

Real Analysis

Homework 8

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1. (Exercise 6.7)

Let F be a closed subset of \mathbb{R}^1 and let $\delta(x) = \delta(x, F)$ be the corresponding distance function. If $\lambda > 0$ and f is nonnegative and integrable over the complement of F , prove that the function

$$\int_{\mathbb{R}^1} \frac{\delta^\lambda(y) f(y)}{|x - y|^{1+\lambda}} dy$$

is integrable over F and so is finite a.e. in F . (In case $f = \chi_{(a,b)}$, this reduces to Theorem 6.17.)

Proof.

$$\begin{aligned} \int_F \left[\int_{\mathbb{R}^1} \frac{\delta^\lambda(y) f(y)}{|x - y|^{1+\lambda}} dy \right] dx &= \int_{\mathbb{R}^1 - F} \left[\int_F \frac{\delta^\lambda(y) f(y)}{|x - y|^{1+\lambda}} dx \right] dy \\ &= \int_{\mathbb{R}^1 - F} \delta^\lambda(y) f(y) \left[\int_F \frac{1}{|x - y|^{1+\lambda}} dx \right] dy \\ &\leq \int_{\mathbb{R}^1 - F} \delta^\lambda(y) f(y) \left[\int_{\{x: \delta(y) \leq |x - y|\}} \frac{1}{|x - y|^{1+\lambda}} dx \right] dy \\ &= 2 \int_{\mathbb{R}^1 - F} \delta^\lambda(y) f(y) \left[\int_{\delta(y)}^{\infty} \frac{1}{t^{1+\lambda}} dt \right] dy \\ &= 2\lambda^{-1} \int_{\mathbb{R}^1 - F} f(y) dy \end{aligned}$$

is integrable since f is integrable over the complement of F , and so is finite a.e. in F .

2. (Exercise 6.8)

Under the hypothesis of Theorem 6.17 and assuming that $b - a < 1$, prove that the function

$$M_0(x) = \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} |x - y|^{-1} dy$$

is finite a.e. in F .

Proof.

Since $b - a < 1$, then $\log(\frac{1}{\delta(y)}) > 0$.

Hence $M_0(x)$ is nonnegative and the integral $\int_F M_0(x)$ exists, then

$$\begin{aligned}
\int_F M_0(x) dx &= \int_F \left\{ \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} |x-y|^{-1} dy \right\} dx \\
&= \int_a^b \left\{ \int_F \left[\log \frac{1}{\delta(y)} \right]^{-1} |x-y|^{-1} dx \right\} dy \\
&\leq \int_a^b \left\{ \int_{\{x: \delta(y) \leq |x-y| \leq 1\}} \left[\log \frac{1}{\delta(y)} \right]^{-1} |x-y|^{-1} dx \right\} dy \\
&= \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} \left[\int_{\{x: \delta(y) \leq |x-y| \leq 1\}} |x-y|^{-1} dx \right] dy \\
&\leq \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} \left[2 \int_{\delta(y)}^1 t^{-1} dt \right] dy \\
&= \int_a^b \left[\log \frac{1}{\delta(y)} \right]^{-1} [-2 \log \delta(y)] dy \\
&= 2(b-a)
\end{aligned}$$

So M_0 is finite a.e. in F .

3. (Exercise 6.9)

- (a) Show that $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.
- (b) Let $F = [c, d]$ be a closed subinterval of a bounded open interval $(a, b) \subset \mathbb{R}^1$, and let M_λ be the corresponding Marcinkiewicz integral, $\lambda > 0$. Show that M_λ is finite for every $x \in (c, d)$ and that $M_\lambda(c) = M_\lambda(d) = \infty$. Show also that $\int_F M_\lambda \leq \lambda^{-1} |G|$, where $G = (a, b) - [c, d]$.

Proof.

- (a) Let $x \notin F$ and $\lambda > 0$. For any $\epsilon > 0$, then $\delta(y) \in B(\delta(x), \epsilon)$ for all $y \in B(x, \epsilon)$, thus

$$\begin{aligned}
M_\lambda(x; F) &= \int_a^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \\
&= \int_a^{x-\epsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy + \int_{x-\epsilon}^{x+\epsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy + \int_{x+\epsilon}^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \\
&\geq \int_{x-\epsilon}^{x+\epsilon} \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \\
&\geq \int_{x-\epsilon}^{x+\epsilon} \frac{\delta^\lambda(x) - \epsilon}{\epsilon^{1+\lambda}} dy \\
&= \frac{2\delta^\lambda(x) - 2\epsilon}{\epsilon^\lambda}
\end{aligned}$$

Let ϵ_n be a sequence on $(0, 1)$ with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, then

$$\frac{(2\delta^\lambda(x) - 2\epsilon_n)}{\epsilon_n^\lambda} \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

So $M_\lambda(x; F) = +\infty$ if $x \notin F$, $\lambda > 0$.

- (b)

$$M_\lambda(x; F) = \int_a^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy$$

i. For any $\epsilon > 0$ and $M = \max\{d - a, b - c\}$, we have

$$\begin{aligned}
\int_a^b \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} dy &= \int_a^{c+\epsilon} \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} dy + \int_{d-\epsilon}^b \frac{\delta^\lambda(y)}{|x - y|^{1+\lambda}} dy \\
&\leq \int_a^{c+\epsilon} \frac{\delta^\lambda(y)}{\epsilon^{1+\lambda}} dy + \int_{d-\epsilon}^b \frac{\delta^\lambda(y)}{\epsilon^{1+\lambda}} dy \\
&\leq \frac{(b-a)^\lambda}{\epsilon^{1+\lambda}} \cdot (c + \epsilon - a) + \frac{(b-c)^\lambda}{\epsilon^{1+\lambda}} \cdot (b - d + \epsilon) \\
&\leq \frac{M^\lambda}{\epsilon^{1+\lambda}} \cdot (c + \epsilon - a) + \frac{M^\lambda}{\epsilon^{1+\lambda}} \cdot (b - d + \epsilon) \\
&= \frac{M^\lambda}{\epsilon^{1+\lambda}} \cdot (c + \epsilon - a + b - d + \epsilon) \\
&= \frac{M^\lambda}{\epsilon^{1+\lambda}} \cdot (|G| + 2\epsilon) \quad \text{is finite, since } \epsilon \neq 0
\end{aligned}$$

Hence, M_λ is finite for every $x \in (c, d)$.

ii.

$$\begin{aligned}
M_\lambda(c) &= \int_a^b \frac{\delta^\lambda(y)}{|c - y|^{1+\lambda}} dy \\
&= \int_a^c \frac{\delta^\lambda(y)}{(c - y)^{1+\lambda}} dy + \int_d^b \frac{\delta^\lambda(y)}{(y - c)^{1+\lambda}} dy \\
&= \int_a^c \frac{(c - y)^\lambda}{(c - y)^{1+\lambda}} dy + \int_d^b \frac{(y - d)^\lambda}{(y - c)^{1+\lambda}} dy \\
&> \int_a^c \frac{1}{c - y} dy \\
&= \lim_{t \rightarrow c} [-\ln(c - y)]_{y=a}^t \\
&= \infty
\end{aligned}$$

$$\begin{aligned}
M_\lambda(d) &= \int_a^b \frac{\delta^\lambda(y)}{|d - y|^{1+\lambda}} dy \\
&= \int_a^c \frac{\delta^\lambda(y)}{(d - y)^{1+\lambda}} dy + \int_d^b \frac{\delta^\lambda(y)}{(y - d)^{1+\lambda}} dy \\
&= \int_a^c \frac{(c - y)^\lambda}{(d - y)^{1+\lambda}} dy + \int_d^b \frac{(y - d)^\lambda}{(y - d)^{1+\lambda}} dy \\
&> \int_d^b \frac{1}{y - d} dy \\
&= \lim_{t \rightarrow d} [\ln(y - d)]_{y=t}^b \\
&= \infty
\end{aligned}$$

iii. If $y \in F$, then $\delta(y) = 0$, thus

$$\begin{aligned}
\int_F M_\lambda(x; F) dx &= \int_F \left[\int_a^b \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \right] dx \\
&= \int_F \left[\int_G \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dy \right] dx \\
&= \int_G \left[\int_F \frac{\delta^\lambda(y)}{|x-y|^{1+\lambda}} dx \right] dy \\
&= \int_G \delta^\lambda(y) \left[\int_F \frac{1}{|x-y|^{1+\lambda}} dx \right] dy \\
&\leq \int_G \delta^\lambda(y) \left[\int_{\{x: \delta(y) \leq |x-y|\}} \frac{1}{|x-y|^{1+\lambda}} dx \right] dy \\
&\leq \int_G \delta^\lambda(y) \left[\int_{\delta(y)}^a \frac{1}{t^{1+\lambda}} dt \right] dy \\
&= \int_G \delta^\lambda(y) \left[\frac{-1}{\lambda} (a^\lambda - \delta^\lambda(y)) \right] dy \\
&\leq \frac{1}{\lambda} \int_G 1 dy \\
&= \frac{|G|}{\lambda}
\end{aligned}$$

4. (Exercise 7.2)

Let $\phi(x)$, $x \in \mathbb{R}^n$, be a bounded measurable function such that $\phi(x) = 0$ for $|x| \geq 1$ and $\int \phi = 1$. For $\epsilon > 0$, let $\phi_\epsilon(x) = \epsilon^{-n} \phi(x/\epsilon)$. (ϕ_ϵ is called an *approximation to the identity*.) If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\epsilon \rightarrow 0} (f * \phi_\epsilon)(x) = f(x)$$

in the Lebesgue set of f . (Note that $\int \phi_\epsilon = 1$, $\epsilon > 0$, so that

$$(f * \phi_\epsilon)(x) - f(x) = \int [f(x-y) - f(x)] \phi_\epsilon(y) dy$$

Use Theorem 7.16.)

Recall Exercise 5.20:

Let $\mathbf{y} = T\mathbf{x}$ be a nonsingular linear transformation of \mathbb{R}^n . If $\int_E f(\mathbf{y}) d\mathbf{y}$ exists, then

$$\int_E f(\mathbf{y}) d\mathbf{y} = |\det T| \int_{T^{-1}E} f(T\mathbf{x}) d\mathbf{x}$$

Recall Theorem 7.16:

Let f be locally integrable in \mathbb{R}^n , then at every point \mathbf{x} of the Lebesgue set of f ,

$$\frac{1}{|S|} \int_S |f(\mathbf{y}) - f(\mathbf{x})| d\mathbf{y} \rightarrow 0$$

for any family $\{S\}$ that shrinks regularly to \mathbf{x} . Thus, also

$$\frac{1}{|S|} \int_S f(\mathbf{y}) d\mathbf{y} \rightarrow f(\mathbf{x}) \text{ a.e.}$$

Proof.

First, we will show that $\int \phi_\epsilon = 1$.

For $\epsilon > 0$, then

$$\int_{\mathbb{R}^n} \phi_\epsilon(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} \epsilon^{-n} \phi(\mathbf{x}/\epsilon) d\mathbf{x} = \int_{\{|\mathbf{x}| < \epsilon\}} \epsilon^{-n} \phi(\mathbf{x}/\epsilon) d\mathbf{x}$$

since $\phi(\mathbf{x}) = 0$ for all $|\mathbf{x}| \geq 1$.

Let $\mathbf{y} = T\mathbf{x} = \frac{\mathbf{x}}{\epsilon}$ be a linear transformation of \mathbb{R}^n , and $T = \text{diag}(\frac{1}{\epsilon}, \dots, \frac{1}{\epsilon})$ so that $|\det T| = \epsilon^{-n}$. If $E = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$, then $T^{-1}E = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < \epsilon\}$, thus by Exercise 5.20

$$\int_E f(\mathbf{y}) d\mathbf{y} = |\det T| \int_{T^{-1}E} f(T\mathbf{x}) d\mathbf{x}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x} &= \epsilon^{-n} \int_{T^{-1}E} \phi(T\mathbf{x}) d\mathbf{x} \\ &= \epsilon^{-n} \cdot \frac{1}{|\det T|} \int_E \phi(\mathbf{y}) d\mathbf{y} \\ &= \int_{\{|\mathbf{y}| < 1\}} \phi(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \phi(\mathbf{y}) d\mathbf{y} \\ &= 1 \end{aligned}$$

Following,

$$\begin{aligned} (f * \phi_\epsilon)(\mathbf{x}) - f(\mathbf{x}) &= \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) d\mathbf{y} - \int_{\mathbb{R}^n} f(\mathbf{x}) \phi_\epsilon(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^n} [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})] \phi_\epsilon(\mathbf{y}) d\mathbf{y} \\ &= \frac{1}{\epsilon^n} \int_{\{|\mathbf{y}| \leq \epsilon\}} [f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})] \phi(\mathbf{y}/\epsilon) d\mathbf{y} \end{aligned}$$

Since $\phi(\mathbf{x})$ is a bounded function on \mathbb{R}^n , then for some $M > 0$, we have $|\phi(\mathbf{x})| \leq M$ and let $Q_{2\epsilon}(\mathbf{x})$ be the cube centered at \mathbf{x} with edge length 2ϵ , then

$$\begin{aligned} |(f * \phi_\epsilon)(\mathbf{x}) - f(\mathbf{x})| &\leq \frac{M}{\epsilon^n} \int_{\{|\mathbf{y}| \leq \epsilon\}} |f(\mathbf{x} - \mathbf{y}) - f(\mathbf{x})| d\mathbf{y} \\ &= \frac{M}{\epsilon^n} \int_{\{|\mathbf{y} - \mathbf{x}| \leq \epsilon\}} |f(\mathbf{y}) - f(\mathbf{x})| d\mathbf{y} \\ &\leq \frac{M}{\epsilon^n} \int_{Q_{2\epsilon}(\mathbf{x})} |f(\mathbf{y}) - f(\mathbf{x})| d\mathbf{y} \\ &\leq \frac{2^n M}{|Q_{2\epsilon}(\mathbf{x})|} \int_{Q_{2\epsilon}(\mathbf{x})} |f(\mathbf{y}) - f(\mathbf{x})| d\mathbf{y} \\ &(|Q_{2\epsilon}(\mathbf{x})| = 2^n \epsilon^n) \end{aligned}$$

Since $f \in L(\mathbb{R}^n)$, by Theorem 7.16, for all points \mathbf{x} of the Lebesgue set of f , we have

$$\frac{1}{|Q_{2\epsilon}(\mathbf{x})|} \int_{Q_{2\epsilon}(\mathbf{x})} |f(\mathbf{y}) - f(\mathbf{x})| d\mathbf{y} \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

Hence,

$$\lim_{\epsilon \rightarrow 0} |(f * \phi_\epsilon)(\mathbf{x}) - f(\mathbf{x})| = 0$$

, which implies

$$\lim_{\epsilon \rightarrow 0} (f * \phi_\epsilon)(\mathbf{x}) = f(\mathbf{x})$$

in the Lebesgue set of f .

5. (Exercise 7.6)

Show that if $\alpha > 0$, then x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Recall Theorem 7.6:

A function f is absolutely continuous on $[a, b]$ if and only if f' exists a.e. in $[a, b]$, f' is integrable on $[a, b]$, and

$$f(x) - f(a) = \int_a^x f', \quad a \leq x \leq b$$

Proof.

Let $f(x) = x^\alpha$.

Since

$$f'(x) = \begin{cases} \alpha x^{\alpha-1}, & \text{if } x \in (0, \infty) \\ \lim_{x \rightarrow 0} \frac{x^\alpha - 0^\alpha}{x - 0} = 0, & \text{if } x = 0 \end{cases}$$

$f(x) = x^\alpha$ is differentiable on $[0, \infty)$, then $f(x) = x^\alpha$ is also differentiable on every bounded subinterval $[a, b]$.

Since f' is a polynomial function, f' is continuous on $[a, b]$.

Since f' is continuous and also bounded on $[a, b]$, then f' is Riemann integrable on $[a, b]$.

If $a \leq x \leq b$, then we have

$$\int_a^x f'(y) dy = \int_a^x \alpha y^{\alpha-1} dy = x^\alpha - a^\alpha = f(x) - f(a)$$

By Theorem 7.6, then we know that $f(x) = x^\alpha$ is absolutely continuous on every bounded subinterval of $[0, \infty)$.

6. (Exercise 7.7)

Prove that f is absolutely continuous on $[a, b]$ if and only if given $\epsilon > 0$, there exists $\delta > 0$ such that $|\sum [f(b_i) - f(a_i)]| < \epsilon$ for every finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$ with $\sum (b_i - a_i) < \delta$.

Proof.

(\Rightarrow)

Since $|\sum [f(b_i) - f(a_i)]| < \sum |f(b_i) - f(a_i)|$, the definition of an absolutely continuous function f immediately leads to this implication.

(\Leftarrow)

For any collection $\{[a_i, b_i]\}$ be a sequence of nonoverlapping subintervals of $[a, b]$ with $\sum (b_i - a_i) < \delta$, we have

$$\sum |f(b_i) - f(a_i)| = \sum [f(b_i) - f(a_i)]^+ + \sum [f(b_i) - f(a_i)]^- < 2\epsilon$$

Hence f is absolutely continuous on $[a, b]$.