Real Analysis
Homework 7

Yueh-Chou Lee

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1. (Exercise 6.1)

   (a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{y : (x,y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $\{x : (x,y) \in E\}$ has measure zero.

   (b) Let $f(x,y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}^1$, $f(x,y)$ is finite for almost every y. Show that for almost every $yin\mathbb{R}^1$, $f(x,y)$ is finite for almost every x.

Proof.

(a) Since $\chi_E(x,y)$ is nonnegative, measurable in $\mathbb{R}^2$ (E is a measurable subset of $\mathbb{R}^2$) and $\{y : (x,y) \in E\}$ has $\mathbb{R}^1$-measure zero, $\int_{\mathbb{R}^1} \chi_E(x,y)dx = 0$, by Tonelli's Theorem, we have

$$\begin{aligned} |E| &= \int\int_{\mathbb{R}^2} \chi_E(x,y)dxdy \\ &= \int_{\mathbb{R}^1}\left[\int_{\mathbb{R}^1} \chi_E(x,y)dy\right]dx \\ &= \int_{\mathbb{R}^1} |\{y : (x,y) \in E\}|\, dx \\ &= 0 \end{aligned}$$

So $|E|$ has measure zero.

$$\begin{aligned} |E| &= \int\int_{\mathbb{R}^2} \chi_E(x,y)dxdy \\ &= \int_{\mathbb{R}^1}\left[\int_{\mathbb{R}^1} \chi_E(x,y)dx\right]dy \\ &= \int_{\mathbb{R}^1} |\{x : (x,y) \in E\}|\, dy \\ &= 0 \end{aligned}$$

So $\{x : (x,y) \in E\}$ has measure zero almost every y.

(b) Since for almost every $x \in \mathbb{R}^1$, $f(x,y)$ is finite for almost every y, then $\{y|f(x,y) = \infty\}$ has measure zero.
Let $Z = \{(x,y)|f(x,y) = \infty\}$, $Z_1 = \{x|f(x,y) = \infty\}$ and $Z_2 = \{y|f(x,y) = \infty\}$, then $Z = Z_1 \times Z_2$.
Since $f(x,y)$ is nonnegative function and measurable in $\mathbb{R}^2$, $\int_{Z_2} dy = |Z_2| = 0$, by Tonelli's theorem, we have

$$\int\int_Z dxdy = \int_{Z_2}\left[\int_{Z_1} dx\right] dy = \int_{Z_1}\left[\int_{Z_2} dy\right] dx = 0$$

Hence $\int_{Z_1} dx = 0$ for almost every y, then $Z_1 = \{x|f(x,y) = \infty\}$ has also measure zero. So $f(x,y)$ is finite for almost every x.

2. (Exercise 6.3)
Let f be measurable and finite a.e. on $[0,1]$. If $f(x) - f(y)$ is integrable over the square $0 \le x \le 1, 0 \le y \le 1$, show that $f \in L[0,1]$.

Proof.

Let $I_1 = (0,1)$ and $I_2 = (0,2)$ such that $I = I_1 \times I_2$.
Since $g(x,y) = f(x) - f(y) \in L(I)$, by Fubini's Theorem, we know that for almost every $x \in I_1$, $g(x,y)$ is measurable and integrable on I_2 as a function of y.
Pick any $x_0 \in (0,1)$ then $g(x_0,y) = f(x_0) - f(y)$ is measurable and integrable on I_2, that is $f(y)$ is integrable on $(0,1)$.
Hence $f \in L(I_2) = L(0,1)$.

3. (Exercise 6.5)

(a) If f is nonnegative and measurable on E and $\omega(y) = |\{x \in E : f(x) > y\}|, y > 0$, use Tonelli's theroem to prove that $\int_E f = \int_0^\infty \omega(y)dy$. (By definition of the integral, $\int_E f = |R(f,E)| = \int\int_{R(f,E)} dxdy$. Use the observation in the proof of Theroem 6.11 that $\{x \in E : f(x) \ge y\} = \{x : (x,y) \in R(f,E)\}$, and recall that $\omega(y) = |\{x \in E : f(x) \ge y\}|$ unless y is a point of discontinuity of ω.)

(b) Deduce from this special case the general formula

$$\int_E f^p = p\int_0^\infty y^{p-1}\omega(y)dy \quad (f \ge 0,\ 0 < p < \infty)$$

Proof.

(a) By definition of the integral and using the observation in the proof of Theroem 6.11 that $\{x \in E : f(x) \ge y\} = \{x : (x,y) \in R(f,E)\}$, we have

$$\begin{aligned}\int_E f = |R(f,E)| &= \int\int_{R(f,E)} dxdy \\ &= \int_0^\infty \left[\int_{\{x:(x,y)\in R(f,E)\}} dx\right] dy \\ &= \int_0^\infty \left[\int_0^\infty \chi_{\{x\in E: f(x)\ge y\}} dx\right] dy \\ &= \int_0^\infty \omega(y)dy\end{aligned}$$

(b) The truth that

$$f^p(x) = \int_0^{f(x)} p \cdot y^{p-1}\, dy$$

for all $x \in E$.

By using the result of part (a), Tonelli's Theorem and the above truth, then we have

$$\begin{aligned}
\int_E f^p(x)dx &= \int_E \int_0^{f(x)} p \cdot y^{p-1}\, dy\, dx \\
&= \iint\limits_{R(f,E)} p \cdot y^{p-1}\, dy\, dx \\
&= \int_0^\infty \left[\int_{\{x \in E : f(x) \geq y\}} p \cdot y^{p-1}\, dx \right] dy \\
&= p \int_0^\infty y^{p-1} \left[\int_{\{x \in E : f(x) \geq y\}} dx \right] dy \\
&= p \int_0^\infty y^{p-1} \omega(y)\, dy
\end{aligned}$$

4. (Exercise 6.10)
   Let v_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theroem that

$$v_n = 2v_{n-1} \int_0^1 \left(1 - t^2\right)^{(n-1)/2} dt$$

(We also observe that by setting $w = t^2$, the integral is a multiple of a classical β-function and so can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt,\ s > 0$.)

Proof.

Using the induction to prove this formula.
Let $v_1 = 2$, that is the length of the interval $[-1, 1]$.
If $n = 2$, v_2 will be the area of the unit circle, then $v_2 = \pi$. Moreover

$$2v_1 \int_0^1 (1 - t^2)^{1/2}\, dt = 2 \cdot 2 \cdot \frac{\pi}{4} = \pi = v_2$$

So it's ture when $n = 2$.
Suppose the formula holds for $n - 1$ and let

$$B_n = \{x \in \mathbb{R}^n : x_1^2 + ... + x_n^2 \leq 1\}$$

be the unit ball in $\mathbb{R}^n$.
Using Tonelli's Theorem, then we have

$$\begin{aligned}
v_n &= \int ... \int_{B_n} 1 \\
&= \int ... \int_{\{x_1^2 + ... + x_n^2 \leq 1\}} 1\, dx_1 \,...\, dx_n \\
&= \int_{-1}^1 \left(\int ... \int_{\{x_2^2 + ... x_n^2 \leq 1 - x_1^2\}} 1\, dx_2 \,...\, dx_n \right) dx_1
\end{aligned}$$

Let $u_j = \frac{x_j}{\sqrt{1 - x_1^2}}$ for $j = 2, ..., n$, then $\frac{du_j}{dx_j} = \frac{1}{\sqrt{1 - x_1^2}}$.

Hence

$$
\begin{aligned}
v_n &= \int_{-1}^{1} \left(\int ... \int_{\{x_2^2+...x_n^2 \le 1-x_1^2\}} 1\, dx_2 \,...\, dx_n \right) dx_1 \\
&= \int_{-1}^{1} \left(\int ... \int_{\{u_1^2+...+u_n^2 \le 1\}} (1-x_1^2)^{\frac{n-1}{2}}\, du_2 \,...\, du_n \right) dx_1 \\
&= \int_{-1}^{1} \left(\int ... \int_{\{u_1^2+...+u_n^2 \le 1\}} du_2 \,...\, du_n \right) (1-x_1^2)^{\frac{n-1}{2}}\, dx_1 \\
&= \int_{-1}^{1} (v_{n-1})(1-x_1^2)^{\frac{n-1}{2}}\, dx_1 \\
&= v_{n-1} \int_{-1}^{1} (1-x_1^2)^{\frac{n-1}{2}}\, dx_1 \\
&= 2v_{n-1} \int_{0}^{1} (1-x_1^2)^{\frac{n-1}{2}}\, dx_1 \\
&= 2v_{n-1} \int_{0}^{1} \left(1-t^2\right)^{(n-1)/2} dt
\end{aligned}
$$

5. (Exercise 6.11)
   Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}$$

(For $n = 1$, write $\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2-y^2} dxdy$ and use polar coordinates. For $n > 1$, use the formula $e^{-|x|^2} = e^{-x_1^2}...e^{-x_n^2}$ and Fubini's theorem to reduce to the case $n = 1$.)

Proof.
By Fubini's Theorem, we know that

$$
\begin{aligned}
\int_{\mathbb{R}^n} e^{|x|^2} dx &= \int ... \int_{\mathbb{R}^n} e^{-x_1^2}...e^{-x_n^2} dx_1...dx_n \\
&= \int_0^{\infty} \left[\int ... \int_{\mathbb{R}^{n-1}} e^{-x_1^2}...e^{-x_n^2} dx_2...dx_n \right] dx_1 \\
&= \int_0^{\infty} e^{-x_1^2} dx_1 \left[\int ... \int_{\mathbb{R}^{n-1}} e^{-x_2^2}...e^{-x_n^2} dx_2...dx_n \right] \\
&= ... \\
&= \int_0^{\infty} e^{-x_1^2} dx_1 \cdot ... \cdot \int_0^{\infty} e^{-x_n^2} dx_n \\
&= \sqrt{\pi} \cdot ... \cdot \sqrt{\pi} \\
&= \pi^{n/2}
\end{aligned}
$$