Real Analysis
Homework 1

Yueh-Chou Lee

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1. (Exercise 8.4)
Let f and g be real-valued and not identically 0 (i.e., neither function equals 0 a.e.), and let $1 < p < \infty$. Prove that equality holds in the inequality $|\int fg| \leq ||f||_p ||g||_{p'}$ if and only if fg has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e. If $||f+g||_p = ||f||_p + ||g||_p$ and $g \neq 0$ in Minkowski's inequality, show that f is a multiple of g. Find analogues of these results for the spaces l^p.

Proof.

(i) ($\Leftarrow$)
Let $|f|^p = c|g|^{p'}$ and $1/p + 1/p' = 1$, then

$$||f||_p ||g||_{p'} = \left(\int |f|^p\right)^{1/p} (\int |g|^{p'})^{1/p'} = c^{1/p} \int |g|^{p'}$$

and

$$|fg| = |f||g| = c^{1/p}|g|^{p'}$$

Since f and g be real-valued and not identically 0, then

$$|\int fg| = \int |f||g| = c^{1/p} \int |g|^{p'} = ||f||_p ||g||_{p'}$$

($\Rightarrow$)
If $|\int fg| = ||f||_p ||g||_{p'}$, $1/p + 1/p' = 1$, f and g be real-valued and not identically 0, then

$$|\int fg| = \int |fg| = ||f||_p ||g||_{p'} \Rightarrow \frac{\int |fg|}{||f||_p ||g||_{p'}} = \int \frac{|f|}{||f||_p} \frac{|g|}{||g||_{p'}} = 1$$

Let $F = \frac{|f|}{||f||_p}$ and $G = \frac{|g|}{||g||_{p'}}$, then

$$\int F^p = 1 \text{ and } \int G^{p'} = 1$$

So

$$\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{p}\int F^p + \frac{1}{p'}\int G^{p'} = \int FG \Rightarrow \int \left(\frac{1}{p}F^p + \frac{1}{p'}G^{p'} - FG\right) = 0$$

Hence

$$FG = \frac{F^p}{p} + \frac{G^{p'}}{p'}$$

By Young's inequality, we know that the equality holds in

$$FG \leq \frac{F^p}{p} + \frac{G^{p'}}{p'}$$

if only if

$$F^p = G^{p'}$$

So

$$\frac{|f|^p}{||f||_p^p} = \frac{|g|^{p'}}{||g||_{p'}^{p'}} \Rightarrow |f|^p = \frac{||f||_p^p}{||g||_{p'}^{p'}} |g|^{p'} = c\,|g|$$

where c is the constant and $c = \frac{||f||_p^p}{||g||_{p'}^{p'}}$

(ii)

$$\begin{aligned}
||f+g||_p^p &= \int |f+g|^p = \int |f+g| \cdot |f+g|^{p-1} \\
&\leq \int (|f| + |g|) \cdot |f+g|^{p-1} \\
&= \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
\text{By Hölder's inequality} \quad &\leq \left(\left(\int |f|^p \right)^{1/p} + \left(\int |g|^p \right)^{1/p} \right) \cdot \left(\int |f+g|^{(p-1)(\frac{p}{p-1})} \right)^{1-\frac{1}{p}} \\
&= (||f||_p + ||g||_p) \frac{||f+g||_p^p}{||f+g||_p}
\end{aligned}$$

Since $||f+g||_p = ||f||_p + ||g||_p$, then the equality will hold in the Hölder's inequality, so we have

$$|f| = c_f \cdot |f+g|^{p-1} \quad \text{and} \quad |g| = c_g \cdot |f+g|^{p-1}$$

where c_f and c_g are the constant. Hence

$$|f| = \frac{c_f}{c_g} \cdot |g| = C \cdot |g|$$

where C is the constant.

(iii) In l^p space, the proof is similar with (i) and (ii), Hölder's inequality states that

$$\sum_k |a_k b_k| \leq \left(\sum_k |a_k|^p \right)^{1/p} \left(\sum_k |b_k|^{p'} \right)^{1/p'}$$

with equality when

$$|b_k|^p = c \cdot |a_k|^{p'}$$

where c is the constant.
Also, Minkowski's inequality states

$$\left(\sum_k |a_k + b_k|^p \right)^{1/p} \leq \left(\sum_k |a_k|^p \right)^{1/p} + \left(\sum_k |b_k|^p \right)^{1/p}$$

with equality when

$$a_k = c_k \cdot b_k$$

where c_k is the constant.

2. (Exercise 8.5)
For $0 < p \le \infty$ and $0 < |E| < +\infty$, define
$$N_p[f] = \left(\frac{1}{|E|}\int_E |f|^p\right)^{1/p},$$
where $N_\infty[f]$ means $||f||_\infty$. Prove that if $p_1 < p_2$, then $N_{p_1}[f] \le N_{p_2}[f]$. Prove also that if $1 \le p \le \infty$, then $N_p[f+g] \le N_p[f] + N_p[g]$, $(1/|E|)\int_E |fg| \le N_p[f]N_{p'}[g]$, $1/p + 1/p' = 1$, and that $\lim_{p\to\infty} N_p[f] = ||f||_\infty$. Thus, N_p behaves like $||\cdot||_p$ but has the advantage of being monotone in p. Recall Exercise 28 of Chapter 5.

Proof.

(i)
$$N_{p_1}[f] = \left(\frac{1}{|E|}\int_E |f|^{p_1}\right)^{1/p_1} \Rightarrow |E|\left(N_{p_1}[f]\right)^{p_1} = \int_E |f|^{p_1}\cdot 1$$
Since $p_1 < p_2$, then $1 \le \frac{p_2}{p_1} \le \infty$, then by Hölder's inequality, we have
$$\int_E |f|^{p_1}\cdot 1 \le \left(\int_E (|f|^{p_1})^{\frac{p_2}{p_1}}\right)^{\frac{p_1}{p_2}} \cdot \left(\int_E 1^{\frac{p_2}{p_2-p_1}}\right)^{\frac{p_2-p_1}{p_2}} = \left(\int_E |f|^{p_2}\right)^{\frac{p_1}{p_2}} \cdot |E|^{\frac{p_2-p_1}{p_2}}$$
$$\Rightarrow N_{p_1}[f] = \left(\frac{1}{|E|}\int_E |f|^{p_1}\right)^{1/p_1} \le |E|^{\frac{-1}{p_2}}\left(\int_E |f|^{p_2}\right)^{1/p_2} = N_{p_2}[f]$$

(ii) Since $1 \le p \le \infty$, then by Minkowski's inequality, we have
$$||f+g||_p \le ||f||_p + ||g||_p \Rightarrow N_p[f+g] \le N_p[f] + N_p[g]$$

(iii) Since $1 \le p \le \infty$ and $1/p + 1/p' = 1$, then by Hölder's inequality, we have
$$||fg||_1 \le ||f||_p + ||g||_{p'} \Rightarrow \frac{1}{|E|}\int_E |fg| \le \left(\frac{1}{|E|}\right)^{\frac{1}{p}} ||f||_p + \left(\frac{1}{|E|}\right)^{\frac{1}{p'}} ||g||_{p'} = N_p[f]N_{p'}[g]$$

(iv) Since $\lim_{p\to\infty} |E|^{-1/p} = 1$, then
$$\lim_{p\to\infty} N_p[f] = \lim_{p\to\infty}\left(\frac{1}{|E|}\int_E |f|^p\right)^{1/p} = \lim_{p\to\infty} |E|^{-1/p}||f||_p = ||f||_\infty$$

3. (Exercise 8.7)
Show that when $0 < p < 1$, the neighborhoods $\{f : ||f||_p < \epsilon\}$ of zero in $L^p(0,1)$ are not convex. (Let $f = \chi_{(0,\epsilon^p)}$, and $g = \chi_{(\epsilon^p, 2\epsilon^p)}$. Show that $||f||_p = ||g||_p = \epsilon$, but that $||\frac{1}{2}f + \frac{1}{2}g||_p > \epsilon$.)

Proof.
For $\epsilon > 0$, let $f = \chi_{(0,\epsilon^p)}$, and $g = \chi_{(\epsilon^p,2\epsilon^p)}$, then
$$||f||_p = \left(\int_0^{\epsilon^p} 1^p dx\right)^{1/p} = \epsilon$$
$$||g||_p = \left(\int_{\epsilon^p}^{2\epsilon^p} 1^p dx\right)^{1/p} = \epsilon$$
$$||\frac{1}{2}f + \frac{1}{2}g||_p^p = \int_0^{2\epsilon^p} |\frac{1}{2}f + \frac{1}{2}g|^p dx = \int_0^{\epsilon^p} \frac{1}{2^p}dx + \int_{\epsilon^p}^{2\epsilon^p} \frac{1}{2^p}dx = \frac{\epsilon^p}{2^{p-1}}$$
Then
$$||\frac{1}{2}f + \frac{1}{2}g||_p = \frac{\epsilon}{2^{1-1/p}} > \frac{1}{2}||f||_p + \frac{1}{2}||g||_p = \epsilon, \quad 0 < p < 1$$
So $\{f : ||f||_p < \epsilon\}$ is not convex for every $\epsilon > 0$ and $0 < p < 1$.

4. (Exercise 8.9)
If f is real-valued and measurable on E, $|E| > 0$, define its *essential infimum* on E by

$$ess_E \inf f = \sup\{\alpha : |\{x \in E : f(x) < \alpha\}| = 0\}.$$

If $f \geq 0$, show that $ess_E \inf f = (ess_E \sup 1/f)^{-1}$.

Proof.

$$\begin{aligned} ess_E \inf f &= \sup\{\alpha : |\{x \in E : f(x) < \alpha\}| = 0\} \\ &= \sup\{\alpha : |\{x \in E : \frac{1}{f(x)} > \frac{1}{\alpha}\}| = 0\} \\ &= \inf\{\frac{1}{\alpha} : |\{x \in E : \frac{1}{f(x)} > \frac{1}{\alpha}\}| = 0\} \\ &= \left(\inf\{\alpha : |\{x \in E : \frac{1}{f(x)} > \alpha\}| = 0\}\right)^{-1} \\ &= (ess_E \sup 1/f)^{-1} \end{aligned}$$

5. (Exercise 8.11)
If $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $||g_k||_\infty \leq M$ for all k, prove that $f_k g_k \to fg$ in L^p.

Proof.
Since $f_k \to f$ in L^p, $1 \leq p < \infty$, then $||f_k - f||^p \to 0$.
Since $g_k \to g$ pointwise, then $|fg_k - fg|^p \to 0$ pointwise.
By Minkowski's Inequality, we have

$$\begin{aligned} ||f_k g_k - fg||_p &\leq ||f_k g_k - fg_k||_p + ||fg_k - fg||_p \\ &\leq M||f_k - f||_p + \left(\int |fg_k - fg|^p\right)^{1/p} \end{aligned}$$

So $||f_k g_k - fg||_p \to 0$, that is $f_k g_k \to fg$ in L^p.

6. (Exercise 8.12)
Let $f, \{f_k\} \in L^p$, $0 < p \leq \infty$. Show that if $||f - f_k||_p \to 0$, then $||f_k||_p \to ||f||_p$. Conversely, if $f_k \to f$ a.e. and $||f_k||_p \to ||f||_p$, $0 < p < \infty$, show that $||f - f_k||_p \to 0$. Show that the converse may fail for $p = \infty$. (For the converse when $0 < p < \infty$, note that $|f - f_k|^p \leq c(|f|^p + |f_k|^p)$ with $c = max\{2^{p-1}, 1\}$; then apply, for example, the sequential version of Lebesgue's dominated convergence theorem given in Exercise 23 of Chapter 5.)

Proof.

(i) For $1 \leq p \leq \infty$, we have

$$|||f_k||_p - ||f||_p| \leq |||f_k - f||_p| \to 0$$

So $||f_k||_p \to ||f||_p$
For $0 < p < 1$, we have

$$|||f_k||_p^p - ||f||_p^p| \leq |||f_k - f||_p^p| \to 0$$

So $||f_k||_p^p \to ||f||_p^p$, hence $||f_k||_p \to ||f||_p$

(ii) Conversely, since $f_k \to f$ a.e., then $|f - f_k| \to 0$ a.e.
Let $c = max\{2^{p-1}, 1\}$, $\phi_k = c(|f|^p + |f_k|^p)$ and $\phi = 2c|f|^p$, then $\phi_k \to \phi$ a.e. and $|f - f_k|^p \leq \phi_k$ a.e. since $f_k \to f$ a.e. and $|f - f_k|^p \leq c(|f|^p + |f_k|^p)$.
$\phi \in L^p(E)$ since $f \in L^p$.
Also, $\int_E \phi_k \to \int_E \phi$ since $||f_k||_p^p \to ||f||_p^p$ By Generalized Lebesgue's Dominated Convergence Theorem, we have

$$\int_E |f - f_k|^p \to 0 \Rightarrow ||f - f_k||_p \to 0$$

7. (Exercise 8.17)
Suppose that $f_k, f \in L^2$ and that $\int f_k g \to \int fg$ for all $g \in L^2$ (i.e., $\{f_k\}$ converges weakly in L^2 to f). If $||f_k||_2 \to ||f||_2$, show that $f_k \to f$ in L^2 norm. The same is true for L^p, $1 < p < \infty$, by a 1913 result of Radon.

Proof.

$$\begin{aligned}
||f_k - f||_2^2 &= \int (f_k - f)\overline{(f_k - f)} \\
&= ||f_k||_2^2 - \int f_k\overline{f} - \int f\overline{f_k} + ||f||_2^2 \\
&= ||f_k||_2^2 - \int f_k\overline{f} - \overline{\int f_k\overline{f}} + ||f||_2^2 \\
&\to ||f_k||_2^2 - \int f\overline{f} - \overline{\int f\overline{f}} + ||f||_2^2 = 0
\end{aligned}$$

So $f_k \to f$ in L^2 norm.

8. (Exercise 8.21)
If $f \in L^p(\mathbb{R}^n), 0 < p < \infty$, show that

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy = 0 \quad \text{a.e.}$$

Note by Exercise 5 that if this condition holds for a given p, then it also holds for all smaller p.

Proof.
Let $\{r_k\}$ be the set of rational numbers, and let Z_k be the set where the formula

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - r_k|^p dy = |f(x) - r_k|^p$$

is not valid.
Since $|f(y) - r_k|^p \leq c(|f(y)|^p + |r_k|^p)$ is locally integrable where $c = \max\{2^{p-1}, 1\}$, by Lebesgue's Differentiation Theorem, we have $|Z_k| = 0$.
Let $Z = \cup Z_k$, then $|Z| = 0$.
For any Q, x and r_k

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy &= \frac{1}{|Q|} \int_Q |[f(y) - r_k] - [f(x) - r_k]|^p dy \\
&\leq c \cdot \frac{1}{|Q|} \int_Q |f(y) - r_k|^p dy + c \cdot \frac{1}{|Q|} \int_Q |f(x) - r_k|^p dy \\
&= c \cdot \frac{1}{|Q|} \int_Q |f(y) - r_k|^p dy + c \cdot |f(x) - r_k|^p
\end{aligned}$$

Therefore, if $x \notin Z$,

$$\limsup_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy \leq 2c \cdot |f(x) - r_k|^p \quad \text{for every } r_k.$$

For any x at which $f(x)$ is finite (in particular, almost everywhere), we can choose r_k such that $|f(x) - r_k|$ is arbitrarily small. This shows that the left side of the last formula is zero a.e., and completes the proof.