Real Analysis
Homework 10

score:10

Yueh-Chou Lee

December 28, 2018

1. (Exercise 7.11)
Prove the following result concerning *changes of variable*. Let $g(t)$ be monotone increasing and absolutely continuous on $[\alpha, \beta]$ and let f be integrable on $[a, b]$, $a = g(\alpha)$, $b = g(\beta)$. Then $f(g(t))g'(t)$ is measurable and integrable on $[\alpha, \beta]$, and

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))g'(t)\,dt$$

(Consider the cases when f is the characteristic function of an interval, an open set, etc.)

Proof.
Since $g(t)$ is monotone increasing and absolutely continuous on $[\alpha, \beta]$, then g is of bounded variatin on $[\alpha, \beta]$, g' also exists a.e. and $g' \in L[\alpha, \beta]$.
Let f and g' be the non-negative function.
Since $f \in L[a, b]$ and $g' \in L[\alpha, \beta]$, then f is measurable on $[a, b]$ and g is measurable on $[\alpha, \beta]$.
Thus, $f(g(t))$ is also measurable on $[\alpha, \beta]$ since $g(t)$ is continuous on $[\alpha, \beta]$.
Then

$$f(g(t))g'(t) = \frac{1}{2}\left\{\left[f(g(t)) + g'(t)\right]^2 - [f(g(t))]^2 - g(t)^2\right\}$$

is measurable.
f and g' are finite on $[a, b]$ and $[\alpha, \beta]$ since $f \in L[a, b]$ and $g' \in L[\alpha, \beta]$, so

$$\int_\alpha^\beta f(g(t))g'(t)dt \leq |[\alpha, \beta]| \sup_{[\alpha,\beta]}\{f(g(t))g'(t)\} < \infty$$

thus, $f(g(t))g'(t)$ is integrable on $[\alpha, \beta]$.

Let $\Gamma = \{t_i\}$ be a partition of $[\alpha, \beta]$ with norm $|\Gamma|$.
Then

$$\begin{aligned}\int_\alpha^\beta f(g(t))g'(t)dt &= \sum \int_{t_{i-1}}^{t_i} f(g(t))g'(t)dt \\ &= \sum f(g(t_{i-1})) \int_{t_{i-1}}^{t_i} g'(t)dt + \sum \int_{t_{i-1}}^{t_i} [f(g(t)) - f(g(t_{i-1}))]g'(t)dt\end{aligned}$$

The first term on the right equals

$$\sum f(g(t_{i-1}))[g(t_i) - g(t_{i-1})]$$

which converges to

$$\int_a^b f dg \quad \text{as } |\Gamma| \to 0$$

The second termon the right is majorized in absolute value by

$$\left[\sup_{|x-y|\leq|\Gamma|}|f(x)-f(y)|\right]\sum\int_{t_{i-1}}^{t_i}|g'|dt=\left[\sup_{|x-y|\leq|\Gamma|}\right]\int_\alpha^\beta|g'|dt$$

Since $f(g(t))$ is uniformly continuous on $[\alpha,\beta]$, the last expression tends to 0 as $|\Gamma|\to 0$.
Thus

$$\int_a^b f(x)\,dx=\int_\alpha^\beta f(g(t))g'(t)\,dt$$

2. (Exercise 7.12)
Use Jensen's inequality to prove that if $a,b\geq 0$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then

$$ab\leq\frac{a^p}{p}+\frac{b^q}{q}$$

More generally, show that

$$a_1...a_N\leq\sum_{j=1}^N\frac{a_j^{p_j}}{p_j}$$

where $a_j\geq 0$, $p_j>1$, $\sum_{j=1}^N\frac{1}{p_j}=1$.
(Write $a_j=e^{x_j/p_j}$ and use the convexity of e^x.)

Proof.
Let $a_j=e^{x_j/p_j}$ and $\phi(x)=e^x$, then $x_j=\ln(a_j^{p_j})$ and ϕ is a convex function.
By using Jensen's inequailty and $\sum_{j=1}^N\frac{1}{p_j}=1$,

$$\begin{aligned}a_1...a_N&=\phi\left(\sum_{i=1}^N\frac{x_j}{p_j}\right)=\phi\left(\frac{\sum_{i=1}^N\frac{1}{p_j}\cdot x_j}{\sum_{i=1}^N\frac{1}{p_j}}\right)\\&\leq\frac{\sum_{i=1}^N\frac{1}{p_j}\phi(x_j)}{\sum_{i=1}^N\frac{1}{p_j}}=\sum_{i=1}^N\frac{e^{x_j}}{p_j}\\&=\sum_{i=1}^N\frac{a_j^{p_j}}{p_j}\end{aligned}$$

If $N=2$, then the inequality $ab\leq\frac{a^p}{p}+\frac{b^q}{q}$ holds.

3. (Exercise 7.14)
Prove that ϕ is convex on (a,b) if and only if it is continuous and

$$\phi\left(\frac{x_1+x_2}{2}\right)\leq\frac{\phi(x_1)+\phi(x_2)}{2}$$

for $x_1,x_2\in(a,b)$.

Proof.
$(\Rightarrow)$
By Theorem 7.40, since ϕ is convex on (a,b), then ϕ is continuous in (a,b).
By the formula 7.34, if ϕ is convex in (a,b), then

$$\phi\left(\frac{p_1x_1+p_2x_2}{p_1+p_2}\right)\leq\frac{p_1\phi(x_1)+p_2\phi(x_2)}{p_1+p_2}\quad\text{holds.}$$

Set $p_1 = p_2 = \frac{1}{2}$, we have

$$\phi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{\phi(x_1) + \phi(x_2)}{2}$$

($\Leftarrow$)
Suppose that ϕ is continuous and satisfies

$$\phi\left(\frac{x_1 + x_2}{2}\right) \leq \frac{\phi(x_1) + \phi(x_2)}{2}$$

for $x_1, x_2 \in (a, b)$. Given $x_1, x_2 \in (a, b)$, then for any $t \in [x_1, x_2]$ can be written as

$$t = (1 - \sum_{k=1}^{\infty} \frac{a_k}{2^k})\, x_1 + (\sum_{k=1}^{\infty} \frac{a_k}{2^k})\, x_2$$

where $a_i \in [0, 1]$ for all i.
Let t_n be the nth partial sum of the series t.
We claim that

$$\phi\left((1 - \sum_{k=1}^{n} \frac{a_k}{2^k})\, x_1 + (\sum_{k=1}^{n} \frac{a_k}{2^k})\, x_2\right) \leq (1 - \sum_{k=1}^{n} \frac{a_k}{2^k})\, \phi(x_1) + (\sum_{k=1}^{n} \frac{a_k}{2^k})\, \phi(x_2)$$

for all n is any positive integer.
For $n = 1$, then

$$\phi((1 - \frac{a_1}{2})\, x_1 + \frac{a_1}{2}\, x_2) \leq (1 - \frac{a_1}{2})\, \phi(x_1) + \frac{a_1}{2}\, \phi(x_2)$$

Suppose that this inequality holds for $n = r$.
For $n = r + 1$, we have

$$\begin{aligned}
&. \quad \phi\left((1 - \sum_{k=1}^{r+1} \frac{a_k}{2^k})\, x_1 + (\sum_{k=1}^{r+1} \frac{a_k}{2^k})\, x_2\right) \\
&\leq \frac{1}{2}\left[\phi((1 - a_1)\, x_1 + a_1 x_2) + \phi(1 - \sum_{k=2}^{r+1} \frac{a_k}{2^k})\, x_1 + \phi(\sum_{k=2}^{n} \frac{a_k}{2^k})\, x_2\right] \\
&\leq \frac{1}{2}\left[\phi((1 - a_1)\, x_1 + a_1 x_2) + (1 - \sum_{k=2}^{r+1} \frac{a_k}{2^k})\, \phi(x_1) + (\sum_{k=2}^{n} \frac{a_k}{2^k})\, \phi(x_2)\right] \\
&= (1 - \sum_{k=1}^{r+1} \frac{a_k}{2^k})\, \phi(x_1) + (\sum_{k=1}^{r+1} \frac{a_k}{2^k})\, \phi(x_2)
\end{aligned}$$

By the induction, then we have

$$\phi(t) = \lim_{n\to\infty} \phi(t_n) \leq (1 - \sum_{k=1}^{\infty} \frac{a_k}{2^k})\, \phi(x_1) + (\sum_{k=1}^{\infty} \frac{a_k}{2^k})\, \phi(x_2)$$

since ϕ is continuous and the above inequality as $n \to \infty$, therefore, ϕ is convex.

4. (Exercise 7.15)
Theorem 7.43 shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\phi(x) = \int_a^x f(t)dt + \phi(a)$ in (a, b) and f is monotone increasing, then ϕ is convex in (a, b). (Use Exercise 14.)

Proof.
Given any interval $[x_1, x_2] \in (a, b)$, since f is monotone increasing, then we have

$$\begin{aligned}
\frac{\phi(x_1)+\phi(x_2)}{2} - \phi(\frac{x_1+x_2}{2}) &= \frac{[\phi(x_2)-\phi(\frac{x_1+x_2}{2})] - [\phi(\frac{x_1+x_2}{2})-\phi(x_1)]}{2} \\
&= \frac{1}{2}\int_{\frac{x_1+x_2}{2}}^{x_2} f(x)dx - \frac{1}{2}\int_{x_1}^{\frac{x_1+x_2}{2}} f(x)dx \\
&\geq \frac{1}{2}\left[\left(x_2 - \frac{x_1+x_2}{2}\right) f(\frac{x_1+x_2}{2}) - \left(\frac{x_1+x_2}{2} - x_1\right) f(\frac{x_1+x_2}{2})\right] \\
&= \left(\frac{x_2-x_1}{4}\right)\left(f(\frac{x_1+x_2}{2}) - f(\frac{x_1+x_2}{2})\right) \\
&= 0
\end{aligned}$$

By Exercise 7.14, ϕ is convex since $\int_a^x f(t)dt$ is continuous.

5. (Exercise 7.16)
Show that the formula

$$\int_{-\infty}^{+\infty} fg' = -\int_{-\infty}^{+\infty} f'g$$

for integration by parts may not hold if f is of bounded variation on $(-\infty, +\infty)$ and g is infinitely differentiable with compact support. (Let f be the Cantor–Lebesgue function on $[0, 1]$, and let $f = 0$ elsewhere.)

Proof.
Follow the hint, let f be the Cantor–Lebesgue function on $[0, 1]$ and $f = 0$ elsewhere. Since $f \in [0, 1]$ on $[0, 1]$ and $f = 0$ elsewhere, then f is of bounded variation on $(-\infty, +\infty)$.
Let

$$g(x) = \begin{cases} e^{\frac{1}{(x-1)^2-1}} & \text{if } x \in (0, 2) \\ 0 & \text{otherwise} \end{cases}$$

Then g vanishes outside the bounded set $[0 + \epsilon, 2 - \epsilon]$ where $\epsilon > 0$ and $g^{(n)} = f_n(x)e^{\frac{1}{(x-1)^2-1}}$, hence, g is infinitely differentiable with compact support.
Since g is also increasing on $(0, 1)$, then

$$\int_{-\infty}^{+\infty} f \cdot g' = \int_0^1 f \cdot g' \geq \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{1}{2} \cdot g' > 0$$

But $f' = 0$ since f is the Cantor–Lebesgue function on $[0, 1]$ and $f = 0$ elsewhere, then $\int_{-\infty}^{+\infty} f'g = 0$.
Thus,

$$\int_{-\infty}^{+\infty} f \cdot g' > 0 = \int_{-\infty}^{+\infty} f'g$$

and then the exercise follows.

6. (Exercise 7.17)
A sequence $\{\phi_k\}$ of set functions is said to be *uniformly absolutely continuous* if given $\epsilon > 0$, there exists $\delta > 0$ such that if E satisfies $|E| < \delta$, then $|\phi_k(E)| < \epsilon$ for all k. If $\{f_k\}$ is a sequence of integrable functions on $(0, 1)$ which converges pointwise a.e. to an integrable f, show that $\int_0^1 |f - f_k| \to 0$ if and only if the indefinite integrals of the f_k are uniformly absolutely continuous. (cf. Exercise 23 of Chapter 10.)

Proof.
Given $\epsilon > 0$.
If the indefinite integrals of the f_k are uniformly absolutely continuous and $f \in L(0,1)$, there exists $\delta > 0$ such that if $E \subseteq (0,1)$ satisfies $|E| < \delta$, then

$$\left|\int_E f_k\right| < \epsilon$$

for all k and

$$\int_E |f| < \epsilon$$

By Egorov's theorem, there is a closed subset F of $(0,1)$ such that

$$|(0,1) - F| < \delta$$

and $\{f_k\}$ converges uniformly to f on F.
Then choose $M > 0$ such that for all $k \geq M$, we have

$$\begin{aligned}
\int_0^1 |f - f_k| &= \int_F |f - f_k| + \int_{(0,1)\setminus F} |f - f_k| \\
&< \epsilon|F| + \int_{(0,1)\setminus F} |f| + \int_{(0,1)\setminus F} |f_k| \\
&< \epsilon + \epsilon + \int_{(0,1)\setminus F} f_k^+ + \int_{(0,1)\setminus F} f_k^- \\
&< 4\epsilon
\end{aligned}$$

So

$$\int_0^1 |f - f_k| \to 0$$

Conversely, given $\epsilon > 0$.
For all k, since the indefinite integral of f_k is absolutely continuous, there exists $\delta_k > 0$ such that for any $E \subseteq (0,1)$ with $|E| < \delta_k$, then we have

$$\left|\int_E f_k\right| < \epsilon$$

Since the indefinite integral of f is absolutely continuous, choose $M > 0$ and $\delta > 0$ such that for any $|E| < \delta$ and $k \geq M + 1$, then we have

$$\left|\int_E f_k\right| \leq \int_E |f_k| \leq \int_E |f_k - f| + \int_E |f| < \epsilon$$

Let $\delta' = \min\{\delta, \delta_1, \delta_2, ..., \delta_M\}$, then

$$\left|\int_E f_k\right| < \epsilon$$

for all k.