Real Analysis
Homework 7

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May 8, 2019

EXERCISE 11.1

(a) Prove the second statements in both parts of Corollary 11.4.

(b) Verify the statements made before Theorem 11.5 about the function $\gamma(A)$ defined on sets $A \subset \mathbb{R}^2$. (One way to see that a set B with $d(\Upsilon, B) > 0$ is not γ-measurable is to denote the mirror reflection of B in the y-axis by B^* and check that the equation $\gamma(B \cup B^*) = \gamma(B) + \gamma(B^*)$ is false.)

(a) **The second statements in both parts of Corollary 11.4:**
Let Γ be an outer measure on $\mathscr{S}$, let $\{E_k\}$ be a collection of measurable sets, and let A be any set.

(i) If $E_k \searrow$ and if $\Gamma(A \cap E_{k_0})$ is finite for some k_0, then $\Gamma(A \cap \lim E_k) = \lim_{k\to\infty} \Gamma(A \cap E_k)$.

(ii) If $\Gamma(A \cap \cup_{k=k_0}^{\infty} E_k)$ is finite for some k_0, then $\Gamma(A \cap \limsup E_k) \geq \limsup_{k\to\infty} \Gamma(A \cap E_k)$.

Proof.

(i) Since $E_k \searrow$, then $\mathscr{S} - E_k \nearrow$. By the first statements in **Corollary 11.4(i)**, thus

$$\Gamma(A \cap \lim_{k\to\infty} (\mathscr{S} - E_k)) = \lim_{k\to\infty} \Gamma(A \cap (\mathscr{S} - E_k)) = \Gamma(A \cap \mathscr{S}) - \lim_{k\to\infty} \Gamma(A \cap E_k).$$

Also, $\Gamma(A \cap \lim_{k\to\infty}(\mathscr{S} - E_k)) = \Gamma(A \cap \mathscr{S}) - \Gamma(A \cap \lim_{k\to\infty} E_k)$, hence we have

$$\Gamma(A \cap \lim E_k) = \lim_{k\to\infty} \Gamma(A \cap E_k).$$

(ii) Let $\{E_k\}$ be measurable and define sets $X_j = \cup_{k=j}^{\infty} E_k$, $j = 1, 2, \cdots$. Then $X_j \searrow \limsup E_k$, so by the second statements in **Corollary 11.4(i)**,

$$\Gamma(A \cap \limsup E_k) = \lim_{j\to\infty} \Gamma(A \cap X_j).$$

But since $A \cap X_j \supset A \cap E_j$, then we have

$$\Gamma(A \cap \limsup E_k) = \lim_{j\to\infty} \Gamma(A \cap X_j) \geq \limsup_{k\to\infty} \Gamma(A \cap E_k).$$

(b) Let d is the usual Euclidean metric in $\mathbb{R}^2$. Define

$$\gamma(A) = \frac{1}{d(A, \Upsilon)}, \quad A \subset \mathbb{R}^2, \quad \text{where } \Upsilon \text{ is the } y\text{-axis},$$

with the conventions $1/0 = \infty$ and $\gamma(\phi) = 0$. To check that γ is an outer measure on $\mathbb{R}^2$ but not a metric outer measure.

Proof.

(1) γ is an outer measure on $\mathbb{R}^2$:

(i) $\gamma(\phi) = 0$ and $d(A, \Upsilon) \geq 0$, $1/0 = \infty$ so $\gamma(A) \geq 0$.

(ii) Let $A_1 \subset A_2$, then $d(A_1, \Upsilon) \geq d(A_2, \Upsilon)$. Thus $\gamma(A_1) \leq \gamma(A_2)$.

(iii) Since d is the usual Euclidean metric in $\mathbb{R}^2$, we can find some k_0 such that $d(A_{k_0}, \Upsilon) = \min\{d(A_k, \Upsilon)\}$ so $\gamma(A_{k_0}) = \gamma(\cup A_k)$. Thus

$$\gamma(\cup A_k) = \gamma(A_{k_0}) \leq \sum \gamma(A_k)$$

(2) γ is not a metric outer measure on $\mathbb{R}^2$:
Let the set B with $d(\Upsilon, B) > 0$ and denote the mirror reflection of B in the y-axis by B^*. Since B^* is the mirror reflection of B in the y-axis, hence

$$\gamma(B) = \gamma(B^*) \quad \text{and} \quad \gamma(B \cup B^*) = \gamma(B).$$

So $\gamma(B \cup B^*) = \gamma(B) + \gamma(B^*)$ is false and then the result follows.

EXERCISE 11.2

Leyt μ be a finite Borel measure on $\mathbb{R}^1$, and define $f_\mu(x) = \mu((-\infty, x])$, $-\infty < x < +\infty$. Show that f_μ is monotone increasing, $\mu((a, b]) = f_\mu(b) - f_\mu(a)$, f_μ is continuous from the right, and $\lim\limits_{x \to -\infty} f_\mu(x) = 0$.

Proof.

(a) Since μ is a finite Borel measure, if $b \geq a$, then

$$\begin{aligned} f_\mu(b) - f_\mu(a) &= \mu((-\infty, b]) - \mu((-\infty, a]) \\ &= \mu((-\infty, a]) + \mu((a, b]) - \mu((-\infty, a]) \\ &= \mu((a, b]) \geq 0 \end{aligned}$$

So f_μ is monotone increasing and $\mu((a, b]) = f_\mu(b) - f_\mu(a)$.

(b)

$$\begin{aligned} \lim_{\varepsilon \to 0} f_\mu(x + \varepsilon) &= \lim_{\varepsilon \to 0} (\mu((-\infty, x]) + \mu((x, x + \varepsilon])) \\ &= \mu((-\infty, x]) + \mu(\mathbb{R} \cap \lim_{\varepsilon \to 0} (x, x + \varepsilon]) \\ &= \mu((-\infty, x]) = f_\mu(x) \end{aligned}$$

So f_μ is continuous from right.

(c)

$$\begin{aligned} \lim_{x \to -\infty} f_\mu(x) &= \lim_{x \to -\infty} \mu((-\infty, x]) \\ &= \lim_{x \to -\infty} \mu(\lim_{k \to -\infty} (k, x]) \\ &= \lim_{x \to -\infty} \lim_{k \to -\infty} \mu((k, x]) \\ &= \lim_{x \to -\infty} \lim_{k \to -\infty} [f_\mu(x) - f_\mu(k)] = 0 \end{aligned}$$

EXERCISE 11.3

Let f be monotone increasing on $\mathbb{R}^1$.

(a) Show that $\Lambda_f(\mathbb{R}^1)$ is finite if and only if f is bounded.

(b) Let f be bounded and right continuous, let $\mu = \Lambda_f$, and let $\bar{f}$ denote the function f_μ defined in **Exercise 11.2**. Show that f and $\bar{f}$ differ by a constant.
Thus, if we make the additional assumption that $\lim_{x \to -\infty} f(x) = 0$, then $f = \bar{f}$.

Proof.

(a) $(\Rightarrow)$
$\forall \varepsilon > 0$ and $a \in \mathbb{R}^1$.
We have $(a, \infty) = \cup_{k=1}^{\infty} (a_k, a_{k+1}]$, where $\{(a_k, a_{k+1}]\}$ disjoint and

$$\sum_{k=1}^{\infty} \lambda(a_k, a_{k+1}] \leq \Lambda_f(a, \infty) + \varepsilon$$

$$\Rightarrow \lim_{k \to +\infty} f(a_k) - f(a) = \lim_{k \to +\infty} \left[\sum_{j=1}^{k} (f(a_{j+1}) - f(a_j)) \right] \leq \Lambda_f(a, \infty) + \varepsilon$$

Hence, we can say that f is bounded.

$(\Leftarrow)$
Suppose f is bounded.
Let $a_k \nearrow \infty$ as $k \nearrow \infty$, then

$$f(a_k) - f(a) = \lambda((a, a_k]) \geq \Lambda_f((a, a_k]) \geq 0$$

So Λ_f is finite.

(b) Let $a \in \mathbb{R}^1$, then

$$\begin{aligned} f(a) - \bar{f}(a) &= f(a) - f_\mu(a) \\ &= f(a) - \mu((-\infty, a]) \\ &= f(a) - \Lambda_f((-\infty, a]) \\ &= f(a) - [f(a) - f(-\infty)] \\ &= f(-\infty) \end{aligned}$$

Since f is bounded, then $f(a) - \bar{f}(a) = f(-\infty) < c$ where c is a constant.

EXERCISE 11.4

If we identify two functions on $\mathbb{R}^1$ which differ by a constant, prove that there is a one-to-one correspondence between the class of finite Borel measures on $\mathbb{R}^1$ and the class of bounded increasing functions that are continuous from the right.

Proof.
Let $S_1 = \{f : f \text{is bounded increasing on } \mathbb{R} \text{ and continuous from right}\}$ and
$S_2 = \{\mu : \mu \text{ is finite Borel measure on } \mathbb{R}\}$.
Let $\varphi : S_1 \to S_2$ and $\varphi(f) = \Lambda_f$.
By **EXERCISE 11.3(a)**, since f is bounded increasing, then Λ_f is finite Borel measure.

(1) one-to-one:
Let $f_1, f_2 \in S_1$, then f_1 and f_2 are bounded increasing and continuous from right. So

$$f_1 - f_\mu = c_1 \quad \text{and} \quad f_2 - f_\mu = c_2 \quad \text{where } c_1 \text{ and } c_2 \text{ are constants.}$$

Hence $\varphi(f_1) = \varphi(f_2)$.

(2) onto:
If $\mu \in S_2$, then by **EXERCISE 11.2**, we know that f_μ is increasing and continuous from right. So by **Theorem 11.10**, if f_μ is increasing and continuous from right, then its Lebsegue-Stieltjes measure Λ satisfies

$$\Lambda_{f_\mu}(a,b] = f_\mu(b) - f_\mu(a) = \mu(-\infty,b] - \mu(-\infty,a] = \mu(a,b]$$

By **Theorem 11.21**, then $\mu = \Lambda_{f_\mu}$ on every Borel sets $B \subset \mathbb{R}^1$. μ is finite so Λ_{f_μ} is also finite. Since Λ_{f_μ} is finite and by **EXERCISE 11.3(a)**, then f_μ is bounded.