Real Analysis
Homework 1

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1. Construct a two-dimensional Cantor set in the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$ as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which forms a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals $C \times C$.

   Proof.
   (i) The resulting set is perfect:
   To prove $C \times C$ is perfect, we have to show that for each point $x \in C \times C$ and for each $\epsilon > 0$, we can find a point $y \in C \times C - \{x\}$ s.t. $|x - y| < \epsilon$

   Choose k so that $\sqrt{2/3^{2k}} < \epsilon$

   Suppose $x \in C \times C$. Let $[a_i, b_i] \times [a_j, b_j]$ be the interval of $C_k \times C_k$ that contains x, for all $i, j \in \mathbb{N}$

   When the area is removed from $[a_i, b_i] \times [a_j, b_j]$, we get four parts $[a_{i1}, b_{i1}] \times [a_{j1}, b_{j1}]$, $[a_{i1}, b_{i1}] \times [a_{j2}, b_{j2}]$, $[a_{i2}, b_{i2}] \times [a_{j1}, b_{j1}]$ and $[a_{i2}, b_{i2}] \times [a_{j2}, b_{j2}]$ of $C_{k+1} \times C_{k+1}$. The point x is contained in one of those four parts, and there is a point $y \in C \times C$ that is contained in the other one of those four intervals. So $y \neq x$ and $|x - y| \leq \sqrt{2/3^{2k}} < \epsilon$

   (ii) The resulting set has measure zero:

$$|C \times C| = \lim_{k \to \infty} (1 - \sum_{k=1}^{\infty} 5 \cdot 2^{2(k-1)} \cdot 3^{-2k}) = 1 - 5 \cdot \lim_{k \to \infty} \sum_{k=1}^{\infty} 2^{2(k-1)} \cdot 3^{-2k} = 1 - 1 = 0$$

   Hence, the resulting set has measure zero.

2. Construct a subset of $[0, 1]$ in the same manner as the Cantor set by removing from each remaining interval a subinterval of relative length δ, $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1 - \delta$, and contains no intervals.

   Proof.
   (i) The resulting set is perfect:
   To prove this Cantor set C is perfect, we have to show that for each $x \in C$ and for each $\epsilon > 0$, we can find a point $y \in C - \{x\}$ such that $|x - y| < \epsilon$.

   To search for y, recall that C is constructed as the intersection of sets $C_0 \supset C_1 \supset C_2 \supset C_3 \supset ...$ where C_k is a union of 2^k disjoint intervals each of length $\delta/3^k$, and recall also that C has nonempty intersection with each of those 2^k intervals.

Choose k so that $\delta/3^k < \epsilon$

Let $[a, b]$ be the interval of C_k that contains x.

When the middle third is removed from $[a, b]$, one gets two intervals $[a, b']$, $[b'', b]$ of C_{k+1}. The point x is contained in one of those two intervals, and there is a point $y \in C$ that is contained in the other one of those two intervals. So $y \neq x$ and $|x - y| < \epsilon$.

(ii) The resulting set has measure $1 - \delta$:
Let E_k be the kth stage of the resulting subinterval, so that the resulting set is $C = \cap_{k=1}^{\infty} C_k$.
By the process of (i), we have

$$|C| = |\cap_{k=1}^{\infty} C_k| = \lim_{k\to\infty} (1 - \sum_{i=1}^{k} 2^{i-1}\delta 3^{-i}) = 1 - \delta$$

3. Construct a Cantor-type subset of $[0, 1]$ by removing from each interval remaining at the kth stage a subinterval of relative length θ_k, $0 < \theta_k < 1$. Show that the remainder has measure zero if and only if $\sum \theta_k = +\infty$.

Proof.
Let E_k be the kth stage of the Cantor-type subinterval.
As we know that Cantor-type set E is equivalent to $\cap_k E_k$, so $|E| = |\cap_k E_k| = \prod_k (1 - \theta_k)$

($\Leftarrow$)
Since $0 < \theta_k < 1$, we have

$$log(1 - \theta_k) = \int_0^{\theta_k} \frac{-1}{1 - x} dx = -\int_0^{\theta_k} 1 + x + x^2 + x^3 + ...dx = -\sum_{n=1}^{k} \frac{\theta_k}{k}$$

$$\Rightarrow -log(1 - \theta_k) > \theta_k$$

Moreover, we know $\sum \theta_k = +\infty$, so

$$\sum \theta_k = +\infty < -\sum log(1 - \theta_k) = -log(\prod_k (1 - \theta_k))$$

Hence, $log(\prod_k (1 - \theta_k)) = -\infty$.
Since $log(\prod_k (1 - \theta_k)) \to -\infty \Rightarrow \prod_k (1 - \theta_k) = |E| \to 0$
Hence, if $\sum \theta_k = +\infty$ then the remainder has measure zero.

($\Rightarrow$)
If $\sum \theta_k = c < +\infty$, then $\theta_k \to 0$ as $k \to \infty$ and $(1 - \theta_k) \to 1$ as $k \to \infty$.
Hence, $|E| = \lim_{k\to\infty} |E_k| = \prod_k (1 - \theta_k) > 0$.
So the remainder has measure zero if $\sum \theta_k = +\infty$.

4. If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $limsup E_k$ has measure zero.

Proof.
Let $\epsilon > 0$. By the definition of limit, since $lim_{n\to\infty} \sum_{k=1}^{n} |E_k|_e = \sum |E_k|_e < +\infty$, there exists N such that for $n \geq$ N,

$$\sum_{k=n+1}^{\infty} |E_k|_e = \sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{n} |E_k|_e < \epsilon$$

Since $limsupE_k = \cap_{j=1}^{\infty} \cup_{k=j}^{\infty} E_k \subseteq \cup_{k=N+1}^{\infty} E_k$,

$$|limsupE_k|_e \leq \left|\cup_{k=N+1}^{\infty} E_k\right|_e \leq \sum_{k=N+1}^{\infty} |E_k|_e < \epsilon$$

$\epsilon > 0$ is arbitrary, so $|limsupE_k|_e = 0$. Hence, $limsupE_k$ has measure zero.

Since $liminfE_k \subseteq limsupE_k$, $|liminfE_k|_e \leq |limsupE_k|_e = 0$.
Hence, $liminfE_k$ has also measure zero.

5. If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

Proof.
Since E_1 is measurable, we have

$$|E_1 \cup E_2| = |(E_1 \cup E_2) \cap E_1| + |(E_1 \cup E_2) \cap E_1^c| = |E_1| + |E_2 \cap E_1^c|$$

Moreover, we know that

$$|E_2| = |E_1 \cap E_2| + |E_1^c \cap E_2|$$
$$\Rightarrow |E_1^c \cap E_2| = |E_2| - |E_1 \cap E_2|$$

Hence,

$$|E_1 \cup E_2| = |E_1| + |E_2 \cap E_1^c| = |E_1| + (|E_2| - |E_1 \cap E_2|)$$
$$\Rightarrow |E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$$

6. If E_1 and E_2 are measurable subsets of R^1, show that $E_1 \times E_2$ is a measurable subset of R^2 and $|E_1 \times E_2| = |E_1|\,|E_2|$.

Proof.
(i) $|E_1 \times E_2|$ is measurable:
Since E_1 and E_2 are measurable, $E_1 = H_1 \cup Z_1$ and $E_2 = H_2 \cup Z_2$, where H_1, H_2 are of type F_σ and $|Z_1| = 0$, $|Z_2| = 0$.
Then

$$E_1 \times E_2 = (H_1 \times H_2) \cup (H_1 \times Z_2) \cup (Z_1 \times H_2) \cup (Z_1 \times Z_2)$$

Since $H_1 \times H_2$ is also of type F_σ, we have to prove other terms have measure zero.

Let $\epsilon > 0$. Since $|Z_2| = 0$, there exists intervals $\{I_k\}$ such that $Z_2 \subseteq \cup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} |I_k| < \epsilon$
Write $H_1^n = H_1 \cap [-n, n]$, then $H_1 = \cup_{n=1}^{\infty} H_1^n$. Note that

$$H_1^n \times Z_2 \subseteq [-n, n] \times \cup_{k=1}^{\infty} I_k = \cup_{k=1}^{\infty}([-n, n] \times I_k)$$

so

$$|H_1^n \times Z_2|_e \leq \sum_{k=1}^{\infty} 2n|I_k| = 2n\epsilon$$

Since $\epsilon > 0$ is arbitrary, $|H_1^n \times Z_2| = 0$ for each n. So

$$|H_1 \times Z_2| = |\cup_{n=1}^{\infty} (H_1^n \times Z_2)|_e \leq \sum_{n=1}^{\infty} |H_1^n \times Z_2|_e = 0$$

Thus $|H_1 \times Z_2| = 0$, similarly, $Z_1 \times H_2$ and $Z_1 \times Z_2$ have also measure zero.
Hence

$$|E_1 \times E_2| = |(H_1 \times H_2) \cup (H_1 \times Z_2) \cup (Z_1 \times H_2) \cup (Z_1 \times Z_2)| = |E_1||E_2|$$

(ii) $|E_1 \times E_2| = |E_1||E_2|$:
Case 1 Suppose $|E_1|$ and $|E_2|$ are both finite:
Since E_1, E_2 are measurable, for each $k \in \mathbb{N}$ there are open sets $S_k \supseteq E_1$, $T_k \supseteq E_2$ such that $|S_k - E_1| < 1/k$, $|T_k - E_2| < 1/k$. We may assume $S_{k+1} \subseteq S_k$, $T_{k+1} \subseteq T_k$.

Since S_k is open, $S_k = \cup_{k \in \mathbb{N} I_i}$ for some non-overlapping closed intervals. Similarly, $T_k = \cup_{j \in \mathbb{N}} J_j$ for some non-overlapping closed intervals.
So

$$|S_k \times T_k| = |\cup_{(i,j) \in \mathbb{N} times \mathbb{N}} (I_i \times J_j)| = \sum_{i,j \in \mathbb{N}} |I_i \times J_j| = \sum_{i,j \in \mathbb{N}} |I_i||J_j| = (\sum_{i \in \mathbb{N}} |I_i|)(\sum_{j \in \mathbb{N}} |J_j|) = |S_k||T_k|$$

Write $S = \cap_{k=1}^{\infty} S_k$, $T = \cap_{k=1}^{\infty} T_k$. then $|S - E_1| = |T - E_2| = 0$.
Hence

$$|E_1 \times E_2| = |S \times T| = \lim_{k \to \infty} |S_k \times T_k| = \lim_{k \to \infty} |S_k||T_k| = |E_1||E_2|$$

where the second equality follows by Monotone Convergence Theorem for measure, since $S_k \times T_k \searrow S \times T$ and $|S_k \times T_k| < \infty$ for some k since $|E_1|$, $|E_2|$ are both finite. The last equality also follows by Monotone Convergence Theorem for measure.

Case 2 Suppose one of $|E_1|$, $|E_2|$ are infinite:
If $|E_1| = \infty$ and $|E_2| > 0$, then write $E_1^n = E_1 \cap [-n, n]$.

$$|E_1 \times E_2| = \lim_{n \to \infty} |E_1^n \times E_2| = \lim_{n \to \infty} |E_1^n||E_2| = |E_1||E_2| = \infty$$

where the first equality follows by Monotone Convergence Theorem for measure, since $E_1^n \times E_2 \nearrow E_1 \times E_2$.
If $|E_1| = \infty$ and $|E_2| = 0$, $|E_1 \times E_2| = 0$ by our first lemma.

7. Motivated by (3.7), define the *inner measure* of E by $|E|_i = sup\,|F|$, where the supremum is taken over all closed subsets F of E. Show that (i) $|E|_i \leq |E|_e$, and (ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$

Proof.
(i)
Since F is closed, we know F is measurable and $|F| = |F|_e$.
Since $F \subset E \Rightarrow |F| = |F|_e \leq |E|_e$.
Moreover, by the definition of *inner measure*, we now have

$$|E|_i = sup|F| \leq sup|E|_e = |E|_e$$

(ii)
By **Lemma 3.22**, E is measurable if and only if give $\epsilon > 0$, there exists a closed set $F \subset E$ such that $|E - F|_e < \epsilon$.

So what we need to do is to prove that $|E-F|_e < \epsilon$ is equivalent to $|E|_i = |E|_e$.

$E = (E-F) \cup F$ and $|E|_e < \infty$, so we have

$$|E|_e = |E-F|_e + |F|_e < \epsilon + |F|_e \Rightarrow |E|_e < sup|F| = |E|_i$$

Since $|E|_e \leq sup|F| = |E|_i$ and (i) $|E|_i \leq |E|_e$, hence, $|E|_i = |E|_e$.

8. Show that the conclusion of part (ii) of Exercise 13 is false if $|E|_e = +\infty$.

Proof.
Given A is a non-measurable set in $(0,1)$ and define $E = (-\infty, 0] \cup A \cup [1, +\infty)$.
Then we will have

$$|E|_e = \infty = |E|_i$$

But E is non-measurable, so the conclusion of part (ii) in previous Exercise will be false if $|E|_e = +\infty$.