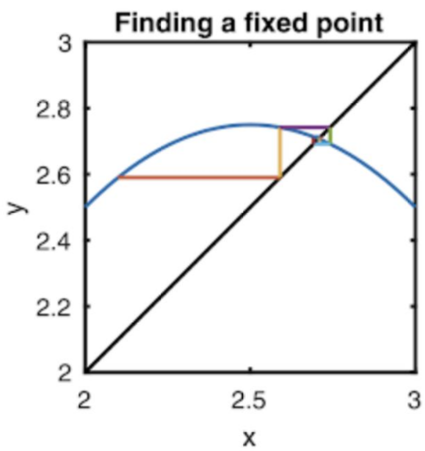
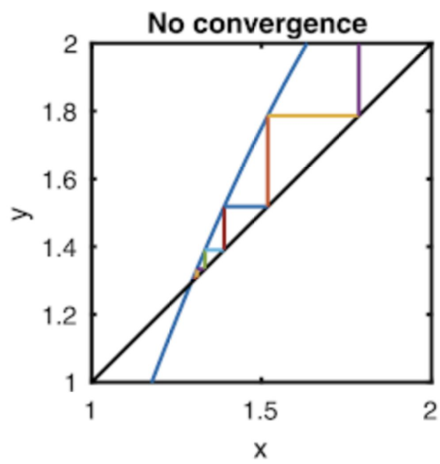
Introduction to Computational Mathematics, Final (2019/1/7)

1. [20 points (10+10)] Let $f(x) = x^2 - 4x + 3.5$ and $g(x) = x - f(x)$.
(a) By using the initial value x=2.1 and x=1.3, the fixed point iteration spirals into a root of $f(x)$ and pushes the iterates away from the root, respectively. Explain these two observations as shown below by referring to the series analysis.
(b) Show that if $g(x) = (x^2 + 3.5)/4$, then any fixed point of $g(x)$ is a root of $f(x)$.

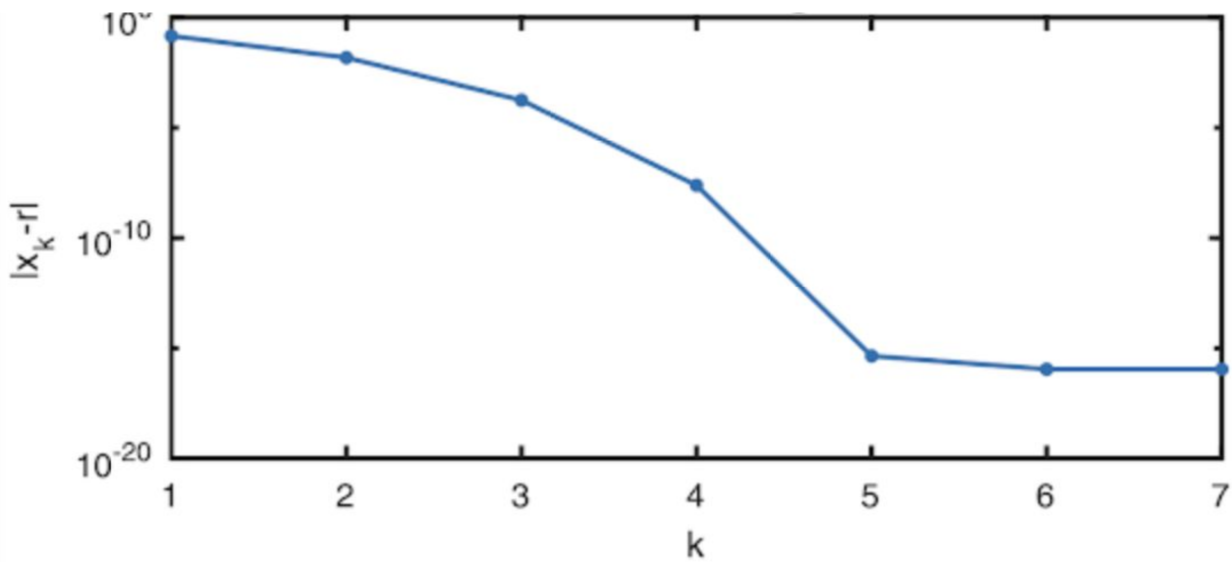


By using initial x=2.1.



By using initial x=1.3.

2. [10 points] Suppose we use an iterative method to find a solution of $f(x) = xe^x$ near the root r=1. The following figure shows the absolute error versus the iteration number. What is the rate of convergence of this method? Why?



3. [15 points (6+9)]

(a) Derive the one-dimensional secant method for finding a root.

(b) Show that the secant method converges in one step for a linear function analytically, regardless of the initialization.

4. [20 points (5+10+5)]

(a) Explain why the Broyden update formula (shown below) is derived. That is, state the motivation for deriving the Broyden update.

$$\boldsymbol{A}_{k+1} = \boldsymbol{A}_k + \frac{1}{\boldsymbol{s}_k^T \boldsymbol{s}_k}(\boldsymbol{f}_{k+1} - \boldsymbol{f}_k - \boldsymbol{A}_k \boldsymbol{s}_k)\boldsymbol{s}_k^T.$$

(b) Explain how the Broyden update formula can be used to solve a nonlinear system.

(c) Explain how you can apply the Sherman-Morrison formula (shown below) while performing the Broyden update.

$$(A + uv^T)^{-1} = A^{-1} - A^{-1}\frac{uv^T}{1 + v^T A^{-1} u}A^{-1}$$

5. [15 points] Let $\boldsymbol{f}(x) = \left[x - 8,\ x^2 - 4\right]^T$. Find the estimate produced by one step of the Gauss-Newton method that starts at $x = 2$.

6. [15 points] Write out the entries of the matrix and right-hand side of the linear system that determines the coefficient for the cubic not-a-knot spline interpolant of the function $cos(\pi^2 x^2)$ and node vector $\boldsymbol{t} = [-1,\ 1,\ 4]^T$.

7. [10 points] Hand-in your one-page CV.

End of the written problems. May you have a nice winter vacation!