Calculus 2 Module-07

NTU 108-1 MATH4007 QUIZ 4 19TH December, 2019

Work out **ALL** questions below. Provide sufficient justification to every step of your arguments. Write your solutions as well as your ID number clearly on the answer sheet.

Time: 17:50 ~ 18:20.

. DEPARTMENT:________________ ID NUMBER:________________ NAME:________________

1. Evaluate the following integrals.

   (a) (15 pts) $\int 2x \arctan x \, dx$.

   Let $u = \arctan x$ and $dv = 2x\,dx$, then

$$du = \frac{1}{1+x^2}\,dx \qquad \text{and} \qquad v = x^2.$$

(Substitution: 3 points)

   Therefore

$$\begin{aligned}
\int 2x \arctan x \, dx = \int u\,dv = uv - \int v\,du &= x^2 \arctan x - \int \frac{x^2}{1+x^2}\,dx \\
&= x^2 \arctan x - \int \frac{(1+x^2)-1}{1+x^2}\,dx \\
&= x^2 \arctan x - \int dx + \int \frac{1}{1+x^2}\,dx \\
&= x^2 \arctan x - x + \arctan x + C \\
&= (x^2+1)\arctan x - x + C, \quad C \in \mathbb{R}
\end{aligned}$$

(Compute the indefinite integral + Answer: 10 + 2 points)

   (b) (20 pts) $\int_0^{\frac{\pi}{2}} \left|\cos^2 x - 3\sin^2 x\right| \, dx$.

   We know that

$$\cos^2 x = \frac{1+\cos 2x}{2}, \ \sin^2 x = \frac{1-\cos 2x}{2}$$

   The indefinite integral can then be computed to obtain

$$\begin{aligned}
\int \cos^2 x - 3\sin^2 x \, dx &= \int \frac{1+\cos 2x}{2} - \frac{3-3\cos 2x}{2}\,dx \\
&= \int -1 + 2\cos 2x \, dx \\
&= -x + \sin 2x + C, \quad C \in \mathbb{R}
\end{aligned}$$

(Compute the indefinite integral: 10 points)

For $x \in [0, \frac{\pi}{2}]$, we have

$$\cos^2 x - 3\sin^2 x \geq 0 \iff 0 \leq x \leq \frac{\pi}{6}$$

(Find the positive or negative inteval : 1 point)

Then

$$\int_0^{\frac{\pi}{2}} \left|\cos^2 x - 3\sin^2 x\right| dx$$

$$= \int_0^{\frac{\pi}{6}} \cos^2 x - 3\sin^2 x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 x \, - \cos^2 x \, dx$$

. (Divided the definite integral into two parts : 2 points)

$$= (-x + \sin 2x)|_0^{\frac{\pi}{6}} + (x - \sin 2x)|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

. (Correcet indefinite integral : 5 points)

$$= \frac{\pi}{6} + \sqrt{3}$$

(Answer : 2 points)

(c) (30 pts) $\int \frac{1}{e^{2x}+e^x+1} \, dx$.

Let $t = e^x \Rightarrow x = \ln t \Rightarrow dx = \frac{dt}{t}$. (Substitution rule : 2 points)

The integral becomes

$$\int \frac{1}{e^{2x} + e^x + 1} \, dx = \int \frac{1}{t^2 + t + 1} \cdot \frac{dt}{t}$$

Suppose that

$$\frac{1}{t(t^2 + t + 1)} = \frac{A}{t} + \frac{P(t)}{t^2 + t + 1} \;\Rightarrow\; A(t^2 + t + 1) + P(t) \cdot t = 1$$

Let $t = 0 \;\Rightarrow\; A = 1$, and then

$$P(t) \cdot t = -t^2 - t \;\Rightarrow\; P(t) = -t - 1$$

(Compute A and $P(t)$: 3 + 5 points)

Consider the integral

$$\begin{aligned}
\int \frac{1}{t^2 + t + 1} \cdot \frac{dt}{t} &= \int \frac{1}{t} \, dt + \int \frac{-t - 1}{t^2 + t + 1} \, dt \\
&= \int \frac{1}{t} \, dt + \left(\frac{-1}{2}\right) \int \frac{2t + 1}{t^2 + t + 1} \, dt + \int \frac{-1/2}{t^2 + t + 1} \, dt \\
&= \ln|t| - \frac{1}{2} \ln|t^2 + t + 1| + \int \frac{-1/2}{t^2 + t + 1} dt
\end{aligned}$$

(Compute the indefinite integral to this step: 10 points)

The thrid term can be computed as

$$\int \frac{-1/2}{t^2+t+1}\,dt = \left(\frac{-1}{2}\right)\int \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}} = \frac{-1}{\sqrt{3}}\arctan(\frac{2t+1}{\sqrt{3}})$$

(Compute the third term: 8 points)

So, the integral in question is given by

$$\begin{aligned}
&\ln|t| - \frac{1}{2}\ln|t^2+t+1| + \int \frac{-1/2}{t^2+t+1}\,dt \\
&= \ln|t| - \frac{1}{2}\ln|t^2+t+1| - \frac{1}{\sqrt{3}}\arctan(\frac{2t+1}{\sqrt{3}}) + C \\
&= x - \frac{1}{2}\ln|e^{2x}+e^x+1| - \frac{1}{\sqrt{3}}\arctan(\frac{2e^x+1}{\sqrt{3}}) + C,
\end{aligned}$$

where C is a constant

(Answer: 2 points)

2. Determine whether the following improper integrals are convergent or divergent. If convergent, please determine its value.

   (a) (17 pts) $\int_0^1 \frac{\cos t}{t^{4/3}}\,dt$.

   Since $0 = \cos\frac{\pi}{2} < \cos 1 \le \cos x \le \cos 0 = 1$ when $x \in [0,1]$, then

$$\frac{\cos t}{t^{4/3}} > \frac{\cos 1}{t^{4/3}}$$

(Inequality: 10 points)

   However $\int_0^1 t^{\frac{-4}{3}}\,dt$ is divergent, thus

$$\int_0^1 \frac{\cos t}{t^{4/3}}\,dt \quad \text{is also divergent.}$$

(Comparison Test + Answer: 5 + 2 points)

   (b) (18 pts) $\int_1^\infty \frac{\arctan x}{x^2}\,dx$.

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\arctan x}{x^2}\,dx$$

(Improper integral: 3 points)

Let $u = \arctan x$ and $dv = \frac{dx}{x^2}$, then $du = \frac{1}{1+x^2}$ and $v = \frac{-1}{x}$. Therefore,

$$\begin{aligned}
\int \frac{\arctan x}{x^2}\,dx = \int u\,dv = uv - \int v\,du \\
&= \frac{-\arctan x}{x} + \int \frac{1}{x}\frac{dx}{1+x^2} \\
&= \frac{-\arctan x}{x} + \int \left[\frac{1}{x} - \frac{x}{1+x^2}\right] dx \\
&= \frac{-\arctan x}{x} + \ln|x| - \frac{1}{2}\ln(x^2+1) + C \\
&= \frac{-\arctan x}{x} + \frac{1}{2}\ln\frac{x^2}{1+x^2} + C, \quad C \in \mathbb{R}
\end{aligned}$$

(Compute the indefinite integral: 10 points)

Thus,

$$\begin{aligned}
\int_1^\infty \frac{\arctan x}{x^2}\,dx &= \lim_{t\to\infty}\left[-\frac{\arctan x}{x} + \frac{1}{2}\ln\frac{x^2}{1+x^2}\right]_{x=1}^{t} \\
&= \lim_{t\to\infty}\left[-\frac{\arctan t}{t} + \frac{1}{2}\ln\frac{t^2}{1+t^2} + \frac{\pi}{4} - \frac{1}{2}\ln\frac{1}{2}\right] \\
&= \frac{\pi}{4} + \frac{\ln 2}{2}
\end{aligned}$$

(Answer: 5 points)

The End.