

Real Analysis Homework 1

Due Date: 9/23

September 19, 2019

1. Please prove the uniqueness of Theorem 1.3 in the textbook.

2. Let R be a rectangle in \mathbb{R}^n . Prove $\sigma(R) = |R| = m_*(R)$.

3. *Theorem 1.3* states that every open set in \mathbb{R} is the disjoint union of open intervals. The analogue in \mathbb{R}^d , $d \geq 2$, is generally false. Prove the following:

(a) An open disc in \mathbb{R}^2 is not the disjoint union of open rectangles.

[Hint: What happens to the boundary of any of these rectangles?]

(b) An open connected set Ω is the disjoint union of open rectangles if and only if Ω is itself an open rectangle.

4. At the start of the theory, one might define the outer measure by taking coverings by rectangles instead of cubes. More precisely, we define

$$m_*^{\mathcal{R}}(E) = \inf \sum_{j=1}^{\infty} |R_j|,$$

where the inf is now taken over all countable coverings $E \subset \cup_{j=1}^{\infty} \mathcal{R}_j$ by (closed) rectangles.

Show that this approach gives rise to the same theory of measure developed in the text, by proving that $m_*(E) = m_*^{\mathcal{R}}(E)$ for every subset E of \mathbb{R}^d .

[Hint: Use *Lemma 1.1*.]