Calculus 1 9/26 Note
Module Class 07

Yueh-Chou Lee

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Section 2.5: Continuity

Definition

A function f is

1. **continuous at a number a** if
$$\lim_{x\to a} f(x) = f(a).$$
2. **continuous from the right at a number a** if
$$\lim_{x\to a^+} f(x) = f(a).$$
3. **continuous from the left at a number a** if
$$\lim_{x\to a^-} f(x) = f(a).$$

Theorem

If f and g are continuous at a and c is a constant, then the function states in the **Limit Laws** are also continuous at a.

Theorem

If f is continuous at b and $\lim\limits_{x\to a} g(x) = b$, then $\lim\limits_{x\to a} f(g(x)) = f(b)$. In other words,
$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

Theorem

If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example:

Explain, using Theorems 4, 5, 7 and 9 in the textbook, why the function is continuous at every number in its domain. State the domain.

$$G(x) = \frac{x^2+1}{2x^2-x-1}$$

Sol.

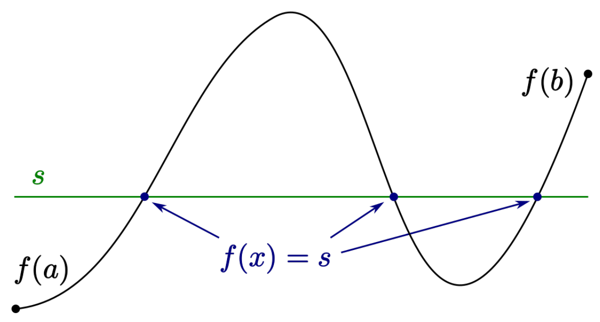
$$G(x) = \frac{x^2+1}{2x^2-x-1} = \frac{x^2+1}{(2x+1)(x-1)}$$

$G(x)$ is a rational function, so it is continuous on its domain, $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$ by Theorem 5(b) in the textbook.

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let s be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = s$.

Graph:



Example:

Use the Intermediate Value Theorem to show that is a root of the given equation in the specified.

$$x^4 + x - 3 = 0, \quad (1, 2)$$

Sol.

$f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$ and $f(2) = 15$.

Since $f(1) = -1 < 0 < 15 = f(2)$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem.

Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

Exercise:

1. Use continuity to evaluate the following limit.

   (a) $\lim_{x \to \pi} \sin(x + \sin x)$

   (b) $\lim_{x \to 1} \ln\left(\frac{5-x^2}{1+x}\right)$

2. Use the Intermediate Value Theorem to show that is a root of the following given equation in the specified.

   (a) $\ln x = x - \sqrt{x}, \quad (2, 3)$

   (b) $\sin x = x^2 - x, \quad (1, 2)$

3. Is there a number that is exactly 1 more than its cube?

Sol.
1. (a) 0 (b) $\ln 2$

2. (a) $f(x) = \ln x - x + \sqrt{x}, \quad f(2) > 0 > f(3)$ (b) $f(x) = \sin x - x^2 + x, \quad f(1) > 0 > f(2)$

3. There is a root between -2 and -1.

Section 2.6: Limits at Infinity; Horizontal Asymptotes

Defintion

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

Theorem

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Example:

Find the following limit or show that it does not exist.

1. $\lim_{x \to \infty} \frac{\sqrt{x}+x^2}{2x-x^2}$

2. $\lim_{x \to \infty} (e^{-x} + 2\cos 3x)$

Sol.

1.

$$\lim_{x\to\infty} \frac{\sqrt{x}+x^2}{2x-x^2} = \lim_{x\to\infty} \frac{(\sqrt{x}+x^2)/x^2}{(2x-x^2)/x^2} = \lim_{x\to\infty} \frac{x^{-3/2}+1}{2x^{-1/2}-1} = -1$$

2. DEN

   $\lim\limits_{x\to\infty} e^{-x} = 0$, but $\lim\limits_{x\to\infty} (2\cos 3x)$ does not exist because the values of $2\cos 3x$ oscillate between the values of -2 and 2 infinitely often, so the given limit does not exist.

Exercise:

1. Find the following limit or show that it does not exist.

   (a) $\lim\limits_{x\to\infty} \dfrac{\sqrt{9x^6-x}}{x^3+1}$

   (b) $\lim\limits_{x\to 0^+} \tan^{-1}(\ln x)$

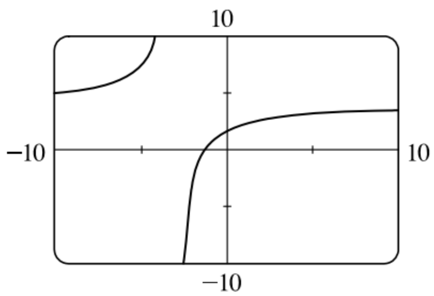
2. Find the horizontal and vertical asymptote of each curve.

   (a) $y = \dfrac{5+4x}{x+3}$

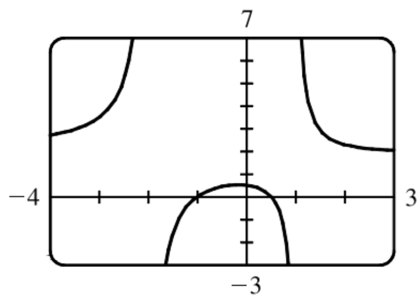
   (b) $y = \dfrac{2x^2+x-1}{x^2+x-2}$

Sol.
1. (a) 3 (b) $-\pi/2$

2. (a) $y = 4$ is a horizontal asymptote and $x = -3$ is a vertical asymptote.



(b)$y = 2$ is a horizontal asymptote; $x = -2$ and $x = 1$ are vertical asymptotes.



Precise Definitions:

Precise Definition of a Limit at Theorem

1. **x approaches to ∞:**
   Let f be a function defined on some interval (a, ∞). Then
   $$\lim_{x\to\infty} f(x) = L$$
   means that for every $\epsilon > 0$ there is a corresponding number N such that
   $$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \epsilon.$$

2. **x approaches to $-\infty$:**
   Let f be a function defined on some interval $(-\infty, a)$. Then
   $$\lim_{x\to-\infty} f(x) = L$$
   means that for every $\epsilon > 0$ there is a corresponding number N such that
   $$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \epsilon$$

Definition of an Infinite Limit at Infinity

Let f be a function defined on some interval (a, ∞). Then
$$\lim_{x\to\infty} f(x) = \infty$$
means that for every positive number M there is a corresponding positive number N such that
$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M.$$

Example:
Use Definition to prove that $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Sol.
For $x < 0$, $|1/x - 0| = -1/x$.

If $\epsilon > 0$ is given, then
$$-\frac{1}{x} < \epsilon \Leftrightarrow x < -\frac{1}{\epsilon}.$$
Take $N = -1/\epsilon$. Then
$$x < N \Rightarrow x < -\frac{1}{\epsilon} \Rightarrow \left|\frac{1}{x} - 0\right| = -\frac{1}{x} < \epsilon,$$

So $\lim_{x\to-\infty} \frac{1}{x} = 0$.

Section 2.7: Derivatives and Rates of Change

Definition

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists.

Definition of Derivatives

The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Example:

1. $f(x) = \dfrac{2x+1}{x+3}$, find $f'(a)$.
2. Each limit represents the derivative of some function f at some number a. State such an f and a in the given case.
$$\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h}$$

Sol.

1.
$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{2(a+h)+1}{(a+h)+3} - \frac{2a+1}{a+3}}{h} \\ &= \lim_{h \to 0} \frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)} \\ &= \lim_{h \to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2} \end{aligned}$$

2. $\lim_{h \to 0} \frac{\sqrt{9+h}-3}{h} = f'(9)$, where $f(x) = \sqrt{x}$ and $a = 9$.

Exercise:

1. (a) $f(x) = 2x^3 + x$, find $f'(a)$.
   (b) $f(x) = \sqrt{1 - 2x}$, find $f'(a)$.
2. Each limit represents the derivative of some function f at some number a. State such an f and a in the given case.
   (a) $\displaystyle\lim_{x \to 1/4} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$.

(b) $\lim_{h \to 0} \dfrac{\cos(\pi + h) + 1}{h}$.

Sol.

1. (a) $6a^2 + 1$ (b) $\dfrac{-1}{\sqrt{1-2a}}$
2. (a) $f(x) = \dfrac{1}{x}$ and $a = \dfrac{1}{4}$ (b) $f(x) = \cos(\pi + x)$ and $a = 0$

Section 2.8: The Derivatives as a Function

Recall:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Definition

A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) if it is differentiable at every number in the interval.

Theorem

If f is differentiable at a, then f is continuous at a.

Note of Higher Derivatives:

$$f' = \frac{df}{dx}, \qquad f'' = (f')' = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2},$$
$$f''' = (f'')' = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) = \frac{d^3 f}{dx^3}, \qquad f^{(4)} = \frac{d^4 f}{dx^4} \qquad \text{and} \ldots.$$

Example:

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

1. $f(x) = 3x - 8$
2. $f(x) = \dfrac{1 - 2x}{3 + x}$

Sol.

1.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h) - 8] - (3x - 8)}{h} = \lim_{h \to 0} \frac{3h}{h} = 3$$

2.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1-2(x+h)}{3+(x+h)} - \frac{1-2x}{3+x}}{h}$$
$$= \lim_{h\to 0} \frac{-6h-h}{h(3+x+h)(3+x)} = \lim_{h\to 0} \frac{-7}{(3+x+h)(3+x)} = \frac{-7}{(3+x)^2}$$

Exercise:

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

1. $f(x) = \sqrt{9-x}$
2. $f(x) = \dfrac{x^2-1}{2x-3}$

Sol.

1. $\dfrac{-1}{2\sqrt{9-x}}$, domain of $f = (-\infty, 9]$ and domain of $f' = (-\infty, 9)$
2. $\dfrac{2x^2-6x+2}{(2x-3)^2}$, domain of $f =$ domain of $f' = (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$