> Please write down your solutions on a separate sheet of paper and submit it to your TA or instructor.
> Submit your solutions to Problems (1) $\sim$ (5) on 16th **November, 2018**.
> Submit your solutions to Problems (6) $\sim$ (9) on 21th **November, 2018**.
> The rest are left for your self-revision.

1. (6 pts) Evaluate $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+...+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)$.

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+...+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)\\
&=\lim_{n\to\infty}\left(\frac{1}{n\sqrt{1+1/n}}+\frac{1}{n\sqrt{1+2/n}}+...+\frac{1}{n\sqrt{1+n/n}}\right)\\
&=\lim_{n\to\infty}\left(\frac{1}{\sqrt{1+1/n}}+\frac{1}{\sqrt{1+2/n}}+...+\frac{1}{\sqrt{1+n/n}}\right)\frac{1}{n}\\
&=\int_0^1\frac{1}{\sqrt{1+x}}dx\\
&=\left[2(1+x)^{1/2}\right]_{x=0}^{x=1}\\
&=2(\sqrt{2}-1)
\end{aligned}
$$

(Identifying the Riemann sum + Evaluation of integral + Answer : 3 + 2 + 1 points)

2. (5 pts) Evaluate the integral by interpreting it in terms of areas.

$$\int_{-3}^{0}(1+\sqrt{9-x^2})dx$$

$\int_{-3}^{0}(1+\sqrt{9-x^2})dx$ can be interpreted as the area under the graph of $f(x)=1+\sqrt{9-x^2}$ between $x=-3$ and $x=0$. (Interpreted it as the correct area : 2 points)
This is equal to one-quarter the area of circle with radius 3, plus the area of rectangle, so

$$\int_{-3}^{0}(1+\sqrt{9-x^2})dx=\frac{1}{4}\pi\cdot 3^2+1\cdot 3=3+\frac{9}{4}\pi$$

(Computing the intergal (Area) + Answer : 2 + 1 points)

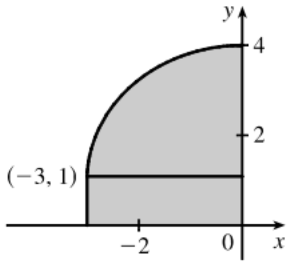


Figure 1: $f(x) = 1 + \sqrt{9 - x^2}$

3. Evaluate the integral.

   (a) (4 pts) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc^2\theta d\theta$

   (b) (4 pts) $\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{4}{\sqrt{1-x^2}} dx$

   (c) (5 pts) $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

(a)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \csc^2\theta d\theta = -\cot\theta\Big|_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} = (-\frac{1}{\sqrt{3}}) - (-1) = 1 - \frac{1}{\sqrt{3}}$$

(Computing the intergal + Answer : 3 + 1 points)

(b)

$$\int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{4}{\sqrt{1-x^2}} dx = 4\arcsin x\Big|_{x=1/2}^{1/\sqrt{2}} = 4(\frac{\pi}{4} - \frac{\pi}{6}) = \frac{\pi}{3}$$

(Computing the intergal + Answer : 3 + 1 points)

(c) Let $u = \ln x$, then $du = \frac{dx}{x}$. When $x = e^4$, $u = 4$; when $x = e$, $u = 1$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_e^{e^4} \frac{1}{\sqrt{\ln x}} \cdot \frac{dx}{x} = \int_1^4 \frac{1}{\sqrt{u}} du = 2u^{\frac{1}{2}}\Big|_{u=1}^{4} = 4 - 2 = 2$$

(Substitution rule + Computing the intergal + Answer : 1 + 3 + 1 points)

4. Find the following values.

   (a) (7 pts) If $x\sin(\pi x) = \int_0^{x^2} f(t)dt$, where f is a continuous function, find $f(4)$. (There is no misprint here.)

   (b) (4 pts) If $f(x) = \int_0^x x^2 \sin(t^2)dt$, find $f'(x)$.

(c) (5 pts) If $\int_0^4 e^{(x-2)^4}dx = k$, find the value of $\int_0^4 xe^{(x-2)^4}dx$.
(Hint: consider the transformation $u := x - 2$.)

(a) To solve for $f(4)$, we have to get the function f out of the integral sign, which can be done via differentiating the equation with respect to x. By the Fundamental Theorem of Calculus, we have

$$\frac{d}{dx}(x\sin(\pi x)) = \frac{d}{dx}\left(\int_0^{x^2} f(t)dt\right) \Rightarrow \sin(\pi x) + \pi x\cos(\pi x) = 2xf(x^2)$$
$$\Rightarrow f(x^2) = \frac{\sin(\pi x) + \pi x\cos(\pi x)}{2x}$$

Plugging $x = 2$ or $x = -2$ into $f(x^2)$, we get

$$\begin{aligned} f(4) &= \frac{\sin(2\pi) + 2\pi\cos(2\pi)}{4} \quad \text{or} \quad \frac{\sin(-2\pi) - 2\pi\cos(-2\pi)}{-4} \\ &= \frac{0 + 2\pi}{4} \quad \text{or} \quad \frac{0 - 2\pi}{-4} \\ &= \frac{\pi}{2} \end{aligned}$$

($\frac{d}{dx}(x\sin(\pi x))$ + $f(4)$: 5 + 2 points)

(b) We can move x^2 out of the integral sign since it is independent of the variable t of integration and can be treated like a constant.
Hence
$$f(x) = x^2\int_0^x \sin(t^2)dt$$
By the Fundamental Theorem of Calculus, we have
$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^2\int_0^x \sin(t^2)dt\right) \\ &= 2x\int_0^x \sin(t^2)dt + x^2\sin(x^2) \end{aligned}$$

(Using the Fundamental Theorem of Calculus + Answer : 3 + 1 points)

(c) For $I = \int_0^4 xe^{(x-2)^4}dx$, let $u = x - 2$ so that $x = u + 2$ and $dx = du$.
Then
$$I = \int_{-2}^2 (u+2)e^{u^4}du = \int_{-2}^2 ue^{u^4}du + \int_{-2}^2 2e^{u^4}du = 0 + 2\int_0^4 e^{(x-2)^4}dx = 2k$$

($\int_{-2}^2 ue^{u^4}du = 0$ because ue^{u^4} is an odd function and $[-2, 2]$ is an interval symmetric about 0.)
(Substitution rule + Computing I + Answer : 1 + 3 + 1 points)

5. Find the general indefinite integral.

   (a) (5 pts) $\int \frac{1+x}{1+x^2} dx$

   (b) (4 pts) $\int (2 + \tan^2 \theta) d\theta$

   (c) (5 pts) $\int \frac{\cos(\ln t)}{t} dt$

   (a) Let $u = 1 + x^2$, then $du = 2x dx$. Thus,

$$\int \frac{1+x}{1+x^2} dx = \int \frac{1}{1+x^2} dx + \int \frac{x}{1+x^2} dx = \tan^{-1} x + C_1 + \int \frac{\frac{1}{2} du}{u}$$

$$= \tan^{-1} x + C_1 + \frac{1}{2} \ln|u| + C_2 = \tan^{-1} x + \frac{1}{2} \ln|1 + x^2| + C$$

where C, C_1 and C_2 are constants

(Substitution rule + Computing the integral + Answer : 1 + 3 + 1 points)

   (b)

$$\int (2 + \tan^2 \theta) d\theta = \int 2 + (\sec^2 \theta - 1) d\theta = \int (1 + \sec^2 \theta) d\theta = \theta + \tan \theta + C$$

where C is a constant

(Computing the integral + Answer : 3 + 1 points)

   (c) Let $u = \ln t$, then $du = \frac{dt}{t}$. Thus,

$$\int \frac{\cos(\ln t)}{t} dt = \int (\cos u) du = \sin u + C = \sin(\ln t) + C \quad \textit{where } C \textit{ is a constant}$$

(Substitution rule + Computing the integral + Answer : 1 + 3 + 1 points)

6. Find the area of the regions bounded by the given curves.

   (a) (7 pts) $y = \sqrt{x}$, $y = -\sqrt[3]{x}$, $y = x - 2$.

   (b) (7 pts) $y = 1/x$, $y = x^2$, $y = 0$, $x = e$.

   (c) (5 pts) $y = \sqrt{x}$, $y = x^2$, $x = 2$.

(a) The line $y = x - 2$ intersects the curve $y = \sqrt{x}$ at $(4, 2)$ and it intersects the curve $y = -\sqrt[3]{x}$ at $(1, -1)$. (Intersection points : 2 points)

$$\begin{aligned}
A &= \int_0^1 \left[\sqrt{x} - (-\sqrt[3]{x})\right] dx + \int_1^4 \left[\sqrt{x} - (x-2)\right] dx \\
&= \left[\frac{2}{3}x^{3/2} + \frac{3}{4}x^{4/3}\right]_{x=0}^{1} + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_{x=1}^{4} \\
&= (\frac{2}{3} + \frac{3}{4}) - 0 + (\frac{16}{3} - 8 + 8) - (\frac{2}{3} - \frac{1}{2} + 2) \\
&= \frac{55}{12}
\end{aligned}$$

Or, integrating with respect to y : $A = \int_{-1}^{0} \left[(y+2) - (-y^3)\right] dy + \int_0^2 \left[(y+2) - y^2\right] dy = \frac{55}{12}$. (Computing the area + Answer : 4 + 1 points)

(b) The curves $y = 1/x$ and $y = x^2$ intersect when $1/x = x^2 \iff x^3 = 1 \iff x = 1$. . (Intersection points : 2 points)

$$\begin{aligned}
A &= \int_0^1 (x^2 - 0)dx + \int_1^e \left(\frac{1}{x} - 0\right) dx \\
&= \left[\frac{1}{3}x^3\right]_{x=0}^{1} + \left[\ln|x|\right]_{x=1}^{e} \\
&= \frac{4}{3}
\end{aligned}$$

(Computing the area + Answer : 4 + 1 points)

(c)

$$\begin{aligned}
A &= \int_0^1 \left(\sqrt{x} - x^2\right) dx + \int_1^2 \left(x^2 - \sqrt{x}\right) dx \\
&= \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_{x=0}^{1} + \left[\frac{1}{3}x^3 - \frac{2}{3}x^{3/2}\right]_{x=1}^{2} \\
&= \frac{10}{3} - \frac{4}{3}\sqrt{2}
\end{aligned}$$

(Area over $x \in (0, 1)$ + Area over $x \in (1, 2)$ + Answer : 2 + 2 + 1 points)

7. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

   (a) (6 pts) $x = 1 + y^2$, $y = x - 3$; about the y-axis.

   (b) (4 pts) $x = 0$, $x = 9 - y^2$; about $x = -1$.

   (c) (7 pts) $x^2 - y^2 = a^2$, $x = a + h$ (where $a > 0$, $h > 0$); about the y-axis.

(a)

$$\begin{aligned} 1+y^2 = y+3 &\iff y^2 - y - 2 = 0 \\ &\iff (y-2)(y+1) = 0 \\ &\iff y = 2 \ \text{ or } \ -1. \end{aligned}$$

(Intersection points : 2 points)

Hence

$$\begin{aligned} V &= \pi \int_{-1}^{2} \left[(y+3)^2 - (1+y^2)^2\right] dy \\ &= \pi \int_{-1}^{2} (8 + 6y - y^2 - y^4) dy \\ &= \pi \left[8y + 3y^2 - \frac{1}{3}y^3 - \frac{1}{5}y^5\right]_{x=-1}^{2} \\ &= \frac{117}{5}\pi \end{aligned}$$

(Computing the volume + Answer : 3 + 1 points)

(b)

$$\begin{aligned} V &= \pi \int_{-3}^{3} \left\{[(9-y^2) - (-1)]^2 - [0 - (-1)]^2\right\} dy \\ &= 2\pi \int_{0}^{3} (99 - 20y^2 + y^4) dy \\ &= 2\pi \left[99y - \frac{20}{3}y^3 + \frac{1}{5}y^5\right]_{x=0}^{3} \\ &= \frac{1656}{5}\pi \end{aligned}$$

(Computing the volume + Answer : 3 + 1 points)

(c) The graph of $x^2 - y^2 = a^2$ is a hyperbola with right and left branches (but only the right branch is considered).
Solving for y gives us $y^2 = x^2 - a^2 \Rightarrow y = \pm\sqrt{x^2 - a^2}$.
We use shell method. The height of each shell is $\sqrt{x^2-a^2} - (-\sqrt{x^2-a^2}) = 2\sqrt{x^2-a^2}$.
The volume is $V = \int_a^{a+h} 2\pi x \cdot 2\sqrt{x^2-a^2}dx$.
. (Find the volume V : 3 points)

To evaluate, let $u = x^2 - a^2$, so $du = 2xdx$ and $xdx = \frac{1}{2}du$.
. (Substitution Rule : 1 point)

When $x = a$, $u = 0$, and when $x = a+h$, $u = (a+h)^2 - a^2 = a^2 + 2ah + h^2 - a^2 = 2ah + h^2$.
Thus,

$$V = 4\pi \int_0^{2ah+h^2} \sqrt{u}\left(\frac{1}{2}du\right) = 2\pi\left[\frac{2}{3}u^{3/2}\right]_{x=0}^{2ah+h^2} = \frac{4}{3}\pi(2ah+h^2)^{3/2}.$$

(Computing the volume + Answer : 2 + 1 points)

8. (a) (4 pts) Find the average value of the function $f(x) = 1/\sqrt{x}$ on the interval $[1, 4]$.

   (b) (3 pts) Find the value c guaranteed by the Mean Value Theorem for Integrals such that $f_{ave} = f(c)$, where $f(x) = 1/\sqrt{x}$.

   (c) (5 pts) If f is a continuous function, what is the limit as $h \to 0$ of the average value of f on the interval $[x, x+h]$?

(a)

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{4-1}\int_1^4 \frac{1}{\sqrt{x}}dx = \frac{1}{3}\int_1^4 x^{-1/2}dx = \frac{1}{3}\left[2\sqrt{x}\right]_{x=1}^4 = \frac{2}{3}$$

(Computing f_{ave} + Answer : 3 + 1 points)

(b)

$$f(c) = f_{ave} \iff \frac{1}{\sqrt{c}} = \frac{2}{3} \iff c = \frac{9}{4}$$

(Computing the value c : 3 points)

(c)

$$\lim_{h\to 0} f_{ave} = \lim_{h\to 0}\frac{1}{(x+h)-x}\int_x^{x+h} f(t)dt = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$$

where $F(x) = \int_a^x f(t)dt$. (Limit : 1 point)

But we recognize that this limit is $F'(x)$ by the definition of derivatives.
. (The definition of derivatives : 2 points)

Therefore, by the Fundamental Theorem of Calculus, we have

$$\lim_{h\to 0} f_{ave} = F'(x) = f(x)$$

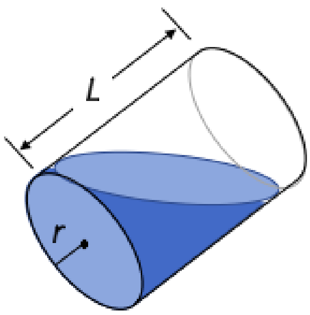
(Answer : 2 points)

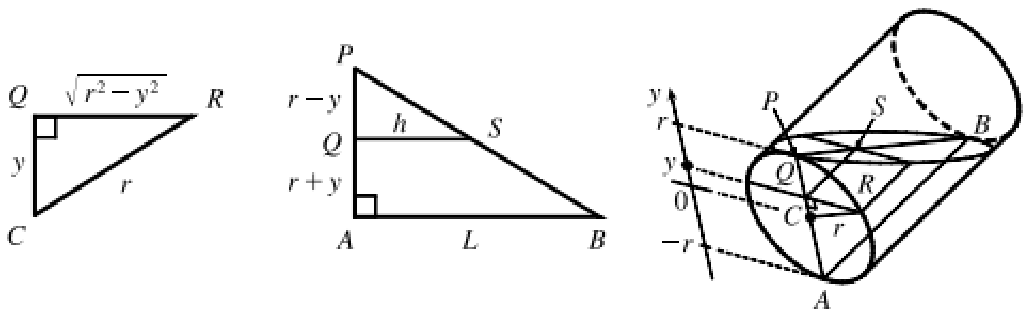
9. A cylindrical glass of radius r and height L is fully filled with water. It is then tilted to let the water flow out until the water remaining in the glass exactly covers its base.

   (a) (8 pts) Find the volume of the water in the glass using calculus.

   (b) (3 pts) Find the volume of the water in the glass from purely geometric consideration.

   (c) (7 pts) Suppose the glass is tilted until the water exactly covers half the base. Find the volume of the water in the glass.





(a) i. If we take slices perpendicular to the line through the center C of the bottom of the glass and the point P where the top surface of the water meets the bottom of the glass.

A typical rectangular cross-section y units above the axis of the glass has width $2|QR| = 2\sqrt{r^2-y^2}$ and length $h = |QS| = \frac{L}{2r}(r-y)$. [Triangles PQS and PAB are similar, so $\frac{h}{L} = \frac{|PQ|}{|PA|} = \frac{r-y}{2r}$.] (Slice area : 4 points)

Thus,

$$\begin{aligned} V &= \int_{-r}^{r} 2\sqrt{r^2-y^2}\cdot\frac{L}{2r}(r-y)dy \\ &= L\int_{-r}^{r}(1-\frac{y}{r})\sqrt{r^2-y^2}dy \\ &= \frac{\pi r^2 L}{2} \end{aligned}$$

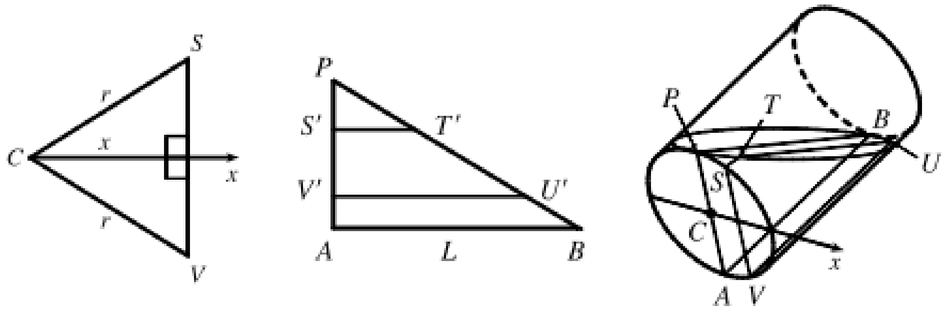
Note:
1. The integral $\int_{-r}^{r}\sqrt{r^2-y^2}dy$ can be intepreted as the area of the semidisc bounded by $x=\sqrt{r^2-y^2}$ and the y-axis.

2. The integral $\int_{-r}^{r} y\sqrt{r^2-y^2}dy = 0$ because the integrand is odd and the interval $[-r,r]$ is symmetric above 0.
. (Computing the volume + Answer : 3 + 1 points)

ii. If we slice parallel to the plane through the axis of the glass and the point of contact P. (This is the plane determined by P, B, and C in the figure.)

Cont.

$STUV$ is a typical trapezoidal slice.
With the respect to the x-axis with origin at C as shown, if S and V have x-coordinate x, then $|SV| = 2\sqrt{r^2 - x^2}$.
Projecting the trapezoid $STUV$ onto the plane of the triangle PAB (call the projection $S'T'U'V'$), we see that $|AP| = 2r$, $|SV| = 2\sqrt{r^2 - x^2}$, and
$|S'P| = |V'A| = \frac{1}{2}(|AP| - |SV|) = r - \sqrt{r^2 - x^2}$.



By similar triangles, $\frac{|ST|}{|S'P|} = \frac{|AB|}{|AP|}$, so $|ST| = (r - \sqrt{r^2 - x^2}) \cdot \frac{L}{2r}$.
In the same way, we find that $\frac{|VU|}{|V'P|} = \frac{|AB|}{|AP|}$, so $|VU| = |V'P| \cdot \frac{L}{2r} = (|AP| - |V'A|) \cdot \frac{L}{2r} = (r + \sqrt{r^2 - x^2}) \cdot \frac{L}{2r}$.
The area $A(x)$ of the trapezoid $STUV$ is $\frac{1}{2}|SV| \cdot (|ST| + |VU|)$; that is,

$$A(x) = \frac{1}{2} \cdot 2\sqrt{r^2 - x^2} \cdot \left[(r - \sqrt{r^2 - x^2}) \cdot \frac{L}{2r} + (r + \sqrt{r^2 - x^2}) \cdot \frac{L}{2r}\right] = L\sqrt{r^2 - x^2}$$

($A(x)$: 4 points)

Thus,

$$V = \int_{-r}^{r} A(x)dx = L\int_{-r}^{r} \sqrt{r^2 - x^2}dx = \frac{\pi r^2 L}{2}$$

Note:
The integral $\int_{-r}^{r} \sqrt{r^2 - y^2}dy$ can be intepreted as the area of the semidisc bounded by $x = \sqrt{r^2 - y^2}$ and the y-axis. (Computing the volume + Answer : 3 + 1 points)

(b) The volume of the water is exactly half the volume of the cylindrical glass, so $V = \frac{1}{2}\pi r^2 L$.
. (Computing the volume + Answer : 2 + 1 points)

(c) Choose x-, y-, and z-axes as shown in the figure.
Then slices perpendicular to the x-axis are triangular, slices perpendicular to the y-axis are rectangular, and slices perpendicular to the z-axis are segments of circles.
Using triangular slices, we find that the area $A(x)$ of a typical slice DEF, where D has x-coordinate x, is given by

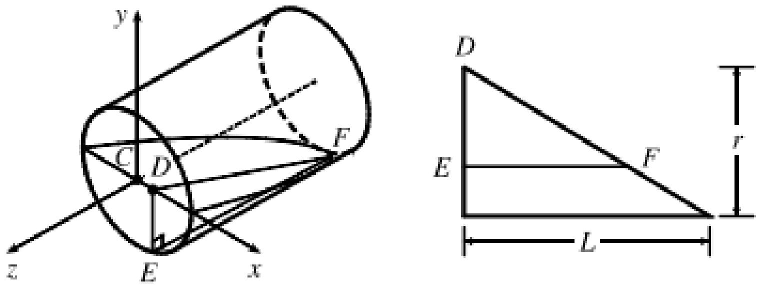
$$A(x) = \frac{1}{2}|DE| \cdot |EF| = \frac{1}{2}|DE| \cdot \left(\frac{L}{r}|DE|\right) = \frac{L}{2r}|DE|^2 = \frac{L}{2r}(r^2 - x^2)$$

($A(x)$: 2 points)

Thus,

$$V = \int_{-r}^{r} A(x)dx = \frac{L}{2r}\int_{-r}^{r}(r^2 - x^2)dx = \frac{L}{r}\int_{0}^{r}(r^2 - x^2)dx = \frac{L}{r}\left[r^2x - \frac{x^3}{3}\right]_{x=0}^{r} = \frac{2}{3}r^2L$$

(Computing the volume + Answer : 4 + 1 points)



The End.