Real Analysis
Homework 8

score:7

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1. (Exercise 6.7)
   Let F be a closed subset of $\mathbb{R}^1$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F, prove that the function

$$\int_{\mathbb{R}^1} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}} \, dy$$

   is integrable over F and so is finite a.e. in F. (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

   Proof.

$$\begin{aligned}
\int_F \left[\int_{\mathbb{R}^1} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}} \, dy \right] dx &= \int_{\mathbb{R}^1 - F} \left[\int_F \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}} \, dx \right] dy \\
&= \int_{\mathbb{R}^1 - F} \delta^\lambda(y) f(y) \left[\int_F \frac{1}{|x-y|^{1+\lambda}} \, dx \right] dy \\
&\leq \int_{\mathbb{R}^1 - F} \delta^\lambda(y) f(y) \left[\int_{\{x : \delta(y) \leq |x-y|\}} \frac{1}{|x-y|^{1+\lambda}} \, dx \right] dy \\
&= 2 \int_{\mathbb{R}^1 - F} \delta^\lambda(y) f(y) \left[\int_{\delta(y)}^\infty \frac{1}{t^{1+\lambda}} \, dt \right] dy \\
&= 2\lambda^{-1} \int_{\mathbb{R}^1 - F} f(y) \, dy
\end{aligned}$$

   is integrable since f is integrable over the complement of F, and so is finite a.e. in F.

2. (Exercise 6.8)
   Under the hypothese of Theorem 6.17 and assuming that $b - a < 1$, prove that the function

$$M_0(x) = \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} |x-y|^{-1} \, dy$$

   is finite a.e. in F.

   Proof.

   Since $b - a < 1$, then $\log(\frac{1}{\delta(y)}) > 0$.

Hence $M_0(x)$ is nonnegative and the integral $\int_F M_0(x)$ exists, then

$$\begin{aligned}
\int_F M_0(x)\,dx &= \int_F \left\{ \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} |x-y|^{-1}\ dy \right\}\ dx \\
&= \int_a^b \left\{ \int_F \left[\log \frac{1}{\delta(y)} \right]^{-1} |x-y|^{-1} dx \right\} dy \\
&\leq \int_a^b \left\{ \int_{\{x:\delta(y)\leq|x-y|\leq 1\}} \left[\log \frac{1}{\delta(y)} \right]^{-1} |x-y|^{-1} dx \right\} dy \\
&= \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} \left[\int_{\{x:\delta(y)\leq|x-y|\leq 1\}} |x-y|^{-1} dx \right] dy \\
&\leq \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} \left[2\int_{\delta(y)}^1 t^{-1} dt \right] dy \\
&= \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} [-2\log\delta(y)] dy \\
&= 2(b-a)
\end{aligned}$$

So M_0 is finite a.e. in F.

3. (Exercise 6.9)
   (a) Show that $M_\lambda(x;F) = +\infty$ if $x \notin F$, $\lambda > 0$.
   (b) Let $F = [c,d]$ be a closed subinterval of a bounded open interval $(a,b) \subset \mathbb{R}^1$, and let M_λ be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c,d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int_F M_\lambda \leq \lambda^{-1}|G|$, where $G = (a,b) - [c,d]$.

Proof.

(a) Let $x \notin F$ and $\lambda > 0$. For any $\epsilon > 0$, then $\delta(y) \in B(\delta(x),\epsilon)$ for all $y \in B(x,\epsilon)$, thus

$$\begin{aligned}
M_\lambda(x;F) &= \int_a^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \\
&= \int_a^{x-\epsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy + \int_{x-\epsilon}^{x+\epsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy + \int_{x+\epsilon}^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \\
&\geq \int_{x-\epsilon}^{x+\epsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \\
&\geq \int_{x-\epsilon}^{x+\epsilon} \frac{\delta^\lambda(x)-\epsilon}{\epsilon^{1+\lambda}} dy \\
&= \frac{2\delta^\lambda(x)-2\epsilon}{\epsilon^\lambda}
\end{aligned}$$

Let ϵ_n be a sequence on $(0,1)$ with $\epsilon_n \to 0$ as $n \to \infty$, then

$$\frac{(2\delta^\lambda(x)-2\epsilon_n)}{\epsilon_n^\lambda} \to \infty \quad \text{as } n \to \infty$$

So $M_\lambda(x;F) = +\infty$ if $x \notin F$, $\lambda > 0$.

(b)

$$M_\lambda(x;F) = \int_a^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy$$

i. For any $\epsilon > 0$ and $M = \max\{d-a, b-c\}$, we have

$$\begin{aligned}
\int_a^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}}dy &= \int_a^{c+\epsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}}dy + \int_{d-\epsilon}^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}}dy \\
&\leq \int_a^{c+\epsilon} \frac{\delta^\lambda(y)}{\epsilon^{1+\lambda}}dy + \int_{d-\epsilon}^b \frac{\delta^\lambda(y)}{\epsilon^{1+\lambda}}dy \\
&\leq \frac{(b-a)^\lambda}{\epsilon^{1+\lambda}} \cdot (c+\epsilon-a) + \frac{(b-c)^\lambda}{\epsilon^{1+\lambda}} \cdot (b-d+\epsilon) \\
&\leq \frac{M^\lambda}{\epsilon^{1+\lambda}} \cdot (c+\epsilon-a) + \frac{M^\lambda}{\epsilon^{1+\lambda}} \cdot (b-d+\epsilon) \\
&= \frac{M^\lambda}{\epsilon^{1+\lambda}} \cdot (c+\epsilon-a+b-d+\epsilon) \\
&= \frac{M^\lambda}{\epsilon^{1+\lambda}} \cdot (|G|+2\epsilon) \quad \text{is finite, since } \epsilon \neq 0
\end{aligned}$$

Hecne, M_λ is finite for every $x \in (c,d)$.

ii.

$$\begin{aligned}
M_\lambda(c) &= \int_a^b \frac{\delta^\lambda(y)}{|c-y|^{1+\lambda}}dy \\
&= \int_a^c \frac{\delta^\lambda(y)}{(c-y)^{1+\lambda}}dy + \int_d^b \frac{\delta^\lambda(y)}{(y-c)^{1+\lambda}}dy \\
&= \int_a^c \frac{(c-y)^\lambda}{(c-y)^{1+\lambda}}dy + \int_d^b \frac{(y-d)^\lambda}{(y-c)^{1+\lambda}}dy \\
&> \int_a^c \frac{1}{c-y}dy \\
&= \lim_{t\to c} \left[-\ln(c-y)\right]_{y=a}^t \\
&= \infty
\end{aligned}$$

$$\begin{aligned}
M_\lambda(d) &= \int_a^b \frac{\delta^\lambda(y)}{|d-y|^{1+\lambda}}dy \\
&= \int_a^c \frac{\delta^\lambda(y)}{(d-y)^{1+\lambda}}dy + \int_d^b \frac{\delta^\lambda(y)}{(y-d)^{1+\lambda}}dy \\
&= \int_a^c \frac{(c-y)^\lambda}{(d-y)^{1+\lambda}}dy + \int_d^b \frac{(y-d)^\lambda}{(y-d)^{1+\lambda}}dy \\
&> \int_d^b \frac{1}{y-d}dy \\
&= \lim_{t\to d} \left[\ln(y-d)\right]_{y=t}^b \\
&= \infty
\end{aligned}$$

iii. If $y \in F$, then $\delta(y) = 0$, thus

$$\begin{aligned}
\int_F M_\lambda(x; F)dx &= \int_F \left[\int_a^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \right] dx \\
&= \int_F \left[\int_G \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \right] dx \\
&= \int_G \left[\int_F \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dx \right] dy \\
&= \int_G \delta^\lambda(y) \left[\int_F \frac{1}{|x-y|^{1+\lambda}} dx \right] dy \\
&\leq \int_G \delta^\lambda(y) \left[\int_{\{x:\delta(y) \leq |x-y|\}} \frac{1}{|x-y|^{1+\lambda}} dx \right] dy \\
&\leq \int_G \delta^\lambda(y) \left[\int_{\delta(y)}^a \frac{1}{t^{1+\lambda}} dt \right] dy \\
&= \int_G \delta^\lambda(y) \left[\frac{-1}{\lambda} (a^\lambda - \delta^\lambda(y)) \right] dy \\
&\leq \frac{1}{\lambda} \int_G 1 \, dy \\
&= \frac{|G|}{\lambda}
\end{aligned}$$

4. (Exercise 7.2)
Let $\phi(x)$, $x \in \mathbb{R}^n$, be a bounded measurable function such that $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi = 1$. For $\epsilon > 0$, let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. ($\phi_\epsilon$ is called an *approximation to the identity*.) If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\epsilon \to 0} (f * \phi_\epsilon)(x) = f(x)$$

in the Lebesgue set of f. (Note that $\int \phi_\epsilon = 1$, $\epsilon > 0$, so that

$$(f * \phi_\epsilon)(x) - f(x) = \int [f(x-y) - f(x)] \, \phi_\epsilon(y) \, dy$$

Use Theorem 7.16.)

Recall Exercise 5.20:
Let $\mathbf{y} = T\mathbf{x}$ be a nonsingular linear transformation of $\mathbb{R}^n$. If $\int_E f(\mathbf{y})d\mathbf{y}$ exists, then

$$\int_E f(\mathbf{y}) \, d\mathbf{y} = |\det T| \int_{T^{-1}E} f(T\mathbf{x}) \, d\mathbf{x}$$

Recall Theorem 7.16:
Let f be locally integrable in $\mathbb{R}^n$, then at every point $\mathbf{x}$ of the Lebesgue set of f,

$$\frac{1}{|S|} \int_S |f(\mathbf{y}) - f(\mathbf{x})| \, d\mathbf{y} \to 0$$

for any family $\{S\}$ that shrinks regularly to $\mathbf{x}$. Thus, also

$$\frac{1}{|S|} \int_S f(\mathbf{y}) \, d\mathbf{y} \to f(\mathbf{x}) \text{ a.e.}$$

Proof.

First, we will show that $\int \phi_\epsilon = 1$.
For $\epsilon > 0$, then

$$\int_{\mathbb{R}^n} \phi_\epsilon(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^n} \epsilon^{-n}\phi(\mathbf{x}/\epsilon)\, d\mathbf{x} = \int_{\{|\mathbf{x}|<\epsilon\}} \epsilon^{-n}\phi(\mathbf{x}/\epsilon)\, d\mathbf{x}$$

since $\phi(\mathbf{x}) = 0$ for all $|\mathbf{x}| \geq 1$.

Let $\mathbf{y} = T\mathbf{x} = \frac{\mathbf{x}}{\epsilon}$ be a linear transformation of $\mathbb{R}^n$, and $T = \operatorname{diag}(\frac{1}{\epsilon}, ..., \frac{1}{\epsilon})$ so that $|\det T| = \epsilon^{-n}$. If $E = \{x \in \mathbb{R}^n : |\mathbf{x}| < 1\}$, then $T^{-1}E = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < \epsilon\}$, thus by Exercise 5.20

$$\int_E f(\mathbf{y})\, d\mathbf{y} = |\det T| \int_{T^{-1}E} f(T\mathbf{x})\, \mathbf{x}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^n} f(\mathbf{x})\, d\mathbf{x} &= \epsilon^{-n} \int_{T^{-1}E} \phi(T\mathbf{x})\, \mathbf{x} \\
&= \epsilon^{-n} \cdot \frac{1}{|\det T|} \int_E \phi(\mathbf{y})\, d\mathbf{y} \\
&= \int_{\{|\mathbf{y}|<1\}} \phi(\mathbf{y})\, \mathbf{y} \\
&= \int_{\mathbb{R}^n} \phi(\mathbf{y})\, d\mathbf{y} \\
&= 1
\end{aligned}$$

Following,

$$\begin{aligned}
(f * \phi_\epsilon)(\mathbf{x}) - f(\mathbf{x}) &= \int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\, d\mathbf{y} - \int_{\mathbb{R}^n} f(\mathbf{x})\phi_\epsilon(\mathbf{y})\, d\mathbf{y} \\
&= \int_{\mathbb{R}^n} [f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x})]\, \phi_\epsilon(\mathbf{y})\, d\mathbf{y} \\
&= \frac{1}{\epsilon^n} \int_{\{|\mathbf{y}|\leq\epsilon\}} [f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x})]\, \phi(y/\epsilon)\, d\mathbf{y}
\end{aligned}$$

Since $\phi(\mathbf{x})$ is a bounded function on $\mathbb{R}^n$, then for some $M > 0$, we have $|\phi(\mathbf{x})| \leq M$ and let $Q_{2\epsilon}(x)$ be the cube centered at $\mathbf{x}$ with edge length 2ϵ, then

$$\begin{aligned}
|(f * \phi_\epsilon)(x) - f(x)| &\leq \frac{M}{\epsilon^n} \int_{\{|\mathbf{y}|\leq\epsilon\}} |f(\mathbf{x}-\mathbf{y}) - f(\mathbf{x})|\, d\mathbf{y} \\
&= \frac{M}{\epsilon^n} \int_{\{|\mathbf{y}-\mathbf{x}|\leq\epsilon\}} |f(\mathbf{y}) - f(\mathbf{x})|\, d\mathbf{y} \\
&\leq \frac{M}{\epsilon^n} \int_{Q_{2\epsilon}(\mathbf{x})} |f(\mathbf{y}) - f(\mathbf{x})|\, d\mathbf{y} \\
&\leq \frac{2^n M}{|Q_{2\epsilon}(\mathbf{x})|} \int_{Q_{2\epsilon}(\mathbf{x})} |f(\mathbf{y}) - f(\mathbf{x})|\, d\mathbf{y} \\
&(|Q_{2\epsilon}(\mathbf{x})| = 2^n\epsilon^n)
\end{aligned}$$

Since $f \in L(\mathbb{R}^n)$, by Theorem 7.16, for all points $\mathbf{x}$ of the Lebesgue set of f, we have

$$\frac{1}{|Q_{2\epsilon}(\mathbf{x})|} \int_{Q_{2\epsilon}(\mathbf{x})} |f(\mathbf{y}) - f(\mathbf{x})|\, d\mathbf{y} \to 0 \text{ as } \epsilon \to 0$$

Hence,

$$\lim_{\epsilon\to 0} |(f * \phi_\epsilon)(\mathbf{x}) - f(\mathbf{x})| = 0$$

, which implies

$$\lim_{\epsilon \to 0} (f * \phi_\epsilon)(\mathbf{x}) = f(\mathbf{x})$$

in the Lebesegue set of f.

5. (Exercise 7.6)
Show that if $\alpha > 0$, then x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Recall Theorem 7.6:
A function f is absolutely continuous an $[a, b]$ if and only if f' exists a.e. in $[a, b]$, f' is integrable on $[a, b]$, and

$$f(x) - f(a) = \int_a^x f', \quad a \le x \le b$$

Proof.
Let $f(x) = x^\alpha$.
Since

$$f'(x) = \begin{cases} \alpha x^{\alpha-1}, & \text{if } x \in (0, \infty) \\ \lim\limits_{x \to 0} \frac{x^\alpha - 0^\alpha}{x - 0} = 0, & \text{if } x = 0 \end{cases}$$

$f(x) = x^\alpha$ is differentiable on $[0, \infty)$, then $f(x) = x^\alpha$ is also differentiable on every bounded subinterval $[a, b]$.

Since f' is a polynomial function, f' is continuous on $[a, b]$.
Since f' is continuous and also bounded on $[a, b]$, then f' is Reimann integrable on $[a, b]$.

If $a \le x \le b$, then we have

$$\int_a^x f'(y)dy = \int_a^x \alpha y^{\alpha-1} dy = x^\alpha - a^\alpha = f(x) - f(a)$$

By Theorem 7.6, then we know that $f(x) = x^\alpha$ is absolutely continuous on every bounded subinterval of $[0, \infty)$.

6. (Exercise 7.7)
Prove that f is absolutely continuous on $[a, b]$ if and only if given $\epsilon > 0$, there exists $\delta > 0$ such that $|\sum [f(b_i) - f(a_i)]| < \epsilon$ for every *finite* collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum (b_i - a_i) < \delta$.

Proof.

$(\Rightarrow)$
Since $|\sum [f(b_i) - f(a_i)]| < \sum |f(b_i) - f(a_i)|$, the definition of an absolutely continuous function f immediately leads to this implication.

$(\Leftarrow)$
For any collection $\{[a_i, b_i]\}$ be a sequence of nonoverlapping subintervals of $[a, b]$ with $\sum (b_i - a_i) < \delta$, we have

$$\sum |f(b_i) - f(a_i)| = \sum [f(b_i) - f(a_i)]^+ + \sum [f(b_i) - f(a_i)]^- < 2\epsilon$$

Hence f is absolutely continuous on $[a, b]$.