Real Analysis
Homework 3

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1. (Exercise 10.2)
   A measure space $(\mathscr{S}, \Sigma, \mu)$ is said to be *complete* if Σ contains all subsets of sets with measure zero; that is, $(\mathscr{S}, \Sigma, \mu)$ is complete if $\Upsilon \in \Sigma$ whenever $\Upsilon \subset \mathrm{Z}$, $\mathrm{Z} \in \Sigma$, and $\mu(\mathrm{Z}) = 0$. In this case, show that if f is measurable and $g = f$ a.e. (μ), then g is also measurable (cf. Theorem 4.5 and Chapter 3, Exercise 34). Is this true if $(\mathscr{S}, \Sigma, \mu)$ is not complete?
   Give an example of an incomplete measure space with a measure that is neither identically infinte nor identically zero.

   Proof.

   (a) Let f and g be measurable functions satisfies $f = g$ a.e. (μ), and let $Z = \{f \neq g\}$, tehn $\mu(Z) = 0$.
   For any constant a, since $\{g > a, f \neq g\}$ is subset of Z, then it has measure zero. Hence $\{g > a\}$ is measurable.

   (b) But if $(\mathscr{S}, \Sigma, \mu)$ is not complete, the set $\{g > a, f \neq g\}$ is maybe nonmeasurable.
   For example, let $\mathscr{S} = \{0, 1, 2\}$. $\Sigma = \{\phi, \{0, 1, 2\}, \{0\}, \{1, 2\}\}$ and let μ be the function with $\mu(\phi) = 0$, $\mu(\{0, 1, 2\}) = 1$, $\mu(\{0\}) = 1$ and $\mu(\{1, 2\}) = 0$, then Σ is a σ-algebra and μ is a measure.
   Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = \{1, 2\} \end{cases}, \qquad g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \end{cases}$$

   Then $\{f \neq g\} = \{1, 2\}$ has measure zero and f is measurable, but $\{g > 2\} = \{2\}$ is nonmeasurable.

2. (Exercise 10.3)
   Theorem 10.14 (Egorov's Theorem)
   Let $(\mathscr{S}, \Sigma, \mu)$ be a measure space, and let E be a measurable set with $\mu(E) < +\infty$. Let $\{f_k\}$ be a sequence of measurable functions on E such that each f_k is finite a.e.(μ) in E and $\{f_k\}$ converges a.e.(μ) in E to a finite limit. Then, given $\epsilon > 0$, there is a measurable set $A \subset E$ with $\mu(E - A) < \epsilon$ such that $\{f_k\}$ converges uniformly on A.

   Proof.
   For $n, k \in \mathbb{N}$, define

$$E_{n,k} = \bigcup_{m \geq n} \left\{ x \in E \,\middle|\, |f_m(x) - f(x)| \geq \frac{1}{k} \right\}$$

   Thus $E_{n+1,k} \subset E_{n,k}$.

   For a point x, the sequence $\{f_m(x)\}$ converges to $f(x)$, but it cannot occur in every set $E_{n,k}$, since $f_m(x)$ has to stay closer to $f(x)$ than $\frac{1}{k}$ eventually.

Hence by the assumption of μ-almost everywhere pointwise convergence on E, then

$$\mu\left(\bigcap_{n\in\mathbb{N}} E_{n,k}\right) = 0, \quad \forall k$$

Since E is of finte measure, we have continuity from above; hence there exists, for each k, and for some $n_k \in \mathbb{N}$ such that

$$\mu(E_{n_k,k}) < \frac{\epsilon}{2^k}$$

Let

$$A = \bigcup_{k\in\mathbb{N}} E_{n_k,k}$$

as the set of all those points x in E.

On the set $E - A$ we therefore have uniform convergence.

Appealing to the σ additivity of μ and using the geometric series, we get

$$\mu(A) \leq \sum_{k\in\mathbb{N}} \mu(E_{n_k,k}) < \sum_{k\in\mathbb{N}} \frac{\epsilon}{2^k} = \epsilon$$

3. (Exercise 10.4)
If $(\mathscr{S}, \Sigma, \mu)$ is a measure space, and if f and $\{f_k\}$ is said to *converge* in μ-measure on E to limit f if

$$\lim_{k\to\infty} \mu\{x \in E : |f(x) - f_k(x)| < \epsilon\} = 0 \text{ for all } \epsilon > 0$$

Formulate and prove analogues of Theorems 4.21 through 4.23.

   (a) Let f and f_k, $k = 1, 2, \cdots$, be measurable and finite a.e. in E. If $f_k \to f$ a.e. on E and $|E| < +\infty$, then $f_k \to f$ in μ-measure on E.

   Proof.
   Given $\epsilon, \eta > 0$, let F be the closed subset of E and $K \in \mathbb{N}$.
   If $k > K$, $\mu\{x \in E : |f(x) - f_k(x)| > \epsilon\} \subset \mu(E - F)$ and since $|E - F| < \eta$, then $f_k \to f$ in μ-measure on E.

   (b) If $f_k \to f$ in μ-measure on E, there is a subsequence $\{f_{k_j}\}$ such that $f_{k_j} \to f$ a.e. in E.

   Proof.
   Since $f_k \to f$ in μ-measure on E, given $j = 1, 2, \cdots$, there exists k_j such that

$$\mu\left\{|f - f_k| > \frac{1}{j}\right\} < \frac{1}{2^j} \quad \text{for } k \geq k_j$$

   We may assume that $k_j \nearrow$. Let $E_j = \{|f - f_{k_j}| > 1/j\}$ and $H_m = \bigcup_{j=m}^{\infty} E_j$.
   Then

$$\mu(E_j) < 2^{-j}, \quad \mu(H_m) \leq \sum_{j=m}^{\infty} 2^{-j} = 2^{-m+1}$$

   and

$$|f - f_{k_j}| \leq \frac{1}{j} \quad \text{in } E - E_j$$

   Thus, if $j \geq m$,

$$|f - f_{k_j}| \leq 1/j \quad \text{in } E - H_m$$

   so that $f_{k_j} \to f$ a.e. in E. This completes the proof.

(c) A necessary and sfficient condition that $\{f_k\}$ converge in μ-measure on E is that for each $\epsilon > 0$,

$$\lim_{k,l\to\infty} u\{x \in E : |f_k(x) - f_l(x)| > \epsilon\} = 0$$

Proof.

The necessity follows from the formula

$$\{|f_k - f_l| > \epsilon\} \subset \left\{|f_k - f| > \frac{\epsilon}{2}\right\} \cup \left\{|f_l - f| > \frac{\epsilon}{2}\right\}$$

and the fact that the measures of the sets on the right tend to zero as $k, l \to \infty$ if $f_k \to f$ in μ-measure.

To prove the converse, choose N_j, $j = 1, 2, \cdots$, so that if $k, l \geq N_j$, then

$$\mu\left\{|f_k - f_j| > \frac{1}{j}\right\} < \frac{1}{2^j}$$

We may assume that $N_j \nearrow$, then

$$|f_{N_{j+1}} - f_{N_j}| \leq \frac{1}{2^j}$$

expect for a set E_j, $|E_j| < 2^{-j}$.

Let $H_i = \bigcup_{j=i}^{\infty} E_j$, $i = 1, 2, \cdots$, then

$$|f_{N_{j+1}}(x) - f_{N_j}(x)| \leq 2^{-j} \quad \text{for } j \geq i \text{ and } x \notin H_i$$

It follows that $\sum(f_{N_{j+1}-f_{N_j}})$ converges uniformly outside H_i for every i and, therefore, that $\{f_{N_j}\}$ converges uniformly outside every H_i.

Since

$$\mu(H_i) \leq \sum_{j\geq i} 2^{-j} = 2^{-i+1}$$

we obtain that $\{f_{N_j}\}$ converges a.e. in E and, letting $f = \lim f_{N_j}$, that $f_{N_j} \to f$ in μ-measure on E, note that

$$\{|f_k - f| > \epsilon\} \subset \left\{|f_k - f_{N_j}| > \frac{\epsilon}{2}\right\} \cup \left\{|f_{N_j} - f| > \frac{\epsilon}{2}\right\} \quad \text{for any } N_j$$

To show that the measure of the set on the left is less than a prescribed $\eta > 0$ for all sufficiently large k, select N_j so that the first term on the right has measure less than $\frac{1}{2}\eta$ for all large k (here, we use the Cauchy condition) and so that the measure of the second term on the right is also less than $\frac{1}{2}\eta$. This completes the proof.

4. (Exercise 10.6)

(a) If $f_1, f_2 \in L(d\mu)$ and $\int_E f_1 d\mu = \int_E f_2 d\mu$ for all measurable E, show that $f_1 = f_2$ a.e. (μ).

(b) Prove the uniqueness of f and σ in Theorem 10.40.

(c) Let μ be σ-finite, and let $f_1, f_2 \in L^{p'}(d\mu)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq p \leq \infty$. If $\int f_1 g d\mu = \int f_2 g d\mu$ for all $g \in L^p(d\mu)$, show that $f_1 = f_2$ a.e. (μ).

Proof.

(a) If $f_2 = 0$, let $E = \{f_1 > 0\}$ and $E_n = \{h \geq \frac{1}{n}\} \nearrow E$.

Since

$$0 \leq f_1 \chi_{E_n} \leq f_1 \chi_E = f_1$$

then

$$\int_{E_n} f_1 d\mu = 0$$

But

$$\int_{E_n} f_1 d\mu \geq \frac{1}{n} \cdot \mu(E_n)$$

so that $\mu(E_n) = 0$ for all n, and thus $\mu(E) = 0$.
For general f_2, let $f = f_1 - f_2$, then

$$\int_E f d\mu = 0$$

Hence

$$\mu(\{f_1 \neq f_2\}) = 0$$

(b) Let

$$v(A) = \int_A f_1 d\mu + \sigma_1(A) = \int_A f_2 d\mu + \sigma_2(A)$$

for every measurable $A \subset E$.
Then

$$\int_A f_1 d\mu - \int_A f_2 d\mu = \sigma_2(A) - \sigma_1(A) = 0$$

since $\sigma_2 - \sigma_1$ and μ are mutually singular and $\sigma_2 - \sigma_1$ is absolutely continuous.
Thus f and σ are unique.

(c) Since $f_1, f_2 \in L^{p'}(d\mu)$ and $g \in L^p d(\mu)$, then $\int_E f_1 g d\mu$ and $\int_E f_2 g d\mu$ are finite.
Since μ is σ-finite, then let $E = \cup_{k=1}^{\infty} E_k$ such that $\mu(E_k) < \infty$ for all k.
For any k, let $g = \chi_{E_k}$, then $\int_A f_1 g d\mu = \int_A f_2 g d\mu$ for any measurable set A.
By (a), we have $f_1 = f_2$ a.e. on E_k, thus $f_1 = f_2$ a.e.

5. (Exercise 10.7)
Prove the integral convergence results in Theorems 10.27 through 10.29 and 10.31.

Proof.
Since $f_k \leq f$ for every $k \geq 1$ and integrals preserve monotonicity, then

$$\int f_k d\mu \leq f d\mu \quad \text{for all } k \geq 1$$

Then we have

$$\lim_{k\to\infty} \int f_k d\mu \leq \int f d\mu$$

On the other hand, for the converse, apply Fatou's lemma, then we have

$$\lim_{k\to\infty} f_k = f$$

by assumption.

Since the limit exists, then we write

$$\liminf_{k\to\infty} f_k = \lim_{k\to\infty} f_k$$

By Fatou's Lemma, so

$$\int \liminf_{k\to\infty} f_k d\mu = \int \lim_{k\to\infty} f_k d\mu \leq \liminf_{k\to\infty} \int f_k d\mu = \lim_{k\to\infty} \int f_k d\mu$$

then we have

$$\int f d\mu \leq \lim_{k\to\infty} f_k d\mu$$

6. (Exercise 10.8)
Show that for $1 \le p < \infty$, the class of simple functions vanishing outside sets of finite measure is dense in $L^p(d\mu)$. See also Exercise 27.

Proof.
If $f \geq 0$ and measurable on $E \in \Sigma$, by Theorem 10.13 (iv), there exists nonnegative, simple measurable $f_k \nearrow f$ on E. Hence $|f_k|^p \nearrow |f|^p$, then $||f_k||_p \nearrow ||f||_p$.

By Exercise 8.12, then $||f_k - f||_p \to 0$.

Suppose there is a simple function f_k on a measurable set E such that $\mu(E) = \infty$. This implies that $||f||_p = \infty$. That is contradiction.

Thus the class of simple functions vanishing outside sets of finite measure is dense in $L^p(d\mu)$.