Calculus 1 10/24 Note
Module Class 07

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Section 4.3: How Derivatives Affect the Shape of Graph

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

(c) If $f'(x) = 0$ on an interval, then f is constant on that interval.

The First Derivative Test

Suppose that c us a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

Example:

1. Determine all intervals where the following function is increasing or decreasing.

$$f(x) = -x^5 + \frac{5}{2}x^4 + \frac{40}{3}x^3 + 5$$

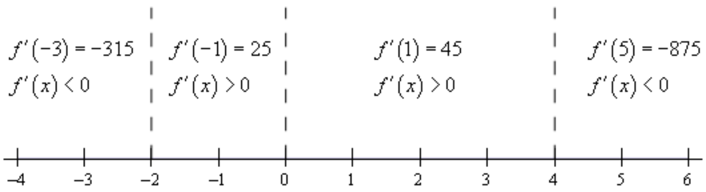
Sol.

To determine if the function is increasing or decreasing we will need the derivative.

$$f'(x) = -5x^4 + 10x^3 + 40x^2 = -5x^2(x-4)(x+2)$$

From the factored form of the derivative we see that we have three critical points

$$x = -2,\ x = 0,\ x = 4$$



Notice that the only place that the derivative may change signs is at the critical points of the function.

So we'll build a number line, graph the critical points and pick test points from each region to see if the derivative is positive or negative in each region.

Here is the number line and the test points for the derivative.

Hence, we'll get the following intervals of increase and decrease.

$$\begin{aligned} &\text{Increase:} \quad -2 < x < 0 \quad \text{and} \quad 0 < x < 4 \\ &\text{Decrease:} \quad -\infty < x < -2 \quad \text{and} \quad 4 < x < \infty \end{aligned}$$

2. Find and classify all the critical points of the following function. Give the intervals where the function is increasing and decreasing.

$$g(t) = t\sqrt[3]{t^2 - 4}$$

Sol.

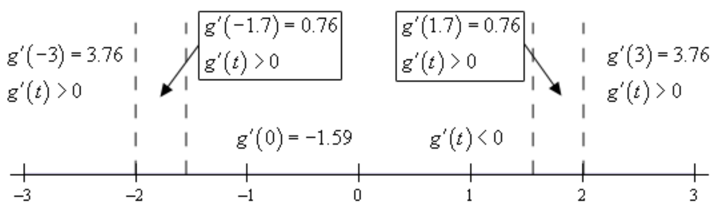
First, we'll need the derivative so we can get our hands on the critical points. Note as well that we'll do some simplification on the derivative to help us find the critical points.

$$\begin{aligned} g'(t) &= (t^2 - 4)^{\frac{1}{3}} + \frac{2}{3}t^2(t^2 - 4)^{\frac{-2}{3}} \qquad \text{(Chain Rule)} \\ &= \frac{5t^2 - 12}{3(t^2 - 4)^{\frac{2}{3}}} \end{aligned}$$

So we'll have four critical points here.

$$\begin{aligned} &t = \pm 2 && \text{The derivative doesn't exist here.} \\ &t = \pm\sqrt{\frac{12}{5}} && \text{The derivative is zero here.} \end{aligned}$$

Finding the intervals of increasing and decreasing will also give the classification of the critical points so let's get those first. Here is a number line with the critical points graphed and test points.



Hence, we'll get the following intervals of increasing and decreasing.

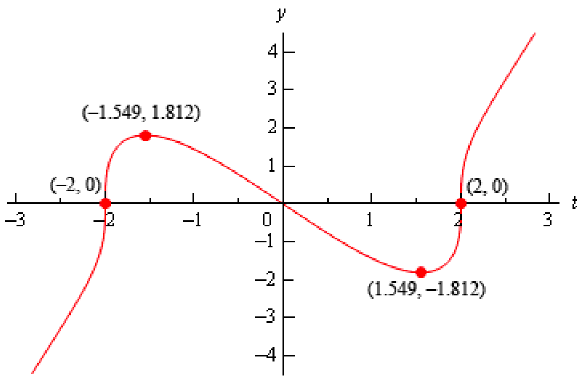
Increase: $-\infty < x < -2,\ -2 < x < -\sqrt{\dfrac{12}{5}},\ \sqrt{\dfrac{12}{5}} < x < 2$ and $2 < x < \infty$

Decrease: $-\sqrt{\dfrac{12}{5}} < x < \sqrt{\dfrac{12}{5}}$

From this it looks like $t = -2$ an $t = 2$ are neither local minimum or local maximum since the function is increasing on both side of them.

On the other hand, $t = -\sqrt{\frac{12}{5}}$ is a local maximum and $t = \sqrt{\frac{12}{5}}$ is a local minimum.

Moreover, the graph is



Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is **concave upward** on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is **concave downward** on I.

Definition

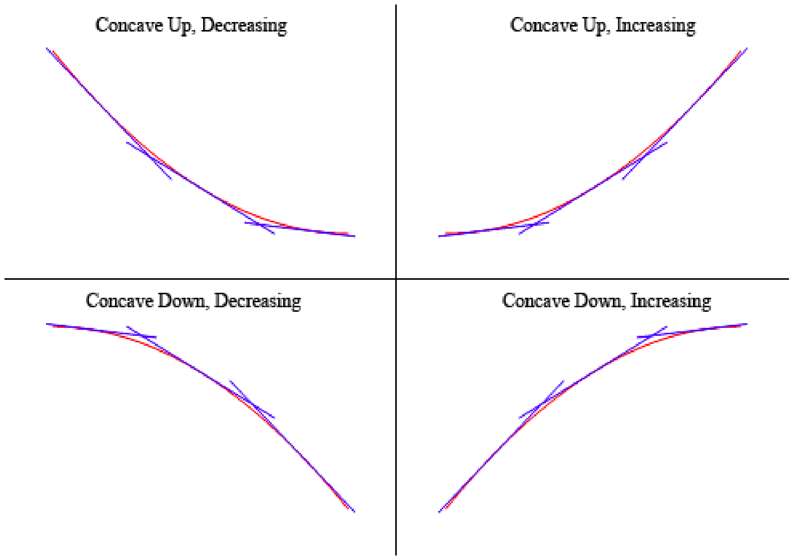
A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test

Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



Example:

For the following function find the inflection points and use the second derivative test, if possible, to classify the critical points. Also, determine the intervals of increase/decrease and the intervals of concave up/concave down and sketch the graph of the function.

$$f(t) = t(6-t)^{\frac{2}{3}}$$

Sol.

We'll need the first and second derivatives to get us started.

$$f'(t) = \frac{18-5t}{3(6-t)^{\frac{1}{3}}}$$

The critical points are

$$t = \frac{18}{5} = 3.6 \qquad t = 6$$

Notice as well that we won't be able to use the second derivative test on $t = 6$ to classify this critical point since the derivative doesn't exist at this point.

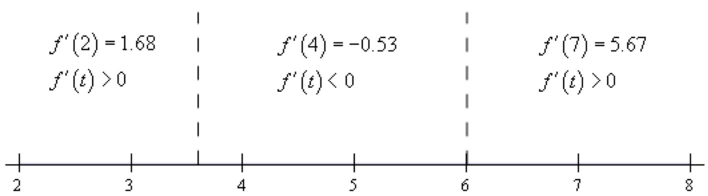
Continue to use the Second Derivative Test to classify the other critical point. Here is the value of the second derivative at $t = 3.6$.

$$f''(3.6) < 0$$

So, according to the second derivative test $t = 3.6$ is a local maximum.

Now let's proceed with the work to get the sketch of the graph and notice that once we have the increasing/decreasing information we'll be able to classify $t = 6$.

Here is the number line for the first derivative.

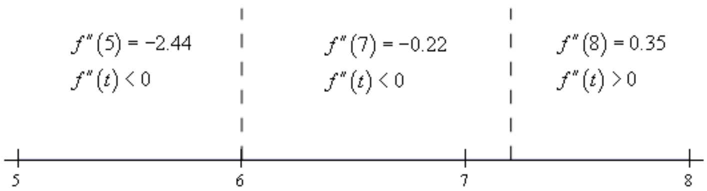


We can also see that $t = 6$ is a local minimum.

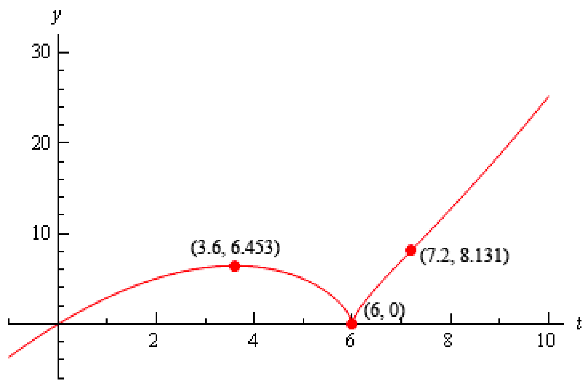
We will need the list of possible inflection points. These are

$$t = 6 \qquad t = \frac{72}{10} = 7.2 \qquad \text{such that } f''(t) = 0 \text{ or DNE}$$

Here is the number line for the second derivative. Note that we will need this to see if the two points above are in fact inflection points.



So the concavity only changes at $t = 7.2$ and this is the only inflection point for this function.
Here is the sketch of the graph.



The change of concavity at $t = 7.2$ is hard to see, but it is there it's just a very subtle change in concavity.

Exercise:

Please find the intervals on which f is increasing, decreasing, concave upward, or concave downward. Also, find the local and absolute maximum and minimum of f on assigned domain.

1. $f(x) = x^2 - x - \ln x$ on $\mathbb{R}^+ := \{x \,|\, x > 0\}$.
2. $f(x) = \sqrt{x}\, e^{-x}$ on $\mathbb{R} \setminus \mathbb{R}^- := \{x \,|\, x \geq 0\}$.

Sol.

1.(a) f is increasing on $(1, \infty)$ and f is decreasing on $(0, 1)$.
(b) $f(1) = 0$ is a local minimum value.
(c) f is concave upward on $(0, \infty)$ and there is no inflection point.

2.(a) f is increasing on $(0, \frac{1}{2})$ and f is decreasing on $(\frac{1}{2}, \infty)$.
(b) $f(\frac{1}{2}) = \frac{1}{\sqrt{2e}}$ is a local maximum value.
(c) f is concave upward on $(\frac{1}{2} + \frac{1}{2}\sqrt{2}, \infty)$, f is concave downward on $(0, \frac{1}{2} + \frac{1}{2}\sqrt{2})$ and there is a inflection point $(\frac{1}{2} + \frac{1}{2}\sqrt{2}, f(\frac{1}{2} + \frac{1}{2}\sqrt{2}))$.

Section 4.4: Indeterminate Forms and L'Hospital's Rule

L'Hospital Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = 0$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example:
Evaluate the following limit.

1. $\displaystyle\lim_{x\to 0^+} x \ln x$

   Sol.

   Now, in the limit, we get the indeterminate form $(0)(\infty)$. L'Hospital's Rule won't work on products, it only works on quotients.

   However, we can turn this into a fraction if we rewrite this limit as

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}$$

   Then the limit is now in the form $-\infty/\infty$ so we can use L'Hospital's Rule.

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

2. $\displaystyle\lim_{x\to\infty} x^{\frac{1}{x}}$

   Sol.

   In the limit this is the indeterminate form ∞^0.

   Now, if we take the natural log of both sides we get

$$\ln\left(x^{\frac{1}{x}}\right) = \frac{\ln x}{x}$$

   Let's now take a look at the following limit.

$$\lim_{x\to\infty} \ln\left(x^{\frac{1}{x}}\right) = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$$

   So

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln(x^{1/x})} = exp(\lim_{x\to\infty} \frac{\ln x}{x}) = e^0 = 1$$

Exercise:
Evaluate the following limits.

1. $\lim\limits_{x\to 0} \frac{x-\sin x}{x-\tan x}$

2. $\lim\limits_{x\to 0} \frac{e^x-e^{-x}-2x}{x-\sin x}$

3. $\lim\limits_{x\to 0^+} \sin x \ln x$

4. $\lim\limits_{x\to \infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$

Sol.
1. $\frac{-1}{2}$ 2. 2 3. 0 4. e^{-8}

Section 4.5: Summary of Curve Sketching

Guidelines for Sketching a Curve

A. Domain

B. Intercepts

C. Symmetry

D. Asymptotes

E. Intervals of Increase or Decrease

F. Local Maximum and Minimum Values

G. Concavity and Points of Inflection

H. Sketch the Curve

Example:
Sketch the following curves.

$$y = \sqrt{x^2+x} - x$$

Sol.

$$y = \sqrt{x^2+x} - x = \sqrt{x(x+1)} - x$$

A. Domain: $(-\infty,-1] \cup [0,\infty)$

B. Intercepts:
y-intercept: $f(0) = 0$.
x-intercept:

$$f(x) = 0 \Rightarrow \sqrt{x^2+x} = x \Rightarrow x^2 + x = x \Rightarrow x = 0$$

C. Symmetry: No symmetry.

D. Asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (\sqrt{x^2+x} \, - \, x)\, \frac{\sqrt{x^2+x} \, + \, x}{\sqrt{x^2+x} \, + \, x} = \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x} \, + \, x}$$

$$= \lim_{x\to\infty} \frac{x/x}{(\sqrt{x^2+x} \, + \, x)/x} = \lim_{x\to\infty} \frac{1}{(\sqrt{1+1/x} \, + \, 1} = \frac{1}{2}$$

So $y = \frac{1}{2}$ is a horizontal asymptote.

Moreover, there is no vertical asymptote.

E. Intervals of Increase or Decrease:

$$f'(x) = \frac{1}{2}(x^2+x)^{-1/2}(2x+1) - 1 = \frac{2x+1}{2\sqrt{x^2+x} - 1} > 0$$

$$\Leftrightarrow 2x+1 > 2\sqrt{x^2+x} \;\;\Leftrightarrow\;\; x + \frac{1}{2} > \sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}$$

Keep in mind that the domain excludes the interval $(-1, 0)$.

When $x + \frac{1}{2}$ is positive (for $x \geq 0$), the last inequality is *true* since the value of the radical is less than $x + \frac{1}{2}$.

When $x + \frac{1}{2}$ is negative (for $x \leq -1$), the last inequality is *false* since the value of the radical is positive.

So f is increasing on $(0, \infty)$ and decreasing on $(-\infty, -1)$.

F. Local Maximum and Minimum Values: No local extrema.

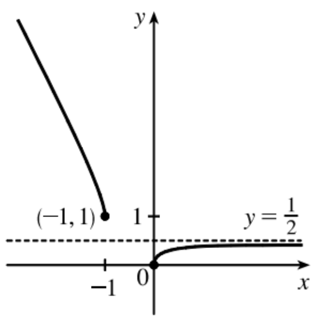
G. Concavity and Points of Inflection:

$$f''(x) = d\left(\frac{2x+1}{2\sqrt{x^2+x}}\right)/dx = \frac{2(x^2+x)^{1/2}(2) - (2x+1)\cdot 2 \cdot \frac{1}{2}(x^2+x)^{-1/2}(2x+1)}{(2\sqrt{x^2+x})^2} = \frac{-1}{4(x^2+x)^{3/2}}$$

$f''(x) < 0$ when it is defined, so f is concave downward on $(-\infty, -1)$ and $(0, \infty)$.

There is no inflection point.

H. Sketch the Curve



Exercise:
Sketch the following curves.

$$y = \arctan\left(\frac{x-1}{x+1}\right)$$

Sol.

A. Domain: $\{x \mid x \neq -1\}$

B. Intercepts:
x-intercept $= 1$, y-intercept $= f(0) = \arctan(-1) = \frac{-\pi}{4}$.

C. Symmetry: No symmetry.

D. Asymptotes: $y = \frac{\pi}{4}$ is a horizontal asymptote, and there is no vertical asymptote.

E. Intervals of Increase or Decrease: f is increasing on $(-\infty, -1)$ and $(-1, \infty)$.

F. Local Maximum and Minimum Values: No extrema.

G. Concavity and Points of Inflection:
f is concave upward on $(-\infty, -1)$ and $(-1, 0)$, and concave downward on $(0, \infty)$.

Inflection point is at $(0, \frac{-\pi}{4})$.

H. Sketch the Curve: