Real Analysis
Homework 6

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EXERCISE 10.17

Let μ be a σ-finite and define $\mathscr{L}^p(d\mu)$ to be the class of complex-valued f with $\int |f|^p \, d\mu < +\infty$. Let l be a complex-valued bounded linear functional on $\mathscr{L}^p(d\mu)$. If $1 \le p < \infty$, show that there is a function $g \in \mathscr{L}^{p'}(d\mu)$ such that $l(f) = \int fg \, d\mu$.
Here, as usual, we define $\int h \, d\mu = \int h_1 \, d\mu + i \int h_2 \, d\mu$ if $h = h_1 + ih_2$ with h_1 and h_2 real-valued.
(Hint: Reduce to the real case.)

Proof.
Let two complex-valued functions be $f = f_1 + if_2$.
Since μ be a σ-finite and $\mathscr{L}^p(d\mu)$ is the class of complex-valued f with $\int |f|^p \, d\mu = \int |f_1 + if_2| \, d\mu < \infty$, then we can deduce that $\int |f_2|^p \, d\mu < \infty$ if we let $f_1 = 0$ and $\int |f_1|^p \, d\mu < \infty$ if we let $f_2 = 0$. So $f_1, f_2 \in L^p$. Since

$$\int fg \, d\mu = \int (f_1 + if_2) \, g \, d\mu = \int f_1 g \, d\mu + i \int f_2 g \, d\mu,$$

by Theorem 10.43 and l is a complex-valued bounded linear functional, we know that

$$\int fg \, d\mu = \int f_1 g \, d\mu + i \int f_2 g \, d\mu = l(f_1) + il(f_2) = l(f_1) + l(if_2) = l(f_1 + if_2) = l(f)$$

EXERCISE 10.18

Give an example to show that $(L^\infty)'$ cannot be identified with L^1 as in Theorem 10.44.
(Consider $L^\infty[-1,1]$ with Lebesgue measure, and let $\mathscr{S}$ be the subspace of continuous functions on $[-1,1]$ with the sup norm. Define $l(f) = f(0)$ for $f \in \mathscr{S}$. Then l is a bounded linear functional on $\mathscr{S}$, so by the Hahn-Banach theorem, l has an extension $l \in (L^\infty[-1,1])'$. If there were a function $g \in L^1[-1,1]$ such that $l(f) = \int_{-1}^1 fg \, dx$ for all $f \in L^\infty[-1,1]$, then we would have $f(0) = \int_{-1}^1 fg \, dx$ for all $f \in \mathscr{S}$. Show that this implies that $g = 0$ a.e., so that $l \equiv 0$.)

Proof.
Let $\mathscr{S}$ be the space of continuous functions on the closed interval $[-1,1]$. Clearly, $\mathscr{S}$ is a subspace of $L^\infty[-1,1]$. Also, $\mathscr{S}$ is a Banach space with norm

$$|| \cdot ||_\infty = \max_{x \in [-1,1]} |f(x)|.$$

Define $l(f) = f(0)$ for $f \in \mathscr{S}$. Then l is a bounded linear functional on $\mathscr{S}$, so by the Hahn-Banach theorem, l has an extension $l \in (L^\infty[-1,1])'$. We want to show that there does not exist $g \in L^1$ such that

$$l(f) = \int_{-1}^{1} fg\, dx \qquad \text{for all } f \in \mathscr{S}.$$

To show this suppose there exists such a function $g \in L^1[-1,1]$. Consider the sequence of functions $\{f_n\}$ such that

$$f_n(x) = \max\{1 - n|x|,\, 0\}.$$

Clearly, $f_n(x)$ converges to 0 pointwise, $|f(x)| \le 1$. Also, we know l is well defined since

$$\int |f_n\, g|\, dx \le \int |g|\, dx < \infty.$$

Hence, by the dominated convergence theorem we have

$$\lim_{n\to\infty} l(f_n) = \lim_{n\to\infty} \int f_n\, g\, dx = \int \lim_{n\to\infty} f_n\, g\, dx = 0.$$

However, $f_n(0) = 1$ for all n, by definition, which leads to a contradiction. This shows that there does not exist such a function $g \in L^1[-1,1]$.

EXERCISE 10.20

Under the hypothesis of Theorem 10.49, prove that

$$\lim_{h\to 0} \frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} |f(\mathbf{y}) - f(\mathbf{x})|\, d\mu(\mathbf{y}) = 0 \quad \text{a.e.}(\mu).$$

Proof.
Since $f \in L(d\mu)$. For $r \in \mathbb{Q}$, we have

$$\begin{aligned}\frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} |f(\mathbf{y}) - f(\mathbf{x})|\, d\mu(\mathbf{y}) &\le \frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} |f(\mathbf{y}) - r|\, d\mu(\mathbf{y}) + \frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} |r - f(\mathbf{x})|\, d\mu(\mathbf{y}) \\ &= \frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} |f(\mathbf{y}) - r|\, d\mu(\mathbf{y}) + |r - f(\mathbf{x})|.\end{aligned}$$

By taking limit on the both sides and Theorem 10.49, we have

$$\lim_{h\to 0} \frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} |f(\mathbf{y}) - f(\mathbf{x})|\, d\mu(\mathbf{y}) \le 2|r - f(\mathbf{x})|$$

Since r can be chosen such that $|r - f(\mathbf{x})|$ is arbitrarily small. Hence

$$\lim_{h\to 0} \frac{1}{\mu(Q_x(h))} \int_{Q_x(h)} |f(\mathbf{y}) - f(\mathbf{x})|\, d\mu(\mathbf{y}) = 0 \quad \text{a.e.}(\mu).$$

EXERCISE 10.21

Derive an analogue of the Besicovitch Covering Lemma for the case of two dimensions (x, y) when the squares $Q_{(x,y)}$ are replaced by rectangles $R_{(x,y)}(h)$ centered at (x, y) whose x and y dimensions are h and h^2, respectively. Use this result to prove that under the hypothesis of Theorem 10.49,

$$\lim_{h\to 0} \frac{1}{\mu(R_{(x,y)}(h))} \int_{R_{(x,y)}(h)} f\, d\mu = f(x, y) \quad \text{a.e.}(\mu).$$

Proof.
Since $f \in L(d\mu)$. For any integrable g, we have

$$\left|\frac{1}{\mu(R_{(x,y)}(h))}\int_{R_{(x,y)}(h)} f\,d\mu - f(x,y)\right| \le \frac{1}{\mu(R_{(x,y)}(h))}\int_{R_{(x,y)}(h)} |f-g|\,d\mu$$
$$. \quad + \left|\frac{1}{\mu(R_{(x,y)}(h))}\int_{R_{(x,y)}(h)} g\,d\mu - f(x,y)\right|.$$

If g is also continuous, the last term on the right converges to $|g(x,y) - f(x,y)|$ as $h \to 0$. Hence, letting $L(x,y)$ denote the lim sup as $h \to 0$ of the term of the left, we obtain

$$L(x,y) \le \sup_{h>0} \frac{1}{\mu(R_{(x,y)}(h))}\int_{R_{(x,y)}(h)} |f-g|\,d\mu + |g(x,y) - f(x,y)|$$

Therefore, the set S_ϵ where $L(x,y) > \epsilon$, $\epsilon > 0$, is contained in the union of the two sets where the corresponding terms on the right side of the last inequality exceed $\frac{\epsilon}{2}$. From Lemma 10.47 and Tchebyshev's inequality, we obtain

$$\mu(S_\epsilon) \le c\left(\frac{\epsilon}{2}\right)^{-1}\int_{\mathbb{R}^n} |f-g|\,d\mu + \left(\frac{\epsilon}{2}\right)^{-1}\int_{\mathbb{R}^n} |f-g|\,d\mu$$

As noted before the proof of Lemma 10.47, g can be chosen such that $\int_{\mathbb{R}^n} |f-g|\,d\mu$ is arbitrarily small. Hence, $\mu(S-\epsilon) = 0$ for every $\epsilon > 0$, and the results follows.

EXERCISE 10.26 (*Hardy's inequality*)

Let $f \ge 0$ on $(0,\infty)$, $1 \le p < \infty$, $d\mu(x) = x^\alpha\,dx$ and $d\nu(x) = x^{\alpha+p}\,dx$ on $(0,\infty)$. Prove there exists a constant c independent of f such that

(i)
$$\int_0^\infty \left(\int_0^x f(t)\,dt\right)^p d\mu(x) \le c\int_0^\infty f^p(x)\,d\nu(x), \quad \alpha < -1,$$

(ii)
$$\int_0^\infty \left(\int_x^\infty f(t)\,dt\right)^p d\mu(x) \le c\int_0^\infty f^p(x)\,d\nu(x), \quad \alpha > -1.$$

For (i), $(\int_0^x f(t)\,dt)^p \le c x^{p-\eta-1}\int_0^x f(t)^p t^\eta\,dt$ by Hölder's inequality, provided $p - \eta - 1 > 0$. Multiply both sides by x^α, integrate over $(0,\infty)$, change the order of integration, and observe that an appropriate η exists since $\alpha < -1$.

Proof.

(i) If $p = 1$, then

$$\begin{aligned}
\int_0^\infty \int_0^x f(t)\,dt\,d\mu(x) &= \int_0^\infty \int_0^x f(t)\,dt\,x^\alpha dx \\
&= \int_0^\infty f(t)\int_t^\infty x^\alpha\,dx\,dt \\
&= \int_0^\infty f(t)\,\frac{t^{\alpha+1}}{\alpha+1}dt, \quad t^{\alpha+1} \to 0 \text{ as } t \to \infty \text{ since } \alpha < -1 \\
&= c\int_0^\infty f(x)\,x^{\alpha+1}dx \\
&= c\int_0^\infty f(x)\,d\nu(x)
\end{aligned}$$

If $1 < p < \infty$ and $\alpha < -1$, then $p + \alpha < p - 1$. So $\exists \eta$ such that $p + \alpha < \eta < p - 1$, then $\alpha + p - \eta - 1 < -1$, $\alpha + p - \eta < 0$. Thus, we let q such that $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality, we have

$$\begin{aligned}\left(\int_0^x f(t)\,dt\right)^p &= \left(\int_0^x f(t)\,t^{\frac{\eta}{p}}\,t^{\frac{-\eta}{p}}\,dt\right)^p \\ &\leq \left(\int_0^x f^p(t)\,t^\eta\,dt\right)\left(\int_0^x t^{\frac{-\eta q}{p}}\,dt\right)^{\frac{p}{q}} \\ &= c_1 x^{p-\eta-1}\left(\int_0^x f^p(t)t^\eta\,dt\right),\end{aligned}$$

$x^{p-\eta-1} \to 0$ as $x \to 0$, since $\frac{-\eta q + p}{p} \cdot \frac{p}{q} = -\eta + \frac{p}{q} = -\eta + p - 1 > 0$. So

$$\begin{aligned}\int_0^\infty (\int_0^x f(t)\,dt)^p\,d\mu(x) &\leq \int_0^\infty c_1 x^{p-\eta-1}\left(\int_0^x f^p(t)t^\eta\,dt\right) x^\alpha\,dx \\ &= c_1 \int_0^\infty x^{\alpha+p-\eta-1}\left(\int_0^x f^p(t)t^\eta\,dt\right)dx \\ &= c_1 \int_0^\infty f^p(t)\,t^\eta \int_t^\infty x^{\alpha+p-\eta-1}\,dx\,dt \\ &= c_1 \int_0^\infty f^p(t)\,t^\eta\,\frac{t^{\alpha+p-\eta}}{\alpha+p-\eta}\,dt \\ &= c\int_0^\infty f^p(x)\,d\nu(x),\end{aligned}$$

$t^{\alpha+p-\eta} \to 0$ as $t \to \infty$ since $\alpha + p - \eta < 0$.

(ii) If $p = 1$, then

$$\begin{aligned}\int_0^\infty \int_x^\infty f(t)\,dt\,d\mu(x) &= \int_0^\infty \int_x^\infty f(t)\,dt\,x^\alpha dx \\ &= \int_0^\infty f(t)\int_0^t x^\alpha\,dx\,dt \\ &= \int_0^\infty f(t)\,\frac{t^{\alpha+1}}{\alpha+1}dt, \quad t^{\alpha+1} \to 0 \text{ as } t \to 0 \text{ since } \alpha > -1 \\ &= c\int_0^\infty f(x)\,x^{\alpha+1}dx \\ &= c\int_0^\infty f(x)\,d\nu(x)\end{aligned}$$

If $1 < p < \infty$ and $\alpha > -1$, then $p - 1 > 0$. So $\exists \eta$ such that $p - \alpha > \eta > p - 1$, then $\alpha + p - \eta - 1 > -1$, $\alpha + p - \eta > 0$. Thus, we let q such that $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality, we have

$$\begin{aligned}\left(\int_x^\infty f(t)\,dt\right)^p &= \left(\int_x^\infty f(t)\,t^{\frac{\eta}{p}}\,t^{\frac{-\eta}{p}}\,dt\right)^p \\ &\leq \left(\int_x^\infty f^p(t)\,t^\eta\,dt\right)\left(\int_x^\infty t^{\frac{-\eta q}{p}}\,dt\right)^{\frac{p}{q}} \\ &= c_1 x^{p-\eta-1}\left(\int_x^\infty f^p(t)t^\eta\,dt\right),\end{aligned}$$

$x^{p-\eta-1} \to 0$ as $x \to \infty$, since $\frac{-\eta q+p}{p} \cdot \frac{p}{q} = -\eta + \frac{p}{q} = -\eta + p - 1 < 0$. So

$$
\begin{aligned}
\int_0^\infty (\int_x^\infty f(t)\,dt)^p \,d\mu(x) &\le \int_0^\infty c_1 x^{p-\eta-1} \left(\int_x^\infty f^p(t) t^\eta \,dt \right) x^\alpha \,dx \\
&= c_1 \int_0^\infty x^{\alpha+p-\eta-1} \left(\int_x^\infty f^p(t) t^\eta \,dt \right) dx \\
&= c_1 \int_0^\infty f^p(t)\, t^\eta \int_0^t x^{\alpha+p-\eta-1} \,dx\,dt \\
&= c_1 \int_0^\infty f^p(t)\, t^\eta \, \frac{t^{\alpha+p-\eta}}{\alpha+p-\eta} \,dt \\
&= c \int_0^\infty f^p(x)\,d\nu(x),
\end{aligned}
$$

$t^{\alpha+p-\eta} \to 0$ as $t \to 0$ since $\alpha + p - \eta > 0$.

EXERCISE 10.27

If μ is a σ-finite regular Borel measure on $\mathbb{R}^n$, show that the class of continuous functions with compact support is dense in $L^p(d\mu)$, $1 \le p < \infty$.
(By **EXERCISE 10.8**, it is enough to approximate χ_E, where E is a Borel set with finite measure. Given $\epsilon > 0$, as shown in Section 10.5 on p. 269, there exist open G and closed F with $F \subset E \subset G$ and $\mu(G - F) < \epsilon$. Now use Urysohn's lemma: if F_1 and F_2 are disjoint closed sets in $\mathbb{R}^n$, there is a continuous f on $\mathbb{R}^n$ with $0 \le f \le 1$, $f = 1$ on F_1, $f = 0$ on F_2.)

Proof.
Follow the hint, we approximate χ_E where E is regular Borel set with $\mu(E) < \infty$.
Let $F \subset E \subset G$, F is closed and G is open such that $\mu(G \setminus F) < \epsilon$, G^c is closed and $F \cap G^c = \phi$. Hence, we can apply Urysohn's lemma to find a continuous f such that $0 \le f \le 1$ where $f = 1$ on F and $f = 0$ on G^c. So

$$
\begin{aligned}
\int_{\mathbb{R}^n} |f - \chi_E|^p \,d\mu &= \int_F |f - \chi_E|^p \,d\mu + \int_{G \setminus F} |f - \chi_E|^p \,d\mu + \int_{G^c} |f - \chi_E|^p \,d\mu \\
&\le 0 + \int_{G \setminus F} 1 \,d\mu + 0 \\
&= \mu(G \setminus F) < \epsilon.
\end{aligned}
$$

Let $S =$ {simple function on E} to approximate χ_E, where $E = \cup_{k=1}^N E_k$ and E_k is regular Borel set, then $S = \sum_{k=1}^N a_k \chi_{E_k}$, so we can find $F_k \subset E_k \subset G_k$ where F_k is closed, G_k is open and $\mu(G_k \setminus F_k) < \dfrac{\epsilon}{N\, a_k^p}$.
By above result, we can find a continuous function f_k with compact support and $f_k = 1$ on F_k, $f_k = 0$ on G_k^c, then

$$
\int_{\mathbb{R}^n} |a_k\, f_k - a_k\, \chi_{E_k}|^p \,d\mu \le a_k^p\, \mu(G_k \setminus F_k) < \frac{\epsilon}{N}
$$

So

$$
\begin{aligned}
\int_{\mathbb{R}^n} |S - \chi_E|^p \,d\mu &= \sum_{k=1}^N \left(\int_{F_k} |f_k - \chi_{E_k}|^p \,d\mu + \int_{G_k \setminus F_k} |f_k - \chi_{E_k}|^p \,d\mu + \int_{G_k^c} |f_k - \chi_{E_k}|^p \,d\mu \right) \\
&\le \sum_{k=1}^N a_k\, \mu(G_k \setminus F_k) < \epsilon.
\end{aligned}
$$