Real Analysis Homework
Chapter 1. Measure theory
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Exercise

If $\{E_k\}$ is a sequence of sets with $\sum m_*(E_k) < \infty$. Show that $\limsup E_k$ has measure zero.

Proof.

Let H_k be a G_δ set, $E_k \subset H_k$ with $m(H_k) = m_*(E_k)$
$\Rightarrow \limsup E_k = \cap_{j=1}^{\infty} \cup_{k=j}^{\infty} E_k \subseteq \cap \cup H_k = \limsup H_k$.

$$\sum m(H_k) = \sum m_*(E_k) < \infty \Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{Z}^+, \forall k \geq N, \sum_{k=N+1}^{\infty} m(H_k) < \varepsilon$$

Let $O_j = \cup_{k=j}^{\infty} H_k$.

$$O_j \searrow \limsup H_k \Rightarrow \begin{cases} m(\limsup H_k) = \lim_{j\to\infty} m(O_j) \leq \lim_{j\to\infty} \sum_{k=j}^{\infty} m(H_k) \leq \sum_{k=N+1}^{\infty} m(H_k) < \varepsilon \\ m_*(\limsup E_k) \leq 0 \end{cases}$$