2 A Glimpse of Measure and Integration

This chapter gives a quick but precise exposition of the essentials of measure and integration so that an overall view of the subject is provided at the outset.

Preliminaries on various types of families of sets and set functions defined on them are covered in the first section, for later use in this chapter as well as in subsequent chapters.

The important L^p spaces are also introduced in this chapter for the reader to have an early appreciation of the power of the basic convergence theorems, which, together with the Egoroff theorem, reveal convincingly the relevance of σ-additivity of measures.

2.1 Families of sets and set functions

Sets considered in this section are subsets of a given fixed set Ω, which is sometimes referred to as a **universal set**; the family of all subsets of Ω is called the **power set** of Ω and is denoted by 2^{Ω}. A function τ defined on a nonempty family Φ of subsets of Ω and taking complex or extended real values is called a **set function**. If the empty set $\phi \in \Phi$, we always require that $\tau(\phi) = 0$. But hereafter in this chapter a set function τ is always assumed to take only nonnegative extended real values; and it is said to be finite if $\tau(A)$ is finite for $A \in \Phi$, while it is **σ-finite** if there is a sequence $\{A_n\} \subset \Phi$ such that $\bigcup \Phi \subset \bigcup_n A_n$ and $\tau(A_n) < \infty$ for each n. A set function τ is **monotone** if $\tau(A) \leq \tau(B)$ for A, B in Φ with $A \subset B$. A monotone set function τ with domain Φ is said to be **continuous from below** at $A \in \Phi$, if for every increasing sequence $\{A_n\} \subset \Phi$ with $A = \bigcup_n A_n$ the equality $\tau(A) = \lim_{n\to\infty} \tau(A_n)$ holds. Note that since τ is monotone, $\lim_{n\to\infty} \tau(A_n)$ exists. The set function τ is **continuous from below on Φ** if it is continuous from below at every $A \in \Phi$. A set function with ϕ in its domain is called a **premeasure** on Ω.

A family $\mathcal{P}$ of subsets of Ω is called a **π-system** on Ω if $A \cap B \in \mathcal{P}$ whenever A and B are in $\mathcal{P}$. The families $\{(-\infty, \alpha] : \alpha \in \mathbb{R}\}$ and $\{(a, b) : -\infty < a \leq b < \infty\}$ are π-systems on $\mathbb{R}$.

A family $\mathcal{A}$ of subsets of Ω is called an **algebra** on Ω if

(a_1) $\Omega \in \mathcal{A}$;

(a_2) if $A \in \mathcal{A}$, then $A^c := \Omega \backslash A$ is in $\mathcal{A}$;

(a_3) $A \cup B \in \mathcal{A}$ whenever A and B are in $\mathcal{A}$.

It is readily seen that if $\{A_1, \ldots, A_n\}$ is any finite subfamily of an algebra $\mathcal{A}$, then $\bigcup_{j=1}^n A_j \in \mathcal{A}$, and consequently $\bigcap_{j=1}^n A_j \in \mathcal{A}$, because $(\bigcap_{j=1}^n A_j)^c = \bigcup_{j=1}^n A_j^c$. One also notes that if A, B are in $\mathcal{A}$, then $A \backslash B := A \cap B^c$ is in $\mathcal{A}$.

A family Σ of subsets of Ω is called a **σ-algebra** on Ω if it is an algebra on Ω and if $\{A_n\}$ is a sequence in Σ; then $\bigcup_n A_n \in \Sigma$. Since $(\bigcap_n A_n)^c = \bigcup_n A_n^c$, $\bigcap_n A_n \in \Sigma$ if $\{A_n\}$ is a sequence in a σ-algebra Σ.

A family $\mathcal{L}$ of subsets of Ω is called a **λ-system** on Ω if the following conditions hold for $\mathcal{L}$:

(λ_1) $\Omega \in \mathcal{L}$;

(λ_2) if $A \in \mathcal{L}$, then $A^c \in \mathcal{L}$;

(λ_3) if $\{A_n\}$ is a disjoint sequence in $\mathcal{L}$, then $\bigcup_n A_n \in \mathcal{L}$.

Observe that if $\mathcal{L}$ is a λ-system on Ω and if A, B are in $\mathcal{L}$ with $A \subset B$, then $B \backslash A \in \mathcal{L}$, because $B \backslash A = A^c \cap B = (A \cup B^c)^c$.

Π-systems, λ-systems, algebras, and σ-algebras on Ω will often be simply referred to as π-systems, λ-systems, algebras, and σ-algebras if Ω is clearly implied in a statement.

We state without proof a trivial lemma for later reference.

Lemma 2.1.1 *A family of subsets of Ω is a σ-algebra on Ω if and only if it is both a π-system and a λ-system on Ω.*

Since the intersection of any collection of λ-systems on Ω is a λ-system, for any family Φ of subsets of Ω the smallest λ-system on Ω containing Φ exists and is denoted by $\lambda(\Phi)$. Similarly, the smallest σ-algebra on Ω containing Φ exists and is denoted by $\sigma(\Phi)$. We note that $\lambda(\Phi) \subset \sigma(\Phi)$ always, because any σ-algebra is a λ-system.

A λ-system satisfies a set of conditions which is a little weaker than that for a σ-algebra; but it turns out that often the set of conditions for λ-systems is much easier to verify than that for σ-algebras. The following theorem was first discovered by W. Sierpinski, and has been shown to be very useful in probability theory by E.B. Dynkin. It is now often referred to as the **(π–λ) Theorem**.

Theorem 2.1.1 (π-λ Theorem) *If $\mathcal{P}$ is a π-system on Ω, then $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$.*

Proof Let $\mathcal{L}_0 = \lambda(\mathcal{P})$. If $\mathcal{L}_0$ is a π-system, then $\mathcal{L}_0$ is a σ-algebra, by Lemma 2.1.1, consequently $\mathcal{L}_0 \supset \sigma(\mathcal{P})$; but since $\mathcal{L}_0 = \lambda(\mathcal{P}) \subset \sigma(\mathcal{P})$, we have $\lambda(\mathcal{P}) = \sigma(\mathcal{P})$. It remains therefore to show that $\mathcal{L}_0$ is a π-system. For $A \in \mathcal{L}_0$, let

$$\mathcal{L}_A = \{B \subset \Omega : A \cap B \in \mathcal{L}_0\}.$$

To show that $\mathcal{L}_0$ is a π-system is to show that $\mathcal{L}_A \supset \mathcal{L}_0$ for every $A \in \mathcal{L}_0$. Clearly, $\mathcal{L}_A$ is a λ-system. Observe then that if $B \in \mathcal{P}$, then $\mathcal{L}_B \supset \mathcal{P}$, since $\mathcal{P}$ is a π-system, and hence $\mathcal{L}_B$ is a λ-system containing $\mathcal{P}$. Therefore, $\mathcal{L}_B \supset \mathcal{L}_0$ if $B \in \mathcal{P}$, this means that $A \cap B \in \mathcal{L}_0$ if $A \in \mathcal{L}_0$ and $B \in \mathcal{P}$, or $\mathcal{L}_A \supset \mathcal{P}$ if $A \in \mathcal{L}_0$. Since $\mathcal{L}_A$ is a λ-system, we then have $\mathcal{L}_A \supset \mathcal{L}_0$ for $A \in \mathcal{L}_0$. Thus $\mathcal{L}_0$ is a π-system and the theorem is proved. ■

We reiterate that hereafter in this chapter set functions are assumed to take nonnegative extended real values.

We shall call a set function μ defined on an algebra $\mathcal{A}$ on Ω an **additive** set function if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \cap B = \phi$. Recall that $\mu(\phi) = 0$. An additive set function μ on an algebra $\mathcal{A}$ is **σ-additive** if

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n),$$

whenever $\{A_n\}$ is a disjoint sequence in $\mathcal{A}$ with $\bigcup_n A_n \in \mathcal{A}$.

Exercise 2.1.1 Let μ be an additive set function defined on an algebra $\mathcal{A}$ on Ω.

(i) Show that μ is monotone.

(ii) Show that if $A_1, \ldots, A_n$ are in $\mathcal{A}$, then $\mu(\bigcup_{j=1}^n A_j) \le \sum_{j=1}^n \mu(A_j)$.

(iii) Show that μ is σ-additive if and only if μ is continuous from below on $\mathcal{A}$.

(iv) Show that if $\{A_j\}_{j=1}^\infty \subset \mathcal{A}$ with $\bigcup_{j=1}^\infty A_j \in \mathcal{A}$, then $\mu(\bigcup_{j=1}^\infty A_j) \le \sum_{j=1}^\infty \mu(A_j)$ if μ is σ-additive on $\mathcal{A}$.

Theorem 2.1.2 *Suppose that Ω is a compact metric space and $\mathcal{A}$ is an algebra of compact subsets of Ω. If μ is an additive set function on $\mathcal{A}$, then μ is σ-additive.*

Proof To show that μ is σ-additive is to show that if $A_1 \subset A_2 \subset \cdots$ is an increasing sequence in $\mathcal{A}$ such that $\bigcup_n A_n \in \mathcal{A}$, then $\mu(\bigcup_n A_n) = \lim_{n\to\infty} \mu(A_n)$ (cf. Exercise 2.1.1 (iii)). Let $A = \bigcup_n A_n$ and put $C_n = A \backslash A_n$ for each n, then $\bigcap_n C_n = \emptyset$. We claim that $\lim_{n\to\infty} \mu(C_n) = 0$. If not, then $\mu(C_n) \ge \lim_{n\to\infty} \mu(C_n) > 0$ implies that $C_n \neq \emptyset$ for all n. Then $\bigcap_n C_n \neq \emptyset$, by Exercise 1.7.2, contradicting the fact that $\bigcap_n C_n = \emptyset$. Now, $\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty}\{\mu(A_n) + \mu(C_n)\} = \mu(A)$. ■

Example 2.1.1 Consider the sequence space $\Omega = \{0,1\} \times \{0,1\} \times \cdots$ and the additive set function P defined on the algebra $\mathcal{Q}$ of all cylinders in Ω (cf. Section 1.3). We have seen in Example 1.7.1 that Ω is compact with a suitable metric and that sets in $\mathcal{Q}$ are compact, hence P is a σ-additive set function on $\mathcal{Q}$, by Theorem 2.1.2.

A σ-additive set function μ defined on a σ-algebra Σ on Ω is called a **measure on Σ**.

Exercise 2.1.2 Let μ be a σ-additive set function defined on an algebra $\mathcal{A}$ on Ω with $\mu(\Omega) < \infty$. Suppose that μ_1 and μ_2 are measures defined on a σ-algebra $\Sigma \supset \mathcal{A}$, with the property that $\mu_1(A) = \mu_2(A) = \mu(A)$ for $A \in \mathcal{A}$. Show that $\mu_1(B) = \mu_2(B)$ for $B \in \sigma(\mathcal{A})$. (Hint: show that $\mathcal{L} = \{B \in \Sigma : \mu_1(B) = \mu_2(B)\}$ is a λ-system.)

2.2 Measurable spaces and measurable functions

A function f defined on a set Ω and taking values in $[-\infty, \infty] := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is said to be **extended real-valued**. The sets $[-\infty, \infty]$ and $[0, \infty] := [0, \infty) \cup \{\infty\}$ will often also be denoted as $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^+$ respectively, while $[0, \infty)$ will also be denoted as $\mathbb{R}^+$. Since, except where explicitly specified otherwise, functions considered are extended real-valued; we shall often call an extended real-valued function defined on Ω simply a function on Ω; while if f takes values in $\mathbb{R}$, f is said to be **real-valued** or **finite-valued**. We recall some usual conventions concerning algebraic operations involving infinity symbols ∞ and $-\infty$: $\infty + \infty = \infty$, $-\infty + (-\infty) = -\infty$, $a + \infty = -(a - \infty) = \infty$ if a is a finite number, while for an extended real number a, $a \cdot \infty = (-a) \cdot (-\infty) = \infty$, or $-\infty$, depending on whether $a > 0$ or $a < 0$, and $0 \cdot \infty = 0 \cdot (-\infty) = 0$. The symbol ∞ is sometimes written $+\infty$ for emphasis. We shall also adopt the convention that $(-\infty)^{-1} = (\infty)^{-1} = 0$, but then $\frac{\infty}{\infty}$, $\frac{-\infty}{-\infty}$, $\frac{\infty}{-\infty}$ and $\frac{-\infty}{\infty}$ are considered not to be defined. We also observe that $\infty - \infty$ and $\infty + (-\infty)$ are not defined.

An ordered pair (Ω, Σ) is called a **measurable space** if Ω is a nonempty set and Σ is a σ-algebra on Ω.

Given a measurable space (Ω, Σ), a function f on Ω is called **Σ-measurable** if $\{x \in \Omega : f(x) > \alpha\} \in \Sigma$ for every $\alpha \in \mathbb{R}$. A Σ-measurable function will simply be called **measurable** if the measurable space (Ω, Σ) is clearly implied. More generally, a function is said to be measurable on $A \in \Sigma$ if its domain of definition contains A and if $\{x \in A : f(x) > \alpha\} \in \Sigma$ for every $\alpha \in \mathbb{R}$. Observe that a function is Σ-measurable if and only if $\{x \in \Omega : f(x) > \alpha\} \in \Sigma$ for all $\alpha \in \overline{\mathbb{R}}$. This is clear, because $\{x \in \Omega : f(x) > \infty\} = \phi$ and $\{x \in \Omega : f(x) > -\infty\} = \bigcup_{n \in \mathbb{N}} \{x \in \Omega : f(x) > -n\}$. For notational simplicity, we shall presently introduce simplified notations for sets like $\{x \in \Omega : f(x) > \alpha\}$. For a set $C \subset \overline{\mathbb{R}}$ and a function f on Ω, the set $\{x \in \Omega : f(x) \in C\}$ will be denoted simply as $\{f \in C\}$. With this notation, f is Σ-measurable if $\{f \in (\alpha, \infty]\} \in \Sigma$ for all $\alpha \in \mathbb{R}$. $\{f \in (\alpha, \infty]\}$ will also be denoted as $\{f > \alpha\}$. Similarly, for $\alpha \leq \beta$ in $\overline{\mathbb{R}}$, the sets $\{f \in (\alpha, \beta)\}$, $\{f \in (\alpha, \beta]\}$, $\{f \in [\alpha, \beta)\}$ and $\{f \in [\alpha, \beta]\}$ in this order will be denoted as $\{\alpha < f < \beta\}$, $\{\alpha < f \leq \beta\}$, $\{\alpha \leq f < \beta\}$, and $\{\alpha \leq f \leq \beta\}$ respectively.

Constant functions are certainly measurable functions; after constant functions, measurable functions of the simplest structure are the simple functions that will now be introduced. For $A \subset \Omega$, we denote by I_A the function defined by $I_A(x) = 1$ or 0, according to whether $x \in A$ or not. The function I_A is called the **indicator** function of the set A; clearly, I_A is measurable if and only if $A \in \Sigma$. A function of the form $\sum_{j=1}^{k} \alpha_j I_{A_j}$, $k \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$, $A_j \in \Sigma$, is called a **simple** function. One can verify directly that simple functions are measurable and form a real vector space of functions.

For a metric space M we shall denote by $\mathcal{B}(M)$ the smallest σ-algebra on M containing all open subsets of M and call a $\mathcal{B}(M)$-measurable function defined on M a **Borel** measurable function (or simply a **Borel** function). It is easily seen that a monotone increasing (decreasing) function defined on an interval of $\mathbb{R}$ is Borel measurable. One also verifies readily that lower semi-continuous functions and upper semi-continuous functions are Borel measurable. Sets in $\mathcal{B}(M)$ are called **Borel** sets in M and $\mathcal{B}(M)$ is usually referred to as the **Borel** field on M. $\mathcal{B}(\mathbb{R})$ will be simply denoted by $\mathcal{B}$. The smallest

σ-algebra on $\overline{\mathbb{R}}$ containing all open subsets of $\mathbb{R}$ as well as sets of the form $(\alpha, \infty]$ for all $\alpha \in \mathbb{R}$ is denoted by $\overline{\mathcal{B}}$. Sets in $\overline{\mathcal{B}}$ are called Borel sets in $\overline{\mathbb{R}}$. For $n \geq 2$, $\mathcal{B}(\mathbb{R}^n)$ is simply denoted by $\mathcal{B}^n$.

Example 2.2.1 Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on a metric space M, and let $C = \{x \in M : \lim_{n\to\infty} f_n(x) \text{ exists}\}$. Then $C = \bigcap_{k\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} \bigcap_{n,m\geq l} A_{nm}^{(k)}$, where for k, m, n in $\mathbb{N}$, $A_{nm}^{(k)} = \{x \in M : |f_n(x) - f_m(x)| < \frac{1}{k}\}$. Since each $A_{nm}^{(k)}$ is open, C is a Borel set in M.

Given a measurable space (Ω, Σ), a function defined on Ω is often referred to as a function on (Ω, Σ), by abuse of language, if the role of Σ is to be emphasized; in particular, a measurable function on (Ω, Σ) means a Σ-measurable function defined on Ω.

Remark If f is a measurable function, then $\{f \geq \alpha\} = \bigcap_{m\in\mathbb{N}} \{f > \alpha - \frac{1}{m}\}$ is in Σ; similarly, $\{f < \alpha\} = \bigcup_{m\in\mathbb{N}} \{f \leq \alpha - \frac{1}{m}\}$ is in Σ, because each $\{f \leq \alpha - \frac{1}{m}\} = \{f > \alpha - \frac{1}{m}\}^c$ is in Σ.

Exercise 2.2.1

(i) Show that $\overline{\mathcal{B}}$ is the smallest σ-algebra on $\overline{\mathbb{R}}$ containing $\{(\alpha, \infty] : \alpha \in \mathbb{R}\}$.

(ii) Let (Ω, Σ) be a measurable space. Show that a function f on Ω is Σ-measurable if and only if $\{f \in B\} \in \Sigma$ for all $B \in \overline{\mathcal{B}}$.

(iii) Let (Ω, Σ) be a measurable space. Show that if f is a finite-valued function on Ω, then f is Σ-measurable if and only if $\{f \in B\} \in \Sigma$ for all $B \in \mathcal{B}$.

For a family $\{f_\alpha\}$ of functions defined on a set Ω, define functions $\inf_\alpha f_\alpha$ and $\sup_\alpha f_\alpha$ by

$$\left(\inf_\alpha f_\alpha\right)(x) = \inf_\alpha f_\alpha(x); \quad \left(\sup_\alpha f_\alpha\right)(x) = \sup_\alpha f_\alpha(x)$$

for $x \in \Omega$. $\inf_\alpha f_\alpha$ and $\sup_\alpha f_\alpha$ are sometimes expressed respectively by $\bigwedge_\alpha f_\alpha$ and $\bigvee_\alpha f_\alpha$. If $\{f_n\}$ is a sequence of functions defined on Ω, define functions $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ by

$$\left(\liminf_{n\to\infty} f_n\right)(x) = \lim_{n\to\infty}\left(\inf_{m\geq n} f_m(x)\right); \quad \left(\limsup_{n\to\infty} f_n\right)(x) = \lim_{n\to\infty}\left(\sup_{m\geq n} f_m(x)\right)$$

for $x \in \Omega$. Since uncertainty is not likely, $(\liminf_{n\to\infty} f_n)(x)$ and $(\limsup_{n\to\infty} f_n)(x)$ will be simply written as $\liminf_{n\to\infty} f_n(x)$ and $\limsup_{n\to\infty} f_n(x)$ respectively.

Naturally, if $\liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$, the common value is denoted by $\lim_{n\to\infty} f_n(x)$ and we say that the sequence $\{f_n\}$ converges at x. If $\{f_n\}$ converges at all $x \in A \subset \Omega$, and if we define a function f on A by $f(x) = \lim_{n\to\infty} f_n(x)$, then we say that the sequence $\{f_n\}$ converges **pointwise** on A to f (notationally, $f_n \to f$ on A).

Exercise 2.2.2 Let (Ω, Σ) be a measurable space and $\{f_n\}$ a sequence of measurable functions on Ω.

(i) Show that both $\inf_n f_n$ and $\sup_n f_n$ are measurable functions on Ω. (Hint: $\{\inf_n f_n > \alpha\} = \bigcup_m \bigcap_n \{f_n > \alpha + \frac{1}{m}\}$.)

(ii) Show that both $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ are measurable functions on Ω.

(iii) Show that $\{x \in \Omega : \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)\} \in \Sigma$.

(iv) Show that if $\lim_{n\to\infty} f_n(x)$ exists for all $x \in \Omega$, then $f = \lim_{n\to\infty} f_n$ is measurable.

Exercise 2.2.3 Let f be measurable. For each positive integer n, let $A^{(n)} = \{f < -n\}$, $C^{(n)} = \{f \geq n\}$, $B_i^{(n)} = \{-n + \frac{i}{n} \leq f < -n + \frac{i+1}{n}\}$, $i = 0, 1, 2, \ldots, 2n^2 - 1$, and let

$$g_n = -nI_{A^{(n)}} + \sum_{i=0}^{2n^2-1} \left(-n + \frac{i}{n}\right) I_{B_i^{(n)}} + nI_{C^{(n)}}.$$

Show that $g_n \to f$ pointwise and show that if f, g are measurable, then fg is measurable; furthermore if $g \neq 0$ everywhere on X, then f/g is also measurable.

Exercise 2.2.4 Let (Ω, Σ) be a measurable space and f, g measurable functions on Ω. Then $f + g$ is defined on Ω if and only if $\{f(x), g(x)\} \neq \{-\infty, \infty\}$ for all $x \in \Omega$. Show that if $f + g$ is defined on Ω, then $f + g$ is measurable. (Hint: $\{f + g > \alpha\} = \bigcup_{q\in\mathbb{Q}} \{f > q\} \cap \{g > \alpha - q\}$ for $\alpha \in \mathbb{R}$, where $\mathbb{Q}$ is the set of all rational numbers.)

Since for a measurable function f on Ω and $\lambda \in \mathbb{R}$, λf is clearly measurable, we infer from Exercise 2.2.4 that the space of all finite-valued measurable functions is a real vector space which contains the space of all simple functions as a vector subspace.

To conclude this section, we present a useful representation for nonnegative measurable functions.

Theorem 2.2.1 *Suppose that (Ω, Σ) is a measurable space and f is a nonnegative measurable function defined on Ω, then there is a sequence $\{A_j\}_{j=1}^\infty \subset \Sigma$ such that*

$$f(\omega) = \sum_{j=1}^{\infty} \frac{1}{j} I_{A_j}(\omega) \tag{2.1}$$

for all $\omega \in \Omega$.

Proof Define sets $A_1, \ldots, A_j, \ldots$ recursively as follows: $A_1 = \{f \geq 1\}$, $A_2 = \{f \geq \frac{1}{2} + I_{A_1}\}, \ldots, A_j = \{f \geq \frac{1}{j} + \sum_{k<j} \frac{1}{k} I_{A_k}\}, \ldots$. Clearly each A_j is in Σ. We now show that (2.1) holds for $\omega \in \Omega$.

Observe first, that $\omega \in \Omega \setminus \bigcup_{j=1}^\infty A_j$ if and only if $f(\omega) = 0$ and that when $\omega \in \Omega \setminus \bigcup_{j=1}^\infty A_j$, both sides of (2.1) are equal to zero. It remains to show that (2.1) holds for $\omega \in \bigcup_{j=1}^\infty A_j$.

For $\omega \in \bigcup_{j=1}^{\infty} A_j$, we distinguish two cases:

[Case 1] $\omega \in A_j$ for only finitely many j.
Let j_0 be the largest j such that $\omega \in A_j$. Then,

$$f(\omega) \geq \frac{1}{j_0} + \sum_{k<j_0} \frac{1}{k} I_{A_k}(\omega) = \sum_{k=1}^{j_0} \frac{1}{k} I_{A_k}(\omega) = \sum_{k=1}^{\infty} \frac{1}{k} I_{A_k}(\omega);$$

on the other hand, for $j > j_0$,

$$f(\omega) < \frac{1}{j} + \sum_{k<j} \frac{1}{k} I_{A_k}(\omega) = \frac{1}{j} + \sum_{k=1}^{\infty} \frac{1}{k} I_{A_k}(\omega);$$

hence, by letting $j \to \infty$, we have

$$f(\omega) \leq \sum_{k=1}^{\infty} \frac{1}{k} I_{A_k}(\omega).$$

Thus (2.1) holds in this case.
[Case 2] $\omega \in A_j$ for infinitely many j.
For infinitely many j, we have

$$f(\omega) \geq \frac{1}{j} + \sum_{k<j} \frac{1}{k} I_{A_k}(\omega) = \sum_{k=1}^{j} \frac{1}{k} I_{A_k}(\omega);$$

let $j \to \infty$ through such j's, it follows that

$$f(\omega) \geq \sum_{k=1}^{\infty} \frac{1}{k} I_{A_k}(\omega). \tag{2.2}$$

Now either $\omega \in A_j$ for $j \geq N$ for some $N \in \mathbb{N}$ or $\omega \notin A_j$ for infinitely many j. In the former case,

$$f(\omega) \geq \sum_{k=1}^{\infty} \frac{1}{k} I_{A_k}(\omega) \geq \sum_{k=N}^{\infty} \frac{1}{k} = \infty,$$

thus $f(\omega) = \infty = \sum_{k=1}^{\infty} \frac{1}{k} I_{A_k}(\omega)$; in the latter case,

$$f(\omega) < \frac{1}{j} + \sum_{k<j} \frac{1}{k} I_{A_k}(\omega)$$

for infinitely many j and hence when $j \to \infty$ through such j's, it follows that

$$f(\omega) \leq \sum_{k=1}^{\infty} \frac{1}{k} I_{A_k}(\omega),$$

which together with (2.2) shows that (2.1) holds. ■

Corollary 2.2.1 *If f is a nonnegative measurable function, then there is a nondecreasing sequence $\{s_n\}$ of nonnegative simple functions which converges to f pointwise.*

Proof Let $\{A_j\}$ be the sequence of measurable sets in Theorem 2.2.1. Choose the sequence $\{s_n\}$ defined by

$$s_n = \sum_{j=1}^{n} \frac{1}{j} I_{A_j}.$$ ■

Exercise 2.2.5 Let f be a measurable function; show that there is a sequence $\{f_n\}$ of simple functions such that $|f_n| \leq |f|$ and $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

2.3 Measure space and integration

A triple (Ω, Σ, μ) is called a **measure space** if (Ω, Σ) is a measurable space and μ is a measure on Σ. When $\mu(\Omega) = 1$, (Ω, Σ, μ) is called a **probability space**, and in this case μ is usually denoted by P.

Example 2.3.1 Let Ω be an arbitrary nonempty set and for $A \subset \Omega$ let $\mu(A)$ be the cardinality of A if A is finite; otherwise let $\mu(A) = \infty$. Obviously μ is a measure on 2^{Ω}, the σ-algebra of all subsets of Ω, and is called the **counting measure** on Ω. The measure space $(\Omega, 2^{\Omega}, \mu)$ will be called the measure space with counting measure on Ω.

Example 2.3.2 Let Ω be a countable set, say $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n, \ldots\}$, and $\{p_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} p_n = 1$. For $A \subset \Omega$, let $\mathbb{N}(A) = \{n \in \mathbb{N} : \omega_n \in A\}$ and $\mu(A) = \sum_{n \in \mathbb{N}(A)} p_n$; then the measure space $(\Omega, 2^{\Omega}, \mu)$ is called a **discrete** probability space.

Given a measure space (Ω, Σ, μ), measurable functions are extended real-valued functions measurable in reference to the measurable space (Ω, Σ).

We now fix a measure space (Ω, Σ, μ) and define the integral for certain measurable functions. Recall that a simple function is a finite linear combination of indicator functions of sets in Σ. Clearly if f is a simple function, then $f = \sum_{i=1}^{k} \alpha_i I_{A_i}$, where $\alpha_1, \ldots, \alpha_k$ are the different values assumed by f and $A_i = \{f = \alpha_i\}$; we define then

$$\int_{\Omega} f d\mu = \sum_{i=1}^{k} \alpha_i \mu(A_i), \tag{2.3}$$

if the right-hand side of (2.3) has a meaning. It is easy to see that if $\int_\Omega f d\mu$ is defined and f is expressed as $f = \sum_{i=1}^{l} \beta_i I_{B_i}$, where $B_1, \ldots, B_l$ are in Σ and are disjoint, then

$$\int_\Omega f d\mu = \sum_{i=1}^{l} \beta_i \mu(B_i).$$

In particular, $\int_\Omega f d\mu$ has a meaning if f is simple and nonnegative, although it is possible that $\int_\Omega f d\mu = +\infty$.

If f is measurable and nonnegative, define

$$\int_\Omega f d\mu = \sup \int_\Omega g d\mu,$$

where the supremum is taken over all simple functions g with $0 \leq g \leq f$. Obviously, if f is nonnegative and simple, this definition coincides with the previously defined $\int_\Omega f d\mu$ for simple functions.

For a function f defined on a set Ω, define nonnegative functions f^+ and f^- by

$$\begin{aligned} f^+(x) &= f(x) \ \text{ if } f(x) \geq 0, \\ &= 0 \text{ otherwise;} \\ f^-(x) &= -f(x) \ \text{ if } f(x) \leq 0, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$; furthermore, if f is measurable on a measurable space (Ω, Σ), then both f^+ and f^- are measurable.

Return now to the discourse interrupted by the last paragraph and let f be a measurable function. Define

$$\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$$

if the right-hand side has a meaning. In this case, $\boldsymbol{\int_\Omega f d\mu}$ is said to **exist** and is called the **integral** of f. One notes that if f is a simple function this definition of $\int_\Omega f d\mu$ coincides with that given by (2.3). If $\int_\Omega f d\mu$ is finite, then f is said to be **integrable**. Integrability and the integral of a measurable function so defined will be referred to more precisely as μ-**integrability** and the μ-**integral** respectively, if the measure μ is to be emphasized. It will be shown later that a measurable function f is integrable if and only if $|f|$ is integrable (see Theorem 2.5.3).

Suppose that f is a measurable function and $A \in \Sigma$; if $\int_\Omega f I_A d\mu$ exists, it is denoted by $\int_A f d\mu$ and is called the integral of f **over** A. Obviously, if $\int_\Omega f d\mu$ exists, then $\int_A f d\mu$ exists for all $A \in \Sigma$.

Example 2.3.3 Let Ω be an arbitrary set and consider the counting measure μ on Ω; then every function f on Ω is measurable and f is integrable if and only if $f(x)$ is finite for $x \in \Omega$ and $\{f(x)\}_{x\in\Omega}$ is summable.

Example 2.3.4 Consider the discrete probability space of Example 2.3.2. Let f be a function on Ω. Since every subset of Ω is measurable, f is measurable and is called a random variable. If $\int_\Omega f d\mu$ exists, it is called the expectation of f. It is easily verified that f is integrable if and only if $\{f(\omega_n)p_n\}_{n\in\mathbb{N}}$ is summable.

Exercise 2.3.1 If f and g are nonnegative simple functions and $\alpha,\ \beta \geq 0$, show that

$$\int_\Omega (\alpha f + \beta g) d\mu = \alpha \int_\Omega f d\mu + \beta \int_\Omega g d\mu.$$

Exercise 2.3.2 If $f \leq g$ are two nonnegative measurable functions, show that $\int_\Omega f d\mu \leq \int_\Omega g d\mu$.

Exercise 2.3.3 Suppose that f and g are measurable functions such that $f \leq g$, and suppose that $\int_\Omega g^+ d\mu < \infty$. Show that $\int_\Omega f d\mu$ exists and $\int_\Omega f d\mu \leq \int_\Omega g d\mu$.

Exercise 2.3.4 Let f be a measurable function on a measure space (Ω, Σ, μ) and for each $k \in \mathbb{N}$ let $A_k = \{2^{k-1} \leq |f| < 2^k\}$. Show that f is integrable if and only if $\sum_{k=1}^{\infty} 2^{k-1}\mu(A_k) < \infty$ and $\mu(\{|f| = \infty\}) = 0$.

Example 2.3.5 Suppose that f is a nonnegative measurable function and $0 \leq p < r < q < \infty$. Then $\int_\Omega f^r d\mu \leq \int_\Omega f^p d\mu + \int_\Omega f^q d\mu$. Actually, if we let $A = \{f \leq 1\}$ and $B = \{f > 1\}$, then $\int_\Omega f^r d\mu = \int_\Omega I_A f^r d\mu + \int_\Omega I_B f^r d\mu \leq \int_\Omega I_A f^p d\mu + \int_\Omega I_B f^q d\mu \leq \int_\Omega f^p d\mu + \int_\Omega f^q d\mu$.

2.4 Egoroff theorem and monotone convergence theorem

Suppose that Ω is a set and $\{A_n\}_{n=1}^{\infty}$ is a sequence of subsets of Ω, define

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k;$$
$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k.$$

If $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$, then we say that the limit of the sequence $\{A_n\}$ exists and has the common set as its limit, which is denoted by $\lim_{n\to\infty} A_n$. In particular, if $A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$ i.e. $\{A_n\}$ is monotone increasing, or $A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$ i.e. $\{A_n\}$ is monotone decreasing, then $\lim_{n\to\infty} A_n$ exists and equals $\bigcup_{n\in\mathbb{N}} A_n$ or $\bigcap_{n\in\mathbb{N}} A_n$ according to whether $\{A_n\}$ is monotone increasing or monotone decreasing. Hence $\limsup_{n\to\infty} A_n = \lim_{n\to\infty} \bigcup_{k\geq n} A_k$ and $\liminf_{n\to\infty} A_n = \lim_{n\to\infty} \bigcap_{k\geq n} A_k$.

Exercise 2.4.1 Let $\{A_n\}_{n=1}^{\infty} \subset 2^{\Omega}$, where Ω is an arbitrary set, and let $B = \liminf_{n\to\infty} A_n$, $C = \limsup_{n\to\infty} A_n$. Show that for each $x \in \Omega$ we have

$$I_B(x) = \liminf_{n\to\infty} I_{A_n}(x) \text{ and } I_C(x) = \limsup_{n\to\infty} I_{A_n}(x).$$

In the following, a measure space (Ω, Σ, μ) is considered and fixed throughout.

Lemma 2.4.1 (Monotone limit lemma) *Let $\{A_n\}_{n=1}^{\infty} \subset \Sigma$ be monotone increasing, then*

$$\mu\left(\lim_{n\to\infty} A_n\right) = \mu\left(\bigcup_n A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof For each positive integer n let $B_n = A_n \setminus A_{n-1}$, where we put $A_0 = \emptyset$, and for convenience let $A = \bigcup_n A_n$. Then $A_n = \bigcup_{k=1}^{n} B_k$ and $A = \bigcup_k B_k$. Since $\{B_k\}$ is disjoint, we have

$$\mu(A) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \mu(A_n).$$

■

Corollary 2.4.1 *Let $\{A_n\}_{n=1}^{\infty} \subset \Sigma$ be monotone decreasing and $\mu(A_1) < \infty$, then*

$$\mu\left(\lim_{n\to\infty} A_n\right) = \mu\left(\bigcap_n A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof For each positive integer n let $B_n = A_1 \setminus A_n$, and for convenience let $A = \bigcap_n A_n$. Then $\{B_k\}$ is monotone increasing and $A_1 \setminus A = \bigcup_k B_k$. From Lemma 2.4.1, we have

$$\mu(A_1 \setminus A) = \mu\left(\bigcup_k B_k\right) = \lim_{n\to\infty} \mu(B_n).$$

But $\mu(A_1 \setminus A) = \mu(A_1) - \mu(A)$ and $\mu(B_n) = \mu(A_1) - \mu(A_n)$; this completes the proof of the corollary. ■

Remark In Corollary 2.4.1 one may assume that $\mu(A_n) < \infty$ for some n, instead of $\mu(A_1) < \infty$.

Exercise 2.4.2 Let (Ω, Σ, μ) be a measure space. Suppose $\{A_n\}_{n\in\mathbb{N}} \subset \Sigma$.

(i) Show that $\mu\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mu(A_n)$.

(ii) If $\mu\left(\bigcup_{j\geq n} A_j\right) < +\infty$ for some n, then show that $\mu\left(\limsup_{n\to\infty} A_n\right) \geq \limsup_{n\to\infty} \mu(A_n)$.

(iii) If the limit of $\{A_n\}$ exists and $\mu\left(\bigcup_{j\geq n} A_j\right) < \infty$ for some n, show that $\lim_{n\to\infty} \mu(A_n)$ exists and

$$\mu\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Theorem 2.4.1 (Egoroff theorem) *If $\{f_n\}$ is a sequence of measurable functions and $f_n \to f$ with finite limit on $A \in \Sigma$, where $\mu(A) < +\infty$, then for any given $\varepsilon > 0$, there is $B \in \Sigma$ with $B \subset A$, such that $\mu(A \setminus B) < \varepsilon$ and $f_n \to f$ uniformly on B.*

Proof

[Step 1] We claim that for $\varepsilon > 0$, $\eta > 0$, there is integer $N > 0$ and $C \in \Sigma$ such that $C \subset A$, $\mu(A \setminus C) < \varepsilon$, and $\sup_{x \in C} |f(x) - f_n(x)| \leq \eta$ whenever $n \geq N$.

To show this, for each n let $C_n = \bigcap_{m \geq n} \{x \in A : |f(x) - f_m(x)| \leq \eta\}$. Then $C_n \nearrow A$. Since $\mu(A) < \infty$, there is N such that $\mu(A \setminus C_N) = \mu(A) - \mu(C_N) < \varepsilon$ by Lemma 2.4.1. Take $C = C_N$.

[Step 2] Now given $\varepsilon > 0$. By [Step 1] for each positive integer m there is integer N_m and $C_m \subset A$ with $C_m \in \Sigma$ such that

$$\mu(A \setminus C_m) < \varepsilon / 2^m$$

and

$$\sup_{x \in C_m} |f(x) - f_n(x)| \leq \frac{1}{m}$$

whenever $n \geq N_m$.

Take $B = \bigcap_{m=1}^{\infty} C_m$, then $\mu(A \setminus B) = \mu(\bigcup_{m=1}^{\infty} (A \setminus C_m)) < \varepsilon$. Given $\sigma > 0$, choose $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} < \sigma$. Then for $n \geq N_{m_0}$, we have $|f(x) - f_n(x)| \leq \frac{1}{m_0} < \sigma$ for all $x \in B$, because $B \subset C_{m_0}$. This shows that $f_n \to f$ uniformly on B. ■

In plain language, Theorem 2.4.1 says that convergence of a sequence of measurable functions on a set of finite measure implies approximate uniform convergence. From its proof, one sees clearly that σ-additivity of μ plays a salient role through Lemma 2.4.1. The following theorem which is called the **monotone convergence theorem** reveals the distinguished feature of σ-additivity of measure μ through integrals.

Example 2.4.1 Suppose $\mu(\Omega) < \infty$ and $\{f_n\}$ is a sequence of real-valued measurable functions such that $\lim_{n \to \infty} f_n(x) = f(x)$ exists and is finite for μ-a.e. x in Ω. For each $k \in \mathbb{N}$, by the Egoroff theorem there is $B_k \in \Sigma$ such that $\mu(\Omega \setminus B_k) < \frac{1}{k}$ and $f_n(x) \to f(x)$ uniformly for $x \in B_k$. Put $\mathbb{Z} = \Omega \setminus \bigcup_k B_k$, then $\mu(\mathbb{Z}) \leq \mu(\Omega \setminus B_k) < \frac{1}{k}$ for all k and hence $\mu(\mathbb{Z}) = 0$. Therefore we have shown that there are $B_1, B_2, \ldots, \mathbb{Z}$ in Σ with $\mu(\mathbb{Z}) = 0$ such that $\Omega = \bigcup_k B_k \cup \mathbb{Z}$ and $\lim_{n \to \infty} f_n(x) = f(x)$ uniformly on each B_k.

Exercise 2.4.3 Show that the conclusion in Example 2.4.1 still holds if (Ω, Σ, μ) is σ-finite.

Theorem 2.4.2 (Monotone convergence theorem) *Let $\{f_n\}$ be a monotone nondecreasing sequence of nonnegative measurable functions. Then*

$$\int_\Omega \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu.$$

Proof Put $f = \lim_{n\to\infty} f_n$; then $f_n \leq f$ for all n. Since $\int_\Omega f_1 d\mu \leq \int_\Omega f_2 d\mu \leq \cdots \leq \int_\Omega f_n d\mu \leq \cdots \leq \int_\Omega f d\mu$, we have

$$\lim_{n\to\infty} \int_\Omega f_n d\mu \leq \int_\Omega f d\mu.$$

It remains to show that $\int_\Omega f d\mu \leq \lim_{n\to\infty} \int_\Omega f_n d\mu$. For this, it suffices to show that $\lim_{n\to\infty} \int_\Omega f_n d\mu \geq \lambda$ for each finite real number $\lambda < \int_\Omega f d\mu$. For such a λ, there is a simple function $g = \sum_{j=1}^{l} \alpha_i I_{A_j}$ such that $0 \leq g \leq f$ and $\int_\Omega g d\mu = \sum_{j=1}^{n} \alpha_j \, \mu\,(A_j) > \lambda$. In the above expression for g, we may assume that $\alpha_1, \ldots, \alpha_l$ are the different positive values taken by g, and hence $A_1, \ldots, A_l$ are disjoint sets in Σ. Then $\alpha_j \leq f$ on each A_j. Choose $\varepsilon > 0$ small enough so that $\alpha_j - \varepsilon > 0$, $j = 1, \ldots, l$. For each $j = 1, \ldots, l$ and positive integer n, let $A_j^{(n)} = \{x \in A_j : f_n(x) > \alpha_j - \varepsilon\}$ and define $g_n = \sum_{j=1}^{l} (\alpha_j - \varepsilon) I_{A_j^{(n)}}$; then $0 \leq g_n \leq f_n$ and hence

$$\lim_{n\to\infty} \int_\Omega f_n d\mu \geq \lim_{n\to\infty} \sum_{j=1}^{l} (\alpha_j - \varepsilon)\mu(A_j^{(n)}) = \sum_{j=1}^{l} (\alpha_j - \varepsilon)\mu(A_j),$$

because for each j, $A_j^{(n)}$ is a nondecreasing sequence with A_j as its limit. It follows then that $\lim_{n\to\infty} \int_\Omega f_n d\mu \geq \sum_{j=1}^{l} \alpha_j \mu(A_j)$ by letting $\varepsilon \searrow 0$. The proof is complete. ∎

Exercise 2.4.4

(i) If f and g are nonnegative measurable functions and $\alpha, \beta \geq 0$, show that

$$\int_\Omega (\alpha f + \beta g) d\mu = \alpha \int_\Omega f d\mu + \beta \int_\Omega g d\mu.$$

(ii) Suppose that f is integrable and $\alpha \in \mathbb{R}$. Show that $\int_\Omega \alpha f d\mu = \alpha \int_\Omega f d\mu$.

2.5 Concepts related to sets of measure zero

We now make some remarks on concepts connected with measure zero sets (as previously, a measure space (Ω, Σ, μ) is considered and fixed). For this purpose, a subset A of Ω is called a **null** set (or more precisely $\boldsymbol{\mu}$**-null** set), if $A \subset B \in \Sigma$ and $\mu(B) = 0$.

Note that countable unions of null sets are null sets. Let $A = \{x \in \Omega : x$ does not have a property $P\}$, if A is a null set, we say that the property P holds **almost everywhere** on Ω (or simply P holds almost everywhere). For example, if outside a null set, f is finite, then we say that f is finite almost everywhere; also if $\lim_{n\to\infty} f_n(x) = f(x)$ exists for each x outside a null set, then we say that f_n converges almost everywhere. If a property P holds almost everywhere, we simply say that P holds a.e. (more precisely, μ-almost everywhere or μ-a.e. if other measures might also be in question). Two measurable functions f and g are said to be **equivalent** if $f = g$ a.e. Clearly, if f and g are equivalent and if the integral of one of them exists, then both of their integrals exist and are equal. If g is equivalent to f, g is sometimes referred to as a **version** of f.

As we shall see, functions which appear naturally are often not defined at every point of Ω. The most important case is when they are defined outside null sets. A function f is said to be defined a.e. on Ω if f is defined on $\Omega\backslash A$, with A being a null set; and f is measurable if f is measurable on $\Omega\backslash N$ for some measurable null set $N \supset A$, or, equivalently, if a new function $\hat{f}$ is defined by $\hat{f}(x) = f(x)$ for $x \in \Omega\backslash N$ and $\hat{f}(x) = 0$ for $x \in N$, then $\hat{f}$ is measurable. Hence, a measurable function f which is defined a.e. on Ω can be considered as defined on Ω if it is replaced by one of $\hat{f}$ defined above; this is legitimate because any pair of such functions $\hat{f}$ are equivalent measurable functions.

Exercise 2.5.1 Show that if f is measurable, then $\{f = +\infty\}$ and $\{f = -\infty\}$ are in Σ. Show also that if f is integrable, then f is finite a.e.

All the results we have established so far remain true if the pointwise conditions are replaced by conditions held almost everywhere. For example:

Theorem 2.5.1 (Egoroff theorem) *If a sequence $\{f_n\}$ of almost everywhere finite measurable functions converges a.e. to a finite function f on A, where $A \in \Sigma$, and $\mu(A) < +\infty$, then for every $\varepsilon > 0$, there is $B \in \Sigma$, $B \subset A$ such that $\mu(A \setminus B) < \varepsilon$ and $f_n \to f$ uniformly on B.*

Theorem 2.5.2 (Monotone convergence theorem) *Let $\{f_n\}$ be a sequence of measurable functions which are nonnegative and nondecreasing a.e., then*

$$\int_\Omega \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu.$$

From Theorem 2.5.2 and Exercise 2.4.4 (i) there follows the following corollary.

Corollary 2.5.1 *If $\{f_n\}$ is a sequence of a.e. nonnegative measurable functions, then*

$$\int_\Omega \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_\Omega f_n d\mu.$$

Exercise 2.5.2 Let f be a measurable function. Prove the following statements:

(i) Suppose that $\int_\Omega f d\mu$ exists, i.e. $\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$, where the right-hand side has a meaning. If $f = f_1 - f_2$ where f_1 and f_2 are nonnegative and measurable, then

$$\int_\Omega f d\mu = \int_\Omega f_1 d\mu - \int_\Omega f_2 d\mu,$$

if the right-hand side has a meaning.

(ii) $\int_\Omega f d\mu$ exists if and only if $f = f_1 - f_2$ for some nonnegative measurable functions f_1 and f_2, such that $\int_\Omega f_1 d\mu - \int_\Omega f_2 d\mu$ is meaningful. (Hint: for f_1 and f_2 as above, observe that $f^+ \leq f_1$ and $f^- \leq f_2$.)

(iii) If f, g are measurable functions such that $\int_\Omega f d\mu$, $\int_\Omega g d\mu$, $\int_\Omega f d\mu + \int_\Omega g d\mu$ are meaningful, then $f + g$ is defined a.e. and

$$\int_\Omega (f + g) d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu.$$

In particular, this holds true if f and g are integrable.

Exercise 2.5.3 Show that Theorem 2.5.2 still holds if $\{f_n\}$ is a sequence of measurable functions bounded from below by an integrable function a.e. and is nondecreasing a.e. (Hint: show first that f_n^- is integrable and hence $\int_\Omega f_n d\mu$ is meaningful for each n.)

Exercise 2.5.4 If $f \geq 0$ a.e. and is measurable, then show that $\int_\Omega f d\mu = 0$ if and only if $f = 0$ a.e.

Exercise 2.5.5 (Beppo–Levi) Let $\{f_n\}$ be a monotone increasing sequence of integrable functions such that $\sup_n \int f_n d\mu < +\infty$. Let $f = \lim_{n\to\infty} f_n$. Show that $-\infty < f < +\infty$ a.e., f is integrable, and $\lim_{n\to\infty} \int |f_n - f| d\mu = 0$.

The following theorems follow from Exercise 2.5.2 (iii) and Exercise 2.4.4 (ii):

Theorem 2.5.3 *A measurable function f is integrable if and only if $|f|$ is integrable.*

Theorem 2.5.4 *Suppose that f and g are integrable and α, β are finite real numbers, then*

$$\int_\Omega (\alpha f + \beta g) d\mu = \alpha \int_\Omega f d\mu + \beta \int_\Omega g d\mu.$$

In particular, if $f \leq g$ a.e. then $\int_\Omega f d\mu \leq \int_\Omega g d\mu$.

Exercise 2.5.6 Suppose that (Ω, Σ, μ) is a finite measure space and f a measurable function on Ω. For $k \in \mathbb{N}$ let $\omega_k := \mu(\{|f| > k\})$. Show that f is integrable if and only if $\sum_{k=1}^\infty \omega_k < \infty$. (Hint: show that $\sum_{k=1}^\infty \omega_k \leq \int_\Omega |f| d\mu \leq \sum_{k=1}^\infty \omega_k + \mu(\Omega)$.)

Exercise 2.5.7 Suppose that f is a nonnegative measurable function. Let $\nu : \Sigma \to [0, +\infty]$ be defined by $\nu(A) = \int_A f d\mu := \int_\Omega f I_A d\mu$; show that (Ω, Σ, ν) is a measure space and if $g \geq 0$ is Σ-measurable, then $\int_\Omega g d\nu = \int_\Omega g f d\mu$ (this fact is usually expressed by $d\nu = fd\mu$). Show also that a measurable function g is ν-integrable if and only if gf is μ-integrable.

Exercise 2.5.8 Suppose that f is a nonnegative integrable function. Show that for every $\varepsilon > 0$, there is $A \in \Sigma$ with $\mu(A) < +\infty$, such that

$$\int_A fd\mu > \int_\Omega fd\mu - \varepsilon.$$

Exercise 2.5.9 Let (Ω, Σ, μ) be a measure space, and $\{A_k\}_{k=1}^\infty \subset \Sigma$.

(i) Show that if $\sum_{k=1}^\infty \mu(A_k) < \infty$, then $\mu\left(\limsup_{k\to\infty} A_k\right) = 0$.

(ii) Show that if f is integrable, then

$$\int_{\limsup\limits_{k\to\infty} A_k} fd\mu = \lim_{k\to\infty} \int_{\bigcup\limits_{j=k}^\infty A_j} fd\mu.$$

(iii) Let f be integrable and $\varepsilon > 0$. Show that there is $\delta > 0$ such that if $A \in \Sigma$ and $\mu(A) < \delta$, then $\int_A |f| d\mu < \varepsilon$.
(Hint: suppose the contrary. Then for each k, there is $A_k \in \Sigma$ such that $\mu(A_k) < \frac{1}{k^2}$ and $\int_{A_k} |f| d\mu \geq \varepsilon$. Then apply (i) and (ii).)

Exercise 2.5.10 Let (Ω, Σ, μ) be a measure space and f a measurable function on Ω. Define a σ-algebra $\sigma(f)$ on Ω by

$$\sigma(f) = \{f^{-1}B : B \in \overline{\mathcal{B}}\}.$$

(i) Suppose that $\int_\Omega fd\mu$ exists and $\int_A fd\mu = 0$ for all $A \in \sigma(f)$. Show that $f = 0$ a.e.

(ii) Suppose now that f is integrable and g is $\sigma(f)$-measurable on Ω such that

$$\int_A gd\mu = \int_A fd\mu$$

for all $A \in \sigma(f)$. Show that there is a null set N in $\sigma(f)$ such that $g = f$ on $\Omega \backslash N$.

2.6 Fatou lemma and Lebesgue dominated convergence theorem

It is indicated in Section 2.4 that the monotone convergence theorem reveals the distinguished feature of σ-additivity of measure through integrals. We now present two consequences of the monotone convergence theorem which manifest behaviors of integral under limit processes. These are the **Fatou lemma** and **Lebesgue dominated convergence theorem** (hereafter abbreviated as **LDCT**).

Theorem 2.6.1 (Fatou lemma) *Let $\{f_n\}$ be a sequence of extended real-valued measurable functions which is bounded from below by an integrable function. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu.$$

Proof Let $g_n = \inf_{k\geq n} f_k$, then g_n is nondecreasing and is bounded from below by an integrable function. By the monotone convergence theorem (see Exercise 2.5.3),

$$\int_\Omega \liminf_{n\to\infty} f_n d\mu = \int_\Omega \lim_{n\to\infty} g_n d\mu$$

$$= \lim_{n\to\infty} \int_\Omega g_n d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu. \qquad \blacksquare$$

Exercise 2.6.1 Show that if $\{f_n\}$ is bounded from above by an integrable function, then

$$\int_\Omega \limsup_{n\to\infty} f_n d\mu \geq \limsup_{n\to\infty} \int_\Omega f_n d\mu.$$

Later, both Theorem 2.6.1 and the statement shown in Exercise 2.6.1 will be referred to as the Fatou lemma. One notes that Theorem 2.6.1 is equivalent to a particular case of itself, with $\{f_n\}$ being a sequence of nonnegative measurable functions. This particular case is the original form of the Fatou lemma.

Theorem 2.6.2 (Lebesgue dominated convergence theorem (LDCT)) *If f_n, $n = 1, 2, \ldots$ and f are measurable functions and $f_n \to f$ a.e. Suppose further that $|f_n| \leq g$ a.e. for all n with g being an integrable function. Then*

$$\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu.$$

Proof $\{f_n\}$ is bounded from below and from above by integrable functions. Hence, by the Fatou lemma,

$$\limsup_{n\to\infty} \int_\Omega f_n d\mu \leq \int_\Omega \lim_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu,$$

and consequently

$$\int_\Omega \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu. \qquad \blacksquare$$

The Lebesgue dominated convergence theorem will henceforth be abbreviated as LDCT.

Exercise 2.6.2 Show that under the same conditions as in LDCT we have

$$\lim_{n\to\infty} \int_\Omega |f_n - f| d\mu = 0.$$

Example 2.6.1 Let $\{f_n\}$ be a sequence of nonnegative integrable functions such that $f_1(x) \geq \cdots \geq f_n(x) \geq f_{n+1}(x) \geq \cdots$ and $\lim_{n\to\infty} f_n(x) = 0$ for μ-a.e. x in Ω; then $\sum_{n=1}^{\infty}(-1)^{n+1} f_n$ is integrable and $\int_\Omega \sum_{n=1}^{\infty}(-1)^{n+1} f_n d\mu = \sum_{n=1}^{\infty}(-1)^{n+1} \int_\Omega f_n d\mu$. Note first, from the well-known alternating series's estimate $|\sum_{n=l}^{l+p}(-1)^{n+1} f_n(x)| \leq f_l(x)$ for μ-a.e. x and any l, p in $\mathbb{N}$, that $\sum_{n=1}^{\infty}(-1)^{n+1} f(x)$ converges for μ-a.e. x. Since $|\sum_{n=1}^{k}(-1)^{n+1} f_n(x)| \leq f_1(x)$ for μ-a.e. x and $k \in \mathbb{N}$, our assertion follows from LDCT.

Exercise 2.6.3 Let $\{f_k\}$ and $\{g_k\}$ be sequences of integrable functions such that $|f_k| \leq g_k$ a.e. on Ω for each $k \in \mathbb{N}$. Suppose that $\{f_k\}$ and $\{g_k\}$ converge a.e. to f and g respectively, and that g is integrable and $\int_\Omega g d\mu = \lim_{k\to\infty} \int_\Omega g_k d\mu$. Show that f is integrable and $\int_\Omega f d\mu = \lim_{k\to\infty} \int_\Omega f_k d\mu$. (Hint: apply the Fatou lemma to the sequences $\{g_k + f_k\}$ and $\{g_k - f_k\}$.)

Exercise 2.6.4 Suppose that $\{f_n\}$ is a sequence of measurable functions on (Ω, Σ, μ). Show that if $\int_\Omega \sum_{n=1}^{\infty} |f_n| d\mu < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges and is finite for a.e. x, $\sum_{n=1}^{\infty} f_n$ is integrable, and

$$\int_\Omega \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_\Omega f_n d\mu.$$

Exercise 2.6.5 A family $\{f_\alpha\}$ of integrable functions on a finite measure space (Ω, Σ, μ) is called uniformly integrable if for any $\varepsilon > 0$, there is $\delta > 0$ such that if $A \subset \Sigma$ with $\mu(A) \leq \delta$, then $\int_A |f_\alpha| d\mu \leq \varepsilon$ for all α. Show that if $\{f_n\}$ is a uniformly integrable sequence of functions on Ω which converges a.e. to an integrable function f on Ω, then

$$\lim_{n\to\infty} \int_\Omega |f_n - f| d\mu = 0.$$

2.7 The space $L^p(\Omega, \Sigma, \mu)$

Associated with a measure space (Ω, Σ, μ) is a family $\{L^p(\Omega, \Sigma, \mu)\}_{p\geq 1}$ of Banach spaces which plays an important role in many fields of mathematics. The introduction and first properties of spaces $L^p(\Omega, \Sigma, \mu)$, $p \geq 1$, are our concern in this section. A more advanced account of these spaces will be given in Chapter 6, when Ω is an open set in $\mathbb{R}^n$.

For a measurable function f, let

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{1/p} \text{ if } 0 < p < +\infty;$$
$$\|f\|_\infty = \inf\{M \geq 0 : |f| \leq M \ a.e.\}.$$

$\|f\|_p$ is called the L^p-**norm** of f; $\|f\|_\infty$ is also called the **essential sup-norm** of f.

Exercise 2.7.1 Show that $|f| \le \|f\|_\infty$ a.e.

Recall that if $p, q \ge 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then they are called conjugate exponents.

Theorem 2.7.1 (Hölder's inequality) *If $p, q \ge 1$ are conjugate exponents, then*

$$\int_\Omega |fg| d\mu = \|fg\|_1 \le \|f\|_p \|g\|_q$$

for any measurable functions f and g.

Proof We may assume that $0 < \|f\|_p, \|g\|_q < +\infty$, hence $|f|$, $|g| < \infty$ a.e. We may further assume that $1 < p, q < \infty$. Now let $\zeta = \left(\frac{|f|}{\|f\|_p}\right)^p$, $\eta = \left(\frac{|g|}{\|g\|_q}\right)^q$, $\alpha = \frac{1}{p}$, and $\beta = \frac{1}{q}$ in Lemma 1.6.1; we have

$$\frac{|f|\,|g|}{\|f\|_p\|g\|_q} \le \frac{1}{p}\frac{|f|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g|^q}{\|g\|_q^q}$$

a.e. on Ω, from which on integrating both sides we complete the proof. ■

Exercise 2.7.2 Suppose that $1 < p, q < \infty$ are conjugate exponents and $\|f\|_p, \|g\|_q$ are both finite. Show that $\|fg\|_1 = \|f\|_p\|g\|_q$ if and only if either $\|f\|_p\|q\|_q = 0$ or $|g|^q = \lambda|f|^p$ a.e. for some $\lambda > 0$. (Hint: use Exercise 1.6.1.)

The following example is a variation of Hölder's inequality.

Example 2.7.1 Let p, q, and r be positive numbers satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and suppose that f and g are measurable functions. Since $1 = \frac{r}{p} + \frac{r}{q}$, $\frac{p}{r}$ and $\frac{q}{r}$ are conjugate exponents; then, $\int_\Omega |fg|^r d\mu = \int_\Omega (|f|^p)^{\frac{r}{p}} (|g|^q)^{\frac{r}{q}} d\mu \le (\int_\Omega |f|^p d\mu)^{\frac{r}{p}} (\int_\Omega |g|^q d\mu)^{\frac{r}{q}}$, by Hölder's inequality. Hence, $\|fg\|_r \le \|f\|_p\|g\|_q$. When $r = 1$, this is Hölder's inequality.

Theorem 2.7.2 (Minkowski's inequality) *Let f, g be measurable, $1 \le p \le +\infty$, then*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p$$

whenever $f + g$ is meaningful a.e. on Ω.

Proof This is obvious when $p = 1$ or $+\infty$. We now consider the case $1 < p < +\infty$, then

$$\begin{aligned}
\|f+g\|_p^p &= \int_\Omega |f+g|^p d\mu = \int_\Omega |f+g|^{p-1}|f+g| d\mu \\
&\le \int_\Omega |f+g|^{p-1}|f| d\mu + \int_\Omega |f+g|^{p-1}|g| d\mu \\
&\le \left[\int_\Omega |f+g|^{(p-1)q} d\mu\right]^{1/q} \{\|f\|_p + \|g\|_p\} \\
&= \|f+g\|_p^{p/q}\{\|f\|_p + \|g\|_p\},
\end{aligned}$$

by Hölder's inequality, where $\frac{1}{p} + \frac{1}{q} = 1$. The theorem follows by dividing extreme ends of the above sequence of inequalities by $\|f+g\|_p^{p-1}$, because we may assume that $0 < \|f+g\|_p < \infty$. ■

Exercise 2.7.3 Verify the last statement of the proof of Theorem 2.7.2. (Hint: show that if $\|f\|_p + \|g\|_p < +\infty$, then $\|f+g\|_p < +\infty$ by using Exercise 1.6.3.)

Exercise 2.7.4 Suppose $1 < p < \infty$ and both $\|f\|_p$ and $\|g\|_p$ are finite. Show that

$$\|f+g\|_p = \|f\|_p + \|g\|_p$$

if and only if either $\|f\|_p\|g\|_p = 0$ or $g = \lambda f$ a.e. for some $\lambda > 0$.

Let now $\mathcal{L}^p(\Omega, \Sigma, \mu)$ be the family of all measurable functions f with $\|f\|_p < +\infty$. From the Minkowski inequality, it is readily seen that $\mathcal{L}^p(\Omega, \Sigma, \mu)$ is a real vector space. If we let

$$\mathcal{N} = \{f \in \mathcal{L}^p(\Omega, \Sigma, \mu) : \|f\|_p = 0\},$$

then $f \in \mathcal{N}$ if and only if $f = 0$ a.e. on Ω. Now consider the space $L^p(\Omega, \Sigma, \mu) = \mathcal{L}^p(\Omega, \Sigma, \mu)/\mathcal{N}$; then $L^p(\Omega, \Sigma, \mu)$ is a vector space which consists of equivalence classes of $\mathcal{L}^p(\Omega, \Sigma, \mu)$ w.r.t. the equivalence relation $\sim$, defined by $f \sim g$ if and only if $f = g$ a.e. on Ω.

We shall allow ourselves the liberty of not distinguishing between a class of functions in $L^p(\Omega, \Sigma, \mu)$ and a function representing the class; hence, by $f \in L^p(\Omega, \Sigma, \mu)$ we shall mean that f is to be considered as a class of equivalent functions in $L^p(\Omega, \Sigma, \mu)$ as well as any function from that class.

For $f \in L^p(\Omega, \Sigma, \mu)$, let

$$\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{1/p} \text{ if } 1 \le p < +\infty,$$

and

$$\|f\|_\infty = \text{ essential sup-norm of } f.$$

Remember that in the definition above, f on the left-hand side is a class of function and f on the right-hand side is a function representing that class. We note that the above definition is well defined. $\|f\|_p$ is called the L^p-norm of f in $L^p(\Omega, \Sigma, \mu)$.

$L^p(\Omega, \Sigma, \mu)$ is called the L^p space of the measure space (Ω, Σ, μ), and is often more compactly denoted by $L^p(\Omega)$ or $L^p(\mu)$ when Σ and μ are assumed to be known, or when Ω and Σ are assumed to be known.

Example 2.7.2 One notes readily that the space $\ell^p(S)$ introduced in the remark at the end of Section 1.6 is the L^p space of the measure space with counting measure on S. It is easily verified that if S is infinite, then $\ell^p(S) \subsetneq \ell^q(S)$ if $1 \leq p < q$.

Exercise 2.7.5 Suppose that the measure space (Ω, Σ, μ) is finite and $f \in L^\infty(\Omega, \Sigma, \mu)$.

(i) Show that $(\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu)^{1/p} \leq (\frac{1}{\mu(\Omega)} \int_\Omega |f|^{p'} d\mu)^{1/p'}$, if $1 \leq p \leq p' < \infty$.

(ii) Show that $\lim_{p\to\infty} \left(\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu\right)^{1/p} = \|f\|_\infty$.

Exercise 2.7.6 Suppose that $\{f_k\}$ is a sequence in $L^p(\Omega, \Sigma, \mu)$ and that $\{f_k\}$ converges a.e. to $f \in L^p(\Omega, \Sigma, \mu)$ with $\|f\|_p = \lim_{k\to\infty} \|f_k\|_p$ $(1 \leq p < \infty)$. Show that $\{f_k\}$ converges in $L^p(\Omega, \Sigma, \mu)$ to f. (Hint: cf. Exercise 2.6.3 or observe that $2^{p-1}(|f|^p + |f_k|^p) - |f - f_k|^p \geq 0$.)

Theorem 2.7.3 *$L^p(\Omega, \Sigma, \mu)$ with norm $\|\cdot\|_p$ is a Banach space.*

Proof This is obvious when $p = +\infty$, if one notes that when $\{f_n\}$ is a Cauchy sequence in $L^\infty(\Omega, \Sigma, \mu)$, there is a measurable null set N such that $\sup_{\omega\in\Omega\setminus N} |f_n(\omega) - f_m(\omega)| \leq \|f_n - f_m\|_\infty$ for all n, m in $\mathbb{N}$.

Assume now that $1 \leq p < +\infty$ and let $\{f_n\}$ be a Cauchy sequence in $L^p(\Omega, \Sigma, \mu)$. There is an increasing sequence $\{n_k\}_{k=1}^\infty$ of positive integers such that $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}$, $k = 1, 2, \ldots$. Put $g = \sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}|$; monotone convergence theorem and Minkowski inequality imply

$$\begin{aligned}
\|g\|_p^p &= \int_\Omega \left(\sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}|\right)^p d\mu = \int_\Omega \lim_{l\to\infty} \left(\sum_{k=1}^l |f_{n_{k+1}} - f_{n_k}|\right)^p d\mu \\
&= \lim_{l\to\infty} \int_\Omega \left(\sum_{k=1}^l |f_{n_{k+1}} - f_{n_k}|\right)^p d\mu = \lim_{l\to\infty} \left\|\sum_{k=1}^l |f_{n_{k+1}} - f_{n_k}|\right\|_p^p \\
&\leq \lim_{l\to\infty} \left(\sum_{k=1}^l \|f_{n_{k+1}} - f_{n_k}\|_p\right)^p = \left(\sum_{k=1}^\infty \|f_{n_{k+1}} - f_{n_k}\|_p\right)^p \leq 1,
\end{aligned}$$

hence, $g \in L^p(\Omega, \Sigma, \mu)$. Observe that if $g(x) < \infty$, then $\sum_{j=1}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| < \infty$ and for $k > l$, we have

$$|f_{n_k}(x) - f_{n_l}(x)| = \left|\sum_{j=l}^{k-1}(f_{n_{j+1}}(x) - f_{n_j}(x))\right| \leq \sum_{j=l}^{k-1} |f_{n_{j+1}}(x) - f_{n_j}(x)| \to 0$$

as $l \to \infty$. This means that $\{f_{n_k}(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Hence, $f_{n_k} \to f$ a.e. with f being finite a.e. But $|f_{n_k}| \leq |f_{n_1}| + g$, $k = 1, 2, \ldots$, implies that $f \in L^p(\Omega, \Sigma, \mu)$. Now $|f_{n_k} - f|^p \leq (|f| + |f_{n_1}| + g)^p$ a.e.; thus by LDCT we know that $\|f_{n_k} - f\|_p \to 0$ as $k \to \infty$; this fact, together with $\{f_n\}$ being a Cauchy sequence, implies that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Hence $L^p(\Omega, \Sigma, \mu)$ is complete. ■

Exercise 2.7.7 Suppose that $\{f_k\}$ is a sequence in $L^p(\Omega, \Sigma, \mu)$, $1 \leq p < \infty$ such that $|f_k| \leq g$ a.e. for each k for some $g \in L^p(\Omega, \Sigma, \mu)$. Assume that $\lim_{k\to\infty} f_k = f$ a.e. Show that $f \in L^p(\Omega, \Sigma, \mu)$ and $\lim_{k\to\infty} \|f_k - f\|_p = 0$.

Exercise 2.7.8 Let $f \in L^p(\Omega, \Sigma, \mu)$, $1 \leq p < \infty$. Show that for any $\varepsilon > 0$, there is a bounded function g in $L^P(\Omega, \Sigma, \mu)$ such that $\|f - g\|_p < \varepsilon$. (Hint: choose g as a truncated function of f, i.e., for some $M > 0$, $g(x) = f(x)$ if $|f(x)| \leq M$, and $g(x) = 0$ otherwise.)

Exercise 2.7.9 Suppose that $\{f_k\}$ is a sequence in $L^p(\Omega, \Sigma, \mu)$ with $\sum_{k=1}^{\infty} \|f_k\|_p < \infty$. Show that $\sum_{k=1}^{\infty} f_k$ converges and is finite a.e. on Ω and is in $L^p(\Omega, \Sigma, \mu)$ with $\|\sum_{k=1}^{\infty} f_k\|_p \leq \sum_{k=1}^{\infty} \|f_k\|_p$.

Exercise 2.7.10 Suppose that $\{f_n\}$ is a convergent sequence in $L^p(\Omega, \Sigma, \mu)$, $p \geq 1$. Show that $\{f_n\}$ has a subsequence which converges a.e. on Ω. (Hint: there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\sum_{k=2}^{\infty} \|f_{n_k} - f_{n_{k-1}}\|_p < \infty$.)

Exercise 2.7.11 If $\mu(\Omega) < \infty$, show that $L^q(\Omega, \Sigma, \mu) \subset L^p(\Omega, \Sigma, \mu)$ for $1 \leq p < q$. Show also that for $f \in L^p(\Omega, \Sigma, \mu)$, $\|f\|_p \leq \|f\|_q \mu(\Omega)^{\frac{1}{p}-\frac{1}{q}}$ for $q \geq p$.

Exercise 2.7.12 Suppose that $1 \leq p < r$. Show that for any q strictly between p and r, $L^q(\Omega, \Sigma, \mu) \subset L^p(\Omega, \Sigma, \mu) + L^r(\Omega, \Sigma, \mu)$.

2.8 Miscellaneous remarks

Some remarks complementing discussions presented so far in this chapter are now in order.

2.8.1 Restriction of measure spaces

If Σ is a σ-algebra on Ω and $A \in \Sigma$, then the family $\Sigma|A := \{B \cap A : B \in \Sigma\}$ is a σ-algebra on A, called the **restriction** of Σ to A. If, further, (Ω, Σ, μ) is a measure space, the measure space $(A, \Sigma|A, \mu)$ is called the restriction to A of the original one. Since

$\Sigma|A \subset \Sigma$, μ is defined on $\Sigma|A$, and hence $(A, \Sigma|A, \mu)$ is indeed a measure space with μ being understood to be restricted to $\Sigma|A$. Suppose now f is a Σ-measurable function on Ω, $f|_A$ is then clearly a $\Sigma|A$-measurable function on A, and if $\int_\Omega f d\mu$ exists, so does $\int_A f|_A d\mu$, and $\int_A f|_A d\mu$ is obviously the same as $\int_A f d\mu := \int_\Omega f I_A d\mu$ (cf. Exercise 2.5.7). But it might happen that $\int_A f|_A d\mu$ exists without $\int_\Omega f d\mu$ being defined, suggesting that it is convenient sometimes to consider $(A, \Sigma|A, \mu)$ instead of (Ω, Σ, μ); when this happens, it will be clear from the context and one does not revert to the formal procedure described previously.

2.8.2 Measurable maps

Suppose (Ω, Σ) and $(\widehat{\Omega}, \widehat{\Sigma})$ are measurable spaces. We say that a map T from Ω into $\widehat{\Omega}$ is **measurable** (more precisely, $\mathbf{\Sigma|\widehat{\Sigma}}$**-measurable**) if $T^{-1}A \in \Sigma$ for every $A \in \widehat{\Sigma}$. In particular, if $\widehat{\Omega} = \overline{\mathbb{R}}$ and $\widehat{\Sigma} = \overline{\mathcal{B}}$, then T is what we call a measurable function on Ω. If (Ω, Σ, μ) and $(\widehat{\Omega}, \widehat{\Sigma}, \hat{\mu})$ are measure spaces, then a measurable map T from Ω into $\widehat{\Omega}$ is **measure preserving** if $\mu(T^{-1}A) = \hat{\mu}(A)$ for every $A \in \widehat{\Sigma}$. Now, if f is a measurable function on $\widehat{\Omega}$ and T is a measure-preserving map from Ω into $\widehat{\Omega}$, then $f \circ T$ is measurable on Ω; furthermore, $\int_{\widehat{\Omega}} f d\hat{\mu}$ exists if and only if $\int_\Omega f \circ T d\mu$ exists, and

$$\int_{\widehat{\Omega}} f d\hat{\mu} = \int_\Omega f \circ T d\mu$$

if either side exists. This is easily verified, if f is nonnegative; in the general case, one needs only to note that $f \circ T = f^+ \circ T - f^- \circ T$.

We note at this point that if the measurable space structure of (Ω, Σ) and $(\widehat{\Omega}, \widehat{\Sigma})$ is to be emphasized, a map $T : \Omega \to \widehat{\Omega}$ will also be called, by abuse of language, a map from (Ω, Σ) to $(\widehat{\Omega}, \widehat{\Sigma})$, and a measurable map from (Ω, Σ) to $(\widehat{\Omega}, \widehat{\Sigma})$ means a $\Sigma|\widehat{\Sigma}$-measurable map from Ω to $\widehat{\Omega}$.

It is readily verified that if (Ω_i, Σ_i) is a measurable space for $i = 1, 2, 3$ and T_i is a measurable map from (Ω_i, Σ_i) to $(\Omega_{i+1}, \Sigma_{i+1})$ for $i = 1, 2$, then $T_2 \circ T_1$ is a measurable map from (Ω_1, Σ_1) to (Ω_3, Σ_3); in particular, if f is a measurable function on (Ω, Σ) and g a Borel function on $\overline{\mathbb{R}}$, then $g \circ f$ is a measurable function on (Ω, Σ). In words, this means that a Borel function of a measurable function is measurable; however, we shall see in Example 4.7.2 that a measurable function of a continuous function may not be measurable.

2.8.3 Complete measure spaces

A measure space (Ω, Σ, μ) is **complete** if every null set is in Σ. One can construct a complete measure space $(\Omega, \overline{\Sigma}, \bar{\mu})$ from a measure space (Ω, Σ, μ) in the following way. Let $\overline{\Sigma} = \{B \subset \Omega : \exists C, D \text{ in } \Sigma \text{ such that } C \subset B \subset D \text{ and } \mu(D \backslash C) = 0\}$. It is clear that $\overline{\Sigma}$ is a σ-algebra on Ω. Now define a set function $\bar{\mu}$ on $\overline{\Sigma}$ by

$$\bar{\mu}(B) = \mu(C), \tag{2.4}$$

if $C \subset B \subset D$, where C and D are in Σ with $\mu(D\backslash C) = 0$. We claim that (2.4) is well defined; this amounts to showing that if $\widehat{C}$, $\widehat{D}$ are in Σ such that $\widehat{C} \subset B \subset \widehat{D}$ and $\mu(\widehat{D}\backslash\widehat{C}) = 0$, then $\mu(\widehat{C}) = \mu(C)$. Now from $C \cup \widehat{C} \subset B \subset D$ and $\mu(D\backslash[C \cup \widehat{C}]) \leq \mu(D\backslash C) = 0$, we infer that $\mu(C \cup \widehat{C}) = \mu(D) = \mu(C)$. Similarly, $\mu(C \cup \widehat{C}) = \mu(\widehat{C})$; hence $\mu(\widehat{C}) = \mu(C)$ as claimed. $\bar{\mu}$ is obviously a measure on $\overline{\Sigma}$. Suppose $B \in \overline{\Sigma}$ with $\bar{\mu}(B) = 0$ and consider $S \subset B$. There are C and D in Σ such that $C \subset B \subset D$, $\mu(D\backslash C) = 0$, and $\mu(C) = 0$. Observe that $\mu(D) = 0$. Since $\emptyset \subset S \subset D$ and $\mu(D\backslash\emptyset) = \mu(D) = 0$, $S \in \overline{\Sigma}$. This means that $(\Omega, \overline{\Sigma}, \bar{\mu})$ is complete. When (Ω, Σ, μ) is complete, one sees readily that $(\Omega, \overline{\Sigma}, \bar{\mu})$ is the same as (Ω, Σ, μ). The measure space $(\Omega, \overline{\Sigma}, \bar{\mu})$ is called the **completion** of (Ω, Σ, μ). Clearly, $\Sigma \subset \overline{\Sigma}$ and $\bar{\mu}$ is an extension of μ.

Exercise 2.8.1 Show that if f is $\overline{\Sigma}$-measurable, then there is a Σ-measurable function $\hat{f}$ such that $f = \hat{f}$ $\bar{\mu}$-a.e. and that f is $\bar{\mu}$-integrable if and only if $\hat{f}$ is μ-integrable.

2.8.4 Integral of complex-valued functions

So far only real-valued functions are considered in regard to measurability and integration; now a brief account will be given for complex-valued functions.

A complex-valued function f defined on a set Ω can be expressed as

$$f = f_1 + if_2,$$

where f_1 and f_2 are finite real-valued functions defined by

$$\begin{aligned} f_1(\omega) &= \text{real part of } f(\omega); \\ f_2(\omega) &= \text{imaginary part of } f(\omega), \end{aligned}$$

for $\omega \in \Omega$. Usually f_1 and f_2 are denoted respectively by $\mathrm{Re} f$ and $\mathrm{Im} f$. If now (Ω, Σ, μ) is a measure space, f is said to be measurable (more precisely, Σ-measurable), if both $\mathrm{Re} f$ and $\mathrm{Im} f$ are measurable.

Exercise 2.8.2 Show that a complex-valued function f defined on Ω is measurable if and only if it is $\Sigma|\mathcal{B}(\mathbb{C})$-measurable; where $\mathcal{B}(\mathbb{C})$ is the σ-algebra generated by the family of all open subsets of the complex field $\mathbb{C}$.

If both $\mathrm{Re} f$ and $\mathrm{Im} f$ are integrable, f is said to be integrable and the integral $\int_\Omega f d\mu$ of f is defined as $\int_\Omega \mathrm{Re} f d\mu + i \int_\Omega \mathrm{Im} f d\mu$. Obviously, f is integrable if and only if $|f|$ is integrable, where $|f|$ is the function defined by $|f|(\omega) = |f(\omega)| = \{\mathrm{Re} f(\omega)^2 + \mathrm{Im} f(\omega)^2\}^{\frac{1}{2}}$ for $\omega \in \Omega$. One verifies easily that $|\int_\Omega f d\mu| \leq \int_\Omega |f| d\mu$, if f is integrable, and that if f and g are integrable, then $\alpha f + \beta g$ are integrable and $\int_\Omega (\alpha f + \beta g) d\mu = \alpha \int_\Omega f d\mu + \beta \int_\Omega g d\mu$ for any complex numbers α and β. For a complex-valued measurable function f, its L^p-norm $\|f\|_p$, $p \geq 1$, is defined by

$$\|f\|_p = \||f|\|_p.$$

Then Hölder inequality holds for complex-valued measurable functions, i.e.

$$\int_{\Omega} |fg| d\mu \leq \|f\|_p \cdot \|g\|_q,$$

where p, q are conjugate exponents; in particular,

$$\left| \int_{\Omega} fg d\mu \right| \leq \|f\|_p \cdot \|g\|_q$$

if fg is integrable. What also holds true is the Minkowski inequality,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

as can easily be verified. It is to be noted that since f and g are complex-valued, $f + g$ is defined on Ω.

Now consider the space $\mathcal{L}^p(\Omega, \Sigma, \mu)$ of all complex-valued measurable functions f such that $\|f\|_p < \infty$. It follows from the Minkowski inequality that $\mathcal{L}^p(\Omega, \Sigma, \mu)$ is a complex vector space. As in Section 2.7, if we let $L^p(\Omega, \Sigma, \mu)$ be the quotient space $\mathcal{L}^p(\Omega, \Sigma, \mu)/\mathcal{N}$, where $\mathcal{N}$ is the vector subspace of $\mathcal{L}^p(\Omega, \Sigma, \mu)$ consisting of all those functions which are zero-valued almost everywhere. For $[f] = f + \mathcal{N}, f \in \mathcal{L}^p(\Omega, \Sigma, \mu)$, let $\|[f]\|_p = \|f\|_p$, then $\|[f]\|_p$ is well defined and $L^p(\Omega, \Sigma, \mu)$ is a complex Banach space with this norm. As before, for $f \in \mathcal{L}^p(\Omega, \Sigma, \mu)$, $[f]$ will also be denoted by f, and $\|[f]\|_p$ by $\|f\|_p$; thus f may denote an element either of $\mathcal{L}^p(\Omega, \Sigma, \mu)$ or of $L^p(\Omega, \Sigma, \mu)$ as occasion prompts, and no confusion is possible.

Henceforth, $L^p(\Omega, \Sigma, \mu)$ will denote a real or complex Banach space as the situation suggests.