Real Analysis Homework
Chapter 1. Measure theory
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Exercise 1.1:

Prove that the Cantor set $\mathcal{C}$ constructed in the text is totally disconnected and perfect. In other words, given two distinct points $x, y \in \mathcal{C}$, there is a point $z \notin \mathcal{C}$ that lies between x and y, and yet $\mathcal{C}$ has no isolated points.

Proof.

1. **Disconnected:**

   Let $x, y \in \mathcal{C}$ and $x \neq y$. Then $x, y \in \mathcal{C}_{\|}$ for all $k \in \mathbb{N}$.

   Since $x \neq y$, we can find $N \in \mathbb{N}$ such that $\dfrac{1}{3^N} < |x - y|$. Hence, x and y belong to different intervals of $\mathcal{C}_{\mathcal{N}}$.

   By the construction of the Cantor set, there must be at least one interval between x and y which does not belong to $\mathcal{C}_{\mathcal{N}}$, and so does not belong to $\mathcal{C}$.

   Select one such interval. Choosing any point z in this interval satisfies that z lies between x and y and $z \notin \mathcal{C}$. Therefore, $\mathcal{C}$ is totally disconnected.

2. **Perfect:**

   let $\varepsilon > 0$ be given and consider $B(x, \varepsilon)$ for any $x \in \mathcal{C}$. Let I_k denote the interval to which x belongs in C_k. We can find $N \in \mathbb{N}$ such that $I_N \subset B(x, \varepsilon)$.

   Now, this interval must have two endpoints a_N and b_N. By the construction of the Cantor set, we know that the endpoints of any interval are never removed, and so $a_N, b_N \in \mathcal{C}$. Furthermore, we have that $a_N, b_N \in I_N \subset B(x, \varepsilon)$. Therefore, x is not isolated.

Exercise 1.2(a):

The Cantor set $\mathcal{C}$ can also be described in terms of ternary expansions.
Every number in $[0, 1]$ has a ternary expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}, \qquad \text{where } a_k = 0, 1, \text{or } 2.$$

Prove that $x \in \mathcal{C}$ if and only if x has a representation as above where every a_k is either 0 or 2.

Proof.
($\Rightarrow$)
Let $x \in \mathcal{C}$. Consider $\mathcal{C}_1$. It must be that x belongs to one of $[0, \dfrac{1}{3}]$ or $[\dfrac{2}{3}, 1]$.

Next, consider $\mathcal{C}_2$. The interval of $\mathcal{C}_1$ to which x currently belongs will be divided into three subintervals, and so we append 0 to the ternary expansion of x if it belongs to the leftmost subinterval or 2 if it belongs to the rightmost subinterval.

Continuing in this way, we see that x has an associated ternary expansion containing only the digits 0 and 2.

($\Leftarrow$)
Let

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}, \qquad \text{where } a_k = 0 \text{ or } 2.$$

If $a_1 = 0$, we choose the left subinterval of $\mathcal{C}_1$. If $a_1 = 2$, we choose the rightmost subinterval of $\mathcal{C}_1$.

When we form $\mathcal{C}_2$, the interval we have just chosen will be subdivided into three subintervals. If $a_2 = 0$, we select the leftmost subinterval. If $a_2 = 2$, we select the rightmost subinterval.

Continue in this way. Since the length of these intervals can be made arbitrarily small, we see that the ternary expansion of x uniquely specifies its location on the real line.

Exercise 1.2(b):

The **Cantor-Lebesgue function** is defined on $\mathcal{C}$ by

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \qquad \text{if } x = \sum_{k=1}^{\infty} a_k 3^{-k}, \quad \text{where } b_k = \frac{a_k}{2}.$$

In this definition, we choose the expansion of x in which $a_k = 0$ or 2.
Show that F is well defined and continuous on $\mathcal{C}$, and moreover $F(0) = 0$ as well as $F(1) = 1$.

Proof.
Let $x, x' \in \mathcal{C}$ with $x = x'$. Denote the k-th digit of the ternary expansion of x and x' by a_k and a'_k, respectively.

Suppose $a_k \neq a'_k$ for all k. Then $a_N \neq a'_N$ for some N. From the construction in the part (a), we see that x and x' must belong to different subintervals in $\mathcal{C}_N$, and so $x \notin x'$, which is a construction.

Now, let $b_k = \dfrac{a_k}{2}$ and $b'_k = \dfrac{a'_k}{2}$. Then $b_k = b'_k$ for all k. Hence

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} = \sum_{k=1}^{\infty} \frac{b'_k}{2^k} = F(x')$$

and so F is well-defined.

To see that F is continuous, let $\varepsilon > 0$ be given and $x, x' \in \mathcal{C}$ so that $|F(x) - F(x')| < \varepsilon$.

Consider the binary expansion of ε. Consider $\delta > 0$ such that $\delta_k = 2\varepsilon_k$ for all k. Let N be the first nonzero digit of δ and ε. Then, $|x - x'| < \delta$ implies that the first $N - 1$ digits of x and x' agree.

Hence, the first $N-1$ digits of $F(x)$ and $F(x')$ agree, and so $|F(x)-F(x')|<\varepsilon$. Therefore, F is continuous.

By the construction in part (a), we know that 0 is represented in ternary form by always choosing the leftmost subinterval, and so for $x=0, b_k=\frac{0}{2}=0$ for all k.

Similarly, 1 is represented in ternary form by always chosing the rightmost subinterval, and so for $x=1, b_k=\frac{2}{2}=1$ for all k. Hence

$$F(0)=\sum_{k=1}^{\infty}\frac{0}{2^k}=0,$$

$$F(1)=\sum_{k=1}^{\infty}\frac{1}{2^k}=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1.$$

Exercise 1.2(c):

Prove that $F:\mathcal{C}\to[0,1]$ is surjective, that is, for every $y\in[0,1]$ there exists $x\in\mathcal{C}$ such that $F(x)=y$.

Proof.

Let $y\in[0,1]$. Then y has a corresponding binary expansion.

Let b_k denote the k-th digit of this expansion.

Construct a string s such that $s_k=2b_k$ for all k, where s_k denotes the k-th digit of s. This construction uniquely identifies some ternary string using only 0 and 2. From part (a), we know that s corresponding uniquely to some $x\in\mathcal{C}$. Now, it is clear from our construction of x that $F(x)=y$.

Exercise 1.2(d):

One can also extend F to be a continuous function on $[0,1]$ as follows. Note that if (a,b) is an open interval of the complement of $\mathcal{C}$, then $F(a)=F(b)$. Hence we may define F to have the constant value $F(a)$ in that interval.

Proof.

A connected component of the complement of $\mathcal{C}$ is of the form

$$\left(\sum_{i=1}^{n}a_i3^{-i}+3^{-n},\sum_{i=1}^{n}a_i3^{-i}+2\cdot3^{-n}\right)$$

for some $a_1,\ldots,a_n\in\{0,2\}$.

Write $r=\sum\limits_{i=1}^{n}a_i3^{-i}+3^{-n}$ so the interval is $(r,r+3^{-n})$.

Note that

$$r=\sum_{i=1}^{n}a_i3^{-i}+\sum_{i=n+1}^{\infty}2\cdot3^{-i}\in\mathcal{C}$$

and

$$\begin{aligned}
F(r) &= \sum_{i=1}^{n}\left(\frac{a_i}{2}\right)2^{-i} + \sum_{i=n+1}^{\infty}2^{-i} = \sum_{i=1}^{n}\left(\frac{a_i}{2}\right)2^{-i} + 2^{-n} \\
&= \sum_{i=1}^{n-1}\left(\frac{a_i}{2}\right)2^{-i} + \frac{a_n+2}{2}\cdot 2^{-n} = F\left(\sum_{i=1}^{n-1}a_i 3^{-i} + (a_n+2)3^{-n}\right) \\
&= F\left(\sum_{i=1}^{n}a_i 3^{-i} + 2\cdot 3^{-n}\right) = F(r+3^{-n})
\end{aligned}$$

as desired.

Exercise 1.4(a):

Cantor-like sets.
Construct a closed set $\hat{\mathcal{C}}$ so that at the k-th stage of the construction one removes 2^{k-1} centrally situated open intervals each of length l_k, with

$$l_1 + 2l_2 + \cdots + 2^{k-1}l_k < 1.$$

If l_j are chosen small enough, then $\sum\limits_{k=1}^{\infty}2^{k-1}l_k < 1$. In this case, show that $m(\hat{\mathcal{C}}) > 0$, and in fact, $m(\hat{\mathcal{C}}) = 1 - \sum\limits_{k=1}^{\infty}2^{k-1}l_k$.

Proof.

We begin by showing that we can choose the l_j such that $\sum\limits_{k=0}^{\infty}2^{k-1}l_k < 1$. This is clear if we choose $l_k \leq (2+\varepsilon)^{-(k-1)}$ for any $\varepsilon > 0$ because then

$$\sum_{k=0}^{\infty}2^{k-1}l_k \leq \sum_{k=0}^{\infty}\left(\frac{2}{2+\varepsilon}\right)^{k-1}$$

and the sum on the right converges and is less than 1 for $l_0 < \frac{1}{2}$. Next, we follow a process similar to the construction of the Cantor set in defining

$$\hat{\mathcal{C}} = \bigcap_{k=0}^{\infty} I_k$$

where $I_0 = [0,1],\ I_1 = [0, \frac{1-l_0}{2}] \cup [\frac{1+l_0}{2}, 1]$ and each subsequent I_k is obtained by taking each of the pieces in the union of I_{k-1} and removing the middle l_k. As before, repeating this procedure yields a sequence of nested compact sets $I_0 \supset I_1 \supset I_2 \supset \cdots$ and their intersection $\hat{\mathcal{C}} \neq \emptyset$. It then follows that $\hat{\mathcal{C}}$ is measurable.

To find its measure, we instead compute the measure of $\hat{\mathcal{C}}^c$, which is also measurable because it's complement of a measurable set. We can see that

$$\hat{\mathcal{C}}^c = \bigcup_{k=0}^{\infty} I_k^c.$$

The complement of each I_k is precisely the 2^{k-1} intervals of length l_k. Hence, themeasure of each of these $m(I_k) = 2^{k-1}l_k$. And so

$$m(\hat{\mathcal{C}}^c) = m\left(\bigcup_{k=0}^{\infty} I_k^c\right) = \sum_{k=0}^{\infty} m(I_k^c) = \sum_{k=0}^{\infty}2^{k-1}l_k.$$

Note that $[0,1] = \hat{\mathcal{C}} \cup \hat{\mathcal{C}}^c$ and so

$$m(\hat{\mathcal{C}}) = 1 - \sum_{k=0}^{\infty} 2^{k-1} l_k.$$

Exercise 1.4(b):

Show that if $x \in \hat{\mathcal{C}}$, then there exists a sequence of points $\{x_n\}_{n=1}^{\infty}$ such that $x_n \notin \hat{\mathcal{C}}$, yet $x_n \to x$ and $x_n \in I_n$, where I_n is a sub-interval in the complement of $\hat{\mathcal{C}}$ with $|I_n| \to 0$.

Proof.

Observe first that since $\sum\limits_{k=1}^{\infty} 2^{k-1} l_k < 1$, the tail of the series must go to zero. That is, for any $\varepsilon > 0$, there exists N such that $l_n < \varepsilon$ for all $n \geq N$.

Now let $x \in \hat{\mathcal{C}}$. Let $\hat{\mathcal{C}}_k$ denote the k stage of the construction.

For each k, x belongs to some closed subset S_k of $\hat{\mathcal{C}}_k$. Let I_k be the open interval removed from S_k to proceed to the next step of the construction.

We take any $x_k \in I_k$ to form our sequence $\{x_n\}_{n=1}^{\infty}$. Clearly, each x_k belongs to an sub-interval in the complement of $\hat{\mathcal{C}}$. Furthermore, $|I_k| = l_k \to 0$. It remains to show that $x_n \to x$.

From the construction of $\hat{\mathcal{C}}_k$ and our selection of x_n, it is clear that

$$|x - x_n| < |I_n| + |S_n|.$$

By our previous observation, we know that $|I_n| = l_n \to 0$. Now

$$|S_n| - \frac{1 - \sum\limits_{k=1}^{n} 2^{k-1} l_k}{2^n} \leq \frac{1}{2^n} \to 0 \quad \text{as } n \to \infty.$$

Hence, $|x - x_n| \to 0$. That is, $\{x_n\}_{n=1}^{\infty}$ converges to x.

Exercise 1.4(c):

Prove as a consequence that $\hat{\mathcal{C}}$ is perfect, and contains no open interval.

Proof.

1. **Perfect:**

   let $\varepsilon > 0$ be given and consider $B(x, \varepsilon)$ for any $x \in \hat{\mathcal{C}}$. We can find $N \in \mathbb{N}$ such that $S_N \subset B(x, \varepsilon)$.

   Now, this interval must have two endpoints a_N and b_N. By the construction of $\hat{\mathcal{C}}$, we know that the endpoints of any interval are never removed, and so $a_N, b_N \in \hat{\mathcal{C}}$. Furthermore, we have that $a_N, b_N \in S_N \subset B(x, \varepsilon)$. Therefore, x is not isolated.

2. **No open interval:**

   We try to prove by contradiction. Suppose that there exists an open interval $O \in \hat{\mathcal{C}}$. Then, for any $x \in O$, there exists ε_0 such that $B(x, \varepsilon_0) \subseteq O$.

   Let $\varepsilon < \varepsilon_0$. Then, there can be no sequence $\{x_n\}_{n=1}^{\infty}$ of the type described in part (b) whose limit is x, since $B(x, \varepsilon_0) \subseteq \hat{\mathcal{C}}$ implies that $|x - x_n| > \varepsilon_0 > \varepsilon$ for all n.

   This contradicts the conclusion of part (b), and so it must be that $\hat{\mathcal{C}}$ contains no open interval.

Exercise 1.4(d):

Show also that $\hat{\mathcal{C}}$ is uncountable.

Proof.
We try to prove by contradiction. Suppose that $\hat{\mathcal{C}}$ is countable and let I_n be an enumeration of the set.
I_1 belongs to exactly one of the two intervals in $\hat{\mathcal{C}}_1$, denote the interval which fails to contain I_1 by F_1.
In $\hat{\mathcal{C}}_2$, 2 disjoint intervals $\subseteq F_1$, one of them say F_2 must fail to contain I_2.
Repeating the process, we have decreasing sequence of closed interval F_n of length f_n s.t. I_n not belongs to F_n and

$$\emptyset \neq \bigcap_{n=1}^{\infty} F_n \subseteq S.$$

Hence

$$\exists I \in \hat{\mathcal{C}} \quad \text{and} \quad I \neq I_n, \ \forall n \in \mathbb{N}.$$

We have a contradiction.