Real Analysis
Homework 6

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1. (Exercise 5.14)
Prove the following result (which is obvious if $|E| < +\infty$), describing the behavior of $a^p\omega(a)$ as $a \to 0+$. If $f \in L^p(E)$, then $\lim_{a\to 0+} a^p\omega(a) = 0$. (If $f \geq 0, \epsilon > 0$, choose $\delta > 0$ so that $\int_{\{f\leq\delta\}} f^p < \epsilon$. Thus, $a^p[\omega(a) - \omega(\delta)] \leq \int_{\{a<f\leq\delta\}} f^p < \epsilon$ for $0 < a < \delta$. Now let $a \to 0$.)

Proof.

Since $\cap_{k=1}^{\infty}\mathrm{R}(f^p, \{0 \leq f \leq \frac{1}{k}\}) = \mathrm{R}(f^p, \{f = 0\})$ and $|\mathrm{R}(f^p, \{0 \leq f \leq 1\})| < \infty$, then

$$\int_{\{0\leq f\leq\frac{1}{k}\}} f^p = |\mathrm{R}(f^p, \{0 \leq f \leq \frac{1}{k}\})| \to |\mathrm{R}(f^p, \{f = 0\})| = 0$$

There exists k_0 such that $\int_{\{0\leq f\leq\frac{1}{k_0}\}} f^p < \epsilon$ for any $\epsilon > 0$.
Thus, for any $a < 1/k_0$, we have

$$a^p[\omega(a) - \omega(\frac{1}{k_0})] \leq \int_{\{a<f\leq\frac{1}{k_0}f^p\}} < \epsilon$$

So

$$\lim_{a\to 0+} a^p\omega(a) = 0$$

2. (Exercise 5.15)
Suppose that f is nonnegative and measurable on E and that ω is finite on $(0, \infty)$. If $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ is finite, show that $\lim_{a\to 0+} a^p\omega(a) = \lim_{b\to+\infty} b^p\omega(b) = 0$. (Consider $\int_{a/2}^{a} and \int_{b/2}^{b}$.)
Recall:

$$\omega(\alpha) = |\{x \in E : f(x) > \alpha\}|$$

Proof.

For every a, the integral

$$\int_{a/2}^{a} \alpha^{p-1}\omega(\alpha)d\alpha \geq \omega(\alpha)\int_{a/2}^{a} \alpha^{p-1}d\alpha \geq \frac{1}{p}\left(\frac{\alpha}{2}\right)^p \omega(\alpha) \geq 0$$

Since $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ is finite, then we know that $\int_{a/2}^{a} \alpha^{p-1}\omega(\alpha)d\alpha \to 0$ as $a \to 0$ or $a \to +\infty$.
Hence

$$\lim_{a\to 0+} a^p\omega(a) = 0$$

and

$$\lim_{b\to+\infty} b^p\omega(b) = 0.$$

3. (Exercise 5.16)
Suppose that f is nonnegative and measurable on E and that ω is finite on $(0,\infty)$. Show that Theorem 5.51 holds without any further restrictions (i.e., f need not be in $L^p(E)$ and $|E|$ need not be finite) if we interpret $\int_0^\infty \alpha^p d\omega(\alpha) = \lim_{\substack{a\to+\infty \\ b\to 0+}} \int_b^a$. (For the first part, use the sets E_{ab} to obtain the relation $\int_E f^p = -\int_0^\infty \alpha^p d\omega(\alpha)$. If either $\int_0^\infty \alpha^p d\omega(\alpha)$ or $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ is finite, use Lemma 5.50 and the results of Exercises 14 or 15 to integrate by parts.)

Recall (Theorem 5.50):
If $0 < p < \infty$ and $f \in L^p(E)$, then

$$\lim_{\alpha\to+\infty} \alpha^p\omega(\alpha) = 0$$

Recall (Theorem 5.51):
If $0 < p < \infty$, $f \geq 0$, and $f \in L^p(E)$, then

$$\int_E f^p = -\int_0^\infty \alpha^p d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$$

where the last integral may be interpreted as either a Lebesgue or an improper Riemann integral.

Proof.

Let $E_{ab} = \{x \in E : a < f(x) \leq b\}$ for $0 < a < b < \infty$.
$|E_{ab}|$ is finite since ω is finite on $(0,\infty)$, then we will have

$$\int_{E_{ab}} f^p = -\int_a^b \alpha^p d\omega(\alpha)$$

Thus

$$\int_E f^p = \lim_{\substack{a\to 0+ \\ b\to+\infty}} \int_{E_{ab}} f^p = \lim_{\substack{a\to 0+ \\ b\to+\infty}} -\int_a^b \alpha^p d\omega(\alpha) = -\int_0^\infty \alpha^p d\omega(\alpha)$$

If $\int_0^\infty \alpha^p d\omega(\alpha)$ and $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ are infinte, then the integral $-\int_0^\infty \alpha^p d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$.

If $\int_0^\infty \alpha^p d\omega(\alpha)$ is finite, then $f \in L^p(E)$.

By Theorem 5.50, we know

$$\lim_{\alpha\to+\infty} \alpha^p\omega(\alpha) = 0$$

By Exercise 5.14, we also know

$$\lim_{\alpha\to 0+} \alpha^p\omega(\alpha) = 0$$

Using integrate by parts, we then have

$$\begin{aligned}
-\int_0^\infty \alpha^p d\omega(\alpha) &= \lim_{\alpha\to+\infty} -\alpha^p\omega(\alpha) + \lim_{\alpha\to 0+} \alpha^p\omega(\alpha) + p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha \\
&= p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha
\end{aligned}$$

If $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ is finite.
By Exercise 5.15, we know

$$\lim_{\alpha\to+\infty} \alpha^p\omega(\alpha) = 0$$

and

$$\lim_{\alpha\to 0+} \alpha^p\omega(\alpha) = 0$$

then

$$-\int_0^\infty \alpha^p d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$$

Hence

$$\int_E f^p = -\int_0^\infty \alpha^p d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$$

4. (Exercise 5.17)
If $f \geq 0$ and $\omega(\alpha) \leq c(1+\alpha)^{-p}$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.

Proof.

If $r = 1$, then

$$\begin{aligned}
\int_E f^r = r\int_0^\infty f^{(r-1)}\omega(\alpha)d\alpha &= \int_0^\infty \omega(\alpha)d\alpha \\
&\leq \int_0^\infty \frac{c}{(1+\alpha)^p}d\alpha \\
&= \lim_{\alpha\to+\infty} c(-p+1)(1+\alpha)^{-p+1} - \lim_{\alpha\to 0+} c(-p+1)(1+\alpha)^{-p+1}
\end{aligned}$$

Since $r = 1 < p$,

$$\lim_{\alpha\to+\infty} c(-p+1)(1+\alpha)^{-p+1} = 0$$

and

$$\lim_{\alpha\to 0+} c(-p+1)(1+\alpha)^{-p+1} = c$$

Then

$$\int_E f \leq \lim_{\alpha\to+\infty} c(-p+1)(1+\alpha)^{-p+1} - \lim_{\alpha\to 0+} c(-p+1)(1+\alpha)^{-p+1} = -c < \infty$$

Hence $f \in L$

If $r \neq 1$, then

$$\begin{aligned}
\int_E f^r &= r\int_0^\infty \alpha^{r-1}\omega(\alpha)d\alpha \\
&\leq r\int_0^\infty \alpha^{r-1}c\cdot(1+\alpha)^{-p}d\alpha \\
&= rc\int_0^\infty \frac{\alpha^{r-1}}{(1+\alpha)^p}d\alpha \\
&= rc\left(\int_0^1 \frac{\alpha^{r-1}}{(1+\alpha)^p}d\alpha + \int_1^\infty \frac{\alpha^{r-1}}{(1+\alpha)^p}d\alpha\right) \\
&\leq rc\left(\int_0^1 \alpha^{r-1}d\alpha + \int_1^\infty \alpha^{r-p-1}d\alpha\right) \\
&\leq rc\left(\frac{1}{r} + \frac{1}{r-p}\right) \\
&< \infty
\end{aligned}$$

Hence $f \in L^r$.

5. (Exercise 5.18)
If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k=-\infty}^{+\infty} 2^{kp}\omega(2^k) < +\infty$. (Use Exercise 16.)

Proof.

If $f \geq 0$.

($\Rightarrow$)
Suppose that $f \in L^p$.
By Exercise 5.16, then

$$\begin{aligned}\int_E f^p &= p\int_0^{\infty} \alpha^{p-1}\omega(\alpha)d\alpha \\ &= p\sum_{k=-\infty}^{+\infty} \int_{2^k}^{2^k+1} \alpha^{p-1}\omega(\alpha)d\alpha \\ &\geq p\sum_{k=-\infty}^{+\infty} 2^{k(p-1)}\omega(2^k)\cdot 2^k \\ &= p\sum_{k=-\infty}^{+\infty} 2^{kp}\omega(2^k)\end{aligned}$$

Then

$$\frac{1}{p}\int_E f^p \geq \sum_{k=-\infty}^{k=+\infty} 2^{kp}\omega(2^k)$$

$\int_E f^p < +\infty$ since $f \in L^p$, hence

$$+\infty > \frac{1}{p}\int_E f^p \geq \sum_{k=-\infty}^{k=+\infty} 2^{kp}\omega(2^k)$$

($\Leftarrow$)
By Exercise 5.16, then

$$\begin{aligned}\int_E f^p &= p\int_0^{\infty} \alpha^{p-1}\omega(\alpha)d\alpha \\ &= p\sum_{k=-\infty}^{+\infty} \int_{2^{k-1}}^{2^k} \alpha^{p-1}\omega(\alpha)d\alpha \\ &\leq p\sum_{k=-\infty}^{+\infty} 2^{k(p-1)}\omega(2^k)\cdot 2^k \\ &= p\sum_{k=-\infty}^{+\infty} 2^{kp}\omega(2^k) \\ &< +\infty\end{aligned}$$

Hence $f \in L^p$.

6. (Exercise 5.20)
Let $\mathbf{y} = T\mathbf{x}$ be a nonsingular linear transformation of $\mathbb{R}^n$. If $\int_E f(\mathbf{y})d\mathbf{y}$ exists, show that

$$\int_E f(\mathbf{y})d\mathbf{y} = |\det T| \int_{T^{-1}E} f(T\mathbf{x})d\mathbf{x}$$

(The case when $f = \chi_{E_1}$, $E_1 \subset E$, follows from integrating the formula $\chi_{E_1}(T\mathbf{x}) = \chi_{T^{-1}E_1}(\mathbf{x})$ over $T^{-1}E$ and then applying Theorem 3.35.)

Proof.

If f is nonnegative simple function, let

$$f = \sum_{i=1}^{n} a_i \chi_{E_i}$$

Then

$$\int_E f(y)dy = \sum_{i=1}^{n} a_i |E_i| = |\det T| \sum_{i=1}^{n} a_i |T^{-1}E_i| = |\det T| \int_{T^{-1}E} f(Tx)dx$$

If f is nonnegative function, there exist $\{f_k\}$ is a sequence of nonnegative simple function such that $f_k \nearrow f$, then

$$\begin{aligned}\int_E f(y)dy &= \lim_{k\to\infty} \int_E f_k(y)dy \\ &= |\det T| \lim_{k\to\infty} \int_{T^{-1}E} f_k(Tx)dx \\ &= |\det T| \int_{T^{-1}E} f(Tx)dx\end{aligned}$$

In general, $f = f^+ - f^-$, then

$$\begin{aligned}\int_E f(y)dy &= \int_E f^+(y)dy - \int_E f^-(y)dy \\ &= |\det T| (\int_{T^{-1}E} f^+(Tx)dx - \int_{T^{-1}E} f^-(Tx)dx) \\ &= |\det T| \int_{T^{-1}E} f(Tx)dx\end{aligned}$$

7. (Exercise 5.21)
If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

Proof.

For any $k \in \mathbb{Z}^+$, the

$$0 = \int_{\{f > \frac{1}{k}\}} f \geq \int_{\{f > \frac{1}{k}\}} \frac{1}{k} = \frac{1}{k} |\{f > \frac{1}{k}\}| \geq 0$$

and

$$0 = \int_{\{f < -\frac{1}{k}\}} f \leq \int_{\{f < -\frac{1}{k}\}} -\frac{1}{k} = -\frac{1}{k} |\{f < -\frac{1}{k}\}| \leq 0$$

Then $\{f > \frac{1}{k}\}$ and $\{f < -\frac{1}{k}\}$ are measure zero for all k.
This implies that

$$\{f > 0\} \cup \{f < 0\} = \bigcup_{k=1}^{\infty} \{f > \frac{1}{k}\} \cup \{f < -\frac{1}{k}\}$$

is measure zero.
Hence $f = 0$ a.e. in E.