REAL VARIABLES: PSET 8

1. Problem 7.5

We know that $\int\limits_a^b \phi dg$ exists because g is absolutely continuous and therefore continuous. We know that $\int\limits_a^b \phi df$ exists by Theorem 2.24 because f is of bounded variation. Then using Theorem 2.16 i and 2.16 iii, we know that:

$$\int\limits_a^b \phi df - \int\limits_a^b \phi dg = \int\limits_a^b \phi d(f-g) = \int\limits_a^b \phi dh$$

Since the two integrals on the left exists, the integral on the right exists. Then using Theorems 2.16 iii and Theorem 7.32:

$$\int\limits_a^b \phi df = \int\limits_a^b \phi d(g+h) = \int\limits_a^b \phi dg + \int\limits_a^b \phi dh = \int\limits_a^b \phi g' dx + \int\limits_a^b \phi dh$$

2. Problem 7.6

One just needs to verify that every condition of 7.29 is satisfied.

3. Problem 7.7

Since $|\sum[f(b_i) - f(a_i)]| < \sum|f(b_i) - f(a_i)|$, the definition of an absolutely continuous function immediately leads to the implication $\Rightarrow$. Next, suppose that given $\epsilon > 0$, there exists a $\delta > 0$ such that $|\sum[f(b_i) - f(a_i)]| < \epsilon$ for any finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of [a,b] with $\sum(b_i - a_i) < \delta$. Assume there exists $\epsilon > 0$ such that for any $\delta > 0$, there exists finite collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of [a,b] with $\sum(b_i - a_i) < \delta$ such that $\sum|f(b_i) - f(a_i)| \geq \epsilon$, then a subcollection can be picked such that $|\sum[f(b_i) - f(a_i)]| \geq \epsilon/2$, a contradiction