Real Analysis Homework# 10

1. Show that if $f \in C^0(\mathbb{R}^n)$, then its support is identical with the support of the distribution

$$\langle f, \phi \rangle = \int f \phi dx, \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

Is this true when $f \in L^1_{loc}(\mathbb{R}^n)$?

2. Show that the principal value integral

$$\text{p.v.} \int \frac{\phi(x)}{x} dx = \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} \frac{\phi(x)}{x} dx + \int_{\varepsilon}^{\infty} \frac{\phi(x)}{x} dx \right)$$

exists for all $\phi \in C_c^\infty(\mathbb{R}^n)$, and is a distribution. What is its order?

3. Find a distribution $u \in \mathcal{D}'(\mathbb{R})$ such that $u = 1/x$ on $(0, \infty)$ and $u = 0$ on $(-\infty, 0)$.

4. Show that

$$\langle u, \phi \rangle = \sum_{k=1}^{\infty} \partial^k \phi(1/k)$$

is a distribution in $(0, \infty)$? What is its order?

5. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ have the property that $\langle u, \phi \rangle \geq 0$ for all real valued nonnegative $\phi \in C_c^\infty(\mathbb{R}^n)$. Show that u is of order 0.

6. Let $\{f_k\}_{k=1}^\infty \in L^1_{loc}(\mathbb{R}^n)$ be a sequence of real valued functions such that

$$\operatorname{supp} f_k \subset \{|x| \leq k^{-1}\}, \quad \int f_k(x) dx = 1, \quad k = 1, 2, \cdots.$$

Show that the sequence $\{f_k^2\}_{k=1}^\infty$ does not converge in $\mathcal{D}'(\mathbb{R}^n)$ as $k \to \infty$.