

# Calculus 2 11/14 Note

## Module Class 07

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### Section 5.1: Areas and Distances

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#### Definition

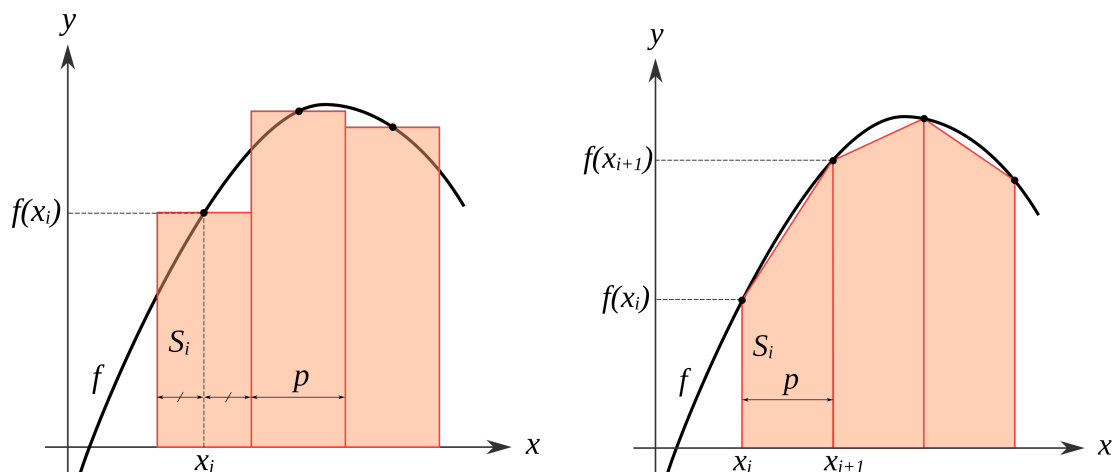
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The **area**  $A$  of the region  $S$  that lies under the graph of the continuous function  $f$  is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} [f(x_1) \Delta x + f(x_2) \Delta x + \cdots + f(x_n) \Delta x]$$


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*Graph:*



*Example:*

Determine a region whose area is equal to the given limit.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\pi}{4n} \tan \frac{i\pi}{4n}$$

*Sol.*

$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\pi}{4n} \tan \frac{i\pi}{4n}$  can be interpreted as the area of the region lying under the graph of  $y = \tan x$  on the interval  $[0, \frac{\pi}{4}]$ , since for  $y = \tan x$  on  $[0, \frac{\pi}{4}]$  with  $\Delta x = \frac{\pi/4 - 0}{n} = \frac{\pi}{4n}$ ,  $x_i = 0 + i\Delta x = \frac{i\pi}{4n}$  and  $x_i^* = x_i$ .

The expression for the area is

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \tan \left( \frac{i\pi}{4n} \right) \frac{\pi}{4n}.$$

## Section 5.2: The Definite Integral

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### Definition of a Definite Integral

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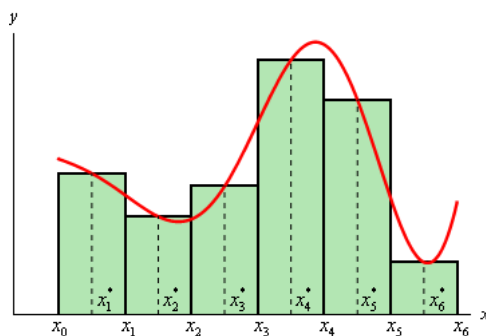
If  $f$  is a function defined for  $a \leq x \leq b$ , we divide the interval  $[a, b]$  into  $n$  sub-intervals of equal width  $\Delta x = \frac{b-a}{n}$ . We let  $x_0 = a, x_1, x_2, \dots, x_n = b$  be the endpoints of these subintervals and we let  $x_1^*, x_2^*, \dots, x_n^*$  be any **sample points** in these subintervals, so  $x_i^*$  lies in the  $i$ -th subinterval  $[x_{i-1}, x_i]$ . Then the **definite Integral of  $f$  from  $a$  to  $b$**  is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that  $f$  is **integrable** on  $[a, b]$ .

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*Graph:*



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### Theorem

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If  $f$  is continuous on  $[a, b]$ , or if  $f$  has only a finite number of jump discontinuities, then  $f$  is integrable on  $[a, b]$ ; that is, the definite integral  $\int_a^b f(x) dx$  exists.

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### Theorem

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If  $f$  is integrable on  $[a, b]$ , then

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

where

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i \Delta x$$

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**Example:**

Express the limit as a definite integral on the given interval.

1.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n x_i \sqrt{1 + x_i^3} \Delta x, \quad [2, 5]$

**Sol.**

$$\int_2^5 x \sqrt{1 + x^3} dx$$

2.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\cos x_i}{x_i} \Delta x, \quad [\pi, 2\pi]$

**Sol.**

$$\int_{\pi}^{2\pi} \frac{\cos x}{x} dx$$

## Midpoint Rule

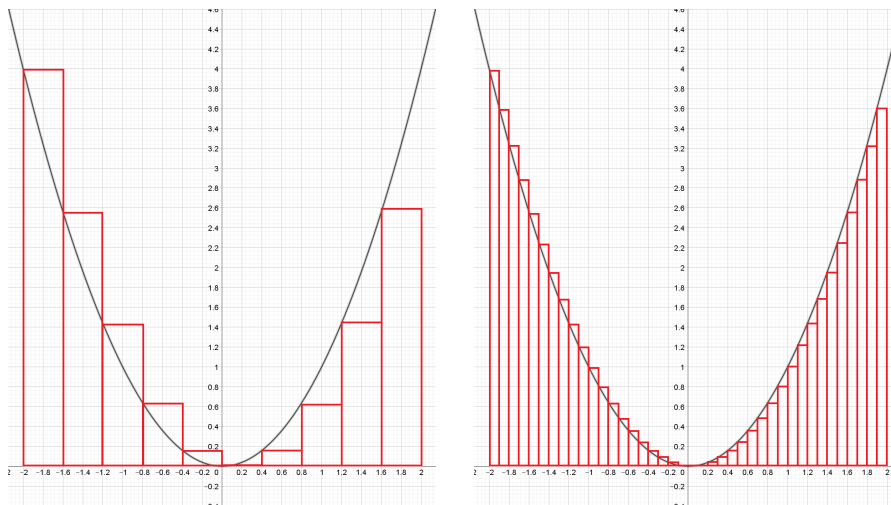
$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(\bar{x}_i) \Delta x = \Delta x [f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where

$$\Delta x = \frac{b - a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

**Graph:**

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## Properties of the Integral

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1.  $\int_a^b c \, dx = c(b - a)$ , where  $c$  is any constant
2.  $\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$
3.  $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$ , where  $c$  is any constant
4.  $\int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$
5.  $\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx$
6. If  $f(x) \geq 0$  for  $a \leq x \leq b$ , then

$$\int_a^b f(x) \, dx \geq 0$$

7. If  $f(x) \geq g(x)$  for  $a \leq x \leq b$ , then

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx$$

8. If  $m \leq f(x) \leq M$  for  $a \leq x \leq b$ , then

$$m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$$

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## Section 5.3: The Fundamental Theorem of Calculus

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### The Fundamental Theorem of Calculus

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Suppose  $f$  is continuous on  $[a, b]$ , then

1. If  $g(x) = \int_a^x f(t) \, dt$ , then  $g'(x) = f(x)$
  2.  $\int_a^b f(x) \, dx = F(b) - F(a)$ , where  $F$  is any antiderivative of  $f$ , that is,  $F' = f$
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**Example:**

Find the derivative of the function.

1.  $g(x) = \int_1^x \ln(1 + t^2) \, dt$

**Sol.**

By Fundamental Theorem of Calculus, then

$$g'(x) = \ln(1 + x^2)$$

2.  $f(x) = \int_x^1 \cos \sqrt{t} dt$

**Sol.**

By Fundamental Theorem of Calculus, since

$$f(x) = \int_x^1 \cos \sqrt{t} dt = - \int_1^x \cos \sqrt{t} dt,$$

then

$$f'(x) = -\cos \sqrt{x}$$

3.  $h(x) = \int_0^{\tan x} \sqrt{t + \sqrt{t}} dt$

**Sol.**

Let  $u = \tan x$ . Then  $\frac{du}{dx} = \sec^2 x$ .

Also  $h'(x) = \frac{h(x)}{dx} = \frac{h(x)}{du} \cdot \frac{du}{dx}$ , so

$$f'(x) = \frac{d}{du} \int_0^u \sqrt{t + \sqrt{t}} dt \cdot \frac{du}{dx} = \sqrt{u + \sqrt{u}} \cdot \sec^2 x = \sqrt{\tan x + \sqrt{\tan x}} \cdot \sec^2 x$$

4.  $F(x) = \int_{1-2x}^{1+2x} t \sin t dt$

**Sol.**

$$\begin{aligned} F(x) &= \int_{1-2x}^{1+2x} t \sin t dt = \int_0^{1+2x} t \sin t dt + \int_{1-2x}^0 t \sin t dt \\ &= \int_0^{1+2x} t \sin t dt - \int_0^{1-2x} t \sin t dt \\ &= [(1+2x) \sin(1+2x)](2) - [(1-2x) \sin(1-2x)](-2) \\ &= (2+4x) \sin(1+2x) + (2-4x) \sin(1-2x) \end{aligned}$$

## Section 5.4: Indefinite Integrals and the Net Change Theorem

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### Formula

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$$\begin{array}{ll}
 \int c f(x) dx = c \int f(x) dx & \int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx \\
 \int k dx = kx + C & \\
 \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) & \int \frac{1}{x} dx = \ln |x| + C \\
 \int e^x dx = e^x + C & \int b^x dx = \frac{b^x}{\ln b} + C \\
 \int \sin x dx = -\cos x + C & \int \cos x dx = \sin x + C \\
 \int \sec^2 x dx = \tan x + C & \int \csc^2 x dx = -\cot x + C \\
 \int \sec x \tan x dx = \sec x + C & \int \csc x \cot x dx = -\csc x + C \\
 \int \frac{1}{x^2 + 1} dx = \tan^{-1} x + C & \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C \\
 \int \sinh x dx = \cosh x + C & \int \cosh x dx = \sinh x + C
 \end{array}$$


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### **Example:**

Evaluate the integral.

1.  $\int_0^1 \frac{4}{1+x^2} dx$

**Sol.**

$$\int_0^1 \frac{4}{1+x^2} dx = [4 \arctan x]_{x=0}^1 = 4 \arctan 1 - 4 \arctan 0 = 4 \cdot \frac{\pi}{4} - 4 \cdot 0 = \pi$$

2.  $\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x} dx$

**Sol.**

$$\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x} dx = \int_{-10}^{10} \frac{2e^x}{\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}} dx = \int_{-10}^{10} 2 dx = [2x]_{x=-10}^{10} = 20$$

### **Exercise:**

Evaluate the integral.

1.  $\int_0^{\pi/3} \frac{\sin \theta + \sin \theta \tan^2 \theta}{\sec^2 \theta} d\theta$

2.  $\int_0^2 |2x - 1| dx$

**Sol.**

1.  $\frac{1}{2}$       2.  $\frac{5}{2}$

## Section 5.5: The Substitution Rule

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### The Substitution Rule

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If  $u = g(x)$  is a differentiable function whose range is an interval  $I$  and  $f$  is continuous on  $I$ , then

$$\int f(g(x))g'(x) dx = \int f(u) du$$

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### The Substitution Rule for Definite Integrals

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If  $g'$  is continuous on  $[a, b]$  and  $f$  is continuous on the range of  $u = g(x)$ , then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

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### Integrals of Symmetric Functions

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Suppose  $f$  is continuous on  $[-a, a]$ .

(a) If  $f$  is even [ $f(-x) = f(x)$ ], then  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

(b) If  $f$  is odd [ $f(-x) = -f(x)$ ], then  $\int_{-a}^a f(x) dx = 0$

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**Example:** Evaluate the integral.

1.  $\int \frac{(\arctan x)^2}{x^2+1} dx$

**Sol.**

Let  $u = \arctan x$ . Then

$$du = \frac{1}{x^2+1} dx,$$

so

$$\int \frac{(\arctan x)^2}{x^2+1} dx = \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} (\arctan x)^3 + C,$$

where  $C$  is the constant.

2.  $\int \frac{\cos(\ln x)}{x} dx$

**Sol.**

Let  $u = \ln x$ . Then

$$du = \frac{1}{x} dx,$$

so

$$\int \frac{\cos(\ln x)}{x} dx = \int \cos u du = \sin u + C = \sin(\ln x) + C,$$

where  $C$  is the constant.

3.  $\int_0^a x\sqrt{a^2 - x^2} dx$

**Sol.**

Assume  $a > 0$ . Let  $u = a^2 - x^2$ , so  $du = -2x dx$ .

When  $x = 0$ ,  $u = a^2$ ; when  $x = a$ ,  $u = 0$ . Thus,

$$\int_0^a x\sqrt{a^2 - x^2} dx = \int_{a^2}^0 u^{1/2} \left(-\frac{1}{2} du\right) = \frac{1}{2} \int_0^{a^2} u^{1/2} du = \frac{a^3}{3}$$

4.  $\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

**Sol.**

Let  $u = \sin^{-1} x$ , so  $du = \frac{dx}{\sqrt{1-x^2}}$ .

When  $x = 0$ ,  $u = 0$ ; when  $x = \frac{1}{2}$ ,  $u = \frac{\pi}{6}$ . Thus,

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\pi/6} u du = \frac{\pi^2}{72}$$

**Exercise:** Evaluate the integral.

1.  $\int x^2\sqrt{2+x} dx$

2.  $\int_0^{T/2} \sin\left(\frac{2\pi x}{T-\alpha}\right) dx$

**Sol.**

1.  $\frac{2}{7}(2+x)^{7/2} - \frac{8}{5}(2+x)^{5/2} + \frac{8}{3}(2+x)^{3/2} + C$ , where  $C$  is the constant.

2.  $\frac{T}{\pi} \cos \alpha$