Real Analysis Homework
Chapter 1. Measure theory
Due Date: 11/4

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November 4, 2019

Exercise 1.27

Suppose E_1 and E_2 are a pair of compact sets in $\mathbb{R}^d$ with $E_1 \subset E_2$, and let $a = m(E_1)$ and $b = m(E_2)$. Prove that for any c with $a < c < b$, there is a compact set E with $E_1 \subset E \subset E_2$ and $m(E) = c$.
[Hint: As an example, if $d = 1$ and E is a measurable subset of $[0, 1]$, consider $m(E \cap [0, t])$ as a function of t.]

Proof.
Since $E_2 \subset E_1$ and E_1, E_2 are compact sets in $\mathbb{R}^d$, $E_2 \setminus E_1$ is a bounded and measurable. For any $t > 0$, the sets $(E_2 \setminus E_1) \cap \overline{B_t(0)}$ are also bounded and measurable.
Let

$$S_t = (E_2 \setminus E_1) \cap \overline{B_t(0)}$$

and define

$$f(t) = m(S_t).$$

Hence, if we can prove f is a continuous function, the proof will be done.

Let $0 \leq \tau < t$. Notice that the function $|f(t) - f(\tau)| = |m(S_t) - m(S_\tau)|$.
Since $S_\tau \subset S_t$,

$$|m(S_t) - m(S_\tau)| = m(S_t) - m(S_\tau) = m(S_t \setminus S_\tau).$$

Now, notice that

$$(S_t \setminus S_\tau) \subset \overline{B_t(0)} \setminus \overline{B_\tau(0)}$$

so

$$m(S_t \setminus S_\tau) \leq m(\overline{B_t(0)} \setminus \overline{B_\tau(0)}) \leq \alpha(d)\,(t^d - \tau^d)$$

where $\alpha(d) = \dfrac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$, the volume of the d-dimensional unit ball.(Exercise 1.6)

Notice that the function $g(t) = \alpha\, t^d$ is continuous, but not uniformly continuous. Thus,

$$|t - \tau| < \delta \Rightarrow \alpha(d)\,(t^d - \tau^d) \leq \varepsilon(t).$$

Therefore, f is a continuous function, and by the Intermediate Value Theorem, given c with $a < c < b$, we can find a $t^* > 0$ such that $m(E) = m(E_1 \cup S_{t^*}) = c$, where $E_1 \cup S_{T^*}$ is compact.

Exercise 1.28

Let E be a subset of $\mathbb{R}$ with $m_*(E) > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$m_*(E \cap I) \geq \alpha\, m_*(I).$$

Loosely speaking, this estimate shows that E contains almost a whole interval.
[Hint: Choose an open set $\mathcal{O}$ that contains E, and such that $m_*(E) \geq \alpha\, m_*(\mathcal{O})$. Write $\mathcal{O}$ as the countable union of disjoint open intervals, and show that one of these intervals must satisfy the desired property.]

Proof.

Let E be a subset of $\mathbb{R}$ with $m_*(E) > 0$ and fix an $\alpha \in (0,1)$. Because E has positive outer measure, we can find a covering of E by closed and almost disjoint interval I_j such that

$$\sum_j |I_j| < m_*(E) + \frac{\varepsilon}{2}.$$

We can expand each of these I_j to an open cube I'_j such that

$$m_*(I'_j - Q_j) < \frac{\varepsilon}{2^{k+1}}$$

and set $\mathcal{O} = \cup_j Q'_j$. So $\mathcal{O}$ is an open set containing E and so we can write

$$E = E \cap \mathcal{O} = \cup_j\, E \cap I'_j$$

By monotonicity we can see that $m_*(E) \leq \sum_j m_*(E \cap I'_j)$.

Now, suppose towards a contradiction that for every $j \in \mathbb{Z}^+$, we have that $m_*(E \cap I'_j) < \alpha\, m_*(I'_j)$. Then

$$m_*(E) \leq \sum_j m_*(E \cap I'_j) < \alpha \sum_j m_*(I'_j) < \alpha(m_*(E) + \varepsilon)$$

But, if we take

$$\varepsilon < \frac{1-\alpha}{\alpha}\, m_*(E)$$

Then we would get that $m_*(E) < m_*(E)$, which is impossible. Hence, we must be able to find some j such that

$$m_*(E \cap I'_j) \geq \alpha\, m_*(I)$$

Exercise 1.29

Suppose E is a measurable subset of $\mathbb{R}$ with $m(E) > 0$. Prove that the **difference set** of E, which is defined by

$$\{z \in \mathbb{R} : z = x - y \text{ for some } x,\, y \in E\},$$

contains an open interval centered at the origin.
If E contains an interval, then the conclusion is straightforward. In general, one may rely on Exercise 1.28.
A more general formulation of this result is as follows.

Proof.

It is enough to prove the claim for a measurable subset of E with positive measure, so we do some reductions by finding some nice subsets of E like this and replacing E with these subsets.

First, since the collection $\{E \cap (n,\, n+1] \,:\, n \in \mathbb{Z}\}$ of disjoint measurable sets cover E.
By countable additivity, there exists $n \in \mathbb{N}$ such that $m(E \cap (n,\, n+1])) > 0$.
So we may assume that E has finite measure.

Second, since $0 < m(E) < \infty$, by **Theorem 3.4 (iii)** of the textbook, there exists a compact set K contained in E such that choose $\varepsilon = \frac{m(E)}{2}$ then

$$m(E \setminus K) \leq \frac{m(E)}{2}.$$

Then by addivity of the measure we get $m(K) \geq \frac{m(E)}{2} > 0$.
So we may assume that E is compact.

Third, since $0 < m(E) < \infty$, by **Theorem 3.4 (iii)** of the textbook again, there exists an open set U containing E such that

$$m(U \setminus E) \leq \frac{m(E)}{2}.$$

Hence, $m(U) \leq \frac{3}{2}\, m(E) < 2m(E)$.

Now E and U^c are disjoint sets where E is compact and U^c is closed. Therefore $\delta := d(E,\, U^c) > 0$.
We claim that $(-\delta,\, \delta)$ lies in the difference set of E.
So let $t \in (-\delta,\, \delta)$. Then by the definition of δ, the set

$$E + t = \{x + t \,:\, x \in E\}$$

does not intersect U^c, therefore $E + t \subseteq U$ and hence $(E + t) \cup E \subseteq U$.

Note that $E + t$ is a measurable set with $m(E + t) = m(E)$.
Suppose $(E + t) \cap E = \emptyset$. Then by additivity, we get

$$m(U) \geq m((E + t) \cup E) = 2m(E)$$

which is contradiction.

Thus, there exists $x,\, y \in E$ such that$x + t = y$, so $t \in (-\delta,\, \delta)$ lies in the difference set of E.

Exercise 1.31

The result in Exercise 1.29 provides an alternate proof of the non-measurability of the set $\mathcal{N}$ studied in the text. In fact, we may also prove the non-measurability of a set in $\mathbb{R}$ that is very closely related to the set $\mathcal{N}$.
Given two real numbers x and y, we shall write as before that $x \sim y$ whenever the difference $x - y$ is rational. Let $\mathcal{N}^*$ denote a set that consists of one element in each equivalence class of $\sim$. Prove that $\mathcal{N}^*$ is non-measurable by using the result in Exercise 1.29.
[Hint: If $\mathcal{N}^*$ is measurable, then so are its translates $\mathcal{N}^*_n = \mathcal{N}^* + r_n$, where $\{r_n\}_{n=1}^{\infty}$ is an enumeration of $\mathbb{Q}$. How does this imply that $m(\mathcal{N}^*) > 0$? Can the difference set of $\mathcal{N}^*$ contain an open interval centered at the origin?]

Proof.

Exercise 1.32

Let $\mathcal{N}$ denote the non-measurable subset of $I = [0, 1]$ constructed at the end of Section 1.3.

(a) Prove that if E is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.

(b) If G is a subset of $\mathbb{R}$ with $m_*(G) > 0$, prove that a subset of G is nonmeasurable.

[Hint: For (a) use the translates of E by the rationals.]

Proof.

(a) Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of the rationals in the interval $[-1, 1]$ and let $E_k = E + r_k$ for each k.

Since $E \subseteq \mathcal{N}$, $E_k \subseteq N_k$. Since each of the $\mathcal{N}_k$ are pairwise disjoint, each of the E_k are pairwise disjoint.

Now, the Lebesgue measure is translation invariant, so $m(E_k) = m(E)$ for each k. We also have that $\cup_{k=1}^{\infty} E_k \subseteq \cup_{k=1}^{\infty} \mathcal{N}_k \subseteq [-1, 2]$. It follows that

$$\sum_{k=1}^{\infty} m(E_k) = m(\cup_{k=1}^{\infty} E_k) \leq 3 \qquad (\text{since } \cup_{k=1}^{\infty} E_k \subseteq [-1,\, 2])$$

But $m(E_k) = m(E)$ for each k. Hence

$$3 \geq \sum_{k=1}^{\infty} m(E_k) = \sum_{k=1}^{\infty} m(E)$$

which implies that $m(E) = 0$.

(b) Since $m(G) > 0$, we can find for any $\varepsilon > 0$ a closed interval $[a,\, b] \subseteq G$ with $m(G \setminus [a,\, b]) \leq \varepsilon$.

Now, consider the set $G - a$ (G translated by $-a$ units). Since the Lebesgue measure is translation invariant, $m(G - a) = m(G)$.

Furthermore, the interval $[0,\, b - a] \subseteq G - a$. Let $A = [0,\, b - a] \cap \mathcal{N}$.
Observe that $A \subseteq G$. Suppose A is measurable A is measurable. Since $A \subseteq \mathcal{N}$, $m(A) = 0$ by part (a). It follows that

$$m(G) = m(G - a) = m((G - a) \setminus A) + m(A) \leq \varepsilon + 0 = \varepsilon$$

Since ε can be chosen arbitrarily small, we conclude that $m(G) = 0$, which is a contradiction. Hence, it must be that A is non-measurable.

Exercise 1.33

Let $\mathcal{N}$ denote the non-measurable set constructed in the text. Recall from the exercise above that measurable subsets of $\mathcal{N}$ have measure zero.
Show that the set $\mathcal{N}^c = I - \mathcal{N}$ satisfies $m_*(\mathcal{N}^c) = 1$, and conclude that if $E_1 = \mathcal{N}$ and $E_2 = \mathcal{N}^c$, then

$$m_*(E_1) + m_*(E_2) \neq m_*(E_1 \cup E_2),$$

although E_1 and E_2 are disjoint.
[Hint: To prove that $m_*(\mathcal{N}^c) = 1$, argue by contradiction and pick a measurable set U such that $U \subset I$, $\mathcal{N}^c \subset U$ and $m_*(U) < 1 - \varepsilon$.]

Proof.

(i) Suppose $m_*(\mathcal{N}^c) < 1$, so there exists $\varepsilon > 0$ such that $m_*(\mathcal{N}^c) < 1 - \varepsilon$. Since

$$m_*(\mathcal{N}^c) = \inf \{m(U) \,:\, \mathcal{N}^c \subseteq U - \text{open}\}$$

there exists an open, hence measurable set U containing $\mathcal{N}^c$ such that $m(U) < 1 - \varepsilon$.

Note that $U \cap I$ is also a measurable containing $\mathcal{N}^c$ with $m(U \cap I) < 1 - \varepsilon$, so we may assume $U \subseteq I$.

Since $\mathcal{N}^c = I - \mathcal{N} \subseteq U \subseteq I$, we have $I - U \subset \mathcal{N}$. But $I - U$ is a measurable set so by additivity of the measure, we have

$$m(I - U) = m(I) - m(U) = 1 - m(U) > \varepsilon.$$

So $I - U$ is a measurable subset of $\mathcal{N}$ with positive measure; a contradiction.

Therefore, $m_*(\mathcal{N}^c) \geq 1$, but on the other hand, $m_*(\mathcal{N}^c) \leq m_*(I) = 1$, hence, $m_*(\mathcal{N}^c) = 1$.

(ii) We know that sets with outer measure zero are measurable, so $m_*(\mathcal{N}) > 0$.
Thus, by part (i), we have

$$m_*(\mathcal{N}) + m_*(\mathcal{N}^c) > 1 = m_*(I) = m_*(\mathcal{N} \cup \mathcal{N}^c).$$

Exercise 1.34

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be any two Cantor sets (constructed in Exercise 1.3). Show that there exists a function $F \,:\, [0,1] \to [0,1]$ with the following properties:

(i) F is continuous and bijective,

(ii) F is monotonically increasing,

(iii) F maps $\mathcal{C}_1$ surjectively onto $\mathcal{C}_2$.

[Hint : Copy the construction of the standard Cantor-Lebesgue function.]

Proof.

Let $\mathcal{C}$ be a Cantor set of constant dissection as in **Exercise 1.3**. By construction, $\mathcal{C}$ is the intersection of a family $\{C_n\}_{n\in\mathbb{N}}$ of closed sets where each $\mathcal{C}_n$ is disjoint union of 2^n closed intervals. So we can label these 2^n intervals from left to right by bit strings of length n, that is, words of length n consisting of 0's and 1's.

For example, $\mathcal{C}_1 = I_0 \cup I_1$ where I_0 is the interval on the left hand side in $\mathcal{C}_1$ and I_1 is the one on the right. Keeping the labeling in a lexicographic order, we have $\mathcal{C}_2 = I_{00} \cup I_{01} \cup I_{10} \cup I_{11}$ and in general $\mathcal{C}_n$ is the union of I_b's where $\mathbf{b}$'s vary over length n bit strings.

Note that $I_b \subseteq I_c$ if and only if $\mathbf{c}$ can be truncated from the right to obtain $\mathbf{b}$. For example, $I_0 \supseteq I_{01} \supseteq I_{010} \supseteq I_{0100}$. In general given an infinite sequence $\mathbf{a} = (a_n)$ of 0's and 1's, if we write $\mathbf{a}|_n$ for its n-truncation $(a_1, \ldots, a_n)$, there is a decreasing sequence

$$I_{\mathbf{a}|_1} \supseteq I_{\mathbf{a}|_2} \supseteq I_{\mathbf{a}|_3} \cdots$$

By compactness, the intersection

$$\cap_{n\in\mathbb{N}} I_{\mathbf{a}|_n}$$

which lies in $\mathcal{C}$, is nonempty.

Yet the diameter of the intersection is zero, hence it must be a singleton. Therefore every infinite sequence $\mathbf{a}$ of 0s and 1s uniquely determines a point in $\mathcal{C}$. So we get a map

$$f : \{0, 1\}^{\mathbb{N}} \to \mathcal{C}$$

which is surjective since points in $\mathcal{C}$ by definition survives the intersection of $\mathcal{C}_n$s, hence, lie in infinitely many (hence in an infinite decreasing chain of) $I_{\mathbf{b}}$s.

If two infinite sequences are distinct, they have different truncations so as the intervals get finer, the two points these sequences determine will fall into different intervals. Hence f is a bijection.

Two points in $\mathcal{C}$ lie in the same $I_{\mathbf{b}}$ where $\mathbf{b}$ is a finite bit string if and only if their inverse images under f both start with $\mathbf{b}$. It follows from this observation (as we did for the middle thirds Cantor set in **Exercise 1.2**) that f is continuous. And since f goes from a compact space to a Hausdorff space, f is a homeomorphism.

Also, observe that if we order $\{0, 1\}^{\mathbb{N}}$ by lexicographic ordering, then f preserves the order. Because if a sequence beats another sequence lexicographically, then at some point it will lie to the right side of a dissection while the other lies on the left side.

So if $\mathcal{C}_1$ and $\mathcal{C}_2$ are Cantor sets, we have order preserving homeomorphisms $f_1 : \{0, 1\}^{\mathbb{N}} \to \mathcal{C}_1$ and $f_2 : \{0, 1\}^{\mathbb{N}} \to \mathcal{C}_2$; thus, $f_2 \circ f_1^{-1}$ gives an order preserving homeomorphism from $\mathcal{C}_1$ to $\mathcal{C}_2$.

Exercise 1.35

Give an example of a measurable function f and a continuous function Φ so that $f \circ \Phi$ is non-measurable.

[Hint: Let $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ as Exercise 1.34, with $m(\mathcal{C}_1) > 0$ and $m(\mathcal{C}_2) = 0$. Let $N \subset \mathcal{C}_1$ be non-measurable, and take $f = \chi_{\Phi(N)}$.]
Use the construction in the hint to show that there exists a Lebesgue measurable set that is not a Borel set.

Proof.
Follow the hint, let $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ as Exercise 1.34, with $m(\mathcal{C}_1) > 0$ and $m(\mathcal{C}_2) = 0$.
Let $N \subset \mathcal{C}_1$ be non-measurable, and take $f = \chi_{\Phi(N)}$. We know such N exists by Exercise 1.32(b).

Since $\Phi(N) \subseteq \mathcal{C}_2$ and $m(\mathcal{C}_2) = 0$, we have $m_*(\Phi(N)) = 0$ and so $\Phi(N)$ is a measurable set.

Therefore, f is a measurable function. However,

$$(f \circ \Phi)^{-1}(1) = \Phi^{-1}(f^{-1}(\{1\})) = \Phi^{-1}(\Phi(N)) = N$$

is not measurable, hence $f \circ \Phi$ is not a measurable function.

Also, the measurable set $\Phi(N)$ cannot be Borel because the inverse images of Borel sets under continuous functions are Borel, but although Φ is continuous, $\Phi^{-1}(\Phi(N)) = N$ is not even measurable.

Exercise 1.37

Suppose Γ is a curve $y = f(x)$ in $\mathbb{R}^2$, where f is continuous. Show that $m(\Gamma) = 0$.
[Hint: Cover Γ by rectangles, using the uniform continuity of f.]

Proof.
Note that since the map $x \mapsto -x$ preserves areas of rectangles, Γ has the same measure with the curve given by $y = |f(x)|$. Therefore we may assume f is nonnegative. Also since

$$\Gamma = \cup_{n\in\mathbb{N}} \{(x,\, f(x)) \,:\, x \in [n,\, n+1]\}$$

and measure is countably sub-additive, it suffices to show that each term in the above union has measure zero.

Thus, we may assume that $f : [a,\, b] \to \mathbb{R}$ where $[a,\, b] \subseteq \mathbb{R}$ is a finite interval. Moverover, by replacing f with $f+1$, we may assume that $f(x) \geq 1$ for every $x \in [a,\, b]$. Then given $0 < \varepsilon < 1$, the set

$$E_\varepsilon = \{(x,\, y) \,:\, a \leq x \leq b,\, f(x) - \varepsilon \leq y \leq f(x) + \varepsilon\}$$

contains Γ.

But since $f \geq 1 > \varepsilon$, both $f + \varepsilon$ and $f - \varepsilon$ are nonnegative and continuous, therefore, the measure of E_ε can be calculated by a definite Riemann integral as

$$m(E_\varepsilon) = \int_a^b (f(x) + \varepsilon)\, dx - \int_a^b (f(x) - \varepsilon)\, dx = \int_a^b 2\varepsilon\, dx = 2\varepsilon(b-a).$$

So $m(\Gamma) \leq 2\varepsilon(b-a)$ for arbitrarily small ε. As a, b is independent from ε, this shows that $m(\Gamma) = 0$.

Exercise 1.38

Prove that $(a+b)^\gamma \geq a^\gamma + b^\gamma$ whenever $\gamma \geq 1$ and $a, b \geq 0$. Also, show that the reverse inequality holds when $0 \leq \gamma \leq 1$.
[Hint: Integrate the inequality between $(a+t)^{\gamma-1}$ and $t^{\gamma-1}$ from 0 to b.]

Proof.

(i) For all a, b, $t \geq 0$ and $\gamma \geq 1$, we have

$$\begin{aligned}
(a+t)^{\gamma-1} \geq t^{\gamma-1} \ &\Rightarrow \int_0^b (a+t)^{\gamma-1}\,dt \ \geq \int_0^b t^{\gamma-1}\,dt \\
&\Rightarrow \frac{1}{\gamma}\left[(a+b)^\gamma - a^\gamma\right] \ \geq \ \frac{1}{\gamma}\, b^\gamma \\
&\Rightarrow (a+b)^\gamma \ \geq \ a^\gamma + b^\gamma
\end{aligned}$$

(ii) For all a, $b \geq 0$ and $\gamma \in [0, 1]$. Let $k = \frac{1}{\gamma}$, $c = a^{\frac{1}{k}} = a^\gamma$ and $d = b^{\frac{1}{k}} = b^\gamma$. Then

$$\begin{aligned}
(a+b)^\gamma \leq a^\gamma + b^\gamma &\Leftrightarrow (c^k + d^k)^{\frac{1}{k}} \leq c + d \\
&\Leftrightarrow c^k + d^k \leq (c+d)^k \\
&\Leftrightarrow 1 + \left(\frac{d}{c}\right)^k \leq \left(1 + \frac{d}{c}\right)^k \\
&\Leftrightarrow \left(\frac{d}{c}\right)^k \leq \left(1 + \frac{d}{c}\right)^k - 1 \\
&\Leftrightarrow k \int_0^{\frac{d}{c}} t^{k-1}\,dt \leq k \int_0^{\frac{d}{c}} (1+t)^{k-1}\,dt
\end{aligned}$$