Since $\|x\|_\infty \le |x|$ for all $x \in \mathbb{R}^n$, thus for $|x| \ge 1$,

$$\begin{aligned} f^*(x) &\ge \frac{b}{(K+2|x|)^n} \\ &\ge \frac{b}{(K|x|+2|x|)^n} && \text{(since } |x| \ge 1) \\ &= \frac{b}{(K+2)^n |x|^n}. \end{aligned}$$

7.2 Q2

Lemma 7.2.1. We show that $\int \phi_\epsilon = 1$.

Proof. For $\epsilon > 0$, note that

$$\int_{\mathbb{R}^n} \phi_\epsilon(x)\, dx = \int_{\mathbb{R}^n} \epsilon^{-n} \phi(x/\epsilon)\, dx = \int_{\{|x|<\epsilon\}} \epsilon^{-n}\phi(x/\epsilon)\, dx$$

since $\phi(x) = 0$ for $|x| \ge 1$.

Let $y = Tx = \frac{1}{\epsilon}x$ be a linear transformation of $\mathbb{R}^n$. Note that $T = \operatorname{diag}(\frac{1}{\epsilon}, \dots, \frac{1}{\epsilon})$ so that $|\det T| = \epsilon^{-n}$. If $E = \{x \in \mathbb{R}^n : |x| < 1\}$, note that $T^{-1}E = \{x \in \mathbb{R}^n : |x| < \epsilon\}$.

Thus using the formula

$$\int_E f(y)\, dy = |\det T| \int_{T^{-1}E} f(Tx)\, dx$$

proved in Chapter 5 Exercise 20, we get

$$\begin{aligned} \int_{\mathbb{R}^n} \phi_\epsilon(x)\, dx &= \epsilon^{-n} \int_{T^{-1}E} \phi(Tx)\, dx \\ &= \epsilon^{-n} \cdot \frac{1}{|\det T|} \int_E \phi(y)\, dy \\ &= \int_{\{|y|<1\}} \phi(y)\, dy \\ &= \int_{\mathbb{R}^n} \phi(y)\, dy \\ &= 1. \end{aligned}$$

Then,

$$
\begin{aligned}
(f * \phi_\epsilon)(x) - f(x) &= \int_{\mathbb{R}^n} f(x-y)\phi_\epsilon(y)\,dy - \int_{\mathbb{R}^n} f(x)\phi_\epsilon(y)\,dy \\
&= \int_{\mathbb{R}^n} [f(x-y) - f(x)]\phi_\epsilon(y)\,dy \\
&= \frac{1}{\epsilon^n} \int_{\{|y|\le\epsilon\}} [f(x-y) - f(x)]\phi(y/\epsilon)\,dy.
\end{aligned}
$$

Since $|\phi(x)| \le M$ for some $M > 0$, we have that

$$
\begin{aligned}
|(f * \phi_\epsilon)(x) - f(x)| &\le \frac{M}{\epsilon^n} \int_{\{|y|\le\epsilon\}} |f(x-y) - f(x)|\,dy \\
&= \frac{M}{\epsilon^n} \int_{\{|y-x|\le\epsilon\}} |f(y) - f(x)|\,dy \\
&\le \frac{M}{\epsilon^n} \int_{Q_{2\epsilon}(x)} |f(y) - f(x)|\,dy \\
& \text{(where } Q_{2\epsilon}(x) \text{ is the cube centered at } x \text{ with edge length } 2\epsilon\text{)} \\
&\le \frac{2^n M}{|Q_{2\epsilon}(x)|} \int_{Q_{2\epsilon}(x)} |f(y) - f(x)|\,dy \\
& \text{(since } |Q_{2\epsilon}(x)| = 2^n\epsilon^n\text{)}.
\end{aligned}
$$

We quote Theorem 7.16:

Theorem (Theorem 7.16)**.** Let f be locally integrable in $\mathbb{R}^n$. Then at every point x of the Lebesgue set of f (in particular, almost everywhere), $\frac{1}{|S|}\int_S |f(y) - f(x)|\,dy \to 0$ for any family $\{S\}$ that shrinks regularly to x. Thus, also $\frac{1}{|S|}\int_S f(y)\,dy \to f(x)$ a.e.

Since $f \in L(\mathbb{R}^n)$, by Theorem 7.16, at every point x of the Lebesgue set of f,

$$
\frac{1}{|Q_{2\epsilon}(x)|} \int_{Q_{2\epsilon}(x)} |f(y) - f(x)|\,dy \to 0
$$

as $\epsilon \to 0$.

Hence $\lim_{\epsilon\to 0}|(f*\phi_\epsilon)(x)-f(x)|=0$, which implies

$$\lim_{\epsilon\to 0}(f*\phi_\epsilon)(x)=f(x)$$

in the Lebesgue set of f. $\square$

7.3 Q5

Lemma 7.3.1. $\int_a^b \phi\, df = \int_a^b \phi\, dg + \int_a^b \phi\, dh.$

Proof. Firstly, note that g is absolutely continuous implies g is of bounded variation on $[a,b]$. Thus $h=f-g$ is also of bounded variation on $[a,b]$. Thus the above three integrals are well-defined.

Then

$$\begin{aligned}
\int_a^b \phi\, df &= \lim_{P\to 0}\sum \phi(\xi_i)(f(x_i)-f(x_{i-1})) \\
&= \lim_{P\to 0}\sum \phi(\xi_i)(g(x_i)+h(x_i)-g(x_{i-1})-h(x_{i-1})) \\
&= \lim_{P\to 0}\sum \phi(\xi_i)(g(x_i)-g(x_{i-1})) + \lim_{P\to 0}\sum \phi(\xi_i)(h(x_i)-h(x_{i-1})) \\
&= \int_a^b \phi\, dg + \int_a^b \phi\, dh.
\end{aligned}$$

$\square$

We quote Theorem 7.32:

Theorem (Theorem 7.32(i))**.** If g is continuous on $[a,b]$ and f is absolutely continuous on $[a,b]$, then $\int_a^b g\, df = \int_a^b g f'\, dx$.

Applying Theorem 7.32, we get

$$\int_a^b \phi\, dg = \int_a^b \phi g'\, dx = \int_a^b \phi f'\, dx$$

since $f'=g'+h'=g'$ a.e. on $[a,b]$.

Combining our results, we have

$$\int_a^b \phi\, df = \int_a^b \phi f'\, dx + \int_a^b \phi\, dh.$$