Real Analysis
Homework 9

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EXERCISE 11.12

Let Γ be an outer measure on $\mathscr{S}$, and let Γ' denote Γ restricted to the Γ-measurable sets. Since Γ' is a measure on an algebra, it induces an outer measure Γ^*. Show that $\Gamma^*(A) \geq \Gamma(A)$ for $A \subset \mathscr{S}$ and that equality holds for a given A if and only if there is a Γ-measurable set E such that $A \subset E$ and $\Gamma(E) = \Gamma(A)$. Thus, $\Gamma = \Gamma^*$ if Γ is regular.

Proof.
Given $A \subset \mathscr{S}$. Let $A_1, A_2, \ldots, A_k$ be a measurable covering of A, then

$$\sum_k \Gamma(A_k) \geq \Gamma(\cup_k A_k) \geq \Gamma(A).$$

This shows that

$$\Gamma^*(A) = \inf \sum_k \Gamma'(A_k) = \inf \sum_k \Gamma(A_k) \geq \Gamma(A).$$

($\Rightarrow$)
Given $A \subset \mathscr{S}$ such that $\Gamma^*(A) = \Gamma(A)$.
Let $\{A_k^{(n)}\}_{k=1}^{K_n}$ be a sequence of coverings such that

$$\lim_{n\to\infty} \Gamma'(A_k^{(n)}) = \Gamma^*(A) = \Gamma(A).$$

and

$$\begin{aligned}\Gamma(A) < \Gamma\left(\cap_{n=1}^{\infty} \cup_{k=1}^{K_n} A_k^{(n)}\right) &= \Gamma'\left(\cap_{n=1}^{\infty} \cup_{k=1}^{K_n} A_k^{(n)}\right) \\ &\leq \Gamma'\left(\cup_{k=1}^{K_n} A_k^{(n)}\right) \quad \forall\, n \\ &\leq \sum_{k=1}^{K_n} \Gamma'(A_k^{(n)}) \to \Gamma(A).\end{aligned}$$

Thus, if $E = \cap_{n=1}^{\infty} \cup_{k=1}^{K_n} A_k^{(n)}$, then $A \subset E$ and $\Gamma(E) = \Gamma(A)$.

($\Leftarrow$)
Given E is measurable and $A \subset E$ such that $\Gamma(E) = \Gamma(A)$. Thus

$$\Gamma(E) = \Gamma(A) \leq \Gamma^*(A) \leq \Gamma^*(E) = \Gamma(E).$$

EXERCISE 11.13

Let λ be a measure on an algebra $\mathscr{A}$, and let λ^* be the corresponding outer measure. Given A, show that there is a set H of the form $\cap_k \cup_j A_{k,j}$ such that $A_{k,j} \in \mathscr{A}$, $A \subset H$ and $\lambda^*(A) = \lambda^*(H)$. Thus, every outer measure that is induced by a measure on an algebra is regular.

Proof.
Given A, let $\{A_{k,j}\}$ be sequence of coverings of A such that $\lim_{k\to\infty} \sum_j \lambda(A_{k,j}) = \lambda^*(A)$. Thus

$$\begin{aligned}\lambda^*(A) \leq \lambda^*\left(\cap_k \cup_j A_{k,j}\right) &\leq \lambda^*\left(\cup_j A_{k,j}\right) \quad \forall\, k\\ &\leq \sum_j \lambda^*(A_{k,j}) \to \lambda^*(A).\end{aligned}$$

So if $H = \cap_k \cup_j A_{k,j}$, then $A \subset H$ and $\lambda^*(A) = \lambda^*(H)$.

EXERCISE 11.15

(a) Show that the intersection of a family of algebras is an algebra.

(b) A collection $\mathscr{C}$ of subsets of $\mathscr{S}$ is called a *subalgebra* if it is closed under finite intersections and if the complement of any set in $\mathscr{C}$ is the union of a finite number of disjoint sets in $\mathscr{C}$. Given an example of a subalgebra. Show that a subalgebra $\mathscr{C}$ generates an algebra by adding $\emptyset$, $\mathscr{S}$, and all finite disjoint unions of sets of $\mathscr{C}$.

Proof.

(a) Let $\{A_k\}$ be a family of algebras and $A = \cap_k A_k$.

   (i) Let X be a set and $X \in A$, then

$$X^c = \cap_k (X \cap A_k)^c \in \mathscr{A},$$

   since A_k is an algebra, $(X \cap A_k)^c \in A_k$ and $A_k \in A$.

   (ii) Let $X_1, X_2, \ldots, X_N$ be the sets and $X_1, X_2 \ldots, X_N \in A$, then

$$\cup_{i=1}^N X_i \in A.$$

(b) Let $\mathscr{C}' = \mathscr{C} \cup \emptyset \cup \mathscr{S} \cup \{\text{all finite disjoint union of sets of } \mathscr{C}\}$.
We need to show that $A^c \in \mathscr{C}'$, if $A \in \mathscr{C}'$.

   (i) Given $A \in \mathscr{C}$, then

$$A^c = \cup_{k=1}^K A_k \in \mathscr{C}' \quad \text{where } A_k \in \mathscr{C}.$$

   (ii) Given $A \in \mathscr{C}' \setminus \mathscr{C}$, may assume that $A = \cup_{k=1}^K A_k$, where $A_k \in \mathscr{C}$, then

$$\begin{aligned}A^c &= \left(\cup_{k=1}^K A_k\right)^c = \cap_{k=1}^K A_k^c\\ &= \cap_{k=1}^K \cup_{l=1}^{L_k} A_{k,l} \quad \text{where } A_{k,l} \in \mathscr{C}\\ &= \cup_l \left(\cap_{k=1}^K A_{k,l}\right) \in \mathscr{C}'.\end{aligned}$$

(iii) Given $A_1, \dots, A_n \in \mathscr{C}'$. W.L.O.G., we can assume $A_1, \dots, A_m \in \mathscr{C}$ and $A_{m+1}, \dots, A_n \in \mathscr{C}' \setminus \mathscr{C}$, then

$$\begin{aligned} \cap_{j=1}^n A_j &= (\cap_{j=1}^m A_j) \cup (\cap_{j=m+1}^n A_j) \\ &= (\cap_{j=1}^m A_j) \cap (\cap_{j=m+1}^n \cup_{k=1}^{K_j} A_{k,j}) \\ &= (\cap_{j=1}^m A_j) \cap (\cup_k \cap_{j=m+1}^n A_{k,j}) \end{aligned}$$

and

$$\cap_{j=1}^m A_j \in \mathscr{C}, \quad \cup_k \cap_{j=m+1}^n A_{k,j} \in \mathscr{C}'.$$

It remains to show that

$$A \in \mathscr{C},\, B \in \mathscr{C}' \Rightarrow A \cap B \in \mathscr{C}',$$

since

$$A \cap B = A \cap (\cup_{k=1}^K B_k) = \cup_{k=1}^K (A \cap B_k) \in \mathscr{C}'.$$

Example of a subalgebra:
The 2×2-matrices over the reals form a unital algebra in the obvious way.
The 2×2-matrices for which all entries are zero, except for the first one on the diagonal, form a subalgebra.

EXERCISE 11.16

If μ is a finite Borel measure on $\mathbb{R}^1$, show that $\mu(B) = \sup \mu(F)$ for every Borel set B, where the sup is taken over all closed subsets F of B.

Proof.
This is to show that μ is inner regular.
Define the collection $\mathscr{C}$ by

$$B \in \mathscr{C} \iff \mu(B) = \sup\{\mu(F) \,:\, F \subset B,\, F \text{ is closed}\}$$

Let $B \in \mathscr{C}$, let $\varepsilon > 0$.
Take F closed with $F \subset B$ and $\mu(B) < \mu(F) + \varepsilon$. Then $B^c \subset F^c$, F^c is open and

$$\mu(B^c) = \mu(\mathbb{R}) - \mu(B) > \mu(\mathbb{R}) - \mu(B) - \varepsilon = \mu(F^c) - \varepsilon.$$

Hence $B^c \in \mathscr{C}$.
Let $B_1, B_2, \dots \in \mathscr{C}$ and let $\varepsilon > 0$. Take for each i, F_i is closed with

$$F_i \subset B, \quad \mu(B_i) - \mu(F_i) < \frac{\varepsilon}{2^{i+1}}.$$

So $\cup_i F_i \subset \cup_i B_i$ and $\mu(\cup_i F_i) = \lim\limits_{k\to\infty} \mu(\cup_{i=1}^k F_i)$, hence for some large N, we have

$$\mu(\cup_i F_i) - \mu(\cup_{i=1}^k F_i) < \frac{\varepsilon}{2}.$$

Then $F := \cup_{i=1}^k F_i \subset \cup_i B_i$, F is closed and

$$\begin{aligned} \mu(\cup_i B_i) - \mu(F) &< \mu(\cup_i B_i) - \mu(\cup_i F_i) + \frac{\varepsilon}{2} \\ &\le \mu(\cup_i B_i \setminus \cup_i F_i) + \frac{\varepsilon}{2} \\ &\le \mu(\cup_i (B_i \setminus F_i)) + \frac{\varepsilon}{2} \\ &\le \sum_i \mu(B_i \setminus F_i) + \frac{\varepsilon}{2} \\ &= \sum_i (\mu(B_i) - \mu(F_i)) + \frac{\varepsilon}{2} \\ &\le \varepsilon. \end{aligned}$$

Hence

$$\mu(B) = \sup \mu(F).$$

EXERCISE 11.17

Show that the conclusions of **Theorems 10.48** and **10.49**, and therefore also the conclusion of **Corollary 10.50**, remain true without the assumption (ii) stated before **Lemma 10.47**. (Show that without this assumption, the conclusions of **Lemma 10.47** are true with μ replaced by μ^*; for example,

$$\mu^* \left\{ \mathbf{x} \in E \,:\, \sup_{h>0} \frac{\nu(Q_{\mathbf{x}}(h))}{\mu(Q_{\mathbf{x}}(h))} > \alpha \right\} \leq c\, \frac{\nu(\mathbb{R}^n)}{\alpha}.\Big)$$

Proof.

It's sufficient to show that without this assumption, the conclusions of **Lemma 10.47** are true with μ replaced by μ^*.

(a) To show

$$\mu^* \left\{ \mathbf{x} \in \mathbb{R}^n \,:\, \sup_{h>0} \frac{\nu(Q_{\mathbf{x}}(h))}{\mu(Q_{\mathbf{x}}(h))} > \alpha \right\} \leq c\, \frac{\nu(\mathbb{R}^n)}{\alpha}$$

Fix $\alpha > 0$, and let

$$S = \left\{ \mathbf{x} \in \mathbb{R}^n \,:\, \sup_{h>0} \frac{\nu(Q_{\mathbf{x}}(h))}{\mu(Q_{\mathbf{x}}(h))} > \alpha \right\}.$$

If B is any bounded Borel set and $\mathbf{x} \in S \cap B$, there is a cube $Q_{\mathbf{x}}$ with center $\mathbf{x}$ such that $\dfrac{\nu(Q_{\mathbf{x}})}{\mu(Q_{\mathbf{x}})} > \alpha$.

Using Besicovitch's lemma, select $\{Q_{\mathbf{x}_k}\}$ and c such that $\nu(Q_{\mathbf{x}_k}) > \alpha\mu(Q_{\mathbf{x}_k})$, $S \cap B \subset \cup Q_{\mathbf{x}_k}$, and $\sum \chi_{Q_{\mathbf{x}_k}} \leq c$.

We then have

$$\mu^*(S \cap B) \leq \mu(S \cap B) \leq \mu(\cup Q_{\mathbf{x}_k}) \leq \sum \mu(Q_{\mathbf{x}_k}) < \frac{1}{\alpha} \sum \nu(Q_{\mathbf{x}_k}),$$

$$\sum \nu(Q_{\mathbf{x}_k}) = \sum \int_{\cup Q_{\mathbf{x}_k}} \chi_{Q_{\mathbf{x}_k}}\, d\nu \leq c \int_{\cup Q_{\mathbf{x}_k}} d\nu = c\, \nu(\cup Q_{\mathbf{x}_k}).$$

Therefore,

$$\mu^*(S \cap B) \leq \frac{c}{\alpha}\, \nu(\cup Q_{\mathbf{x}_k}),$$

so that

$$\mu^*(S \cap B) \leq \frac{c}{\alpha}\, \nu(\mathbb{R}^n).$$

Letting $B \nearrow \mathbb{R}^n$, we obtain $\mu^*(S) \leq \dfrac{c}{\alpha}\, \nu(\mathbb{R}^n)$, as desired.

(b) To show

$$\mu^* \left\{ \mathbf{x} \in E \,:\, \limsup_{h\to 0} \frac{\nu(Q_{\mathbf{x}}(h))}{\mu(Q_{\mathbf{x}}(h))} > \alpha \right\} \leq c\, \frac{\nu(E)}{\alpha}$$

Fix $\alpha > 0$, and let

$$T = \left\{ \mathbf{x} \in E \,:\, \limsup_{h\to 0} \frac{\nu(Q_{\mathbf{x}}(h))}{\mu(Q_{\mathbf{x}}(h))} > \alpha \right\}.$$

If $\nu(E) = +\infty$, there is nothing to prove.

Otherwise, choose an open set $G \supset E$ with $\nu(G) < \nu(E) + \varepsilon$, and let B be a bounded Borel set.

If $\mathbf{x} \in T \cap B$, there is a cube $Q_{\mathbf{x}}$ such that

$$Q_{\mathbf{x}} \subset G \quad \text{and} \quad \frac{\nu(Q_{\mathbf{x}})}{\mu(Q_{\mathbf{x}})} > \alpha.$$

By again using Besicovitch's lemma, there exists $\{Q_{\mathbf{x}_k}\}$, $Q_{\mathbf{x}_k} \subset G$, such that

$$\mu^*(T \cap B) \leq \mu(T \cap B) \leq \frac{c}{\alpha}\,\nu(G) \leq \frac{c}{\alpha}\,[\nu(E) + \varepsilon]$$

The result now follows by first letting $\varepsilon \to 0$ and then letting $B \nearrow \mathbb{R}^n$.