Real Analysis
9/11 Note

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Chapter 1. Measure theory

measure: length, area, volume

$E \subset \mathbb{R}^n$, $v(E) = n$-dim volume excepted properties:

(1) $v(E) \geq 0$

(2) $E_1 \subset E_2 \subset \mathbb{R}^n \;\Rightarrow\; v(E_1) \leq v(E_2)$

(3) $E_1 \cap E_2 = \emptyset \;\Rightarrow v(E_1 \cup E_2) = v(E_1) + v(E_2)$

$((3) + (4) \;\Rightarrow\; (2) \;:\; v(E_2 = E_1 \cup (E_2 \setminus E_1)) = v(E_1) + v(E_2 \setminus E_1) \geq v(E_1))$

1. Rectangles:

n-dim closed rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n] = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i, i = 1, \ldots, n\}$

cube $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$, $b_1 - a_1 = \cdots = b_n - a_n$

Def:

$$\text{vol}(R) = |R| = (b_1 - a_1) \times \cdots \times (b_n - a_n) \quad (|Q| = (b_1 - a_1)^n)$$

Question: open rectangle $H = (a_1, b_1) \times \cdots \times (a_n, b_n)$, how about $\text{vol}(H)$?
Ans: We should have

$$\begin{aligned} &|R| = \text{vol}(H) \leq |\bar{H}|, \\ &\bar{H} = [a_1, b_1] \times \cdots \times [a_n, b_n] = \text{closure}(H) \\ &\epsilon \to 0^+ \;:\; |\bar{H}| \leq \text{vol}(H) = |R| \end{aligned}$$

HW (deadline: 9/23):
Prove the uniqueness part of Theorem 1.3 in Stein.