1081 Calculus2 模組 07 Quiz 1

Time: 17:50~ 18:20 Date: Nov 28, 2019

DEPARTMENT: ______________ ID NUMBER: ______________ NAME: ______________

It's necessary to explain all the reasons in detail and show all of your work on the answer sheet. Or you will NOT get any credits. If you used any theorems in textbook or proved in class, state it carefully and explicitly.

1. (20%) **True/False** (Write down **T** or **F** only. Any explanations are not required.)

   (a) If f and g are continuous on $[a, b]$, then

   $$\int_a^b (f(x)g(x))\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

   (b) For any given real number t, $\displaystyle\int_{-t}^{t} \sin tx\,dx = 0$

   (c) Suppose a and b are positive numbers, $\displaystyle\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$

   (d) Suppose f is differentiable, then $\displaystyle\int (\frac{d}{dx} f(x))\,dx = f(x)$

   Answer: (a)__F__ (b)__T__ (c)__T__ (d)__F__

   Solution:

   (a)Consider $f(x) = g(x) = -1$, then we have

   $$\int_a^b (-1)(-1)\,dx = \int_a^b 1\,dx = b - a \neq \left(\int_a^b -1\,dx\right)\left(\int_a^b -1\,dx\right)$$
   $$= (-(b-a))\,(-(b-a)) = (b-a)^2$$

   (b)Since for any given t, $\sin tx$ is an odd function, so $\displaystyle\int_{-t}^{t} \sin tx\,dx = 0$

   (c)$\displaystyle\int_0^1 x^a(1-x)^b\,dx \overset{let\ (1-x)=u}{=} \int_1^0 u^b(1-u)^a\,(-du) = \int_0^1 x^b(1-x)^a\,dx$

   (d)Consider $f(x) = x$, then $\displaystyle\int (\frac{d}{dx}x)\,dx = \int 1\,dx = x + C$

2. Evaluate the following integrals,summation, or the volume of the solid.

(a) (10%) $\int (\tan^{\frac{1}{4}} x)(\sec^2 x)\,dx$

Solution.
Let $\tan x = u, \sec^2 x\,dx = du$

$$\int (\tan^{\frac{1}{4}} x)(\sec^2 x)\,dx = \int u^{\frac{1}{4}}\,du = \frac{4}{5}u^{\frac{5}{4}} + C = \frac{4}{5}\tan^{\frac{5}{4}} x + C$$

(b) (10%) $\int \frac{\sin t}{\cos^2 t\sqrt{1+\sec t}}dt$

Solution.

$$\begin{aligned}\int \frac{\sin t}{\cos^2 t\sqrt{1+\sec t}}dt &= \int \frac{\sec t\tan t}{\sqrt{1+\sec t}}dt \\ &= \int \frac{1}{\sqrt{u}}\,du \qquad (\mathit{Let}\ 1+\sec t = u, \tan t\sec t\,dt = du) \\ &= 2u^{\frac{1}{2}} + C = 2(1+\sec t)^{\frac{1}{2}} + C\end{aligned}$$

(c) (10%) $\lim_{n\to\infty}\sum_{i=1}^{n}(\frac{2i}{n})^2\sqrt{1+\left(\frac{i}{n}\right)^3}\frac{1}{n}$

Solution.

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}(\frac{2i}{n})^2\sqrt{1+\left(\frac{i}{n}\right)^3}\frac{1}{n} &= \lim_{n\to\infty}\sum_{i=1}^{n}4(\frac{i}{n})^2\sqrt{1+\left(\frac{i}{n}\right)^3}\frac{1}{n} \\ &= \int_0^1 4x^2\sqrt{1+x^3}dx \qquad (\mathit{Let}\ x^3 = u, 3x^2dx = du) \\ &= \frac{4}{3}\int_0^1 \sqrt{1+u}\,du \\ &= \frac{4}{3}\Big[\frac{2}{3}(1+u)^{\frac{3}{2}}\Big]_0^1 = \frac{8}{9}(2\sqrt{2}-1)\end{aligned}$$

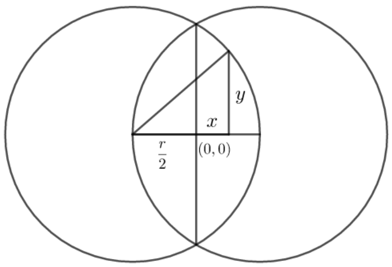
(d) (10%) $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta$

Solution.

$$\begin{aligned}\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta &= \int_0^{\pi/3} \frac{\sin\theta(1+\tan^2\theta)}{\sec^2\theta}\,d\theta \qquad (1+\tan^2\theta = \sec^2\theta) \\ &= \int_0^{\pi/3}\sin\theta\,d\theta = \Big[-\cos\theta\Big]_0^{\frac{\pi}{3}} = \frac{1}{2}\end{aligned}$$

(e) (20%)Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

Solution. Now slicing with the line which connect two sphere's center. Then we could find we only need to compute two caps of height $\frac{r}{2}$.

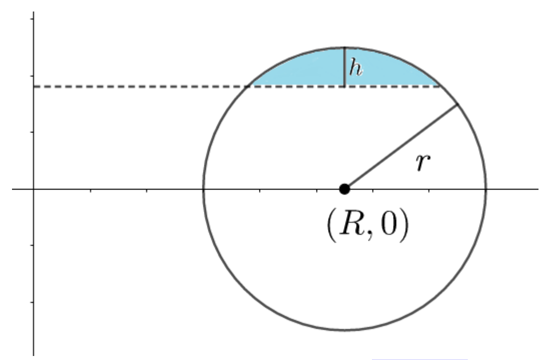


$$\int_0^{\frac{r}{2}} \pi f(x)^2\,dx = \int_0^{\frac{r}{2}} \pi(r^2 - (\frac{r}{2}+x)^2)\,dx$$
$$= \frac{\pi}{2}r^3 - \int_0^{\frac{r}{2}} \pi(\frac{r}{2}+x)^2\,dx = \frac{\pi}{2}r^3 - \pi\Big[\frac{1}{3}(\frac{r}{2}+x)^3\Big]_0^{\frac{r}{2}}$$
$$= \frac{1}{2}\pi r^3 - \frac{1}{3}\pi r^3 + \frac{1}{24}\pi r^3 = \frac{5}{24}\pi r^3 \qquad (One\ cap)$$
$$\frac{5}{24}\pi \times 2 = \frac{5}{12}\pi r^3 \qquad (Two\ caps)$$

(f) (20%)Find the volume of a cap of a sphere which is centered at $(R,0,0)$ with radius r and height h, rotating with z-axis

Solution.

Slicing with the x-z-plane and replace the coordinate, we could find that the volume we need to compute is the blue area rotating with the y-axis.



The equation of the circle is $(x-R)^2 + y^2 = r^2$, the right half is $x = R + \sqrt{r^2 - y^2}$, and the left half is $x = R - \sqrt{r^2-y^2}$. So we need to compute is

$$\int_{r-h}^{r} \pi(R+\sqrt{r^2-y^2})^2 - \pi(R-\sqrt{r^2-y^2})^2\,dy$$
$$= \int_{r-h}^{r} 4\pi R\sqrt{r^2-y^2}\,dy = 4\pi Rr\int_{r-h}^{r}\sqrt{1-(\frac{y}{r})^2}\,dy \quad (Let\ \frac{y}{r} = \sin u, \frac{1}{r}dy = \cos u du)$$
$$= 4\pi Rr\int_{\sin^{-1}(\frac{r-h}{r})}^{\frac{\pi}{2}} r\cos u\cos u\,du = 4\pi Rr^2\int_{\sin^{-1}(\frac{r-h}{r})}^{\frac{\pi}{2}}\cos^2 u\,du \quad (\cos^2 u = \frac{1-\cos 2u}{2})$$
$$= 4\pi Rr^2\int_{\sin^{-1}(\frac{r-h}{r})}^{\frac{\pi}{2}} \frac{1-\cos 2u}{2}\,du = 2\pi Rr^2\int_{\sin^{-1}(\frac{r-h}{r})}^{\frac{\pi}{2}} 1\,du - \pi Rr^2\int_{\sin^{-1}(\frac{r-h}{r})}^{\frac{\pi}{2}} 2\cos 2u\,du$$
$$= 2\pi Rr^2\Big(\frac{\pi}{2} - \sin^{-1}(\frac{r-h}{r})\Big) - \pi Rr^2\Big[\sin 2u\Big]_{\sin^{-1}(\frac{r-h}{r})}^{\frac{\pi}{2}}$$