# 1 Introduction and Preliminaries

This chapter serves two purposes. The first purpose is to prepare the reader for a more systematic development in later chapters of the methods of real analysis through some introductory accounts of a few specific topics. The second purpose is, in view of the possible situation where some readers might not be conversant with basic concepts in elementary abstract analysis, to acquaint them with the fundamentals of abstract analysis. Nevertheless, readers are assumed to have some basic training in rigorous analysis as usually offered by courses in advanced calculus, and to have some acquaintance with the rudiments of linear algebra.

Throughout the book, the field of real numbers and that of complex numbers are denoted, respectively, by $\mathbb{R}$ and $\mathbb{C}$, while the set of all positive integers and the set of all integers are denoted by $\mathbb{N}$ and $\mathbb{Z}$ respectively.

The standard set-theoretical terminology is assumed; but terminology and notations regarding mappings will now be briefly recalled. If $T$ is a mapping from a set $A$ into a set $B$ (expressed by $T : A \to B$), $T(a)$ denotes the element in $B$ which is associated with $a \in A$ under the mapping $T$; for a subset **S** of **A**, the set $\{T(x) : x \in S\}$ is denoted by $TS$ and is called the **image of $S$ under $T$**; thus $T\{a\} = \{T(a)\}$. $T(a)$ is sometimes simply written as $Ta$ if no confusion is possible, and at times, an element $a$ of a set and the set $\{a\}$ consisting of an element are not clearly distinguished as different objects. For example, $Ta$ and $T\{a\}$ may not be distinguished and $Ta$ is also called the image of $a$ under $T$. A mapping $T : A \to B$ is said to be **one-to-one** or **injective** if $Ta = Ta'$ leads to $a = a'$, and is said to be **surjective** if $TA = B$; $T$ is **bijective** if it is both injective and surjective. If $TA = B$, $T$ is also referred to as a mapping from $A$ onto $B$. Mappings are also called maps. Synonyms for maps are operators and transformations. As usual, a map from a set into $\mathbb{R}$ or $\mathbb{C}$ is called a function.

Some convenient notations for operations on sets are now introduced. Regarding a family $\mathcal{F} = \{A_\alpha\}_{\alpha \in I}$ of sets indexed by an index set $I$, the union $\bigcup_{\alpha \in I} A_\alpha$ is also expressed by $\bigcup \mathcal{F}$; if $A$ and $B$ are sets in a vector space and $\alpha$ a scalar, the set $\{x + y : x \in A, y \in B\}$ is denoted by $A + B$, and the set $\{\alpha x : x \in A\}$ by $\alpha A$.

## 1.1 Summability of systems of real numbers

Summability of systems of real numbers is a special case in the theory of integration, to be treated in Chapter 2, but it reveals many essential points of the theory.

For a set $S$, the family of all nonempty finite subsets of $S$ will be denoted by $F(S)$. Consider now a system $\{c_\alpha\}_{\alpha\in I}$ of real numbers indexed by an index set $I$. The system $\{c_\alpha\}_{\alpha\in I}$ will be denoted simply by $\{c_\alpha\}$ if the index set $I$ is assumed either explicitly or implicitly. The system is said to be **summable** if there is $\ell \in \mathbb{R}$, such that for any $\varepsilon > 0$ there is $A \in F(I)$, with the property that whenever $B \in F(I)$ and $B \supset A$, then

$$\left|\sum_{\alpha\in B} c_\alpha - \ell\right| < \varepsilon. \tag{1.1}$$

**Exercise 1.1.1** Show that if $\ell$ in the preceding definition exists, then it is unique.

If $\{c_\alpha\}$ is summable, the uniquely determined $\ell$ in the above definition is called the sum of $\{c_\alpha\}$ and is denoted by $\sum_{\alpha\in I} c_\alpha$.

Before we go further it is worthwhile remarking that the convergence of the series $\sum_{n=1}^{\infty} c_n$ depends on the order $1 < 2 < 3 < \cdots$ and $\sum_{n\in\mathbb{N}} c_n$, if it exists, does not depend on how $\mathbb{N}$ is ordered. Hence $\sum_{n\in\mathbb{N}} c_n$ may not exist while $\sum_{n=1}^{\infty} c_n$ exists. We will come back to this remark in Exercise 1.1.5.

**Theorem 1.1.1** *If $\{c_\alpha^{(1)}\}_{\alpha\in I}$ and $\{c_\alpha^{(2)}\}_{\alpha\in I}$ are summable, then so is $\{ac_\alpha^{(1)} + bc_\alpha^{(2)}\}_{\alpha\in I}$ for fixed real numbers a and b, and*

$$\sum_{\alpha\in I}\left(ac_\alpha^{(1)} + bc_\alpha^{(2)}\right) = a\sum_{\alpha\in I} c_\alpha^{(1)} + b\sum_{\alpha\in I} c_\alpha^{(2)}.$$

***Proof*** We may assume that $|a| + |b| > 0$, and for convenience put $\sum_{\alpha\in I} c_\alpha^{(1)} = l_1$, $\sum_{\alpha\in I} c_\alpha^{(2)} = l_2$. Let $\varepsilon > 0$ be given, there are $A_1$ and $A_2$ in $F(I)$ such that when $B_1$, $B_2$ are in $F(I)$ with $B_1 \supset A_1$, $B_2 \supset A_2$, we have $|\sum_{\alpha\in B_1} c_\alpha^{(1)} - l_1| < \frac{\varepsilon}{|a|+|b|}$ and $|\sum_{\alpha\in B_2} c_\alpha^{(2)} - l_2| < \frac{\varepsilon}{|a|+|b|}$. Choose now $A = A_1 \cup A_2$, then for $B \in F(I)$ with $B \supset A$, we have $|\sum_{\alpha\in B}(ac_\alpha^{(1)} + bc_\alpha^{(2)}) - (al_1 + bl_2)| \le |a||\sum_{\alpha\in B} c_\alpha^{(1)} - l_1| + |b||\sum_{\alpha\in B} c_\alpha^{(2)} - l_2| < \frac{|a|\varepsilon}{|a|+|b|} + \frac{|b|\varepsilon}{|a|+|b|} = \varepsilon$. This shows that $\{ac_\alpha^{(1)} + bc_\alpha^{(2)}\}$ is summable and $\sum_{\alpha\in I}(ac_\alpha^{(1)} + bc_\alpha^{(2)}) = al_1 + bl_2$. ■

**Theorem 1.1.2** *If $c_\alpha \ge 0\ \forall\alpha \in I$, then $\{c_\alpha\}$ is summable if and only if*

$$\left\{\sum_{\alpha\in A} c_\alpha : A \in F(I)\right\} \tag{1.2}$$

*is bounded.*

***Proof*** That boundedness of (1.2) is necessary for $\{c_\alpha\}$ to be summable is left as an exercise. Now we show that boundedness of (1.2) is sufficient for $\{c_\alpha\}$ to be summable. Let $\ell$ be the least upper bound of $\{\sum_{\alpha\in A} c_\alpha : A \in F(I)\}$; for any $\varepsilon > 0$ there is $A \in F(I)$ such that

$$0 \leq \ell - \sum_{\alpha\in A} c_\alpha < \varepsilon. \tag{1.3}$$

Let now $B \in F(I)$ and $B \supset A$, then

$$\left|\sum_{\alpha\in B} c_\alpha - \ell\right| = \ell - \sum_{\alpha\in B} c_\alpha \leq \ell - \sum_{\alpha\in A} c_\alpha < \varepsilon.$$

■

We note before moving on that if a subset $S$ of $\mathbb{R}$ is bounded from above, then the least upper bound of $S$ exists uniquely and is denoted by $\sup S$; similarly, if $S$ is bounded from below, then the greatest lower bound exists uniquely and is denoted by $\inf S$. If $S = \{s_\alpha : \alpha \in I\}$, then $\inf S$ and $\sup S$ are also expressed, respectively, by $\inf_{\alpha\in I} s_\alpha$ and $\sup_{\alpha\in I} s_\alpha$.

**Exercise 1.1.2** Show that boundedness of (1.2) is necessary for $\{c_\alpha\}$ to be summable.

Because of Theorem 1.1.2, if $\{c_\alpha\}$ is a system of nonnegative real numbers and is not summable, then we write $\sum_{\alpha\in I} c_\alpha = +\infty$. Hence, $\sum_{\alpha\in I} c_\alpha$ always has a meaning if $\{c_\alpha\}$ is a system of nonnegative numbers.

**Theorem 1.1.3** (Cauchy criterion) *A system $\{c_\alpha\}$ is summable if and only if for any $\varepsilon > 0$ there is $A \in F(I)$, such that $|\sum_{\alpha\in B} c_\alpha| < \varepsilon$ whenever $B \in F(I)$ and $A \cap B = \emptyset$.*

***Proof*** Sufficiency: Choose $A \in F(I)$ such that $|\sum_{\alpha\in B} c_\alpha| < 1$ for $B \in F(I)$, satisfying $A \cap B = \emptyset$, then obviously if $B \in F(I)$ with $B \cap A = \emptyset$, we have $\sum_{\alpha\in B} c_\alpha^+ < 1$, where $c_\alpha^+ = c_\alpha$ or 0 according to whether $c_\alpha \geq 0$ or $< 0$. Now, for $B \in F(I)$, we have

$$\sum_{\alpha\in B} c_\alpha^+ = \sum_{\alpha\in B\cap A} c_\alpha^+ + \sum_{\alpha\in B\setminus A} c_\alpha^+ < \sum_{\alpha\in A} c_\alpha^+ + 1,$$

i.e., $\{\sum_{\alpha\in B} c_\alpha^+ : B \in F(I)\}$ is bounded; hence by Theorem 1.1.2 $\{c_\alpha^+\}$ is summable.

Similarly $\{c_\alpha^-\}$ is summable, where $c_\alpha^- = -c_\alpha$ or 0 according to whether $c_\alpha \leq 0$ or $> 0$. Now $c_\alpha = c_\alpha^+ - c_\alpha^-$, hence $\{c_\alpha\}$ is summable by Theorem (1.1).

The necessary part is left for the reader to verify. ■

**Exercise 1.1.3** Suppose that $\{c_\alpha\}_{\alpha\in I}$ is summable and that $J$ is a nonempty subset of $I$. Show that (i) $\{c_\alpha\}_{\alpha\in J}$ is summable, and (ii) $\sum_{\alpha\in I} c_\alpha = \sum_{\alpha\in J} c_\alpha + \sum_{\alpha\in I\setminus J} c_\alpha$.

**Exercise 1.1.4** Show that $\{c_\alpha\}$ is summable if and only if $\{|c_\alpha|\}$ is summable; show also that $\{c_\alpha\}$ is summable if and only if

$$\left\{\left|\sum_{\alpha\in A} c_\alpha\right| : A \in F(I)\right\}$$

is bounded.

**Exercise 1.1.5** Show that $\{c_\alpha\}_{\alpha\in\mathbb{N}}$ is summable if and only if the series $\sum_{\alpha=1}^{\infty} c_\alpha$ is absolutely convergent. Show also that $\sum_{\alpha\in\mathbb{N}} c_\alpha = \sum_{\alpha=1}^{\infty} c_\alpha$ if $\{c_\alpha\}_{\alpha\in\mathbb{N}}$ is summable.

**Exercise 1.1.6** Show that $\{c_\alpha\}_{\alpha\in I}$ is summable if and and only if (i) $\{\alpha \in I : c_\alpha \neq 0\}$ is finite or countable; and (ii) if $\{\alpha \in I : c_\alpha \neq 0\} = \{\alpha_1, \alpha_2, \ldots\}$ is infinite; then the series $\sum_{k=1}^{\infty} c_{\alpha_k}$ converges absolutely.

**Exercise 1.1.7** Suppose that for each $n = 1, 2, 3, \ldots$, there is $A_n \in F(I)$, with the property that for each $A \in F(I)$, there is a positive integer $N$ such that $A \subset A_n$ for all $n \geq N$. Show that if $\{c_\alpha\}_{\alpha\in I}$ is summable, then

$$\sum_{\alpha\in I} c_\alpha = \lim_{n\to\infty} \sum_{\alpha\in A_n} c_\alpha.$$

Give an example to show that it is possible that $\lim_{n\to\infty} \sum_{\alpha\in A_n} c_\alpha$ exists and is finite, but $\{c_\alpha\}$ is not summable.

**Example 1.1.1** Suppose that $I = \bigcup_{n\in\mathbb{N}} I_n$, where $I_n$'s are pairwise disjoint. Let $\{c_\alpha\}_{\alpha\in I}$ be summable, then $\sum_{\alpha\in I} c_\alpha = \sum_{n\in\mathbb{N}} (\sum_{\alpha\in I_n} c_\alpha)$. By Exercise 1.1.4, we may assume that $c_\alpha \geq 0$ for all $\alpha \in I$. It follows from $\sum_{\alpha\in I} c_\alpha = \sup\{\sum_{\alpha\in A} c_\alpha : A \in F(I)\}$ that $\sum_{\alpha\in I} c_\alpha \leq \sum_{n\in\mathbb{N}} (\sum_{\alpha\in I_n} c_\alpha)$. It remains to be seen that $\sum_{\alpha\in I} c_\alpha \geq \sum_{n\in\mathbb{N}} (\sum_{\alpha\in I_n} c_\alpha)$. Let $k \in \mathbb{N}$ and $\varepsilon > 0$. For each $n = 1, \ldots, k$, there is a finite set $A_n \subset I_n$ such that $\sum_{\alpha\in I_n} c_\alpha < \sum_{\alpha\in A_n} c_\alpha + \frac{\varepsilon}{k}$. Then, if we put $B_k = \bigcup_{n=1}^{k} A_n$, we have $\sum_{\alpha\in I} c_\alpha \geq \sum_{\alpha\in B_k} c_\alpha > \sum_{n=1}^{k} (\sum_{\alpha\in I_n} c_\alpha - \frac{\varepsilon}{k}) = \sum_{n=1}^{k} (\sum_{\alpha\in I_n} c_\alpha) - \varepsilon$; since $\varepsilon > 0$ is arbitrary, $\sum_{\alpha\in I} c_\alpha \geq \sum_{n=1}^{k} (\sum_{\alpha\in I_n} c_\alpha)$ for each $k \in \mathbb{N}$. Now let $k \to \infty$ to obtain $\sum_{\alpha\in I} c_\alpha \geq \sum_{n\in\mathbb{N}} (\sum_{\alpha\in I_n} c_\alpha)$. Observe from the proof that $\{\sum_{\alpha\in I_n} c_\alpha\}_{n\in\mathbb{N}}$ is summable.

We shall recognize in Example 2.3.3 that summability considered in this section is the integrability with respect to the counting measure on $I$.

## 1.2 Double series

Let $I = \mathbb{N} \times \mathbb{N} = \{(i, j) : i, j = 1, 2, \ldots\}$ and write $c_{ij}$ for $c_{(i,j)}$. When the summability of the system $\{c_{ij}\}$ is in question, the system $\{c_{ij}\}$ is referred to as a **double series** and is denoted by $\sum c_{ij}$. Hence the double series $\sum c_{ij}$ is summable if $\{c_{ij}\} = \{c_{(i,j)}\}$ is summable, and $\sum_{(i,j)\in I} c_{ij}$ is called the sum of the double series $\sum c_{ij}$.

For a double sequence $\{a_{mn}\}$, we say that $\lim_{m,n\to\infty} a_{mn} = \ell$, if for any $\varepsilon > 0$ there is a positive integer $N$ such that $|a_{mn} - \ell| < \varepsilon$ whenever $m, n \geq N$.

**Theorem 1.2.1** *If the double series $\sum c_{ij}$ is summable, then*

$$\sum_{(i,j)\in I} c_{ij} = \lim_{m,n\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} c_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij}.$$

***Proof*** We show first that $\sum_{(i,j)\in I} c_{ij} = \lim_{n,m\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}$. Let $\ell = \sum_{(i,j)\in I} c_{ij}$. Given $\varepsilon > 0$, there is $A \in F(I)$ such that

$$\left| \sum_{(i,j)\in B} c_{ij} - \ell \right| < \varepsilon$$

whenever $B \in F(I)$ and $B \supset A$. Let $N = \max\{i \vee j : (i,j) \in A\}$, where $i \vee j$ is the larger of $i$ and $j$. For $n, m \geq N$, let $B_{mn} = \{(i,j) \in I : 1 \leq i \leq m,\ 1 \leq j \leq n\}$, then $B_{mn} \in F(I)$ and $B_{mn} \supset A$, hence

$$\left| \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} - \ell \right| = \left| \sum_{(i,j)\in B_{mn}} c_{ij} - \ell \right| < \varepsilon.$$

This means that $\ell = \lim_{m,n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}$.

Since $\sum_{(i,j)\in I} c_{ij} = \sum_{(i,j)\in I} c_{ij}^{+} - \sum_{(i,j)\in I} c_{ij}^{-}$, in the remaining part of the proof, we may assume that $c_{ij} \geq 0$ for all $(i,j) \in I$. Observe then that

$$\ell = \sup_{n,m\geq 1} \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}.$$

Hence,

$$\ell \geq \lim_{m\to\infty} \left( \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} \right) = \sum_{j=1}^{n} \sum_{i=1}^{\infty} c_{ij}$$

for each $n$ and consequently

$$\ell \geq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij}.$$

On the other hand,

$$\begin{aligned} \ell &= \sup_{n,m\geq 1} \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} \leq \sup_{n\geq 1} \left( \sum_{j=1}^{n} \sum_{i=1}^{\infty} c_{ij} \right) = \lim_{n\to\infty} \left( \sum_{j=1}^{n} \sum_{i=1}^{\infty} c_{ij} \right) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij}. \end{aligned}$$

We have shown that $\ell = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij}$; similarly,

$$\ell = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij}. \qquad \blacksquare$$

**Example 1.2.1** If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are summable, then the double series $\sum a_n b_m$ is summable and $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} a_n b_m = (\sum_{n\in\mathbb{N}} a_n)(\sum_{m\in\mathbb{N}} b_m)$. That $\sum a_n b_m$ is summable follows from Exercise 1.1.4 and the observation that $\{\sum_{(n,m)\in A} |a_n b_m| : A \in F(\mathbb{N}\times\mathbb{N})\}$ is bounded from above by $(\sum_{n\in\mathbb{N}} |a_n|)\cdot(\sum_{m\in\mathbb{N}} |b_m|)$. Then, by Theorem 1.2.1, $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} a_n b_m = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} a_n b_m = (\sum_{n\in\mathbb{N}} a_n)(\sum_{m\in\mathbb{N}} b_m)$. For $k \geq 2$ in $\mathbb{N}$, put $A_k = \{(n,m)\in\mathbb{N}\times\mathbb{N} : n+m=k\}$; then $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} a_n b_m = \sum_{\substack{k\in\mathbb{N}\\k\geq 2}} (\sum_{(n,m)\in A_k} a_{nm})$ from Example 1.1.1. The system $\{\sum_{(n,m)\in A_k} a_n b_m\}_{k\geq 2}$ is called the product of $\{a_n\}$ and $\{b_n\}$; we have shown that the sum of the product is the product of the sums.

The following exercise complements Theorem 1.2.1.

**Exercise 1.2.1** Copy the proof of Theorem 1.2.1 to show that if $c_{ij} \geq 0$ for all $i$ and $j$ in $\mathbb{N}$, then the conclusion of Theorem 1.2.1 still holds, even if $\sum_{(i,j)\in I} c_{ij} = \infty$ (recall that for a system $\{c_\alpha\}$ of nonnegative numbers, $\sum_\alpha c_\alpha = \infty$ means that $\{c_\alpha\}$ is not summable).

**Remark** For $i, j$ in $\mathbb{N}$, let

$$c_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -1 & \text{if } j = i+1; \\ 0 & \text{otherwise,} \end{cases}$$

then $\sum c_{ij}$ is not summable and $0 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij} \neq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} c_{ij} = 1$.

## 1.3 Coin tossing

A pair of symbols $H$ and $T$, associated, respectively, with nonnegative numbers $p$ and $q$ such that $p + q = 1$ is called a **Bernoulli trial** and is denoted by $B(p, q)$. A Bernoulli trial $B(p, q)$ is a mathematical model for the tossing of a coin, of which heads occur with probability $p$ and tails turn out with probability $q$; this explains the symbols $H$ and $T$. In particular, $B(\frac{1}{2}, \frac{1}{2})$ models the tossing of a fair coin.

In this section, we consider the first step towards construction of a mathematical model for a sequence of tossing of a fair coin. For convenience, we replace $H$ and $T$ by 1 and 0 in this order; then an infinite sequence $\omega = (\omega_1, \omega_2, \ldots, \omega_k, \ldots)$ of 0's and 1's represents a realization of a sequence of coin tossing. Let

$$\Omega = \{0, 1\}^\infty := \{\omega = (\omega_k),\ \omega_k = 0 \text{ or } 1 \text{ for each } k\},$$

where we adopt the usual convention of expressing an infinite sequence $(\omega_1, \ldots, \omega_k, \ldots)$ by $(\omega_k)$ with the understanding that $\omega_k$ is the entry at the $k$-th position of the sequence. In terminology of probability theory, elements in $\Omega$ are called **sample points** of a sequence of coin tossings and $\Omega$ is called the **sample space** of the sequence of tossings. Subsets of $\Omega$ will often be referred to as **events**. Now for $n \in \mathbb{N}$, let

$$\Omega_n = \{0,1\}^n := \{(\varepsilon_1, \ldots, \varepsilon_n) : \varepsilon_j \in \{0,1\},\ j = 1, \ldots, n\},$$

and for $(\varepsilon_1, \ldots, \varepsilon_n) \in \Omega_n$, call the set

$$E(\varepsilon_1, \ldots, \varepsilon_n) = \{\omega = (\omega_k) \in \Omega : \omega_k = \varepsilon_k,\ k = 1, \ldots, n\}$$

an **elementary cylinder**; but if $n$ is to be emphasized, it is called an **elementary cylinder of rank n**. A finite union of elementary cylinders is called a **cylinder** in $\Omega$. Since intersection of two elementary cylinders is either empty or an elementary cylinder, every cylinder in $\Omega$ can be expressed as a disjoint union of elementary cylinders; in fact, if $Z$ is a cylinder in $\Omega$, there is $n \in \mathbb{N}$ and $H \subset \Omega_n$ such that

$$Z = \bigcup \{E(\varepsilon_1, \ldots, \varepsilon_n) : (\varepsilon_1, \ldots, \varepsilon_n) \in H\},$$

of which one notes that $E(\varepsilon_1, \ldots, \varepsilon_n)$'s are mutually disjoint. Of course, a cylinder $Z$ can be expressed as above in many ways. We denote by $\mathcal{Q}$ the family of all cylinders in $\Omega$. Since $\Omega = E(0) \cup E(1)$, $\Omega \in \mathcal{Q}$; $\emptyset$ is also in $\mathcal{Q}$, because it is the union of an empty family of elementary cylinders.

**Exercise 1.3.1** Show that $\mathcal{Q}$ is an algebra of subsets of $\Omega$, in the sense that $\mathcal{Q}$ satisfies the following conditions: (i) $\Omega \in \mathcal{Q}$; (ii) if $Z \in \mathcal{Q}$, then $Z^c = \Omega \backslash Z$ is in $\mathcal{Q}$; and (iii) if $Z_1, Z_2$ are in $\mathcal{Q}$, then $Z_1 \cup Z_2$ is in $\mathcal{Q}$.

For an event $Z$ in $\mathcal{Q}$, we define its probability $P(Z)$ as follows. First, for an elementary cylinder $C = E(\varepsilon_1, \ldots, \varepsilon_n)$, define $P(C) = (\frac{1}{2})^n$; intuitively, this definition of $P(C)$ means that we consider the modeling of a sequence of independent tossing of a fair coin. Now if $Z \in \mathcal{Q}$ is given by

$$Z = \bigcup \{E(\varepsilon_1, \ldots, \varepsilon_n) : (\varepsilon_1, \ldots, \varepsilon_n) \in H\},$$

where $H \subset \Omega_n$, then define

$$P(Z) = \sum_{(\varepsilon_1, \ldots, \varepsilon_n) \in H} P(E(\varepsilon_1, \ldots, \varepsilon_n)) = \#H \cdot 2^{-n},$$

where $\#H$ is the number of elements in $H$. We claim that $P(Z)$ is well defined. Actually if $Z$ is also given by

$$Z = \bigcup \{E(\varepsilon_1, \ldots, \varepsilon_m) : (\varepsilon_1, \ldots, \varepsilon_m) \in H'\},$$

where $H' \subset \Omega_m$, then (assuming $m \geq n$) $H' = \{(\varepsilon_1, \ldots, \varepsilon_m) \in \Omega_m : (\varepsilon_1, \ldots, \varepsilon_n) \in H\}$ and therefore $\#H' = \#H \cdot 2^{m-n}$; consequently

$$\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in H'} P(E(\varepsilon_1, \ldots, \varepsilon_m)) = \#H' \cdot 2^{-m} = \#H \cdot 2^{m-n} \cdot 2^{-m}$$
$$= \#H \cdot 2^{-n} = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in H} P(E(\varepsilon_1, \ldots, \varepsilon_n)),$$

implying that the definition of $P(Z)$ is independent of how $Z$ is expressed as a finite disjoint union of elementary cylinders of a given rank. We complete the definition of $P$ by letting $P(\emptyset) = 0$. Note that $P(\Omega) = 1$.

**Exercise 1.3.2**

(i) Show that $P$ is additive on $\mathcal{Q}$, i.e. $P(Z_1 \cup Z_2) = P(Z_1) + P(Z_2)$ if $Z_1$, $Z_2$ are disjoint elements of $\mathcal{Q}$.

(ii) For $k \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}$, put $E^k_\varepsilon = \{\omega \in \Omega : \omega_k = \varepsilon\}$. Show that

$$P(E^{k_1}_{\varepsilon_1} \cap \cdots \cap E^{k_n}_{\varepsilon_n}) = \prod_{j=1}^{n} P(E^{k_j}_{\varepsilon_j}) = 2^{-n}$$

for any finite sequence $k_1 < k_2 < \cdots < k_n$ in $\mathbb{N}$.

From now on we write $d_j(\omega) = \omega_j, j = 1, 2, \ldots,$ if $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$; and for each $n$ define a function $S_n$ on $\Omega$ by

$$S_n(\omega) = \sum_{j=1}^{n} d_j(\omega).$$

**Exercise 1.3.3** Show that, for each $k = 0, 1, 2, \ldots, n$, the set $\{S_n = k\} := \{\omega \in \Omega : S_n(\omega) = k\}$ is in $\mathcal{Q}$ and

$$P(\{S_n = k\}) = \binom{n}{k} \frac{1}{2^n},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

For a given realization $\omega$ of a sequence of independent coin tossing, $S_n(\omega)$ is the number of heads that appear in the first $n$ tosses and $\frac{S_n(\omega)}{n}$ measures the relative frequency of appearance of heads in the first $n$ tosses. Let

$$E = \left\{\omega \in \Omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n} = \frac{1}{2}\right\};$$

$E$ is easily seen to be not in $\mathcal{Q}$. Nevertheless, we expect that $P$ can be extended to be defined on a larger family of sets than $\mathcal{Q}$ in such a way that $P(A)$ can be interpreted as the probability of event $A$, and such that $P(E)$ is defined with value 1. We expect $P(E) = 1$, because this is what a fair coin is accounted for intuitively. Discussion of the subject matter of this section will be continued in Example 1.7.1, Example 2.1.1, Example 3.4.6, and Example 7.5.2; and eventually we shall answer positively to this expectation in the paragraph following Corollary 7.5.3.

## 1.4 Metric spaces and normed vector spaces

The usefulness of the concept of continuity has already surfaced in elementary analysis of functions defined on an interval. This section considers a structure on a set which allows one to speak of "nearness" for elements in the set, so that a concept of continuity can be defined for functions defined on the set, parallel to that for functions defined on an interval of the real line. We shall not treat the most general situation; instead, we consider the situation where an abstract concept of distance can be defined between elements of the set, because this situation abounds sufficiently for our purposes later. When the set considered is a vector space, it is natural to consider the case where the distance defined and the linear structure of the set mingle well, as in the case of a real line or Euclidean plane. This leads to the concept of normed vector spaces.

Let $M$ be a nonempty set and let $\rho : M \times M \to [0, +\infty)$ satisfy (i) $\rho(x, y) = \rho(y, x) \geq 0$ for all $x, y \in M$ and $\rho(x, y) = 0$ if and only if $x = y$; (ii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x$, $y$, and $z$ in $M$. Such a $\rho$ is then called a **metric** on $M$, and $(M, \rho)$ is called a **metric space**. Usually we say that $M$ is a metric space with metric $\rho$, or simply that $M$ is a metric space when a certain metric $\rho$ is explicitly or implicitly implied. For a nonempty subset $S$ of $M$ the restriction of $\rho$ to $S \times S$ is a metric on $S$ which will also be denoted by $\rho$. The metric space $(S, \rho)$ is called a subspace of $(M, \rho)$ and $\rho$ is called the metric on $S$ inherited from $M$. Unless stated otherwise, if $S$ is a subset of a metric space $M$, $S$ is equipped with the metric inherited from $M$. For a nonempty subset $A$ of $M$, the **diameter** of $A$, denoted diam $A$, is defined by

$$\operatorname{diam} A := \sup_{x,y \in A} \rho(x, y);$$

while $\operatorname{diam} A = 0$ if $A = \emptyset$.

A subset $A$ of $M$ is said to be **bounded** if $\operatorname{diam} A < \infty$. In other words, $A$ is bounded if $\{\rho(x, x_0) : x \in A\}$ is a bounded set in $\mathbb{R}$ for every $x_0 \in M$.

Elements of a metric space are often called **points** of the space.

**Example 1.4.1** Let $M = \mathbb{R}^n$ and for $x, y \in \mathbb{R}^n$ let $\rho(x, y) = |x - y|$, where $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. To show that $\rho$ is a metric on $\mathbb{R}^n$ we first establish the well-known **Schwarz inequality**: $|x \cdot y| \leq |x||y|$ if $x, y \in \mathbb{R}^n$, where, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, $x \cdot y := \sum_{i=1}^n x_i y_i$ is called the inner product of $x$ and $y$. For this purpose we note first that for $x \in \mathbb{R}^n$, $|x|^2 = x \cdot x$ and that we may assume that $x \neq 0$ and $y \neq 0$, hence $|x| > 0$ and $|y| > 0$. For $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq |x + ty|^2 &= (x + ty) \cdot (x + ty) = |x|^2 + 2t(x \cdot y) + t^2|y|^2 \\ &= \big(|x| + t|y|\big)^2 + 2t\big(x \cdot y - |x||y|\big), \end{aligned}$$

from which by taking $t = -|x|/|y|$ we obtain $x \cdot y \leq |x||y|$. Then $|x \cdot y| \leq |x||y|$ follows, because $-(x \cdot y) \leq |x|| - y| = |x||y|$. Now for $x$, $y$, and $z$ in $\mathbb{R}^n$, we have

$$\begin{aligned} \rho(x, z)^2 &= |x - z|^2 = |x - y + y - z|^2 = |x - y|^2 + 2(x - y) \cdot (y - z) + |y - z|^2 \\ &\leq |x - y|^2 + 2|x - y||y - z| + |y - z|^2 = \big(|x - y| + |y - z|\big)^2 \\ &= \big[\rho(x, y) + \rho(y, z)\big]^2, \end{aligned}$$

i.e.

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z).$$

Hence $\mathbb{R}^n$ is a metric space with metric $\rho$ defined above. This metric is called the **Euclidean metric** on $\mathbb{R}^n$. Unless stated otherwise, $\mathbb{R}^n$ is considered as a metric space with this metric, then $\mathbb{R}^n$ is called the $n$-dimensional Euclidean space.

Similarly, $\mathbb{C}^n$ is a metric space, with the metric $\rho$ defined by $\rho(\zeta, \eta) = (\sum_{j=1}^n |\zeta_j - \eta_j|^2)^{1/2}$ for $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\eta = (\eta_1, \ldots, \eta_2)$ in $\mathbb{C}^n$. $\mathbb{C}^n$ with this metric is called the **$n$-dimensional unitary space**. This follows, as in the case of the Euclidean metric for $\mathbb{R}^n$, from the Schwarz inequality $|\zeta \cdot \eta| \leq |\zeta||\eta|$ for $\zeta$, $\eta$ in $\mathbb{C}^n$, where $\zeta \cdot \eta = \sum_{j=1}^n \zeta_j \bar{\eta}_j$ and $|\zeta| = (\sum_{j=1}^n |\zeta_j|^2)^{\frac{1}{2}}$. As before, if $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq |\zeta + t\eta|^2 &= (\zeta + t\eta) \cdot (\zeta + t\eta) = |\zeta|^2 + 2t\,\mathrm{Re}\zeta \cdot \eta + t^2|\eta|^2 \\ &= (|\zeta| + t|\eta|)^2 + 2t\{\mathrm{Re}\,\zeta \cdot \eta - |\zeta||\eta|\}, \end{aligned}$$

from which we infer that $\mathrm{Re}\,\zeta \cdot \eta \leq |\zeta||\eta|$ by choosing $t = -|\zeta||\eta|^{-1}$ if $\eta \neq 0$. Then, $|\zeta \cdot \eta| \leq |\zeta||\eta|$ follows from replacing $\zeta$ by $e^{-i\theta}\zeta$ if $\zeta \cdot \eta = |\zeta \cdot \eta|e^{i\theta}$. Note that for a complex number $\alpha$, $\bar{\alpha}$ denotes the conjugate of $\alpha$, while $\mathrm{Re}\,\alpha$ denotes the real part of $\alpha$.

**Example 1.4.2** For a closed finite interval $[a, b]$ in $\mathbb{R}$, let $C[a, b]$ denote the space of all real-valued continuous functions defined on $[a, b]$. For $f, g \in C[a, b]$, let $\rho(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|$. It is easily verified that $C[a, b]$ is a metric space with metric $\rho$ so defined. Unless stated otherwise, $C[a, b]$ is equipped with this metric, which is often referred to as the **uniform metric** on $C[a, b]$. $C[a, b]$ is also used to denote the space of all complex-valued continuous functions on $[a, b]$ with metric defined similarly. When $C[a, b]$ denotes the latter space, it shall be explicitly indicated.

**Exercise 1.4.1** Show that $\mathbb{R}^n$ is also a metric space, with metric $\rho$ defined by $\rho(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$ if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

A map from $\mathbb{N}$, the set of all positive integers, to a set $M$ is called a **sequence** in $M$ or a **sequence of elements** of $M$. Such a sequence will be denoted by $\{x_n\}$, where $x_n$

is the image of the positive integer $n$ under the mapping. If $\{x_n\}$ is a sequence in $M$, then $\{x_{n_k}\}$ is called a subsequence of $\{x_n\}$ if $n_1 < n_2 < \cdots < n_k < \cdots$ is a subsequence of $\{n\}$. A sequence $\{x_n\}$ in a metric space $M$ is said to converge to $x \in M$ if for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\rho(x_n, x) < \varepsilon$ whenever $n \geq n_0$. Since $x$ is uniquely determined, $x$ is called the **limit** of $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n$. That $x = \lim_{n\to\infty} x_n$ is often expressed by $x_n \to x$. If $\lim_{n\to\infty} x_n$ exists, then we say that $\{x_n\}$ converges in $M$ and $\{x_n\}$ is referred to as a **convergent sequence**. A sequence $\{x_n\}$ in $M$ is usually expressed by $\{x_n\} \subset M$ by abuse of notation, and therefore $\{x_n\}$ also denotes the range of the sequence $\{x_n\}$. A sequence in $M$ is said to be bounded if its range is bounded.

**Example 1.4.3** $\{f_n\} \subset C[a, b]$ converges if and only if $f_n(x)$ converges uniformly for $x \in [a, b]$.

A sequence $\{x_n\} \subset M$ is called a **Cauchy sequence** if for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\rho(x_n, x_m) < \varepsilon$ whenever $n, m \geq n_0$. Clearly, a Cauchy sequence is bounded.

**Exercise 1.4.2** Show that if $\{x_n\} \subset M$ converges, then $\{x_n\}$ is a Cauchy sequence.

**Exercise 1.4.3** Let $\{x_n\}$ be a Cauchy sequence. Show that if $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ converges.

A metric space $M$ is called ***complete*** if every Cauchy sequence in $M$ converges in $M$.

**Exercise 1.4.4** Show that both $\mathbb{R}^n$ and $C[a, b]$ are complete.

**Exercise 1.4.5** If instead of the uniform metric we equip $C[a, b]$ with a new metric $\rho'$, defined by

$$\rho'(f, g) = \int_a^b |f(t) - g(t)|dt$$

for $f, g$ in $C[a, b]$, show that $C[a, b]$ is not complete when considered as a metric space with metric $\rho'$.

**Exercise 1.4.6** Show that any nonempty set $M$ can be considered as a complete metric space by defining $\rho(x, y) = 0$ or $1$ depending on $x = y$ or $x \neq y$. Such a metric $\rho$ is said to be discrete.

Let $M_1, M_2$ be metric spaces with metrics $\rho_1$ and $\rho_2$ respectively. A map $T : M_1 \to M_2$ is said to be **continuous at** $x \in M_1$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that $\rho_2(T(x), T(y)) < \varepsilon$ whenever $\rho_1(x, y) < \delta$. If $T$ is continuous at every point of $M_1$, then $T$ is said to be continuous on $M_1$ and is called a **continuous map** from $M_1$ into $M_2$. A continuous map from a metric space $M$ into $\mathbb{R}$ or $\mathbb{C}$ is called a **continuous function** on $M$ and is generically denoted by $f$. The space of all continuous real(complex)-valued functions on a metric space $M$ is denoted by $C(M)$; $C(M)$ is a real- or complex vector space depending on whether the functions in question are real- or complex-valued.

A point $x$ of a set $A$ in a metric space is called an **interior** point of $A$ if there is $\varepsilon > 0$ such that $y \in A$ whenever $\rho(x, y) < \varepsilon$; the set of all interior points of $A$ is denoted by $\mathring{A}$. A set $G$ in a metric space $M$ is said to be **open** if $\mathring{G} = G$. The complement of an open set is called a **closed** set. For $x \in M$ and $r > 0$, let $B_r(x) = \{y \in M : \rho(y,x) < r\}$ and $C_r(x) = \{y \in M : \rho(y,x) \leq r\}$. It is easily verified that $B_r(x)$ is an open set and $C_r(x)$ is a closed set. $B_r(x)$ $(C_r(x))$ is usually referred to as the **open** (**closed**) ball centered at $x$ and with radius $r$. A point $x \in M$ is said to be **isolated** if $B_r(x) = \{x\}$ for some $r > 0$. A set $N \subset M$ is called a **neighborhood** of $x \in M$ if $N$ contains an open set which contains $x$; similarly, if $N$ contains an open set which contains a set $A$, then $N$ is called a **neighborhood** of $A$. It is clear that a sequence $\{x_n\}$ in $M$ converges to $x \in M$ if and only if, for any neighborhood $N$ of $x$, there is $n_0 \in \mathbb{N}$ such that $x_n \in N$ whenever $n \geq n_0$. One notes that if $x_0$ is an isolated point of $M$, then any map $T$ from $M$ into any metric space is continuous at $x_0$.

Note that open sets depend on the metric $\rho$, and when $\rho$ is to be emphasized, an open set in a metric space with metric $\rho$ is more precisely said to be open w.r.t. $\rho$.

**Exercise 1.4.7** Let $M_1, M_2$ be metric spaces and let $T : M_1 \to M_2$.

(i) Show that $T$ is continuous at $x \in M_1$ if and only if, for any sequence $\{x_n\} \subset M_1$ with $\lim_{n\to\infty} x_n = x$, it holds that $\lim_{n\to\infty} T(x_n) = T(x)$ in $M_2$; also show that $T$ is continuous at $x \in M_1$ if and only if, for every sequence $\{x_n\} \subset M_1$ with $\lim_{n\to\infty} x_n = x$, it holds that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} T(x_{n_k}) = T(x)$.

(ii) Show that $T$ is continuous at $x \in M_1$ if and only if, for any neighborhood $N$ of $T(x)$ in $M_2$, the set $T^{-1}N = \{y \in M_1 : T(y) \in N\}$ is a neighborhood of $x$ in $M_1$.

(iii) Show that $T$ is continuous on $M_1$ if and only if for any open set $G_2 \subset M_2$, $T^{-1}G_2$ is an open subset of $M_1$.

**Exercise 1.4.8** Let $\mathcal{T}$ be the family of all open subsets of a metric space $M$. Show that:

(i) $\emptyset$ and $M$ are in $\mathcal{T}$;

(ii) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$;

(iii) if $\{A_i\}_{i\in I} \subset \mathcal{T}$, then $\bigcup_{i\in I} A_i \in \mathcal{T}$, where $I$ is any index set.

Suppose that $(M_1, \rho_1)$ and $(M_2, \rho_2)$ are metric spaces. Let $M_1 \times M_2 := \{(x,y) : x \in M_1, y \in M_2\}$ be the **Cartesian product** of $M_1$ and $M_2$; define a metric $\rho$ on $M_1 \times M_2$ by

$$\rho((x,y),(x',y')) = \rho_1(x,x') + \rho_2(y,y')$$

for $(x,y)$ and $(x',y')$ in $M_1 \times M_2$. It is easily verified that $\rho$ is actually a metric on $M_1 \times M_2$. With this metric $\rho$, $M_1 \times M_2$ is called the **product space** of $M_1$ and $M_2$ as metric space.

**Exercise 1.4.9** Let $M_1 \times M_2$ be the product space of metric spaces $M_1$ and $M_2$.

(i) For $A \subset M_1$ and $B \subset M_2$, show that $A \times B$ is open in $M_1 \times M_2$ if and only if both $A$ and $B$ are open in $M_1$ and $M_2$ respectively.

(ii) Let $G$ be an open set in $M_1 \times M_2$; show that $G_1 := \{x \in M_1 : (x,y) \in G$ for some $y$ in $M_2\}$ and $G_2 := \{y \in M_2 : (x,y) \in G$ for some $x$ in $M_1\}$ are open in $M_1$ and $M_2$ respectively.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $E$ be a vector space over $\mathbb{K}$. Elements of $\mathbb{K}$ are called scalars. Suppose that for each $x \in E$, there is a nonnegative number $\|x\|$ associated with it so that:

(i) $\|x\| = 0$ if and only if $x$ is the zero element of $E$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{K}$ and $x \in E$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y$ in $E$ (**triangle inequality**).

Then $E$ is called a **normed vector space** (abbreviated as **n.v.s.**) with **norm** $\|\cdot\|$, and $\|\cdot\|$ is called a **norm on** $E$.

If $E$ is a n.v.s., for $x, y$ in $E$, let

$$\rho(x, y) = \|x - y\|,$$

then $\rho$ is a metric on $E$ and is called the metric associated with norm $\|\cdot\|$. Unless stated otherwise, we always consider this metric for a n.v.s.. The n.v.s. $E$ with norm $\|\cdot\|$ is denoted by $(E, \|\cdot\|)$ if the norm $\|\cdot\|$ is to be emphasized.

**Lemma 1.4.1** *Suppose that $E$ is a n.v.s. and $x_n \to x$ in $E$, then $\|x\| = \lim_{n\to\infty} \|x_n\|$. In other words, $\|\cdot\|$ is a continuous function on $E$.*

***Proof*** The lemma follows from the following sequence of triangle inequalities:

$$\|x_n\| - \|x_n - x\| \leq \|x\| \leq \|x_n\| + \|x_n - x\|. \qquad \blacksquare$$

A *normed vector space* is called a **Banach space** if it is a complete metric space.

Both $\mathbb{R}^n$ and $C[a, b]$ are Banach spaces, with norms given respectively by $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\|f\| = \max_{a\leq t\leq b} |f(t)|$ for $f \in C[a, b]$. Similarly, the unitary space $\mathbb{C}^n$ is a Banach space with norm $\|z\| = (\sum_{j=1}^n |z_j|^2)^{\frac{1}{2}}$ for $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$. The norms defined above for $\mathbb{R}^n$ and $\mathbb{C}^n$ are called respectively the **Euclidean norm** and the **unitary norm** and are denoted by $|\cdot|$ in both cases, in accordance with the notations introduced in Example 1.4.1; note that their associated metrics are the metrics introduced for $\mathbb{R}^n$ and $C^n$ in Example 1.4.1. The norm defined for $C[a, b]$ is called the **uniform norm**; its associated metric is the uniform metric defined in Example 1.4.2.

A class of well-known Banach spaces, the $l^p$ spaces, will be introduced in §1.6. This class of Banach spaces anticipates the important and more general class of $L^p$ spaces treated in Section 2.7 and in Chapter 6.

In the remaining part of this section, linear maps from a normed vector space $E$ into a normed vector space $F$ over the same field $\mathbb{R}$ or $\mathbb{C}$ are considered. Recall that a map $T$ from a vector space $E$ into a vector space $F$ over the same field is said to be **linear** if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, for all $x, y$ in $E$ and all scalars $\alpha, \beta$. Linear maps are more often called **linear transformations** or **linear operators**.

**Exercise 1.4.10** Suppose that $T$ is a linear transformation from $E$ into $F$. Show that $T$ is continuous on $E$ if and only if it is continuous at one point.

**Theorem 1.4.1** *Let $T$ be a linear transformation from $E$ into $F$, then $T$ is continuous if and only if there is $C \geq 0$ such that*

$$\|Tx\| \leq C\|x\|$$

*for all $x \in E$.*

***Proof*** If there is $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ holds for all $x \in E$, then $T$ is obviously continuous at $x = 0$ and hence by Exercise 1.4.10 is continuous on $E$.

Conversely, suppose that $T$ is continuous on $E$, and is hence continuous at $x = 0$. There is then $\delta > 0$ such that if $\|x\| \leq \delta$, then $\|Tx\| \leq 1$. Let now $x \in E$ and $x \neq 0$, then $\left\| \frac{\delta}{\|x\|} x \right\| = \delta$, so $\left\| T\left(\frac{\delta}{\|x\|} x\right) \right\| \leq 1$. Thus $\|Tx\| \leq \frac{1}{\delta}\|x\|$. If we choose $C = \frac{1}{\delta}$, then $\|Tx\| \leq C\|x\|$ for all $x \in E$. ∎

From this theorem it follows that if $T$ is a continuous linear transformation from $E$ into $F$, then

$$\|T\| := \sup_{x \in E, x \neq 0} \frac{\|Tx\|}{\|x\|} < +\infty,$$

and is the smallest $C$ for which $\|Tx\| \leq C\|x\|$ for all $x \in E$. $\|T\|$ is called the norm of $T$. Of course, $\|T\|$ can be defined for any linear transformation $T$ from $E$ into $F$; then $\|Tx\| \leq \|T\|\|x\|$ holds always and $T$ is continuous if and only if $\|T\| < +\infty$. Hence a continuous linear transformation is also called a **bounded** linear transformation.

**Exercise 1.4.11** Show that $\|T\| = \sup_{x \in E, \|x\|=1} \|Tx\|$.

**Exercise 1.4.12** Let $L(E, F)$ be the space of all bounded linear transformations from $E$ into $F$. Show that it is a normed vector space with norm $\|T\|$ for $T \in L(E, F)$ as previously defined.

**Remark** Any linear map $T$ from a Euclidean space $\mathbb{R}^n$ into a Euclidean space $\mathbb{R}^m$ is continuous. This follows from the representation of $T$ by a matrix $(a_{jk})$, $1 \leq j \leq m$, $1 \leq k \leq n$, of real entries, in the sense that if $y = Tx$, then $y_j = \sum_{k=1}^{n} a_{jk}x_k$, $j = 1, \ldots, m$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$, by observing that

$$|y|^2 = \sum_{j=1}^{m} \left( \sum_{k=1}^{n} a_{jk}x_k \right)^2 \leq \left( \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk}^2 \right) |x|^2.$$

**Theorem 1.4.2** *If $F$ is a Banach space, then $L(E, F)$ is a Banach space.*

***Proof*** Let $\{T_n\}$ be a Cauchy sequence in $L(E, F)$. Since

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \cdot \|x\|,$$

$\{T_n x\}$ is a Cauchy sequence in $F$ for each $x \in E$. Since $F$ is complete, $\lim_{n\to\infty} T_n x$ exists. Put $Tx = \lim_{n\to\infty} T_n x$. $T$ is obviously a linear transformation from $E$ into $F$.

We claim now $T \in L(E, F)$. Since $\{T_n\}$ is Cauchy, $\|T_n\| \leq C$ for some $C > 0$, and for all $n$. Now, from Lemma 1.4.1,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \left(\sup_n \|T_n\|\right)\|x\| \leq C\|x\|$$

for each $x \in E$. Hence $T$ is a bounded linear transformation.

We show next, $\lim_{n\to\infty} \|T_n - T\| = 0$. Given $\varepsilon > 0$, there is $n_0$ such that $\|T_n - T_m\| < \varepsilon$ if $n, m \geq n_0$. Let $n \geq n_0$, we have

$$\begin{aligned}
\|T_n - T\| &= \sup_{x\in E,\|x\|=1} \|T_n x - Tx\| \\
&= \sup_{x\in E,\|x\|=1} \lim_{m\to\infty} \|T_n x - T_m x\| \\
&\leq \sup_{x\in E,\|x\|=1} \left(\sup_{m\geq n_0} \|T_n - T_m\|\right)\|x\| \\
&\leq \sup_{x\in E,\|x\|=1} \varepsilon\|x\| = \varepsilon;
\end{aligned}$$

this shows that $\lim_{n\to\infty} \|T_n - T\| = 0$, or $\lim_{n\to\infty} T_n = T$. Thus the sequence $\{T_n\}$ has a limit in $L(E, F)$. Therefore $L(E, F)$ is complete. ∎

$L(E, \mathbb{C})$, or $L(E, \mathbb{R})$, depending on whether $E$ is a complex or a real vector space, is called the topological dual of $E$ and is denoted by $E^*$; $E^*$ is a Banach space. Elements of $E^*$ are called bounded linear functionals on $E$.

When $E = F$, $L(E, F)$ is usually abbreviated to $L(E)$. For $S, T$ in $L(E)$, $S \circ T$ is in $L(E)$ and $\|S \circ T\| \leq \|S\| \cdot \|T\|$, as follows directly from definitions. Usually, we shall denote $S \circ T$ by $ST$; then for $S, T$, and $U$ in $L(E)$, $(ST)U = S(TU)$, and we may therefore denote $TT$ by $T^2$, $(TT)T$ by $T^3$, $\ldots$ etc. for $T \in L(E)$ free of misinterpretation. Note that $\|T^k\| \leq \|T\|^k$ for $T \in L(E)$ and $k \in \mathbb{N}$. For convenience, we put $T^\circ = 1$, the identity map on $E$.

**Exercise 1.4.13** Let $S$ be a nonempty set and consider the vector space $B(S)$ of all bounded real(complex)-valued functions on $S$. Addition and multiplication by scalar in $B(S)$ are usual for functions. For $f \in B(S)$, let $\|f\| = \sup_{s\in S} |f(s)|$.

(i) Show that $(B(S), \|\cdot\|)$ is a Banach space.

(ii) For $a \in B(S)$, define $A : B(S) \to B(S)$ by $(Af)(s) = a(s)f(s)$, $s \in S$. Show that $A$ is a bounded linear transformation from $B(S)$ into itself and that $\|A\| = \|a\|$.

**Exercise 1.4.14** Consider $C[0,1]$ and let $g \in C[0,1]$. Define a linear functional $\ell$ on $C[0,1]$ by

$$\ell(f) = \int_0^1 f(x)g(x)dx.$$

Show that $\ell \in C[0,1]^*$ and $\|\ell\| = \int_0^1 |g(x)|dx$.

**Exercise 1.4.15** Let $g$ be a continuous function on $[0,1] \times [0,1]$ and for $f \in C[0,1]$, let the function $Tf$ be defined by $Tf(x) = \int_0^1 g(x,y)f(y)dy$. Show that $T \in L(C[0,1])$ and $\|T\| = \max_{x\in[0,1]} \int_0^1 |g(x,y)|dy$.

We now consider a series of elements in a n.v.s. $E$. A symbol of the form $\sum_{k=1}^{\infty} x_k$ with each $x_k$ in $E$ is called a **series**. For each $n \in \mathbb{N}$, $\sum_{k=1}^{n} x_k$ is called the $n$-th partial **sum** of the series $\sum_{k=1}^{\infty} x_k$. If it happens that $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists in $E$, say $x$, then the series $\sum_{k=1}^{\infty} x_k$ is said to be convergent in $E$ and $x$ is called the **sum** of the series, $\sum_{k=1}^{\infty} x_k$, symbolically expressed by $x = \sum_{k=1}^{\infty} x_k$, i.e. when $\sum_{k=1}^{\infty} x_k$ converges, we attach a meaning to the symbol $\sum_{k=1}^{\infty} x_k$ by referring to it as $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$, or the sum of the series.

**Theorem 1.4.3** *Let $\{x_k\}$ be a sequence in a Banach space $E$ such that $\sum_{k=1}^{\infty} \|x_k\| < \infty$. Then $\sum_{k=1}^{\infty} x_k$ converges in $E$.*

***Proof*** For $n \in \mathbb{N}$, let $y_n = \sum_{k=1}^{n} x_k$. Then for $m > n$ in $\mathbb{N}$,

$$\|y_m - y_n\| = \left\| \sum_{k=n+1}^{m} x_k \right\| \le \sum_{k=n+1}^{m} \|x_k\| \to 0$$

as $n \to \infty$. This means that $\{y_n\}$ is a Cauchy sequence in $E$, but the fact that $E$ is complete implies that $\{y_n\}$ converges in $E$, i.e. $\lim_{n\to\infty} \sum_{k=1}^{n} x_k$ exists in $E$. ■

**Exercise 1.4.16** Suppose that $\sum_{k=1}^{\infty} x_k$ is a convergent series in a n.v.s. $E$. Show that

$$\left\| \sum_{k=1}^{\infty} x_k \right\| \le \sum_{k=1}^{\infty} \|x_k\|.$$

**Exercise 1.4.17** Suppose that $\sum_{k=1}^{\infty} \alpha_k$ is a convergent series in $\mathbb{R}$.

(i) If $x$ is an element of a n.v.s. $E$, show that $\sum_{k=1}^{\infty} \alpha_k x$ converges in $E$.

(ii) If $\{x_k\}$ is a bounded sequence in a Banach space $E$ and $\sum_{k=1}^{\infty} \alpha_k$ is absolutely convergent, show that $\sum_{k=1}^{\infty} \alpha_k x_k$ converges in $E$.

The following example, which complements Theorem 1.4.3, illustrates a method to extract a convergent subsequence from a given sequence.

**Example 1.4.4** If a series $\sum_{n=1}^{\infty} x_n$ in a n.v.s. $E$ converges whenever $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $E$ is a Banach space. To show this, let $\{y_n\}$ be a Cauchy sequence in $E$. Since $\{y_n\}$ is Cauchy, there is an increasing sequence $n_1 < n_2 < \cdots < n_k < \cdots$ in $\mathbb{N}$ such that $\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{k^2}$ for each $k$. Then $\sum_{k=1}^{\infty} \|y_{n_{k+1}} - y_{n_k}\| < \infty$ and hence

$\sum_{k=1}^{\infty}(y_{n_{k+1}} - y_{n_k})$ converges, which is equivalent to $\{y_{n_k}\}$ being a convergent sequence. We have shown that $\{y_n\}$ has a convergent subsequence; thus $\{y_n\}$ converges by Exercise 1.4.3 and $E$ is therefore complete.

**Remark** We conclude this section with a remark on norms on a vector space $E$. Suppose that $\|\cdot\|'$ and $\|\cdot\|''$ are different norms on a vector space $E$, in general, $\|\cdot\|'$ and $\|\cdot\|''$ will generate different families of open sets; but a moment's reflection convinces us that $\|\cdot\|'$ and $\|\cdot\|''$generate the same family of open sets if and only if there is $c > 0$ such that

$$c\|x\|'' \leq \|x\|' \leq \frac{1}{c}\|x\|''$$

for all $x$ in $E$ (in this case $\|\cdot\|'$ and $\|\cdot\|''$ are said to be equivalent). We shall see in Proposition 1.7.2 that all norms on a finite-dimensional vector space are equivalent.

## 1.5 Semi-continuities

For real-valued functions, the fact that the real field $\mathbb{R}$ is ordered plays an important role in the analysis of functions. In particular, for real-valued functions defined on a metric space, lower semi-continuity and upper semi-continuity are useful concepts that owe their existence to $\mathbb{R}$ being ordered. Semi-continuities are our concern in this section. For a subset $S$ of $\mathbb{R}$ we shall adopt the convention that $\inf S = \infty$ and $\sup S = -\infty$ if $S$ is empty; and that $\inf S = -\infty$ if $S$ is not bounded from below, while $\sup S = \infty$ if $S$ is not bounded from above.

For a sequence $x_n$, $n = 1, 2, \ldots$, of real numbers, let

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right), \tag{1.4}$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right). \tag{1.5}$$

Notice that $\inf_{k\geq n} x_k$ is increasing and $\sup_{k\geq n} x_k$ is decreasing as $n$ increases, hence both limits on the right-hand sides of (1.4) and (1.5) exist, although they may not be finite. Thus $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ always exist, and are called respectively the **inferior limit** and the **superior limit** of $\{x_n\}$. Clearly, $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

### Exercise 1.5.1

(i) Show that $\lim_{n\to\infty} x_n$ exists if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, and $\lim_{n\to\infty} x_n$ is the common value $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$ if it exists.

(ii) Show that $\liminf_{n\to\infty}(x_n + y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$ ($\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$), if $\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$ ($\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$) is meaningful. Note that $\alpha + \beta$ is meaningful if at least one of $\alpha$ and $\beta$ is finite, or if both $\alpha$ and $\beta$ are either $\infty$ or $-\infty$.

(iii) Show that $\liminf_{n\to\infty}(x_n + y_n) \le \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$ if the right-hand side is meaningful and that $\limsup_{n\to\infty}(x_n + y_n) \ge \liminf_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$ if the right-hand side is meaningful.

A real-valued function $f$ defined on a metric space $M$ with metric $\rho$ is said to be **lower semi-continuous (upper semi-continuous)** at $x \in M$ if, for every sequence $\{x_n\}$ in $M$ with $x = \lim_{n\to\infty} x_n$, $f(x) \le \liminf_{n\to\infty} f(x_n)$ $(f(x) \ge \limsup_{n\to\infty} f(x_n))$ holds. Lower semi-continuity and upper semi-continuity will often be abbreviated as l.s.c. and u.s.c. respectively. It is clear that a function $f$ is l.s.c. (u.s.c.) at $x$ if and only if for any given $\varepsilon > 0$ there is $\delta > 0$ such that $f(y) > f(x) - \varepsilon$ $(f(y) < f(x) + \varepsilon)$ if $\rho(y, x) < \delta$.

### Exercise 1.5.2

(i) Show that $f$ is lower semi-continuous (upper semi-continuous) at $x$ if and only if

$$f(x) = \lim_{\delta\searrow 0}\left[\inf_{y\in M,\ \rho(x,y)<\delta} f(y)\right] \left(f(x) = \lim_{\delta\searrow 0}\left[\sup_{y\in M,\ \rho(x,y)<\delta} f(y)\right]\right);$$

(ii) show that $f$ is continuous at $x$ if and only if $f$ is both lower semi-continuous and upper semi-continuous at $x$.

Because of the assertions of Exercise 1.5.2, if $x$ is not an isolated point of $M$, we define $\liminf_{y\to x} f(y)$ and $\limsup_{y\to x} f(y)$ by

$$\liminf_{y\to x} f(y) = \lim_{\delta\searrow 0}\left[\inf_{y\in M,\ 0<\rho(x,y)<\delta} f(y)\right];$$

$$\limsup_{y\to x} f(y) = \lim_{\delta\searrow 0}\left[\sup_{y\in M,\ 0<\rho(x,y)<\delta} f(y)\right],$$

since $\inf_{y\in M,\ 0<\rho(x,y)<\delta} f(y)$ increases as $\delta$ decreases and $\sup_{y\in M,\ 0<\rho(x,y)<\delta} f(y)$ decreases as $\delta$ decreases, both $\liminf_{y\to x} f(y)$ and $\limsup_{y\to x} f(y)$ exist, although they may not be finite. If $\liminf_{y\to x} f(y) = \limsup_{y\to x} f(y)$, the common value is called the limit of $f(y)$ as $y \to x$ and is denoted by $\lim_{y\to x} f(y)$. Usually, $\lim_{y\to x} f(y)$ is simply called the limit of the function $f$ at $x$. Note that $\liminf_{y\to x} f(y)$ and $\limsup_{y\to x} f(y)$ are defined if $f$ is defined on a neighborhood of $x$ with $x$ excluded. If $x$ is an isolated point of $M$ and $f$ is defined at $x$, then $\liminf_{y\to x} f(y) = \limsup_{y\to x} f(y) = \lim_{y\to x} f(y) = f(x)$ by definition.

### Exercise 1.5.3

(i) Show that $\liminf_{y\to x} f(y) \le \limsup_{y\to x} f(y)$ and that $f$ is continuous at $x$ if and only if $\lim_{y\to x} f(y) = f(x)$.

(ii) Show that $f$ is l.s.c. (u.s.c.) at $x$ if and only if $f(x) \le \liminf_{y\to x} f(y)$ $(f(x) \ge \limsup_{y\to x} f(y))$.

If $f$ is lower semi-continuous (upper semi-continuous) at every point of $M$, then $f$ is said to be lower semi-continuous (upper semi-continuous) on $M$.

**Exercise 1.5.4** Show that $f$ is lower semi-continuous (upper semi-continuous) on $M$ if and only if $\{x \in M : f(x) > \alpha\}$ ($\{x \in M : f(x) < \alpha\}$) is open for every $\alpha \in \mathbb{R}$.

**Exercise 1.5.5** Let $f_\alpha$, $\alpha \in I$, be a family of real-valued continuous functions defined on $M$ and assume that $\sup_{\alpha\in I} f_\alpha(x)$ ($\inf_{\alpha\in I} f_\alpha(x)$) is finite for each $x \in M$; show that $\sup_{\alpha\in I} f_\alpha(x)$ ($\inf_{\alpha\in I} f(x)$) is lower (upper) semi-continuous on $M$.

**Exercise 1.5.6** Suppose that $f$ is a real-valued function defined on a metric space and assume that $f$ is bounded from below on $M$, i.e. there is $c \in \mathbb{R}$ such that $f(z) \geq c$ for all $z \in M$. For each $k \in \mathbb{N}$ is defined a function $f_k$ on $M$ by

$$f_k(x) = \inf_{z\in M}\{f(z) + k\rho(x,z)\}, \quad x \in M.$$

(i) Show that $f_k(x)$ is finite for all $x \in M$ and

$$|f_k(x) - f_k(y)| \leq k\rho(x,y)$$

for all $x, y$ in $M$.

(ii) Suppose that $f$ is l.s.c. on $M$. Show that

$$f(x) = \lim_{k\to\infty} f_k(x), \quad x \in M.$$

(iii) Show that $f$ is l.s.c. on $M$ if and only if there is an increasing sequence $\{f_k\}$ of continuous functions on $M$ such that

$$f(x) = \lim_{k\to\infty} f_k(x)$$

for all $x \in M$.

**Exercise 1.5.7** A metric space $M$ is called a compact space if every sequence in $M$ has a subsequence which converges in $M$. Show that if $f$ is lower semi–continuous (upper semi-continuous) on a compact metric space $M$, then $f$ assumes its minimum (maximum) on $M$. (Hint: There is a sequence $\{x_n\}$ in $M$ such that $\lim_{n\to\infty} f(x_n) = \inf_{x\in M} f(x)$)

## 1.6 The space $\ell^p(\mathbb{Z})$

The Banach spaces considered in this section are included in the more general class of $L^p$ spaces, to be introduced in Section 2.7; but it is expedient to give a separate and direct treatment here without recourse to general theory of measure and integration.

Let $\mathbb{Z}$ be the set of all integers and consider the space $L$ of all real-valued functions defined on $\mathbb{Z}$. With the usual definition of addition of functions and multiplication of a function by a scalar, $L$ is a real vector space. For $f \in L$ and $j \in \mathbb{Z}$, if we denote $f(j)$ by $f_j$, then $f$ can be identified with the two-way sequence $(f_j)_{j\in\mathbb{Z}}$ of real numbers and $L$ is the space of all sequences $(a_j)_{j\in\mathbb{Z}}$ of real numbers. For $f \in L$ and $1 \le p \le \infty$, let

$$\|f\|_p = \begin{cases} \left(\sum_{j\in\mathbb{Z}} |f(j)|^p\right)^{\frac{1}{p}} & \text{if } p < \infty; \\ \sup_{j\in\mathbb{Z}} |f(j)| & \text{if } p = \infty. \end{cases}$$

Now consider the space $\ell^p(\mathbb{Z})$, $1 \le p \le \infty$, defined by

$$\ell^p(\mathbb{Z}) = \{f \in L : \|f\|_p < \infty\}.$$

Presently we shall prove that $\ell^p(\mathbb{Z})$ is a vector space and $\|\cdot\|_p$ is a norm on $\ell^p(\mathbb{Z})$, but for this purpose we first show an inequality which is a generalization of the Schwarz inequality and is called Hölder's inequality. Two extended real numbers $p, q \ge 1$ are called **conjugate** exponents if $\frac{1}{p} + \frac{1}{q} = 1$ ($\frac{1}{\infty} = 0$; for further arithmetic conventions regarding $\infty$ and $-\infty$, see the first paragraph of Section 2.2), while two nonnegative numbers $\alpha$ and $\beta$ will be called a **convex** pair if $\alpha + \beta = 1$.

**Lemma 1.6.1** *If $\alpha$ and $\beta$ is a convex pair, then for any $0 \le \zeta, \eta < \infty$ the following inequality holds:*

$$\zeta^\alpha \eta^\beta \le \alpha\zeta + \beta\eta. \tag{1.6}$$

***Proof*** We may assume that $0 < \alpha, \beta < 1$ and $\zeta, \eta > 0$.

Since $(1 + x)^\alpha \le \alpha x + 1$, for $x \ge 0$, we have

$$y^\alpha \le \alpha y + \beta, \;\; y \ge 1. \tag{1.7}$$

Now either $\zeta\eta^{-1} \ge 1$ or $\zeta^{-1}\eta \ge 1$; if $\zeta\eta^{-1} \ge 1$, take $y = \zeta\eta^{-1}$ in (1.7), while if $\zeta^{-1}\eta \ge 1$, take $y = \zeta^{-1}\eta$ in (1.7) with $\alpha$ and $\beta$ interchanged, then proceed to (1.6). ■

**Lemma 1.6.2** (**Hölder's inequality**) *If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are in $\mathbb{R}^n$, then for conjugate exponents $p$ and $q$ we have*

$$\sum_{j=1}^{n} |x_j y_j| \le \|x\|_p \|y\|_q.$$

**Remark** Since an element $x$ of $\mathbb{R}^n$ can be identified with an element $f$ of $L$ by $f(1) = x_1, \ldots, f(n) = x_n$, and $f(j) = 0$ for other $j$, $\|x\|_p$ is defined.

**Proof of Lemma 1.6.2** It is clear that if one of $p$ and $q$ is $\infty$, the lemma is trivial, hence we suppose that $1 < p, q < \infty$. Since $\|x\|_p = 0$ if and only if $x = 0$, we may assume that $\|x\|_p > 0$ and $\|y\|_p > 0$. For $1 \le j \le n$, choose $\zeta = \left(\frac{|x_j|}{\|x\|_p}\right)^p$ and $\eta = \left(\frac{|y_j|}{\|y\|_q}\right)^q$ in Lemma 1.6.1. with $\alpha = \frac{1}{p}$ and $\beta = \frac{1}{q}$, then

$$\frac{|x_j y_j|}{\|x\|_p \|y\|_q} \le \frac{1}{p}\frac{|x_j|^p}{\|x\|_p^p} + \frac{1}{q}\frac{|y_j|^q}{\|y\|_q^q},$$

and consequently

$$\sum_{j=1}^n |x_j y_j| \le \|x\|_p \|y\|_q \left(\frac{1}{p} + \frac{1}{q}\right) = \|x\|_p \|y\|_q. \qquad \blacksquare$$

**Exercise 1.6.1** Suppose that $\alpha > 0$ and $\beta > 0$ is a convex pair. Show that

$$\zeta^\alpha \eta^\beta = \alpha\zeta + \beta\eta,\ \zeta \ge 0,\ \eta \ge 0$$

if and only if $\zeta = \eta$.

We are now in a position to prove that $\ell^p(\mathbb{Z})$ is a vector space and $\|\cdot\|_p$ is a norm on $\ell^p(\mathbb{Z})$. That $\|f\|_p = 0$ if and only if $f = 0$ and that $\lambda f \in \ell^p(\mathbb{Z})$ and $\|\lambda f\|_p = |\lambda|\|f\|_p$ for $\lambda \in \mathbb{R}$ and $f \in \ell^p(\mathbb{Z})$ are obvious. It only remains to show that $\|f+g\|_p \le \|f\|_p + \|g\|_p$ for $f, g$ in $\ell^p(\mathbb{Z})$. For this purpose, we may assume that $1 < p < \infty$ and $\|f+g\|_p > 0$. Under this assumption, there is $A \in F(\mathbb{Z})$ such that $\sum_{j\in A} |f(j)+g(j)|^p > 0$. For such $A$, we have

$$0 < \sum_{j\in A} |f(j)+g(j)|^p \le \sum_{j\in A} |f(j)+g(j)|^{p-1}\big(|f(j)| + |g(j)|\big),$$

from which, by using Hölder's inequality (see Lemma 1.6.2.), we have

$$\begin{aligned} 0 &< \sum_{j\in A} |f(j)+g(j)|^p \\ &\le \left(\sum_{j\in A} |f(j)+g(j)|^{(p-1)q}\right)^{\frac{1}{q}} \left\{\left(\sum_{j\in A}|f(j)|^p\right)^{\frac{1}{p}} + \left(\sum_{j\in A}|g(j)|^p\right)^{\frac{1}{p}}\right\} \\ &\le \left(\sum_{j\in A} |f(j)+g(j)|^p\right)^{\frac{1}{q}} \big(\|f\|_p + \|q\|_p\big), \end{aligned}$$

and thus, on dividing the last sequence of inequalities by $\left(\sum_{j\in A} |f(j)+g(j)|^p\right)^{\frac{1}{q}}$, we obtain

$$\left(\sum_{j\in A} |f(j)+g(j)|^p\right)^{\frac{1}{p}} \le \|f\|_p + \|g\|_p. \tag{1.8}$$

Now observe that (1.8) holds for any $A \in F(\mathbb{Z})$. Taking the supremum on the left-hand side of (1.8) over $A \in F(\mathbb{Z})$, we see that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. Therefore, $\ell^p(\mathbb{Z})$ is a vector space and $\|\cdot\|_p$ is a norm on $\ell^p(\mathbb{Z})$. We shall always refer to $\ell^p(\mathbb{Z})$ as a normed vector space with this norm.

**Exercise 1.6.2** Let $k_1 < \cdots < k_n$ be a finite sequence in $\mathbb{Z}$ of length $n$; define a map $T$ from $\ell^p(\mathbb{Z})$ to the $n$-dimensional Euclidean $\mathbb{R}^n$ by

$$T(f) = \big(f(k_1), \ldots, f(k_n)\big), \ f \in \ell^p(\mathbb{Z}).$$

Show that $T$ is continuous from $\ell^p(\mathbb{Z})$ onto $\mathbb{R}^n$ and that the image under $T$ of any open set in $\ell^p(\mathbb{Z})$ is an open set in $\mathbb{R}^n$.

**Exercise 1.6.3** Suppose $1 \leq p < \infty$; show that $|a_1 + \cdots + a_n|^p \leq n^{p-1} \sum_{j=1}^n |a_j|^p$ for $a_1, \ldots, a_n$ in $\mathbb{R}$.

**Exercise 1.6.4** Let $f_1, f_2, \ldots, f_n, \ldots$ be a Cauchy sequence in $\ell^p(\mathbb{Z})$; show that $\lim_{n\to\infty} f_n(j)$ exists and is finite for every $j \in \mathbb{Z}$.

**Exercise 1.6.5** Show that $\ell^\infty(\mathbb{Z})$ is a Banach space.

**Theorem 1.6.1** *$\ell^p(\mathbb{Z})$ is a Banach space for $1 \leq p \leq \infty$.*

***Proof*** The case $p = \infty$ is relatively easy and is left as an exercise (see Exercise 1.6.5). Consider now the case $1 \leq p < \infty$. Let $f_1, f_2, \ldots, f_n, \ldots$ be a Cauchy sequence in $\ell^p(\mathbb{Z})$, then $\lim_{n\to\infty} f_n(j)$ exists and is finite for each $j \in \mathbb{Z}$ (see Exercise 1.6.4), say $f(j) = \lim_{n\to\infty} f_n(j)$. We show first that $f \in \ell^p(\mathbb{Z})$. Since $f_1, f_2, \ldots, f_n, \ldots$ is a Cauchy sequence, it is necessarily bounded. Let $\|f_n\|_p \leq M$ for all $n$. There is $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p < 1, \ \ n, m \geq n_0.$$

Now fix $m \geq n_0$ and let $A \in F(\mathbb{Z})$, then

$$\begin{aligned}\sum_{j\in A} |f(j)|^p &= \lim_{n\to\infty} \sum_{j\in A} |f_n(j)|^p = \lim_{n\to\infty} \sum_{j\in A} |f_n(j) - f_m(j) + f_m(j)|^p \\ &\leq \limsup_{n\to\infty} \sum_{j\in A} \{|f_n(j) - f_m(j)| + |f_m(j)|\}^p,\end{aligned}$$

from which, by Exercise 1.6.3, we have

$$\begin{aligned}\sum_{j\in A} |f(j)|^p &\leq \limsup_{n\to\infty} 2^{p-1} \left\{ \sum_{j\in A} |f_n(j) - f_m(j)|^p + \sum_{j\in A} |f_m(j)|^p \right\} \\ &\leq 2^{p-1} \left\{ \limsup_{n\to\infty} \|f_n - f_m\|_p^p + \|f_m\|_p^p \right\} \\ &\leq 2^{p-1}\{1 + M^p\}.\end{aligned}$$

Thus,

$$\sum_{j\in\mathbb{Z}}|f(j)|^p = \sup_{A\in F(\mathbb{Z})}\sum_{j\in A}|f(j)|^p \le 2^{p-1}(1+M^p) < \infty,$$

which shows $f \in \ell^p(\mathbb{Z})$. We now claim $\lim_{n\to\infty} f_n = f$ in $\ell^p(\mathbb{Z})$. Actually, given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p < \varepsilon, \ \ n, m \ge N.$$

Now, for $n \ge N$ and $A \in F(\mathbb{Z})$,

$$\begin{aligned}\sum_{j\in A}|f(j) - f_n(j)|^p &= \lim_{m\to\infty}\sum_{j\in A}|f_m(j) - f_n(j)|^p \\ &\le \liminf_{m\to\infty}\|f_m - f_n\|_p^p \le \varepsilon^p,\end{aligned}$$

which implies

$$\|f - f_n\|_p^p = \sup_{A\in F(\mathbb{Z})}\sum_{j\in A}|f(j) - f_n(j)|^p \le \varepsilon^p,$$

or

$$\|f - f_n\|_p \le \varepsilon, \ \ n \ge N.$$

In other words, $\lim_{n\to\infty} f_n = f$ in $\ell^p(\mathbb{Z})$. This shows that $\ell^p(\mathbb{Z})$ is complete and hence is a Banach space. ■

**Exercise 1.6.6** Let $f, g$ be in $\ell^1(\mathbb{Z})$.

(i) Show that $\{f(n-m)g(m)\}_{(n,m)\in\mathbb{Z}\times\mathbb{Z}}$ is summable and

$$\sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} f(n-m)g(m) = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} f(n-m)g(m).$$

(ii) Define $f * g(n) = \sum_{m\in\mathbb{Z}} f(n-m)g(m), \ n \in \mathbb{Z}$. Show that $f * g \in \ell^1(\mathbb{Z})$, $f * g = g * f$, and $\|f * g\|_1 \le \|f\|_1\|g\|_1$.

**Exercise 1.6.7** Suppose that $f \in l^p(\mathbb{Z})$ and $g \in l^1(\mathbb{Z})$. Show that $f * g$ can be defined similarly as in Exercise 1.6.6 (ii); then show that $f * g = g * f$, and

$$\|f * g\|_p \le \|f\|_p\|g\|_1.$$

**Remark** For any nonempty set $S$ and $1 \le p \le \infty$, the Banach space $\ell^p(S)$ can be defined in the same way that $\ell^p(\mathbb{Z})$ is defined. The first such space is the space $\ell^2(\mathbb{N})$ introduced by D. Hilbert in his study of the Fredholm theory of integral equations.

## 1.7 Compactness

This section is devoted to a study of compactness, introduced in Exercise 1.5.7. Existence of mathematical objects in analysis often involves arguments of compactness: for example, Exercise 1.5.7 guarantees that if $f$ is a lower semi-continuous function defined on a compact metric space $M$, then there exists $x_0 \in M$ such that

$$f(x_0) = \min_{x \in M} f(x).$$

Recall from Exercise 1.5.7 that a metric space $M$ is called a **compact** space if every sequence in $M$ has a subsequence which converges in $M$. One observes readily that a compact metric space is necessarily complete. There is a characterization of compact metric spaces which is often useful. To prepare for the statement of such a characterization, we call a point $x_0$ of a metric space $M$ a **limit point** of a set $A \subset M$ if every neighborhood of $x_0$ contains a point of $A$ other than $x_0$.

**Exercise 1.7.1** Let $A$ be a subset of a metric space $M$.

(i) Show that a point $x_0$ is a limit point of $A$ if and only if every neighborhood of $x_0$ contains infinitely many points of $A$;

(ii) show that $A$ is closed if and only if it contains all its limit points. Infer in particular that a finite set is closed.

**Theorem 1.7.1** *A metric space $M$ is compact if and only if every infinite subset of $M$ has a limit point.*

***Proof*** Suppose first that $M$ is compact and let $A$ be an infinite subset of $M$. We shall show that $A$ has a limit point. Since $A$ is infinite, there is a sequence $\{x_n\}$ in $A$ formed of mutually different points. As $M$ is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to $x \in M$. Since $\{x_{n_k}\}$ is formed of mutually different points in $A$ and $x = \lim_{k\to\infty} x_{n_k}$, $x$ is a limit point of $A$. We have shown that if $M$ is compact, then every infinite subset of $M$ has a limit point.

Next, suppose that every infinite subset of $M$ has a limit point. Let us show that $M$ is compact. Suppose that $\{x_n\}$ is a sequence in $M$. If the range of the sequence $\{x_n\}$ is a finite set, then $x_{n_1} = x_{n_2} = \cdots = x_{n_k} = \cdots$ for some subsequence $\{n_k\}$ of $\{n\}$, and hence the subsequence $\{x_{n_k}\}$ of $\{x_n\}$, being a constant sequence, converges. On the other hand, if the range of $\{x_n\}$ is infinite, then it has a limit point $x$. It is clear that $x$ is the limit of a subsequence of $\{x_n\}$. Thus $M$ is compact. ■

A subset $K$ of a metric space is said to be compact if $K$ is a compact metric space with metric inherited from $M$. From the **Bolzano–Weierstrass theorem**, which states that every bounded infinite subset of $\mathbb{R}$ has a limit point, it follows that every bounded closed subset of $\mathbb{R}$ is compact. Historically, the Bolzano–Weierstrass theorem is the genesis of the concept of compact sets.

**Exercise 1.7.2** Suppose that $K_1 \supset K_2 \supset \cdots \supset K_n \supset K_{n+1} \supset \cdots$ is a decreasing sequence of nonempty compact sets in a metric space. Show that $\bigcap_n K_n \neq \emptyset$.

**Exercise 1.7.3** Show that the Bolzano–Weierstrass theorem holds also for $\mathbb{R}^k$, $k \geq 2$ and then infer that every bounded closed subset of $\mathbb{R}^k$ is compact. Show also that every bounded closed set in the unitary space $\mathbb{C}^k$ is compact.

**Exercise 1.7.4**

(i) Show that compact subsets of a metric space are both bounded and closed.

(ii) Show that a subset of the Euclidean space $\mathbb{R}^k$ or of the unitary space $\mathbb{C}^n$ is compact if and only if it is both bounded and closed.

(iii) Let, for each $n \in \mathbb{Z}$, $e_n$ be the element of $l^2(\mathbb{Z})$ (see Section 1.6) such that $e_n(j) = \delta_{nj}$, $j \in \mathbb{Z}$. Show that $\{e_n\}_{n\in\mathbb{Z}}$ is a bounded and closed subset of $l^2(\mathbb{Z})$, but it is not compact. Recall that $\delta_{nj}$ is the Kronecker delta, defined by $\delta_{nj} = 1$ or 0 according to whether $n = j$ or $n \neq j$.

**Proposition 1.7.1** *If $T$ is a continuous map from a metric space $M_1$ into a metric space $M_2$, then for every compact set $K$ in $M_1$, $TK$ is a compact set in $M_2$, i.e. continuous images of compact sets are compact.*

***Proof*** Let $K$ be a compact set in $M_1$; we may assume that $K$ is nonempty. Suppose that $\{y_n\}$ is a sequence in $TK$; we have to show that $\{y_n\}$ has a subsequence which converges to an element in $TK$. For each $n \in \mathbb{N}$, pick $x_n \in K$ such that $y_n = Tx_n$. Since $K$ is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x \in K$. Since $T$ is continuous, $y_{n_k} = Tx_{n_k} \to Tx$. Thus the subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converges to an element in $TK$. ■

An interesting consequence of Proposition 1.7.1 is the following proposition concerning norms on a finite-dimensional vector space.

**Proposition 1.7.2** *If $E$ is a finite-dimensional vector space, then any two norms $\|\cdot\|'$ and $\|\cdot\|''$ on $E$ are equivalent, in the sense that there is $c > 0$ such that $c\|v\|'' \leq \|v\|' \leq \frac{1}{c}\|v\|''$ for all $v \in E$.*

***Proof*** For definiteness we assume that $E$ is a complex vector space. Let $n = \dim E$, and choose a basis $\{v_1, \ldots, v_n\}$ of $E$. Define a norm $\|\cdot\|$ on $E$ by

$$\|v\| = \left\{ \sum_{j=1}^{n} |\alpha_j|^2 \right\}^{1/2}$$

if $v = \sum_{j=1}^{n} \alpha_j v_j$, where each $\alpha_j \in \mathbb{C}$. Let $\Gamma$ be the set $\{v = \sum_{j=1}^{n} \alpha_j v_j : \sum_{j=1}^{n} |\alpha_j|^2 = 1\}$ in $E$. Define a map $T : \mathbb{C}^n \to E$ by

$$T(\zeta) = \sum_{j=1}^{n} \zeta_j v_j, \quad \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n.$$

From $\|T(\zeta) - T(\eta)\|' \leq \sum_{j=1}^{n} |\zeta_j - \eta_j| \|v_j\|' \leq \sqrt{n} \max_{1 \leq j \leq n} \|v_j\|' |\zeta - \eta|$, where $|\zeta - \eta|$ is the norm of $\zeta - \eta$ in the unitary space $\mathbb{C}^n$, it follows that $T$ is continuous from the unitary space $\mathbb{C}^n$ into $(E, \|\cdot\|')$. Note that $T$ is bijective. Since $\Gamma$ is the image under $T$ of the compact set $\{\zeta \in \mathbb{C}^n : \sum_{j=1}^{n} |\zeta_j|^2 = 1\}$ in $\mathbb{C}^n$, $\Gamma$ is compact in $(E, \|\cdot\|')$, by Proposition 1.7.1. Now let $r = \inf_{v \in \Gamma} \|v\|'$ and observe that since $\Gamma$ is compact in $(E, \|\cdot\|')$ and $\Gamma$ does not contain the zero element of $E$, $r = \min_{v \in \Gamma} \|v\|' > 0$; in other words, $\|v\|' \geq r > 0$ for all $v$ with $\|v\| = 1$. Now let $v \in E$, $v \neq 0$, then $\|\frac{v}{\|v\|}\|' \geq r$ or $r\|v\| \leq \|v\|'$. On the other hand, $\|v\|' \leq \sum_{j=1}^{n} |\alpha_j| \|v_j\|' \leq \sqrt{n}(\max_{1 \leq j \leq n} \|v_j\|')\{\sum_{j=1}^{n} |\alpha_j|^2\}^{1/2} = \sqrt{n}(\max_{1 \leq j \leq n} \|v_j\|')\|v\|$, or, if we let $\sqrt{n}(\max_{1 \leq j \leq n} \|v_j\|') = R$, we have

$$\|v\|' \leq R\|v\|$$

for all $v \in E$ (note: we write $v = \sum_{j=1}^{n} \alpha_j v_j$ for $v \in E$). We choose then $c' > 0$ such that $c' \leq r$ and $\frac{1}{c'} \geq R$, then

$$c'\|v\| \leq \|v\|' \leq \frac{1}{c'}\|v\|, \quad v \in E.$$

Similarly, there is $c'' > 0$ such that

$$c''\|v\| \leq \|v\|'' \leq \frac{1}{c''}\|v\|, \quad v \in E.$$

Then, for $v \in E$,

$$c'c''\|v\|'' \leq c'\|v\| \leq \|v\|' \leq \frac{1}{c'}\|v\| \leq \frac{1}{c'c''}\|v\|'',$$

or

$$c\|v\|'' \leq \|v\|' \leq \frac{1}{c}\|v\|'',$$

where $c = c'c'' > 0$. ■

**Corollary 1.7.1** *Finite-dimensional vector subspaces of a n.v.s. E are all closed.*

***Proof*** For definiteness, assume that $E$ is a real n.v.s. with norm $\|\cdot\|$. Consider any finite-dimensional vector subspace $F$ of $E$, put $n$ = dimension of $F$ and choose a basis $\{v_1, \ldots, v_n\}$ of $F$. Define a new norm $\|\cdot\|'$ on $F$ as follows: for $u = \sum_{j=1}^{n} \alpha_j v_j$ where $\alpha_1, \ldots, \alpha_n$ are real numbers, let $\|u\|' = (\sum_{j=1}^{n} \alpha_j^2)^{1/2}$. Clearly, $\|\cdot\|'$ is a norm on $F$. Let $T$ be the linear map from the Euclidean space $\mathbb{R}^n$ onto $F$, defined by $Tx = \sum_{j=1}^{n} x_j v_j$ for $x = (x_1, \ldots, x_n)$. If we denote by $|\cdot|$ the Euclidean norm for $\mathbb{R}^n$, then $\|Tx\|' = |x|$. By Proposition 1.7.2, there is $c > 0$ such that $c\|u\|' \leq \|u\| \leq c^{-1}\|u\|'$

for $u \in F$; consequently, $\|Tx\| \le c^{-1}\|Tx\|' = c^{-1}|x|$ for $x \in \mathbb{R}^n$ and hence $T$ is a continuous map from $\mathbb{R}^n$ into $E$. To show that $F$ is closed in $E$, we have to show that if $\{u_k\}$ is a sequence in $F$ which converges in $E$, then the limit is in $F$. Since $\{u_k\}$ converges, it is bounded, say $\|u_k\| \le A$ for all $k$ for some $A > 0$. Now write $u_k = \sum_{j=1}^n \alpha_j^{(k)} v_j$ and put $\alpha^{(k)} = (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)})$, then $u_k = T\alpha^{(k)}$ and $|\alpha^{(k)}| = \|u_k\|' \le c^{-1}\|u_k\| \le c^{-1}A$ for each $k$. Thus $\{u_k\}$ is contained in the image $K \subset F$ of the closed ball $\{x \in \mathbb{R}^n : |x| \le c^{-1}A\}$ under $T$. Since closed balls in $\mathbb{R}^n$ are compact, $K$ is compact by Proposition 1.7.1 and is therefore closed in $E$. Now $\{u_k\} \subset K$ implies that its limit is in $K \subset F$. This shows that $F$ is closed. ■

**Corollary 1.7.2** *Suppose that F is an affine subspace of* $\mathbb{R}^n$, *then for each* $x \in \mathbb{R}^n$, *there is unique y in F such that* $|x - y| = \min_{z \in F} |x - z|$. *Furthermore, y is characterized by the condition that* $(x - y) \cdot (z - y) = 0$ *for all* $z \in F$.

***Proof*** We need only consider the case that $F$ is a proper affine subspace of $\mathbb{R}^n$ and $x$ is not in $F$. Since $F$ is closed by Corollary 1.7.1, $\inf_{z \in F} |x - z| = l > 0$. Let $K = \{z \in F : |x - z| \le 2l\}$, then $\inf_{z \in F} |x - z| = \inf_{z \in K} |x - z|$; but, since $K$ is compact, there is $y \in K$ such that $l = \min_{z \in F} |x - z| = \min_{z \in K} |x - z| = |x - y|$. Consider now $z \in F$ and let $f(t) = |x - y + t(z - y)|^2 = |x - y|^2 + 2t(x - y) \cdot (z - y) + t^2|z - y|^2$ for $t \in \mathbb{R}$. Since $f$ assumes minimum $l^2$ at $t = 0$, $f'(0) = 2(x - y) \cdot (z - y) = 0$. Hence $y$ satisfies the condition that $(x - y) \cdot (z - y) = 0$ for all $z \in F$; on the other hand, if $y \in F$ satisfies the condition that $(x - y) \cdot (z - y) = 0$ for all $z \in F$, then for any $z \in F$ we have $|x - z|^2 = |x - y + y - z|^2 = |x - y|^2 + 2(x - y) \cdot (y - z) + |y - z|^2 = |x - y|^2 + |y - z|^2 \ge |x - y|^2$, i.e. $|x - y| = \min_{z \in F} |x - z|$. Thus, we have shown that there is $y \in F$ such that $|x - y| = \min_{z \in F} |x - z|$ and that $y$ is characterized by the condition that $(x - y) \cdot (z - y) = 0$ for all $z \in F$. It remains to show that $y$ is unique. Let $y$ and $y'$ in $F$ satisfy $|x - y| = |x - y'| = \min_{z \in F} |x - z|$, then

$$(x - y) \cdot (z - y) = 0, \quad (x - y') \cdot (z - y') = 0$$

for all $z$ in $F$. Choose $z = y'$ and $y$ respectively in these equalities; we have

$$(x - y) \cdot (y - y') = 0, \quad (x - y') \cdot (y - y') = 0;$$

subtract the first equality from the second; we have $(y - y') \cdot (y - y') = 0 = |y - y'|^2$, implying $y = y'$. ■

The map $x \mapsto y$, as asserted by Corollary 1.7.2, is called the **orthogonal** projection from $\mathbb{R}^n$ onto $F$. If this map is denoted by $P$, then (1) $Px = x$ if and only if $x \in F$; (2) $P^2 = P$; and (3) $|Px - Px'| \le |x - x'|$. That (1) and (2) hold is fairly obvious. To see that (3) holds, observe firstly that

$$(x - x' - Px + Px') \cdot (Px - Px') = 0,$$

from which it follows that $|Px - Px'|^2 = (x - x') \cdot (Px - Px') \le |x - x'||Px - Px'|$ and hence (3) holds. It follows from (3) that $P$ is a continuous map.

**Remark** If $F$ is a vector subspace of $\mathbb{R}^n$, then

(i) $P$ is actually a linear map, as follows easily from the characterization that $(x - Px) \cdot z = 0$ for all $z \in F$;

(ii) since $(x - Px) \cdot Px = 0$, $|x|^2 = |Px|^2 + |x - Px|^2$ for every $x \in \mathbb{R}^n$; this last equality is called the **Pythagoras relation**.

**Proposition 1.7.3** *Suppose that $T$ is an injective and continuous map from a compact metric space $M_1$ into a metric space $M_2$. Then $T^{-1} : TM_1 \to M_1$ is continuous.*

***Proof*** Let $y \in TM_1$ and $\{y_n\}$ be a sequence in $TM_1$ with $y = \lim_{n\to\infty} y_n$. To show that $T^{-1}$ is continuous at $y$, we have to show that $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ such that $\lim_{k\to\infty} T^{-1}y_{n_k} = T^{-1}y$ (cf. Exercise 1.4.7 (i)). Let $x_n = T^{-1}y_n$. Since $M_1$ is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to $x$ in $M_1$. Now $y_{n_k} = Tx_{n_k} \to Tx$ entails that $Tx = y$ and hence $\lim_{k\to\infty} T^{-1}y_{n_k} = \lim_{k\to\infty} x_{n_k} = x = T^{-1}y$. ■

We shall presently give a useful characterization of compact sets in a complete metric space corresponding to the characterization of compact sets in $\mathbb{R}^k$ as bounded and closed sets (see Exercise 1.7.4 (ii)).

A finite family of open balls with radius $\varepsilon > 0$ in a metric space $M$ is called an **$\varepsilon$-net** for a subset $A$ of $M$ if its union contains $A$. A set $A$ in a metric space is said to be **totally bounded** if for any $\varepsilon > 0$ there is an $\varepsilon$-net for $A$.

**Exercise 1.7.5**

(i) Show that a set in $\mathbb{R}^n$ is totally bounded if and only if it is bounded.

(ii) Show that a set $A$ in a metric space is totally bounded if and only if for any $\varepsilon > 0$ there is an $\varepsilon$-net for $A$ whose balls have their centers in $A$.

**Lemma 1.7.1** *A subset $A$ of a metric space $M$ is totally bounded if and only if every sequence in $A$ has a Cauchy subsequence. In particular, compact sets are totally bounded.*

***Proof*** Suppose that $A$ is totally bounded and let $\{x_n\}$ be a sequence in $A$. There is a $\frac{1}{2}$-net for $A$ and hence one of its balls contains a subsequence $\{x_n^{(1)}\}$ of $\{x_n\}$. After the sequence $\{x_n^{(1)}\}$ is chosen, we then choose a $\frac{1}{4}$-net for $A$. As before one of the balls of this $\frac{1}{4}$-net contains a subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$. We proceed in this way to obtain a sequence of subsequences, $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(k)}\}, \ldots$ of $\{x_n\}$, each of which is a subsequence of the preceding one, and for each $k$ the sequence $\{x_n^{(k)}\}$ is contained in a ball of radius $2^{-k}$. Now, $\{x_n^{(n)}\}$ is a subsequence of $\{x_n\}$. For each positive integer $n_0$, if $n > m \geq n_0$, both $x_n^{(n)}$ and $x_m^{(m)}$ are in a ball of radius $2^{-n_0}$, hence $\rho(x_n^{(n)}, x_m^{(m)}) \leq 2^{-n_0+1}$, from which it follows that $\{x_n^{(n)}\}$ is a Cauchy sequence. Thus each sequence in $A$ has a Cauchy subsequence.

Next, suppose that each sequence in $A$ has a Cauchy subsequence. We are going to show that $A$ is totally bounded. Suppose to the contrary that for some $\varepsilon_0 > 0$, no $\varepsilon_0$-net for $A$ exists. Choose $x_1 \in A$, since $B_{\varepsilon_0}(x_1)$ does not cover $A$ there is $x_2 \in A \backslash B_{\varepsilon_0}(x_1)$. Suppose that $x_1, \ldots, x_n$ in $A$ have been chosen so that $\rho(x_i, x_j) \geq \varepsilon_0$ for

$i, j \le n$ and $i \ne j$, then choose $x_{n+1} \in A \setminus \bigcup_{i=1}^n B_{\varepsilon_0}(x_i)$. Such an $x_{n+1}$ exists because $\{B_{\varepsilon_0}(x_0), \ldots, B_{\varepsilon_0}(x_n)\}$ is not an $\varepsilon_0$-net for $A$. But then $\rho(x_i, x_j) \ge \varepsilon_0$ for $i, j \le n+1$ and $i \ne j$. By mathematical induction we have thus exhibited a sequence $\{x_n\}$ in $A$ such that $\rho(x_i, x_j) \ge \varepsilon_0$ when $i \ne j$. Such a sequence can not have a Cauchy subsequence, this contradicts our assumption about $A$. Thus $A$ is totally bounded. ∎

**Theorem 1.7.2** *A subset $K$ of a complete metric space $M$ is compact if and only if $K$ is closed and totally bounded.*

***Proof*** Suppose that $K$ is compact, then $K$ is closed. Since each sequence in $K$ has a convergent subsequence which is therefore Cauchy, Lemma 1.7.1 implies that $K$ is totally bounded. Next, suppose $K$ is closed and totally bounded and let $\{x_n\}$ be a sequence in $K$, then $\{x_n\}$ has a Cauchy subsequence $\{x'_n\}$ by Lemma 1.7.1. But since $K$ is a closed subset of a complete metric space, it is complete and hence $\{x'_n\}$ converges in $K$. This shows that $K$ is compact. ∎

Let $A$ be a subset of a metric space; the smallest closed set which contains $A$ is called the **closure** of $A$ and is denoted by $\overline{A}$. Obviously, $\overline{A}$ is the intersection of all those closed sets containing $A$. If $\overline{A} = M$, we say that $A$ is dense in $M$, or that $A$ is a dense subset of $M$. A metric space $M$ is said to be **separable** if it contains a countable dense subset. A subset of a metric space is separable, if it is separable as a metric space; it is **precompact**, if its closure is compact.

Since the closure of a totally bounded set is totally bounded, Corollary 1.7.3 follows from Theorem 1.7.2 (see Exercise 1.7.6 and Exercise 1.7.7):

**Corollary 1.7.3** *A set in a complete metric space is precompact if and only if it is totally bounded.*

**Exercise 1.7.6** Show that the closure of a totally bounded set is totally bounded.

**Exercise 1.7.7** Show that a set in a complete metric space is precompact if and only if it is totally bounded.

**Exercise 1.7.8** Show that a totally bounded subset of a metric space is separable. In particular, a compact subset of a metric space is separable.

**Example 1.7.1** (Sequence space) This example illustrates a method to construct a compact space from a sequence $(M_k, \rho_k)$, $k = 1, 2, \ldots,$ of compact metric spaces with $diam M_k \le C$ for all $k$. For such a sequence, put $M = \prod_{k=1}^\infty M_k = \{x = (x_1, \ldots, x_k, \ldots) : x_k \in M_k, k = 1, 2, \ldots\}$. We shall often denote $x = (x_1, \ldots, x_k, \ldots)$ by $(x_k)$. For $x = (x_k), y = (y_k)$ in $M$, let

$$\rho(x, y) = \sum_{k=1}^\infty \frac{1}{k^2} \rho_k(x_k, y_k). \tag{1.9}$$

It is clear that $\rho$ is a metric on $M$, and with this metric diam $M \le 2C$. If $\{x^{(n)}\}_{n \in \mathbb{N}}$ is a sequence in $M$, and $x \in M$, then $\rho_k(x_k^{(n)}, x_k) \le k^2 \rho(x^{(n)}, x)$ for each $k$, from which it follows that if $\lim_{n\to\infty} x^{(n)} = x$ in $M$, then $\lim_{n\to\infty} x_k^{(n)} = x_k$ in $M_k$ for each $k$. Conversely, if $\lim_{n\to\infty} x_k^{(n)} = x_k$ for each $k$, we claim that $\lim_{n\to\infty} x^{(n)} = x$ in $M$. Let $\varepsilon > 0$ be given. There is $k_0 \in \mathbb{N}$ such that $\sum_{k=k_0+1}^{\infty} \frac{1}{k^2}\rho_k(x_k^{(n)}, x_k) \le C\sum_{k=k_0+1}^{\infty}\frac{1}{k^2} < \frac{\varepsilon}{2}$. Now, since $\lim_{n\to\infty}\rho_k(x_k^{(n)}, x_k) = 0$ for $k = 1, \ldots, k_0$, there is $L \in \mathbb{N}$ such that $\rho_k(x_k^{(n)}, x_k) < \frac{\varepsilon}{4}$ for $k = 1, \ldots, k_0$, whenever $n \ge L$. Consequently, when $n \ge L$, we have

$$\rho(x^{(n)}, x) = \sum_{k=1}^{k_0}\frac{1}{k^2}\rho_k(x_k^{(n)}, x_k) + \sum_{k=k_0+1}^{\infty}\frac{1}{k^2}\rho_k(x_k^{(n)}, x_k) < \frac{\varepsilon}{4}\sum_{k=1}^{k_0}\frac{1}{k^2} + \frac{\varepsilon}{2} < \varepsilon;$$

this means $\lim_{n\to\infty} x^{(n)} = x$. Thus, we have shown that $\lim_{n\to\infty} x^{(n)} = x$ in $M$ if and only if $\lim_{n\to\infty} x_k^{(n)} = x_k$ in $M_k$ for each $k$. We show now that $M$ is compact. Suppose that $\{x^{(n)}\}$ is a sequence in $M$; we have to show that $\{x^{(n)}\}$ has a subsequence which converges in $M$. We achieve this by the well-known **diagonalization procedure**. Since $M_1$ is compact $\{x_1^{(n)}\}$ has a subsequence $\{x_1^{(n_j^{(1)})}\}$ which converges in $M_1$ to, say, $x_1$; then $\{x_2^{(n_j^{(1)})}\}$ has a subsequence $\{x_2^{(n_j^{(2)})}\}$ which converges in $M_2$ to $x_2$; continuing in this fashion, we obtain an array of subsequences of $\{x^{(n)}\}$:

$$\begin{array}{l} x^{(n_1^{(1)})},\ x^{(n_2^{(1)})},\ \ldots,\ x^{(n_j^{(1)})},\ \ldots \\ x^{(n_1^{(2)})},\ x^{(n_2^{(2)})},\ \ldots,\ x^{(n_j^{(2)})},\ \ldots \\ \vdots \\ x^{(n_1^{(j)})},\ x^{(n_2^{(j)})},\ \ldots,\ x^{(n_j^{(j)})},\ \ldots \\ \vdots \end{array} \tag{1.10}$$

where each low contains the next one as a subsequence, and for each $k \in \mathbb{N}$,

$$\lim_{j\to\infty} x_k^{(n_j^{(k)})} = x_k \quad (\text{in } M_k). \tag{1.11}$$

Now, put $n_j = n_j^{(j)}$, $j = 1, 2, \ldots$. $\{x^{(n_j)}\}$ is a subsequence of $\{x^{(n)}\}$ formed of the diagonal elements of the array $(1.10)$. Observe that $\{x^{(n_j)}\}_{j\ge k}$ is a subsequence of $\{x^{(n_j^{(k)})}\}$ for each $k$, therefore $\lim_{j\to\infty} x_k^{(n_j)} = x_k$ by $(1.11)$ for each $k$, and consequently $\{x^{(n_j)}\}$ converges in $M$ to $(x_k)$, as we have shown previously in this example. We have shown that $\{x^{(n)}\}$ has a converging subsequence in $M$. Thus $M$ is compact. In particular, if each $M_k$ is a finite set with discrete metric (see Exercise 1.4.6), then $M$ is compact with metric given by $(1.9)$. We have encountered such a space $\Omega = \{0,1\} \times \{0,1\} \times \cdots$ in Section 1.3, of which one observes readily that each set in the algebra $\mathcal{Q}$ is a closed subset of $\Omega$ and is hence compact.

**Remark** In Example 1.7.1, the assumption that $\operatorname{diam} M_k \le C$ for all $k$ is not necessary, because, if we replace each $\rho_k$ by $\rho_k' = (\operatorname{diam} M_k)^{-1}\rho_k$, then each $(M_k, \rho_k')$ is compact and diam $M_k \leq 1$ w.r.t. the new metric $\rho'_k$. Hence from any sequence $(M_k, \rho_k)$ of compact metric spaces, one can construct a compact sequence space as in Example 1.7.1.

Now we give a characterization of compact sets which is usually taken as the definition for compact sets in topological spaces.

A family $\{S_\alpha\}$ of subsets of a given set $S$ is called a **covering** of a subset $A$ of $S$ if $A \subset \bigcup_\alpha S_\alpha$; then we also say that $\{S_\alpha\}$ covers $A$. If $S$ is a metric space and each set $S_\alpha$ is open, $\{S_\alpha\}$ is called an **open covering** of $A$ if it covers $A$. A subset $A$ of a metric space is said to have the **finite covering property** if every open covering of $A$ has a finite subfamily which covers $A$.

**Lemma 1.7.2** *Let $K$ be a compact subset of a metric space and suppose that $\{G_\alpha\}_{\alpha \in I}$ is an open covering of $K$, then there is $\delta > 0$, called a **Lebesgue** number of $K$ relative to $\{G_\alpha\}$, such that any subset $A$ of $K$ with* diam $A \leq \delta$ *is contained in $G_\alpha$ for some $\alpha \in I$.*

***Proof*** Suppose the contrary. Then for each $n \in \mathbb{N}$ there is a subset $A_n$ of $K$ with diam $A_n \leq \frac{1}{n}$ such that $A_n$ is contained in no $G_\alpha$. Then choose $x_n \in A_n$. Since $K$ is compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to $x \in K$. Let $x \in G_{\alpha_0}$, $\alpha_0 \in I$, and choose $r > 0$ so that $B_r(x) \subset G_{\alpha_0}$. If $k$ is sufficiently large, $\frac{1}{n_k} < \frac{r}{2}$ and $x_{n_k} \in B_{\frac{r}{2}}(x)$; consequently $A_{n_k} \subset B_r(x) \subset G_{\alpha_0}$. This contradicts the fact that $A_{n_k}$ is contained in no $G_\alpha$. The contradiction proves the lemma. ■

**Theorem 1.7.3** *A subset $K$ of a metric space $M$ is compact if and only if $K$ has the finite covering property.*

***Proof*** Suppose first that $K$ has the finite covering property. Consider a sequence $\{x_n\}$ in $K$; we shall show that $\{x_n\}$ has a subsequence which converges to a point in $K$. Suppose the contrary, then for each $x \in K$, there is an open ball $B_x$ centered at $x$ such that $x_n \in B_x$ for only finitely many $n$. $\{B_x\}_{x \in K}$ is an open covering of $K$, hence has a finite subfamily $\{B_1, \ldots, B_l\}$ which also covers $K$. Since $\bigcup_{j=1}^{l} B_j \supset K$ and $x_n \in B_j$ for only finitely many $n$ for each $j$, $x_n \in K$ for only finitely many $n$, contradicting the fact that $\{x_n\}$ is a sequence in $K$. Thus $\{x_n\}$ has a subsequence which converges in $K$, showing that $K$ is compact.

Next, suppose that $K$ is compact. Let $\{G_\alpha\}$ be an open covering of $K$; we are going to show that $\{G_\alpha\}$ has a finite subfamily which also covers $K$. Choose a Lebesgue number $\delta > 0$ of $K$ relative to $\{G_\alpha\}$ according to Lemma 1.7.2. Since $K$ is totally bounded by Lemma 1.7.1, there is an $\frac{\delta}{2}$-net $\{B_1, \ldots, B_k\}$ containing $K$. For $j = 1, \ldots, k$, diam $K \cap B_j \leq \delta$ implies $K \cap B_j \subset G_{\alpha_j}$ for some $\alpha_j$, and consequently $K \subset \bigcup_{j=1}^{k} G_{\alpha_j}$ i.e. $\{G_{\alpha_1}, \ldots, G_{\alpha_k}\}$ is a finite subfamily of $\{G_\alpha\}$ and it covers $K$. This shows that $K$ has the finite covering property. ■

**Corollary 1.7.4** (Finite intersection property) *Let $\{K_\alpha\}_{\alpha \in I}$ be a family of compact sets in a metric space $M$ with the property that intersection of any finite subfamily of $\{K_\alpha\}$ is nonempty. Then $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$.*

***Proof*** Suppose the contrary, that $\bigcap_{\alpha\in I} K_\alpha = \emptyset$. Choose and fix $\alpha_0 \in I$. Then for $x \in K_{\alpha_0}$, there is $\alpha_x \in I$ such that $x \in K^c_{\alpha_x}$; hence $\{K^c_\alpha\}_{\alpha\in I}$ is an open covering of $K_{\alpha_0}$. There is therefore a finite set $\{\alpha_1, \ldots, \alpha_k\} \subset I$ such that $\bigcup_{j=1}^k K^c_{\alpha_j} \supset K_{\alpha_0}$, by Theorem 1.7.3; this last inclusion relation means that $K_{\alpha_0} \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_k}$ is empty, contradicting our assumption about the family $\{K_\alpha\}$. The contradiction proves the corollary. ■

Two applications of Theorem 1.7.3 will now be given; both concerned with the uniformity concept. Suppose that $T$ is a map from a metric space $M_1$ with metric $\rho_1$ into a metric space $M_2$ with metric $\rho_2$. $T$ is said to be **uniformly** continuous on $M_1$ if for any given $\varepsilon > 0$, there is $\delta > 0$ such that $\rho_2(Tx, Ty) < \varepsilon$ whenever $x$ and $y$ are in $M_1$ with $\rho_1(x, y) < \delta$. Obviously, if $T$ is uniformly continuous on $M_1$, it is, *a fortiori*, continuous on $M_1$. A sequence $\{T_n\}$ of maps from $M_1$ into $M_2$ is said to converge pointwise to a map $T$ from $M_1$ into $M_2$ if $Tx = \lim_{n\to\infty} T_n x$ for each $x \in M_1$; it is said to converge **uniformly** to $T$ on $M_1$ if for any given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\rho_2(T_n x, Tx) \le \varepsilon$ for every $x \in M_1$ whenever $n \ge n_0$.

**Theorem 1.7.4** *If $T$ is a continuous map from a compact metric space $M_1$ into a metric space $M_2$, then $T$ is uniformly continuous on $M_1$.*

***Proof*** Let $\varepsilon > 0$ be given, and let $x \in M_1$. Since $T$ is continuous at $x$, there is $\delta_x > 0$ such that $\rho_2(Ty, Tx) < \varepsilon/2$ if $\rho_1(y, x) < \delta_x$. Consider $\{B_{\frac{1}{2}\delta_x}(x)\}_{x\in M_1}$; it is an open covering of $M_1$; by Theorem 1.7.3, it contains a finite subfamily, say $\{B_{\frac{1}{2}\delta_{x_1}}(x_1), \ldots, B_{\frac{1}{2}\delta_{x_l}}(x_l)\}$, which also covers $M_1$. Choose $\delta = \frac{1}{2}\min\{\delta_{x_1}, \ldots, \delta_{x_l}\}$. Suppose now that $x, y \in M_1$ with $\rho_1(x, y) < \delta$, and let $x \in B_{\frac{1}{2}\delta_{x_j}}(x_j)$, $1 \le j \le l$. Then $\rho_1(y, x_j) \le \rho_1(x, y) + \rho_1(x, x_j) < \delta + \frac{1}{2}\delta_{x_j} \le \delta_{x_j}$, hence $\rho_2(Ty, Tx_j) < \frac{\varepsilon}{2}$; since $x \in B_{\frac{1}{2}\delta_{x_j}}(x_j)$, $\rho_2(Tx, Tx_j) < \frac{\varepsilon}{2}$. Therefore, $\rho_2(Tx, Ty) \le \rho_2(Tx, Tx_j) + \rho_2(Tx_j, Ty) < \varepsilon$. This shows that $T$ is uniformly continuous. ■

**Theorem 1.7.5** (Dini) *Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on a compact metric space $M$ such that $f_1(x) \le f_2(x) \le \cdots \le f_n(x) \le \cdots$ and converges to a finite real number $f(x)$ for each $x \in M$. If, further, $f$ is continuous on $M$, then the sequence $\{f_n\}$ converges uniformly to $f$ on $M$.*

***Proof*** Given $\varepsilon > 0$ and $x \in M$, there is $k_x \in \mathbb{N}$ such that $0 \le f(x) - f_{k_x}(x) < \frac{\varepsilon}{3}$. Because both $f$ and $f_{k_x}$ are continuous, there is an open ball $B(x)$ centered at $x$ such that $|f(y) - f(x)| < \frac{\varepsilon}{3}$ and $|f_{k_x}(y) - f_{k_x}(x)| < \frac{\varepsilon}{3}$ whenever $y \in B(x)$; as a consequence, we have

$$0 \le f(y) - f_{k_x}(y) \le |f(y) - f(x)| + |f(x) - f_{k_x}(x)| + |f_{k_x}(x) - f_{k_x}(y)| < \varepsilon$$

whenever $y \in B(x)$, or

$$0 \le f(y) - f_k(y) < \varepsilon \tag{1.12}$$

whenever $y \in B(x)$ and $k \geq k_x$. Now $\{B(x) : x \in M\}$ is an open covering of $M$; by Theorem 1.7.3 it has a finite subfamily, say $\{B(x_1), \ldots, B(x_l)\}$, which also covers $M$. Let $k_0 = \max\{k_{x_1}, \ldots, k_{x_l}\}$; then for $y \in M$ and $k \geq k_0$, it follows from (1.12) that

$$0 \leq f(y) - f_k(y) < \varepsilon,$$

because $y \in B(x_j)$ for some $1 \leq j \leq l$ and $k \geq k_0 \geq k_{x_j}$. Thus the sequence $\{f_n\}$ converges to $f$ uniformly on $M$. ■

We come now, in the final part of this section, to prove a historically important theorem characterizing precompact sets in the n.v.s. $C(X)$ of all continuous real(complex)-valued functions defined on a compact metric space $X$ with norm given by

$$\|f\| = \sup_{x\in X} |f(x)| = \max_{x\in X} |f(x)|$$

for $f \in C(X)$, where $\sup_{x\in X} |f(x)| = \max_{x\in X} |f(x)|$ is a consequence of Exercise 1.5.7. Clearly, $C(X)$ is a n.v.s. with norm given as such. For a compact metric space $X$, the norm given previously on $C(X)$ is implicitly assumed without further notice. Actually $C(X)$ is a Banach space; to show this we need a lemma.

**Lemma 1.7.3** *Let $\{f_n\}$ be a sequence of continuous functions defined on a metric space $M$. Suppose that $\{f_n\}$ converges uniformly to a function $f$ on $M$, then $f$ is continuous on $M$.*

***Proof*** Let $x \in M$. We shall show that $f$ is continuous at $x$. Given $\varepsilon > 0$, by the uniform convergence of $\{f_n\}$ to $f$ on $M$ there is $n_0 \in \mathbb{N}$ such that $|f_{n_0}(y) - f(y)| < \frac{\varepsilon}{3}$ for all $y$ in $M$. Since $f_{n_0}$ is continuous at $x$, there is $\delta > 0$ such that $|f_{n_0}(y) - f_{n_0}(x)| < \frac{\varepsilon}{3}$ whenever $\rho(x, y) < \delta$. Hence if $\rho(x, y) < \delta$, then

$$\begin{aligned} |f(y) - f(x)| &\leq |f_{n_0}(y) - f(y)| + |f_{n_0}(y) - f_{n_0}(x)| + |f_{n_0}(x) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which shows that $f$ is continuous at $x$. ■

**Proposition 1.7.4** *$C(X)$ is a Banach space.*

***Proof*** Let $\{f_n\}$ be a Cauchy sequence in $C(X)$; we have to show that $\{f_n\}$ converges in $C(X)$. Since $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|$ for $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence of scalars and hence converges to a scalar $f(x)$ for every $x$ in $X$; thus as a sequence of functions, $\{f_n\}$ converges pointwise to a function $f$ on $X$. Actually $\{f_n\}$ converges uniformly to $f$ on $X$. Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\| < \varepsilon$ whenever $n, m \geq n_0$, hence $|f_n(x) - f_m(x)| < \varepsilon$ for all $x$ in $X$ and $n, m \geq n_0$, and thus $|f_n(x) - f(x)| \leq \varepsilon$ for all $x$ in $X$ if $n \geq n_0$, by letting $m \to \infty$. It follows then from Lemma 1.7.3 that $f \in C(X)$. We claim finally that $\lim_{n\to\infty} \|f_n - f\| = 0$, i.e. $\{f_n\}$ converges to $f$ in $C(X)$. To see this, for $\varepsilon > 0$ given choose $n_0 \in \mathbb{N}$ as above, then $|f_n(x) - f(x)| \leq \varepsilon$

for all $x \in X$ and $n \geq n_0$; this means that $\sup_{x \in X} |f_n(x) - f(x)| \leq \varepsilon$ when $n \geq n_0$, or $\|f_n - f\| \leq \varepsilon$ when $n \geq n_0$. Thus $\lim_{n\to\infty} \|f_n - f\| = 0$. ■

A family $\mathcal{F}$ of functions defined on a metric space $M$ is called an **equicontinuous** family if for each given $\varepsilon > 0$ there is $\delta > 0$ such that whenever $\rho(x, y) < \delta$, then $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$. Note that functions in an equicontinuous family are necessarily uniformly continuous.

The theorem that follows is not only historically important, but is also useful in the theory of differential equations.

**Theorem 1.7.6** (Arzelà–Ascoli) *If $X$ is a compact metric space, a subset $K$ of $C(X)$ is precompact if and only if it is bounded in $C(X)$ and equicontinuous as a family of functions on $X$.*

***Proof*** Suppose that $K$ is precompact. Since $C(X)$ is complete, as asserted by Proposition 1.7.4, $K$ is totally bounded by Corollary 1.7.3. Let $\varepsilon > 0$ and let $f_1, \ldots, f_n$ be the centers of an $\frac{\varepsilon}{3}$-net for $K$. Since $f_1, \ldots, f_n$ are uniformly continuous on $X$, by Theorem 1.7.4, there is $\delta > 0$ such that

$$|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$$

for $i = 1, \ldots, n$ when $\rho(x, y) < \delta$. Consider now $f \in K$ and choose $j \in \{1, \ldots, n\}$ so that

$$\sup_{x \in X} |f(x) - f_j(x)| < \frac{\varepsilon}{3};$$

such $j$ exists because $f_1, \ldots, f_n$ are centers of an $\frac{\varepsilon}{3}$-net for $K$. Then if $\rho(x, y) < \delta$, we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &< \frac{2}{3}\varepsilon + |f_j(x) - f_j(y)| < \varepsilon, \end{aligned}$$

and therefor $K$ is equicontinuous. Since $K$ is totally bounded, it is bounded in $C(X)$.

Conversely, suppose that $K$ is bounded in $C(X)$ and is equicontinuous as a family of functions on $X$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{4}$ for $f \in K$ when $f(x, y) < \delta$. As $X$ is compact, there is a $\delta$-net for $X$ with centers $x_1, \ldots, x_n$. For simplicity's sake, in the argument that follows we assume that functions in $C(X)$ are real-valued; the corresponding argument when $C(X)$ consists of complex-valued functions will be clear. Since $K$ is bounded in $C(X)$, there is $L > 0$ so that $|f(x)| \leq L$ for all $f \in K$ and all $x \in X$. Divide the interval $[-L, L]$ into $k$ equal parts by the partition

$$y_0 = -L < y_1 < \cdots < y_k = L,$$

where $k$ is chosen so that $|y_i - y_{i+1}| < \frac{\varepsilon}{4}$ for $i = 0, \ldots, k-1$. We say that an $n$-tuple $(y_{i_1}, \ldots, y_{i_n})$ of numbers $y_0, \ldots, y_k$ is admissible if for some $f \in K$ the following inequalities hold:

$$|f(x_j) - y_{i_j}| < \frac{\varepsilon}{4}, \quad j = 1, \ldots, n. \tag{1.13}$$

Clearly, for each $f \in K$ there is an $n$-tuple $(y_{i_1}, \ldots, y_{i_n})$ so that (1.13) holds. Hence the set $Y$ of all admissible $n$-tuples is nonempty. Note that $Y$ is finite. For each $n$-tuple $y = (y_{i_1}, \ldots, y_{i_n})$ in $Y$ choose and fix an $f_y \in K$ so that (1.13) holds, with $f$ replaced by $f_y$. Let now $f \in K$. Choose $y = (y_{i_1}, \ldots, y_{i_n})$ in $Y$ such that (1.13) holds. For $x \in X$ choose $x_j$, $1 \leq j \leq n$, so that $\rho(x, x_j) < \delta$. Then

$$|f(x) - f_y(x)| \leq |f(x) - f(x_j)| + |f(x_j) - y_{i_j}| + |y_{i_j} - f_y(x_j)| + |f_y(x_j) - f_y(x)|,$$

from which we infer that $\|f - f_y\| < \varepsilon$ from the fact that both $f$ and $f_y$ satisfy (1.13) as well as from the way $\delta > 0$ is chosen. Thus $\{B_\varepsilon(f_y) : y \in Y\}$ is an $\varepsilon$-net for $K$. We have shown that $K$ is totally bounded. Hence $K$ is precompact by Corollary 1.7.3. ■

**Example 1.7.2** Let $K = \{f \in C^1[0,1] : f(0) = a \text{ and } |f'| \leq g\}$, where $a \in \mathbb{R}$ and $g$ is a nonnegative continuous function on $[0, 1]$. It is clear from Theorem 1.7.6 that $K$ is a precompact set in $C[0, 1]$.

## 1.8 Extension of continuous functions

We consider in this section the question of when a continuous real-valued function defined on a subset of a metric space can be extended continuously to the whole space.

**Lemma 1.8.1** (Uryson) *Let $A$, $B$ be nonempty disjoint closed sets in a metric space $M$, then there is a continuous function defined on $M$ such that $0 \leq f \leq 1$, $f = 0$ on $A$, and $f = 1$ on $B$.*

***Proof*** For a set $S \subset M$, the function $x \mapsto \rho(x, S) := \inf_{z \in S} \rho(x, z)$ is continuous on $M$. This follows from the obvious inequality

$$|\rho(x, S) - \rho(y, S)| \leq \rho(x, y)$$

for $x, y$ in $M$. Since $A$ and $B$ are disjoint closed sets, $\rho(x, A) + \rho(x, B) > 0$ for $x \in M$, we may then define $f : M \to \mathbb{R}$ by

$$f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)}, \quad x \in M.$$

Clearly $f$ is continuous and is the function to be sought. ■

**Corollary 1.8.1** *Let $A$ and $B$ be nonempty disjoint closed sets in a metric space $M$; then for any pair $\alpha < \beta$ of real numbers, there is a continuous function $f$ defined on $M$ such that $\alpha \leq f \leq \beta$, $f = \alpha$ on $A$, and $f = \beta$ on $B$.*

**Exercise 1.8.1** Prove Corollary 1.8.1.

**Theorem 1.8.1** (Tietze) *Suppose that $g$ is a bounded continuous function defined on a closed set $C$ in a metric space $M$, and let $\gamma = \sup_{x\in C} |g(x)|$. Then there is a continuous function $f$ defined on $M$ such that $f = g$ on $C$ and $\sup_{x\in M} |f(x)| = \gamma$.*

***Proof*** We may assume that $M\backslash C$ contains infinitely many points, because otherwise $M$ consists only of points from $C$ and a finite number of isolated points, in which case the theorem is trivial. Then we may pick any two points $x_1$ and $x_2$ outside $C$, define $g(x_1) = -\gamma$, $g(x_2) = \gamma$, and replace $C$ by $C \cup \{x_1, x_2\}$. Thus we may assume that $\min_{x\in C} g(x) = -\gamma$ and $\max_{x\in C} g(x) = \gamma$.

Now let $A = \{x \in C : g(x) \leq -\frac{\gamma}{3}\}$, $B = \{x \in C : g(x) \geq \frac{\gamma}{3}\}$, then $A$ and $B$ are disjoint nonempty closed sets. By Corollary 1.8.1 there is a continuous function $f_1$ on $M$ such that $|f_1| \leq \frac{\gamma}{3}$, $f_1 = -\frac{\gamma}{3}$ on $A$ and $f_1 = \frac{\gamma}{3}$ on $B$. It is readily verified that $|g - f_1| \leq \frac{2}{3}\gamma$ on $C$. Note that $\min_{x\in C}\{g(x) - f_1(x)\} = -\frac{2}{3}\gamma$ and $\max_{x\in C}\{g(x) - f_1(x)\} = \frac{2}{3}\gamma$.

Repeat the argument of the last paragraph with $g$ replaced by $g - f_1$ and $\gamma$ by $\frac{2}{3}\gamma$; we obtain a continuous function $f_2$ on $M$ such that $|f_2| \leq \frac{1}{3} \cdot \frac{2}{3}\gamma$ and $|g - f_1 - f_2| \leq (\frac{2}{3})^2\gamma$ on $C$. Continuing in this fashion, we obtain a sequence $\{f_n\}$ of continuous functions on $M$ such that $|f_n| \leq \frac{1}{3}(\frac{2}{3})^{n-1}\gamma$ and $|g - \sum_{j=1}^{n} f_j| \leq (\frac{2}{3})^n\gamma$ on $C$. It follows then that $\sum_n f_n$ converges uniformly to a continuous function $f$ on $M$ and $f = g$ on $C$. Now, $|f| \leq \sum_{j=1}^{\infty} |f_j| \leq \sum_{j=1}^{\infty} \frac{1}{3}(\frac{2}{3})^{j-1}\gamma = \gamma$. ■

**Remark** The function $g$ in Theorem 1.8.1 is usually called an **extension** of the function $f$, while $f$ is called the **restriction** of $g$ on $C$ and is often denoted as $g|_C$.

## 1.9 Connectedness

A metric space $M$ is said to be **connected** if any nonempty subset of $M$ which is both open and closed is $M$ itself. Obviously any discrete space cannot be connected except when it consists of only one point. A subset of a metric space $M$ is called connected if it is connected as a metric space with its metric inherited from $M$.

**Exercise 1.9.1** Show that a metric space $M$ is connected if and only if it cannot be expressed as a disjoint union of two nonempty subsets, both of which are open.

**Theorem 1.9.1** *A finite closed interval in $\mathbb{R}$ is connected.*

***Proof*** Let the interval be $I = [a, b]$, $-\infty < a, b < \infty$. Suppose that $I$ is not connected, then $I = A \cup B$, where $A \cap B = \emptyset$ and both $A$ and $B$ are nonempty open and closed in $I$. We may suppose $a \in A$. Since $B$ is bounded below by $a$, $\inf B \in I$. Since $B$ is closed in $I$, $\inf B \in B$ and hence cannot be in $A$, which implies $a < \inf B$. Thus $(a, \inf B) \subset A$, and $\inf B$ is a limit point of $A$, but that $A$ is closed implies $\inf B$ is in $A$, a contradiction. ■

**Exercise 1.9.2**

(i) Modify the arguments in the proof of Theorem 1.9.1 to show that any interval in $\mathbb{R}$ is connected whether it is finite or infinite and whether it is closed, open, or half-open.

(ii) Show that a subset $A$ of $\mathbb{R}$ is connected if and only if for any pair $x < y$ of elements in $A$, $[x, y] \subset A$. Conclude then that connected sets in $\mathbb{R}$ are intervals.

**Exercise 1.9.3** Show that every open set in $\mathbb{R}$ is a disjoint union of at most countably many open intervals.

## 1.10 Locally compact spaces

An account of compact sets in a locally compact metric space will now be given in regard to construction of some useful continuous functions relating to compact sets.

A metric space $X$ is called a **locally compact** space if every $x$ in $X$ has a compact neighborhood. Clearly, $\mathbb{R}^n$ with the Euclidean metric is a locally compact space. We observe the following two facts for a locally compact space $X$:

(i) If $K$ is a compact subset of $X$, then $K$ has a compact neighborhood.

(ii) If $K$ is a compact subset of $X$ and $x \in X\backslash K$, then $K$ has a compact neighborhood $W_x$ not containing $x$.

To see (i), consider the open covering $\{\overset{\circ}{U}_x\}_{x\in K}$, where $U_x$ is a compact neighborhood of $x$, and extract from it a finite subcovering $\{\overset{\circ}{U}_{x_1}, \ldots, \overset{\circ}{U}_{x_k}\}$ of $K$; then $\bigcup_{j=1}^k U_{x_j}$ is a compact neighborhood of $K$. Now if $x \in X\backslash K$, put $\delta = \text{dist}(x, K) > 0$; then $W_x = V \cap \{y \in X : \text{dist}(y, K) \leq \frac{1}{2}\delta\}$ is a compact neighborhood of $K$ not containing $x$, where $V$ is a compact neighborhood of $K$ as asserted in (i); thus (ii) holds.

**Lemma 1.10.1** *Suppose that $K$ is a compact subset of a locally compact space $X$ and is contained in an open set $G$. Then $K$ has a compact neighborhood $V$ contained in $G$.*

***Proof*** Because of (i) we may assume that $X\backslash G \neq \emptyset$. For each $x \in X$ let $W_x$ be a compact neighborhood of $K$ not containing $x$, as in (ii), and consider the family $\mathcal{F} = \{W_x \cap G^c : x \in G^c\}$ of compact sets; Clearly, $\bigcap \mathcal{F} = \emptyset$ and by the finite intersection property (Corollary 1.7.4) there are $x_1, \ldots, x_k$ in $G^c$ such that $\bigcap_{j=1}^k \{W_{x_j} \cap G^c\} = \left[\bigcap_{j=1}^k W_{x_j}\right] \cap G^c = \emptyset$. We infer then from the last set relation that $V = \bigcap_{j=1}^k W_{x_j}$ is a compact neighborhood of $K$ contained in $G$. ■

**Lemma 1.10.2** *Let $\mathcal{F} = \{G_1, \ldots, G_n\}$ be a finite open covering of a compact set $K$ in a locally compact space $X$; then there are compact sets $K_1, \ldots, K_n$ in $X$ such that $K_j \subset G_j$ for each $j = 1, \ldots, n$ and $K \subset \bigcup_{j=1}^n K_j$.*

***Proof*** For $x \in K$, there is $j$, $1 \leq j \leq n$, such that $x \in G_j$; then Lemma 1.10.1 implies that $x$ has a compact neighborhood $V_x \subset G_j$. Since $\{\overset{\circ}{V}_x : x \in K\}$ is an open covering of $K$, there are $x_1, \ldots, x_k$ in $K$ such that $\bigcup_{j=1}^{k} \overset{\circ}{V}_{x_j} \supset K$. For each $j = 1, \ldots, n$, let $\mathcal{F}_j = \{V_{x_i} : V_{x_i} \subset G_j\}$ and put $K_j = \bigcup \mathcal{F}_j$; then $K_j$ is a compact set $\subset G_j$ and $\bigcup_{j=1}^{n} K_j = \bigcup_{i=1}^{k} V_{x_i} \supset K$. ■

**Remark** In Lemma 1.10.2, some of the $K_j$'s might be empty; but if $\mathcal{F}$ has the property that every one of its proper subfamily is not a covering of $K$, then each $K_j$ is nonempty.

For a function $f$ defined on a metric space $X$, we shall denote by **supp *f*** the closure of the set $\{x \in X : f(x) \neq 0\}$. If *suppf* (which is called the **support** of $f$) is compact, $f$ is called a function with **compact support**. The family of all continuous functions with compact support in a metric space $X$ is denoted by $C_c(X)$. Note that $C_c(X)$ is a real or complex vector space depending on whether real-valued or complex-valued functions are considered. For an open set $G$ in a metric space $X$, the family of all continuous functions $f$ on $X$ with compact support such that $0 \leq f \leq 1$ and supp$f \subset G$ is to be denoted by $U_c(G)$.

**Corollary 1.10.1** *Suppose that $K$ is a compact set contained in an open set $G$ of a locally compact space $X$. Then there is $f$ in $U_c(G)$ such that $f = 1$ on $K$.*

***Proof*** $K$ has a compact neighborhood $V$ contained in $G$ by Lemma 1.10.1; then $K$ and $\overset{\circ}{V}{}^c$ are disjoint closed subsets of $X$. Using the Uryson lemma (Lemma 1.8.1), we find a continuous function $f$ on $X$ such that $0 \leq f \leq 1$, $f = 0$ on $\overset{\circ}{V}{}^c$, and $f = 1$ on $K$. Since supp$f \subset V \subset G$, $f \in U_c(G)$. ■

Suppose now that $K$ is a compact set in a metric space $X$ and $\mathcal{F} = \{G_1, \ldots, G_n\}$ is a finite open covering of $K$, then a collection $\{u_1, \ldots, u_n\}$ of continuous functions is called a **partition of unity** of $K$ **subordinate** to $\mathcal{F}$ if $u_j \in U_c(G_j)$ for each $j = 1, \ldots, n$ and $\sum_{j=1}^{n} u_j(x) = 1$ for all $x \in K$.

**Theorem 1.10.1** (Partition of unity) *Suppose that $K$ is a compact set in a locally compact metric space $X$ and that $\mathcal{F}$ is a finite open covering of $K$. Then $K$ has a partition of unity subordinate to $\mathcal{F}$.*

***Proof*** Let $\mathcal{F} = \{G_1, \ldots, G_n\}$. There are compact sets $K_1, \ldots, K_n$ such that $K_j \subset G_j$ for each $j$ and $K \subset \bigcup_{j=1}^{n} K_j$, by Lemma 1.10.2. For each $j = 1, \ldots, n$, it then follows from Corollary 1.10.1 that there is a $f_j \in U_c(G_j)$ such that $f_j = 1$ on $K_j$. Define functions $u_1, \ldots, u_n$ by

$$u_1 = f_1,\ u_2 = (1 - f_1)f_2, \ldots,\ u_n = (1 - f_1)\cdots(1 - f_{n-1})f_n.$$

Then, $u_j \in U_c(G_j)$, $j = 1, \ldots, n$. Now

$$\sum_{j=1}^{n} u_j = 1 - (1 - f_1) \cdots (1 - f_n), \tag{1.14}$$

as can be verified from $u_1 = 1 - (1 - f_1)$, $u_1 + u_2 = 1 - (1 - f_1)(1 - f_2)$, and so on. If $x \in K$, then $x \in K_j$ for some $j$ and therefore $(1 - f_1(x)) \cdots (1 - f_n(x)) = 0$; consequently $\sum_{j=1}^{n} u_j(x) = 1$, by (1.14). ■