Real Analysis
Homework 4

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1. (Exercise 5.3)
   Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \leq f$ a.e. on E, show that $\int_E f_k \to \int_E f$.

   Proof.
   Since $f_k \to f$ a.e. and measurable in E, then f is also measurable.
   By Lebesgue Dominated Convergence Theorem for Nonnegative Functions, since $0 \leq f_k$, $f_k \leq f$ $\forall k$ with $\int_E f dx < +\infty$ and $f_k \to f$ a.e. in E, then $\int_E f_k(x)dx \to \int_E f(x)dx$.

2. (Exercise 5.4)
   If $f \in L(0,1)$, show that $x^k f(x)$ in $L(0,1)$ for $k = 1, 2, ...$, and that $\int_0^1 x^k f(x)dx \to 0$.

   Proof.
   Since $f \in L(0,1)$ and $x \in (0,1)$, then $|f|$ is also measurable and $|x^k f(x)| \leq |f(x)|$ in $(0,1)$.
   $x^k f(x) \to 0$ a.e. as $k \to 0$.
   By Lebesgue Dominated Convergence Theorem, since $x^k f(x) \to 0$ a.e. in $(0,1)$, $|x^k f(x)| \leq |f(x)|$ $\forall k$ and $|f|$ is also measurable, then $\int_{(0,1)} f_k(x)dx \to \int_{(0,1)} 0dx = 0$. ✔

3. (Exercise 5.5)
   Use Egorov's theorem to prove the bounded convergence theorem.

   Recall (Egorov's Theorem):
   Suppose that $\{f_k\}$ is a sequence of measurable functions that converges a.e. in a set E of finite measure to a finite limit f. Then given $\epsilon > 0$, there is a closed subset F of E such that $|E - F| < \epsilon$ and $\{f_k\}$ converge uniformly to F.

   Recall (Bounded Convergence Theorem):
   Let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If $|E| < +\infty$ and there is a finite constant M such that $|f_k| \leq M$ a.e. in E, then $\int_E f_k \to \int_E f$.

   Proof.
   By Egorov's theorem, for any ϵ, there exists a closed set $F \subseteq E$ such that $\{f_k\}$ converges uniformly on F and $|E - F| < \frac{M\epsilon}{4}$.

Since $|f_k| \le M$ a.e. and $M|E| < \infty$, by Fatou's lemma, we have

$$\begin{aligned}\int_F f &= \int_F \liminf_{k\to\infty} f_k \\ &\le \liminf_{k\to\infty} \int_F f_k \\ &\le \limsup_{k\to\infty} \int_F f_K \\ &\le \int_F \limsup_{k\to\infty} f_k \\ &= \int_F f\end{aligned}$$

Then $\int_F f_k \to \int_F f$.
There exists $N > 0$ such that for all $k \ge N$, we have $\left|\int_F f - \int_F f_k\right| < \frac{\epsilon}{2}$.
Hence, for $k \ge N$

$$\left|\int_E f - \int_E f_k\right| \le \left|\int_F f - \int_F f_k\right| + \left|\int_{E-F} f\right| + \left|\int_{E-F} f_k\right| < \epsilon$$

Then $\int_E f_k \to \int_E f$.

4. (Exercise 5.6)
Let $f(x,y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $(\partial f(x,y)/\partial x)$ is a bounded function of (x,y). Show that $(\partial f(x,y)/\partial x)$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)dy$$

Proof.

(a) $(\partial f(x,y)/\partial x)$ is a measurable function of y for each x:
By definition, we know for every x

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y) - f(x,y)}{h}$$

Since $f(x,y)$ is an integrable function of y for every x, $f(x,y)$ is measurable function of y for every x, then $\frac{\partial f(x,y)}{\partial x}$ is also measurable for every x.

(b)

$$\begin{aligned}\frac{d}{dx}\int_0^1 f(x,y)dy &= \lim_{h\to 0}\frac{\int_0^1 f(x+h,y)dy - \int_0^1 f(x,y)dy}{h} \\ &= \lim_{h\to 0}\int_0^1 \frac{f(x+h,y) - f(x,y)}{h}dy\end{aligned}$$

By Mean Value Theorem, there exists $0 < h' \le h$ such that

$$\frac{f(x+h,y) - f(x,y)}{h} = \frac{\partial}{\partial x} f(x+h',y)$$

which is a bounded function of (x,y), then by Bounded Convergence Theorem

$$\frac{d}{dx}\int_0^1 f(x,y)dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)dy$$

5. (Exercise 5.7)
Give an example of an f that is not integrable, but whose improper Riemann integral exists and is finite.

Proof.
Let f be a function on $[1,\infty)$ with $f(x) = (-1)^n\frac{1}{n}$ if $x \in [n, n+1)$ where $n \in \mathbb{Z}^+$, then

$$\int_{[1,\infty)} f^+ = \int_{[1,\infty)} \max\{f, 0\} = \sum_{k=1}^{\infty} \frac{1}{2k}|[2k, 2k+1)| = \infty$$

and

$$\int_{[1,\infty)} f^- = \int_{[1,\infty)} -\min\{f, 0\} = \sum_{k=1}^{\infty} \frac{1}{2k-1}|[2k-1, 2k)| = \infty$$

f is said to be integrable in $[1,\infty)$
$\iff \ |\int_[1,\infty)f(x)dx| = |\int_[1,\infty)f^+(x)dx - \int_[1,\infty)f^-(x)dx| < \infty.$

Since $\int_{[1,\infty)} f^+(x)dx = \infty$ and $\int_{[1,\infty)} f^-(x)dx = \infty$, hence, f is not integrable.

But

$$(R)\int_{[1,\infty)} f(x)dx = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n} < \infty$$

It implies that f is Riemann integrable.

6. (Exercise 5.9)
If $p > 0$ and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that $f_k \xrightarrow{m} f$ on E (and thus that there is a subsequence $f_{k_j} \to f$ a.e. in E).

Proof.
Let $\omega(\alpha) = |\{x \in E : f(x) > \alpha\}|$ where $\alpha > 0$.
We first need to prove that $\omega(\alpha) \leq \frac{1}{\alpha^p}\int_{\{f>\alpha\}} f^p(x)dx$.
Let $g(x) = \begin{cases} \alpha, & \text{if } f(x) > \alpha \\ 0, & \text{o.w.} \end{cases}$ Then

$$\int_{\{f>\alpha\}} f^p \geq \int_{\{f>\alpha\}} g^p = \int_{\{f>\alpha\}} \alpha^p = \alpha^p|\{f > \alpha\}| = \alpha^p\omega(\alpha)$$

Hence,

$$\omega(\alpha) \leq \frac{1}{\alpha^p}\int_{\{f>\alpha\}} f^p(x)dx$$

Now, we let

$$\omega'(\alpha) = |\{x \in E : |f(x) - f_k(x)|^p > \alpha\}|$$

By above, we then have

$$\omega'(\alpha) = |\{x \in E : |f(x) - f_k(x)|^p > \alpha\}| \leq \frac{1}{\alpha^p}\int_E |f - f_k|^p$$

That is

$$|\{x \in E : |f(x) - f_k(x)| > \alpha^{1/p}\}| \leq \frac{1}{\alpha^p}\int_E |f - f_k|^p$$

Hence,

$$0 \le \lim_{k\to\infty} |\{x \in E : |f(x) - f_k(x)| > \alpha^{1/p}\}| \le \frac{1}{\alpha^p} \cdot \lim_{k\to\infty} \int_E |f - f_k|^p = 0$$

Thus,

$$\lim_{k\to\infty} |\{x \in E : |f(x) - f_k(x)| > \alpha^{1/p}\}| = 0$$

Since $\alpha^{1/p}$ can be any positive real number, we have that $f_k \xrightarrow{m} f$.

7. (Exercise 5.10)
   If $p > 0$, $\int_E |f - f_k|^p \to 0$ and $\int_E |f_k|^p \le M$ for all k, show that $\int_E |f|^p \le M$.

   Proof.
   By Exercise 5.9, since $\int_E |f - f_k|^p \to 0$, $\forall p > 0$, then $f_k \xrightarrow{m} f$ on E.
   So we can find the subsequence $\{f_{k_j}\}$ such that $f_{k_j} \to f$ a.e. in E.
   Then $|f_{k_j}|^p \to |f|^p$ a.e. in E.
   By Fatou's Lemma, we have

$$\int_E |f|^p = \int_E \liminf_{j\to\infty} |f_{k_j}|^p \le \liminf_{j\to\infty} \int_E |f_{k_j}|^p \le \liminf_{j\to\infty} M = M$$

8. (Exercise 5.13)

   (a) Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum f_k$ converges absolutely a.e. in E if $\sum \int_E |f_k| < +\infty$. (Use Theorem 5.16 and 5.22.)

   (b) If $\{r_k\}$ denotes the rational numbers in $[0, 1]$ and $\{a_k\}$ satisfies $\sum |a_k| < +\infty$, show that $\sum a_k |x - r_k|^{-1/2}$ converges absolutely a.e. in $[0, 1]$.

   Recall (Theorem 5.16):
   If f_k, $k = 1, 2, ...$, are nonnegative and measurable, then

$$\int_E \left(\sum_{k=1}^{\infty} f_k \right) = \sum_{k=1}^{\infty} \int_E f_k$$

   Recall (Theorem 5.22):
   If $f \in L(E)$, then f is finite a.e. in E.

   Proof.

   (a) If $\int_E |\sum f_k| < \infty$, then $\sum f_k$ converges absolutely a.e. in E.

$$\int_E \left|\sum f_k\right| = \int_E \sum |f_k|$$

   $|f_k|$ is measurable on E, since f_k is measurable on E.
   By Theorem 5.16, since $|f_k| \ge 0$ and measurable on E, then

$$\int_E \left|\sum_{k=1}^{\infty} f_k\right| = \int_E \sum_{k=1}^{\infty} |f_k| = \sum_{k=1}^{\infty} \int_E |f_k| < +\infty$$

   Hence, $\sum f_k$ converges absolutely a.e. in E.

(b) If $\int_{[0,1]} \left|\sum a_k|x-r_k|^{-1/2}\right| dx < \infty$, then $\sum a_k|x-r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$.

$$\begin{aligned}
\int_{[0,1]} \left|\sum a_k|x-r_k|^{-1/2}\right| dx &\leq \int_{[0,1]} \sum |a_k||x-r_k|^{-1/2} dx \\
&= \sum \int_{[0,1]} |a_k||x-r_k|^{-1/2} dx \\
&= \sum |a_k| \int_{[0,1]} |x-r_k|^{-1/2} dx \\
&= \sum |a_k|(2r_k^{1/2} + 2(1-r_k)^{1/2}) dx \\
&< \infty
\end{aligned}$$

Hence, $\sum a_k|x-r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$.