Real Analysis
Homework 4

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October 24, 2018

1. (Exercise 4.11)
Let f be defined on $\mathbb{R}^n$ and let $B(x)$ denote the open ball $\{y : |x-y| < r\}$ with center x and fixed radius r. Show that the function $g(x) = \sup\{f(y) : y \in B(x)\}$ is lsc and that the function $h(x) = \inf\{f(y) : y \in B(x)\}$ is usc on $\mathbb{R}^n$. Is the same true for the *closed* ball $\{y : |x-y| \leq r\}$?

Proof.

(a) Let x_0 be the limit point of $\mathbb{R}^n$.
Since $g(x_0) = \sup\{f(y) : y \in B(x_0)\}$, then there will exist $x_1 \in B(x_0)$ such that $f(x_1) > M$ for any $M < g(x_0)$.
Let $\delta = r - |x_0 - x_1| > 0$, then for all $x \in B(x_0, \delta)$, we have $x_1 \in B(x)$.
See the function f in the ball $B(x)$, $f(x_1)$ may not be the superior value, therefore,

$$g(x) = \sup\{f(y) : y \in B(x)\} \geq f(x_1) > M,$$

then

$$\liminf_{x \to x_0} g(x) \geq g(x_0).$$

Hence, $g(x)$ is lsc.

(b) Similarly, let x_0' be the limit point of $\mathbb{R}^n$.
Since $h(x_0') = \inf\{f(y) : y \in B(x_0')\}$,then there will exist $x_1' \in B(x_0')$ such that $f(x_1') < M'$ for any $M' > h(x_0')$.
Let $\delta = r - |x_0' - x_1'| > 0$, then for all $x \in B(x_0', \delta)$, we have $x_1' \in B(x)$.
See the function f in the ball $B(x)$, $f(x_1)$ may not be the inferior value, therefore,

$$h(x) = \inf\{f(y) : y \in B(x)\} \leq f(x_1') < M',$$

then

$$\limsup_{x \to x_0'} h(x) \leq h(x_0').$$

Hence, $h(x)$ is usc.

(c) False!
Let f be a function on $\mathbb{R}^1$ with $f(1) = 1, f(2) = 2andf(x) = 0$ as $x \neq 1$ and $x \neq 2$.
Let $r = 1$, then $g(1) = 2$ but $\lim\limits_{x \to 1^-} g(x) = 1$, so g is not lsc.
Similarly, let f be a function on $\mathbb{R}^1$ with $f(1) = -1, f(2) = -2andf(x) = 0$ as $x \neq 1$ and $x \neq 2$.
Let $r = 1$, then $h(1) = -2$ but $\lim\limits_{x \to 1^-} h(x) = -1$, so h is not usc.

2. (Exercise 4.12)
If $f(x), x \in \mathbb{R}^1$, is continuous at almost every point of an interval $[a, b]$, show that f is measurable on $[a, b]$. Generalize this to functions defined in $\mathbb{R}^n$. (For a constructive proof, use the subintervals of a sequence of partitions to define a sequence of simple measurable functions converging to f a.e. in $[a, b]$. Use Theorem 4.12. See also the proof of Theorem 5.54.)

Proof.

(a) f is measurable on $[a, b]$:

Note: Part(a) is proved if part(b) has been proved.

Let E be the subset of $[a, b]$ such that $Z = [a, b] \setminus E$ then Z is meausre zero.
The set E is also measurable since $[a, b]$ and Z are measurable.
For any α and $+\infty > \alpha > -\infty$, we then have

$$\{x \in [a, b] : f(x) > \alpha\} = \{x \in E : f(x) > \alpha\} \cup \{x \in Z : f(x) > \alpha\}$$

$\{x \in E : f(x) > \alpha\}$ is measurable since E is measurable and f is contiunous on $E \subseteq [a, b]$.
Due to $\{x \in Z : f(x) > \alpha\} \subseteq Z$ and Z is measure zero, so $\{x \in Z : f(x) > \alpha\}$ is also measurable (measurable zero).
By the above, we know that $\{x \in E : f(x) > \alpha\}$ and $\{x \in Z : f(x) > \alpha\}$ are measurable, therefore, $\{x \in [a, b] : f(x) > \alpha\}$ is also measurable.
Hence, f is measurable on the interval $[a, b]$.

(b) Generalize:

Assume $f(x)$ is continuous at almost every point of an interval I where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^1$.
Let E be the subset of $I \subseteq \mathbb{R}^n$ such that $Z = I \setminus E$ then Z is meausre zero.
The set E is also measurable since I and Z are measurable.
For any α and $+\infty > \alpha > -\infty$, we then have

$$\{x \in I : f(x) > \alpha\} = \{x \in E : f(x) > \alpha\} \cup \{x \in Z : f(x) > \alpha\}$$

$\{x \in E : f(x) > \alpha\}$ is measurable since E is measurable and f is contiunous on $E \subseteq I$.
Due to $\{x \in Z : f(x) > \alpha\} \subseteq Z$ and Z is measure zero, so $\{x \in Z : f(x) > \alpha\}$ is also measurable (measurable zero).
By the above, we know that $\{x \in E : f(x) > \alpha\}$ and $\{x \in Z : f(x) > \alpha\}$ are measurable, therefore, $\{x \in I : f(x) > \alpha\}$ is also measurable.
Hence, f is measurable on the interval $I \subseteq \mathbb{R}^n$.

3. (Exercise 4.14)
Let $f(x, y)$ be as in Exercise 13. Show that given $\epsilon > 0$, there exists a closed $F \subset E$ with $|E - F| < \epsilon$ such that $f(x, y)$ converges uniformly for $x \in F$ to $f(x)$ as $y \to 0$. (Follow the proof of Egorov's theorem, using the sets $E_{\epsilon,1/m}$ defined in Exercise 13 in place of the sets E_m in the proof of Lemma 4.18.)

Proof.
By Exercise 4.13 and the hint, let

$$E_{\epsilon,\frac{1}{m}} = \{x \in E : |f(x, y) - f(x)| \le \epsilon \text{ for all } y < \frac{1}{m}\}$$

for $m \in \mathbb{Z}^+$.
By Exercise 4.13, we also know that $\lim\limits_{y \to 0} f(x, y)$, so there exists $M' \in \mathbb{Z}^+$ such that for $y < 1/M'$, we have $|f(x, y) - f(x)| \leq \epsilon$, then $E_{\epsilon,1/m} \nearrow E$.
By Lemma 3.26, since $E_{\epsilon,1/m} \nearrow E$, then $|E_{\epsilon,1/m}| \to |E|$.
Follow the proof of Egorov's Theorem, for any $\epsilon > 0$, there exists $M \in \mathbb{Z}^+$ such that $|E - E_{\epsilon,1/M}| < \epsilon 2^{-m-1}$.
By Egorov's Theorem, since $E_{\epsilon,1/M}$ is measurable, there exists a closed set F_m such that $F_m \subseteq E_{\epsilon,1/M}$ and $|E_{\epsilon,1/M} - F_m| < \epsilon 2^{-m-1}$.
Hence

$$|E - F_m| \leq |E - E_{\epsilon,1/M}| + |E_{\epsilon,1/M} - F_m| < \epsilon 2^{-m}$$

Let $F = \bigcap\limits_{m} F_m$, then

$$|E - F| \leq |E - \cap_{m=1}^{\infty} F_m| \leq |\cup_{m=1}^{\infty} (E - F_m)| \leq \sum_{m=1}^{\infty} |E - F_m| < \sum_{m=1}^{\infty} \epsilon 2^{-m} < \epsilon$$

and also $f(x, y)$ converges uniformly to $f(x)$ on F as $y \to 0$.

4. (Exercise 4.15)
Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable E with $|E| < +\infty$. If $|f_k(x)| \leq M_x < +\infty$ for all k for each $x \in E$, show that given $\epsilon > 0$, there is closed $F \subset E$ and a finite M such that $|E - F| < \epsilon$ and $|f_k(x)| \leq M$ for all k and all $x \in F$.

Proof.
Let $\epsilon > 0$ and $f(x) = \sup\limits_{k \in \mathbb{N}} f_k(x)$.
Since each f_k is measurable, then f is measurable and $f(x) \leq M_x$ for all $x \in E$.
Since f is measurable on E, by Lusin's Theorem, then for all $\epsilon > 0$, there will exist a closed $F \subseteq E$ such that $|E - F| < \epsilon$ and f is continuous relative to F.
Since $|E| < \infty$ and F is closed, we can find a compact set $F^* \subseteq F$ such that $|E - F^*| < \epsilon$.
Since f is continuous relative to F and F^*, hence, f will have the maximum, so there will exist a constant M such that $f(x) \leq M$ for all $x \in F^* \subseteq F \subseteq E$.

5. (Exercise 4.16)
Prove that $f_k \xrightarrow{m} f$ on E if and only if give $\epsilon > 0$, there exists K such that $|\{|f - f_k| > \epsilon\}| < \epsilon$ if $k > K$. Give an analogous Cauchy criterion.

Proof.
($\Rightarrow$)
By definition, since $f_k \xrightarrow{m} f$, then for all $\epsilon, \delta > 0$, there will exist $K \in \mathbb{N}$ such that $|\{x \in E : |f(x) - f_k(x)| > \delta\}| < \epsilon$ for all $k > K$.
Take $\delta = \epsilon$, then $|\{x \in E : |f(x) - f_k(x)| > \epsilon\}| < \epsilon$ if $k > K$.

($\Leftarrow$)
Given $\delta, \epsilon > 0$, then there will exist $K_\delta, K_\epsilon \in \mathbb{N}$ such that $|\{x \in E : |f(x) - f_k(x)| > \delta\}| < \delta$ for all $k > K_\delta$ and $|\{x \in E : |f(x) - f_k(x)| > \epsilon\}| < \epsilon$ for all $k > K_\epsilon$.
Let $\eta = \min\{\delta, \epsilon\}$ and take $K = \max\{K_\delta, K_\epsilon\}$, we then have

$$\{x \in E : |f(x) - f_k(x)| > \epsilon\} \subseteq \{x \in E : |f(x) - f_k(x)| > \eta\}$$

That is

$$|\{x \in E : |f(x) - f_k(x)| > \epsilon\}| \leq |\{x \in E : |f(x) - f_k(x)| > \eta\}| < \eta \leq \delta.$$

Hence,
$$f_k \overset{m}{\to} f \text{ on } E.$$

(Cauchy criterion)
By the course's note, we know the Cauchy criterion is:
$f_k \overset{m}{\to} f$ if and only if for all $\epsilon, \delta > 0$ there exists $K \in \mathbb{N}$ such that $|\{x \in E : |f_k(x) - f_l(x)| > \delta\}| < \epsilon$ for all $k, l > K$.

6. (Exercise 4.17)
Suppose that $f_k \overset{m}{\to}$ and $g_k \overset{m}{\to} g$ on E. Show that $f_k + g_k \overset{m}{\to} f + g$ on E and, if $|E| < +\infty$, that $f_k g_k \overset{m}{\to} fg$ on E. If, in addition, $g_k \to g$ on E, $g \neq 0$ a.e., and $|E| < +\infty$, show that $f_k/g_k \overset{m}{\to} f/g$ on E. (For the product $f_k g_k$, write $f_k g_k - fg = (f_k - f)(g_k - g) + f(g_k - g) + g(f_k - f)$. Consider each term separately, using the fact that a function that is finite on E, $|E| < +\infty$ is bounded outside a subset of E with small measure.)

Proof.

(a) $f_k + g_k \overset{m}{\to} f + g$ on E:

Since $f_k \overset{m}{\to}$ on E, then for all $\epsilon > 0$ there will exist $M_1 \in \mathbb{N}$ such that $|\{x \in E : |f_k(x) - f(x)| > \epsilon/2\}| < \epsilon/2$ for all $k \geq M_1$.
Similarly, since $g_k \overset{m}{\to} g$ on E, then for all $\epsilon > 0$ there will exist $M_2 \in \mathbb{N}$ such that $|\{x \in E : |g_k(x) - g(x)| > \epsilon/2\}| < \epsilon/2$ for all $k \geq M_2$.
Consider Triangle Inequality, we then have
$$\begin{aligned}\{x \in E : |(f_k(x) - f(x)) + (g_k(x) - g(x))| < \epsilon\} \subseteq \{&x \in E : |f_k(x) - f(x)| < \epsilon/2\} \\ &\cup \{x \in E : |g_k(x) - g(x)| < \epsilon/2\}.\end{aligned}$$

So
$$\begin{aligned}|\{x \in E : |(f_k(x) - f(x)) + (g_k(x) - g(x))| < \epsilon\}| &= |\{x \in E : |(f_k(x) + g_k(x)) - (f(x) + g(x))| < \epsilon\}| \\ &\leq \begin{array}{l}|\{x \in E : |f_k(x) - f(x)| < \epsilon/2\}| \\ + |\{x \in E : |g_k(x) - g(x)| < \epsilon/2\}|\end{array} \\ &< \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

Take $k > M = \max\{M_1, M_2\}$, then we will have
$$f_k + g_k \overset{m}{\to} f + g \text{ on } E$$

(b) $f_k g_k \overset{m}{\to} fg$ on E:

Follow the hint, since $|E| < +\infty$, we can re-write $f_k g_k - fg$ as
$$(f_k - f)(g_k - g) + f(g_k - g) + g(f_k - f)$$
Since $f_k \overset{m}{\to}$ on E, then for all $\epsilon > 0$ there will exist $M_3 \in \mathbb{N}$ such that $|\{x \in E : |f_k(x) - f(x)| > \sqrt{\epsilon}\}| < \epsilon/2$ for all $k \geq M_3$.
Similarly, since $g_k \overset{m}{\to} g$ on E, then for all $\epsilon > 0$ there will exist $M_4 \in \mathbb{N}$ such that $|\{x \in E : |g_k(x) - g(x)| > \sqrt{\epsilon}\}| < \epsilon/2$ for all $k \geq M_4$.
Take $k > M = \max\{M_3, M_4\}$, we then have
$$\begin{aligned}|\{|x \in E : (f_k(x) - f(x))(g_k(x) - g(x))| > \epsilon\}| &\leq |\{x \in E : |f_k(x) - f(x)| > \sqrt{\epsilon}\}| \\ &\quad + |\{x \in E : |g_k(x) - g(x)| > \sqrt{\epsilon}\}| \\ &< \epsilon\end{aligned}$$

Hence, $(f_k - f)(g_k - g) \xrightarrow{m} 0$.

Following, we will show that $f(g_k - g) \xrightarrow{m} 0$ and $g(f_k - f) \xrightarrow{m} 0$.
By Exercise 4.15, for the sequence of measurable function $\{f\}$, there is a closed $F \subseteq E$ and a finite n such that $|E - F| < \epsilon/2$ and $|f(x)| \leq n$ for all $x \in F$.
Since $g_k \xrightarrow{m} g$ on E, then for all $\epsilon > 0$ there will exist $M_5 \in \mathbb{N}$ such that $|\{x \in E : |g_k(x) - g(x)| > \epsilon/n\}| < \epsilon/2$ for all $k \geq M_5$.
So

$$\begin{aligned}|\{x \in E : |f(g_k - g)| > \epsilon\}| &= |\{x \in F : |f(g_k - g)| > \epsilon\}| + |\{x \in E \setminus F : |f(g_k - g)| > \epsilon\}| \\ &\leq |\{x \in F : |g_k - g| > \epsilon/M\}| + |E \setminus F| \\ &< \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

for all $k > M_5$.
Therefore, $f(g_k - g) \xrightarrow{m} 0$.
Similarly, $g(f_k - f) \xrightarrow{m} 0$.
Hence, by above all, we will know that $f_k g_k - fg \xrightarrow{m} 0$, that is

$$f_k g_k \xrightarrow{m} fg \text{ on } E.$$

(c) $f_k/g_k \xrightarrow{m} f/g$ on E:

Since part(b), it suffices to only show that $1/g_k \xrightarrow{m} 1/g$ on E.
$g \neq 0$ a.e., then $1/g$ is measurable and finite a.e. in E.
Since $g_k \to g$ on E for sufficiently large k then $g_k \neq 0$ a.e., so that $1/g_k$ is also measurable and finite a.e. in E.
By Theorem 4.21, since $1/g_k \to 1/g$ a.e. on E and $|E| < +\infty$, then $1/g_k \xrightarrow{m} 1/g$ on E.
Hence, $f_k/g_k \xrightarrow{m} f/g$ on E.

7. (Exercise 4.18)
If f is measurable on E, define $\omega_f(a) = |\{f > a\}|$ for $-\infty < a < +\infty$. If $f_k \nearrow f$, show that $\omega_{f_k} \nearrow \omega_f$. If $f_k \xrightarrow{m} f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f. (For the second part, show that if $f_k \xrightarrow{m} f$, then $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a-\epsilon)$ and $\liminf_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a+\epsilon)$ for every $\epsilon > 0$.)

Proof.
Since $\omega_f(a) = \{f > a\} = \cup_{i=1}^{\infty}\{f_i > a\}$ and $\{f_i > a\} \subseteq \{f_{i+1} > a\}$ for all i, then

$$\{f_k > a\} = \cup_{i=1}^{k}\{f_i > a\} \nearrow \cup_{i=1}^{\infty}\{f_i > a\} = \{f > a\}$$

as $k \to \infty$.
Hence, $|\{f_k > a\}| \to |\{f > a\}|$ and $|\{f_k > a\}| \leq |\{f_{k+1} > a\}|$ for all k, so $\omega_{f_k} \nearrow \omega_f$.
Suppose that $f_k \xrightarrow{m} f$.
Let a be a point of continuity of ω_f.
Given any $\epsilon, \eta > 0$, there exists $M_1 > 0$ such that for all $k \geq M_1$, we then have

$$\begin{aligned}|\{f_k > a\}| &\leq |\{f_k > a\} - \{f_k > a\} \cap \{f > a - \epsilon\}| + |\{f > a - \epsilon\}| \\ &\leq |\{|f - f_k| > \epsilon\}| + |\{f > a - \epsilon\}| \\ &\leq \eta + |\{f > a - \epsilon\}|.\end{aligned}$$

That is $\limsup\limits_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \epsilon)$, and there exists $M_2 > 0$ such that for all $k \geq M_2$, we then have

$$\begin{aligned}|\{f > a + \epsilon\}| &\leq |\{f > a + \epsilon\} - \{f > a + \epsilon\} \cap \{f_k > a\}| + |\{f_k > a\}| \\ &\leq |\{|f - f_k| > \epsilon\}| + |\{f_k > a\}| \\ &\leq \eta + |\{f_k > a\}|.\end{aligned}$$

That is $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a+\epsilon)$.
Since ω_f is continuous at a, then we have

$$\limsup_{k\to\infty} \omega_{f_k}(a) \leq \lim_{\epsilon\to 0} \omega_f(a-\epsilon) = \omega_f(a) = \lim_{\epsilon\to 0} \omega_f(a+\epsilon) \leq \liminf_{k\to\infty} \omega_{f_k}(a).$$

Therefore, $\lim_{k\to\infty} \omega_{f_k}(a) = \omega_f(a)$, so $\omega_{f_k} \to \omega_f$ at each point of contiunity of ω_f.