Real Analysis
Homework 4

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1. (Exercise 10.9)
   The *symmetric difference* of two sets E_1 and E_2 is defined as

$$E_1 \triangle E_2 = (E_1 - E_2) \cup (E_2 - E_1)$$

Let $(\mathscr{S}, \Sigma, \mu)$ be a measure space, and identify measurable sets E_1 and E_2 if $\mu(E_1 \triangle E_2) = 0$. Show that Σ is a metric space with distance $d(E_1, E_2) = \mu(E_1 \triangle E_2)$ and if μ is finite, then $L^p(\mathscr{S}, \Sigma, \mu)$ is separable if and only if Σ is $1 \leq p < \infty$. (For the sufficiency in the second part, Exercise 10.8 may be helpful; for the necessity, let $\{f_k\}$ be a countable dense set in $L^p(\mathscr{S}, \Sigma, \mu)$ and consider the sets $\{1/2 < f_k \leq 3/2\}$.)

Proof.

(a) *Show that Σ is a metric space with distance $d(E_1, E_2) = \mu(E_1 \triangle E_2)$.*

   i. $\mu \geq 0$ for all measurable set.
   ii. $E_1 \triangle E_2 = E_2 \triangle E_1$.
   iii. $E_1 \triangle E_2 \subseteq (E_1 \triangle E_3) \cup (E_3 \triangle E_2)$.

   By above three, then Σ is a metric space with distance d.

(b) $(\Rightarrow)$
   If μ is finite and $L^p(\mathscr{S}, \Sigma, \mu)$ is separable, let A be a countable dense subset of L^p.
   By exercise 10.8, we know that for any $f \in A$, there exists the sequence of simple functions $\{f_k\}$ vanishing outside sets of finite measure such that $f_k \to f$ in L^p.

   Let B is class of the disjoint union $\underset{i}{\cup} E_i$ satisfies $f_k = \sum\limits_i a_i E_i$ is for any k and any $f \in A$. Then $B \subset \Sigma$ is countable.

   For any $E \in \Sigma$, there exists $f_n \in A$ such that $f_n \to \chi_E$ in L^p and exists the sample functions $f_{nm} \to f_n$ in L^p.
   Hence there exists the sequence of simple functions $\{f_j\}$ with $f_j = \sum\limits_l a_{jl} E_{jl}$ where $\cup_l E_{jl} \in B$ such that $f_j \to \chi_E$ in L^p.

   For any $\epsilon > 0$, there exists $M > 0$ such that for any $j \geq M$, we have $||f_j - \chi_E||_p < \epsilon$.
   Note that $a_{jl} \to 1$ as $j \to \infty$ for any l. Since μ is finite, then for any $j \geq M$, we have

$$\begin{aligned}
\mu^{1/p}(\cup_l E_{jl} \triangle E) &= ||\chi_{\cup_l E_{jl}} - \chi_E||_p \\
&\leq ||\chi_{\cup_l E_{jl}} - \sum a_{jl} E_{jl}||_p + ||f_j - \chi_E||_p \\
&\leq ||\chi_{\cup_l E_{jl}} - \sum a_{jl} E_{jl}||_p + \epsilon
\end{aligned}$$

Note that $a_{jl} \to 1$ as $j \to \infty$ for any l. Thus $\mu(\cup_l E_{jl} \triangle E) \to 0$ as $j \to \infty$.

$(\Leftarrow)$
Let $B = \{E_k\}$ be a countable dense subset of Σ.
Let A be the set of all linear combinations of characteristic functions of these E_k, the coefficients being complex numbers with rational real and imaginary parts.
Then A is a countable subset of $L^p(\mathscr{S}, \Sigma, \mu)$.

To see that A is dense, let f be any function in L^p, by exercise 10.8, we know that there exists the sequence of simple functions $\{f_k\}$ vanishing outside sets of finite measure such that $f_k \to f$ in L^p.

For any f_k, let $f_{kl} \to f_k$ with $f_{kl} \in A$ since B is dense, then $||f_{kl} - f_k||_p \to 0$ as $l \to \infty$. Hence there exists $\{f_j\} \subset A$ such that $f_j \to f$ in L^p. Thus A is dense in L^p.

2. (Exercise 10.10)
If ϕ is a set function whose Jordan decomposition is $\phi = \overline{V} - \underline{V}$, define

$$\int_E f d\phi = \int_E f d\overline{V} - \int_E f d\underline{V},$$

provided not both integrals on the right are infinite with the same sign. If V is the total variation of ϕ on E, and if $|f| \le M$, prove that $|\int_E f d\phi| \le MV$.

Proof.
The functions $\overline{V}$ and $\underline{V}$ are measure since $\overline{V}$, $\underline{V} \ge 0$ and countably additive. We have

$$\begin{aligned} |\int_E f d\phi| &= |\int_E f d(\overline{V} - \underline{V})| \\ &= |\int_E f d\overline{V} - \int_E f d\underline{V}| \\ &\le \int_E |f| d\overline{V} + \int_E |f| d\underline{V} \\ &\le M\overline{V}(E) + M\underline{V}(E) \\ &= MV(E) \end{aligned}$$

3. (Exercise 10.13)
Show that the set P of the Hahn decomposition is unique up to null sets. (By a null set for ϕ, we mean a set N such that ϕ, we mean a set N such that $\phi(A) = 0$ for every measurable $A \subset N$.)

Proof.
If $P_1 \cup E - P_1$ and $P_2 \cup E - P_2$ are Hahn decompositions of E, then $\phi(A) \ge 0$ if $A \subset P_1$ or $A \subset P_2$, $\phi(A) \le 0$ if $A \subset E - P_1$ or $A \subset E - P_2$.
We said that Hahn decomposition is unique if $\phi(A) = 0$ for any $A \subset P_1 \triangle P_2$, that is

$$\phi(A) = \phi(P_1 \cap A) + \phi(P_2 \cap A) = 0$$

Let $N_1 = E - P_1$ and $N_2 = E - P_2$ be two null sets, such that $\phi(N_1) = 0$ and $\phi(N_2) = 0$. Then for any $A \subset P_1 \triangle P_2$, we have

$$\phi(A) = \phi(P_1 \cap A) + \phi(P_2 \cap A) = \phi((E - P_2) \cap A) + \phi((E - P_1) \cap A) = 0 + 0 = 0,$$

since $\phi(N_1) = 0$ and $\phi(N_2) = 0$.

Assume that $P_1, E-P_1, P_2$ and $E-P_2$ are not in null set. For any $A \subset P_1 \triangle P_2$, then

$$\phi(A) = \phi(P_1 \cap A) + \phi(P_2 \cap A) = \phi((E-P_2) \cap A) + \phi(P_2 \cap A)$$

If $\phi(P_1 \cap A) = -\phi(P_2 \cap A)$, then $\phi(A) = 0$.

Since $\phi(A) \geq 0$ if $A \subset P_1$ or $A \subset P_2$, we have $\phi(P_1 \cap A) = \phi(P_2 \cap A) = 0$. This is a contradiction since P_1, P_2 are not in null set.

Hence the set P of the Hahn decomposition is unique up to null sets.

4. (Exercise 10.15)
(Converse of Hölder's inequality) Let μ be a σ-finite measure and $1 \leq p \leq \infty$.

   (a) Show that

   $$||f||_p = \sup |\int fgd\mu|,$$

   where the supremum is taken over all bounded measurable functions g that vanish outside a set (depending on g) of finite measure, and for which $||g||_{p'} \leq 1$ and $\int fgd\mu$ exists. (If $1 < p \leq \infty$ and $||f||_p < \infty$, this can be deduced from Theorem 10.44.)

   (b) Show that a real-valued measurable f belongs to L^p if $fg \in L^1$ for all $g \in L^{p'}$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Proof.

(a) For all $||g||_{p'} \leq 1$ then $\int fgd\mu$ exists, so for $1 \leq p \leq \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have

$$|\int fgd\mu| \leq \int |fg|d\mu \underset{\text{By Hölder's inequality}}{\leq} ||f||_p ||g||_{p'} \leq ||f||_p$$

so $\sup |\int fgd\mu| \leq ||f||_p$.

Since μ is σ-finite $\quad \Rightarrow \quad \exists E_j \nearrow \mathscr{S}, \quad \mu(E_j) < \infty$ for all j.
If $p = 1$, then $p' = \infty$, let

$$g_j = \begin{cases} 1 & \text{, if } x \in E_j,\ f \geq 0 \\ -1 & \text{, if } x \in E_j,\ f < 0 \\ 0 & \text{, if } x \notin E_j \end{cases} \quad \Rightarrow \quad 0 \leq fg_j \nearrow |f|$$

By Monotone Convergence Theorem, we have

$$|\int fg_j d\mu| = \int fg_j \nearrow \int |f| \quad \Rightarrow \quad \sup |\int fgd\mu| \geq ||f||_1$$

Thus

$$||f||_1 = \sup |\int fgd\mu|$$

If $1 < p \leq \infty$, then $1 \leq p' < \infty$.
By Theorem 10.44, we know that for $f \in L^p$, $\forall g \in L^{p'}$ and let $l(g) = \int gfd\mu$, then

$$||f||_p = ||l|| = \sup_{||g||_{p'} \leq 1} |l(g)| = \sup_{||g||_{p'} \leq 1} |\int fgd\mu|$$

But if $||f||_p = +\infty$, let

$$f_j(x) = \begin{cases} \min\{|f|, j\} & \text{if } x \in E_j \\ 0 & \text{if } x \notin E_j \end{cases}$$

$$\Rightarrow \quad f_j \in L^p, \quad 0 \leq f_j \nearrow |f|, \quad ||f_j||_p \nearrow ||f||_p$$

(b) Suppose that we have a sequence of L^p functions $\{g_k : ||g_k||_{p'} = 1\}$ where $\int |fg_k|dx > 3^k$ Set $g = \sum_{k=1}^{\infty} 2^{-k} g_k$ so

$$||g||_{p'} \leq 1 \quad \text{yet} \quad fg \notin L^1$$

Thus, by Theorem 12.88 Riesz's Theorem, there must be a constant C so that

$$||fg||_1 \leq C||g||_{p'}$$

5. (Exercise 10.16)
Consider a convolution operator $Tf(\mathbf{x}) = \int_{\mathbb{R}^n} f(\mathbf{y})K(\mathbf{x}-\mathbf{y})d\mathbf{y}$ with $K \geq 0$. If $1 \leq p \leq \infty$ and $||Tf||_p \leq M||f||_p$ for all f, show that $||Tf||_{p'} \leq M||f||_{p'}$ for all f, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. (Use Exercise 10.15 to write $||Tf||_{p'} = \sup_{||g||_p \leq 1} |\int_{\mathbb{R}^n} (Tf)gd\mathbf{x}|$, and note that

$$\int_{\mathbf{R}^n} (Tf)(\mathbf{x})g(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^n} (T\widetilde{g})(-\mathbf{y})f(\mathbf{y})d\mathbf{y}$$

where $\widetilde{g}(\mathbf{x}) = g(-\mathbf{x})$.

Find a generalization if the hypothesis is instead that $||Tf||_q \leq M||f||_p$ for all f, where q is a fixed index with $1 \leq q \leq \infty$ and $q \neq p$.

Proof.
By Exercise 10.15, we can write

$$||Tf||_{p'} = \sup_{||g||_p \leq 1} |\int_{\mathbb{R}^n} (Tf)gd\mathbf{x}|,$$

so

$$\begin{aligned}
||Tf||_{p'} &= \sup_{||g||_p \leq 1} |\int_{\mathbb{R}^n} (Tf)(\mathbf{x})g(\mathbf{x})d\mathbf{x}| \\
&= \sup_{||\widetilde{g}||_p \leq 1} |\int_{\mathbb{R}^n} (T\widetilde{g})(-\mathbf{y})f(\mathbf{y})d\mathbf{y}| \\
\text{By Hölder inequality} \quad &\leq \sup_{||\widetilde{g}||_p \leq 1} ||T\widetilde{g}||_p||f||_{p'} \\
&\leq \sup_{||\widetilde{g}||_p \leq 1} M||\widetilde{g}||_p||f||_{p'} \\
&\leq M||f||_{p'}
\end{aligned}$$

Generalization:
By Exercise 10.15, we can write

$$||Tf||_q = \sup_{||g||_p \leq 1} |\int_{\mathbb{R}^n} (Tf)gd\mathbf{x}|,$$

so

$$\begin{aligned}
||Tf||_q &= \sup_{||g||_p \leq 1} |\int_{\mathbb{R}^n} (Tf)(\mathbf{x})g(\mathbf{x})d\mathbf{x}| \\
&= \sup_{||\widetilde{g}||_p \leq 1} |\int_{\mathbb{R}^n} (T\widetilde{g})(-\mathbf{y})f(\mathbf{y})d\mathbf{y}| \\
\text{By Hölder inequality} \quad &\leq \sup_{||\widetilde{g}||_p \leq 1} ||T\widetilde{g}||_q||f||_p \\
&\leq \sup_{||\widetilde{g}||_p \leq 1} M||\widetilde{g}||_p||f||_p \\
&\leq M||f||_p
\end{aligned}$$