Real Analysis Homework
Chapter 3. Construction of Measures
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Exercise 1(a)

$\Omega = \mathbb{R}^2$, $\mathcal{G}_1 = \{[0,\,1] \times [0,\,1],\, \emptyset\}$ and $\mu_0([0,\,1] \times [0,\,1]) = 2$, $\mu_0(\emptyset) = 0$. Find μ_* and Σ_C.

Recall:
If Ω is a set, $\mathcal{G} \subset 2^\Omega$ and $\mu_0 : \mathcal{G} \to [0,\,\infty]$, then

$$\mu_*(E) = \inf_{\substack{A_k \subset \mathcal{G} \\ E \subset \cup A_k}} \Sigma\, \mu_0(A_k)$$

Recall:
E is called Caratheodory measurable if

$$\mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c) \qquad \text{for any } A \subset \Omega,$$

then

$$\Sigma_C = \{E \subset \Omega \,|\, E \text{ is Caratheodory measurable}\}$$

Proof.

1. Find μ_*.

   (a) $\mu_*(\emptyset) = \mu_0(\emptyset) = 0$.

   (b) For any set $E \subseteq [0,\,1] \times [0,\,1] \subset \Omega$, then

   $$\mu_*(E) = \inf_{\substack{A_k \subset \mathcal{G}_1 \\ E \subset \cup A_k}} \Sigma\, \mu_0(A_k) = \mu_0([0,\,1] \times [0,\,1]) = 2.$$

   (c) Let $E \cap [0,\,1] \times [0,\,1] = \emptyset$, then $\mu_*(E) = \infty$.

2. Find Σ_C.

   (a) Let $E \cap [0,\,1] \times [0,\,1] = \emptyset$, then

   $$\mu_*(A \cap E) + \mu_*(A \cap E^c) = \mu_*(\emptyset) + \mu_*(A) = \mu_*(A) \qquad \text{for any } A \subset \Omega$$

   So $E \in \Sigma_C$ for all $E \cap [0,\,1] \times [0,\,1] = \emptyset$.

(b) Let $E \supseteq [0, 1] \times [0, 1]$.
If $A \cap E = \emptyset$, then

$$\mu_*(A) \leq \mu_*(A \cap E) + \mu_*(A \cap E^c) = \mu_*(\emptyset) + \mu_*(A \cap E^c) \leq \mu_*(A)$$

If $A \cap E \neq \emptyset$, then $A \cap E^c \cap \mathcal{G}_1 \subset \emptyset$, thus

$$\mu_*(A) \leq \mu_*(A \cap E) + \mu_*(A \cap E^c) = \mu_*(A \cap E) + \mu_*(\emptyset) = \mu_*(A \cap E) \leq \mu_*(A)$$

So $E \in \Sigma_C$ for all $E \supseteq [0, 1] \times [0, 1]$.

(c) Let $E \subset [0, 1] \times [0, 1]$.
Suppose to the contrary that $E \in \Sigma_C$. Take $A = [0, 1] \times [0, 1]$, then

$$2 = \mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c) = \mu_*(E) + \mu_*(A \setminus E) = \mu_0(A) + \mu_0(A) = 2 + 2 = 4$$

Therefore we get a contradiction. So $E \notin \Sigma_C$ for all $E \subset [0, 1] \times [0, 1]$.

Exercise 1(b)

Let $\Omega = \mathbb{R}^2$, $\mathcal{G}_2 = \left\{Q_1 = [0, 1] \times [0, 1],\, Q_2 = \left[\frac{1}{2}, 2\right] \times \left[\frac{1}{2}, 2\right],\, \emptyset\right\}$ and $\mu_0(Q_1) = 2$, $\mu_0(Q_2) = \frac{9}{4}$, $\mu_0(\emptyset) = 0$. Find μ_* and Σ_C.

Proof.

1. Find μ_*.

(a) $\mu_*(\emptyset) = \mu_0(\emptyset) = 0$.

(b) For any set $E \subseteq Q_1 \subset \Omega$, then

$$\mu_*(E) = \inf_{\substack{A_k \subset \mathcal{G}_2 \\ E \subset \cup A_k}} \Sigma\, \mu_0(A_k) = \mu_0(Q_1) = 2.$$

(c) For any set $E \subseteq Q_2 \setminus Q_1 \subset \Omega$, then

$$\mu_*(E) = \inf_{\substack{A_k \subset \mathcal{G}_2 \\ E \subset \cup A_k}} \Sigma\, \mu_0(A_k) = \mu_0(Q_2) = \frac{9}{4}.$$

(d) For any set $E \subseteq Q_1 \cup Q_2$ and $E \cap Q_1 \neq \emptyset$, $E \cap Q_2 \setminus Q_1 \neq \emptyset$, then

$$\mu_*(E) = \inf_{\substack{A_k \subset \mathcal{G}_2 \\ E \subset \cup A_k}} \Sigma\, \mu_0(A_k) = \mu_0(Q_1) + \mu_0(Q_2) = 2 + \frac{9}{4} = 4\frac{1}{4}.$$

(e) Let $E \cap (Q_1 \cup Q_2) = \emptyset$, then $\mu_*(E) = \infty$.

2. Find Σ_C.

   (a) Let $E \cap (Q_1 \cup Q_2) = \emptyset$, then

   $$\mu_*(A \cap E) + \mu_*(A \cap E^c) = \mu_*(\emptyset) + \mu_*(A) = \mu_*(A) \qquad \text{for any } A \subset \Omega$$

   So $E \in \Sigma_C$ for all $E \cap (Q_1 \cup Q_2) = \emptyset$.

   (b) Let $E \supseteq Q_1 \cup Q_2$.
   If $A \cap E = \emptyset$, then

   $$\mu_*(A) \leq \mu_*(A \cap E) + \mu_*(A \cap E^c) = \mu_*(\emptyset) + \mu_*(A \cap E^c) \leq \mu_*(A)$$

   If $A \cap E \neq \emptyset$, then $A \cap E^c \cap \mathcal{G}_2 \subset \emptyset$, thus

   $$\mu_*(A) \leq \mu_*(A \cap E) + \mu_*(A \cap E^c) = \mu_*(A \cap E) + \mu_*(\emptyset) = \mu_*(A \cap E) \leq \mu_*(A)$$

   So $E \in \Sigma_C$ for all $E \supseteq Q_1 \cup Q_2$.

   (c) Let $E \subset Q_1 \cup Q_2$ where $E \cap Q_1 \neq \emptyset$ or $E \cap Q_2 \setminus Q_1 \neq \emptyset$.
   Suppose to the contrary that $E \in \Sigma_C$. Take $A = Q_1 \cup Q_2$, then

   $$\begin{aligned} 4\frac{1}{4} &= \mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c) \\ &= \mu_*(E) + \mu_*(A \setminus E) \\ &= \mu_0(A) + \mu_*((A \setminus E) \cap Q_1) + \mu_*((A \setminus E) \cap (Q_2 \setminus Q_1)) \\ &> \mu_0(A) = 4\frac{1}{4} \end{aligned}$$

   Therefore we get a contradiction. So $E \notin \Sigma_C$ for all $E \subset Q_1 \subset Q_2$.

Exercise 2

Let $\Omega = \mathbb{R}^2$. Assume E is Lebesgue measurable with finite measure, $E_1 \cup E_2 = E$, $E_1 \cap E_2 = \emptyset$, and $\mu_*(E) = \mu_*(E_1) + \mu_*(E_2)$. Prove that E_1 and E_2 are Lebesgue measurable.

Proof.
Choose a G_δ set G such that $E_1 \subseteq G$ and $\mu_*(E_1) = \mu(G)$. In particular, G is measurable.

Now let $H = (E_1 \cup E_2) \cap G$ so H is measurable.
Since $E_1 \subseteq H \subseteq G$, then $\mu_*(E_1) \leq \mu(H) \leq \mu(G) = \mu_*(E_1)$, so $\mu_*(E_1) = \mu(H)$.
Thus

$$\begin{aligned} \mu_*(E_1) + \mu_*(E_2) = \mu_*(E_1 \cup E_2) & \\ &= \mu_*((E_1 \cup E_2) \cap H) + \mu_*((E_1 \cup E_2) \setminus H) \\ &= \mu_*(H) + \mu_*((E_1 \cup E_2) \setminus H) \\ &= \mu_*(E_1) + \mu_*((E_1 \cup E_2) \setminus H). \end{aligned}$$

So

$$\mu_*(E_2) = \mu((E_1 \cup E_2) \setminus H).$$

On the other hand, we have $(E_1 \cup E_2) \setminus H \subseteq E_2$. Thus

$$\begin{aligned}
\mu_*(E_2) &= \mu_*(E_2 \cap ((E_1 \cup E_2) \setminus H)) + \mu_*(E_2 \setminus ((E_1 \cup E_2) \setminus H)) \\
&= \mu_*((E_1 \cup E_2) \setminus H) + \mu_*(E_2 \setminus ((E_1 \cup E_2) \setminus H)) \\
&= \mu_*(E_2) + \mu_*(E_2 \setminus ((E_1 \cup E_2) \setminus H)),
\end{aligned}$$

so $\mu_*(E_2 \setminus ((E_1 \cup E_2) \setminus H)) = 0$, then $E_2 \setminus ((E_1 \cup E_2) \setminus H)$ is measurable.
Hence $E_2 = ((E_1 \cup E_2) \setminus H) \cup (E_2 \setminus ((E_1 \cup E_2) \setminus H))$ is also measurable.

Exercise 3

Prove the following lemma.
Lemma:
Assume f, g are measurable on Ω. Then

(1) fg is measurable.

(2) f/g is measurable if $g(x) \neq 0$ for each $x \in \Omega$.

(3) If $\{f(x), g(x)\} \neq \{\infty, -\infty\}$ for each $x \in \Omega$, then $f + g$ is measurable.

Proof.

(1) Since f is measurable and $\{f^2 > a\} = \{f > \sqrt{a}\} \cup \{f < -\sqrt{a}\}$ for all $a \geq 0$, then f^2 is measurable.

By (3), since f and g are measurable, so are $f + g$ and $f - g$. Also, since $f + g$ and $f - g$ are measurable, so are $(f + g)^2$ and $(f - g)^2$.
Hence, $fg = \dfrac{(f+g)^2 - (f-g)^2}{4}$ is measurable.

(2) First, we need to prove that if $g \neq 0$ and is measurable, then $1/g$ is measurable.

(a) If $a > 0$, then $\{1/g > a\} = \{0 < g < 1/a\}$.
(b) If $a = 0$, then $\{1/g > a\} = \{g > 0\}$.
(c) If $a < 0$, then $\{1/g > a\} = \{g > 0\} \cup \{g < 1/a\}$.

So if $g \neq 0$ and is measurable, then $1/g$ is measurable.

By (1), if $g(x) \neq 0$ for each $x \in \Omega$, since f and $1/g$ are measurable, then f/g is measurable.

(3) If g is measurable and a is finite, then $\{f > \lambda - a\}$ is measurablefor each finite λ. So $\{f + a > \lambda\}$ is measurable for each finite λ.
And since g is measurable, so is $a - g$ for any finite a.

If f and $a - g$ are measurable for each finte a. Let $\{r_k\}$ be the rational numbers, then

$$\{f > a - g\} = \bigcup_k \{f > r_k > a - g\} = \bigcup_k (\{f > r_k\} \cap \{r_k > a - g\})$$

So $\{f > a - g\}$ is measurable.

Also, since $\{f + g > a\} = \{f > a - g\}$ and $\{f(x),\, g(x)\} \neq \{\infty,\, -\infty\}$, then $f + g$ is measurable.

Exercise 4

Assume $(\Omega,\, \Sigma,\, \mu)$ is a metric space. Let $\widetilde{\Sigma} = \{E \cup Z \,|\, E \in \Sigma,\ Z \text{ is a null set}\}$ and $\widetilde{\mu}(E \cup Z) = \mu(E)$ if $E \in \Sigma$ and Z is null. Prove $(\Omega,\, \widetilde{\Sigma},\, \widetilde{\mu})$ is a complete measure space.

Proof.
First, we need to check $\widetilde{\mu}$ is a measure of $\widetilde{\Sigma}$.
Notice that $\widetilde{\mu}(\emptyset) = \mu(\emptyset) = 0$.
If $A_1 \subset A_2$, $A_1 = E_1 \cup Z_1$ and $A_2 = E_2 \cup Z_2$, where $E_1,\, E_2 \in \Sigma$ and $Z_1,\, Z_2 \in \mathrm{Null}(\Sigma)$, then

$$\widetilde{\mu}(A_1) = \mu(E_1) \leq \mu(E_2) = \widetilde{\mu}(A_2).$$

Lastly, if $A_i = E_i \cup Z_i$ where $E_i \in \Sigma$ and $Z_i \in \mathrm{Null}(\Sigma)$ for all $i \in \mathbb{N}$, then

$$\widetilde{\mu}\left(\cup_{i=1}^{n} A_i\right) = \mu(\cup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{n} \widetilde{\mu}(A_i).$$

So $\widetilde{\mu}$ is a measure of $\widetilde{\Sigma}$.

Next for completion, we need to show if $Z \in \widetilde{\Sigma}$ and $\widetilde{\mu}(Z) = 0$, if $E \subset Z$ then $E \subset \widetilde{\Sigma}$.
Let $F \subset Z \in \mathrm{Null}(\Sigma)$.
Without loss of generality, if $E \cup F \in \widetilde{\Sigma}$, while $F \subset Z \in \mathrm{Null}(\Sigma)$, one can assume that $E \cap Z = \emptyset$.
Indeed, otherwise one could write

$$E \cup F = (E \setminus Z) \cup [(Z \cap E) \cup F].$$

Also, we have

$$E \setminus Z \in \Sigma \ \text{ and } \ (Z \cap E) \cup F \subset Z \in \mathrm{Null}(\Sigma)$$

So $E^c = E^c \setminus Z \cup Z \in \widetilde{\Sigma}$, since $E^c \setminus Z \in \Sigma$, $Z \in \mathrm{Null}(\Sigma)$. Hence $(\Omega,\, \widetilde{\Sigma},\, \widetilde{\mu})$ is a complete.