Real Analysis
Homework 8

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EXERCISE 11.5

Let f be monotone increasing and right continuous on $\mathbb{R}^1$.

(a) Show that Λ_f is absolutely continuous with respect to Lebesgue measure if and only if f is absolutely continuous on $\mathbb{R}^1$. (By absolutely continuous on $\mathbb{R}^1$, we mean absolutely continuous on every compact interval.)

(b) If Λ_f is absolutely continuous with respect to Lebesgue measure, show that its Radon–Nikodym derivative equals df/dx.

Proof.

(a) $(\Rightarrow)$
By **Theorem 10.34**, since Λ_f is absolutely continuous on $\mathbb{R}^1$.
Let $[a,b] \subset \mathbb{R}^1$, then $\forall \varepsilon > 0$, there exists $\delta > 0$ such that $\Lambda_f(A) < \varepsilon$ for any measurable $A \subset [a,b]$ with $|A| < \delta$.
Let $\{[a_k, b_k]\}$ be nonoverlapping subintervals of $[a,b]$ and $\sum_k (b_k - a_k) < \delta$. Then

$$\sum_k |f(b_k) - f(a_k)| = \sum_k \Lambda_f((a_k, b_k]) = \Lambda_f(\cup(a_k, b_k]) < \varepsilon$$

Hence, f is absolutely continuous.

$(\Leftarrow)$
Let $[a,b] \subset \mathbb{R}^1$ and $\{[a_k, b_k]\}$ be nonoverlapping subintervals of $[a,b]$.
Since f is absolutely continuous, then $\forall \varepsilon > 0$, there exists $\delta > 0$ such that $\sum_k |f(b_k) - f(a_k)| < \varepsilon$ with $\sum_k (b_k - a_k) < \delta$.
Let $|A| < \delta$ such that $A \subset \cup_k [a_k, b_k]$.

$$\begin{aligned}\Lambda_f(A) \le \Lambda_f(\cup_k [a_k, b_k]) &\le \sum_k \Lambda_f([a_k, b_k]) \\ &= \sum_k (f(b_k) - f(a_k^-)) = \sum_k (f(b_k) - f(a_k)) < \varepsilon\end{aligned}$$

Hence, Λ_f is absolutely continuous.

(b) Let $[a,b] \subset \mathbb{R}^1$.
By **Theorem 10.39 (Radon-Nikodym)**, we know that there exists a unique $g \ge 0$ such that $\Lambda_f(A) = \int_A g\,dx$, $\forall A \subset \mathbb{R}^1$.
By **Exercise 11.5 (a)**, since Λ_f is absolutely continuous, then f is also absolutely continuous.

Also, by **Theorem 7.29**, since f is absolutely continuous, then f' exists a.e. in $[a,b]$ and f' is integrable on $[a,b]$. So

$$\Lambda_f([a,b]) = f(b) - f(a) = \int_a^b f'(x)\,dx$$

Hence, Radon-Nikodym derivative of Λ_f is $f' = df/dx$.

EXERCISE 11.7

If f is monotone increasing and continuous from the right on $\mathbb{R}^1$, show that $\Lambda_f^*(A) = \Lambda_f^{o*}(A)$, where Λ_f^{o*} is defined in the same way as Λ_f^* except that we use *open* intervals (a_k, b_k).

Proof.
Let the countable collections $\{(a_k, b_k]\}$ such that $A \subset \cup_k (a_k, b_k]$. So for $\Lambda_f^*(A)$, we know that

$$\Lambda_f^*(A) = \inf \sum_k \lambda(a_k, b_k] = \inf \sum_k [f(b_k) - f(a_k)].$$

For all collections $\{(a_k, b_k^+)\}$, we have

$$\sum_k \lambda(a_k, b_k^+) = \sum_k [\lambda(a_k, b_k] + \lambda(b_k, b_k^+)].$$

Since f is monotone increasing and continuous from the right, then $\lambda \geq 0$ and $f(x) = f(x^+)$ for all $x \in \mathbb{R}^1$. Also, $0 \leq \lambda(b_k, b_k^+) \leq \lambda(b_k, b_k^+] = f(b_k^+) - f(b_k) = 0 \;\Rightarrow\; \lambda(b_k, b_k^+) = 0 \quad$ for all k. Thus,

$$\sum_k \lambda(a_k, b_k^+) = \sum_k \lambda(a_k, b_k].$$

Hence

$$\Lambda_f^*(A) = \inf \sum_k \lambda(a_k, b_k] = \inf \sum_k \lambda(a_k, b_k^+) = \Lambda_f^{o*}(A).$$

EXERCISE 11.8

If f is monotone increasing and continuous from the right, derive formulas for $\Lambda_f([a,b])$ and $\Lambda_f((a,b))$.

Proof.

(i) By **Theorem 11.10**, since f is monotone increasing and continuous from the right, then

$$\Lambda_f((a,b]) = f(b) - f(a).$$

In particular, $\Lambda(\{a\}) = f(a) - f(a^-)$. So

$$\Lambda_f([a,b]) = \Lambda_f(\{a\}) + \Lambda((a,b]) = [f(a) - f(a^-)] + [f(b) - f(a)] = f(b) - f(a^-).$$

(ii) Also,

$$\Lambda_f((a,b)) = \Lambda_f((a,b]) - \Lambda_f(\{b\}) = [f(b) - f(a)] - [f(b) - f(b^-)] = f(b^-) - f(a).$$

EXERCISE 11.10

Show that in $\mathbb{R}^n$, $n > 1$, the Hausdorff outer measure H_n is not identical to Lebesgue outer measure. (For example, let $n = 2$, and write $A = \cup A_k$, $\delta(A_k) < \varepsilon$. Enclose A_k in a circle C_k with the same diameter, and show that $\sum \delta(A_k)^2 \geq (4/\pi)|A|_e$. Thus, $H_2^\varepsilon(A) \geq (4/\pi)|A|_e$.)

Proof.
Follow the hint, let $n = 2$, and write $A = \cup A_k$, $\delta(A_k) < \varepsilon$. Enclose A_k in a circle C_k with the same diameter. Then

$$|C_k| = \left(\frac{\delta(C_k)}{2}\right)^2 \pi = \left(\frac{\delta(A_k)}{2}\right)^2 \pi.$$

So

$$\sum_k \delta(A_k)^2 = \sum_k \frac{4}{\pi}\,|C_k| \geq \frac{4}{\pi}\sum_k |A_k|_e \geq \frac{4}{\pi}\,|\cup_k A_k|_e = \frac{4}{\pi}\,|A|_e.$$

Thus,

$$H_2^\varepsilon(A) \geq (4/\pi)|A|_e.$$

This counterexample is sufficient to show that in $\mathbb{R}^n$, $n > 1$, the Hausdorff outer measure H_n is not identical to Lebesgue outer measure.

EXERCISE 11.11

If A is a subset of $\mathbb{R}^n$, define the *Hausdorff dimension* of A as follows: If $H_\alpha(A) = 0$ for all $\alpha > 0$, let $\dim A = 0$; otherwise, let

$$\dim A = \sup\{\alpha : H_\alpha(A) = +\infty\}.$$

(a) Show that $H_\alpha(A) = 0$ if $\alpha > \dim A$ and that $H_\alpha(A) = +\infty$ if $\alpha < \dim A$. Show that in $\mathbb{R}^n$ we have $\dim A \leq n$. See **Exercise 11.19** in order to determine the Hausdorff dimension of the Cantor set.

(b) If $\dim A_k = d$ for each A_k in a countable collection $\{A_k\}$, show that $\dim(\cup A_k) = d$. Hence, show that every countable set has Hausdorff dimension 0.

Proof.

(a) (i) Let $\alpha > \dim A$.
If $H_\alpha(A) = +\infty \quad \Rightarrow \quad \dim A \geq \alpha > \dim A$ $(\rightarrow\leftarrow)$.
So $H_\alpha(A) < +\infty$. Then we choose $\alpha > \alpha_0 > \dim A \quad \Rightarrow \quad H_{\alpha_0}(A) < +\infty$.
By **Theorem 11.13 (i)**, if $H_{\alpha_0}(A) < +\infty$, then

$$H_\alpha(A) = 0 \quad \text{for } \alpha > \alpha_0.$$

(ii) Since $\dim A = \sup\{\alpha : H_\alpha(A) = +\infty\}$, then for all $\alpha > 0$ with $H_\alpha(A) = +\infty$, we have $\dim A \geq \alpha$. Hence,

$$H_\alpha(A) = +\infty \quad \text{for all } \alpha < \dim A.$$

(iii) By **Theorem 11.16 (ii)**, if $\alpha > n$, then $H_\alpha(A) = 0$, $\forall A \subset \mathbb{R}^n$. So we have $\dim A \leq n$.

(b) (i) If $\alpha > d = \dim A_k$, then by (a) we have $H_\alpha(A_k) = 0$. So

$$0 \leq H_\alpha(\cup_k A_k) \leq \sum_k H_\alpha(A_k) = 0 \quad \Rightarrow \quad \dim(\cup_k A_k) \leq d.$$

If $\alpha < d = \dim A_k$, then by (a) we have $H_\alpha(A_k) = \infty \leq H_\alpha(\cup_k A_k)$. So

$$\dim(\cup_k A_k) \geq d.$$

Hence, $\dim(\cup_k A_k) = d$.

(ii) If $x \in \mathbb{R}^n$, then $x \in B_\varepsilon(x)$. So

$$H_\alpha(\{x\}) = \liminf_{\varepsilon \to 0} \sum_k \delta(A_k)^\alpha \leq 0.$$