

# Real Analysis

## Homework 9

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1. (Exercise 7.4)

If  $E_1$  and  $E_2$  are measurable subsets of  $\mathbb{R}^1$  with  $|E_1| > 0$  and  $|E_2| > 0$ , prove that the set  $\{x : x = x_1 - x_2, x_1 \in E_1, x_2 \in E_2\}$  contains an interval. (cf. Lemma 3.37.)

**Proof.**

Consider the set  $-E_1$  and  $E_2$ , with positive finite measure, then both  $\chi_{-E_1}$  and  $\chi_{E_2}$  are integrable, and

$$\begin{aligned}\int_x \chi_{-E_1} * \chi_{E_2} dx &= \int_x \int_y \chi_{-E_1}(x-y) \chi_{E_2}(y) dy dx \\ &\stackrel{\text{Fubini}}{=} \int_y \int_x \chi_{-E_1}(x-y) \chi_{E_2}(y) dx dy \\ &= \int_y \chi_{E_2}(y) \int_x \chi_{-E_1}(x-y) dx dy \\ &= \int_y \chi_{E_2}(y) | -E_1 | dy \\ &= |E_2| | -E_1 | > 0\end{aligned}$$

So there must exist some point where  $\chi_{-E_1} * \chi_{E_2} > 0$ . Convolution is continuous, so  $\chi_{-E_1} * \chi_{E_2} > 0$  on the interval,  $(x_1, x_2)$ , then for all  $t \in (x_1, x_2)$ ,

$$\chi_{-E_1} * \chi_{E_2}(t) = \int_x \chi_{-E_1}(t-x) \chi_{E_2}(x) dx > 0$$

there must be some  $x \in \mathbb{R}$  for which  $\chi_{-E_1}(t-x) \chi_{E_2}(x) > 0$ , then we have  $x \in E_2$  and  $t-x \in -E_1$ ,

$$t = (t-x) + (x) \in -E_1 + E_2 \Rightarrow t \in E_2 - E_1$$

Hence the set  $\{x : x = x_1 - x_2, x_1 \in E_1, x_2 \in E_2\}$  contains an interval.

2. (Exercise 7.5)

Let  $f$  be of bounded variation on  $[a, b]$ . If  $f = g + h$ , where  $g$  is absolutely continuous and  $h$  is singular, show that

$$\int_a^b \phi df = \int_a^b \phi f' dx + \int_a^b \phi dh$$

for any continuous  $\phi$ .

**Proof.**

$\int_a^b \phi dg$  exists because  $g$  is absolutely continuous and therefore continuous.

$\int_a^b \phi df$  exists because  $f$  is of bounded variation.

Since  $\int_a^b \phi dg$ ,  $\int_a^b \phi df$  exist and

$$\int_a^b \phi df - \int_a^b \phi dg = \int_a^b \phi d(f-g) = \int_a^b \phi dh$$

then  $\int_a^b \phi dh$  also exists.

By Theorem 7.32, so

$$\begin{aligned}\int_a^b \phi df &= \int_a^b \phi d(g+h) \\ &= \int_a^b \phi dg + \int_a^b \phi dh \\ &= \int_a^b \phi g' dx + \int_a^b \phi dh \\ &= \int_a^b \phi f' dx + \int_a^b \phi dh\end{aligned}$$

where  $\int_a^b \phi f' dx = \int_a^b \phi g' dx$  since  $h$  is singular ( $h' = 0$ ), then  $f' = g'$  a.e.

3. (Exercise 7.8)

Prove the following converse of Theorem 7.31: If  $f$  is of bounded variation on  $[a, b]$ , and if the function  $V(x) = V[a, x]$  is absolutely continuous on  $[a, b]$ , then  $f$  is absolutely continuous on  $[a, b]$ .

**Proof.**

Since  $f$  is of bounded variation on  $[a, b]$  and the function  $V(x) = V[a, x]$ ,

$$V(x) = V[a, x] \leq V[a, b] < \infty$$

for all  $x \in [a, b]$ .

Since  $V(x)$  is absolutely continuous on  $[a, b]$ , for given  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any collection  $\{[a_i, b_i]\}$  of nonoverlapping subintervals of  $[a, b]$ ,

$$\sum |V(b_i) - V(a_i)| < \epsilon, \quad \text{if } \sum (b_i - a_i) < \delta$$

$V(b_i) - V(a_i)$  is well-defined since  $V(x) < \infty$  for all  $x \in [a, b]$ , then if  $\sum (b_i - a_i) < \delta$ ,

$$\begin{aligned}\sum |f(b_i) - f(a_i)| &\leq \sum V[a_i, b_i] \\ &= \sum |V[a, b_i] - V[a, a_i]| \\ &= \sum |V(b_i) - V(a_i)| \\ &< \epsilon\end{aligned}$$

Hence  $f$  is absolutely continuous on  $[a, b]$ .

4. (Exercise 7.9)

If  $f$  is of bounded variation on  $[a, b]$ , show that

$$\int_a^b |f'| \leq V[a, b]$$

Show that if equality holds in this inequality, then  $f$  is absolutely continuous on  $[a, b]$ . (For the second part, use Theorems 2.2(ii) and 7.24 to show that  $V(x)$  is absolutely continuous and then use the result of Exercise 8.)

**Proof.**

(i) .

By Theorem 7.24, since  $f$  is of bounded variation on  $[a, b]$ , then

$$V'(x) = |f'(x)| \text{ for a.e. } x \in [a, b]$$

The integral

$$\int_a^b |f'| = \int_a^b V' \leq V(b^-) - V(a^+) \leq V[a, b]$$

(ii) .

If the equality holds in this inequality.

By Theorem 7.24, we have

$$\begin{aligned} \int_a^x V' &= \int_a^x |f'| \\ &= \int_a^b |f'| - \int_x^b |f'| \\ &= V[a, b] - \int_x^b |f'| \\ &= V[a, x] + V[x, b] - \int_x^b |f'| \\ &\geq V[a, x] \end{aligned}$$

for all  $x \in [a, b]$ .

This completes the prove by Theorem 7.29 and Exercise 7.8.

5. (Exercise 7.10)

- (a) Show that if  $f$  is absolutely continuous on  $[a, b]$  and  $Z$  is a subset of  $[a, b]$  of measure zero, then the image set defined by  $f(Z) = \{w : w = f(z), z \in Z\}$  also has measure zero. Deduce that the image under  $f$  of any measurable subset of  $[a, b]$  is measurable. (Compare Theorem 3.33.) (Hint: use the fact that the image of an interval  $[a_i, b_i]$  is an interval of length at most  $V(b_i) - V(a_i)$ .)
- (b) Give an example of a strictly increasing Lipschitz continuous function  $f$  and a set  $Z$  with measure 0 such that  $f^{-1}(Z)$  does not have measure 0 (and consequently,  $f^{-1}$  is not absolutely continuous). (Let  $f^{-1}(x) = x + C(x)$  on  $[0, 1]$ , where  $C(x)$  is the Cantor–Lebesgue function.)

**Proof.**

(a) .

Let  $\epsilon > 0$ , since  $f$  is absolutely continuous on  $[a, b]$ , so is the variation  $V$  of  $f$  over  $[a, b]$ , then there exists  $\delta > 0$  such that

$$\sum_i |V(b_i) - V(a_i)| < \epsilon$$

for any nonoverlapping subintervals  $[a_i, b_i]$  of  $[a, b]$  the sum of whose length  $\sum_i (b_i - a_i)$  is less than  $\delta$ .

Let  $Z$  be any subset of  $[a, b]$  with measure zero, there exists an open set  $G$  contains  $Z$  such that  $|G| < \delta$ .

The open set  $G$  can be written as the countable union of nonoverlapping subintervals  $[a'_i, b'_i]$  of  $[a, b]$ .

Thus

$$\sum_i (b'_i - a'_i) < \delta$$

This implies that

$$\begin{aligned} |f(Z)|_e &\leq |f(\cup_i [a'_i, b'_i])|_e \\ &\leq \sum_i |f([a'_i, b'_i])|_e \\ &\leq \sum_i \left[ \sup_{x \in [a'_i, b'_i]} f(x) - \inf_{x \in [a'_i, b'_i]} f(x) \right] \\ &\leq \sum_i [V(b'_i) - V(a'_i)] \\ &< \epsilon \end{aligned}$$

So  $f(Z)$  is measure zero.

For  $E$  be any measurable subset of  $[a, b]$ , written as  $E = F \cup Z$  where  $F$  is of type  $F_\sigma$ ,  $Z$  is a set with measure zero and  $F \cap Z = \phi$ .

Note that  $F$  is union of compact subsets of  $[a, b]$ , then  $f(F)$  is measurable since  $f$  is continuous on  $[a, b]$ .

Hence  $f(E)$  is measurable.

(b) .

Let  $f^{-1}(x) = x + C(x)$  on  $[0, 1]$ .

Since  $f^{-1}(x)$  is strictly increasing, its inverse  $f(x)$  exists and  $f$  is strictly increasing.

Let  $x, y \in f^{-1}([0, 1]) = [0, 2]$ .

Suppose  $x < y$  and write  $x = p + C(p)$ ,  $y = q + C(q)$  where  $p < q$ .

Since  $f^{-1}(x) = x + C(x) \Rightarrow x = f(x + C(x))$  and the Cantor function  $C(x)$  is increasing, then

$$\begin{aligned} f(y) - f(x) &= f(q + C(q)) - f(p + C(p)) \\ &= q - p \\ &\leq q + C(q) - p - C(p) \\ &= y - x \end{aligned}$$

So  $f$  is Lipschitz continuous.

Let  $Z$  be the Cantor set, which has measure zero.

Since  $C(x)$  is constant on each disjoint interval in  $[0, 1] \setminus Z$ ,  $f^{-1}(x)$  maps each interval to an interval of the same length.

Thus

$$|f^{-1}([0, 1] \setminus Z)| = |[0, 1] \setminus Z| = 1$$

Since  $f^{-1}([0, 1]) = [0, 2]$ , thus  $|f^{-1}(Z)| = 1 > 0$ .