**] Write $f(x)$ as $f(x) = e^{\cos x\ln x}$. Then, differentiating $f(x)$ with chain rule yields

$$f'(x) = e^{\cos x\ln x}(\cos x\ln x)' = e^{\cos x\ln x}\left(-\sin x\ln x + \frac{\cos x}{x}\right) \ .$$

[**Method 2**] Taking logarithm of f yields

$$\ln f(x) = \cos x \ln x \ .$$

Differentiating both sides gives

$$\frac{f'(x)}{f(x)} = (-\sin x \ln x + \frac{\cos x}{x}) \ .$$

Thus,

$$f'(x) = x^{\cos x}(-\sin x \ln x + \frac{\cos x}{x}) \ .$$

(d) Use logarithmic differentiation. As

$$\ln y = 5\ln(2x+1) + 3\ln(x^2+1) - 6\ln(3x-2) - 4\ln(x^3+1) \ ,$$

we obtain

$$\begin{aligned} \frac{y'}{y} &= \frac{10}{2x+1} + \frac{6x}{x^2+1} - \frac{18}{3x-2} - \frac{12x^2}{x^3+1} \\ \Rightarrow \quad y' &= y(\frac{10}{2x+1} + \frac{6x}{x^2+1} - \frac{18}{3x-2} - \frac{12x^2}{x^3+1}) \end{aligned}$$

Therefore,

$$y'(0) = y(0)(10+0+9-0) = \frac{1^5 \cdot 1^3}{(-2)^6 \cdot 1^4} \cdot (19) = \frac{19}{64} \ .$$

9. Suppose that $f(x)$ is a twice differentiable function such that

$$\lim_{x\to 1} \frac{(f(x))^3 - 8}{x-1} = 18 \quad \text{and} \quad \lim_{t\to 0} \frac{f'(1+t) - f'(1-3t)}{t} = 1 \ .$$

(a) (15 points) Find $f(1)$, $f'(1)$ and $f''(1)$.

(b) (15 points) Suppose that $g(x) = f(e^{2x})$ is an one-to-one function and $h(x) = g^{-1}(x)$, the inverse function of $g(x)$. Find $h(2)$, $h'(2)$ and $h''(2)$.

Solution.

(a) As the limit $\lim_{x\to 1} \frac{(f(x))^3-8}{x-1}$ exists while the denominator tends to 0 in the limit, we have $\lim_{x\to 1}(f(x))^3 - 8 = 0$. (2 points)
Furthermore, f being differentiable implies that it is continuous. (2 points)
We therefore obtain $(f(1))^3 - 8 = 0$, which yields $f(1) = 2$. (1 point)
Then,

$$\begin{aligned} 18 = \lim_{x\to 1} \frac{(f(1))^3 - 8}{x-1} &= \lim_{x\to 1} \frac{(f(x)-2)[(f(x))^2 + 2f(x) + 4]}{x-1} \\ &= \lim_{x\to 1}[\frac{f(x)-f(1)}{x-1} \cdot ((f(x))^2 + 2f(x) + 4)] \\ &= f'(1) \cdot ((f(1))^2 + 2f(1) + 4) \\ &= f'(1) \cdot 12 \\ \Rightarrow \qquad f'(1) &= \frac{3}{2} \ . \end{aligned}$$

(factoring out the difference quotient of f: 4 points)

(answer: 1 point)

Moreover,

$$\begin{aligned}
\lim_{t\to 0}\frac{f'(1+t)-f'(1-3t)}{t} &= \lim_{t\to 0}\frac{f'(1+t)-f'(1)+f'(1)-f'(1-3t)}{t} \\
&= \lim_{t\to 0}\left[\frac{f'(1+t)-f'(1)}{t}+3\cdot\frac{f'(1-3t)-f'(1)}{-3t}\right] \\
&= f''(1)+3f''(1)=4f''(1)=1 \\
\Rightarrow \qquad f''(1) &= \frac{1}{4}\,.
\end{aligned}$$

(rearranging terms to obtain difference quotients of f': 4 points)

(answer: 1 point)

(b) Notice that $g(0)=f(e^{2\cdot 0})=f(1)=2$, so $h(2)=g^{-1}(2)=0$. (2 points)

Differentiating successively, we obtain

$$\begin{aligned}
& g(h(x)) = x \\
\Rightarrow\quad & g'(h(x))\cdot h'(x) = 1 \qquad (*') \\
\Rightarrow\quad & g''(h(x))\cdot (h'(x))^2 + g'(h(x))\cdot h''(x) = 0\,. \qquad (*'')
\end{aligned}$$

(relations between h and g and their derivatives: 5 points)

Moreover, we have

$$\begin{aligned}
& g'(x) = f'(e^{2x})\cdot e^{2x}\cdot 2 \\
\Rightarrow\quad & g'(0) = f'(1)\cdot 2 = \frac{3}{2}\cdot 2 = 3\,; \\
& g''(x) = f''(e^{2x})\cdot (e^{2x})^2\cdot 2^2 + f'(e^{2x})\cdot e^{2x}\cdot 2^2 \\
\Rightarrow\quad & g''(0) = f''(1)\cdot 4 + f'(1)\cdot 4 = 7\,.
\end{aligned}$$

(computation of $g'(0)$ and $g''(0)$: 3+3 points)

Therefore, by putting $x=2$ in $(*')$ and substituting the value of $g'(h(2))=g'(0)$ by 3, we get that $h'(2)=\frac{1}{3}$. (1 point)

Putting $x=2$ in $(*'')$, we also get

$$h''(2)=\frac{-g''(0)\cdot(h'(2))^2}{g'(0)} = -\frac{-7\cdot\frac{1}{3^2}}{3} = -\frac{7}{27}\,. \qquad \text{(1 point)}$$

10. (20 points) A lamp located 4 units to the right of the y-axis and a shadow created by the elliptical region $x^2+5y^2\le 6$. If the point $(-6,0)$ is on the edge of the shadow, how far above the x-axis is the lamp located? (106-1 Midterm 4)

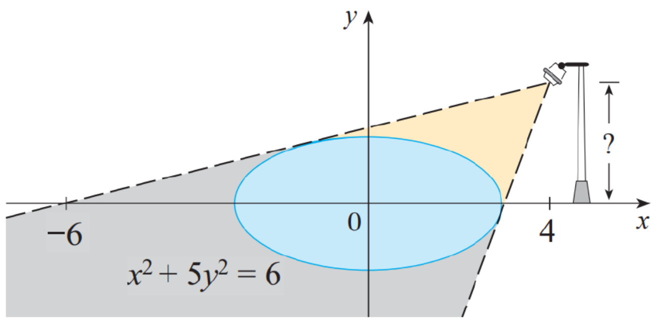

Figure of Problem 10

Solution.

[Method 1] By implicitly differentiating the equation $x^2 + 5y^2 = 6$ of the ellipse (the boundary of the given region) with respect to x, we have

$$2x + 10yy' = 0 \quad \Rightarrow \quad y' = -\frac{x}{5y} \; .$$

Suppose that the point of tangency between the light ray and the ellipse is (x_0, y_0). Then the path of the light ray can be described by $y = -\frac{x_0}{5y_0}(x - x_0) + y_0$.

(correct derivation of equation of tangent: 6 points)

Suppose that the light ray is also the one passing through $(-6, 0)$, plugging $(-6, 0)$ into the above equation of the path of the light ray gives us

$${x_0}^2 + 5{y_0}^2 = -6x_0 \; .$$

Moreover, since (x_0, y_0) is on the ellipse,

$${x_0}^2 + 5{y_0}^2 = 6 \; .$$

(equations for (x_0, y_0): 4+4 points)

Solving the above two equations simultaneously we get,

$$x_0 = -1, \quad y_0 = 1 \quad \text{(as } y_0 \text{ has to be positive)} \; . \qquad \text{(2 points)}$$

The path of the light ray is thus given by $y = \frac{1}{5}x + \frac{6}{5}$, so $y\,|_{x=4} = 2$, i.e. the lamp is 2 units above the $x-$axis. (conclusion: 4 points)

[Method 2] Suppose the lamp is located at $(4, h)$. Then the tangent line is given by $y = \frac{h}{10}(x+6)$.

(5 points)

Since it is tangent to the ellipse, the equation

$$\begin{cases} y = \dfrac{h}{10}(x+6) \\ x^2 + 5y^2 = 6 \end{cases}$$

should have only one zero (repeated roots), or equivalently, the discriminant of $x^2 + 5(\frac{h}{10}(x+6))^2 = 6$ should be zero. (geometric insight: 6 points)

Thus we have $h = 2$. (computation + answer: 7+2 points)

[Method 3] Note that a point on the upper half ellipse is give by $(x, \sqrt{\frac{6-x^2}{5}})$. Suppose that the lamp is at $(4, h)$. (coordinates of points on the upper half ellipse: 2 points)

If $(x, \sqrt{\frac{6-x^2}{5}})$, $(4, h)$ and $(-6, 0)$ are collinear points, then we have

$$\frac{h-0}{4+6} = \frac{\sqrt{\frac{6-x^2}{5}} - 0}{x+6}$$

$$\Rightarrow \qquad h = \frac{2\sqrt{30 - 5x^2}}{x+6}$$

(relation between h and x: 4 points)

If the line containing the three collinear points is a tangent line to the ellipse, then h attains its maximum as a function of x. (geometric insight: 6 points)

We can find h by solving the equation $\frac{dh}{dx} = 0$. Indeed, logarithmic differentiation yields

$$\frac{dh}{dx} = h(\frac{-1}{2}\frac{(-10x)}{30-5x^2} - \frac{1}{x+6}) = -h(\frac{x}{6-x^2} + \frac{1}{x+6}) \ .$$

(computing h': 3 points)

Note that $h = 0 \iff x = \pm\sqrt{6}$. Then, for $-\sqrt{6} < x < \sqrt{6}$,

$$\frac{dh}{dx} = 0 \iff \frac{x}{6-x^2} + \frac{1}{x+6} = 0 \iff -x(x+6) = 6 - x^2 \ ,$$

which has the unique solution $x = -1$. (solving equation: 3 points)

Therefore, $h|_{x=-1} = 2$, and the lamp is located at $(4, 2)$, i.e. it is 2 units above the x-axis. (conclusion: 2 points)