REAL VARIABLES: PSET 7

1. Problem 6.1

a) Consider χ_E. We know that $E_x = \{y : (x, y) \in E\}$ has measure 0 in $\mathbb{R}^1$. Theorem 5.11 says that the integral of a non-negative function is zero only if the function is equal to zero a.e. Clearly, $\chi_E = 0$ a.e. only if E has measure 0. Suppose that $|E| < \infty$. For every $x \in E$:

$$\chi_{E_x} = 0 \implies \int\limits_{E_x} \chi_{E_x} dy = \int\limits_{E_x} \chi_E dy = 0 \implies \int\limits_R \left[\int\limits_{E_x} \chi_E dy \right] dx = 0$$

So by Theorem 6.8, $\iint_E \chi_E dxdy = 0$. So $\chi_E = 0$ a.e., but since it is the characteristic function of E, this means that $|E| = 0$. If $|E| = \infty$, then for every compact subset of E, the argument above holds. Then taking a sequence of closed balls of radius k centered at the origin, the measure of $E \cap B_k(0) = 0 \ \forall k$. Taking the limit as $k \to \infty$, this shows that $|E| = 0$. Since E has measure 0, by Theorem 6.5, for almost every $y \in \mathbb{R}^n$, $\{x : (x, y) \in E\}$ has measure zero.

b) Let E be the set where $f(x, y) = \infty$. By assumption, for a.e. $x \in \mathbb{R}$, $\{y : (x, y) \in E\}$ has $\mathbb{R}$ measure 0. Then by the result of part 1, E has measure 0, and for almost every $y \in \mathbb{R}$, $\{x : (x, y) \in E\}$ has measure zero.

2. Problem 6.2

Since both f and g are measurable, $f^{-1}(A)$ and $g^{-1}(B)$ are measurable for any measurable sets A and B. Take any bounded, measurable set $E \subset \mathbb{R}$. Then $f^{-1}(E) = E_1$ and $g^{-1}(E) = E_2$ are both measurable. We can apply Lusin's Theorem to show that there exist closed subsets $F_1 \subset E_1$ and $F_2 \subset E_2$ such that for any $\epsilon > 0$, f and g are continuous on F_1 and F_2 respectively, and $|E_1 - F_1| < \epsilon/2$ and $|E_2 - F_2| < \epsilon/2$. Then $f(x)g(y)$ is continuous on $F_1 \times F_2$, and $|(E_1 \times E_2) - (F_1 \times F_2)| < \epsilon/2|E_2| + \epsilon/2|E_1|$. Since we can approximate any open set in $\mathbb{R}^n \times \mathbb{R}^n$ by a countable union of cubes, and we can write any measurable set as the union of a set of measure zero and a G_δ set, we can approximate any measurable set in $\mathbb{R}^n \times \mathbb{R}^n$ by the product of measurable sets in $\mathbb{R}^n$. So for any measurable set in $\mathbb{R}^n \times \mathbb{R}^n$, f(x)g(y) has property $\mathcal{C}$, so by Lusin's Theorem, f(x)g(y) is measurable on $\mathbb{R}^n \times \mathbb{R}^n$.

From the fact that the characteristic function of any set is measurable if and only if the set is measurable, if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$, then their product is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

Since $E_1 \times E_2$ is measurable, $\chi_{E_1 \times E_2}$ is integrable, and by Theorem 6.8:

$$|E_1 \times E_2| = \iint_{E_1 \times E_2} \chi_{E_1 \times E_2} = \int\limits_{E_1} \chi_{E_1} \int\limits_{E_2} \chi_{E_2} = \int\limits_{E_1} \chi_{E_1} |E_2| = |E_1||E_2|$$

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3. Problem 6.3

Since $f(x) - f(y) \in L(I)$, where $I = (0,1) \times (0,1)$, by Fubini's theorem, we have:

for almost every $y \in (0,1)$, $f(x) - f(y)$ is integrable. In particular, for such y, $f(y)$ is finite, which implies $f(x)$ is integrable.

4. Problem 6.5

a) Split the integral $\int_0^\infty \omega(y)dy$ into two parts, one over the set A where ω is continuous, and one over the set B where ω is discontinuous. Then $|B| = 0$, so $\int_B \omega(y)dy = 0$. So:

$$\int\limits_0^\infty \omega(y)dy = \int_A \omega(y)dy = \int\limits_0^\infty |x \in E : f(x) \geq y|dy = \iint\limits_{R(f,E)} dxdy = \int\limits_E f$$

The equivalence of the total integral to the iterated integrals in the second to last step is guaranteed by Tonelli's Theorem.

b) The set $\{x \in E : f(x) \geq y\} = \{x \in E : f(x)^p \geq y^p\}$. So examining the measure of the region in each differential element: $|\{x \in E : f(x) \geq y\}|dy = \omega(y)dy \implies |\{x \in E : f(x)^p \geq y^p\}|dy^p = \omega(y)dy^p$.

$$\implies \boxed{\int\limits_E f^p = \int\limits_0^\infty \omega(y)d(y^p) = p\int\limits_0^\infty y^{p-1}\omega(y)d(y)}$$

5. Problem 6.6

$$(f*g)\check{}(x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u-t)g(t)dte^{-ixu}du$$

by Fubini's Theorem: $$= \int_{-\infty}^{+\infty} g(t) \int_{-\infty}^{+\infty} f(u-t)e^{-ixu}dudt = \int_{-\infty}^{+\infty} g(t) \int_{-\infty}^{+\infty} f(u-t)e^{-ixu}e^{-ixt}e^{ixt}dudt$$

$$= \int_{-\infty}^{+\infty} g(t)e^{-ixt} \int_{-\infty}^{+\infty} f(u-t)e^{-ix(u-t)}dudt \qquad v = u-t, \quad dv = du$$

$$= \int_{-\infty}^{+\infty} g(t)e^{-ixt} \int_{-\infty}^{+\infty} f(v)e^{-ixv}dvdt = \int_{-\infty}^{+\infty} g(t)e^{-ixt}\hat{f}(x)dt = \hat{f}(x)\int_{-\infty}^{+\infty} g(t)e^{-ixt}dt = \boxed{\hat{f}(x)\hat{g}(x)}$$

6. Problem 7.1

Suppose that $|f| > 0$ on some set $E \subset \mathbb{R}^n$ with $|E| > 0$. Then there exists $c > 0$ such that $E_c := \{|f| > c\}$ has positive measure. Therefore $c\chi_{E_c}(x) \leq |f(x)|$. Then $c\chi^*_{E_c} \leq |f^*(x)|$ for all $x \in \mathbb{R}^n$. Applying (7.7) from the text, we find that for all large x,

$$c_1 c\frac{|E_c|}{|x|^n} \leq c\chi^*_{E_c} \leq f^*(x).$$

For small x, since f^* is positive and lower semicontinuous, it attains its minimum, which is positive.

7. Problem 7.2

By Theorem 7.3, we can approximate f by a sequence of continuous functions $\{C_k\}$.

$$|(f*\phi_\epsilon)(x) - f(x)| = \left| \int_{R^n} [f(x-y) - f(x)]\phi_\epsilon(y)dy \right| \leq \int_{R^n} |[f(x-y) - f(x)]|\phi_\epsilon(y)dy$$

In the last step we assumed without loss of generality, that $\phi_\epsilon(y) > 0$.

$$\leq \int_{R^n} [|f(x-y) - C_k(x-y)| + |C_k(x-y) - C_k(x)| + |C_k(x) - f(x)|]\phi_\epsilon(y)dy$$

Since C_k is continuous, and $y \to x$ as the support of the integrand shrinks as $\epsilon \to 0$, the middle term vanishes.

$$= \int_{R^n} [|f(x-y) - C_k(x-y)| + |C_k(x) - f(x)|]\phi_\epsilon(y)dy$$

$$= \int_{R^n} |f(x-y) - C_k(x-y)|\phi_\epsilon(y)dy + \int_{R^n} |C_k(x) - f(x)|\phi_\epsilon(y)dy$$

Since the continuous functions converge to f in the L_1 norm, not necessarily pointwise, we need to ensure that some subsequence of these functions simultaneously makes both of these integrals arbitrarily small. To do this, note that:

$$\int_{R^n} |C_m(x-y) - C_n(x)|\phi_\epsilon(y)dy \implies \int_{R^n}\int_{R^n} |C_m(x-y) - C_n(x)|\phi_\epsilon(y)dydx$$

$$= \int_{R^n} \phi_\epsilon(y) \int_{R^n} |C_m(x-y) - C_n(x)|dxdy$$

As $\epsilon \to 0$, $C_m(x-y) \to C_m(x)$, and since the space of continuous functions is complete under the L_1 norm, $\int_{R^n} |C_m(x-y) - C_n(x)|dx < \delta$ for small ϵ and large enough m and n. Then there exists a single subsequence of $\{C_k\}$ that simultaneously makes both integrals converge to zero, so

$$\boxed{(f * \phi_\epsilon)(x) - f(x) \to 0 \quad \text{as} \quad \epsilon \to 0}$$

8. Problem 7.3

The only place in the whole argument that the relative lengths of the sides of the cubes comes into play is in making sure that all cubes overlapping any given Q_i are contained in Q_i^* and in determining the constant $|Q_i^*| = c|Q_i|$. So for rectangles in $\mathbb{R}^2$ with dimensions h and h^2, the entire argument follows the same as in the book, except that $|Q_i^*| = 5^3|Q_i|$, so $\beta = 5^{-3}$.

In general, for two dimensions and any increasing function f(h), let h_i^* be the supremum of the x side length of any rectangle in the given collection, as in the book. However, in order to ensure that R_i^*, the large rectangle at the ith step, contains all rectangles intersecting $R_i = Q_i$, choose the side length h of R_i such that $h > \frac{1}{2}h^*$ and $f(h) > \frac{1}{2}f(h^*)$. Then $\beta = (hf(h))^{-1}$. For n dimensions, add a restraint for each dimension such that $f_i(h) > f_i(h^*)$. Then $\beta = (hf_1(h)f_2(h)...)^{-1}$.

9. Problem 7.4

Consider sets $-E_1$ and E_2, with measure > 0 and $< \infty$. Then both χ_{-E1} and χ_{E2} are integrable, and:

$$\int_x \chi_{-E1} * \chi_{E2} dx = \int_x \int_y \chi_{-E1}(x-y)\chi_{E2}(y)dydx = \int_y \int_x \chi_{-E1}(x-y)\chi_{E2}(y)dxdy$$
$$= \int_y \chi_{E2}(y) \int_x \chi_{-E1}(x-y)dxdy \quad \text{by Fubini}$$

Measure is invariant under translation, so:

$$= \int_y \chi_{E2}(y)|-E1|dy = |E2||-E1| > 0$$

So there must exist some point where $\chi_{-E1} * \chi_{E2} > 0$. Convolution is continuous, so $\chi_{-E1} * \chi_{E2} > 0$ on an interval, (x_1, x_2). Then for all $t \in (x_1, x_2)$, $\chi_{-E1} * \chi_{E2}(t) = \int_x \chi_{-E1}(t-x)\chi_{E2}(x)dx > 0$, there must be some $x \in \mathbb{R}$ for which $\chi_{-E1}(t-x)\chi_{E2}(x) > 0$. Then we have $x \in E2$ and $t - x \in -E1$, $t = t - x + x \in -E1 + E2 \implies t \in E2 - E1$.