Chapter 1

Preliminaries

Exercise 1.1 *Prove the following facts, which were left as exercises above.*

(a) *For a sequence of sets* $\{E_k\}$, $\limsup E_k$ *consists of those points which belong to infinitely many* E_k, *and* $\liminf E_k$ *consists of those points which belong to all* E_k *from some* k *on.*

(b) *The De Morgan laws.*

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Solution.

(a) For any $x \in \limsup E_k$, suppose x belong to only finitely many E_k. Let x belong to $E_{k_1}, E_{k_2}, \cdots, E_{k_n}$. Let $k' = \min\{k_1, k_2, \cdots, k_n\}$, then

$$x \notin \bigcap_{n=k'+1}^{\infty} \bigcup_{k=n}^{\infty} E_k \supseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \limsup E_k.$$

That is a contradiction. So x belong to infinitely many E_k. Next, For any $x \in \liminf E_k$, there $x \in \bigcap_{k=n'}^{\infty} E_k$ for some n'. That is $x \in E_k$ for all $k \geq n'$.

(b) First we can show that $(\bigcup_{E\in\mathscr{F}} E)^c \subseteq \bigcap_{E\in\mathscr{F}} E^c$. For any $x \in (\bigcup_{E\in\mathscr{F}} E)^c$. That is $x \in \mathbb{R}^n$ and $x \notin \bigcup_{E\in\mathscr{F}} E$. So that $x \notin E$ for all $E \in \mathscr{F}$. Hence $x \in E^c$ for all $E \in \mathscr{F}$. So $x \in \bigcap_{E\in\mathscr{F}} E^c$. It implies $(\bigcup_{E\in\mathscr{F}} E)^c \subseteq \bigcap_{E\in\mathscr{F}} E^c$. Next, we will show that $(\bigcup_{E\in\mathscr{F}} E)^c \supseteq \bigcap_{E\in\mathscr{F}} E^c$. For any $x \in \bigcap_{E\in\mathscr{F}} E^c$, that is $x \in E^c$ for all $E \in \mathscr{F}$. So $x \notin E$ for all $E \in \mathscr{F}$. Hence $x \in \bigcup_{E\in\mathscr{F}} E)^c$. It implies $(\bigcup_{E\in\mathscr{F}} E)^c \supseteq \bigcap_{E\in\mathscr{F}} E^c$. So we show that $(\bigcup_{E\in\mathscr{F}} E)^c = \bigcap_{E\in\mathscr{F}} E^c$. It follows that $(\bigcap_{E\in\mathscr{F}} E)^c = \bigcup_{E\in\mathscr{F}} E^c$. This complete the proof.

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Chapter 2

Functions of Bounded Variation; the Riemann-Stieltjes Integral

Exercise 2.1 *Let $f(x) = x\sin(1/x)$ for $0 < x \le 1$ and $f(0) = 0$. Show that f is bounded and continuous on $[0,1]$, but that $V[f;0,1] = +\infty$.*

Solution. First, we show that f is continuous on $[0,1]$. Let f_0, f_1 and f_2 be functions with $f_0(x) = x$, $f_1(x) = \sin(x)$ and $f_2(x) = \frac{1}{x}$. Then f is continuous on $(0,1]$ since f_0, f_1 and f_2 are continuous on $(0,1]$ and $f = f_0(f_1 \circ f_2)$. If $x = 0$, for every $\varepsilon > 0$, let $\delta = \min\{\varepsilon, 1\}$ such that for all $y \in [0,\delta)$, we have $y\sin(\frac{1}{y}) \le |y| < \delta \le \varepsilon$. So f is continuous on $[0,1]$. Obviously $|f(x)| < 2$ for all $x \in [0,1]$. So f is bounded. Next, we show that $V[f;0,1] = +\infty$. Let $\Gamma = \{0,1\} \cup \{\frac{1}{(k+\frac{1}{2})\pi}\}_{k=0}^{n}$ be a partition. Since $\frac{1}{k+\frac{1}{2}\pi}\sin(k+\frac{1}{2})\pi = \frac{1}{(k+\frac{1}{2})\pi}$ for all $k \in \mathbb{Z}^+$, Let $S_\Gamma \ge \sum_{i=0}^{n} |\frac{1}{(k+\frac{1}{2})\pi} + \frac{1}{(k-\frac{1}{2})\pi}| = \frac{1}{\pi}\sum_{i=0}^{n} |\frac{2k}{k^2-\frac{1}{4}}| > \frac{1}{\pi}\sum_{i=0}^{n} \frac{2}{k} \to \infty$ as $n \to \infty$. So that $V[f;0,1] = +\infty$.

Exercise 2.2 *Prove theorem (2.1), that is*

(i) *If f is of bounded variation on [a,b], then f is bounded on $[a,b]$.*

(ii) *Let f and g be of bounded variation on $[a,b]$. Then cf (for any real constant), $f+g$ and fg are of bounded variation on $[a,b]$. Moreover, f/g is of bounded variation on $[a,b]$ if there exists an $\varepsilon > 0$ such that $|g(x)| \ge \varepsilon$ for $x \in [a,b]$.*

Solution.

(i) For every $x \in [a,b]$, then $M > |f(x) - f(a)| + |f(b) - f(x)| \ge |2f(x) - (f(a) + f(b))| \ge |2f(x)| - |f(a) + f(b)|$ for some $M > 0$. So $|f(x)| < \frac{1}{2}(M + |f(a) + f(b)|)$. That is f is bounded on $[a,b]$.

(ii) Let $V(f)$ and $V(g)$ be total variation of f and g on [a,b], then $V(f) < M$ and $V(g) < M$ for some $M > 0$. By (i), we let $|f(x)| < N$ and $|g(x)| < N$

for all x and some $N > 0$. For every partition $\Gamma = \{x_i\}_{i=0}^n$, the

$$\begin{aligned}
V(cf) &= \sup_\Gamma\{\sum_{i=0}^n |cf(x_i) - cf(x_{i-1})|\} \\
&= |c|\sup_\Gamma\{\sum_{i=0}^n |f(x_i) - f(x_{i-1})|\} \\
&< |c|M. \\
V(f+g) &= \sup_\Gamma\{\sum_{i=0}^n |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})|\} \\
&\le \sup_\Gamma\{\sum_{i=0}^n |f(x_i) - f(x_{i-1})|\} + \sup_\Gamma\{\sum_{i=0}^n |g(x_i) - g(x_{i-1})|\}\} \\
&< 2M. \\
V(fg) &= \sup_\Gamma\{\sum_{i=0}^n |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})|\} \\
&= \sup_\Gamma\{\sum_{i=0}^n |f(x_i) - f(x_{i-1})||g(x_i) - g(x_{i-1})|\} \\
&\le \sup_\Gamma\{\sum_{i=0}^n |f(x_i) - f(x_{i-1})| \sum_{i=0}^n |g(x_i) - g(x_{i-1})|\} \\
&= < M^2. \\
V(\frac{f}{g}) &= \sup_\Gamma\{\sum_{i=0}^n |\frac{f(x_i)}{g(x_i)} - \frac{f(x_{i-1})}{g(x_{i-1})}|\} \\
&= \sup_\Gamma\{\sum_{i=0}^n |\frac{f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i)}{g(x_i)g(x_{i-1})}|\} \\
&\le \frac{1}{\varepsilon^2}\sup_\Gamma\{\sum_{i=0}^n |(f(x_i) - f(x_{i-1}))||g(x_{i-1})| \\
&\quad +|f(x_{i-1})||(g(x_i) - g(x_{i-1})|\} \\
&\le \frac{N}{\varepsilon^2}\sup_\Gamma\{\sum_{i=0}^n |(f(x_i) - f(x_{i-1}))| + |(g(x_i) - g(x_{i-1})|\} \\
&< \frac{2NM}{\varepsilon^2}.
\end{aligned}$$

Then this theorem is clearly.

Exercise 2.3 *If $[a', b']$ is a subinterval of $[a, b]$, show that $P[a', b'] \le P[a, b]$ and $N[a', b'] \le N[a, b]$.*

Solution. Let Γ be a partition of $[a', b']$, then

$$\begin{aligned}
\sum_{i=1}^n [f(x_i) - f(x_{i-1})]^+ &\le \sum_{i=1}^n [f(x_i) - f(x_{i-1})]^+ + [f(x_0) - f(a)]^+ \\
&\quad +[f(b) - f(x_n)]^+ \\
&\le P[a, b].
\end{aligned}$$

Hence $P[a',b'] \leq P[a,b]$. Similarly, we show that $N[a',b'] \leq N[a,b]$.

Exercise 2.4 *Let $\{f_k\}$ be a sequence of functions of bounded variation on $[a,b]$. If $V[f_k;a,b] \leq M < +\infty$ for all k and if $f_k \to f$ pointwise on $[a,b]$, show that f is of bounded variation and that $V[f;a,b] \leq M$. Give an example of a convergent sequence of functions of bounded variation whose limit is not of bounded variation.*

Solution. For every $\Gamma = \{x_i\}_{i=1}^n$ be a partition. The

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| = \lim_{k\to\infty} \sum_{i=1}^n |f_k(x_i) - f_k(x_{i-1})|.$$

Since $V[f_k;a,b] \leq M < +\infty$, then $\lim_{k\to\infty} \sum_{i=1}^n |f_k(x_i) - f_k(x_{i-1})| \leq M$. That is $\sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq M$. It means f is of bounded variation and that $V[f;a,b] \leq M$. We give an example. Let $\mathbb{Q} \cap [a,b] = \{r_k\}_{k=1}^\infty$ and let

$$f_k(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \cdots, r_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f_k is of bounded variation for any k. But $f_k \to f$ where

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

The function f is not of bounded variation.

Exercise 2.5 *Suppose f is finite on $[a,b]$ and of bounded variation on every interval $[a+\varepsilon,b]$, $\varepsilon > 0$, with $V[f;a+\varepsilon,b] \leq M < +\infty$. Show that $V[f;a,b] < +\infty$. Is $V[f;a,b] \leq M$? If not, what additional assumption will make it so?*

Solution. For every partition $\Gamma = \{x_i\}_{i=0}^n$ of $[a,b]$ and every $\varepsilon < x_1 - x_0$, the

$$\begin{aligned} & \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \\ \leq\ & |f(a+\varepsilon) - f(x_0)| + |f(x_1) - f(a+\varepsilon)| + \sum_{i=2}^n |f(x_i) - f(x_{i-1})| \\ \leq\ & |f(a+\varepsilon) - f(x_0)| + M \\ <\ & +\infty. \end{aligned}$$

If f is continuous at a, there exists $\delta > 0$ such that for all $x \in [a, a+\delta)$, we have $|f(x) - f(a)| < \eta$ for every $\eta > 0$. For $\varepsilon < \delta$, then

$$\begin{aligned} & \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \\ \leq\ & |f(a+\varepsilon) - f(x_0)| + M \\ <\ & \eta + M. \end{aligned}$$

Hence $V[f;a,b] \leq M$.

Exercise 2.6 *Let $f(x) = x^2 \sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$. Show that $V[f;a,b] < +\infty$.* [Examine the graph of f, or use Exercise 5 and (2.10).]

Solution. For every $\varepsilon > 0$, the function f is continuous on $[0+\varepsilon, 1]$. Then for all $x \in [0+\varepsilon, 1]$, the

$$|f'(x)| = |2x \sin \frac{1}{x} - \cos \frac{1}{x}| \leq |2x| + 1 \leq 3.$$

So for every partition $\Gamma = \{x_k\}_{k=0}^n$ of $[0+\varepsilon, 1]$, we have

$$\begin{aligned} \sum_{k=1}^n |f(x_k) - f(x_{k-1})| &= \sum_{k=1}^n |f'(x)||x_k - x_{k-1}| \\ &\leq 3(x_n - x_0) \\ &< 3. \end{aligned}$$

So $V[f; a+\varepsilon, b] \leq 3$, this implies $V[f; a+\varepsilon, b] < +\infty$ by Exercise 2.5.

Exercise 2.7 *Suppose f is of bounded variation on $[a, b]$. If f is continuous on $[a, b]$, show that $V(x)$, $P(x)$, and $N(x)$ are also continuous.* (If $\Gamma = \{x_i\}$, note that $V[x_{i-1}, x_i] - |f(x_{i-1}) - f(x_i)| \leq V[a, b] - S_\Gamma$.)

Solution. The

$$\begin{aligned} S_\Gamma - |f(x_{i-1}) - f(x_i)| &= \sum_{k=1}^n |f(x_{k-1}) - f(x_k)| - |f(x_{i-1}) - f(x_i)| \\ &= \sum_{k=1}^{i-1} |f(x_{k-1}) - f(x_k)| - \sum_{k=i+1}^{n} |f(x_{k-1}) - f(x_k)| \\ &\leq V[a, x_{i-1}] + V[x_i, b] \\ &= V[a, b] - V[x_{i-1}, x_i]. \end{aligned}$$

So $V[x_{i-1}, x_i] - |f(x_{i-1}) - f(x_i)| \leq V[a, b] - S_\Gamma$ for every $\Gamma = \{x_i\}$. For $x \in [a, b]$, since f is continuous, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in (x-\delta, x+\delta) \cap [a, b]$, we have $|f(y) - f(x)| < \frac{\varepsilon}{2}$. Then for every $y \in (x-\delta, x+\delta) \cap [a, b]$. We may assume $y < x$, then let $\Gamma = \{x_i\}$ be a partition satisfies $|V[a, b] - S_\Gamma| < \frac{\varepsilon}{2}$ and $x_{i_0 - 1} = y$, $x_{i_0} = x$ for some i_0. Thus

$$\begin{aligned} |V(y) - V(x)| &= V[y, x] \\ &\leq V[a, b] - S_\Gamma + |f(y) - f(x)| \\ &< \varepsilon. \end{aligned}$$

Then V is continuous. Since $P = \frac{1}{2}[V + f(b) - f(a)]$ and $N = \frac{1}{2}[V - f(b) + f(a)]$, then P and N are continuous.

Exercise 2.8 *The main results about functions of bounded variation on a closed interval remain true for open or partly open intervals and for infinite intervals. Prove, for example, that if f is of bounded variation on $(-\infty, \infty)$, then f is the difference of two increasing bounded functions.*

Solution. For all $M \in \mathbb{Z}^+$, $V[0, M] < \infty$ and $V[-M, 0] < \infty$ since $V(-\infty, \infty)$. By Jordan's Theorem, if f is on $[0, M]$, we have

$$f(x) = P_1[f; 0, x] - N_1[f; 0, x] + f(0)$$

and if f is on $[-M,0]$, we have

$$f(x) = -P_2[f;x,0] + N_2[f;x,0] + f(0).$$

Let

$$P(x) = \begin{cases} P_1[f;0,x] & \text{if } x \geq 0, \\ -P_2[f;x,0] & \text{if } x < 0, \end{cases}$$

and

$$N(x) = \begin{cases} N_1[f;0,x] - f(0) & \text{if } x \geq 0, \\ -N_2[f;x,0] - f(0) & \text{if } x < 0. \end{cases}$$

First, we claim that P and N are increasing. For $0 < x < y$, the

$$P(x) = P_1[f;0,x] \leq P_1[f;0,y] = P(y).$$

For $x < y < 0$, the

$$P(x) = -P_2[f;x,0] \leq -P_2[f;y,0] = P(y).$$

And for $x < 0 < y$, the

$$P(x) = -P_2[f;x,0] \leq P_1[f;0,y] = P(y).$$

Then P is increasing. Similarly, N is increasing. Next, for $x \in [0,M]$, the

$$0 \leq P(x) = P_1[f;0,x] \leq P_1[f;0,M] \leq V[f;0,M] \leq V(-\infty,\infty) < \infty$$

and for $x \in [-M,0]$, the

$$0 \geq P(x) = -P_2[f;x,0] \geq -P_2[f;-M,0] \geq -V[f;-M,0] > -\infty.$$

So P is bounded. Similarly, N is bounded. Thus $f = P - N$ on $[-M,M]$, this implies $f = P - N$ on $(-\infty,\infty)$ as $M \to \infty$.

Exercise 2.9 *Let C be a curve with parametric equations $x = \phi(t)$ and $y = \psi(t)$, $a \leq t \leq b$.*

(a) *If ϕ and ψ are of bounded variation and continuous, show that $L = \lim_{|\Gamma|\to 0} l(\Gamma)$.*

(b) *If ϕ and ψ are continuously differentiable, show that $L = \int_a^b ([\phi'(t)]^2 + [\psi'(t)]^2)^{1/2} dt$.*

Solution.

(a) Since ϕ and ψ are continuous on $[a,b]$, then ϕ and ψ are uniform continuous on $[a,b]$. That is for any $\varepsilon > 0$, there exists $\delta' > 0$ such that for all $|x-y| < \delta'$, we have $|\phi(x) - \phi(y)| < \frac{\sqrt{2}\varepsilon}{4(k+1)}$ and $|\psi(x) - \psi(y)| < \frac{\sqrt{2}\varepsilon}{4(k+1)}$. Let $\bar{\Gamma} = \{\bar{x}_i\}_{i=0}^k$ be a partition on $[a,b]$ with $l(\bar{\Gamma}) > M + \varepsilon$ and let $\delta = \min\{\delta', \bar{x}_1 - \bar{x}_0, \bar{x}_2 - \bar{x}_1, \cdots, \bar{x}_k - \bar{x}_{k-1}\}$. Then for all $|\Gamma| < \delta$, write $\Gamma = \{x_i\}$, we have

$$\begin{aligned} l(\Gamma) &= \sum_{i=1}^{n} ((\phi(x_i) - \phi(x_{i-1}))^2 + (\psi(x_i) - \psi(x_{i-1}))^2)^{\frac{1}{2}} \\ &= \Sigma' + \Sigma'' \\ &\leq \Sigma' + \Sigma''' \\ &= l(\Gamma \cup \bar{\Gamma}) \end{aligned}$$

where Σ'' is extended over all i such that (x_{i-1}, x_i) contains some $\bar{x}_j$. Any (x_{i-1}, x_i) can contain at most one $\bar{x}_j$ since $|\Gamma| < \delta \le |\bar{\Gamma}|$. So Σ'' has at most $k+1$ terms. And Σ''' is obtained from Σ'' by replacing each from by $((\phi(x_i) - \phi(\bar{x}_j))^2 + (\psi(x_i) - \psi(\bar{x}_j))^2)^{\frac{1}{2}} + ((\phi(\bar{x}_j) - \phi(x_{i-1}))^2 + (\psi(\bar{x}_j) - \psi(x_{i-1}))^2)^{\frac{1}{2}}$. Therefore $\Sigma''' < 2(k+1)(2(\frac{\sqrt{2}\varepsilon}{4(k+1)})^2)^{\frac{1}{2}} = \varepsilon$. Hence

$$l(\Gamma) \ge \Sigma' = l(\Gamma \cup \bar{\Gamma}) - \Sigma''' > l(\Gamma \cup \bar{\Gamma}) - \varepsilon \ge l(\bar{\Gamma}) - \varepsilon > M.$$

So $L = \lim_{|\Gamma| \to 0} l(\Gamma)$.

(b) Since ϕ' and ψ' are continuous on $[a, b]$, $\int_a^b ([\phi'(t)]^2 + [\psi'(t)]^2)^{\frac{1}{2}} dt$ exists. By mean value theorem,

$$\begin{aligned}
L &= \lim_{|\Gamma| \to 0} l(\Gamma) \\
&= \lim_{|\Gamma| \to 0} \sum_{i=1}^{n} ([\phi(x_i) - \phi(x_{i-1})]^2 + [\psi(x_i) - \psi(x_{i-1})]^2)^{\frac{1}{2}} \\
&= \lim_{|\Gamma| \to 0} \sum_{i=1}^{n} ([\phi'(y_i)(x_i - x_{i-1})]^2 + [\psi(z_i)(x_i - x_{i-1})]^2)^{\frac{1}{2}} \\
&= \lim_{|\Gamma| \to 0} \sum_{i=1}^{n} ([\phi'(y_i)]^2 + [\psi(z_i)]^2)^{\frac{1}{2}} (x_i - x_{i-1}) \\
&= \int_a^b ([\phi'(t)]^2 + [\psi'(t)]^2)^{1/2} dt
\end{aligned}$$

for some $y_i, z_i \in (x_{i-1}, x_i)$.

Exercise 2.10 *If $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ is a finite sequence and $-\infty < s < \infty$, write $\sum_k a_k e^{s\lambda_k}$ as a Riemann-Stieltjes integral.* [Take $f(x) = e^{-sx}$, ϕ to be an appropriate step function, and $[a, b]$ to contain all the λ_k in its interior.]

Solution. Let $f(x) = e^{-sx}$ be continuous function, for $x \in [a, b]$ and

$$\phi(x) = \begin{cases} a_0 & \text{if } x \in [a, \lambda_1], \\ \sum_{i=0}^{k} a_i & \text{if } x \in (\lambda_k, \lambda_{k+1}) \text{ for } k = 1, 2, \cdots, m-1, \\ \sum_{i=0}^{m} a_i & \text{if } x \in (\lambda_m, b]. \end{cases}$$

Then ϕ is a step function on $[a, b]$. So

$$\begin{aligned}
\int_a^b f d\phi &= f(a)[\phi(a^+) - \phi(a)] + \sum_{k=1}^{m} f(\lambda_k)[\phi(\lambda_k^+) - \phi(\lambda_k^-)] \\
&\quad + f(b)[\phi(b) - \phi(b^-)] \\
&= \sum_{k=1}^{m} f(\lambda_k)[\phi(\lambda_k^+) - \phi(\lambda_k^-)] \\
&= \sum_{k=1}^{m} a_k e^{s\lambda_k}.
\end{aligned}$$

Exercise 2.11 *Show that $\int_a^b f d\phi$ exists if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that $|R_\Gamma - R_{\Gamma'}| < \varepsilon$ if $|\Gamma|$, $|\Gamma'| < \delta$.*

Solution. First, suppose that $\int_a^b f d\phi$ exists. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition Γ of $[a, b]$ with $|\Gamma| < \delta$, we have $|R_\Gamma - I| < \frac{\varepsilon}{2}$ for some $I \in \mathbb{R}$. Then

$$|R_\Gamma - R_{\Gamma'}| \leq |R_\Gamma - I| + |I - R_{\Gamma'}| < \varepsilon$$

if $|\Gamma|$, $|\Gamma'| < \delta$. Conversely, let $\Gamma_n = \{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < \cdots < b\}$ be a partition of $[a, b]$ and let

$$t_n = \sum_{k=1}^{n} f(a + k\frac{b-a}{2})(\phi(a + k\frac{b-a}{2}) - \phi(a + (k-1)(\frac{b-a}{2}))).$$

Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $|\Gamma_n|$, $|\Gamma_m| < \delta$, we have $|t_n - t_m| < \varepsilon$. Hence the $\{t_n\}$ is a Cauchy sequence. We let $\lim_{n\to\infty} t_n = \alpha$, for every Γ be a partition of $[a, b]$ with $|\Gamma| < \delta$ and if $|\Gamma_n| < \delta$, we have

$$|R_\Gamma - t_n| < \varepsilon.$$

Then

$$|R_\Gamma - \alpha| \leq \varepsilon.$$

Hence there exists $\delta' > 0$ such that for all $|\Gamma| < \delta'$, we have

$$|R_\Gamma - \alpha| \leq \frac{\varepsilon}{2} < \varepsilon.$$

So that $\int_a^b f d\phi$ exists.

Exercise 2.12 *Prove that the conclusion of* (2.30) *is valid if the assumption that ϕ is continuous is replaced by the assumption that f and ϕ have no common discontinuities. (Instead of the uniform continuity of ϕ, use the fact that either f or ϕ is continuous at each point $\bar{x}$ of $\bar{\Gamma}$.)*

Solution. Let $|f(x)| < M_0$ for some M_0 and all $x \in [a, b]$ since f is bounded, and let $\sup_\Gamma L_\Gamma = L$, then for any $M < L$, there exists $\bar{\Gamma}$ be a partition of $[a, b]$ such that $M + \varepsilon_0 < L_{\bar{\Gamma}} < L$ for some $\varepsilon_0 > 0$. We write $\bar{\Gamma} = \{\bar{x}_i\}_{i=0}^k$. For all $\bar{x}_j$, if f or ϕ is continuous at $\bar{x}_j$, there exists δ_j such that for all $x \in (x_j - \delta_j, x_j + \delta_j) \cap [a, b]$, we have

$$|f(x) - f(\bar{x}_j)| < \frac{\varepsilon_0}{4(k+1)(\phi(b) - \phi(a) + 1)} \ \textit{or}\ |\phi(x) - \phi(\bar{x}_j)| < \frac{\varepsilon_0}{2M_0(k+1)}.$$

Let $\delta = \min\{\delta_0, \delta_1, \cdots, \delta_k, |\bar{\Gamma}|\}$, for all $\Gamma = \{x_i\}_{i=0}^n$ be a partition of $[a, b]$ with $|\Gamma| < \delta$, then

$$L_\Gamma = \sum_{i=1}^{n} m_i(\phi(x_i) - \phi(x_{i-1})) = \Sigma_1 + \Sigma_2$$

where Σ_2 is extended over all i such that (x_{i-1}, x_i) contains some $\bar{x}_j$. Any (x_{i-1}, x_i) can contain at most one $\bar{x}_j$, we write $\bar{x}_j = \bar{x}_{j_i} \in (x_{i-1}, x_i)$. Since $|\Gamma| < \delta \leq |\bar{\Gamma}|$, then Σ_2 has at most $k + 1$ terms. Let $L_{\Gamma\cup\bar{\Gamma}} = \Sigma_1 + \Sigma_2'$ where Σ_2' is extended as Σ_2. So

$$\begin{aligned} L_{\Gamma\cup\bar{\Gamma}} - L_\Gamma &= \Sigma_2' - \Sigma_2 \\ &= \sum_i (m_i' - m_i)(\phi(x_i) - \phi(\bar{x}_{j_i})) + (m_i'' - m_i)(\phi(x_i) - \phi(\bar{x}_{j_i})) \end{aligned}$$

where $m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$, $m_i' = \inf\{f(x) \mid \bar{x}_{j_i} \le x \le x_i\}$ and $m_i'' = \inf\{f(x) \mid x_{i-1} \le x \le \bar{x}_{j_i}\}$. For every term, if ϕ is continuous at $\bar{x}_{j_i}$, then

$$(m_i'-m_i)(\phi(x_i)-\phi(\bar{x}_{j_i}))+(m_i''-m_i)(\phi(x_i)-\phi(\bar{x}_{j_i})) < 2M_0\frac{\varepsilon_0}{2M_0(k+1)} = \frac{\varepsilon_0}{k+1}.$$

If ϕ is discontinuous at $\bar{x}_{j_i}$, then f is continuous at $\bar{x}_{j_i}$, the

$$\begin{aligned}
& (m_i' - m_i)(\phi(x_i) - \phi(\bar{x}_{j_i})) + (m_i'' - m_i)(\phi(x_i) - \phi(\bar{x}_{j_i})) \\
\le\ & [(m_i' - m_i) + (m_i'' - m_i)](\phi(b) - \phi(a)) \\
=\ & (m_i' - f(\bar{x}_{j_i}) + f(\bar{x}_{j_i}) - m_i + m_i'' - f(\bar{x}_{j_i}) + f(\bar{x}_{j_i}) - m_i)(\phi(b) - \phi(a)) \\
<\ & \frac{4\varepsilon_0}{4(k+1)(\phi(b) - \phi(a) + 1)}(\phi(b) - \phi(a)) < \frac{\varepsilon_0}{k+1}.
\end{aligned}$$

This implies $0 \le L_{\Gamma\cup\bar{\Gamma}} - L_\Gamma < \varepsilon_0$, then $L_\Gamma > L_{\Gamma\cup\bar{\Gamma}} - \varepsilon_0 > L_{\bar{\Gamma}} - \varepsilon_0 > M$ for all $|\Gamma| < \delta$. That is $\lim_{|\Gamma|\to 0} L_\Gamma = \sup_\Gamma L_\Gamma$. Similarly, we know that $\lim_{|\Gamma|\to 0} U_\Gamma = \inf_\Gamma U_\Gamma$.

Exercise 2.13 *Prove theorem* (2.16). *That is*

(i) *If $\int_a^b f d\phi$ exists, then so do $\int_a^b cf d\phi$ and $\int_a^b f d(c\phi)$ for any constant c, and*

$$\int_a^b cf d\phi = \int_a^b f d(c\phi) = c\int_a^b f d\phi.$$

(ii) *If $\int_a^b f_1 d\phi$ and $\int_a^b f_2 d\phi$ both exist, so does $\int_a^b (f_1 + f_2) d\phi$, and*

$$\int_a^b (f_1 + f_2) d\phi = \int_a^b f_1 d\phi + \int_a^b f_2 d\phi.$$

(iii) *If $\int_a^b f d\phi_1$ and $\int_a^b f d\phi_2$ both exist, so does $\int_a^b f d(\phi_1 + \phi_2)$, and*

$$\int_a^b f d(\phi_1 + \phi_2) = \int_a^b f d\phi_1 + \int_a^b f d\phi_2.$$

Solution.

(i) For every partition $\Gamma = \{x_i\}$, the

$$\begin{aligned}
\sum_{i=1}^n cf(y_i)(\phi(x_i) - \phi(x_{i-1})) &= \sum_{i=1}^n f(y_i)(c\phi(x_i) - c\phi(x_{i-1})) \\
&= c\sum_{i=1}^n f(y_i)(\phi(x_i) - \phi(x_{i-1}))
\end{aligned}$$

where $y_i \in [x_{i-1}, x_i]$. Then $\int_a^b cf d\phi$ and $\int_a^b f d(c\phi)$ exists, Hence $\int_a^b cf d\phi = \int_a^b f d(c\phi) = c\int_a^b f d\phi$.

(ii) For every partition $\Gamma = \{x_i\}$, the

$$\begin{aligned} & \sum_{i=1}^{n}(f_1(y_i)+f_2(y_i))(\phi(x_i)-\phi(x_{i-1})) \\ = & \sum_{i=1}^{n} f_1(y_i)(\phi(x_i)-\phi(x_{i-1})) + \sum_{i=1}^{n} f_2(y_i)(\phi(x_i)-\phi(x_{i-1})) \end{aligned}$$

where $y_i \in [x_{i-1}, x_i]$. Then $\int_a^b (f_1+f_2)d\phi$ exists. Hence $\int_a^b (f_1+f_2)d\phi = \int_a^b f_1 d\phi + \int_a^b f_2 d\phi$.

(iii) For every partition $\Gamma = \{x_i\}$, the

$$\begin{aligned} & \sum_{i=1}^{n} f(y_i)((\phi_1+\phi_2)(x_i)-(\phi_1+\phi_2)(x_{i-1})) \\ = & \sum_{i=1}^{n} f(y_i)(\phi_1(x_i)-\phi_1(x_{i-1})) + \sum_{i=1}^{n} f(y_i)(\phi_2(x_i)-\phi_2(x_{i-1})) \end{aligned}$$

where $y_i \in [x_{i-1}, x_i]$. Then $\int_a^b fd(\phi_1+\phi_2)$ exists. Hence $\int_a^b fd(\phi_1+\phi_2) = \int_a^b fd\phi_1 + \int_a^b fd\phi_2$.

Exercise 2.14 *Give an example which shows that for $a < c < b$, $\int_a^c fd\phi$ and $\int_c^b$ may both exist but $\int_a^b fd\phi$ may not. Compare* (2.17). [Take $[a,b] = [-1,1]$, $c = 0$, and f and ϕ as in the example following (2.28)]

Solution. Let

$$f(x) = \begin{cases} 0 & \text{if } -1 \le x < 0, \\ 1 & \text{if } 0 \le x \le 1, \end{cases}$$

and

$$\phi(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0, \\ 1 & \text{if } 0 < x \le 1, \end{cases}$$

Then $\int_{-1}^1 fd\phi$ doesn't exist since f and ϕ has common discontinuous point. For any $1 > \varepsilon > 0$, then $\int_{-1}^{0-\varepsilon} fd\phi = 0$. So

$$\int_{-1}^{0} fd\phi = \lim_{\varepsilon\to 0}\int_{-1}^{0-\varepsilon} fd\phi = 0.$$

Similarly, we can show that $\int_0^1 fd\phi = 0$.

Exercise 2.15 *Suppose f is continuous and ϕ is of bounded variation in $[a,b]$. Show that the function $\psi(x) = \int_a^x fd\phi$ is of bounded variation on $[a,b]$. If g is continuous on $[a,b]$, show that $\int_a^b gd\psi = \int_a^b gfd\phi$.*

Solution. Let $\Gamma = \{x_i\}_{i=0}^n$ be every partition of $[a, b]$. then

$$\begin{aligned}
V[\psi; a, b] &= \sup_\Gamma \{\sum_{i=1}^n |\psi_(i) - \psi(x_{i-1})|\} \\
&= \sup_\Gamma \{\sum_{i=1}^n |\int_a^{x_i} f d\phi - \int_a^{x_{i-1}} f d\phi|\} \\
&= \sup_\Gamma \{\sum_{i=1}^n |\int_{x_{i-1}}^{x_i} f d\phi|\} \\
&\leq \sup_\Gamma \{\sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f| d|\phi|\} \\
&= \sup_\Gamma \{\int_a^b |f| d|\phi|\} \\
&= \int_a^b |f| d|\phi|.
\end{aligned}$$

Since $|f|$ is continuous on $[a, b]$ and $|\phi|$ is of bounded variation, then $\int_a^b |f| d|\phi| < \infty$. This implies ψ is of bounded variation. Now, we assume ϕ is increasing. If g is continuous on $[a, b]$, then $\int_a^b g d\psi$ and $\int_a^b fg d\phi$ exists. For every $\varepsilon > 0$, there exist $\delta > 0$ such that for partition $|\Gamma| < \delta$, we have

$$|\sum_{i=1}^n g(y_i)[\psi(x_i) - \psi(x_{i-1})] - \int_a^b g d\psi| < \varepsilon$$

where y_i is between x_{i-1} and x_i. Since f and g are uniform continuous on [a,b], then

$$\begin{aligned}
& |\sum_{i=1}^n g(y_i)[\psi(x_i) - \psi(x_{i-1})] - \int_a^b gf d\phi| \\
= \ & |\sum_{i=1}^n g(y_i) f(z_i)[\phi(x_i) - \phi(x_{i-1})] - \int_a^b gf d\phi| \\
\leq \ & |\sum_{i=1}^n g(y_i) f(z_i)[\phi(x_i) - \phi(x_{i-1})] - \sum_{i=1}^n g(y_i) f(y_i)[\phi(x_i) - \phi(x_{i-1})]| \\
& + |\sum_{i=1}^n g(y_i) f(y_i)[\phi(x_i) - \phi(x_{i-1})] - \int_a^b gf d\phi| \\
< \ & \varepsilon.
\end{aligned}$$

for some z_i between x_{i-1} and x_i. So $\int_a^b g d\psi = \int_a^b gf d\phi$.

Exercise 2.16 *Suppose that ϕ is of bounded variation on $[a, b]$ and that f is bounded and continuous except for a finite number of jump discontinuous in $[a, b]$. If ϕ is continuous at each discontinuity of f, show that $\int_a^b f d\phi$ exists.*

Exercise 2.17 *If ϕ is of bounded variation on $(-\infty, +\infty)$, f is continuous on $(-\infty, +\infty)$, and $\lim_{|x| \to +\infty} f(x) = 0$, show that $\int_{-\infty}^{+\infty} f d\phi$ exists.*

Solution. First, we shows that $\int_0^\infty f d\phi$ exists. Since ϕ is of bounded variation on $(-\infty, \infty)$, then we let $\phi(x) = c$ as $x \to \infty$ and $\phi(x) = d$ as $x \to -\infty$. The $\lim_{|x| \to +\infty} f(x) = 0$ implies for any $\varepsilon > 0$, there exist $M > 0$ such that for all $x \geq M$, we have $|f(x)| < \frac{\varepsilon}{c-d}$. For any $n > m \geq M$, the

$$\begin{aligned}
|\int_0^n f d\phi - \int_0^m f d\phi| &= |\int_m^n f d\phi| \\
&\leq (\sup_{[m,n]} |f|) V[\phi; n, m] \\
&< \frac{\varepsilon}{c-d}(c-d) \\
&< \varepsilon.
\end{aligned}$$

So $\{\int_0^n f d\phi\}$ is a Cauchy sequence, this implies $\int_0^\infty f d\phi$ exist. Similarly, $\int_{-\infty}^0 f d\phi$ exist. Then $\int_{-\infty}^\infty f d\phi$ exist since f is continuous on $(-\infty.\infty)$ and ϕ is of bounded variation on $(-\infty.\infty)$.

Exercise 2.18 *Let $f(z) = \sum_{k=0}^\infty a_k z^k$ be a power series. Show that if $\sum |a_k| < +\infty$, then $f(z)$ is of bounded variation on every radius of the circle $|z| = 1$.* If for example the radius is $0 \leq x \leq 1$ and the a_k are real, then $f(x) = \sum a_k^+ x^k - \sum a_k^- x^k$.

Solution. Given any radius of the circle $|z| = 1$, Let θ be the angle between this radius and positive x-axis and let $\{z_i\}_{i=1}^n$ be the partition of this radius. So

$$\begin{aligned}
\sum_{i=1}^n |f(z_i) - f(z_{i-1})| &= \sum_{i=1}^n |\sum_{k=0}^\infty a_k z_i^k - \sum_{k=0}^\infty a_k z_{i-1}^k| \\
&= \sum_{i=1}^n |\sum_{k=0}^\infty a_k (z_i^k - z_{i-1}^k)| \\
&\leq \sum_{i=1}^n \sum_{k=0}^\infty |a_k||(z_i^k - z_{i-1}^k)| \\
&\leq \sum_{i=1}^n \sum_{k=0}^\infty |a_k|(|z_i|^k + |z_{i-1}|^k) \\
&\leq \sum_{i=1}^n \sum_{k=0}^\infty 2|a_k| \\
&< +\infty.
\end{aligned}$$

Then $f(z)$ is of bounded variation on every radius of the circle $|z| = 1$.

Chapter 3

Lebesgue Measure and Outer Measure

Exercise 3.1

Exercise 3.2

Exercise 3.3

Exercise 3.4

Exercise 3.5

Exercise 3.6

Exercise 3.7 *Prove* (3.15)*, that is if* $\{I_k\}_{k=1}^N$ *is a finite collection of nonoverlapping intervals, then* $\bigcup I_k$ *is measurable and* $|\bigcup I_k| = \sum |I_k|$*.*

Solution. We assume that is in $\mathbb{R}^n$. Let

$$I_k = \{(x_1, x_2, \cdots, x_n) \mid a_{ki} \le x_i \le b_{ki} \text{ for all } i\},$$

and let

$$I_k' = \{(x_1, x_2, \cdots, x_n) \mid a_{ki} - \varepsilon < x_i < b_{ki} + \varepsilon \text{ for all } i\}$$

for every $\varepsilon > 0$. Then

$$|\bigcup_{k=1}^N I_k' - \bigcup_{k=1}^N I_k|_e \le |\bigcup_{k=1}^N (I_k' - I_k)|_e \le \sum_{k=1}^N |I_k' - I_k|_e < \sum_{k=1}^N \varepsilon M_k = \varepsilon M$$

for some M_k and which $M = \sum_{k=1}^N M_k$. Since ε is every element in $\mathbb{R}^+$, then $|\bigcup I_k' - \bigcup I_k|_e < \varepsilon$. That is $\bigcup I_k$ is measurable. Next, the $|\bigcup I_k| \le \sum |I_k|$ by Theorem 3.4. Let I_k^o is the set of interior points of I_k. We choose $I_k^* \subset I_k^o$ such that $|I_k^*|_e > |I_k^o| - \frac{\varepsilon}{N}$ for all k. Since I_k^* is compact for all k and $I_i^* \cap I_j^* = \phi$ for all $i \ne j$, the $d(I_i^*, I_j^*) > 0$. So we have

$$|\bigcup_{k=1}^N I_k^*| = \sum |I_k^*|.$$

Hence,

$$\sum_{k=1}^{N}|I_k| \le \sum_{k=1}^{N}(|I_k^*| + \frac{\varepsilon}{N}) = \sum_{k=1}^{N}|I_k^*| + \varepsilon = |\bigcup_{k=1}^{N} I_k^*| + \varepsilon \le |\bigcup_{k=1}^{N} I_k| + \varepsilon.$$

Since ε is every element in $\mathbb{R}^+$, then $\sum |I_k| \le |\bigcup I_k|$. Therefore, $|\bigcup I_k| = \sum |I_k|$.

Exercise 3.8 *Show that the Borel σ-algebra $\mathscr{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.*

Solution. Since F^c is open for every closed set F, then $F^c \in \mathscr{B}$. This implies $F \in \mathscr{B}$. Let $\mathscr{A}$ be a σ-algebra containing all closed sets in $\mathbb{R}^n$. Since G^c is closed for every open set G, then $G^c \in \mathscr{A}$. This implies $G \in \mathscr{A}$. Hence $\mathscr{A}$ is containing all open sets in $\mathbb{R}^n$. Since $\mathscr{B}$ is the smallest σ-algebra containing the open sets in $\mathbb{R}^n$. So $\mathscr{B} \subseteq \mathscr{A}$. Then $\mathscr{B}$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.

Exercise 3.9 *If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ (and so also $\liminf E_k$) has measure zero.*

Solution. Since $\sum |E_k|_e < +\infty$, for every $\varepsilon > 0$, there exist $M > 0$ such that

$$\sum_{k=M}^{\infty} |E_k|_e < \varepsilon.$$

Then

$$|\limsup E_k|_e = |\bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} E_k|_e \le |\bigcup_{k=M}^{\infty} E_k|_e \le \sum_{k=M}^{\infty} |E_k|_e < \varepsilon.$$

So $\limsup E_k$ have measure zero. Since $\liminf E_k \subseteq \limsup E_k$ and $|\limsup E_k| = 0$, then $|\liminf E_k| = 0$.

Exercise 3.10 *If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.*

Solution. The

$$\begin{aligned}
|E_1| + |E_2| &= |(E_1 - (E_1 \cap E_2)) \cup (E_1 \cap E_2)| + |E_2| \\
&= |(E_1 - (E_1 \cap E_2))| + |E_1 \cap E_2| + |E_2| \\
&= |(E_1 - (E_1 \cap E_2)) \cup E_2| + |E_1 \cap E_2| \\
&= |E_1 \cup E_2| + |E_1 \cap E_2|.
\end{aligned}$$

Exercise 3.11 *Prove* (3.29). *That is suppose that $|E|_e < +\infty$. Then E is measurable if and only if given $\varepsilon > 0$, $E = (S \cup N_1) - N_2$, where S is a finite union of nonoverlapping intervals and $|N_1|_e$, $|N_2|_e < \varepsilon$.*

Solution. First, we suppose that E is measurable. Given every $\varepsilon > 0$, there exists an open set G with $E \subset G$ such that $|G - E| < \varepsilon$, and we have $\{I_k\}$ is countable nonoverlapping such that $\bigcup I_k = G$. Hence $\sum |I_k|_e$ is convergence. There exists $M \in \mathbb{Z}^+$ such that

$$\sum_{k=M}^{\infty} |I_k|_e < \varepsilon.$$

Let

$$S = \bigcup_{k=1}^{M-1} I_k,\ N_1 = \bigcup_{k=M}^{\infty} I_k \text{ and } N_2 = (\bigcup_{k=1}^{\infty} I_k) - E,$$

then

$$|N_1|_e = |\bigcup_{k=M}^{\infty} I_k|_e \leq \sum_{k=M}^{\infty} |I_k|_e < \varepsilon$$

and

$$|N_2|_e = |(\bigcup_{k=1}^{\infty} I_k) - E|_e = |G - E|_e < \varepsilon$$

since E is measurable. The

$$(S \cup N_1) - N_2 = (\bigcup_{k=1}^{M-1} I_k) \cup (\bigcup_{k=M}^{\infty} I_k) - ((\bigcup_{k=1}^{\infty} I_k) - E) = E.$$

Conversely, suppose $E = (S \cup N_1) - N_2 = (S - N_2) \cup (N_1 - N_2)$, where S is a finite union of nonoverlapping intervals and $|N_1|_e,\ |N_2|_e < \frac{\varepsilon}{3}$ for all $\varepsilon > 0$. Let G is open with $G \supset N_2$ such that $|G|_e \leq |N_2|_e + \frac{\varepsilon}{3} < \frac{2\varepsilon}{3}$. And let $F = S - G = S \cap G^c$, then F is closed and $F \subseteq E$. The

$$\begin{aligned}
|E - F|_e &= |E - (S - G)|_e \\
&= |E \cap (S \cap G^c)^c|_e \\
&= |E \cap (S^c \cup G)|_e \\
&= |(E \cap S^c) \cup (E \cap G)|_e \\
&= |(E - S) \cup (E \cap G)|_e \\
&\leq |E - S| + |E \cap G|_e \\
&\leq |N_1|_e + |G|_e \\
&< \varepsilon.
\end{aligned}$$

So E is a measurable set.

Exercise 3.12 *If E_1 and E_2 are measurable subsets of $\mathbb{R}^1$, show that $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1||E_2|$.* (Interpret $0 \cdot \infty$ as 0)(Hint:Use a characterization of measurability.)

Solution. Since E_1 and E_2 are measurable, there exists H_1 and H_2 are of type G_δ such that $|H_1 - E_1| = 0$ and $|H_2 - E_2| = 0$. We can find nonoverlapping intervals $\bigcup_{k=1}^{\infty} I_k \supset H_1 - E_1$ such that $\sum_{k=1}^{\infty} |I_k| < \varepsilon$. Let $H_k' = H_2 \cap B_k$ and $\bigcup_{j=1}^{\infty} I_{k,j}' \supset H_k'$ with $\sum_j |I_{k,j}'| < |H_k'| + \frac{\varepsilon}{2^k}$, then $\bigcup_{k=1}^{\infty} H_k' = H_2$ and $|H_k'| < \infty$.

Thus

$$\begin{aligned}
|(H_1 - E_1) \times H_2|_e &\leq |\bigcup_{k=1}^{\infty} I_k \times \bigcup_{k'=1}^{\infty} H'_{k'}| \\
&= |\bigcup_{k=1}^{\infty} \bigcup_{k'=1}^{\infty} I_k \times H'_{k'}| \\
&\leq \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} |I_k \times H'_{k'}| \\
&\leq \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} |\bigcup_{j=1}^{\infty} I_k \times I'_{k',j}| \\
&\leq \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \sum_{j=1}^{\infty} |I_k \times I'_{k',j}| \\
&= \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \sum_{j=1}^{\infty} |I_k||I'_{k',j}| \\
&= \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} (|I_k| \sum_{j=1}^{\infty} |I'_{k',j}|) \\
&\leq \sum_{k=1}^{\infty} |I_k| \sum_{k'=1}^{\infty} |H'_{k'}| + \frac{\varepsilon}{2^{k'}} \\
&\leq \varepsilon \sum_{k'=1}^{\infty} |H'_{k'}| + \frac{\varepsilon^2}{2^{k'}}
\end{aligned}$$

Then $|(H_1 - E_1) \times H_2|_e \leq 0 \sum_{k'=1}^{\infty} |H'_{k'}| = 0$ as $\varepsilon \to 0$. Similarly, $H_1 \times (H_2 - E_2) \cup (H_1 - E_1) \times H_2$ is measurable. So

$$E_1 \times E_2 = H_1 \times H_2 - (H_1 \times (H_2 - E_2) \cup (H_1 - E_1) \times H_2)$$

is measurable.

Exercise 3.13 *Motivated by (3.7), define the inner measure of E by $|E|_i = \sup |F|$, where the supremum is taken over all closed subsets F of E. Show that*

(i) $|E|_i \leq |E|_e$.

(ii) *If $|E|_e < \infty$, then E is measurable if and only if $|E|_i = |E|_e$.* [Use (3.22)]

Solution.

(i) If $|E|_e = +\infty$, then $|E|_i \leq |E|_e$. If $|E|_e < +\infty$, for all $F \subseteq E$ be closed set, the $|F| \leq |E|_e$. So $|E|_i = \sup\{|F| \mid F \subseteq E \textit{ and } F \textit{ is closed}\} \leq |E|_e$.

(ii) First, we suppose that E is measurable. That is for any $\varepsilon > 0$, there exist $F \subseteq E$ is close set such that $|E - F|_e < \varepsilon$. Since E and F are measurable, then $|E|_e - |F|_e = |E - F|_e$. The

$$|E|_i = \sup\{|F| \mid F \subseteq E \text{ is closed}\} \geq |F| = |F|_e > |E|_e - \varepsilon.$$

Hence $|E|_i \geq |E|_e$. This implies $|E|_i = |E|_e$ by (i). Conversely, for every $\varepsilon > 0$, there exist F such that $|F| > |E|_i - \varepsilon = |E|_e - \varepsilon$. Then

$$|E - F|_e = |E|_e - |F| < \varepsilon$$

by (3.31). So E is measurable.

Exercise 3.14 *Show that the conclusion of part* (ii) *of* Exercise 13 *is false if* $|E|_e = +\infty$.

Solution. Let $E \subset [0, 1]$ be a nonmeasurable set. Then $\mathbb{R} - E$ is nonmeasurable set. Since $[2, \infty) \subset \mathbb{R} - E$, then $|\mathbb{R} - E|_e = |\mathbb{R} - E|_i = \infty$.

Exercise 3.15

Exercise 3.16

Exercise 3.17 *Give an example which shows that the image of a measurable set under a continuous transformation may not be measurable.* (*Consider the Cantor-Lebesque function and the pre-image of an appropriate nonmeasurable subset of its range.*)

Solution. Let f be Cantor-Lebesque function and C be Cantor set. We claim that $f(C) = [0, 1]$. For all $x \in C$, let

$$x = \sum_{k=1}^{\infty} c_k 3^{-k}, \ c_i = 0 \text{ or } 2 \text{ for all } i.$$

Then

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2} c_k 2^{-k} \in [0, 1]$$

since $c_k = 0$ or 2. Hence $f(C) \subseteq [0, 1]$. Conversely, for every $y \in [0, 1]$, let

$$y = \sum_{k=1}^{\infty} a_k 2^{-k}$$

where $a_k = 0$ or 1. Let

$$x = \sum_{k=1}^{\infty} 2a_k 3^{-k} \in C$$

since $2a_k = 0$ or 2. Then

$$f(x) = f(\sum_{k=1}^{\infty} 2a_k 3^{-k}) = \sum_{k=1}^{\infty} a_k 2^{-k} = y.$$

This implies $f(C) = [0, 1]$. Since $|[0, 1]| = 1 > 0$, there exists $B \subseteq [0, 1]$ such that B is nonmeasurable set. Let

$$A = \{x \in C \mid f(x) \in B\},$$

then $A \subseteq C$. A is measure zero since C is measure zero. So f is continuous and $f(A) = B$ where A is measurable set and B is nonmeasurable set.

Exercise 3.18

Exercise 3.19 *Carry out the details of the construction of a nonmeasurable subset of* $\mathbb{R}^n$*,* $n > 1$*.*

Solution. For every $x, y \in \mathbb{R}^n$, write $x \sim y$ if and only if $x - y = r$ for some $r \in \mathbb{Q}^n$. For every $x, y, z \in \mathbb{R}^n$, $x \sim x$ since $x - x = 0 \in \mathbb{Q}^n$. If $x \sim y$, that is $x - y = r$ for some $r \in \mathbb{Q}^n$. Then $y - x = -r \in \mathbb{Q}^n$ implies $y \sim x$. And If $x \sim y$ and $y \sim z$, that is $x - y = r_1$ and $y - z = r_2$ for some $r_1, r_2 \in \mathbb{Q}^n$. Then $x - z = r_1 + r_2 \in \mathbb{Q}^n$ implies $X \sim Z$. Hence $\sim$ is an equivalence relation. Let E be a subset of $\prod_{k=1}^n (0,1)$ containing only one element in every equivalence class. Suppose that E is measurable. Let $|E| = \alpha$ and $E + r = \{x + r \mid x \in E\}$. For every $r \neq s$, the $(E + r) \cap (E + s) = \phi$ since we choose only one element in every equivalence class. For all $x \in \prod_{k=1}^n (0,1)$, there exists $y \in E$ such that $x - y \in \mathbb{Q}^n$. Let $r = x - y$, then $x = y + r \in E + r$ and $r \in \prod_{k=1}^n (-1,1)$. Hence

$$\prod_{k=1}^{n} (0,1) \subseteq \bigcup_{k=1}^{\infty} E + r_k$$

where $\{r_k\} = \prod_{k=1}^n (0,1) \cap \mathbb{Q}^n$. Then

$$1 = |\prod_{k=1}^{n} (0,1)| \leq |\bigcup_{k=1}^{\infty} E + r_k| = \sum_{k=1}^{\infty} |E + r_k| = \alpha \cdot \infty.$$

But

$$\bigcup_{k=1}^{\infty} E + r_k \subseteq \prod_{k=1}^{n} (-1,2).$$

This implies $\alpha = 0$. That is a contradiction since $1 \leq 0 \cdot \infty = 0$. So E is nonmeasurable.

Exercise 3.20 *Show that there exist disjoint* $E_1, E_2, \cdots, E_k, \cdots$ *such that* $|\bigcup E_k|_e < \sum |E_k|_e$ *with strict inequality. (Let* E *be a nonmeasurable subset of* $[0,1]$ *whose rational translate are disjoint. Consider the translate of* E *by all rational numbers* r*,* $0 < r < 1$*, and use* Exercise 18.)

Exercise 3.21 *Show that there exist sets* $E_1, E_2, \cdots, E_k, \cdots$ *such that* $E_k \searrow E$*,* $|E_k|_e < +\infty$*, and* $\lim_{k\to\infty} |E_k|_e > |E|_e$ *with strict inequality.*

Solution. Take $A \subset [0,1]$ be the nonmeasurable set of 3.39, then $|A|_e = \alpha > 0$. Let $\{r_i\}_{i=1}^{\infty} = \mathbb{Q} \cap [0,1]$, $A_{r_i} = \{a + r_i \mid a \in A\}$ and

$$E_k = \bigcup_{i=k}^{\infty} A_{r_i}.$$

Since $A_{r_i} \cap A_{r_j} = \phi$ for $i \neq j$, then $E_k \searrow \phi$. And $E_k \subset [0,2]$ implies

$$|E_k|_e \leq 2 < +\infty.$$

Hence

$$|E_k|_e = |\bigcup_{i=k}^{\infty} A_{r_i}|_e \geq |A_{r_k}|_e = |A|_e = \alpha > 0$$

for all k. But $|\phi|_e$=0.

Exercise 3.22

Exercise 3.23 *Let Z be a subset of $\mathbb{R}^1$ with measure zero. Show that the set $\{x^2 : x \in Z\}$ also has measure zero.*

Solution. Let $\mathbb{R}^1 = \bigcup_{k=1}^{\infty}[-k,k]$, and let $Z_k = [-k,k] \cap Z \subseteq Z$ and $|x^2 - y^2| = |x+y||x-y| \leq 2k|x-y|$, then $f(x) = x^2$ on $[-k,k]$ is Lipschitz transformation. So $f(Z_k)$ is measure zero. Hence

$$|f(Z)| = |f(\bigcup_{k=1}^{\infty} Z_k)| = |\bigcup_{k=1}^{\infty} f(Z_k)| \leq \sum_{k=1}^{\infty} |f(Z_k)| = 0.$$

So $f(Z)$ is measure zero.

Exercise 3.24

Exercise 3.25 *Construct a measurable subset E of $[0,1]$ such that for every subinterval I, both $E \cap I$ and $I - E$ have positive measure.* [Take a Cantor-type subset of $[0,1]$ with positive measure (see Exercise 5), and on each subinterval of the complement of this set, construct another such set, and so on. The measures can be arranged so that the union of all the sets has the desired property.]

Solution. Take the Cantor-type subset of $[0,1]$ with positive measure in Exercise 5, and on each subinterval of the complement of this set, construct another such set, and so on. Let E be this set. Then for every interval $I \subseteq [0,1]$, the set $I \cap E \neq \phi$. Let $\alpha = \inf I \cap E$ and $\beta = \sup I \cap E$, then $[\alpha,\beta]$ is not a subset of E. So we was removed an subinterval I' of $[\alpha,\beta]$ with $|I'| \lneqq |[\alpha,\beta]|$ for some step. By hypothesis, we construct another Cantor-type subset of I'. Then

$$|I \cap E| \geq |[\alpha,\beta] \cap E| \gneqq |I' \cap E| > 0$$

and

$$|I - E| = |I \cap E^c| \gneqq |I' \cap E^c| > 0.$$

So the set E satisfies $E \cap I$ and $I - E$ have positive measure for every subinterval $I \subseteq [0,1]$.

Exercise 3.26

Exercise 3.27

Chapter 4

Lebesgue Measure Functions

Exercise 4.1 *Prove that*

(4.2) *If f is measurable, then $\{f > -\infty\}$, $\{f < +\infty\}$, $\{f = +\infty\}$, $\{a \leq f \leq b\}$, $\{f = a\}$, etc., are all measurable. Moreover, f is measurable if and only if $\{a < f < +\infty\}$ is measurable for every finite a.*

(4.8) *If f is measurable and λ is any real number, then $f + \lambda$ and λf are measurable.*

Solution.

(4.2) (i) The set

$$\{f > -\infty\} = \bigcup_{k=1}^{\infty} \{f > -k\}.$$

Since $\{f > -k\}$ is measurable for all k, then $\{f > -\infty\}$ is measurable.

(ii) The set

$$\{f < +\infty\} = E - \bigcap_{k=1}^{\infty} \{f > k\}.$$

So the set $\{f < +\infty\}$ is measurable since the sets E and $\{f > k\}$ for all k are measurable.

(iii) The set

$$\{f = +\infty\} = \bigcap_{k=1}^{\infty} \{f > k\}.$$

Since the set $\{f > k\}$ for all k is measurable, then the set $\{f = +\infty\}$ is measurable.

(iv) The set

$$\{a \leq f \leq b\} = \bigcap_{k=1}^{\infty} \{f > a - \frac{1}{k}\} - \bigcup_{k=1}^{\infty} \{f > b + \frac{1}{k}\}.$$

Then $\{a \le f \le b\}$ is measurable since $\{f > a - \frac{1}{k}\}$ and $\{f > b + \frac{1}{k}\}$ are measurable for all k.

(v) The set

$$\{f = a\} = \bigcap_{k=1}^{\infty}\{f > a - \frac{1}{k}\} - \bigcup_{k=1}^{\infty}\{f > a + \frac{1}{k}\}.$$

Then $\{f = a\}$ is measurable since $\{f > a - \frac{1}{k}\}$ and $\{f > a + \frac{1}{k}\}$ are measurable for all k.

(vi) The set

$$\{a < f < +\infty\} = \{f > a\} - \bigcap_{k=1}^{\infty}\{f > k\}.$$

Then $\{a < f < +\infty\}$ is measurable since $\{f > a\}$ and $\{f > k\}$ are measurable for all k. Conversely, for all a,

$$\{f > a\} = \{a < f < +\infty\} \cup (E - \bigcup_{k=1}^{\infty}\{-k < f < +\infty\} \cup \{f = -\infty\}).$$

Then the set $\{f > a\}$ is measurable since $\{a < f < +\infty\}$, E, $\{-k < f < +\infty\}$ and $\{f = -\infty\}$ are measurable for all k. That is f is measurable.

(4.8) Given any $a \in (-\infty, \infty)$, then $\{f + \lambda > a\} = \{f > a - \lambda\}$ is measurable since f is measurable. If $\lambda > 0$, then $\{\lambda f > a\} = \{f > \frac{a}{\lambda}\}$ is measurable since f is measurable. If $\lambda < 0$, then $\{\lambda f > a\} = \{f < \frac{a}{\lambda}\}$ is measurable since f is measurable. If $\lambda = 0$, then $\{\lambda f > a\} = \{f_0 > a\} = E$ or ϕ where f_0 be a zero function. Hence λf is measurable.

Exercise 4.2 *Let f be a simple function, taking its distinct values on disjoint sets $E_1, \ldots, E_N$. Show that f is measurable if and only if $E_1, \ldots, E_N$ are measurable.*

Solution. First, suppose that f is measurable. We may assume $a_1 < a_2 < \cdots < a_N$ which $a_i = f(x)$ for $x \in E_i$. Let

$$\varepsilon = \frac{1}{2}\min\{a_2 - a_1, a_3 - a_2, \cdots, a_N - a_{N-1}\}.$$

For all i, the set

$$\begin{aligned} E_i &= \{x \mid f(x) = a_i\} \\ &= \{x \mid f(x) > a_i - \varepsilon\} - \{x \mid f(x) > a_i + \varepsilon\}. \end{aligned}$$

Then E_i is measurable since f is measurable. Conversely, for all $a \in \mathbb{R}$, let $a_1 < a_2 < \cdots \le a_{i_0-1} < a < a_{i_0} < a_{i_0+1} < \cdots < a_N$. Then

$$\begin{aligned} \{x \mid f(x) > a\} &= \{x \mid f(x) \in \{a_{i_0}, a_{i_0+1}, \cdots, a_N\}\} \\ &= E_{i_0} \cup E_{i_0+1} \cup \cdots \cup E_N. \end{aligned}$$

Hence $\{x \mid f(x) > a\}$ is measurable. So f is measurable.

Exercise 4.3 *Theorem (4.3) can be used to define measurability for vector-value* (e.g., complex-valued) *functions. Suppose, for example, that f and g are real-valued and defined in $\mathbb{R}^n$, and let $F(x) = (f(x), g(x))$. Then F is said to be measurable if $F^{-1}(G)$ is measurable for every open $G \subset \mathbb{R}^2$. Prove that F is measurable if and only if both f and g are measurable in $\mathbb{R}^n$.*

Solution. Suppose that F is measurable, let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be projections onto the first and second factors, respectively. These maps are continuous. So $f = \pi_1 \circ F$ and $g = \pi_2 \circ F$ are measurable. Conversely, let $[a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$, then the inverse image

$$F^{-1}([a_1, b_1] \times [a_2, b_2]) = f^{-1}([a_1, b_1]) \cap g^{-1}([a_2, b_2]).$$

Since $f^{-1}([a_1, b_1])$ and $g^{-1}([a_2, b_2])$ are measurable, then $F^{-1}([a_1, b_1] \times [a_2, b_2])$ is measurable. For every open subset G of $\mathbb{R}^2$, let $G = \bigcup_{k=1}^{\infty} I_k$ be written as a countable union of nonoverlapping cubes, then

$$F^{-1}(G) = \bigcup_{k=1}^{\infty} F^{-1}(I_k).$$

Hence $F^{-1}(G)$ is measurable. So F is measurable.

Exercise 4.4 *Let f be defined and measurable in $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(Tx)$ is measurable.* [If $E_1 = \{x : f(x) > a\}$ and $E_2 = \{x : f(Tx) > a\}$, show that $E_2 = T^{-1}E_1$.]

Solution. Now, we show that $E_2 = T^{-1}E_1$. For every $x \in E_2$, there exists y such that $Tx = y$, then $y \in E_1$. Hence $x \in T^{-1}E_1$. Conversely, let $x \in T^{-1}E_1$, there exists $y' \in E_1$ such that $x = T^{-1}y'$. Then $x \in E_1$. So $E_2 = T^{-1}E_1$. Since E_1 is measurable and T^{-1} is linear transformation, then E_2 is measurable. Thus that $f(Tx)$ is measurable.

Exercise 4.5 *Give an example to show that $\phi(f(x))$ may not be measurable if ϕ and f are measurable.* (Let F be the Cantor-Lebesgue function and let f be its inverse, suitably defined. Let ϕ be the characteristic function of a set of measure zero whose image under F is not measurable.)

Solution. Let C be Cantor set and let g be Cantor-Lebesgue function and $h : [0, 1] \to [0, 2]$ with $h(x) = g(x) + x$. Since h is one-to-one and onto function, let $f : [0, 2] \to [0, 1]$ be the inverse function of h. Use the proof of Exercise 3.17, there exists $A \subseteq C$ such that $h(A) = B$ where $B \subset [0, 2]$ be nonmeasurable set. We let $\phi : [0, 1] \to [0, 2]$ be the function with

$$\phi(x) = \begin{cases} h(x) & \text{if } x \in A, \\ 0 & \text{if } x \in [0, 1] - A. \end{cases}$$

The function f is continuous since h is one-to-one, onto and continuous function and $[0, 1]$ is compact set. This implies f is a measurable function. Thus for any a, the set $\{x \in [0, 1] \mid \phi(x) > a\} \subseteq C$ or $\{x \in [0, 1] \mid \phi(x) > a\} = [0, 1]$. So the set is measurable. Hence ϕ is also measurable. But

$$\begin{aligned} \{x \mid \phi \circ f(x) > 0\} &= \{x \in B \mid x \neq 0\} \\ &= B - \{0\}. \end{aligned}$$

Since B is nonmeasurable set, then $\{x \mid \phi \circ f(x) > 0\}$ is also. That is $\phi \circ f$ is not a measurable function.

Exercise 4.6 *Let f and g be measurable functions on E.*

(a) *If f and g are finite a.e. in E, show that $f+g$ is measurable no matter how we define it at the points when it has the form $+\infty + (-\infty)$ or $-\infty + \infty$.*

(b) *Show that fg is measurable without restriction on the finiteness of f and g. Show that $f+g$ is measurable if it is defined to have the same value at every point where it has the form $+\infty + (-\infty)$ or $-\infty + \infty$.* (Note that a function h defined on E is measurable if and only if both $\{h = +\infty\}$ and $\{h = -\infty\}$ are measurable and the restriction of h to the subset of E where h is finite is measurable.)

Solution.

(a) For all $a \in \mathbb{R}$, let $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$, then

$$\begin{aligned}\{f+g>a, f,g \text{ finite}\} &= \{-\infty < a-g < f < \infty\} \\ &= \bigcup_k \{-\infty < a-g < r_k < f < \infty\} \\ &= \bigcup_k \{r_k < f < \infty\} \cap \{-\infty < a-g < r_k\}.\end{aligned}$$

Thus $\{f+g>a, f,g \text{ finite}\}$ is measurable. Then

$$\{f+g>a\} = \{f+g>a, f,g \text{ finite}\} \cup \{f+g>a,\ f \text{ or } g \text{ infinite}\}$$

Since $\{f+g>a,\ f \text{ or } g \text{ is infinite}\}$ is measurable zero, then $\{f+g>a\}$ is measurable. So $f+g$ is measurable.

(b) (i) For all $a \in \mathbb{R}$, let $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$, then

$$\begin{aligned}\{> fg > a, +\infty > g > 0\} &= \{f > \frac{a}{g}, +\infty > g > 0\} \\ &= \bigcup_k \{f > r_k > \frac{a}{g}, +\infty > g > 0\} \\ &= \bigcup_k \{f > r_k\} \cap \{r_k > \frac{a}{g}, +\infty > g > 0\} \\ &= \bigcup_k \{f > r_k\} \cap \{r_k g > a, +\infty > g > 0\} \\ &= \bigcup_k \{f > r_k\} \cap \{r_k g > a\} \cap \{+\infty > g > 0\},\end{aligned}$$

$$\begin{aligned}\{> fg > a, -\infty < g < 0\} &= \{f < \frac{a}{g}, -\infty < g < 0\} \\ &= \bigcup_k \{f < r_k < \frac{a}{g}, -\infty < g < 0\} \\ &= \bigcup_k \{f < r_k\} \cap \{r_k < \frac{a}{g}, -\infty < g < 0\} \\ &= \bigcup_k \{f < r_k\} \cap \{r_k g > a, -\infty < g < 0\} \\ &= \bigcup_k \{f < r_k\} \cap \{r_k g > a\} \cap \{-\infty < g < 0\}.\end{aligned}$$

The $\{fg > a, g = 0\}$ is equal to ϕ or $\{g = 0\}$. Since $\{fg > a, g = +\infty\} = \phi$, $\{f = 0\} \cap \{g = +\infty\}$ or E, then $\{fg > a, g = +\infty\}$ is measurable. This implies $\{fg > a, g = -\infty\}$ is measurable. Hence $\{fg > a, g > 0\}$, $\{fg > a, g < 0\}$ and $\{fg > a, g = 0\}$ are measurable sets. Thus $\{fg > a\}$ is measurable for all $a \in \mathbb{R}$. So fg is measurable.

(ii) Let $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$, $A = \{f = +\infty\} \cap \{g = -\infty\}$, $B = \{f = -\infty\} \cap \{g = +\infty\}$ and $E = A \cup B$. Then A, B and E are measurable. For every $a \in \mathbb{R}$, then $\{x \in E \mid f + g > a\}$ is measurable since this set is equal to ϕ, A, B or E. Hence

$$\begin{aligned}
\{f+g>a\} &= \{f+g>a, x \notin E\} \cup \{f+g>a, x \in E\} \\
&= (\bigcup_k \{f > r_k > a-g\} \cap E^c) \cup \{f+g>a, x\in E\} \\
&= (\bigcup_k \{f > r_k\} \cap \{r_k > a-g\} \cap E^c) \\
&\quad \cup \{f+g>a, x \in E\}.
\end{aligned}$$

So $\{f + g > a\}$ is measurable implies $f + g$ is measurable.

Exercise 4.7 *Let f be usc and less than $+\infty$ on a compact set E. Show that f is bounded above on E. Show that also f assumes its maximum on E, i.e., that there exists $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.*

Solution. Since f is usc, there exists $\delta_x > 0$ such that for all $y \in B(x, \delta) \cap E$, we have

$$f(y) < f(x) + 1$$

for all $x \in E$. The set E is a compact set, so we have

$$\bigcup_{i=1}^{n} B(x_i, \delta_{x_i}) \supseteq E.$$

Let $M = \max\{f(x_1) + 1, f(x_2) + 1, \ldots, f(x_n) + 1\}$, then f is bounded above by M on E. Now let $\alpha = \sup f(E)$, there exists a sequence $\{f(x_k)\}$ such that $f(x_k) \to \alpha$. Hence it has convergent subsequence $\{x_{k_i}\}$ in $\{x_k\}$. Let $x_{k_i} \to x_0$, then for every $\varepsilon > 0$, there exists $M' > 0$ such that for $i \geq M'$, we have

$$f(x_{k_i}) < f(x_0) + \varepsilon.$$

Thus that

$$\alpha \leq f(x_0) + \varepsilon$$

for any $\varepsilon > 0$. Then $\alpha \leq f(x_0)$. Since $\alpha = \sup f(E)$, then $\alpha = f(x_0)$. So f assumes its maximum on E.

Exercise 4.8 (a) *Let f and g be two functions which are usc at x_0. Show that $f + g$ is usc at x_0. Is $f - g$ usc at x_0? When is fg usc at x_0?*

(b) *If $\{f_k\}$ is a sequence of functions which are usc at x_0, show that* $\inf_k f_k(x)$ *is usc at x_0.*

(c) *If $\{f_k\}$ is a sequence of functions which are usc at x_0 and which converge uniformly near x_0, show that* $\lim f_k$ *is usc at x_0.*

Solution.

(a) (i) The

$$\begin{aligned}\limsup_{x\to x_0;x\in E} f(x)+g(x) &\leq \limsup_{x\to x_0;x\in E} f(x)+\limsup_{x\to x_0;x\in E} g(x)\\ &\leq f(x_0)+g(x_0).\end{aligned}$$

So $f+g$ is usc at x_0.

(ii) Let $f:[0,1]\to\mathbb{R}$ be a zero function and $g:[0,1]\to\mathbb{R}$ be a function with

$$g(x)=\begin{cases}0 & \text{if } x\in[0,\frac{1}{2}],\\ -1 & \text{if } x\in(\frac{1}{2},1].\end{cases}$$

Then f and g are usc at $\frac{1}{2}$. The function $f-g$ with

$$(f-g)(x)=\begin{cases}0 & \text{if } x\in[0,\frac{1}{2}],\\ 1 & \text{if } x\in(\frac{1}{2},1].\end{cases}$$

Thus $f-g$ is not usc at $\frac{1}{2}$.

(iii) Suppose there exist $\delta_0>0$ such that $f(x)\geq 0$ and $g(x)\geq 0$ for all $x\in B'(x_0,\delta_0)$. Since f and g are usc at x_0, for every $\varepsilon>0$, let $\varepsilon'=\min\{\varepsilon,1\}$, there exist $\delta_1>0$ such that for all $x\in B'(x_0,\delta_1)\cap E$, we have

$$f(x)<f(x_0)+\frac{\varepsilon'}{3(g(x_0)+1)}$$

and

$$g(x)<g(x_0)+\frac{\varepsilon'}{3(f(x_0)+1)}.$$

Let $\delta'=\min\{\delta_0,\delta_1\}$, then for all $\delta<\delta'$ such that $x\in B'(x_0,\delta)$, we have

$$\begin{aligned}f(x)g(x) &< (f(x_0)+\frac{\varepsilon'}{g(x_0)+1})(g(x_0)+\frac{\varepsilon'}{f(x_0)+1})\\ &< f(x_0)g(x_0)+\frac{\varepsilon'}{3}+\frac{\varepsilon'}{3}+\frac{\varepsilon'^2}{9(f(x_0)+1)(g(x_0)+1)}\\ &< f(x_0)g(x_0)+\varepsilon.\end{aligned}$$

Hence

$$\limsup_{x\to x_0;x\in E} f(x)g(x)\leq f(x_0)g(x_0).$$

So fg usc at x_0 when $f\geq 0$ and $g\geq 0$ near x_0.

(b) For every $\varepsilon>0$, there exist k_0 such that $f_{k_0}(x_0)<\inf_k f_k(x_0)+\varepsilon$. Then

$$\begin{aligned}\limsup_{x\to x_0;x\in E}\inf_k f_k(x) &\leq \limsup_{x\to x_0;x\in E} f_{k_0}(x)\\ &\leq f_{k_0}(x_0)\\ &< \inf_k f_k(x_0)+\varepsilon.\end{aligned}$$

Hence

$$\limsup_{x\to x_0;x\in E}\inf_k f_k(x)\leq\inf_k f_k(x_0)$$

So $\inf_k f_k(x)$ is usc at x_0.

(c) Since $\{f_k\}$ converge uniformly near x_0, there exist $\delta_1 > 0$ such that $f_k \to f$ on $B'(x_0, \delta_1)$ converge uniformly. For every $\delta < \delta_1$ and $\varepsilon > 0$, there exist $M > 0$ such that for all $k \geq M$ and $x \in B'(x_0, \delta)$, we have $|f_k(x) - f(x)| < \varepsilon$. This implies $f_k(x) < f(x) + \varepsilon$ and $f(x) < f_k(x) + \varepsilon$. So for $k \geq M$,

$$\sup_{x \in B'(x_0,\delta) \cap E} f(x) \leq \sup_{x \in B'(x_0,\delta) \cap E} f_k(x) + \varepsilon$$

Hence

$$\limsup_{x \to x_0; x \in E} f(x) \leq \limsup_{x \to x_0; x \in E} f_k(x) \leq f_k(x_0)$$

Thus

$$\limsup_{x \to x_0; x \in E} f(x) \leq f(x_0)$$

So $\lim f_k$ is usc at x_0.

Exercise 4.9 (a) *Show that the limit of a decreasing (increasing) sequence of functions usc (lsc) at x_0 is usc (lsc) at x_0. In particular, the limit of a decreasing (increasing) sequence of functions at x_0 is usc (lsc) at x_0.*

(b) *Let f be usc and less than $+\infty$ on $[a, b]$. Show that there exist continuous f_k on $[a, b]$ such that $f_k \searrow f$.*

Solution.

(a) Let $f_k \searrow f$, then

$$\limsup_{x \to x_0; x \in E} f(x) \leq \limsup_{x \to x_0; x \in E} f_k(x) \leq f_k(x_0).$$

Hence

$$\limsup_{x \to x_0; x \in E} f(x) \leq f(x_0).$$

So f is usc at x_0. If $f_k \nearrow f$, then $-f_k \searrow -f$. Thus

$$\limsup_{x \to x_0; x \in E} -f(x) \leq -f(x_0).$$

Hence

$$\liminf_{x \to x_0; x \in E} f(x) \geq f(x_0).$$

So f is lsc at x_0. In particular, if every f_k is continuous at x_0, it follows that $|f(x_0)| < +\infty$ and f is both usc and lsc at x_0. If $f_k \searrow f$, then f is usc at x_0. If $f_k \nearrow f$, then f is lsc at x_0.

(b) First we assume $f \leq 0$ and finite-valued, then let $f_n : [a, b] \to \mathbb{R}$ with

$$f_n(x) = \sup\{f(t) - n|x - t| \mid t \in [a, b]\}.$$

Then for all n,

$$\begin{aligned} f_n(x) &= \sup\{f(t) - n|x - t| \mid t \in [a, b]\} \\ &\geq f(x) - n|x - x| \\ &= f(x) \end{aligned}$$

and for every $\varepsilon > 0$ and $x, y \in [a, b]$ with $|x - y| < \frac{\varepsilon}{n}$,

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |\sup\{f(t) - n|x - t| - f(t) + n|y - t| \mid t \in [a, b]\}| \\ &\leq n|x - y| \\ &< \varepsilon. \end{aligned}$$

For every $\varepsilon > 0$, for each n we can choose $t_n \in [a, b]$ such that

$$f(x) \leq f_n(x) < f(t_n) - n|x - t_n| + \frac{\varepsilon}{2} \leq -n|x - t_n| + \frac{\varepsilon}{2}$$

Then $|x - t_n| \to 0$ as $n \to \infty$. Since f is usc, then $\limsup_{n\to\infty} f(t_n) \leq f(x)$. There is $M > 0$ such that for all $n \geq M$, we have $f(t_n) < f(x) + \varepsilon$. So for $n \geq M$, the

$$\begin{aligned} f_n(x) - f(x) &< f(t_n) - n|x - t_n| + \frac{\varepsilon}{2} - f(x) \\ &\leq f(t_n) + \frac{\varepsilon}{2} - f(x) \\ &< f(x) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} - f(x) \\ &= \varepsilon. \end{aligned}$$

Thus $\{f_n\}$ is decreasing sequence of continuous functions with $f_n \searrow f$. In general f, let $h(x) = -\frac{1}{2} + \arctan x$, $x \in \overline{\mathbb{R}}$, then $h \circ f$ is finite-valued , usc and $f \leq 0$. So we can find continuous functions g_n such that $g_n \searrow h \circ f$. Let $f_n = h^{-1} \circ g_n$, then $f_n \searrow f$.

Exercise 4.10 (a) *If f is defined and continuous on E, show that $\{a < f < b\}$ is relatively open, and that $\{a \leq f \leq b\}$ and $\{f = a\}$ are relatively closed.*

(b) *Let f be a finite function on $\mathbb{R}^n$. Show that f is continuous on $\mathbb{R}^n$ if and only if $f^{-1}(G)$ is open for every open G in $\mathbb{R}^1$, or if and only if $f^{-1}(F)$ is closed for every closed F in $\mathbb{R}^1$.*

Solution.

(a) For any limit point $f(x_0) \in \{a < f < b\}$, there exists $\varepsilon_0 > 0$ such that $B(f(x_0), \varepsilon_0) \cap f(E) \subseteq \{a < f < b\}$, then exists $\delta > 0$ such that for $x \in B(x_0, \delta) \cap E$, we have $f(x) \in B(f(x_0), \varepsilon_0) \cap f(E)$. So $\{a < f < b\}$ is relatively open. Next, let x_0 be any limit point of $\{a \leq f \leq b\}$. Since f is continuous, then $f(x_0) \in [a, b]$. That is $x_0 \in \{a \leq f \leq b\}$. So $\{a \leq f \leq b\}$ is closed. And if $a = b$, then $\{f = a\}$ is closed.

(b) For any limit point $f(x_0) \in G$, there exists $\varepsilon_0 > 0$ such that $B(f(x_0), \varepsilon_0) \cap f(E) \subseteq G$, then exists $\delta > 0$ such that for $x \in B(x_0, \delta) \cap E$, we have $f(x) \in B(f(x_0), \varepsilon_0) \cap f(E)$. So $f^{-1}(G)$ is relatively open. Next, let x_0 be any limit point of $f^{-1}(F)$. Since f is continuous, then $f(x_0) \in F$. That is $x_0 \in f^{-1}(F)$. So $f^{-1}(F)$ is closed.

Exercise 4.11 *Let f be defined on $\mathbb{R}^n$ and let $B(x)$ denote the open ball $\{y : |x - y| < r\}$ with center x and fixed radius r. Show that the function $g(x) = \sup\{f(y) : y \in B(x)\}$ is lsc and that the function $h(x) = \inf\{f(y) : y \in B(x)\}$ is usc on $\mathbb{R}^n$. Is the same true for the closed ball $\{y : |x - y| \leq r\}$.*

Solution. Let x_0 be limit point of $\mathbb{R}^n$. Since $g(x_0) = \sup\{f(y) : y \in B(x_0)\}$, there exists $x_1 \in B(x_0)$ such that

$$f(x_1) > M$$

for any $M < g(x_0)$. Let $\delta = r - |x_0 - x_1| > 0$, then for every $x \in B(x_0, \delta)$, we have $x_1 \in B(x)$. Hence

$$g(x) = \sup\{f(y) : y \in B(x)\} \geq f(x_1) > M.$$

That is $\liminf_{x \to x_0} g(x) \geq g(x_0)$. So g is lsc. Next, let x_0' be limit point of $\mathbb{R}^n$. Since $h(x_0') = \inf\{f(y) : y \in B(x_0')\}$, there exists $x_1' \in B(x_0')$ such that

$$f(x_1') < M'$$

for any $M' > h(x_0')$. Let $\delta = r - |x_0 - x_1| > 0$, then for every $x \in B(x_0', \delta)$, we have $x_1' \in B(x)$. Hence

$$g(x) = \inf\{f(y) : y \in B(x)\} \leq f(x_1') < M'.$$

That is $\limsup_{x \to x_0'} g(x) \leq g(x_0')$. So g is usc. That is false for the closed ball. For example, let f be a function on $\mathbb{R}^1$ with $f(1) = 1$, $f(2) = 2$ and $f(x) = 0$ as $x \neq 1$ and $x \neq 2$. And let $r = 1$, then $g(1) = 2$ and $\lim_{x \to 1^-} g(x) = 1$. So g is not lsc. Similarly, let f be a function on $\mathbb{R}^1$ with $f(1) = -1$, $f(2) = -2$ and $f(x) = 0$ as $x \neq 1$ and $x \neq 2$. And let $r = 1$, then $h(1) = -2$ and $\lim_{x \to 1^-} h(x) = -1$. So h is not usc.

Exercise 4.12 *If $f(x), x \in \mathbb{R}^1$, is continuous at almost every point of an interval $[a, b]$, show that f is measurable on $[a, b]$. Generalize this to functions defined in $\mathbb{R}^n$.* [For a constructive proof, use the subintervals of a sequence of partitions to defined a sequence of simple measurable functions converging to f a.e. in $[a, b]$. Use (4.12). See also the proof of (5.54) in Chapter 5.]

Solution. Let the set I be an interval of $\mathbb{R}^n$ and the set E be the subset of I with $|I - E| = 0$ such that f is continuous on E. For every open set G, then $G \cap f(E)$ is open in $f(E)$. It follows that $f^{-1}(G) \cap E$ is open in E. There exists a open set G' of $[0, 1]$ such that $G' \cap E = f^{-1}(G) \cap E$. Then

$$\begin{aligned} f^{-1}(G) &= (f^{-1}(G) \cap E) \cup (f^{-1}(G) - E) \\ &= (G' \cap E) \cup (f^{-1}(G) - E) \end{aligned}$$

is measurable since E is measurable and $I - E$ has measure zero. So f is measurable.

Exercise 4.13 *One difficulty encountered in trying to extend the proof of Egorov's Theorem to the continuous parameter case $f_y(x) \to f(x)$ as $y \to y_0$ is showing that the analogues of the sets E_m in* (4.18) *are measurable. This difficulty can often be overcome in individual cases. Suppose, for example, that $f(x, y)$ is defined and continuous in the square $0 \leq x \leq 1$, $0 < y \leq 1$, and that $f(x) = \lim_{y \to 0} f(x, y)$ exists and is finite for x in a measurable subset E of $[0, 1]$. Show that if ε and δ satisfy $0 < \varepsilon, \delta < 1$, the set $E_{\varepsilon\delta} = \{x \in E : |f(x, y) - f(x)| \leq \varepsilon$ for all $y < \delta\}$ is measurable.* [If y_k, $k = 1, 2, \ldots$, is a dense subset of $(0, \delta)$, show that $E_{\varepsilon\delta} = \bigcap_k \{x \in E : |f(x, y_k) - f(x)| \leq \varepsilon\}$.]

Solution. First, we show that $E_{\varepsilon\delta} = \bigcap_k \{x \in E : |f(x, y_k) - f(x)| \le \varepsilon\}$. Take any $x \in E_{\varepsilon\delta}$. That is for every $y < \delta$, we have $|f(x,y) - f(x)| \le \varepsilon$. Hence $x \in \{x \in E : |f(x, y_k) - f(x)| \le \varepsilon\}$ for all k. So $E_{\varepsilon\delta} \subseteq \bigcap_k \{x \in E : |f(x, y_k) - f(x)| \le \varepsilon\}$. Conversely, Let $x \in \bigcap_k \{x \in E : |f(x, y_k) - f(x)| \le \varepsilon\}$, then for all $y < \delta$, there exists a subsequence $\{y_{k_i}\}$ of $\{y_k\}$ such that $y_{k_i} \to y$ since $\{y_k\}$ is dense in $(0, \delta)$. Then

$$\bigcap_i \{x \in E : |f(x, y_{k_i}) - f(x)| \le \varepsilon\} \subseteq E_{\varepsilon\delta}.$$

Hence $E_{\varepsilon\delta} = \bigcap_k \{x \in E : |f(x, y_k) - f(x)| \le \varepsilon\}$. It follows that $E_{\varepsilon\delta}$ is measurable.

Exercise 4.14 *Let $f(x,y)$ be as in* Exercise 13. *Show that given $\varepsilon > 0$, there exists a closed $F \subset E$ with $|E - F| < \varepsilon$ such that $f(x,y)$ converges uniformly for $x \in F$ to $f(x)$ as $y \to 0$.* [Follow the proof of Egorov's Theorem, using the set $E_{\varepsilon,1/m}$ defined in Exercise 13 for the sets E_m of (4.18).]

Solution. Let

$$E_{\varepsilon,\frac{1}{m}} = \{x \in E : |f(x,y) - f(x)| \le \varepsilon \text{ for all } y < \frac{1}{m}\}$$

for $m \in \mathbb{Z}^+$. Since $\lim_{y\to 0} f(x,y) = f(x)$ exists and finite, there exists $M' \in \mathbb{Z}^+$ such that for $y < 1/M'$, we have $|f(x,y) - f(x)| \le \varepsilon$. Then $E_{\varepsilon,1/m} \nearrow E$. By (3.26), then $|E_{\varepsilon,1/m}| \to |E|$. For any $\varepsilon > 0$, there exists $M \in \mathbb{Z}^+$ such that $|E - E_{\varepsilon,1/M}| < \varepsilon 2^{-m-1}$. Since $E_{\varepsilon,1/M}$ is measurable, there exists a closed set F_m such that $F_m \subseteq E_{\varepsilon,1/M}$ and $|E_{\varepsilon,1/M} - F_m| < \varepsilon 2^{-m-1}$. Hence

$$|E - F_m| \le |E - E_{\varepsilon,1/M}| + |E_{\varepsilon,1/M} - F_m| < \varepsilon 2^{-m}$$

Let $F = \bigcap_m F_m$, then

$$|E - F| \le |E - \bigcap_{m=1}^{\infty} F_m| \le |\bigcup_{m=1}^{\infty} (E - F_m)| \le \sum_{m=1}^{\infty} |(E - F_m)| < \varepsilon.$$

And $f(x,y)$ converges uniformly to $f(x)$ on F as $y \to 0$.

Exercise 4.15 *Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable E with $|E| < +\infty$. If $|f_k(x)| \le M_x < +\infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there is a closed $F \subset E$ and a finite M such that $|E - F| < \varepsilon$ and $|f_k(x)| \le M$ for all k and for all $x \in F$.*

Solution. Let

$$E_m = \{x \mid |f_k(x)| < m \text{ for all } k\}.$$

Since for every $x \in E$, we have $|f_k(x)| \le M_x < m_x$ for some $m_x \in \mathbb{Z}^+$ and all k, then $E_m \nearrow E$. It follows that $|E_m| \to |E|$. That is for every $\varepsilon > 0$, there exists m_0 such that $|E - E_{m_0}| = |E| - |E_{m_0}| < \varepsilon/2$. Let $F \subseteq E_{m_0}$ be a closed set with $|E_{m_0} - F| < \frac{\varepsilon}{2}$, then $|E - F| \le |E - E_{m_0}| + |E_{m_0} - F| < \varepsilon$. And for all $x \in F \subseteq E_{m_0}$, then $|f_k(x)| < m_0$ for all k.

Exercise 4.16 *Prove that $f_k \xrightarrow{m} f$ on E if and only if given $\varepsilon > 0$, there exists K such that $|\{|f - f_k| > \varepsilon\}| < \varepsilon$ if $k > K$. Give an analogous Cauchy criterion.*

Solution. First, we suppose that $f_k \xrightarrow{m} f$ on E. That is for every $\varepsilon, \eta > 0$, there exists K such that $|\{|f - f_k| > \varepsilon\}| < \eta$ for all $k \geq K$. So there exists K' such that $|\{|f - f_k| > \varepsilon\}| < \varepsilon$ for all $k > K'$. Conversely, for every $\varepsilon, \eta > 0$, if $\varepsilon \leq \eta$, there exists K_1 such that

$$|\{|f - f_k| > \varepsilon\}| < \varepsilon \leq \eta$$

if $k > K_1$. If $\varepsilon > \eta$, there exists K_2 such that

$$|\{|f - f_k| > \varepsilon\}| \leq |\{|f - f_k| > \eta\}| < \eta$$

if $k > K_2$. So $f_k \xrightarrow{m} f$ on E. Now, we give an analogous Cauchy criterion. We show that $f_k \xrightarrow{m} f$ on E if and only if given $\varepsilon > 0$, there exists K such that $|\{|f_n - f_m| > \varepsilon\}| < \varepsilon$ if $n, m > K$. First, we suppose $f_k \xrightarrow{m} f$ on E. By Theorem 4.23, for every $\varepsilon, \eta > 0$, there exists K such that $|\{|f_n - f_m| > \varepsilon\}| < \eta$ if $n, m \geq K$. Then there exists K' such that $|\{|f_n - f_m| > \varepsilon\}| < \varepsilon$ if $n, m > K'$. Conversely, for every $\varepsilon, \eta > 0$, If $\varepsilon \leq \eta$, there exists K_1 such that

$$|\{|f_n - f_m| > \varepsilon\}| < \varepsilon \leq \eta$$

if $n, m > K_1$. If $\varepsilon > \eta$, there exists K_2 such that

$$|\{|f_n - f_m| > \varepsilon\}| \leq |\{|f_n - f_m| > \eta\}| < \eta$$

if $n, m > K_2$. By Theorem 4.23, then $f_k \xrightarrow{m} f$ on E.

Exercise 4.17 *Suppose that $f_k \xrightarrow{m} f$ and $g_k \xrightarrow{m} g$ on E. Show that $f_k + g_k \xrightarrow{m} f + g$ on E and, if $|E| < +\infty$, that $f_k g_k \xrightarrow{m} fg$ on E. If, in addition, $g_k \to g$ on E, $g \neq 0$ a.e., and $|E| < +\infty$, show that $f_k/g_k \xrightarrow{m} f/g$ on E.* [For the product f_k/g_k, write $f_k g_k - fg = (f_k - f)(g_k - g) + f(g_k - g) + g(f_k - f)$. Consider each term separately, using the fact that a function which is finite on E, $|E| < +\infty$, is bounded outside a subset of E with small measure.]

Solution.

(i) For every $\varepsilon, \eta > 0$, there K such that for all $k \geq K$, we have

$$|\{|f_k - f| > \frac{\varepsilon}{2}\}| < \frac{\eta}{2}$$

and

$$|\{|g_k - g| > \frac{\varepsilon}{2}\}| < \frac{\eta}{2}.$$

Then

$$|\{|f_k + g_k - f - g| > \varepsilon\}| \leq |\{|f_k - f| > \frac{\varepsilon}{2}\}| + |\{|g_k - g| > \frac{\varepsilon}{2}\}| < \eta.$$

So $f_k + g_k \xrightarrow{m} f + g$ on E.

(ii) Since f and g are finite a.e, then $|\{|f| = +\infty\}| = 0$ and $|\{|g| = +\infty\}| = 0$. The

$$\{|f| > n\} = \bigcap_{k=1}^{n} \{|f| > k\} \searrow \bigcap_{k=1}^{\infty} \{|f| > k\} = \{|f| = +\infty\}$$

and $|\{|f|>1\}|<+\infty$ since $|E|<+\infty$. Thus

$$\lim_{n\to\infty}|\{|f|>n\}|=|\{|f|=+\infty\}|=0.$$

Similarly,

$$\lim_{n\to\infty}|\{|g|>n\}|=0.$$

Then for every $\eta>0$, there exists $M>0$ such that

$$|\{|f|>M\}|<\frac{\eta}{8}$$

and

$$|\{|g|>M\}|<\frac{\eta}{8}.$$

Since $f_k \xrightarrow{m} f$ and $g_k \xrightarrow{m} g$ on E, there exists $N>0$ such that for all $k\geq N$, we have

$$|\{|f_k-f|>\frac{\varepsilon}{3M}\}|<\frac{\eta}{8},$$
$$|\{|g_k-g|>\frac{\varepsilon}{3M}\}|<\frac{\eta}{8},$$
$$|\{|f_k-f|>\sqrt{\frac{\varepsilon}{3}}\}|<\frac{\eta}{8}$$

and

$$|\{|g_k-g|>\sqrt{\frac{\varepsilon}{3}}\}|<\frac{\eta}{8}.$$

Then for $k\geq N$,

$$\begin{aligned}
|\{|f(g_k-g)|>\frac{\varepsilon}{3}\}| &\leq |\{|f(g_k-g)|>\frac{\varepsilon}{3},|f|>M\}| \\
&\quad +|\{|f(g_k-g)|>\frac{\varepsilon}{3},0\leq|f|\leq M\}| \\
&< \frac{\eta}{8}+|\{|(g_k-g)|>\frac{\varepsilon}{3M},0\leq|f|\leq M\}| \\
&< \frac{\eta}{4}
\end{aligned}$$

and

$$\begin{aligned}
|\{|g(f_k-f)|>\frac{\varepsilon}{3}\}| &\leq |\{|g(f_k-f)|>\frac{\varepsilon}{3},|g|>M\}| \\
&\quad +|\{|g(f_k-f)|>\frac{\varepsilon}{3},0\leq|g|\leq M\}| \\
&< \frac{\eta}{8}+|\{|(f_k-f)|>\frac{\varepsilon}{3M},0\leq|g|\leq M\}| \\
&< \frac{\eta}{4}.
\end{aligned}$$

Then

$$\begin{aligned}
|\{|f_kg_k-fg|>\varepsilon\}| &\leq |\{|(f_k-f)(g_k-g)|>\frac{\varepsilon}{3}\}|+|\{|f(g_k-g)|>\frac{\varepsilon}{3}\}| \\
&\quad +|\{|g(f_k-f)|>\frac{\varepsilon}{3}\}| \\
&< |\{|(f_k-f)|>\sqrt{\frac{\varepsilon}{3}}\}|+|\{|(g_k-g)|>\sqrt{\frac{\varepsilon}{3}}\}|+\frac{\eta}{4}+\frac{\eta}{4} \\
&< \eta.
\end{aligned}$$

So $f_kg_k \xrightarrow{m} fg$ on E.

(iii) Since $g \neq 0$ a.e. and g is finite a.e., let $Z = \{g = 0\} \cup \{|g| = +\infty\}$, then $|Z| = 0$. Thus $\frac{1}{g_k} \to \frac{1}{g}$ on $E - Z$. Then $\frac{1}{g_k} \to \frac{1}{g}$ a.e.. Since $|E| < +\infty$, then $\frac{1}{g_k} \xrightarrow{m} \frac{1}{g}$ by Theorem 4.21. So $f_k/g_k \xrightarrow{m} f/g$ on E by the above.

Exercise 4.18 *If f is measurable on E, defined $\omega_f(a) = |\{f > a\}|$ for $-\infty < a < +\infty$. If $f_k \nearrow f$, show that $\omega_{f_k} \nearrow \omega_f$. If $f_k \xrightarrow{m} f$, show that $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f.* [For the second part, show that if $f_k \xrightarrow{m} f$, then $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$ and $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$ for every $\varepsilon > 0$.]

Solution. Since $\{f > a\} = \bigcup_{i=1}^{\infty}\{f_i > a\}$ and $\{f_i > a\} \subseteq \{f_{i+1} > a\}$ for all i, then

$$\{f_k > a\} = \bigcup_{i=1}^{k}\{f_i > a\} \nearrow \bigcup_{i=1}^{\infty}\{f_i > a\} = \{f > a\}$$

as $k \to \infty$. Hence $|\{f_k > a\}| \to |\{f > a\}|$ and $|\{f_k > a\}| \leq |\{f_{k+1} > a\}|$ for all k. So $\omega_{f_k} \nearrow \omega_f$. Suppose that $f_k \xrightarrow{m} f$. Let a be a point of continuity of ω_f. Given any $\varepsilon, \eta > 0$, there exists $M_1 > 0$ such that for all $k \geq M_1$, we have

$$\begin{aligned}
|\{f_k > a\}| &\leq |\{f_k > a\} - \{f_k > a\} \cap \{f > a - \varepsilon\}| + |\{f > a - \varepsilon\}| \\
&\leq |\{|f - f_k| > \varepsilon\}| + |\{f > a - \varepsilon\}| \\
&\leq \eta + |\{f > a - \varepsilon\}|.
\end{aligned}$$

That is $\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a - \varepsilon)$. And there exists $M_2 > 0$ such that for all $k \geq M_2$, we have

$$\begin{aligned}
|\{f > a + \varepsilon\}| &\leq |\{f > a + \varepsilon\} - \{f > a + \varepsilon\} \cap \{f_k > a\}| + |\{f_k > a\}| \\
&\leq |\{|f - f_k| > \varepsilon\}| + |\{f_k > a\}| \\
&\leq \eta + |\{f_k > a\}|.
\end{aligned}$$

That is $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a + \varepsilon)$. Since ω_f is continuous at a, then

$$\limsup_{k\to\infty} \omega_{f_k}(a) \leq \lim_{\varepsilon\to 0} \omega_f(a - \varepsilon) = \omega_f(a) = \lim_{\varepsilon\to 0} \omega_f(a + \varepsilon) \leq \liminf_{k\to\infty} \omega_{f_k}(a).$$

This implies that $\lim_{k\to\infty} \omega_{f_k}(a) = \omega_f(a)$. So $\omega_{f_k} \to \omega_f$ at each point of continuity of ω_f.

Exercise 4.19 *Let $f(x, y)$ be a function defined on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ which is continuous in each variable separately. Show that f is a measurable function of (x, y).*

Solution. Let $f_k(x, y) = f(x, \frac{i}{k})$ for any $y \in (\frac{i-1}{k}, \frac{i}{k}]$ for some i and $f_k(x, 0) = f(x, \frac{1}{k})$. For all $a \in \mathbb{R}$, the

$$\begin{aligned}
\{f > a\} &= \bigcap_{k=1}^{\infty}\{f_k > a\} \\
&= \bigcap_{k=1}^{\infty}\bigcup_{i=1}^{k}\{x \mid f(x, \frac{i}{k}) > a\} \times ([0, \frac{i}{k}] - [0, \frac{i-1}{k}])
\end{aligned}$$

Since f is continuous in each variable separately, then $\{x \mid f(x, \frac{i}{k}) > a\}$ is measurable for all k, i. This implies $\{f > a\}$ is measurable. So f is measurable.

Exercise 4.20 *If f is measurable on $[a,b]$, show that given $\varepsilon > 0$, there is a continuous g on $[a,b]$ such that $|\{x : f(x) \neq g(x)\}| < \varepsilon$.* (See Exercise 18 of Chapter 1.)

Solution. If f is finite a.e., let $Z = \{|f| = \infty\}$, then $f|_{E-Z}$ is measurable. By Lusin's Theorem, the function $f|_{E-Z}$ has property $\mathscr{C}$. Then for every $\varepsilon > 0$, there is a closed set $F \subset E - Z$ such that $|E - F| = |(E - Z) - F| < \varepsilon$ and f is continuous relative to F. By Exercise 18 of Chapter 1, there is a continuous function g on E which equals f in F. Then $|\{x : f(x) \neq g(x)\}| \leq |E - F| < \varepsilon$.

Chapter 5

The Lebesgue Integral

Exercise 5.1 *If f is a simple measurable function (not necessarily positive) takeing values a_j on E_j, $j = 1, 2, \ldots, N$, show that $\int_E f = \sum_{j=1}^N a_j|E_j|$.* [Use (5.24)]

Solution. By (5.4), the

$$\int_E f^+ = \sum_{j=1}^N a_j^+|E_j|$$

and

$$\int_E f^- = \sum_{j=1}^N a_j^-|E_j|$$

where $a_j^+ = \max\{a_j, 0\}$ and $a_j^- = -\min\{a_j, 0\}$ for all j. We assume that $\int_E f^+$ or $\int_E f^-$ is finite. So $\int_E f$ exists and

$$\begin{aligned}
\int_E f &= \int_E f^+ - \int_E f^- \\
&= \sum_{j=1}^N a_j^+|E_j| - \sum_{j=1}^N a_j^-|E_j| \\
&= \sum_{j=1}^N (a_j^+ - a_j^-)|E_j| \\
&= \sum_{j=1}^N a_j|E_j|.
\end{aligned}$$

Exercise 5.2 *Show that the conclusions of* (5.32) *are not true without the assumption that $\phi \in L(E)$.* [In part (ii), for example, take $f_k = \chi_{(k,+\infty)}$.]

Solution. In part (i), we take $f_k = -\chi_{(k,+\infty)}$ and $\phi = -10$, then $f_k \geq \phi$ for all k and $f_k \nearrow 0$, but $\int_E f_k \to -\infty \neq 0$. In part (ii), we take $f_k = \chi_{(k,+\infty)}$ and $\phi = 10$, then $f_k \leq \phi$ for all k and $f_k \searrow 0$, but $\int_E f_k \to \infty \neq 0$.

Exercise 5.3 *Let $\{f_k\}$ be a sequence of nonnegative measurable functions defined on E. If $f_k \to f$ and $f_k \leq f$ a.e. on E, show that $\int_E f_k \to \int_E f$.*

Solution. We use Fatou's Lemma, the

$$\int_E f = \int_E \liminf_{k\to\infty} f_k \le \liminf_{k\to\infty} \int_E f_k.$$

And since $f_k \le f$ a.e., then $\int_E f_k \le \int_E f$. Hence

$$\int_E f \le \liminf_{k\to\infty} \int_E f_k \le \limsup_{k\to\infty} \int_E f_k \le \int_E f.$$

So $\int_E f_k \to \int_E f$.

Exercise 5.4 *If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots,$ and $\int_0^1 x^k f(x)dx \to 0$.*

Solution. Since $x \in (0,1)$, then $|x^k f(x)| \le |f(x)|$. That is

$$\int_0^1 |x^k f(x)|dx \le \int_0^1 |f(x)|dx < \infty.$$

So $x^k f(x) \in L(0,1)$. Since $f \in L(0,1)$, then f is finite a.e.. This implies that $x^k f(x) \to 0$ a.e. as $k \to \infty$. By Lebesgue's Dominated Convergence Theorem, then $\int_0^1 x^k f(x)dx \to 0$.

Exercise 5.5 *Use Egorov's theorem to prove the bounded convergence theorem. It means let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If $|E| < +\infty$ and there is a finite constant M such that $|f_k| \le M$ a.e. in E, then $\int_E f_k \to \int_E f$.*

Solution. By Egorov's theorem, for any ε, there exists a closed set $F \subseteq E$ such that $\{f_k\}$ converges uniformly on F and $|E - F| < \frac{M\varepsilon}{4}$. Since $|f_k| \le M$ a.e. and $M|E| < \infty$, by Fatou's lemma, we have

$$\begin{aligned}
\int_F f &= \int_F \liminf_{k\to\infty} f_k \\
&\le \liminf_{k\to\infty} \int_F f_k \\
&\le \limsup_{k\to\infty} \int_F f_k \\
&\le \int_F \limsup_{k\to\infty} f_k \\
&= \int_F f.
\end{aligned}$$

Then $\int_F f_k \to \int_F f$. There exists $N > 0$ such that for all $k \ge N$, we have $|\int_F f - \int_F f_k| < \frac{\varepsilon}{2}$. Hence for $k \ge N$, the

$$|\int_E f - \int_E f_k| \le |\int_F f - \int_F f_k| + |\int_{E-F} f| + |\int_{E-F} f_k| < \varepsilon.$$

Then $\int_E f_k \to \int_E f$.

Exercise 5.6 *Let $f(x,y)$, $0 \leq x,y \leq 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $(\partial f(x,y)/\partial x)$ is a bounded function of (x,y). Show that $(\partial f(x,y)/\partial x)$ is a measurable function of y for each x and*

$$\frac{d}{dx}\int_0^1 f(x,y)dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)dy.$$

Solution. For every x, the

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

is measurable since $f(x,y)$ is measurable function of y. Next, the

$$\begin{aligned}
\frac{d}{dx}\int_0^1 f(x,y)dy &= \lim_{h\to 0} \frac{\int_0^1 f(x+h,y)dy - \int_0^1 f(x,y)dy}{h} \\
&= \lim_{h\to 0} \int_0^1 \frac{f(x+h,y) - f(x,y)}{h} dy \\
&= \lim_{h\to 0} \int_0^1 \frac{f(x+h,y) - f(x,y)}{h} dy
\end{aligned}$$

where $0 < h_1 \leq h$. Since

$$\frac{f(x+h,y) - f(x,y)}{h} = \frac{\partial}{\partial x} f(x+h_1,y)$$

which is bounded function of (x,y), then

$$\frac{d}{dx}\int_0^1 f(x,y)dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)dy$$

by Bounbed Convergence Theorem.

Exercise 5.7 *Give an example of an f which is not integrable, but whose improper Riemann integral exists and is finite.*

Solution. Let f be a function on $[1,\infty)$ with $f(x) = (-1)^n \frac{1}{n}$ if $x \in [n, n+1)$ where $n \in \mathbb{Z}^+$. Then

$$\int_{[1,\infty)} f^+ = \sum_{k=1}^{\infty} \frac{1}{2k} |[2k, 2k+1)| = \infty$$

and

$$\int_{[1,\infty)} f^- = \sum_{k=1}^{\infty} \frac{1}{2k-1} |[2k-1, 2k)| = \infty.$$

It implies f is not Lebesgue integrable. But

$$(R)\int_1^{\infty} f = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} < \infty.$$

It follows that f is Riemann integrable.

Exercise 5.8 *Prove* (5.49). *That is for measurable f there is an L^p version of Tchebyshev's inequality*

$$\omega(\alpha) \leq \frac{1}{\alpha^p} \int_{\{f>\alpha\}} f^p, \alpha > 0.$$

Solution. The

$$\frac{1}{\alpha^p} \int_{\{f>\alpha\}} f^p \geq \frac{1}{\alpha^p} \int_{\{f>\alpha\}} \alpha^p = |\{f > a\}| = \omega(\alpha).$$

Then we complete this proof.

Exercise 5.9 *If $p > 0$ and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that $f_k \xrightarrow{m} f$ on* E (and thus that there is a subsequence $f_{k_j} \to f$ a.e. in E).

Solution. For every $\varepsilon, \eta > 0$, there exists $K > 0$ such that for all $k \geq K$, we have

$$\begin{aligned} \varepsilon^p |\{|f - f_k| > \varepsilon\}| &= \int_{\{|f-f_k|>\varepsilon\}} \varepsilon^p \\ &\leq \int_{\{|f-f_k|>\varepsilon\}} |f - f_k|^p \\ &\leq \int_E |f - f_k|^p < \eta\varepsilon^p. \end{aligned}$$

So $f_k \xrightarrow{m} f$ on E.

Exercise 5.10 *If $p > 0$, $\int_E |f - f_k|^p \to 0$, and $\int_E |f_k|^p \leq M$ for all k, show that $\int_E |f|^p \leq M$.*

Solution. Since $\int_E |f - f_k|^p \to 0$, by Exercise 9, the $f_k \xrightarrow{m} f$ on E. Thus that there is a subsequence $f_{k_j} \to f$ a.e. in E. Then $|f_{k_j}|^p \to |f|^p$ a.e.. Let $F = \{|f_{k_j}|^p \to |f|^p\}$. By Fatou's lemma, the

$$\int_E |f|^p = \int_F |f|^p = \int_F \liminf_{k\to\infty} |f_{k_j}|^p \leq \liminf_{k\to\infty} \int_F |f_{k_j}|^p \leq \liminf_{k\to\infty} \int_E |f_{k_j}|^p \leq M.$$

That completes the proof.

Exercise 5.11 *For which $p > 0$ does $1/x \in L^p(0,1)$?$L^p(1,\infty)$?$L^p(0,\infty)$?*

Solution. If $0 < p < 1$, then the integral

$$\int_0^1 x^{-p} dx = \frac{1}{-p+1}$$

which is finite and if $p \geq 1$, then

$$\int_0^1 x^{-p} dx \geq \int_0^1 x^{-1} dx = \infty.$$

If $0 < p < 1$, then

$$\int_1^\infty x^{-p} dx = \lim_{b\to\infty} \int_1^b x^{-p} dx = \lim_{b\to\infty} \frac{b^{1-p}}{-p+1} - \frac{1}{-p+1} = \infty.$$

If $p = 1$ then $\int_1^\infty x^{-1}dx = \infty$ and if $1 < p < \infty$, then

$$\int_1^\infty x^{-p}dx = \lim_{b\to\infty} \frac{b^{1-p}}{-p+1} - \frac{1}{-p+1} = -\frac{1}{-p+1}.$$

For every $0 < p < \infty$, the integral $\int_0^\infty x^{-p}dx$ is infinite. So the $1/x \in L^p(0,1)$ as $0 < p < 1$, the $1/x \in L^p(1,\infty)$ as $1 < p < \infty$. And for every $0 < p < \infty$, the $1/x \notin L^p(0,\infty)$.

Exercise 5.12 *Given an example of a bounded continuous f on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0,\infty)$ for any $p > 0$.*

Solution. Let $f(x) = \frac{1}{\ln(x+2)}$, then f is a bounded continuous on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$. But for any $p > 0$, since $\frac{(\ln(x+2))^p}{x} \to 0$ as $x \to \infty$. There exists $M > 0$ such that $(\ln(x+2))^p < x$ for all $x \geq M$. Then the integral

$$\int_{(0,\infty)} \frac{1}{(\ln(x+2))^p} \geq \int_{(M,\infty)} \frac{1}{(\ln(x+2))^p} \geq \int_{(M,\infty)} \frac{1}{x} = \infty.$$

So $f \notin L^p(0,\infty)$.

Exercise 5.13 (a) *Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum f_k$ converges absolutely a.e. in E if $\sum \int_E |f_k| < +\infty$.* [Use theorems (5.16) and (5.22).]

(b) *If $\{r_k\}$ denotes the rational numbers in $[0,1]$ and $\{a_k\}$ satisfies $\sum |a_k| < \infty$, show that $\sum a_k|x - r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$.*

Solution.

(a) Since $|f_k| \geq 0$ and measurable for every k, then $\int_E \sum |f_k| = \sum \int_E |f_k| < +\infty$ by (5.16). Hence $\sum |f_k| < \infty$ a.e.. So $\sum f_k$ converges absolutely a.e. in E.

(b) The integral

$$\begin{aligned}
\sum \int_0^1 |a_k||x - r_k|^{-\frac{1}{2}}dx &= \sum |a_k| \int_0^1 |x - r_k|^{-\frac{1}{2}}dx \\
&\leq \sum |a_k|(2r_k^{\frac{1}{2}} + 2(1 - r_k)^{\frac{1}{2}})dx \\
&< \infty.
\end{aligned}$$

This implies $\sum a_k|x - r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$ by (a).

Exercise 5.14 *Prove the following result (which is obvious if $|E| < +\infty$), describing the behavior of $a^p\omega(a)$ as $a \to 0+$. If $f \in L^p(E)$, then $\lim_{a\to 0+} a^p\omega(a) = 0$.* (If $f \geq 0$, $\varepsilon > 0$, choose $\delta > 0$ so that $\int_{\{f\leq\delta\}} f^p < \varepsilon$. Thus, $a^p[\omega(a) - \omega(\delta)] \leq \int_{\{a<f\leq\delta\}} f^p < \varepsilon$ for $0 < a < \delta$. Now let $a \to 0$.)

Solution. Since $\bigcap_{k=1}^\infty R(f^p, \{0 \leq f \leq \frac{1}{k}\}) = R(f^p, \{f = 0\})$ and $|R(f^p, \{0 \leq f \leq 1\})| < \infty$, then

$$\int_{\{0\leq f\leq\frac{1}{k}\}} f^p = |R(f^p, \{0 \leq f \leq \frac{1}{k}\})| \to |R(f^p, \{f = 0\})| = 0.$$

There exists k_0 such that $\int_{\{0\le f\le \frac{1}{k_0}\}} f^p < \varepsilon$ for any $\varepsilon > 0$. Thus, for any $a < 1/k_0$, we have

$$a^p[\omega(a) - \omega(\frac{1}{k_0})] \le \int_{\{a<f\le \frac{1}{k_0}\}} f^p < \varepsilon.$$

So $\lim_{a\to 0+} a^p\omega(a) = 0$.

Exercise 5.15 *Suppose that f is nonnegative and measurable on E and that ω is finite on $(0,\infty)$. If $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ is finite, show that $\lim_{a\to 0+} a^p\omega(a) = \lim_{b\to+\infty} b^p\omega(b) = 0$. (Consider $\int_{a/2}^a$ and $\int_{b/2}^b$.)*

Solution. For every a, the integral

$$\int_{\frac{a}{2}}^a \alpha^{p-1}\omega(\alpha)d\alpha \ge (\frac{a}{2})^p\omega(a) \ge 0.$$

Since $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ is finite, then $\int_{\frac{a}{2}}^a \alpha^{p-1}\omega(\alpha)d\alpha \to 0$ as $a \to 0$ and $a \to +\infty$. Hence $\lim_{a\to 0} a^p\omega(a) = 0$ and $\lim_{b\to+\infty} b^p\omega(b) = 0$.

Exercise 5.16 *Suppose that f is nonnegative and measurable on E and that ω is finite on $(0,\infty)$. Show that* (5.51) *holds without any further restrictions* [that is, f need not be in $L^p(E)$ and $|E|$ need not be finite] *if we interpret $\int_0^\infty \alpha^p d\omega(\alpha) = \lim_{\substack{a\to 0^+\\ b\to+\infty}} \int_a^b$.* [Use E_{ab} to obtain the relation $\int_E f^p = -\int_0^\infty \alpha^p d\omega(\alpha)$. If either $\int_0^\infty \alpha^p d\omega(\alpha)$ or $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ is finite, use (5.50) and the results of Exercise 14 or 15 to integrate by parts.]

Solution. Let $E_{ab} = \{a < f(x) \le b\}$, then $|E_{ab}|$ is finite since ω is finite. Then we have $\int_{E_{ab}} f^p = -\int_a^b \alpha^p d\omega(\alpha)$. Thus

$$\int_E f^p = \lim_{\substack{a\to 0^+\\ b\to+\infty}} \int_{E_{ab}} f^p = \lim_{\substack{a\to 0^+\\ b\to+\infty}} -\int_a^b \alpha^p d\omega(\alpha) = -\int_0^\infty \alpha^p d\omega(\alpha).$$

If $\int_0^\infty \alpha^p d\omega(\alpha)$ and $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ are infinite, then the integral $-\int_0^\infty \alpha^p d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$. If $\int_0^\infty \alpha^p d\omega(\alpha)$ is finite, then $f \in L^p(E)$. By (5.50) and Exercise 14, we have

$$\begin{aligned} -\int_0^\infty \alpha^p d\omega(\alpha) &= \lim_{b\to\infty} -b^p\omega(b) + \lim_{a\to 0} a^p\omega(a) + p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha \\ &= p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha. \end{aligned}$$

If $\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$ is finite, by Exercise 15, then $-\int_0^\infty \alpha^p d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha$.

Exercise 5.17 *If $f \ge 0$ and $\omega(\alpha) \le c(1+\alpha)^{-p}$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.*

Solution. If $r = 1$, then $f \in L^1$. If $r \neq 1$, then the integral

$$\begin{aligned}
\int_E f^r &= r\int_0^\infty \alpha^{r-1}\omega(\alpha)d\alpha \\
&\leq r\int_0^\infty \alpha^{r-1}c(1+\alpha)^{-p}d\alpha \\
&= rc\int_0^\infty \frac{\alpha^{r-1}}{(1+\alpha)^p}d\alpha \\
&= rc(\int_0^1 \frac{\alpha^{r-1}}{(1+\alpha)^p}d\alpha + \int_1^\infty \frac{\alpha^{r-1}}{1+\alpha^p}d\alpha) \\
&\leq rc(\int_0^1 \alpha^{r-1}d\alpha + \int_1^\infty \alpha^{r-p-1}d\alpha) \\
&\leq rc(\frac{1}{r} + \frac{1}{r-p}) \\
&< \infty.
\end{aligned}$$

So $f \in L^r$.

Exercise 5.18 *If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k=-\infty}^{+\infty} 2^{kp}\omega(2^k) < +\infty$.* (Use Exercise 16.)

Solution. First, we suppose that $f \in L^p$, then

$$\begin{aligned}
\int_E f^p &= p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha \\
&= p\sum_{k=-\infty}^{+\infty}\int_{2^k}^{2^{k+1}} \alpha^{p-1}\omega(\alpha)d\alpha \\
&\geq p\sum_{k=-\infty}^{+\infty} 2^{k(p-1)}\omega(2^k)2^k \\
&= p\sum_{k=-\infty}^{+\infty} 2^{kp}\omega(2^k).
\end{aligned}$$

This implies $\sum_{k=-\infty}^{+\infty} 2^{kp}\omega(2^k) < +\infty$. Conversely, by Exercise 16, we have

$$\begin{aligned}
\int_E f^p &= p\int_0^\infty \alpha^{p-1}\omega(\alpha)d\alpha \\
&= p\sum_{k=-\infty}^{+\infty}\int_{2^{k-1}}^{2^k} \alpha^{p-1}\omega(\alpha)d\alpha \\
&\leq p\sum_{k=-\infty}^{+\infty} 2^{k(p-1)}\omega(2^k)2^k \\
&= p\sum_{k=-\infty}^{+\infty} 2^{kp}\omega(2^k).
\end{aligned}$$

This implies $f \in L^p$. Then we complete this proof.

Exercise 5.19 *Derive analogues of* (5.52) *and* (5.54) *for integrals over intervals in* $\mathbb{R}^n$*,* $n > 1$*.*

Solution. First, we show that (5.52), Let $\{\Gamma_k\}$ be a sequence of partitions of $I = \prod[a_i, b_i]$ with norm tending to zero. For each k, define two simple functions as follows: if $x_{1j}^{(k)} < x_{2j}^{(k)} < \cdots$ are the partitioning points of Γ_k on the j^{th}-axis, let $l_k(x)$ and u_k be defined in each semiopen interval $\prod[x_{ij}^{(k)}, x_{(i+1)j}^{(k)})$ as the lower and upper bounds of f on $\prod[x_{ij}^{(k)}, x_{(i+1)j}^{(k)}]$, respectively. Then l_k and u_k are uniformly bounded and measurable in $\prod[a_i, b_i)$, and if L_k and U_k denote the lower and upper Riemann sums of f corresponding to Γ_k, we have

$$\int_a^b l_k = L_k, \ \int_a^b u_k = U_k.$$

Note also that $l_k \leq f \leq u_k$ and, if we assume that Γ_{k+1} is a refinement of Γ_k, that $l_k \nearrow$ and $u_k \searrow$. Let $l = \lim_{k\to\infty} l_k$ and $u = \lim_{k\to\infty} u_k$. Then l and u are measurable, $l \leq f \leq u$ and, by the bounded convergence theorem, $L_k \to \int_a^b l$ and $U_k \to \int_a^b u$. But since f is Riemann integrable, L_k and U_k both converge to $(R)\int_a^b f$. Therefore,

$$(R)\int_a^b f = \int_a^b l = \int_a^b u.$$

Since $u - l \geq 0$, (5.11) implies that $l = f = u$ a.e. in I. Therefore, f is measurable and $(R)\int_a^b f = \int_a^b f$, which completes the proof. Now, we show that (5.54), suppose that f is bounded and Riemann integrable. Let Γ_k, l_k, u_k, etc. be as in the proof of above. Let Z be the set of measurable zero outside which $l = f = u$. We claim that if x is not a partitioning point of any Γ_k and if $x \notin Z$, then f is continuous at x. In fact, if f is not continuous at x and x is never a partitioning point, there exists $\varepsilon > 0$, depending on x but not on k, such that $u_k(x) - l_k(x) \geq \varepsilon$. This implies that $u(x) - l(x) \geq \varepsilon$, which is impossible if $x \notin Z$. Therefore, f is continuous a.e. in I. To prove the converse, let f be a bounded function which is continuous a.e. in I. Let $\{\Gamma'_k\}$ be any sequence of partitions with norms tending to zero, and define the corresponding l'_k, u'_k, L'_k and U'_k as in (5.52). Note that $\{l'_k\}$ and $\{u'_k\}$ may not be monotone since Γ'_{k+1} may not be a refinement of Γ'_k. However, by the continuity of f, both l'_k and u'_k converge a.e. to f. Hence, by the bounded convergence theorem, $\int_a^b l'_k$ and $\int_a^b u'_k$ both converge to $\int_a^b f$. Since $L'_k = \int_a^b l'_k$ and $U'_k = \int_a^b u'_k$, it follows that the upper and lower Riemann sums converge to the same limit. Therefore, f is Riemann integrable.

Exercise 5.20 *Let* $y = Tx$ *be a nonsingular linear transformation of* $\mathbb{R}^n$*. If* $\int_E f(y)dy$ *exists, show that*

$$\int_E f(y)dy = |\det T| \int_{T^{-1}E} f(Tx)dx.$$

[The case when $f = \chi_{E_1}$, $E_1 \subset E$, follows from integrating the formula $\chi_{E_1}(Tx) = \chi_{T^{-1}E_1}(x)$ over $T^{-1}E$, and then applying (3.35)]

Solution. If f is nonnegative simple function, let

$$f = \sum_{i=1}^{n} a_i \chi_{E_i}.$$

Then

$$\int_E f(y)dy = \sum_{i=1}^{n} a_i |E_i| = |\det T| \sum_{i=1}^{n} a_i |T^{-1}E_i| = |\det T| \int_{T^{-1}E} f(Tx)dx.$$

If f is nonnegative function, there exists $\{f_k\}$ is a sequence of nonnegative simple function such that $f_k \nearrow f$, then

$$\begin{aligned}
\int_E f(y)dy &= \lim_{k\to\infty} \int_E f_k(y)dy \\
&= |\det T| \lim_{k\to\infty} \int_{T^{-1}E} f_k(Tx)dx \\
&= |\det T| \int_{T^{-1}E} f(Tx)dx.
\end{aligned}$$

In general, $f = f^+ - f^-$, then

$$\begin{aligned}
\int_E f(y)dy &= \int_E f^+(y)dy - \int_E f^-(y)dy \\
&= |\det T| (\int_{T^{-1}E} f^+(Tx)dx - \int_{T^{-1}E} f^-(Tx)dx) \\
&= |\det T| \int_{T^{-1}E} f(Tx)dx.
\end{aligned}$$

This complete the proof.

Exercise 5.21 *If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.*

Solution. For any $k \in \mathbb{Z}^+$, the

$$0 = \int_{\{f > \frac{1}{k}\}} f \geq \int_{\{f > \frac{1}{k}\}} \frac{1}{k} = \frac{1}{k} |\{f > \frac{1}{k}\}| \geq 0$$

and

$$0 = \int_{\{f < -\frac{1}{k}\}} f \leq \int_{\{f < -\frac{1}{k}\}} -\frac{1}{k} = -\frac{1}{k} |\{f < -\frac{1}{k}\}| \leq 0.$$

Then $\{f > \frac{1}{k}\}$ and $|\{f < \frac{1}{k}\}$ are measure zero for all k. It follows that

$$\{f > 0\} \cup \{f < 0\} = \bigcup_{k=1}^{\infty} \{f > \frac{1}{k}\} \cup \{f < -\frac{1}{k}\}$$

is measure zero. So $f = 0$ a.e. in E.

Chapter 6

Repeated Integration

Exercise 6.1 (a) *Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{y : (x, y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero, and that for almost every $y \in \mathbb{R}^1$, $\{x : (x, y) \in E\}$ has measure zero.*

(b) *Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every x.*

Solution.

(a) By Tonelli's theorem, the measure

$$\begin{aligned} |E| &= \iint_{\mathbb{R}^2} \chi_E(x, y)dxdy \\ &= \int_{\mathbb{R}^1} [\int_{\mathbb{R}^1} \chi_E(x, y)dy]dx \\ &= \int_{\mathbb{R}^1} |\{y : (x, y) \in E\}|dx \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} |E| &= \iint_{\mathbb{R}^2} \chi_E(x, y)dxdy \\ &= \int_{\mathbb{R}^1} [\int_{\mathbb{R}^1} \chi_E(x, y)dx]dy \\ &= \int_{\mathbb{R}^1} |\{x : (x, y) \in E\}|dy \\ &= 0 \end{aligned}$$

So $\{x : (x, y) \in E\}$ has measure zero almost every y.

(b) Let $Z = \{(x, y) \mid f(x, y) = \infty\}$, $Z_1 = \{x \mid f(x, y) = \infty\}$ and $Z_2 = \{y \mid f(x, y) = \infty\}$, then $Z = Z_1 \times Z_2$. By Tonelli's theorem, the integral

$$\int_{Z_2} [\int_{Z_1} dx]dy = \iint_Z dxdy = \int_{Z_1} [\int_{Z_2} dy]dx = 0.$$

This implies $\int_{Z_1} dx = 0$ almost every y. That is $\{x \mid f(x,y) = \infty\}$ has measure zero. So $f(x,y)$ is finite for almost every x.

Exercise 6.2 *If f and g are measurable in $\mathbb{R}^n$, show that the function $h(x,y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. Deduce that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$, then their Cartesian product $E_1 \times E_2 = \{(x,y) : x \in E_1, y \in E_2\}$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$, and $|E_1 \times E_2| = |E_1||E_2|$.*

Solution. Let $h_1(x,y) = f(x)$ and $h_2(x,y) = g(y)$, then h_1 and h_2 is measurable in $\mathbb{R}^{2n}$ by (6.15). This implies h is measurable in $\mathbb{R}^{2n}$. By Tonelli's theorem, we have $\chi_{E_1 \times E_2} = \chi_{E_1}\chi_{E_2}$ is measurable and

$$|E_1 \times E_2| = \iint_{\mathbb{R}^{2n}} \chi_{E_1 \times E_2} dxdy = \int_{\mathbb{R}^n} [\int_{\mathbb{R}^n} \chi_{E_1}\chi_{E_2} dx]dy = |E_1||E_2|.$$

Exercise 6.3 *Let f be measurable on $(0,1)$. If $f(x) - f(y)$ is integrable over the square $0 \le x \le 1$, $0 \le y \le 1$, show that $f \in L(0,1)$.*

Solution. By Fubini's theorem, $f(x) - f(y)$ is integrable on $(0,1)$ as a function of y for almost every $x \in (0,1)$. Let $x_0 \in (0,1)$ satisfy $f(x_0) - f(y)$ is integrable on $(0,1)$. This implies $f(y)$ is integrable on $(0,1)$. So $f \in L(0,1)$.

Exercise 6.4 *Let f be measurable and periodic with period 1:$f(t+1) = f(t)$. Suppose that there is a finite c such that*

$$\int_0^1 |f(a+t) - f(b+t)|dt \le c$$

for all a and b. Show that $f \in L(0,1)$. (Set $a = x$, $b = -x$, integrate with respect to x, and make the change variable $\xi = x+t$, $\eta = -x+t$.)

Solution. Let $T(x,t) = (x+t, -x+t)$ and $g(\xi,\eta) = f(\xi) - f(\eta)$, then T is a nonsingular linear transformation. Since $T^{-1}(0,1)^2$ is bounded. Let $T^{-1}(0,1)^2 \subseteq (-M,M)^2$, then

$$\begin{aligned}
\iint_{(0,1)^2} |f(\xi) - f(\eta)| d\xi d\eta &= \iint_{(0,1)^2} |g(\xi,\eta)| d\xi d\eta \\
&= |\det T| \iint_{T^{-1}(0,1)^2} |g(T(x,t))| dxdt \\
&\le |\det T| \iint_{(-M,M)^2} |g(T(x,t))| dxdt \\
&= |\det T| \int_{-M}^{M} [\int_{-M}^{M} |f(x+t) - f(-x+t)| dt] dx \\
&\le 4M^2 |\det T| c.
\end{aligned}$$

That is $f(x) - f(y)$ is integrable over $[0,1] \times [0,1]$. By exercise 3, then $f \in L(0,1)$.

Exercise 6.5 (a) *If f is nonnegative and measurable on E and $\omega(y) = |\{x \in E : f(x) > y\}|$, $y > 0$, use Tonelli's theorem to prove that $\int_E f = \int_0^\infty \omega(y)dy$.* [By definition of the integral, we have $\int_E f = |R(f,E)| = \iint_{R(f,E)} dxdy$. Use the observation in the proof of (6.11) that $\{x \in E : f(x) \ge y\} = \{x : (x,y) \in R(f,E)\}$, and recall that $\omega(y) = |\{x \in E : f(x) \ge y\}|$ unless y is a point of discontinuity of ω.]

(b) *Deduce from this special case the general formula*

$$\int_E f^p = p\int_0^\infty y^{p-1}\omega(y)dy \ (f \geq 0, 0 < p < \infty).$$

Solution.

(a) The integral

$$\begin{aligned}
\int_E f &= |R(f,E)| \\
&= \iint_{R(f,E)} dxdy \\
&= \int_0^\infty [\int_0^\infty \chi_{\{x\in E: f(x)\geq y\}} dx]dy \\
&= \int_0^\infty \omega(y)dy
\end{aligned}$$

Then we complete this proof.

(b) By the result of part (a), the

$$\begin{aligned}
\int_E f^p &= \int_0^\infty |\{f^p > y\}|dy \\
&= \int_0^\infty |\{f > y^{1/p}\}|dy \\
&= \int_0^\infty \omega(y^{1/p})dy \\
&= \int_0^\infty pt^{p-1}\omega(t)dt \\
&= p\int_0^\infty y^{p-1}\omega(y)dy.
\end{aligned}$$

Then we complete this proof.

Exercise 6.6 *For $f \in L(R^1)$, define the Fourier transform $\hat{f}$ of f by*

$$\hat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{ixt}dt \ (x \in \mathbb{R}^1).$$

(For a complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i\int F_1$.) *Show that if f and g belong to $L(\mathbb{R}^1)$, then*

$$(f * g)^\wedge(x) = \hat{f}(x)\hat{g}(x).$$

Solution. The function $(f * g)^\wedge$ is integrable since f and g are integrable. We use Fubini's theorem to show that

$$\begin{aligned}
(f*g)^\wedge(x) &= \int_{-\infty}^{+\infty} f*g(t)e^{ixt}dt \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t-y)*g(y)dye^{ixt}dt \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t-y)g(y)e^{ixt}dydt \\
&= \int_{-\infty}^{+\infty}[\int_{-\infty}^{+\infty} f(t-y)g(y)e^{ixt}dt]dy \\
&= \int_{-\infty}^{+\infty} g(y)[\int_{-\infty}^{+\infty} f(u)e^{ix(u+y)}du]dy \\
&= \int_{-\infty}^{+\infty} g(y)e^{ix(y)}[\int_{-\infty}^{+\infty} f(u)e^{ix(u)}du]dy \\
&= \int_{-\infty}^{+\infty} g(y)e^{ixy}dy\int_{-\infty}^{+\infty} f(u)e^{ixu}du \\
&= \hat{f}(x)\hat{g}(x).
\end{aligned}$$

Then we complete this proof.

Exercise 6.7 *Let F be a closed subset of $\mathbb{R}^1$ and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F, prove that the function*

$$\int_{\mathbb{R}^1} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}}dy$$

is integrable over F, and so is finite a.e. in F. [In case $f = \chi_{(a,b)}$, this reduces to (6.17.)]

Solution. The integral

$$\begin{aligned}
\int_F[\int_{\mathbb{R}^1} \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}}dy]dx &= \int_{\mathbb{R}^1-F}[\int_F \frac{\delta^\lambda(y)f(y)}{|x-y|^{1+\lambda}}dx]dy \\
&= \int_{\mathbb{R}^1-F} \delta^\lambda(y)f(y)[\int_F \frac{1}{|x-y|^{1+\lambda}}dx]dy \\
&\le \int_{\mathbb{R}^1-F} \delta^\lambda(y)f(y)[\int_{\{x:\delta(y)\le|x-y|\}} \frac{1}{|x-y|^{1+\lambda}}dx]dy \\
&= 2\int_{\mathbb{R}^1-F} \delta^\lambda(y)f(y)[\int_{\delta(y)}^{\infty} \frac{1}{t^{1+\lambda}}dt]dy \\
&= 2\lambda^{-1}\int_{\mathbb{R}^1-F} f(y)dy
\end{aligned}$$

is integrable since f is integrable over the complement of F, and so is finite a.e. in F.

Exercise 6.8 *Under the hypotheses of* (6.17) *and assuming that $b-a < 1$,prove that the function*

$$M_0(x) = \int_a^b [\log 1/\delta(y)]^{-1}|x-y|^{-1}dy$$

is finite a.e. in F.

Solution. Since $b-a<1$, then $\log 1/\delta(y)>0$. Hence $M_0(x)$ is nonnegative and the integral $\int_F M_0(x)$ exists. Then

$$\begin{aligned}
&\int_F\Big[\int_a^b [\log 1/\delta(y)]^{-1}|x-y|^{-1}dy\Big]dx \\
=\ & \int_a^b\Big[\int_F [\log 1/\delta(y)]^{-1}|x-y|^{-1}dx\Big]dy \\
\le\ & \int_a^b\Big[\int_{\{x:\delta(y)\le|x-y|\le 1\}} [\log 1/\delta(y)]^{-1}|x-y|^{-1}dx\Big]dy \\
=\ & \int_a^b [\log 1/\delta(y)]^{-1}\Big[\int_{\{x:\delta(y)\le|x-y|\le 1\}} |x-y|^{-1}dx\Big]dy \\
\le\ & \int_a^b [\log 1/\delta(y)]^{-1}\Big[2\int_{\delta(y)}^1 t^{-1}dt\Big]dy \\
=\ & \int_a^b [\log 1/\delta(y)]^{-1}[-2\log\delta(y)]dy \\
=\ & 2(b-a)
\end{aligned}$$

So M_0 is finite a.e. in F.

Exercise 6.9 *Show that $M^\lambda(x;F)=+\infty$ if $x\notin F$, $\lambda>0$.*

Solution. Let $x\notin F$ and $\lambda>0$. For any $\varepsilon>0$, then $\delta(y)\in B(\delta(x),\varepsilon)$ for all $y\in B(x,\varepsilon)$. Thus

$$\begin{aligned}
M^\lambda(x;F) &= \int_a^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}}dy \\
&= \int_a^{x-\varepsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}}dy + \int_{x-\varepsilon}^{x+\varepsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}}dy + \int_{x+\varepsilon}^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}}dy \\
&\ge \int_{x-\varepsilon}^{x+\varepsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}}dy \\
&\ge \int_{x-\varepsilon}^{x+\varepsilon} \frac{\delta^\lambda(x)-\varepsilon}{\varepsilon^{1+\lambda}}dy \\
&= \frac{2\delta^\lambda(x)-2\varepsilon}{\varepsilon^\lambda}
\end{aligned}$$

Let ε_n be a sequence on $(0,1)$ with $\varepsilon_n\to 0$ as $n\to\infty$, then

$$(2\delta^\lambda(x)-2\varepsilon_n)/\varepsilon_n^\lambda\to\infty$$

as $n\to\infty$. So $M^\lambda(x;F)=+\infty$ if $x\notin F$, $\lambda>0$.

Exercise 6.10 *Let v_n be the volume of the unit ball in $\mathbb{R}^n$. Show that by using Fubini's theorem that*

$$v_n = 2v_{n-1}\int_0^1 (1-t^2)^{(n-1)/2}dt.$$

(We also observe that the integral can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$, $s > 0$.]

Solution. Let $v_n[r]$ be the volume of the ball with radius r in $\mathbb{R}^n$. We use Fubini's theorem to show that

$$\begin{aligned}
v_n &= \int\!\!\cdots\!\!\int_{\{x_1^2+\cdots+x_n^2\le 1\}} dx_1\cdots dx_n \\
&= \int_{-1}^{1}[\int\!\!\cdots\!\!\int_{\{x_1^2+\cdots+x_{n-1}^2\le 1-x_n^2\}} dx_1\cdots dx_{n-1}]dx_n \\
&= \int_{-1}^{1} v_{n-1}[\sqrt{1-x_n^2}]dx_n \\
&= 2\int_0^1 v_{n-1}[\sqrt{1-x_n^2}]dx_n \\
&= 2v_{n-1}\int_0^1 (\sqrt{1-x_n^2})^{n-1}dx_n \\
&= 2v_{n-1}\int_0^1 (1-t^2)^{(n-1)/2}dt
\end{aligned}$$

Exercise 6.11 *Use Fubini's theorem to prove that*

$$\int_{\mathbb{R}^n} e^{|x|^2}dx = \pi^{n/2}.$$

[For $n = 1$, write $(\int_{-\infty}^{+\infty} e^{-x^2}dx)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}dxdy$ and use polar coordinates. For $n > 1$, use the formula $e^{|x|^2} = e^{-x_1^2}\cdots e^{-x_n^2}$ and Fubini's theorem to reduce to the case $n = 1$.]

Solution. We use Fubini's theorem to show that

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{|x|^2}dx &= \int\!\!\cdots\!\!\int_{\mathbb{R}^n} e^{-x_1^2}\cdots e^{-x_n^2}dx_1\cdots dx_n \\
&= \int_{\mathbb{R}^1} e^{-x_1^2}dx_1\cdots\int_{\mathbb{R}^1} e^{-x_n^2}dx_n \\
&= \pi^{n/2}.
\end{aligned}$$

Then we complete this proof.

Chapter 7

Differentiation

Exercise 7.1 *Let f be measurable in $\mathbb{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(x) \geq c|x|^{-n}$ for $|x| \geq 1$.*

Solution. Since f is different from zero in some set of positive measure, there exists $\varepsilon > 0$ such that $|E| > 0$ which $E = \{f \geq \varepsilon\}$ and $Q(k) = [-k, k]^n$ such that $0 < |Q(k) \cap E| < \infty$ for k large enough. For any $x \in \mathbb{R}^n$ with $|x| \geq 1$, let Q_x be a cube with center x and edge length $4k|x|$, then

$$\begin{aligned} f^*(x) &= \sup_Q \frac{1}{|Q|} \int_Q |f(y)| dy \\ &\geq \frac{1}{|Q_x|} \int_{Q_x} |f(y)| dy \\ &\geq \frac{1}{|Q_x|} \int_{Q(k) \cap E} |f(y)| dy \\ &\geq \frac{1}{|Q_x|} \int_{Q(k) \cap E} \varepsilon dy \\ &\geq \frac{\varepsilon |Q(k) \cap E|}{|Q_x|} \\ &= \frac{\varepsilon |Q(k) \cap E|}{4^n k^n |x|^n}. \end{aligned}$$

So let $c = \varepsilon |Q(k) \cap E| / 4^n k^n$, then $f^*(x) \geq c|x|^{-n}$ for $|x| \geq 1$.

Exercise 7.2 *Let $\phi(x)$, $x \in \mathbb{R}^n$, be a bounded measurable function such that $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi = 1$. For $\varepsilon > 0$, let $\phi_\varepsilon(x) = \varepsilon^{-n} \phi(x/\varepsilon)$.($\phi_\varepsilon$ is* called an approximation to the identity.) *If $f \in L(\mathbb{R}^n)$, show that*

$$\lim_{\varepsilon \to 0} (f * \phi_\varepsilon)(x) = f(x)$$

in the Lebesgue set of f. [Note that $\int \phi_\varepsilon = 1, \varepsilon > 0$, so that

$$(f * \phi_\varepsilon)(x) - f(x) = \int [f(x - y) - f(x)] \phi_\varepsilon(y) dy.$$

Use (7.16)]

Solution. Let $|\phi(x)| < M$ for all $x \in \mathbb{R}^n$ and some $M > 0$. For any x is in the Lebesgue set of f, the

$$\begin{aligned}
|(f * \phi_\varepsilon)(x) - f(x)| &= |\int_{\mathbb{R}^n} [f(x-y) - f(x)]\phi_\varepsilon(y)dy| \\
&\leq \frac{1}{\varepsilon^n} \int_{|y|<\varepsilon} |f(x-y) - f(x)||\phi(\frac{y}{\varepsilon})|dy \\
&\leq \frac{1}{\varepsilon^n} M \int_{|y|\leq\varepsilon} |f(x-y) - f(x)|dy \\
&= \frac{1}{\varepsilon^n} M \int_{|x-t|\leq\varepsilon} |f(t) - f(x)|dt \\
&\leq 2^n M \frac{1}{(2\varepsilon)^n} \int_{Q(x,2\varepsilon)} |f(t) - f(x)|dt \to 0.
\end{aligned}$$

So $\lim_{\varepsilon\to 0}(f * \phi_\varepsilon)(x) = f(x)$.

Exercise 7.3 *Show that the conclusion of (7.4) remains true for the case of two dimensions if instead of being squares, the sets Q covering E are rectangles with x-dimension equal to h and y-dimension equal to h^2. (Of course, h varies with Q.)*
Show that the same conclusion is valid if the y-dimension is any increasing function of h. Generalize this to higher dimensions.

Solution. For general, let E be a subset of $\mathbb{R}^n$ with $|E|_e < +\infty$, and let K be a collection of rectangles R with x_1-dimension equal to h and x_i-dimension equal to $f_i(h)$ for $i = 1, \dots, n-1$ covering E, where $f_1, f_2, \dots, f_{n-1}$ are increasing functions. We show that there exist a positive constant β, depending only on n, and a finite number of disjoint rectangles $R_1, \dots, R_N$ in K such that

$$\sum_{j=1}^{N} |R_j| \geq \beta |E|_e.$$

We will index the size of a rectangle $R \in K$ by writing $R = R(h)$, where h is the edge length of x_1-dimension. Let $K_1 = K$ and

$$h_1^* = \sup\{h : R = R(h) \in K_1\}.$$

If $h_1^* = +\infty$, then K_1 contains a sequence of rectangles R with $|R| \to +\infty$. In this case, given $\beta > 0$, we simply choose one $R \in K_1$ with $|R| \geq \beta|E|_e$. If $h^* < +\infty$, we can choose $R_1 = R_1(h_1) \in K_1$ such that $h_1 > \frac{1}{2}h_1^*$ and $f_i(h_1) > \frac{1}{2}f_i(h_1^*)$ since f_i is increasing for all i. Now split $K_1 = K_2 \cup K_2'$ where K_2 consists of those rectangles in K_1 which are disjoint from R_1, and K_2' of those which intersect R_1. Let R_1^* denote the rectangle concentric with R_1 whose edge length is 5 times of edge length of R_1. Thus, $|R_1^*| = 5^n|R_1|$, and since $2h_1 > h_1^*$, every rectangle in K_2' is contained in R_1^*. Starting with $j = 2$, continue this selection process for $j = 2, 3, \dots$, by letting

$$h_j^* = \sup\{h : R = R(h) \in K_j\},$$

choosing a rectangle $R_j = R_j(h_j) \in K_j$ with $h_j > \frac{1}{2}h_j^*$ and $f_i(h_j) > \frac{1}{2}f_i(h_j^*)$ since f_i is increasing for all i. Now split $K_j = K_{j+1} \cup K_{j+1}'$, where K_{j+1} consists of all those rectangles of K_j which are disjoint from R_j. If K_{j+1} is empty, the process ends. We have $h_j^* \geq h_{j+1}^*$; moreover, for each j, the $R_1, \ldots, R_j$ are disjoint from one another and from every rectangle in K_{j+1} and every rectangle in K'_{j+1} is contained in the rectangle R_j^* concentric with R_j whose edge length is 5 times of edge length of R_j. Note that $|R_j^*| = 5^n|R_j|$. Consider the sequence $h_1^* \geq h_2^* \geq \cdots$. If some K_{N+1} is empty, then since

$$K_1 = K_2 \cup K'_2 \cdots = K_{N+1} \cup K'_{N+1} \cup \cdots \cup K'_2,$$

and E is covered by the rectangles in K_1, it follows that E is covered by the rectangles in $K'_{N+1} \cup \cdots \cup K'_2$. Hence, $E \subset \bigcup_{j=1}^N R_j^*$, so that

$$|E|_e \leq \sum_{j=1}^N |R_j^*| = 5^n \sum_{j=1}^N |R_j|.$$

This proves the exercise with $\beta = 5^{-n}$. On the other hand, if no h_j^* is zero, then either there exists a $\delta > 0$ such that $h_j^* \geq \delta$ for all j, or $h_j^* \to 0$. In the first case, $h_j \geq \frac{1}{2}\delta$ for all j and, therefore, $\sum_{j=1}^N |R_j| \to +\infty$ as $N \to \infty$. Given any $\beta > 0$, the lemma follows in this case by choosing N sufficiently large. Finally, if $h_j^* \to 0$, we claim that every rectangle in K_1 is contained in $\bigcup_j R_j^*$. Otherwise, there would be a rectangle $R = R(h)$ not intersecting any R_j. Since this rectangle would belong to every K_j, h would satisfy $h \leq h_j^*$ for every j and, therefore, $h = 0$. This contradiction establishes the claim. Since E is covered by the rectangles in K_1, it follows that

$$|E|_e \leq \sum_j |R_j^*| = 5^n \sum_j |R_j|.$$

Hence, given β with $0 < \beta < 5^{-n}$, there exists an N such that $\sum_{j=1}^N |R_j| \geq \beta|E|_e$. This completes the proof. Thus, we clear this exercise about the y-dimension equal to h^2 or increasing function of h for the case of two dimensions.

Exercise 7.4 *If E_1 and E_2 are measurable subsets of $\mathbb{R}^1$ with $|E_1| > 0$ and $|E_2| > 0$, prove that the set $\{x : x = x_1 - x_2, x_1 \in E_1, x_2 \in E_2\}$ contains an interval. [cf. (3.37).]*

Exercise 7.5 *Let f be of bounded variation on $[a, b]$. If $f = g + h$, where g is absolutely continuous and h is singular, show that*

$$\int_a^b \phi df = \int_a^b \phi f' dx + \int_a^b \phi dh,$$

for any continuous ϕ.

Solution. The $\int_a^b \phi df$, $\int_a^b \phi dg$ and $\int_a^b \phi dh$ are exist since ϕ is continuous and both f and g are of bounded variation, so is h. Then we have

$$\begin{aligned}\int_a^b \phi df &= \int_a^b \phi d(g+h) \\ &= \int_a^b \phi dg + \int_a^b \phi dh \\ &= \int_a^b \phi g' dx + \int_a^b \phi dh.\end{aligned}$$

Note that $\int_a^b \phi f' dx = \int_a^b \phi g' dx$ since $f' = g'$ a.e.. Then we complete this proof.

Exercise 7.6 *Show that if $\alpha > 0$, x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.*

Solution. For any subinterval $[a, b]$ of $[0, \infty)$, the function x^α is absolutely continuous on $[a, b]$ since the function x^α is differentiable on $[a, b]$ except for $x = 0$ and

$$\int_a^x \alpha y^{\alpha-1} = x^\alpha - a^\alpha$$

whether a is equal to zero or not.

Exercise 7.7 *Prove that f is absolutely continuous on $[a, b]$ if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \varepsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) < \delta$.*

Solution. It is obvious if the function f is absolutely continuous on $[a, b]$. Conversely, for any collection $\{[a_i, b_i]\}$ be a sequence of nonoverlapping subintervals of $[a, b]$ with $\sum_i (b_i - a_i) < \delta$, we have

$$\sum_{i=1}^n |f(b_i) - f(a_i)| = \sum_{i=1}^n [f(b_i) - f(a_i)]^+ + \sum_{i=1}^n [f(b_i) - f(a_i)]^- < 2\varepsilon$$

for all $n \in \mathbb{Z}^+$. Then we complete this proof since this inequality is hold for every finite n.

Exercise 7.8 *Prove the following converse of (7.31): If f of bounded variation on $[a, b]$, and if the function $V(x) = V[a, x]$ is absolutely continuous on $[a, b]$, then f is absolutely continuous on $[a, b]$.*

Solution. Since the function V is absolutely continuous on $[a, b]$, then

$$\sum_i |f(b_i) - f(a_i)| \leq \sum_i [V(b_i) - V(a_i)] < \varepsilon$$

for any sequence $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum_i (b_i - a_i) < \delta$. This completes this proof.

Exercise 7.9 *If f is of bounded variation on $[a, b]$, show that*

$$\int_a^b |f'| \leq V[a, b].$$

Show that if equality holds in this inequality, then f is absolutely continuous on $[a, b]$. (For the second part, use (2.2)(ii) and (7.24) to show that $V(x)$ is absolutely continuous, and then use the result of Exercise 8.)

Solution. The integral

$$\int_a^b |f'| = \int_a^b V' \leq V(b-) - V(a+) \leq V[a, b].$$

If the equality holds in this inequality. By Theorem 7.24, we have

$$\begin{aligned}
\int_a^x V' &= \int_a^x |f'| \\
&= \int_a^b |f'| - \int_x^b |f'| \\
&= V[a,b] - \int_x^b |f'| \\
&= V[a,x] + V[x,b] - \int_x^b |f'| \\
&\geq V[a,x].
\end{aligned}$$

for all $x \in [a,b]$. Note that $\int_a^x V' = \int_a^x |f'| \leq V[a,x]$. This completes the prove by Theorem 7.29 and Exercise 8.

Exercise 7.10 *Show that if f is absolutely continuous on $[a,b]$ and Z is a subset of $[a,b]$ of measure zero, Deduce that the image under f of any measurable subset of $[a,b]$ is measurable.* [Compare (3.33).][Hint: use the fact that the image of an intervals $[a_i,b_i]$ is an interval of length at most $V(b_i) - V(a_i)$.]

Solution. Let $\varepsilon > 0$ be given. Since f is absolutely continuous on $[a,b]$, so is the variation V of f over $[a,b]$ by (7.31). Then there exists $\delta > 0$ such that

$$\sum_i |V(b_i) - V(a_i)| < \varepsilon$$

for any nonoverlapping subintervals $[a_i,b_i]$ of $[a,b]$ the sum of whose length $\sum_i (b_i - a_i)$ is less than δ. Let Z be any subset of $[a,b]$ with measure zero, there exists an open set G contains Z such that $|G| < \delta$. The open set G can be written as the countable union of nonoverlapping subintervals $[a_i', b_i']$ of $[a,b]$. Thus $\sum_i (b_i' - a_i') < \delta$. This implies that

$$\begin{aligned}
|f(Z)|_e &\leq |f(\bigcup_i [a_i', b_i'])|_e \\
&\leq \sum_i |f([a_i', b_i'])|_e \\
&\leq \sum_i [\sup_{x \in [a_i', b_i']} f(x) - \inf_{x \in [a_i', b_i']} f(x)] \\
&\leq \sum_i [V(b_i') - V(a_i')] \\
&< \varepsilon.
\end{aligned}$$

So $f(Z)$ is measure zero. For E be any measurable subset of $[a,b]$, written as $E = F \cup Z$ where F is of type F_σ, Z is a set with measure zero and $F \cap Z = \phi$. Note that F is union of compact subsets of $[a,b]$. Then $f(F)$ is measurable since f is continuous on $[a,b]$. Hence $f(E)$ is measurable.

Exercise 7.11

Exercise 7.12 *Use Jensen's inequality to prove that for* $a, b \geq 0$, $p, q > 1$, $(1/p) + (1/q) = 1$, *we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

More generally, show that

$$a_1 \cdots a_N \leq \sum_{j=1}^{N} \frac{a_j^{p_j}}{p_j}$$

where $a_j \geq 0$, $p_j > 1$, $\sum_{j=1}^{N}(1/p_j) = 1$. (Write $a_j = e^{x_j/p_j}$ and use the convexity of e^x.)

Solution. This is trivial if $a_j = 0$ for some j. Otherwise, let $a_j = e^{x_j/p_j}$ and $\phi(x) = e^x$, then ϕ is a convex function and following that

$$a_1 \cdots a_N = \phi(\sum_{j=1}^{N} \frac{x_j}{p_j}) \leq \sum_{j=1}^{N} \frac{\phi(x_j)}{p_j} = \sum_{j=1}^{N} \frac{a_j^{p_j}}{p_j}$$

by Jensen's inequality. If we take $N = 2$, then the first inequality holds.

Exercise 7.13 *Prove theorem (7.36).*

(i) If ϕ_1 *and* ϕ_2 *are convex in* (a, b), *then* $\phi_1 + \phi_2$ *is convex in* (a, b).

(ii) If ϕ *is convex in* (a, b) *and* c *is a positive constant, then* $c\phi$ *is convex in* (a, b).

(iii) If ϕ_k, $k = 1, 2, \ldots,$ *are convex in* (a, b) *and* $\phi_k \to \phi$ *in* (a, b), *then* ϕ *is convex in* (a, b).

Exercise 7.14 *Prove that* ϕ *is convex on* (a, b) *if and only if it is continuous and*

$$\phi(\frac{x_1 + x_2}{2}) \leq \frac{\phi(x_1) + \phi(x_2)}{2}$$

for $x_1, x_2 \leq (a, b)$.

Solution. It is clearly by Theorem 7.40 if ϕ is convex. Now, we suppose that ϕ is continuous and satisfies $\phi((x_1+x_2)/2) \leq (\phi(x_1)+\phi(x_2))/2$ for $x_1, x_2 \leq (a, b)$. Given $x_1, x_2 \in (a, b)$, For any $t \in [x_1, x_2]$ can be written as

$$t = (1 - \sum_{k=1}^{\infty} \frac{a_k}{2^k})x_1 + (\sum_{k=1}^{\infty} \frac{a_k}{2^k})x_2$$

where $a_i \in \{0, 1\}$ for all i. Let t_n be the nth partial sum of the series t. We claim that

$$\phi((1 - \sum_{k=1}^{n} \frac{a_k}{2^k})x_1 + (\sum_{k=1}^{n} \frac{a_k}{2^k})x_2) \leq (1 - \sum_{k=1}^{n} \frac{a_k}{2^k})\phi(x_1) + (\sum_{k=1}^{n} \frac{a_k}{2^k})\phi(x_2)$$

for all n is any positive integer. For $n = 1$, then

$$\phi((1 - \frac{a_1}{2})x_1 + \frac{a_1}{2}x_2) \leq (1 - \frac{a_1}{2})\phi(x_1) + \frac{a_1}{2}\phi(x_2).$$

Suppose that this inequality holds for $n = r$. For $n = r + 1$, we have

$$\begin{aligned}
& \phi((1-\sum_{k=1}^{r+1} a_k 2^{-k})x_1 + (\sum_{k=1}^{r+1} a_k 2^{-k})x_2) \\
\leq \; & \frac{1}{2}[\phi((1-a_1)x_1 + a_1 x_2) + \phi((1-\sum_{k=2}^{r+1}\frac{a_k}{2^{k-1}})x_1 + (\sum_{k=2}^{r+1}\frac{a_k}{2^{k-1}})x_2)] \\
\leq \; & \frac{1}{2}[(1-a_1)\phi(x_1) + a_1\phi(x_2) + (1-\sum_{k=2}^{r+1}\frac{a_k}{2^{k-1}})\phi(x_1) + (\sum_{k=2}^{r+1}\frac{a_k}{2^{k-1}})\phi(x_2)] \\
= \; & (1-\sum_{k=1}^{r+1} a_k 2^{-k})\phi(x_1) + (\sum_{k=1}^{r+1} a_k 2^{-k})\phi(x_2).
\end{aligned}$$

This claim is established by induction. Then

$$\phi(t) = \lim_{n\to\infty}\phi(t_n) \leq (1-\sum_{k=1}^{\infty} a_k 2^{-k})\phi(x_1) + (\sum_{k=1}^{\infty} a_k 2^{-k})\phi(x_2)$$

since ϕ is continuous. This complete the proof.

Exercise 7.15 *Theorem (7.43) shows that a convex function is the indefinite integral of a monotone increasing function. Prove the converse: If $\phi(x) = \int_a^x f(t)dt + \phi(a)$ in (a,b) and f is monotone increasing, then ϕ is convex in (a,b).* (Use Exercise 14.)

Solution. Given any interval $[x_1, x_2] \in (a,b)$, we have

$$\begin{aligned}
\frac{\phi(x_1)+\phi(x_2)}{2} - \phi(\frac{x_1+x_2}{2}) &= \frac{1}{2}\int_{\frac{x_1+x_2}{2}}^{x_2} f(x)dx - \frac{1}{2}\int_{x_1}^{\frac{x_1+x_2}{2}} f(x)dx \\
&\geq (\frac{x_2-x_1}{4})(f(\frac{x_1+x_2}{2}) - f(\frac{x_1+x_2}{2})) \\
&= 0.
\end{aligned}$$

So ϕ is convex by Exercise 14 since $\int_a^x f(t)dt$ is continuous.

Exercise 7.16 *Show that the formula*

$$\int_{-\infty}^{\infty} fg' = -\int_{-\infty}^{+\infty} f'g$$

for integration by parts may not hold if f is of bounded variation on $(-\infty, +\infty)$ and g is infinitely differentiable with compact support. (Let f be the Cantor-Lebesgue function on $[0,1]$, and let $f = 0$ elsewhere.)

Solution. Let f be the Cantor-Lebesgue function on $[0,1]$ and $f = 0$ elsewhere, then f is of bounded variation on $(-\infty, +\infty)$. Let

$$g(x) = \begin{cases} e^{\frac{1}{(x-1)^2-1}} & \text{if } x \in (0,2), \\ 0 & \text{otherwise,} \end{cases}$$

then g is infinitely differentiable with compact support. Since g is increasing on (0,1), then

$$\int_{-\infty}^{\infty} fg' \geq \frac{1}{2}\int_{\frac{1}{3}}^{\frac{2}{3}} g' > 0.$$

But $\int_{-\infty}^{+\infty} f'g = 0$ and the exercise follows.

Exercise 7.17 *A sequence $\{\phi_k\}$ of set functions is said to be uniformly absolutely continuous if given $\varepsilon > 0$, there exists $\delta > 0$ such that if E satisfies $|E| < \delta$, then $|\phi_k(E)| < \varepsilon$ for all k. If $\{f_k\}$ is a sequence of integrable functions on $(0,1)$ which converges pointwise a.e. to an integrable f, show that $\int_0^1 |f - f_k| \to 0$ if and only if the indefinite integrals of the f_k are uniformly absolutely continuous.*

Solution. Given $\varepsilon > 0$. If the indefinite integrals of the f_k are uniformly absolutely continuous and $f \in L(0,1)$, there exists $\delta > 0$ such that if $E \subseteq (0,1)$ satisfies $|E| < \delta$, then $|\int_E f_k| < \varepsilon$ for all k and $\int_E |f| < \varepsilon$. By Egorov's theorem, there is a closed subset F of $(0,1)$ such that $|(0,1) - F| < \delta$ and $\{f_k\}$ converges uniformly to f on F. Then choose $M > 0$ such that for all $k \geq M$, we have

$$\begin{aligned}
\int_0^1 |f - f_k| &= \int_F |f - f_k| + \int_{(0,1)-F} |f - f_k| \\
&< \varepsilon|F| + \int_{(0,1)-F} |f| + \int_{(0,1)-F} |f_k| \\
&< \varepsilon + \varepsilon + \int_{(0,1)-F} f_k^+ + \int_{(0,1)-F} f_k^- \\
&< 4\varepsilon.
\end{aligned}$$

So $\int_0^1 |f - f_k| \to 0$. Conversely, given $\varepsilon > 0$. For all k, since the indefinite integral of f_k is absolutely continuous, there exists $\delta_k > 0$ such that for any $E \subseteq (0,1)$ with $|E| < \delta_k$, we have $|\int_E f_k| < \varepsilon$. Since the indefinite integral of f is absolutely continuous, choose $M > 0$ and $\delta > 0$ such that for any $|E| < \delta$ and $k \geq M + 1$, we have

$$|\int_E f_k| \leq \int_E |f_k| \leq \int_E |f_k - f| + \int_E |f| < \varepsilon.$$

Let $\delta' = \min\{\delta, \delta_1, \delta_2, \ldots, \delta_M\}$, then $|\int_E f_k| < \varepsilon$ for all k and this theorem follows.

Chapter 8

L^p Class

Exercise 8.1 *For complex-valued, measurable f, $f = f_1 + if_2$ with f_1 and f_2 real-valued and measurable, we have $\int_E f = \int_E f_1 + i\int_E f_2$. Prove that $\int_E f$ is finite if and only if $\int_E |f|$ is finite, and $|\int_E f| \leq \int_E |f|$.* (Note that $|\int_E f| = [(\int_E f_1)^2 + (\int_E f_2)^2]$, and use the fact that $(a^2+b^2)^{1/2} = a\cos\alpha + b\sin\alpha$ for an appropriate α, while $(a^2 + b^2)$)

Solution. In fact, we choose α such that

$$\begin{aligned} |\int_E f| &= [(\int_E f_1)^2 + (\int_E f_2)^2]^{1/2} \\ &= (\int_E f_1)\cos\alpha + (\int_E f_2)\sin\alpha \\ &= \int_E |f_1\cos\alpha + f_2\sin\alpha| \\ &\leq \int_E ((f_1)^2 + (f_2)^2)^{1/2} \\ &= \int_E |f|. \end{aligned}$$

Thus, the integral $\int_E f$ is finite if $\int_E |f|$ is finite. Conversely, we have

$$|\int_E f_1| \leq [(\int_E f_1)^2 + (\int_E f_2)^2]^{1/2} = |\int_E f| < \infty.$$

It follows that $\int_E |f_1| < \infty$ and $\int_E |f_2| < \infty$ similarly. Then

$$\int_E |f| = \int_E (f_1^2 + f_2^2)^{1/2} \leq \int_E |f_1| + |f_2| < \infty.$$

Then we complete this exercise.

Exercise 8.2 *Prove the converse of Hölder's inequality for $p = 1$ and ∞. Show also that for real-valued $f \notin L^p(E)$, there exists a function $g \in L^{p'}(E)$, $1/p + 1/p' = 1$, such that $fg \notin L^1(E)$. (Construct g of the form $\sum a_k g_k$ for appropriate a_k and g_k, with g_k satisfying $\int_E fg_k \to +\infty$.)*

Solution. It's obvious to show $\|f\|_p \geq \sup \int_E fg$. And we have

$$\|f\|_1 = \int_E f\mathrm{sign}(f).$$

So we consider $p = \infty$. If $\|f\|_\infty = 0$, then $\|f\|_\infty = \sup \int_E fg$ for every g since $f = 0$ a.e.. Given any $0 < M < \|f\|_\infty$, let $E' = \{f > M\}$, then $|E'| > 0$ and there exists $r > 0$ such that $E'' = E' \cap B(0,r)$ satisfies $0 < |E''| < \infty$. Let $g_1 = \mathrm{sign}(f)\chi_{E''}/|E''|$, then $\|g\|_1 = 1$ and

$$\sup_{\|g\|_1 \leq 1} \int_E fg \geq \int_E fg_1 = \int_{E''} |f|/|E''| > M.$$

Hence $\sup_{\|g\|_1 \leq 1} \int_E fg \geq \|f\|_\infty$. So $\|f\|_\infty = \sup \int_E fg$. Now we show that the part two. If $p = 1$, let $g = \mathrm{sign} f$, then $g \in L^\infty(E)$ and $fg = |f| \notin L^1$. If $1 < p < \infty$, let

$$f_k(x) = \begin{cases} 0 & \text{if } |x| > k, \\ \min\{|f(x)|, k\} & \text{if } |x| \leq k, \end{cases}$$

then $\|f_k\|_p \nearrow \|f\|_p = \infty$ and $f_k \in L^p$ for all k. There exists $\|f_{k_j}\|_p > 2^{2j}$, $k_{j+1} > k_j$ and g_j with $\|g_j\|_{p'} \leq 1$ such that $\|f_{k_j}\|_p = \int_E f_{k_j} g_j$ for all j. Let $g = \sum_j 2^{-j} g_j$, then $\|g\|_{p'} \leq \sum_j 2^{-j}\|g_j\|_{p'} \leq 1$ and for all j, we have

$$\int_E |fg| \geq 2^{-j} \int_E f_{k_j} g_j = 2^{-j}\|f_{k_j}\|_p \geq 2^j.$$

Hence $fg \notin L^1(E)$. If $p = \infty$, let

$$f_k(x) = \begin{cases} |f(x)| & \text{if } |f(x)| \leq k, \\ k & \text{if } |f(x)| > k, \end{cases}$$

then $f_k \in L^\infty$ and $\|f_k\|_\infty \nearrow \|f\|_\infty$. There exists $g_k \geq 0$ with $\|g_k\|_1 = 1$ such that $\int_E f_k g_k \geq \|f_k\|_\infty$. And we have $\int_E f_{k_j} g_{k_j} \geq \|f_{k_j}\|_\infty > 2^{2j}$ which $k_{j+1} > k_j$. Let $g = \sum_j 2^{-j} g_{k_j}$, then $\|g\|_1 \leq 1$, and for all j,

$$\int_E |fg| \geq 2^{-j} \int_E f_{k_j} g_j = 2^{-j}\|f_{k_j}\|_\infty \geq 2^j.$$

So $fg \notin L^1(E)$.

Exercise 8.3

Exercise 8.4

Exercise 8.5 *For $1 \leq p < \infty$ and $0 < |E| < \infty$, define*

$$N_p[f] = (\frac{1}{|E|}\int_E |f|^p)^{1/p}.$$

Prove that if $p_1 < p_2$, then $N_{p_1}[f] \leq N_{p_2}[f]$.
Prove also that $N_p[f+g] \leq N_p[f] + N_p[g]$, $(1/|E|)\int_E |fg| \leq N_p[f]N_{p'}[g]$, $1/p + 1/p' = 1$, and that $\lim_{p\to\infty} N_p[f] = \|f\|_\infty$. Thus, N_p behaves like $\|\cdot\|_p$, but has the advantage of being monotone in p.

Solution.

(i) We use Hölder's inequality to show that

$$\begin{aligned}
N_{p_1}[f] &= (\frac{1}{|E|}\int_E |f|^{p_1})^{1/p_1}\\
&\leq (\frac{1}{|E|})^{1/p_1}(\|f^{p_1}\|_{\frac{p_2}{p_1}}\|1\|_{\frac{p_2}{p_2-p_1}})^{1/p_1}\\
&= (\frac{1}{|E|})^{1/p_1}\|f\|_{p_2}|E|^{\frac{p_2-p_1}{p_1p_2}}\\
&= |E|^{-1/p_2}\|f\|_{p_2}\\
&= N_{p_2}[f].
\end{aligned}$$

(ii) Use Minkowski's inequality, we have

$$N_p[f+g] = |E|^{-1/p_1}\|f+g\|_p \leq |E|^{-1/p}(\|f\|_p + \|g\|_p) = N_p[f] + N_p[g].$$

(iii) Use Hölder's inequality, we have

$$(1/|E|)\int_E |fg| \leq |E|^{-1/p}|E|^{-1/p'}\|f\|_p\|g\|_{p'} = N_p[f]N_{p'}[g].$$

(iv) By Theorem 8.1, we have

$$\lim_{p\to\infty} N_p[f] = \lim_{p\to\infty} |E|^{-1/p}\|f\|_p = \|f\|_\infty.$$

Then we complete this proof.

Exercise 8.6 *Prove the following generalization of Hölder's inequality. If* $\sum_{i=1}^k 1/p_i = 1/r$*,* $p_i, r \geq 1$*, then*

$$\|f_1\cdots f_k\|_r \leq \|f_1\|p_1\cdots\|f_k\|_{p_k}.$$

(See also Exercise 12 of Chapter 7.)

Solution. It is obvious to $k=1$. If $k=2$, we have

$$\|f_1f_2\|_r^r = \|f_1^rf_2^r\|_1 \leq \|f_1^r\|_{p_1/r}\|f_2^r\|_{p_2/r} = \|f_1\|_{p_1}^r\|f_2\|_{p_2}^r.$$

Then we complete this proof for $k=2$. Suppose that $k=n$ holds, consider $k=n+1$, we have

$$\begin{aligned}
\|f_1f_2\cdots f_{n+1}\|_r &\leq \|f_1f_2\cdots f_n\|_{p'}\|f_{n+1}\|_{p_{k+1}}\\
&\leq \|f_1\|_{p_1}\|f_2\|_{p_2}\cdots\|f_{n+1}\|_{p_{n+1}}
\end{aligned}$$

where $1/p' = \sum_{i=1}^n 1/p_k$ and $p' \geq r \geq 1$. This complete the proof by induction.

Exercise 8.7 *Show that when* $0<p<1$*, the neighborhoods* $\{f : \|f\|_p < \varepsilon\}$ *of zero in* $L^p(0,1)$ *are not convex.* (Let $f = \chi_{(0,\varepsilon^p)}$ and $g = \chi_{(\varepsilon^p,2\varepsilon^p)}$. Show that $\|f\|_p = \|g\|_p = \varepsilon$, but that $\|\frac{1}{2}f + \frac{1}{2}g\|_p > \varepsilon$.)

Solution. For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $2(\frac{\varepsilon}{N})^p < 1$ and $\frac{N-1}{N}2^{1/p-1} > 1$. Let $f = (N-1)\chi_{(0,(\frac{\varepsilon}{N})^p)}$ and $g = (N-1)\chi_{((\frac{\varepsilon}{N})^p,2(\frac{\varepsilon}{N})^p)}$, then we have

$$\|f\|_p = (\int_{(0,(\frac{\varepsilon}{N})^p)} (N-1)^p)^{1/p} = \frac{N-1}{N}\varepsilon < \varepsilon$$

and

$$\|g\|_p = (\int_{((\frac{\varepsilon}{N})^p,2(\frac{\varepsilon}{N})^p)} (N-1)^p)^{1/p} = \frac{N-1}{N}\varepsilon < \varepsilon.$$

But

$$\|\frac{1}{2}f + \frac{1}{2}g\|_p = (\int_{(0,2(\frac{\varepsilon}{N})^p)} (\frac{N-1}{2})^p)^{1/p} = \frac{N-1}{N}2^{1/p-1}\varepsilon > \varepsilon.$$

So $\{f : \|f\|_p < \varepsilon\}$ is not convex for every $\varepsilon > 0$ and $0 < p < 1$.

Exercise 8.8 *Prove the following integral version of Minkowski's inequality for* $1 \le p < \infty$*:*

$$[\int |\int f(x,y)dx|^p dy]^{1/p} \le \int [\int |f(x,y)|^p dy]^{1/p} dx.$$

[For $1 < p < \infty$, note that pth power of the left-hand side is majorized by $\iint [\int |f(z,y)|dz]^{p-1}|f(x,y)|dxdy$. Integrate first with respect to y and apply Hölder's inequality.]

Solution. If $p = 1$ then this inequality is true. If $1 < p < \infty$, then

$$\begin{aligned}
\int |\int f(x,y)dx|^p dy &= \int |\int f(z,y)dz|^{p-1}|\int f(x,y)dx|dy \\
&\le \int [|\int f(z,y)dz|^{p-1}\int |f(x,y)|dx]dy \\
&= \int [\int |\int f(z,y)dz|^{p-1}|f(x,y)|dx]dy \\
&= \int [\int |\int f(z,y)dz|^{p-1}|f(x,y)|dy]dx \\
&\le \int (\int |\int f(z,y)dz|^p dy)^{\frac{p-1}{p}} (\int |f(x,y)|^p dy)^{\frac{1}{p}} dx \\
&= (\int |\int f(x,y)dx|^p dy)^{\frac{p-1}{p}} \int (\int |f(x,y)|^p dy)^{\frac{1}{p}} dx.
\end{aligned}$$

If $\int |\int f(x,y)dx|^p dy \ne 0$ then

$$(\int |\int f(x,y)dx|^p dy)^{\frac{1}{p}} \le \int (\int |f(x,y)|^p dy)^{\frac{1}{p}} dx.$$

If $\int |\int f(x,y)dx|^p dy = 0$ then $\int [\int |f(x,y)|^p dy]^{1/p} dx \ge 0$. If $\int |\int f(x,y)dx|^p dy = \infty$, there exists $f_k \nearrow |f|$ with compact support, $f_k \ge 0$ and that is bounded. Then $\int |f_k(x,y)|^p dx \nearrow \int |f(x,y)|^p dx$. Hence $\int [\int |f_k(x,y)|^p dy]^{1/p} dx \nearrow \int [\int |f(x,y)|^p dy]^{1/p} dx$. Use the result of above, we have

$$\begin{aligned}
[\int |\int \liminf f_k(x,y)dx|^p dy]^{1/p} &\le \liminf [\int |\int f_k(x,y)dx|^p dy]^{1/p} \\
&\le \liminf \int [\int |f_k(x,y)|^p dy]^{1/p} dx \\
&= \int [\int |f(x,y)|^p dy]^{1/p} dx.
\end{aligned}$$

This complete the proof.

Exercise 8.9 *If f is real-valued and measurable on E, define its essential infimum on E by*

$$\operatorname*{ess\,inf}_E = \sup\{\alpha : |\{x \in E : f(x) < \alpha\}| = 0\}.$$

If $f \geq 0$, show that $\mathrm{ess}_E \inf f = (\mathrm{ess}_E \sup 1/f)^{-1}$.

Solution. We have

$$\begin{aligned}
\operatorname*{ess\,inf}_E f &= \sup\{\alpha : |\{x \in E : f(x) < \alpha\}| = 0\} \\
&= \sup\{\alpha : |\{x \in E : \frac{1}{f(x)} > \frac{1}{\alpha}\}| = 0\} \\
&= \inf\{\frac{1}{\alpha} : |\{x \in E : \frac{1}{f(x)} > \frac{1}{\alpha}\}| = 0\} \\
&= (\inf\{\alpha : |\{x \in E : \frac{1}{f(x)} > \alpha\}|\})^{-1} \\
&= (\operatorname*{ess\,sup}_E \frac{1}{f})^{-1}.
\end{aligned}$$

Exercise 8.10 *Prove that $L^\infty(E)$ is not separable for any E with $|E| > 0$.* (Construct a sequence of decreasing subsets of E whose measures strictly decrease. Consider the characteristic function of the class of sets obtained by taking all possible unions of the differences of these subsets.)

Solution. Since $|E| > 0$, there exists $r > 0$ such that $0 < |Q(0, r) \cap E| < \infty$ by writing $E_1 = Q(0, r) \cap E$. We can choose $E_2 \subset E_1$ such that $0 < |E_2| < |E_1|/2$. Then we find a sequence $\{E_k\}$ of subset of E such that $|E_{k+1}| < |E_k|/2$ for all $k \geq 1$. Let $F_k = E_k - E_{k+1}$, then $F_1, F_2, \ldots$ are disjoint with $0 < |F_k| < \infty$ for all k. Let

$$A = \{f : f = \sum_{j=1}^{\infty} \chi_{F_{k_j}}, \{F_{k_j}\} \text{ is the subsequence of } \{F_k\}\}.$$

Then A is a uncountable set. Suppose that $L^\infty(E)$ is separable, let $\{g_k\}$ be the countable dense subset of L^∞. There exists $f_1, f_2 \in A$ such that $\|g_{i_0} - f_1\|_\infty < 1/3$ and $\|g_{i_0} - f_2\|_\infty < 1/3$, then

$$\|f_1 - f_2\|_\infty \leq \|f_1 - g_{i_0}\|_\infty + \|g_{i_0} - f_2\|_\infty < 2/3.$$

But $\|f - g\|_\infty = 1$ for any $f, g \in A$ with $f \neq g$ a.e.. That is a contradiction and the theorem follows.

Exercise 8.11 *If $f_k \to f$ in L^p, $1 \leq p < \infty$, $g_k \to g$ pointwise, and $\|g_k\|_\infty \leq M$ for all k, prove that $f_k g_k \to fg$ in L^p.*

Solution. Since $g_k \to g$ pointwise, then $|fg_k - fg|^p \to 0$ pointwise and $|fg_k - fg|^p \leq |f|^p (2M)^p$ with $\int_E |f|^p (2M)^p$ is finite. Use Minkowski's inequality,

$$\begin{aligned}
\|f_k g_k - fg\|_p &\leq \|f_k g_k - fg_k\|_p + \|fg_k - fg\|_p \\
&\leq M\|f_k - f\|_p + (\int_E |f|^p |g_k - g|^p)^{1/p}
\end{aligned}$$

By Lebesgue's Dominated Convergence Theorem, we have $\|f_kg_k - fg\|_p \to 0$. That complete this proof.

Exercise 8.12 *Let $f, \{f_k\} \in L^p$. Show that if $\|f - f_k\|_p \to 0$, then $\|f_k\| \to \|f\|$. Conversely, if $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$, $1 \le p < \infty$, show that $\|f - f_k\| \to 0$.*

Solution. For $1 \le p \le \infty$, if $\|f - f_k\|_p \to 0$, then

$$|\|f_k\|_p - \|f\|_p| \le \|f_k - f\|_p \to 0.$$

So $\|f_k\|_p \to \|f\|_p$. For $0 < p < 1$, we have

$$|\|f_k\|_p^p - \|f\|_p^p| \le \|f_k - f\|_p^p \to 0.$$

Then $|\|f_k\|_p^p \to \|f\|_p^p$. Hence $\|f_k\|^p \to \|f\|^p$. Conversely, since $|f_k - f|^p \le 2^p(|f_k|^p + |f|^p)$, use Fatou's lemma, we have

$$\begin{aligned} \int 2^{p+1}|f|^p &\le \liminf_{k\to\infty} \int 2^p(|f_k|^p + |f|^p) - |f_k - f|^p \\ &\le \int 2^{p+1}|f|^p - \limsup_{k\to\infty} \int |f_k - f|^p. \end{aligned}$$

Then $\limsup \int |f_k - f|^p \le 0$. It implies $\int |f_k - f|^p \to 0$. So $\|f_k - f\|_p \to 0$.

Exercise 8.13 *Suppose that $f_k \to f$ a.e. and that $f_k, f \in L^p$, $1 < p < \infty$. If $\|f_k\|_p \le M < +\infty$, show that $\int f_k g \to \int fg$ for all $g \in L^{p'}$, $1/p + 1/p' = 1$.*

Solution. Since $g^{p'} \in L^1$, given any $\varepsilon > 0$, there exists a set E with finite measure such that $\int_{E^c} |g|^{p'} < \varepsilon$ and $\delta > 0$ such that for any measurable set A with $|A| < \delta$, we have $\int_A |g|^{p'} < \varepsilon$. By Egorov's Theorem, choose a subset F of E such that $|E - F| < \delta$ and $f_k \to f$ uniformly on F. Then we find $N > 0$ such that for all $k \ge N$, we have

$$\begin{aligned} |\int f_k g - fg| &\le \int |f_k - f||g| \\ &= \int_E |f_k - f||g| + \int_{E^c} |f_k - f||g| \\ &\le \int_F |f_k - f||g| + \int_{E-F} |f_k - f||g| + \|f_k - f\|_{p,E^c}\|g\|_{p',E^c} \\ &< \|f_k - f\|_{p,F}\|g\|_{p',F} + \|f_k - f\|_{p,E-F}\|g\|_{p',E-F} + 2M\varepsilon^{1/p'} \\ &< (\varepsilon/|E|^{1/p})|F|^{1/p}\|g\|_{p',F} + 2M\varepsilon^{1/p'} + 2M\varepsilon^{1/p'}. \end{aligned}$$

So $\int f_k g \to \int fg$ for all $g \in L^{p'}$, $1/p + 1/p' = 1$.

Exercise 8.14 *Verify that the following systems are orthogonal:*

(a) $\{\frac{1}{2}, \cos x, \sin x, \ldots, \cos kx, \sin kx, \ldots\}$ *on any interval of length 2π.*

(b) $\{e^{2\pi ikx/(b-a)}; k = 0, \pm 1, \pm 2, \ldots\}$ *on (a, b).*

Solution.

(a) Since for any $h, k \in \mathbb{N}$ with $h \geq k$, we have

$$\int_0^{2\pi} \sin kx dx = \int_0^{2\pi} \cos kx dx = 0.$$

Thus, $\langle \frac{1}{2}, \sin kx \rangle = \langle \frac{1}{2}, \cos kx \rangle = 0$, and for any $a \in \mathbb{R}$,

$$\langle \sin hx, \cos kx \rangle = \frac{1}{2} \int_a^{a+2\pi} [\sin(h+k)x + \sin(h-k)x] = 0.$$

Similarly, $\langle \cos hx, \sin kx \rangle$, $\langle \sin hx, \sin kx \rangle$, $\langle \cos hx, \cos kx \rangle$ are equal to zero. So the system is orthogonal.

(b) For any $h, k \in \mathbb{Z}$ with $h \neq k$, we have

$$\begin{aligned} \langle e^{\frac{2\pi i h x}{b-a}}, e^{\frac{2\pi i k x}{b-a}} \rangle &= \int_a^b e^{\frac{2\pi i (n-m)x}{b-a}} dx \\ &= \frac{b-a}{2\pi i (n-m)} e^{\frac{2\pi i (n-m)a}{b-a}} (e^{2\pi i (n-m)} - 1) \\ &= 0. \end{aligned}$$

So the system is orthogonal.

Exercise 8.15 *If $f \in L^2(0, 2\pi)$, show that*

$$\lim_{k \to \infty} \int_0^{2\pi} f(x) \cos kx dx = \lim_{k \to \infty} \int_0^{2\pi} f(x) \sin kx dx = 0.$$

Prove that the same is true if $f \in L^1(0, 2\pi)$ (This last statement is the Riemann-Lebesgue lemma. To prove it, approximate f in L^1 norm by L^2 functions. See (12.21).)

Solution. We have the system $\{e^{ikx}/\sqrt{2\pi}\}_{k=1}^{\infty}$ is orthonormal by Exercise 14(b) in $L^2(0, 2\pi)$. Thus, by Bessel's inequality, we have

$$\sum_{k=1}^{\infty} |\langle f, e^{ikx}/\sqrt{2\pi} \rangle|^2 \leq \|f\|^2 < \infty.$$

Hence

$$\lim_{k \to \infty} |\int_0^{2\pi} f(x) \cos kx - i f(x) \sin kx| = 0.$$

So

$$\lim_{k \to \infty} \int_0^{2\pi} f(x) \cos kx dx = \lim_{k \to \infty} \int_0^{2\pi} f(x) \sin kx dx = 0.$$

Next, if $f \in L^1(0, 2\pi)$ with $f \geq 0$, let

$$f_k(x) = \begin{cases} f(x) & \text{if } f(x) \leq k, \\ 0 & \text{if } f(x) > k. \end{cases}$$

Then $f_k \nearrow f$. Hence $\int f_k \nearrow \int f$. Given any $\varepsilon > 0$, there exists k' such that $\int f - f_{k'} < \varepsilon$. Note that $f_{k'} \in L^2(0, 2\pi)$ and we have

$$\begin{aligned} |\int_0^{2\pi} f(x)\cos kxdx| &\leq |\int_0^{2\pi} (f(x) - f_{k'}(x))\cos kxdx| + |\int_0^{2\pi} f_{k'}(x)\cos kxdx| \\ &\leq \int_0^{2\pi} f(x) - f_{k'}(x)dx + |\int_0^{2\pi} f_{k'}(x)\cos kxdx| \\ &\leq \varepsilon. \end{aligned}$$

For general $f \in L^1(0, 2\pi)$, let $f = f^+ - f^-$, then

$$|\int_0^{2\pi} f(x)\cos kx| \leq |\int_0^{2\pi} f^+(x)\cos kx| + |\int_0^{2\pi} f^-(x)\cos kx| \to 0$$

Similarly, we have $\lim_{k\to\infty} \int_0^{2\pi} f(x)\cos kxdx = \lim_{k\to\infty} \int_0^{2\pi} f(x)\sin kxdx = 0$.

Exercise 8.16 *A sequence $\{f_k\}$ in L^p is said to converge weakly to a function f in L^p if $\int f_k g \to \int fg$ for all $g \in L^{p'}$. Prove that if $f_k \to f$ in L^p norm, $1 \leq p \leq \infty$, then $\{f_k\}$ converges weakly to f in L^p. Note by Exercise 15 that the converse is not true.*

Solution. For any $g \in L^{p'}$, we have

$$|\int f_k g - \int fg| \leq \int |f_k - f||g| \leq \|f_k - f\|_p \|g\|_{p'} \to 0.$$

So $\{f_k\}$ converges weakly to f in L^p.

Exercise 8.17 *Suppose that $f_k, f \in L^2$ and that $\int f_k g \to \int fg$ for all $g \in L^2$* (that is, $\{f_k\}$ converges weakly to f in L^2). *If $\|f_k\|_2 \to \|f\|_2$, show that $f_k \to f$ in L^2 norm.*

Solution. We have

$$\begin{aligned} \|f_k - f\|_2^2 &= \int (f_k - f)\overline{(f_k - f)} \\ &= \|f_k\|_2^2 - \int \bar{f}_k f - \int \bar{f} f_k + \|f\|_2^2 \\ &= \|f_k\|_2^2 - \overline{\int f_k \bar{f}} - \int \bar{f} f_k + \|f\|_2^2 \\ &\to \|f\|_2^2 - \overline{\int f\bar{f}} - \int \bar{f} f + \|f\|_2^2 \\ &= 0. \end{aligned}$$

Then $f_k \to f$ in L^2 norm.

Exercise 8.18 *Prove the parallelogram law for L^2:*

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2.$$

Is this true for L^p when $p \neq 2$? The geometric interpretation is that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides.

Solution. We have

$$\begin{aligned}\|f+g\|^2+\|f-g\|^2 &= 2\|f\|^2+2\|g\|^2+2\operatorname{Re}\langle f,g\rangle-2\operatorname{Re}\langle f,g\rangle\\ &= 2\|f\|^2+2\|g\|^2.\end{aligned}$$

Let $f=\chi_{(0,1)}$ and $g=\chi_{(1,2)}$. If p is finite, then $\|f+g\|_p^2=\|f-g\|_p^2=2^{2/p}$ and $\|f\|_p^2=\|g\|_p^2=1$. If $p=\infty$, then $\|f+g\|_\infty^2=\|f-g\|_\infty^2=1$ and $\|f\|_\infty^2=\|g\|_\infty^2=1$. So we have no the parallelogram law for L^p when $p\neq 2$.

Exercise 8.19 *Prove that a finite dimensional Hilbert space is isometric with $\mathbb{R}^n$ for some n.*

Solution. Let H be a finite dimensional Hilbert space and $\{\phi_1,\phi_2,\ldots,\phi_n\}$ be an orthonormal basis of H. Let a mapping $T:H\to\mathbb{R}^{2n}$ given by

$$f=\sum_{k=1}^n a_k\phi_k\mapsto(\operatorname{Re}a_1,\operatorname{Im}a_1,\operatorname{Re}a_2,\operatorname{Im}a_2,\ldots,\operatorname{Re}a_n,\operatorname{Im}a_n).$$

Then T is well-defined and onto since

$$T(\sum_{k=1}^n(x_k+y_ki)\phi_k)=(x_1,y_1,x_2,y_2,\ldots,x_n,y_n)$$

for any $x_k,y_k\in\mathbb{R}$ is given. Every $f,g\in H$ are written as $f=\sum_{k=1}^n a_k\phi_k$ and $g=\sum_{k=1}^n a_k'\phi_k$, we have

$$\begin{aligned}\|f-g\|_H^2 &= \langle f-g,f-g\rangle\\ &= \sum_{k=1}^n|a_k-a_k'|^2\\ &= \sum_{k=1}^n|\operatorname{Re}a_k-\operatorname{Re}a_k'|^2+|\operatorname{Im}a_k-\operatorname{Im}a_k'|^2\\ &= \|Tf-Tg\|_{\mathbb{R}^{2n}}^2.\end{aligned}$$

Then we complete this proof.

Exercise 8.20

Exercise 8.21 *If $f\in L^p(\mathbb{R}^n)$, $0<p<\infty$, show that*

$$\lim_{Q\searrow x}\frac{1}{|Q|}\int_Q|f(y)-f(x)|^p dy=0\ \textit{a.e.}$$

Note by Exercise 5 that this condition for a given p implies it for all smaller p.

Solution. Let $\{r_k\}$ be the rational numbers, and let Z_k be the set where the formula

$$\lim_{Q\searrow x}\frac{1}{|Q|}\int_Q|f(y)-r_k|^p=|f(x)-r_k|^p$$

is not valid. Since $|f(y) - r_k|^p$ is locally integrable, we have $|Z_k| = 0$. Let $Z = \bigcup Z_k$; then $|Z| = 0$. For any Q, x, and r_k,

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f(y) - f(x)|^p dy &\leq 2^p \frac{1}{|Q|}\int_Q |f(y) - r_k|^p dy + 2^p \frac{1}{|Q|}\int_Q |f(x) - r_k|^p dy \\
&= 2^p \frac{1}{|Q|}\int_Q |f(y) - r_k|^p dy + 2^p |f(x) - r_k|^p.
\end{aligned}$$

Therefore, if $x \notin Z$,

$$\limsup_{Q \searrow x} \frac{1}{|Q|}\int_Q |f(y) - f(x)|^p dy \leq 2^{p+1}|f(x) - r_k|^p$$

for every r_k. For an x at with $f(x)$ is finite, we can choose r_k such that $|f(x)-r_k|$ is arbitrarily small. This shows that the left side of the last formula is zero a.e., and completes the proof.

Exercise 8.22 *Let $\{\phi_k\}$ be a complete orthonormal system in L^2, and let $m = \{m_k\}$ be a given sequence of numbers. If $f \in L^2$, $f \sim \sum c_k \phi_k$, define Tf by $Tf \sim \sum m_k c_k \phi_k$. Show that T is bounded on L^2, i.e., that there is a constant c independent of f such that $\|Tf\|_2 \leq c\|f\|_2$ for all $f \in L^2$, if and only if $m \in l^\infty$. Show that the smallest possible choice for c is $\|m\|_{l^\infty}$.*

Solution. Suppose that $m \in l^\infty$. Use Parserval's formula, we have

$$\|Tf\|_2^2 = \sum_{k=1}^{\infty} |m_k c_k|^2 \leq \|m\|_\infty^2 \sum_{k=1}^{\infty} |c_k|^2 \leq \|m\|_\infty^2 \|f\|_2^2.$$

Conversely, for any k, we have

$$|m_k|^2 = \sum_{j=1}^{\infty} |m_j \langle \phi_k, \phi_j \rangle|^2 \leq \|T\phi_k\|_2^2 \leq c^2 \|\phi_k\|_2^2 = c^2$$

for some c. Hence $\|m\|_\infty \leq c$. It follows that $m \in l^\infty$ and the smallest possible choice for c is $\|m\|_{l^\infty}$. Then we complete this proof.

Chapter 9

Approximations of the Identity; Maximal Functions

Exercise 9.1 *Use Minkowski's integral inequality* (see Exercise 8, Chapter 8) *to prove* (9.1)*, which is let* $1 \leq p \leq \infty$*,* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^1(\mathbb{R}^n)$*. Then* $f * g \in L^p(\mathbb{R}^n)$ *and*

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

Solution. If $p = \infty$, then for any $x \in \mathbb{R}^n$, we have

$$|\int_{\mathbb{R}^n} f(x-y)g(y)dy| \leq \|f\|_\infty \|g\|_1.$$

Hence $\|f * g\|_\infty \leq \|f\|_\infty \|g\|_1$. If $1 \leq p < \infty$, we have

$$\|f * g\|_p = \| \int_{\mathbb{R}^n} f(\cdot - y)g(y)dy\|_p \leq \int_{\mathbb{R}^n} \|f(\cdot - y)\|_p |g(y)|dy \leq \|f\|_p \|g\|_1,$$

which complete the proof.

Exercise 9.2 *Prove Young's theorem* (9.2). [For $f, g \geq 0$ and $p, q, r < \infty$, write

$$(f * g)(x) = \int f(t)^{p/r} g(x-t)^{q/r} \cdot f(t)^{p(1/p-1/r)} \cdot g(x-t)^{q(1/q-1/r)} dt,$$

and apply Hölder's inequality for three functions (Exercise 6, Chapter 8) with exponents r, p_1, and p_2, where $1/p_1 = 1/p - 1/r$, $1/q - 1/r$.]

Solution. Since $1/p_1 + 1/p_2 + 1/r = 1$, then

$$\begin{aligned}
& \|(f * g)(x)\|_r \\
= & \ \|\int f(t)^{p/r} g(x-t)^{q/r} \cdot f(t)^{p(1/p-1/r)} \cdot g(x-t)^{q(1/q-1/r)} dt\|_r \\
\le & \ (\int\int |f(t)|^p |g(x-t)|^q dt dx)^{1/r} \|f\|_p^{p/p_1} \|g\|_q^{q/p_2} \\
= & \ (\int\int |f(t)|^p |g(x-t)|^q dx dt)^{1/r} \|f\|_p^{p/p_1} \|g\|_q^{q/p_2} \\
\le & \ \|f\|_p^{p/r} \|g\|_q^{q/r} \|f\|_p^{p/p_1} \|g\|_q^{q/p_2} \\
= & \ \|f\|_p \|g\|_q.
\end{aligned}$$

Then we complete this proof.

Exercise 9.3 *Show that if $f \in L^p(\mathbb{R}^n)$ and $K \in L^{p'}(\mathbb{R}^n)$, $1 \le p \le \infty$, $1/p + 1/p' = 1$, then $f * K$ is bounded and continuous in $\mathbb{R}^n$.*

Solution. Obviously, $f * K$ is bounded in $\mathbb{R}^n$ since

$$f * K \le \|f\|_p \|K\|_{p'}.$$

Now we show that $f * K$ is continuous in $\mathbb{R}^n$. If $1 \le p < \infty$, then

$$|f * K(x+h) - f * K(x)| \le \|K\|_{p'} \|f(x+h) - f(x)\|_p \to 0 \text{ as } h \to 0$$

since $f \in L^p(\mathbb{R}^n)$. If $p = \infty$, then $p' = 1$, we have

$$|f * K(x+h) - f * K(x)| \le \|K(x+h) - K(x)\|_1 \|f\|_\infty \to 0 \text{ as } h \to 0$$

since $K \in L^1(\mathbb{R}^n)$. Then we complete this proof.

Exercise 9.4 (a) *Show that the function h defined by $h(x) = e^{-1/x^2}$ for $x > 0$ and $h(x) = 0$ for $x \le 0$ is in C^∞.*

(b) *Show that the function $g(x) = h(x-a)h(b-x)$, $a < b$, is C^∞ with support $[a, b]$.*

(c) *Construct a function in $C_0^\infty(\mathbb{R}^n)$ whose support is a ball or an interval.*

Solution.

(a) Let $y = 1/x$ and $P_m(y)$ be a polynomial such that $h^{(m)}(x) = P_m(y)/e^{y^2}$, then $P_m(y)/e^{y^2} \to 0$ as $y \to \infty$. Hence $h \in C^\infty$.

(b) Let $g(x) = h(x-a)h(b-x)$, then $g \in C^\infty$ with support $[a, b]$ since $h \in C^\infty$, $h(x-a) = 0$ if $x \le a$ and $h(b-x) = 0$ if $x \ge b$.

(c) Let $f(x) = e^{-1/(1-|x|)^2}$ if $|x| < 1$ and $f(x) = 0$ if $|x| \ge 1$, then $f \in C_0^\infty(\mathbb{R}^n)$.

Exercise 9.5

Exercise 9.6 *Prove theorem (9.4). That is, if $f \in L(\mathbb{R}^n)$ and K is bounded and uniformly continuous on $\mathbb{R}^n$, then $f * K$ is bounded and uniformly continuous on $\mathbb{R}^n$.*

Solution. The $f * K$ is bounded since $|f * K(x)| \leq M\|f\|_1$ where $|K(x)| < M$ for every $x \in R^n$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $|x-y| < \delta$, we have

$$|K(x) - K(y)| < \varepsilon.$$

Then

$$|K * f(x) - K * f(y)| \leq \int_{\mathbb{R}^n} |f(z)||K(x-z) - K(y-z)|dz < \varepsilon\|f\|_1$$

for any $|x-y| < \delta$. Then we complete this proof.

Exercise 9.7 *Let $f \in L^p(-\infty, +\infty)$, $1 \leq p \leq \infty$. Show that the Poisson integral of f, $f(x,y)$ is harmonic in the upper half-plane $y > 0$.* [Show that $((\partial^2/\partial x^2) + (\partial^2/\partial y^2))f(x,y) = \int_{-\infty}^{+\infty} f(t)((\partial^2/\partial x^2) + (\partial^2/\partial y^2))P_y(x-t)dt$.]

Solution. Since $\frac{\partial}{\partial x}P(x,y)$, $\frac{\partial^2}{\partial x^2}P(x,y)$, $\frac{\partial}{\partial y}P(x,y)$, $\frac{\partial^2}{\partial y^2}P(x,y)$ are continuous uniformly. We have

$$\begin{aligned}\frac{f(x+h,y) - f(x,y)}{h} &= \int_{-\infty}^{+\infty} f(t)\frac{P_y(x+h-t) - P_y(x-t)}{h}dt \\ &= \int_{-\infty}^{+\infty} f(t)\frac{\partial}{\partial x}P_y(x+h'-t)dt\end{aligned}$$

Exercise 9.8 (Schur's lemma) *For $s,t \geq 0$, let $K(s,t)$ satisfy $K \geq 0$ and $K(\lambda s, \lambda t) = \lambda^{-1}K(s,t)$ for all $\lambda > 0$, and suppose that $\int_0^\infty t^{-1/p}K(1,t)dt = \gamma < +\infty$ for some p, $1 \leq p \leq \infty$. For example, $K(s,t) = 1/(s+t)$ has these properties. Show that if*

$$(Tf)(s) = \int_0^\infty f(t)K(s,t)dt \qquad (f \geq 0),$$

then $\|Tf\|_p \leq \gamma\|f\|_p$. [Note that $K(s,t) = s^{-1}K(1,t/s)$, so that $(Tf)(s) = \int_0^\infty f(ts)K(1,t)dt$. Now apply Minkowski's inequality (see Exercise 8, Chapter 8).]

Solution. Since

$$(Tf)(s) = \int_0^\infty f(t)K(s,t)dt = \int_0^\infty f(t)K(1,t/s)dt/s = \int_0^\infty f(su)K(1,u)du.$$

If $1 \leq p < \infty$, then

$$\begin{aligned}\|Tf\|_p &= (\int_{\mathbb{R}^n} |\int_0^\infty f(su)K(1,u)du|^p ds)^{1/p} \\ &\leq \int_0^\infty (\int_{\mathbb{R}^n} |f(su)K(1,u)|^p ds)^{1/p}du \\ &= \int_0^\infty (\int_{\mathbb{R}^n} |f(su)|^p d(su))^{1/p}u^{-1/p}K(1,u)du \\ &\leq \gamma\|f\|_p.\end{aligned}$$

If $p = \infty$, we have

$$|Tf(s)| = |\int_0^\infty f(su)K(1,u)du| \leq \int_0^\infty |f(su)K(1,u)|du \leq \gamma\|f\|_\infty.$$

Hence $\|Tf\|_\infty \leq \gamma\|f\|_\infty$ and the exercise follows.

Exercise 9.9 *The maximal function is defined as* $f^*(x) = \sup |Q|^{-1} \int_Q |f|$, *where the supremum is taken over cubes* Q *with center* x. *Let* $f^{**}(x)$ *be defined similarly, but with the supremum taken over all* Q *containing* x. *Thus,* $f^*(x) \leq f^{**}(x)$. *Show that there is a positive constant* c *depending only on the dimension such that* $f^{**}(x) \leq cf^*(x)$.

Solution. We will index the size of a cube Q with center x by writing Q_x and $Q(r)$, where r is the edge length of Q. Then for any x and any $Q(r)$ containing x, we have

$$\frac{1}{|Q(r)|}\int_{Q(r)} |f| \leq \frac{|Q_x(2r)|}{|Q(r)|}\frac{1}{|Q_x(2r)|}\int_{Q_x(2r)} |f| \leq 2^n f^*(x).$$

So $f^{**}(x) \leq 2^n f^*(x)$.

Exercise 9.10 *Let* $T : f \to Tf$ *be a function transformation which is sublinear; that is,* T *has the property that if* Tf_1 *and* Tf_2 *are defined, then so is* $T(f_1+f_2)$, *and*

$$|T(f_1 + f_2)(x)| \leq |(Tf_1)(x)| + |(Tf_2)(x)|.$$

Suppose also that there are constant c_1 *and* c_2 *such that* T *satisfies* $\|Tf\|_\infty \leq c_1\|f\|_\infty$ *and* $|\{x : |(Tf)(x)| > \alpha\}| \leq c_2 a^{-1}\|f\|_1$, $\alpha > 0$. *Show that for* $1 < p < \infty$, *there is a constant* c_3 *such that* $\|Tf\|_p \leq c_3\|f\|_p$. *This is a special case of an interpolation result due to Marcinkiewicz.* (An example of such a T is the maximal function operator $Tf = f^*$, and the proof in the general case is like that for f^*.)

Solution. For any $\alpha > 0$ and f be a function. Let

$$f_1(x) = \begin{cases} f(x) & \text{if } |f(x)| > \alpha/2c_1, \\ 0 & \text{if } |f(x)| \leq \alpha/2c_1. \end{cases}$$

Then

$$|T(f)| \leq |T(f_1)| + |T(f - f_1)| \leq |T(f_1)| + \alpha/2$$

since $\|T(f - f_1)\|_\infty \leq c_1\|f - f_1\|_\infty$. Thus,

$$\begin{aligned}
\int_{\mathbb{R}^n} |T(f)(x)|^p dx &= p\int_0^\infty \alpha^{p-1}|\{x : |T(f)(x)| > \alpha\}|d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}|\{x : |T(f_1)(x)| > \alpha/2\}|d\alpha \\
&\leq p\int_0^\infty \alpha^{p-1}\frac{2c_2}{\alpha}(\int_{\{2c_1|f|>\alpha\}} |f(x)|dx)d\alpha \\
&= 2c_2 p\int_{\mathbb{R}^n} |f(x)|\int_0^{2c_1|f(x)|} \alpha^{p-2}d\alpha dx \\
&= \frac{2^p c_1^{p-1} c_2 p}{p-1}\|f\|_p^p,
\end{aligned}$$

and the exercise follows.

Exercise 9.11

Exercise 9.12

Exercise 9.13 *Let $f \in L^p(0,1)$, $1 \le p < \infty$, and for each $k = 1, 2, \ldots$, define a function f_k on $(0,1)$ by letting $I_{k,j} = \{x : (j-1)2^{-k} \le x < j2^{-k}\}$, $j = 1, 2, \ldots, 2^k$, and setting $f_k(x)$ equal to $|I_{k,j}|^{-1} \int_{I_{k,j}} f$ for $x \in I_{k,j}$. Prove that $f_k \to f$ in $L^p(0,1)$ norm. [Exercise 18 of Chapter 7 may be helpful for the case $p = 1$.]*

Solution. If $1 < p < \infty$, let $f \in L^p(0,1)$, then $f_k \to f$ a.e. since $f \in L(0,1)$. Given any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $|A| < \delta$, we have $\int_A |f|^p < \varepsilon$ and $\int_A |f^*|^p < \varepsilon$. By Egorov's Theorem, there exists a closed set $F \subset (0,1)$ such that $|(0,1) - F| < \delta$ and $f_k \to f$ uniformly on F. Then

$$\begin{aligned}
\int_{(0,1)} |f_k - f|^p &= \int_{(0,1)-F} |f_k - f|^p + \int_F |f_k - f|^p \\
&\le 2^p \int_{(0,1)-F} |f_k|^p + 2^p \int_{(0,1)-F} |f|^p + \int_F |f_k - f|^p \\
&\le 2^p \int_{(0,1)-F} |f^*|^p + 2^p \int_{(0,1)-F} |f|^p + \int_F |f_k - f|^p \\
&\to 2^{p+1}\varepsilon,
\end{aligned}$$

and the exercise follows.

Chapter 10

Abstract Integration

Exercise 10.1

Exercise 10.2 *A measure space* $(\mathscr{S}, \Sigma, \mu)$ *is said to be complete if* Σ *contains all subsets of sets with measure measure zero; i.e.,* $(\mathscr{S}, \Sigma, \mu)$ *is complete if* $Y \in \Sigma$ *whenever* $Y \subset Z$, $Z \in \Sigma$, *and* $\mu(Z) = 0$. *In this case, show that if* f *is measurable and* $g = f$ *a.e., then* g *is also measurable. Is this true if* $(\mathscr{S}, \Sigma, \mu)$ *is not complete?*

Solution. Let f and g be measurable functions satisfies $f = g$ a.e., and let $Z = \{f \neq g\}$, then $\mu(Z) = 0$. For any constant a, since $\{g > a, f \neq g\}$ is subset of Z, then it has measure zero. Hence $\{g > a\}$ is measurable. But if $(\mathscr{S}, \Sigma, \mu)$ is not complete, the set $\{g > a, f \neq g\}$ is maybe nonmeasurable. For example, let $\mathscr{S} = \{0, 1, 2\}$, $\Sigma = \{\phi, \{0, 1, 2\}, \{0\}, \{1, 2\}\}$ and let μ be the function with $\mu(\phi) = 0$, $\mu(\{0, 1, 2\}) = 1$, $\mu(\{0\}) = 1$ and $\mu(\{1, 2\}) = 0$. Then Σ is a σ-algebra and μ is a measure. Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in \{1, 2\}, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2 & \text{if } x = 1, \\ 3 & \text{if } x = 2. \end{cases}$$

Then $\{f \neq g\} = \{1, 2\}$ has measure zero and f is measurable, but $\{g > 2\} = \{2\}$ is nonmeasurable.

Exercise 10.3

Exercise 10.4 *If* $(\mathscr{S}, \Sigma, \mu)$ *is a measure space, and if* f

Exercise 10.5 *Complete the proof of lemma* (10.18). *That is,*

(i) *If* f *and* g *are nonnegative, simple measurable functions on* E, *and if* c *is a nonnegative constant, then* $\int_E (f + g) d\mu = \int_E f d\mu + \int_E g d\mu$ *and* $\int_E c f d\mu = c \int_E f d\mu$.

(ii) *If* f *is a nonnegative, simple measurable function on* E, *and* $E = E_1 \cup E_2$ *is the union of two disjoint measurable sets, then* $\int_E f d\mu = \int_{E_1} f d\mu + \int_{E_2} f d\mu$.

Solution.

(i) We have

$$\begin{aligned}&\sup\sum_j[\inf_{x\in E_j}(f+g)(x)]\mu(E_j)\\=\;&\sup\sum_j[\inf_{x\in E_j}f(x)]\mu(E_j)+\sup\sum_j[\inf_{x\in E_j}g(x)]\mu(E_j)\end{aligned}$$

and

$$\sup\sum_j[\inf_{x\in E_j}cf(x)]\mu(E_j)=c\sup\sum_j[\inf_{x\in E_j}f(x)]\mu(E_j).$$

Then $\int_E(f+g)d\mu=\int_E fd\mu+\int_E gd\mu$ and $\int_E cfd\mu=c\int_E fd\mu$.

(ii) We have

$$\begin{aligned}&\sup\sum_j[\inf_{x\in E_j}f(x)]\mu(E_j)\\=\;&\sup\sum_j[\inf_{x\in E_j}f(x)]\mu(E_j\cap E_1)+\sup\sum_j[\inf_{x\in E_j}f(x)]\mu(E_j\cap E_2).\end{aligned}$$

Then $\int_E fd\mu=\int_{E_1}fd\mu+\int_{E_2}fd\mu$.

Exercise 10.6 (a) *If $f_1,f_2\in L(d\mu)$ and $\int_E f_1d\mu=\int_E f_2d\mu$ for all measurable E, show that $f_1=f_2$ a.e. (μ).*

(b) *Prove the uniqueness of f and σ in (10.40).*

(c) *Let μ be σ-finite, and let $f_1,f_2\in L^{p'}(d\mu)$, $1/p+1/p'=1$, $1\le p\le\infty$. If $\int f_1gd\mu=\int f_2gd\mu$ for all $g\in L^p(d\mu)$, show that $f_1=f_2$ a.e.*

Solution.

(a) If $f_2=0$, let $B=\{f_1>0\}$ and $B_n=\{h\ge 1/n\}\nearrow B$. Since $0\le f_1\chi_{B_n}\le f_1\chi_B=f_1$, then $\int_{B_n}f_1d\mu=0$. But $\int_{B_n}f_1d\mu\ge(1/n)\mu(B_n)$, so that $\mu(B_n)=0$ for all n, and thus $\mu(B)=0$. For general f_2. Let $f=f_1-f_2$, then $\int_E fd\mu=0$. Hence $\mu(\{f_1\ne f_2\})=0$.

(b) Let $v(A)=\int_A f_1d\mu+\sigma_1(A)=\int_A f_2d\mu+\sigma_2(A)$ for every measurable $A\subset E$. Then

$$\int_A f_1d\mu-\int_A f_2d\mu=\sigma_2(A)-\sigma_1(A)=0$$

since $\sigma_2-\sigma_1$ and μ are mutually singular and $\sigma_2-\sigma_1$ is absolutely continuous. Thus f and σ are unique.

(c) Since $f_1,f_2\in L^{p'}(d\mu)$ and $g\in L^p(d\mu)$, then $\int_E f_1gd\mu$ and $\int_E f_2gd\mu$ are finite. Since μ is σ-finite, then let $E=\bigcup_{k=1}^\infty E_k$ such that $\mu(E_k)<\infty$ for all k. For any k, let $g=\chi_{E_k}$, then $\int_A f_1gd\mu=\int_A f_2gd\mu$ for any measurable set A. By (a), we have $f_1=f_2$ a.e. on E_k. Thus $f_1=f_2$ a.e..

Exercise 10.7

Exercise 10.8 *Show that for $1\le p<\infty$, the class of simple functions vanishing outside sets of finite measure is dense in $L^p(d\mu)$.*

Solution. If $f \geq 0$. By (10.13)(iv), there exist nonnegative, simple measurable $f_k \nearrow f$ on E. Hence $|f_k|^p \nearrow |f|^p$. Then $\|f_k\|_p \to \|f\|_p$. Thus $\|f_k - f\|_p \to 0$ since $f_k \to f$. Suppose there is a simple function f_k on a measurable set E such that $\mu(E) = \infty$. This implies that $\|f\|_p = \infty$. That is a contradiction. Thus the class of simple functions vanishing outside sets of finite measure is dense in $L^p(d\mu)$.

Exercise 10.9 *The symmetric difference of two sets E_1 and E_2 is defined as*

$$E_1 \Delta E_2 = (E_1 - E_2) \cup (E_2 - E_1).$$

Let $(\mathscr{S}, \Sigma, \mu)$ be a measure space, and identify E_1 and E_2 if $\mu(E_1 \Delta E_2) = 0$. Show that Σ is a metric space with distance $d(E_1, E_2) = \mu(E_1 \Delta E_2)$, and that if μ is finite $L^p(\mathscr{S}, \Sigma, \mu)$ is separable if and only if Σ is, $1 \leq p < \infty$.

Solution. Since $\mu \geq 0$, $E_1 \Delta E_2 = E_2 \Delta E_1$ and $E_1 \Delta E_2 \subseteq (E_1 \Delta E_3) \cup (E_3 \Delta E_2)$ then Σ is a metric space with distance d. If μ is finite and $L^p(\mathscr{S}, \Sigma, \mu)$ is separable, let A be a countable dense subset of L^p. For any $f \in A$, there exist the sequence of simple functions $\{f_k\}$ vanishing outside sets of finite measure such that $f_k \to f$ in L^p by Exercise 8. Let B is class of the disjoint union $\bigcup E_i$ satisfies $f_k = \sum a_i E_i$ is for any k and any $f \in A$. Then $B \subset \Sigma$ is countable. For any $E \in \Sigma$, there exists $f_n \in A$ such that $f_n \to \chi_E$ in L^p and exists the simple functions $f_{nm} \to f_n$ in L^p. Hence there exists the sequence of simple functions $\{f_j\}$ with $f_j = \sum a_{jl} E_{jl}$ where $\bigcup_l E_l \in B$ such that $f_j \to \chi_E$ in L^p. For any $\varepsilon > 0$, there exists $M > 0$ such that for any $j \geq M$, we have $\|f_j - \chi_E\|_p < \varepsilon$. Note that $a_{jl} \to 1$ as $j \to \infty$ for any l. Since μ is finite, then for any $j \geq M$, we have

$$\begin{aligned}
\mu^{1/p}(\bigcup_l E_{jl} \Delta E) &= \|\chi_{\bigcup_l E_{jl}} - \chi_E\|_p \\
&\leq \|\chi_{\bigcup_l E_{jl}} - \sum a_{jl} E_{jl}\|_p + \|f_j - \chi_E\|_p \\
&\leq \|\chi_{\bigcup_l E_{jl}} - \sum a_{jl} E_{jl}\|_p + \varepsilon.
\end{aligned}$$

Note that $a_{jl} \to 1$ as $j \to \infty$ for any l. Thus $\mu(\bigcup_l E_{jl} \Delta E) \to 0$ as $j \to \infty$. Then we complete this proof. Conversely, let $B = \{E_k\}$ be a countable dense subset of Σ. Let A be the set of all linear combinations of characteristic functions of these E_k, the coefficients being complex numbers with rational real and imaginary parts Then A is a countable subset of $L^p(\mathscr{S}, \Sigma, \mu)$. To see that A is dense, let f be any function in L^p, there exists the sequence of simple functions $\{f_k\}$ vanishing outside sets of finite measure such that $f_k \to f$ in L^p by Exercise 8. For any f_k, let $f_{kl} \to f_k$ with $f_{kl} \in A$ since B is dense, then $\|f_{kl} - f_k\|_p \to 0$ as $l \to \infty$. Hence there exists $\{f_j\} \subset A$ such that $f_j \to f$ in L^p. Thus A is dense in L^p and the Exercise follows.

Exercise 10.10 *If ϕ is a set function whose Jordan decomposition is $\phi = \overline{V} - \underline{V}$, define*

$$\int_E f d\phi = \int_E f d\overline{V} - \int_E f d\underline{V},$$

provided not both integrals on the right are infinite with the same sign. If V is the total variation of ϕ on E, and if $|f| \leq M$, prove that $|\int_E f d\phi| \leq MV$.

Solution. The functions $\overline{V}$ and $\underline{V}$ are measure since $\overline{V}, \underline{V} \geq 0$ and countably additive. And we have

$$\begin{aligned}|\int_E f d\phi| &= |\int_E f d\overline{V} - \int_E f d\underline{V}| \\ &\leq \int_E |f| d\overline{V} + \int_E |f| d\underline{V} \\ &\leq M\overline{V}(E) + M\underline{V}(E) \\ &= MV(E).\end{aligned}$$

Then we complete this proof.

Exercise 10.11

Exercise 10.12 *Give an example of a pair of measures ν and μ such that ν is absolutely continuous with respect to μ, but given $\varepsilon > 0$, there is no $\delta > 0$ such that $\nu(A) < \varepsilon$ for every A with $\mu(A) < \delta$.* [Thus, the analogue for measures of (10.34) may fail.]
Prove the analogue of (10.35) for mutually singular measures ν and μ.

Solution. Let μ be Lebesgue measure and

$$\mu(A) = \int_A \frac{1}{t} dt$$

for any measurable set A. Then for any $\varepsilon, \delta > 0$, let $A = [0, \delta]$, then $\mu(A) = \infty > \varepsilon$. But ν is absolutely continuous with respect to μ. Next, if ν is mutually singular on E with respect to μ, there exists a measurable set A such that $\phi(A) = 0$ and

Exercise 10.13 *Show that the set P of the Hahn decomposition is unique up to null sets.* [By a null set for ϕ, we mean a set N such that $\phi(A) = 0$ for every measurable $A \subset N$.]

Solution. Let P_1, P_2 be a set such that $\phi(A) \geq 0$ for $A \subset P_1$ or $A \subset P_2$ and $\phi(A) \leq 0$ for $A \subset E - P_1$ or $A \subset E - P_2$. Then for any measurable subset $A \subset P_1 \Delta P_2$, we have

$$\phi(A) = \phi(P_1 \cap A) + \phi(P_2 \cap A).$$

It follows that $\phi(A) = 0$. This completes this proof.

Exercise 10.14

Exercise 10.15

Exercise 10.16 *Consider a convolution operator $Tf(x) = \int_{\mathbb{R}^n} f(y)K(x-y)dy$ with $K \geq 0$. If $\|Tf\|_p \leq M\|f\|_p$, $1 \leq p \leq \infty$, for $f \in L^p(\mathbb{R}^n, dx)$, show that $\|Tf\|_{p'} \leq M\|f\|_{p'}$, $1/p + 1/p' = 1$* [Use Exercise 15 to write $\|Tf\|_{p'} = \sup_{\|g\|_{p \leq 1}} |\int_{\mathbb{R}^n} (Tf)g dx|$, and note that

$$\int_{\mathbb{R}^n} (Tf)(x)g(x)dx = \int_{\mathbb{R}^n} (T\tilde{g})(-y)f(y)dy,$$

where $\tilde{g}(x) = g(-x)$.]

Solution. By Exercise 15, we have

$$\begin{aligned}
\|Tf\|_{p'} &= \sup_{\|g\|_p\leq 1} |\int_{\mathbb{R}^n}(Tf)gdx| \\
&= \sup_{\|g\|_p\leq 1} |\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)K(x-y)g(x)dydx| \\
&= \sup_{\|g\|_p\leq 1} |\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)K(x-y)g(x)dxdy| \\
&= \sup_{\|g\|_p\leq 1} |\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)K(-y-x)g(-x)dxdy| \\
&= \sup_{\|g\|_p\leq 1} |\int_{\mathbb{R}^n}(T\tilde{g})(-y)f(y)dy| \\
&\leq \sup_{\|g\|_p\leq 1} \|T\tilde{g}\|_p\|f\|_{p'} \\
&\leq \sup_{\|g\|_p\leq 1} M\|\tilde{g}\|_p\|f\|_{p'} \\
&\leq M\|f\|_{p'},
\end{aligned}$$

and the exercise follows.

Exercise 10.17

Exercise 10.18

Exercise 10.19

Exercise 10.20

Exercise 10.21