NTU 107-1 MATH1201 | CALCULUS A-05 QUIZ 2 SOLUTION | 26TH OCTOBER, 2018

> Work out **ALL** questions below. Provide sufficient justification to every step of your arguments.
> Write your solutions as well as your ID number clearly on A4-sized paper and submit them to *Instructor's office* **before 4pm (GMT +8) on 29th October, 2018**.
> Recommended time limit: 150 minutes.

1. Let a and b be two real numbers and

$$f(x) := \begin{cases} \dfrac{e^{-x^{-2}} - a}{x} & \text{for } x \neq 0 \, , \\ b & \text{for } x = 0 \, . \end{cases}$$

(a) (8 points) If f is continuous everywhere on $\mathbb{R}$, what are the values of a and b?

(b) (8 points) If f is continuous everywhere on $\mathbb{R}$, determine whether $f'(0)$ and $f''(0)$ exist. Find their values if they do.

Solution.

(a) If f is continuous on $\mathbb{R}$, it is continuous at 0 in particular, and we have

$$\lim_{x \to 0} f(x) = b \, . \qquad \text{(1 point)}$$

Since the limit of $f(x)$ exists when $x \to 0$, we must have

$$\lim_{x \to 0} \left(e^{-x^{-2}} - a \right) = \lim_{x \to 0} x f(x) = 0 \, . \qquad \text{(1 point)}$$

Notice that, under the transformation $t := x^{-2}$,

$$\lim_{x \to 0} e^{-x^{-2}} = \lim_{t \to \infty} e^{-t} = 0 \, . \qquad \text{(2 points)}$$

It follows that $a = 0$. (1 point)

Moreover, using l'Hospital's rule, it can be seen that

$$b = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^{-x^{-2}}}{x} = \lim_{t \to \infty} \frac{\sqrt{t}}{e^t} \underset{(\frac{\infty}{\infty})}{\overset{\text{l'H.R.}}{=}} \lim_{t \to \infty} \frac{1}{2\sqrt{t}\, e^t} = 0 \, .$$

(correct use of l'Hospital's rule: 2 points)

(value of b : 1 point)

(Note that the transformation $t := x^{-2}$ instead of $t := x^{-1}$ is used here in order to avoid the need to consider separately the sided limits $\lim_{x \to 0^+}$ and $\lim_{x \to 0^-}$.)

(b) As f is continuous in particular at 0, we have $a = b = 0$ as found in previous question. From the definition of derivatives, it follows that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-x^{-2}}}{x^2} = \lim_{t \to \infty} \frac{t}{e^t} \underset{(\frac{\infty}{\infty})}{\overset{\text{l'H.R.}}{=}} \lim_{t \to \infty} \frac{1}{e^t} = 0 \, .$$

This shows that $f'(0)$ exists. (definition of derivative: 1 point)
(correct use of l'Hospital's rule: 2 points)
(value of $f'(0)$: 1 point)

Note also that

$$f'(x) = \frac{2-x^2}{x^4} e^{-x^{-2}} \quad \text{for } x \neq 0 \,. \qquad \text{(1 point)}$$

We then have

$$f''(0) = \lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0} \frac{\left(2-x^2\right) e^{-x^{-2}}}{x^5} \overset{(*)}{=} \lim_{t\to\infty} \left(2 - \frac{1}{t}\right) \lim_{t\to\infty} \frac{t^{\frac{5}{2}}}{e^t}$$

$$\underset{(\frac{\infty}{\infty})}{\overset{\text{l'H.R.}}{=}} 2 \lim_{t\to\infty} \frac{5t^{\frac{3}{2}}}{2e^t} \underset{(\frac{\infty}{\infty})}{\overset{\text{l'H.R.}}{=}} 5 \lim_{t\to\infty} \frac{3t^{\frac{1}{2}}}{2e^t} \underset{(\frac{\infty}{\infty})}{\overset{\text{l'H.R.}}{=}} \frac{15}{2} \lim_{t\to\infty} \frac{1}{2t^{\frac{1}{2}} e^t} = 0 \,.$$

($(*)$: the equality holds after knowing that both limits on the right-hand-side exist.)
Therefore, $f''(0)$ also exists. (correct use of l'Hospital's rule: 2 points)
(value of $f''(0)$: 1 point)

2. (a) (6 points) Let $f(x) = \dfrac{x(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)}$. Find $f'(0)$.
   (b) (8 points) Suppose that the function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If $k := f'(0)$, show that $f'(x) = kf(x)$.
   (Hint: you may have to consider the cases $f(0) = 0$ and $f(0) \neq 0$ separately.)

Solution.

(a) Note that $f(0) = 0$. (1 point)
By the definition of derivatives, we have

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} && \text{(definition: 2 points)} \\ &= \lim_{x\to 0} \frac{1}{x} \cdot \frac{x(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)} \\ &= \lim_{x\to 0} \frac{(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)} && \text{(computations: 2 points)} \\ &= \frac{(-1)(-2)\cdots(-n)}{(1)(2)\cdots(n)} \\ &= (-1)^n \,. && \text{(answer: 1 point)} \end{aligned}$$

(b) **[Method 1]** If $f(0) = 0$, then $f(x) = f(x+0) = f(x)f(0) = 0$ for all $x \in \mathbb{R}$. Therefore, f is the constant function which is constantly equal to 0, and thus the equality $f'(x) = kf(x)$ holds trivially in this case. (2 points)
If $f(0) \neq 0$, we have

$$f(0) = f(0+0) = f(0)f(0) \quad \Rightarrow \quad f(0) = 1 \,. \qquad \text{(1 point)}$$

Therefore, from the definition of derivatives, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-x} \qquad \text{(definition: 2 points)}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x)(f(h)-1)}{h} && \text{(derivation: 3 points)} \\
&= f(x) \cdot \lim_{h\to 0} \frac{(f(h)-f(0))}{h-0} = f(x)f'(0) = kf(x) \ .
\end{aligned}
$$

[Method 2] (Assume that f is differentiable.) Treating x as a constant and differentiating the equation $f(x+y) = f(x)f(y)$ with respect to y yield (strategy: 5 points)

$$f'(x+y) = f'(x+y) \cdot \frac{d(x+y)}{dy} = f(x)f'(y) \ .$$

Substituting $y = 0$ then gives

$$f'(x) = f(x)f'(0) = kf(x) \ .$$

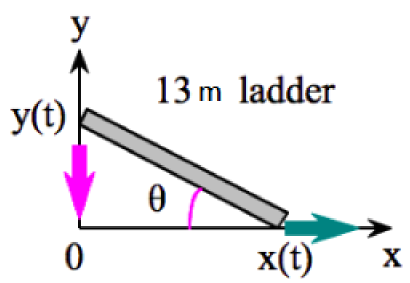
This holds true no matter whether $f(0) = 0$ or not. (computation: 3 points)

3. (8 points) Find $\dfrac{dy}{dx}$ if $\tan^{-1}\left(\frac{y}{x}\right) = \ln\left(\sqrt{x^2+y^2}\right)$.

Solution. Differentiating both sides of the equality with respect to x gives

$$
\begin{aligned}
& \frac{1}{1+(\frac{y}{x})^2} \frac{y'x - y}{x^2} = \frac{1}{\sqrt{x^2+y^2}} \left(\frac{2x+2yy'}{2\sqrt{x^2+y^2}} \right) && \text{(5 points)} \\
\Rightarrow \quad & \frac{y'x-y}{x^2+y^2} = \frac{x+yy'}{x^2+y^2} \\
\Rightarrow \quad & y'x - y = x + yy' \quad (\text{since} \quad x^2+y^2 > 0) && \text{(derivation: 2 points)} \\
\Rightarrow \quad & \frac{dy}{dx} = y' = \frac{x+y}{x-y} \ . && \text{(answer: 1 point)}
\end{aligned}
$$

4. (12 points) A ladder 13 metres long is leaning against a wall when its base starts to slide away. By the time the base is 12 metres from the wall, the base is moving at the rate of 0.5 m/s. At what rate is the area of the triangle formed by the ladder, the wall and the ground changing at that moment?



Solution. The area of the triangle in question is given by

$$A(t) = \frac{1}{2}x(t)y(t) \ . \qquad \text{(relation between } A, x \text{ and } y\text{: 2 points)}$$

Cont.

At the moment $t = t_0$ when $x(t_0) = 12$ (m), we have $\left.\frac{dx}{dt}\right|_{t=t_0} = \frac{1}{2}$ (m/s). To find $\left.\frac{dA}{dt}\right|_{t=t_0}$, we first differentiate $A(t)$ with respect to t to obtain

$$A'(t) = \frac{1}{2}\left(\frac{dx}{dt}y + x\frac{dy}{dt}\right) . \qquad \text{(3 points)}$$

To find $y(t_0)$ and $\left.\frac{dy}{dt}\right|_{t=t_0}$, we notice the relation $x^2 + y^2 = 13^2$ as the triangle in question is a right-angled triangle. (relation between x and y: 2 points)

Therefore, we obtain

$$y(t_0) = \sqrt{13^2 - (x(t_0))^2} = \sqrt{13^2 - 12^2} = 5 \text{ (m)} . \qquad \text{(1 point)}$$

By differentiating the equation $x^2 + y^2 = 13^2$, we also obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad \left.\frac{dy}{dt}\right|_{t=t_0} = \left.\left(-\frac{x}{y}\cdot\frac{dx}{dt}\right)\right|_{t=t_0} = -\frac{12}{5}\cdot\frac{1}{2} = -\frac{6}{5} \text{ (m/s)} . \qquad \text{(3 points)}$$

Therefore, substituting these values back into $A'(t_0)$, we get

$$A'(t_0) = \frac{1}{2}\left.\left(\frac{dx}{dt}y + x\frac{dy}{dt}\right)\right|_{t=t_0} = \frac{1}{2}\left(\frac{1}{2}\cdot 5 - 12\cdot\frac{6}{5}\right) = -\frac{119}{20} = -5.95 \text{ (m}^2\text{/s)} . \qquad \text{(1 point)}$$

That is, the area is shrinking at the rate of $5.95 \text{ m}^2/\text{s}$ at that moment.

5. (a) (10 points) Evaluate $\lim_{x\to\infty}(\sin((x+2)^{\frac{3}{4}}) - \sin(x^{\frac{3}{4}}))$ using the Mean Value Theorem.

   (b) (6 points) Applying the Mean Value Theorem to show that $\frac{1}{1+x} < \frac{\ln(1+x)}{x} < 1$ for $x > 0$.

Solution.

(a) Let $f(z) := \sin(z^{\frac{3}{4}})$, then $f'(z) = \cos(z^{\frac{3}{4}})\cdot\frac{3}{4}\cdot z^{-\frac{1}{4}}$. (2 points)

As f is differentiable (and thus continuous) on $(0, \infty)$, the Mean Value Theorem can be applied to assure that, for every $x > 0$, there is a number $c \in (x, x+2)$ such that

$$\sin((x+2)^{\frac{3}{4}}) - \sin(x^{\frac{3}{4}}) = \frac{3}{4c^{\frac{1}{4}}}\cos(c^{\frac{3}{4}})\cdot(x+2-x) = \frac{3}{2c^{\frac{1}{4}}}\cos(c^{\frac{3}{4}}) . \qquad \text{(2 points)}$$

When $x \to \infty$, we have $c \to \infty$ (by the Squeeze Theorem). (2 points)

Note also that

$$0 \le \left|\frac{3}{2c^{\frac{1}{4}}}\cos(c^{\frac{3}{4}})\right| \le \frac{3}{2c^{\frac{1}{4}}} . \qquad \text{(1 point)}$$

Since $\lim_{x\to\infty}\frac{1}{c^{\frac{1}{4}}} = 0$, the Squeeze Theorem yields $\lim_{x\to\infty}\left|\frac{3}{2c^{\frac{1}{4}}}\cos(c^{\frac{3}{4}})\right| = 0$. (2 points)

Therefore, $\lim_{x\to\infty}\frac{3}{2c^{\frac{1}{4}}}\cos(c^{\frac{3}{4}}) = 0$, i.e. $\lim_{x\to\infty}(\sin((x+2)^{\frac{3}{4}}) - \sin(x^{\frac{3}{4}})) = 0$. (1 point)

Cont.

(b) We know that $\ln(1+x)$ is continuous at every $x \in [0,\infty)$ and differentiable at every $x \in (0,\infty)$. (1 point)
The Mean Value Theorem can then be applied to assure that, for every $x > 0$, there exists a constant $c \in (0,x)$ such that

$$\frac{\ln(1+x)}{x} = \frac{\ln(1+x)-\ln(1)}{x} = \frac{1}{1+c} < 1\,.$$

(Mean Value Thm. + inequality: 3+1 points)

Moreover, we have

$$\frac{\ln(1+x)}{x} = \frac{1}{1+c} > \frac{1}{1+x}\,.$$ (1 point)

The required inequalities are thus proved.

6. Determine if the following limits exist or not. Evaluate them if they do.

(a) (5 points) $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x}$

(b) (8 points) $\displaystyle\lim_{x\to 0}\left(\frac{\sin^{-1}x}{x}\right)^{\frac{1}{x^2}}$
(Hint: avoid differentiating quotients whenever you want to apply l'Hospital's rule.)

(c) (8 points) $\displaystyle\lim_{x\to\infty} x\left(\left(1+\frac{1}{x}\right)^{x} - e\right)$

Solution.

(a) Put $y := \sin^{-1}x$, then we have $x = \sin y$ and thus $y \to 0$ when $x \to 0$. (1 point)
It therefore follows that

$$\lim_{x\to 0}\frac{\sin^{-1}x}{x} = \lim_{y\to 0}\frac{y}{\sin y} = \frac{1}{\lim_{y\to 0}\frac{\sin y}{y}} = 1\,.$$ (derivation + answer: 3+1 points)

(b) Put $y := \sin^{-1}x$, then we have $x = \sin y$ and thus $y \to 0$ when $x \to 0$. Notice that $\frac{\sin^{-1}x}{x} = \frac{y}{\sin y}$ is positive when x (hence y) is close to 0. Substituting x by $\sin y$ and taking logarithm of the expression in the limit, we are led to consider the limit

$$\begin{aligned}
\lim_{y\to 0}\frac{\ln\frac{y}{\sin y}}{\sin^2 y} &= \lim_{y\to 0}\frac{\ln|y| - \ln|\sin y|}{\sin^2 y} = \underbrace{\lim_{y\to 0}\frac{y^2}{\sin^2 y}}_{=1}\cdot\lim_{y\to 0}\frac{\ln|y|-\ln|\sin y|}{y^2} \\
&\overset{\text{l'H.R.}}{\underset{(\frac{0}{0})}{=}} \lim_{y\to 0}\frac{\frac{1}{y}-\frac{\cos y}{\sin y}}{2y} = \underbrace{\lim_{y\to 0}\frac{y}{\sin y}}_{=1}\cdot\lim_{y\to 0}\frac{\sin y - y\cos y}{2y^3} \\
&\overset{\text{l'H.R.}}{\underset{(\frac{0}{0})}{=}} \lim_{y\to 0}\frac{\cos y - \cos y + y\sin y}{6y^2} = \lim_{y\to 0}\frac{\sin y}{6y} = \frac{1}{6}\,.
\end{aligned}$$

As a result,

$$\lim_{x\to 0}\left(\frac{\sin^{-1}x}{x}\right)^{\frac{1}{x^2}} = \lim_{y\to 0} e^{\frac{1}{\sin^2 y}\ln\frac{y}{\sin y}} = e^{\frac{1}{6}}\,.$$

(correct use of l'Hospital's rule: 4 points)
(derivation + answer: 3+1 points)

Cont.

(c) It is known that

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to 0^+}(1+y)^{\frac{1}{y}} \ , \qquad \text{(2 points)}$$

so l'Hospital's rule can be applied to obtain

$$\begin{aligned}
\lim_{x\to\infty} x\left(\left(1+\frac{1}{x}\right)^x - e\right) &= \lim_{y\to 0^+}\frac{(1+y)^{\frac{1}{y}}-e}{y} \\
&\overset{\text{l'H.R.}}{\underset{(\frac{0}{0})}{=}} \lim_{y\to 0^+}(1+y)^{\frac{1}{y}}\left(\underbrace{\frac{1}{y(1+y)}}_{=\frac{1}{y}-\frac{1}{1+y}} - \frac{\ln(1+y)}{y^2}\right) \\
&= \underbrace{\lim_{y\to 0^+}(1+y)^{\frac{1}{y}}}_{=e}\left(\lim_{y\to 0^+}\frac{y-\ln(1+y)}{y^2} - \lim_{y\to 0^+}\frac{1}{(1+y)}\right) \\
&\overset{\text{l'H.R.}}{\underset{(\frac{0}{0})}{=}} e\left(\lim_{y\to 0^+}\frac{1-\frac{1}{1+y}}{2y} - 1\right) \\
&= e\left(\lim_{y\to 0^+}\frac{1}{2(1+y)} - 1\right) = -\frac{e}{2} \ .
\end{aligned}$$

(correct use of l'Hospital's rule: 3 points)

(computation + answer: 2+1 points)

7. Let

$$f(x) := \frac{\sqrt{|x|}\,(x-2)}{\sqrt{x+1}} \ .$$

(a) (1 point) What is the domain of f ?

(b) (3 points) Find all vertical asymptotes of the graph of f.

(c) (6 points) Evaluate $\lim\limits_{x\to\infty}(f(x)-x)$. Thus find all slant asymptotes of the graph of f.

(d) (6 points) Find all critical points of f (in its domain). Identify also the intervals on which f is increasing or decreasing.

(e) (6 points) Identify all the intervals on which the graph of f is concave upward or downward. Is there any inflection point?

(f) (4 points) Sketch the graph of f using the results above. Label all local extrema and inflection points with their coordinates.

Solution.

(a) The denominator $\sqrt{x+1}$ of $f(x)$ is well-defined only when $x+1 \geq 0$, so $f(x)$ is well-defined only when $x+1>0$. Therefore, the domain of f is $(-1,\infty)$. (1 point)

(b) As the numerator of $f(x)$ converges to a finite value when x converges to any number $x_0 \in (-1,\infty)$, it follows that $f(x)\to\infty$ or $-\infty$ only when the denominator $\sqrt{x+1}\to 0^+$. This happens only when $x\to -1^+$. (1 point)

In that case,

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}\frac{\sqrt{|x|}\,(x-2)}{\sqrt{x+1}} = -\infty \ . \qquad \text{(1 point)}$$

Therefore, the line $x=-1$ is the only vertical asymptote of the graph of f. (1 point)

(c) Notice that, as $|x| = x$ when $x > 0$, we have

$$\begin{aligned}\lim_{x\to\infty}(f(x)-x) &= \lim_{x\to\infty}\left(\frac{\sqrt{|x|}\,(x-2)}{\sqrt{x+1}} - x\right)\\ &= \lim_{x\to\infty}\left(x\left(\frac{1}{\sqrt{1+\frac{1}{x}}}-1\right) - \frac{2}{\sqrt{1+\frac{1}{x}}}\right). \qquad \text{(treatment to } |x|\text{: 1 point)}\end{aligned}$$

While it can be seen easily that

$$-\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x}}} = -2\,, \qquad \text{(1 point)}$$

the limit of the first term can be computed via the transformation $u := \frac{1}{x}$ and the use of l'Hospital's rule to give

$$\lim_{x\to\infty} x\left(\frac{1}{\sqrt{1+\frac{1}{x}}}-1\right) = \lim_{u\to 0^+}\frac{\frac{1}{\sqrt{1+u}}-1}{u} \overset{\text{l'H.R.}}{\underset{(\frac{0}{0})}{=}} -\lim_{u\to 0^+}\frac{1}{2(1+u)^{\frac{3}{2}}} = -\frac{1}{2}\,. \qquad \text{(2 points)}$$

As a result,

$$\lim_{x\to\infty}(f(x)-x) = -\frac{1}{2} - 2 = -\frac{5}{2}\,. \qquad \text{(1 point)}$$

This shows that the line $y = x - \frac{5}{2}$ is a slant asymptote (which is the only one) of the graph of f. (1 point)

(d) Via logarithmic differentiation, we obtain

$$\begin{aligned}\frac{f'(x)}{f(x)} = \frac{d}{dx}\ln|f(x)| &= \frac{1}{2x} + \frac{1}{x-2} - \frac{1}{2(x+1)}\\ &= \frac{2x^2+3x-2}{2x(x+1)(x-2)} = \frac{(2x-1)(x+2)}{2x(x+1)(x-2)}\\ \Rightarrow \quad f'(x) &= \frac{\sqrt{|x|}\,(2x-1)(x+2)}{2x(x+1)^{\frac{3}{2}}} \qquad \text{(2 points)}\end{aligned}$$

for all $x \in (-1,0)\cup(0,\infty)$. (Note that we have to take absolute value before taking logarithm in this case.)

Since

$$\lim_{x\to 0^+}\frac{\sqrt{|x|}}{x} = \lim_{x\to 0^+}\frac{1}{\sqrt{x}} = \infty$$

(or $\lim_{x\to 0^-}\frac{\sqrt{|x|}}{x} = -\lim_{x\to 0^-}\frac{1}{\sqrt{|x|}} = -\infty$), we see that $f'(0)$ does not exist. (1 point)

(This claim follows from l'Hospital's rule which gives

$$\lim_{x\to 0^\pm}\frac{f(x)-f(0)}{x-0} \overset{\text{l'H.R.}}{\underset{(\frac{0}{0})}{=}} \lim_{x\to 0^\pm} f'(x) = \pm\infty\,,$$

where $\lim_{x\to 0^\pm}$ denotes either $\lim_{x\to 0^+}$ or $\lim_{x\to 0^-}$.) It is easy to see that, among all $x \in (-1,\infty)$, $x = 0$ is the only point such that f' is not defined, and $f'(x) = 0$ if and only if $x = \frac{1}{2}$. (1 point)

Cont.

As a result, 0 and $\frac{1}{2}$ are the only critical points of f.
It can be seen that the sign of $f'(x)$ on $(-1,0)\cup(0,\infty)$ is the same as the sign of $\frac{2x-1}{x}$, and we thus see that

$$f'(x)\begin{cases} >0 & \text{for } x\in(-1,0)\cup(\frac{1}{2},\infty)\ , \\ <0 & \text{for } x\in(0,\frac{1}{2})\ . \end{cases} \qquad \text{(1 point)}$$

Therefore, f is increasing on $(-1,0)$ and on $(\frac{1}{2},\infty)$ while it is decreasing on $(0,\frac{1}{2})$.
(1 point)

(e) Via logarithmic differentiation again, we obtain

$$\begin{aligned} \frac{f''(x)}{f'(x)} = \frac{d}{dx}\ln\left|f'(x)\right| &= \frac{1}{2x}+\frac{2}{2x-1}+\frac{1}{x+2}-\frac{1}{x}-\frac{3}{2(x+1)} \\ &= \frac{11x+2}{2x(2x-1)(x+2)(x+1)} \\ \Rightarrow \qquad f''(x) = \frac{\sqrt{|x|}\,(11x+2)}{4x^2(x+1)^{\frac{5}{2}}} & \qquad \text{(2 points)} \end{aligned}$$

for $x\in(-1,0)\cup(0,\infty)$.
Notice that the sign of $f''(x)$ is the same as the sign of $11x+2$ on the domain of f'', and thus

$$f''(x)\begin{cases} >0 & \text{for } x\in(-\frac{2}{11},0)\cup(0,\infty)\ , \\ <0 & \text{for } x\in(-1,-\frac{2}{11})\ . \end{cases} \qquad \text{(1 point)}$$

Therefore, the graph of f is concave upward on $(-\frac{2}{11},0)$ and on $(0,\infty)$ while it is concave downward on $(-1,-\frac{2}{11})$. (1 point)
As the concavity of the graph of f has changed when x increases across $-\frac{2}{11}$, there is an inflection point of the graph of f at the point $P(-\frac{2}{11},f(-\frac{2}{11}))=P(-\frac{2}{11},-\frac{8\sqrt{2}}{11})$.
(definition of inflection points: 1 point)
(coordinates of the inflection point: 1 point)

(f) The curve $y=f(x)$ is shown in Figure 1.
($y=f(x)$ approaching asymptotes: 1 point)
(corner at $(0,0)$: 0.5 points)
(correct extrema: 0.5 points)
(concavity: 1 point)
(labelling extrema and the inflection point: 1 point)

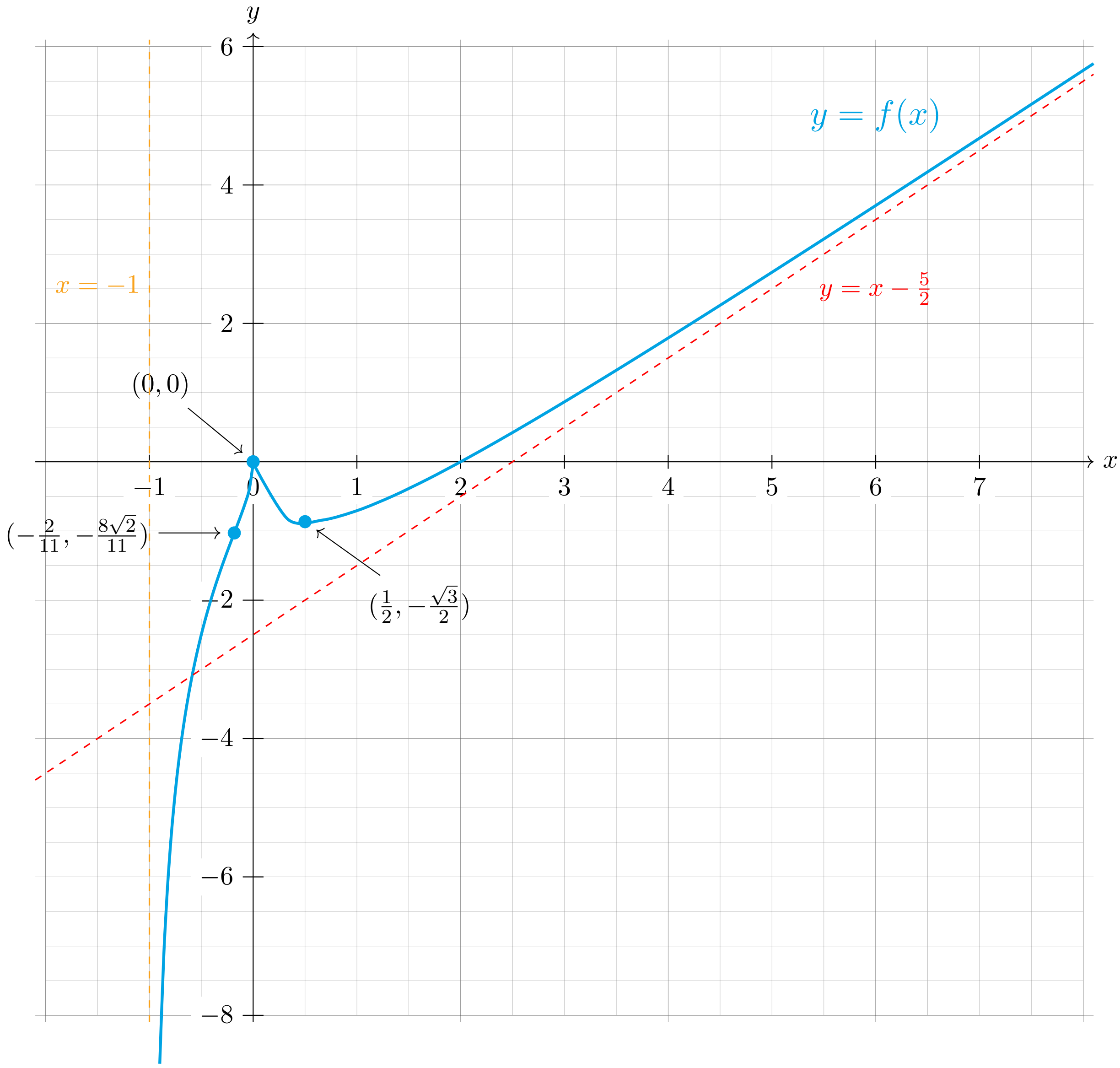


Figure 1: The curve $y = f(x)$ of Problem (7)

The End.