REAL VARIABLES: PSET 8

1. Problem 7.5

We know that $\int_a^b \phi dg$ exists because g is absolutely continuous and therefore continuous. We know that $\int_a^b \phi df$ exists by Theorem 2.24 because f is of bounded variation. Then using Theorem 2.16 i and 2.16 iii, we know that:

$$\int_a^b \phi df - \int_a^b \phi dg = \int_a^b \phi d(f-g) = \int_a^b \phi dh$$

Since the two integrals on the left exists, the integral on the right exists. Then using Theorems 2.16 iii and Theorem 7.32:

$$\int_a^b \phi df = \int_a^b \phi d(g+h) = \int_a^b \phi dg + \int_a^b \phi dh = \int_a^b \phi g' dx + \int_a^b \phi dh$$

2. Problem 7.6

One just needs to verify that every condition of 7.29 is satisfied.

3. Problem 7.7

Since $|\sum[f(b_i) - f(a_i)]| < \sum|f(b_i) - f(a_i)|$, the definition of an absolutely continuous function immediately leads to the implication $\Rightarrow$. Next, suppose that given $\epsilon > 0$, there exists a $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of [a,b] with $\sum(b_i - a_i) < \delta$. Assume there exists $\epsilon > 0$ such that for any $\delta > 0$, there exists finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of [a,b] with $\sum(b_i - a_i) < \delta$ such that $\sum|f(b_i) - f(a_i)| \geq \epsilon$, then a subcollection can be picked such that $|\sum[f(b_i) - f(a_i)]| \geq \epsilon/2$, a contradiction

4. Problem 7.8

Note the result of Problem 7.7. Since V(x) is bounded and absolutely continuous, $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that for any finite N:

$$\sum_{j=1}^{N}(b_j - a_j) < \delta \implies \sum_{j=1}^{N}|V(a_j) - V(b_j)| = \sum_{j=1}^{N} V(a_j, b_j) < \epsilon$$

$$\implies \sum_{j=1}^{N}\left(\sum_{i=1}^{m}|f(a_{j_i}) - f(b_{j_i})|\right) \leq \sum_{j=1}^{N}\left(\sup_{\Gamma}\sum_{i=1}^{m}|f(a_{j_i}) - f(b_{j_i})|\right) < \epsilon$$

The inner sum is taken over all i's in the partition Γ. Since the outer sum is taken over disjoint intervals, the two sums on the left can be combined into a single sum over the index $k = i_j$. Obviously $\sum_{j=1}^{N}|a_j - b_j| = \sum_{k=1}^{mN}|a_k - b_k|$, so $\sum_{j=1}^{N}|a_j - b_j| < \delta \implies \sum_{k=1}^{mN}|a_k - b_k| < \delta$. So for any finite collection of nonoverlapping subintervals $[a_i, b_i]$ of [a,b], and any $\epsilon > 0$, $\exists\ \delta > 0$ such that:

$$\sum_{i=1}^{N}|f(b_i) - f(a_i)| = \sum_{i=1}^{N}\left(\sum_{j=1}^{m}|f(b_{i_j}) - f(a_{i_j})|\right) \leq \sum_{i=1}^{N}\left(\sup_{\Gamma}\sum_{j=1}^{m}|f(b_{i_j}) - f(a_{i_j})|\right) < \epsilon$$

when $\sum_{i=1}^{N}(b_i - a_i) < \delta$.

5. Problem 7.12

The inequality is obvious if either a or b = 0, so suppose that $a, b > 0$. We know that the natural log and exponential functions are convex. For $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, consider:

$$ab = e^{log(ab)} = e^{\frac{1}{p}log(a^p) + \frac{1}{q}log(b^q)}$$

By the definition of convexity, we can take $\theta = \frac{1}{p}$, $(1 - \theta) = \frac{1}{q}$, $x_1 = log(a^p)$, and $x_2 = log(b^q)$. Since $f(x) = e^x$ is a convex function for all x, we can use $f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2)$ to get:

$$e^{\frac{1}{p}log(a^p) + \frac{1}{q}log(b^q)} \leq \frac{1}{p}e^{log(a^p)} + \frac{1}{q}e^{log(b^q)} \implies \boxed{ab \leq \frac{a^p}{p} + \frac{b^q}{q}}$$

Following the exact same argument, and employing Jensen's Inequality (Theorem 7.35):

$$a_1 \cdot a_2 \cdot \ldots \cdot a_n = e^{log(a_1)+\ldots+log(a_n)} = e^{\sum_{j=1}^{n} \frac{1}{p_j} log(a_j^{p_j})} \leq \frac{\sum_{j=1}^{n} \frac{1}{p_j} e^{log(a_j^{p_j})}}{\sum_{j=1}^{n} \frac{1}{p_j}} = \sum_{j=1}^{n} \frac{a_j^{p_j}}{p_j}$$

6. Problem 7.14

First prove $\Rightarrow$. Assume that ϕ is convex. By Theorem 7.40, ϕ is continuous. The desired inequality follows immediately from the formulation of convexity in 7.33 setting $\theta = \frac{1}{2}$ or in 7.34 by setting $p_1 = p_2 = 1$.

Next prove $\Leftarrow$. Assume that ϕ is continuous and $\phi(\frac{x_1+x_2}{2}) \leq \frac{\phi(x_1)+\phi(x_2)}{2}$ for all $x_1,\ x_2 \in (a,b)$. This means that ϕ satisfies 7.33 for $\theta = \frac{1}{2}$ and any $x_1,\ x_2 \in (a,b)$. It remains to show that ϕ satisfies 7.33 for any $0 \leq \theta \leq 1$. The equality is obviously satisfied for $\theta = 0$ or $\theta = 1$.

Choose some point x_0 and points $y_0 = x_0 - \delta$, $z_0 = x_0 + \delta$. Then $\phi(x_0) \leq \frac{1}{2}(\phi(y_0) - \phi(z_0))$. Define the line segment connecting y_0 and z_0 as L. Then choose $x_1 = \frac{1}{2}(z_0 + x_0)$. x_1 lies below the line segment connecting x_0 and z_0. Since x_0 lies below L, this means that the line segment connecting x_0 and z_0 lies below L. So x_1 is also below L. By the same argument, the point $x_2 = \frac{1}{2}(y_0 + x_0)$ also lies below L. We can iterate this process to show that the midpoint of any similar interval between y_0 and z_0 lies below L. Because x_0 was chosen arbitrarily, this means that for any number $\theta \in A = (0,1) \cap \{m/2^n :\ m,n \in \mathbb{N}\}$, $\phi(\theta x_1 + (1-\theta)x_2) \leq \theta\phi(x_1) + (1-\theta)\phi(x_2)$. The set A is dense in (0,1). Since ϕ and L', the extension of L to a line, are both continuous, and $\phi > L'$ on the set $\{\theta(y_0) + (1-\theta)z_0 : \theta \in A\}$, which is dense in (y_0, z_0), $\phi < L'$ for all $x \in (y_0, z_0)$. So ϕ is convex.

7. Problem 7.15

By Theorem 7.1, since $f \in L(a,b)$, $\phi(x)$ is absolutely continuous, and therefore continuous.

$$\forall\ x_1,\ x_2 \in (a,b) \phi\left(\frac{x_1+x_2}{2}\right) = \int_a^x f(t)dt + \phi(a)$$

$$\frac{\phi(x_1)+\phi(x_2)}{2} = \frac{1}{2}\left[\int_a^{x_1} f(t)dt + \phi(a)\right] + \frac{1}{2}\left[\int_a^{x_2} f(t)dt + \phi(a)\right]$$

$$= \frac{1}{2}\left[2\int_a^{x_1} f(t)dt + \int_{x_1}^{\frac{x_1+x_2}{2}} f(t)dt + \int_{\frac{x_1+x_2}{2}}^{x_2} f(t)dt\right] + \phi(a) = \int_a^{x_1} f(t)dt + \phi(a) + \frac{1}{2}\left[\int_{x_1}^{\frac{x_1+x_2}{2}} f(t)dt + \int_{\frac{x_1+x_2}{2}}^{x_2} f(t)dt\right]$$

$$\geq \int_a^{x_1} f(t)dt + \phi(a) + \frac{1}{2}\left[2\int_{x_1}^{\frac{x_1+x_2}{2}} f(t)dt\right] = \int_a^{\frac{x_1+x_2}{2}} f(t)dt + \phi(a) = \phi\left(\frac{x_1+x_2}{2}\right)$$

So by Exercise 7.14, since ϕ is both continuous and midpoint convex, it is convex.