Calculus 2 12/5 Note
Module Class 07

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Section 7.4: Integration of Rational Function by Partial Fractions

Consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. If f is *improper*, that is $deg(P) \geq deg(Q)$, then we will rewrite as

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where S and R are also polynomials.

Example:

Write out the form of the partial fraction decomposition of the function.

1. $\frac{x^3+1}{x^3-3x^2+2x}$

Sol.

$$\begin{aligned}\frac{x^3+1}{x^3-3x^2+2x} &= 1 + \frac{3x^2-2x+1}{x^3-3x^2+2x} = 1 + \frac{3x^2-2x+1}{x(x-1)(x-2)} \\ &= 1 + \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}\end{aligned}$$

where A, B and C are the certain constants.

CASE I. The denominator $Q(x)$ is a product of distinct linear factors

This means that we can write

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$$

where no factor is repeated. Then there exist constants A_1, $A_2, \ldots,$ A_k such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}$$

Example:

Evaluate the following integrals.

1. $\int_0^1 \frac{2}{2x^2+3x+1}\,dx$

Sol.

$$\frac{2}{2x^2+3x+1} = \frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$

Multiply both side by $(2x+1)(x+1) \;\Rightarrow\; 2 = A(x+1) + B(2x+1) = (A+2B)x + (A+B)$

So $A+2B = 0$ and $A+B = 2$, then we have $A = 4$ and $B = -2$. Thus,

$$\begin{aligned}\int_0^1 \frac{2}{2x^2+3x+1}\,dx &= \int_0^1 \left(\frac{4}{2x+1} - \frac{2}{x+1}\right) dx \\ &= \left[\frac{4}{2}\ln|2x+1| - 2\ln|x+1|\right]_{x=0}^{1} \\ &= 2\ln\frac{3}{2}\end{aligned}$$

CASE II. $Q(x)$ **is a product of linear factors, some of which are repeated**

Suppose the first linear factor (a_1x+b_1) is repeated r times; that is $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then we would use

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_r}{(a_1x+b_1)^r}$$

For example, we could write

$$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Example:

Evaluate the following integrals.

1. $\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$

Sol.

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

Multiply both side by $(x+1)^2(x+2)$

$$\begin{aligned}\Rightarrow x^2+x+1 &= A(x+1)(x+2) + B(x+2) + C(x+1)^2 \\ &= (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)\end{aligned}$$

So $A+C=1$, $3A+B+2C=1$ and $2A+2B+C=1$, then we have $A=-2$, $B=1$ and $C=3$. Thus,

$$\begin{aligned}\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx &= \int_0^1 \left(\frac{-2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}\right)dx \\ &= \left[-2\ln|x+1|-\frac{1}{x+1}+3\ln|x+2|\right]_{x=0}^{1} \\ &= \frac{1}{2}-5\ln 2+3\ln 3 \quad \text{or} \quad \frac{1}{2}+\ln\frac{27}{32}\end{aligned}$$

CASE III. $Q(x)$ contains irreducible quadratic factors, none of which is repeated

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If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then

$$\frac{R(x)}{Q(x)}=\frac{Ax+B}{ax^2+bx+c}$$

where A and B are constants. If necessary, we could use the formula

$$\int \frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

For example, we could write

$$\frac{x}{(x-2)(x^2+1)(x^2+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{x^2+4}$$

Example:

Evaluate the following integrals.

1. $\int \frac{10}{(x-1)(x^2+9)}\,dx$

Sol.

$$\frac{10}{(x-1)(x^2+9)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+9}$$

Multiply both side by $(x-1)(x^2+9)$

$$\Rightarrow\ 10=A(x^2+9)+(Bx+C)(x-1)=(A+B)x^2+(-B+C)x+(9A-C)$$

So $A+B=0$, $-B+C=0$ and $9A-C=10$, then we have $A=1$, $B=-1$ and $C=-1$. Thus,

$$\begin{aligned}\int \frac{10}{(x-1)(x^2+9)}\,dx &= \int\left(\frac{1}{x-1}+\frac{-x-1}{x^2+9}\right)dx = \int\left(\frac{1}{x-1}-\frac{x}{x^2+9}-\frac{1}{x^2+9}\right)dx \\ &= \ln|x-1|-\frac{1}{2}\ln(x^2+9)-\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+\text{constant}\end{aligned}$$

Note: $\dfrac{d\tan^{-1}x}{dx}=\dfrac{1}{x^2+1}$.

CASE III. $Q(x)$ contains a repeated irreducible quadratic factor

If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then we would use

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

For example, we could write

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}$$

Example:

Evaluate the following integrals.

1. $\int \frac{5x^4+7x^2+x+2}{x(x^2+1)^2}\,dx$

Sol.

$$\frac{5x^4+7x^2+x+2}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Multiply both side by $x(x^2+1)^2$, then

$$\begin{aligned}
5x^4+7x^2+x+2&=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x \;\Leftrightarrow\\
5x^4+7x^2+x+2&=A(x^4+2x^2+1)+(Bx^2+Cx)(x^2+1)+Dx^2+Ex \;\Leftrightarrow\\
5x^4+7x^2+x+2&=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A
\end{aligned}$$

So $A=2$, $B=3$, $C=0$, $D=0$ and $E=1$. Thus

$$\begin{aligned}
\int \frac{5x^4+7x^2+x+2}{x(x^2+1)^2}\,dx&=\int\left[\frac{2}{x}+\frac{3x}{x^2+1}+\frac{1}{(x^2+1)^2}\right]dx\\
&=2\ln|x|+\frac{3}{2}\ln(x^2+1)+\int\frac{1}{(x^2+1)^2}\,dx
\end{aligned}$$

Moverover, let $x=\tan\theta$, then $dx=\sec^2\theta\,d\theta$, hence

$$\begin{aligned}
\int\frac{1}{(x^2+1)^2}\,dx&=\int\frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2}\\
&=\int\cos^2\theta\,d\theta=\int\frac{1+\cos(2\theta)}{2}\,d\theta\\
&=\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)+\text{constant}\\
&=\frac{1}{2}\theta+\frac{1}{2}\sin\theta\,\cos\theta+\text{constant}\\
&=\frac{1}{2}\tan^{-1}x+\frac{1}{2}\frac{x}{\sqrt{x^2+1}}\,\frac{1}{\sqrt{x^2+1}}+\text{constant}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int\frac{5x^4+7x^2+x+2}{x(x^2+1)^2}\,dx&=2\ln|x|+\frac{3}{2}\ln(x^2+1)+\int\frac{1}{(x^2+1)^2}\,dx\\
&=2\ln|x|+\frac{3}{2}\ln(x^2+1)+\frac{1}{2}\tan^{-1}x+\frac{x}{2(x^2+1)}+\text{constant}
\end{aligned}$$

Exercise:

Evaluate the following integrals.

1. $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

2. $\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx$

3. $\int \frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx$

Sol.
1. $\frac{3}{2}+\ln\frac{3}{2}$ 2. $\frac{1}{2}+\ln 6$
3. $\frac{1}{4}\ln(x^2+1)-\frac{3}{2}\tan^{-1}x+\frac{1}{4}\ln(x^2+3)-\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\text{constant}$

Section 7.5: Strategy for Integration

Table of Integration Formulas

Constants of integration have been omitted.

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$
3. $\displaystyle\int e^x\,dx = e^x$
4. $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b}$
5. $\displaystyle\int \sin x\,dx = -\cos x$
6. $\displaystyle\int \cos x\,dx = \sin x$
7. $\displaystyle\int \sec^2 x\,dx = \tan x$
8. $\displaystyle\int csc^2 x\,dx = -\cot x$
9. $\displaystyle\int \sec x\,\tan x\,dx = \sec x$
10. $\displaystyle\int \csc x\,\cot x\,dx = -\csc x$
11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$
12. $\csc x\,dx = \ln|\csc x - \cot x|$
13. $\displaystyle\int \tan x\,dx = \ln|\sec x|$
14. $\displaystyle\int \cot x\,dx = \ln|\sin x|$
15. $\displaystyle\int \sinh x\,dx = \cosh x$
16. $\displaystyle\int \cosh x\,dx = \sinh x$
17. $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
18. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right),\ a > 0$

*19. $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}|$

Example:

Evaluate the following integrals.

1. $\int \ln\left(x + \sqrt{x^2-1}\right)\,dx$

Sol.

Use integration by parts with $u = \ln(x + \sqrt{x^2-1})$, $dv = dx$, then

$$du = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)dx = \frac{1}{\sqrt{x^2-1}}\,dx \qquad \text{and} \qquad v = x$$

Thus

$$\begin{aligned}\int \ln\left(x+\sqrt{x^2-1}\right)dx &= x\ln\left(x+\sqrt{x^2-1}\right) - \int \frac{x}{\sqrt{x^2-1}}\,dx \\ &= x\ln\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1} + \text{constant}\end{aligned}$$

2. $\int \sqrt{1-\sin x}\,dx$

Sol.

$$\begin{aligned}\int \sqrt{1-\sin x}\,dx &= \int \sqrt{\frac{1-\sin x}{1}\cdot\frac{1+\sin x}{1+\sin x}}\,dx = \int \sqrt{\frac{1-\sin^2 x}{1+\sin x}}\,dx \\ &= \int \sqrt{\frac{\cos^2 x}{1+\sin x}}\,dx = \int \frac{\cos x}{\sqrt{1+\sin x}}\,dx \qquad (\text{assume } \cos x > 0) \\ &= \int \frac{d\sin x}{\sqrt{1+\sin x}} \\ &= 2\sqrt{1+\sin x} + \text{constant}\end{aligned}$$

Exercise:

Evaluate the following integrals.

1. $\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\, dx$

2. $\int_0^\pi \sin(6x)\, \cos(3x)\, dx$

3. $\int \frac{4^x+10^x}{2^x}\, dx$

4. $\int \frac{x^2}{\sqrt{x^2+1}}\, dx$

Sol.
1. $\sqrt{1+(\ln x)^2} + \text{constant}$ 2. $\frac{4}{9}$ 3. $\frac{2^x}{\ln 2} + \frac{5^x}{\ln 5} + \text{constant}$

4. $\frac{1}{2}[x\sqrt{x^2+1} - \ln(\sqrt{x^2+1} + x)] + \text{constant}$