Real Analysis Homework
Chapter 1. Measure theory
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Exercise 1.9

Given an example of an open set $\mathcal{O}$ with the following property: the boundary of the closure of $\mathcal{O}$ has positive Lebesgue measure.
[Hint: Consider the set obtained by taking the union of open intervals which are deleted at the odd steps in the construction of a Cantor-like set.]

Sol.

We begin with a Cantor-like set $\hat{\mathcal{C}}$ as described in **Exercise 1.4**. Our plan is to construct an open set whose closure has boundary $\hat{\mathcal{C}}$. At the k-th stage of the construction we remove 2^{k-1} intevals each of length l_j with the property that

$$\sum_{j=1}^{k} 2^{j-1} l_j < 1$$

for each k.

We now consider the set

$$\mathcal{I} \ = \ \cup_{k=1}^{\infty} \cup_{j=1}^{2^k} \ I_{2j-1,\,k}$$

which is the union of those intervals I_k that are removed at odd steps in the construction.

We have following,

Claim. *Every $x \in \hat{\mathcal{C}}$ is a limit point of a sequence in $\mathcal{I}$.*

Proof.
We contruct $\hat{\mathcal{C}}$ iteratively as described in **Exercise 1.4**. Let $\mathcal{C}_j$ denote the remaining set after j iteratioins of the removal step and $\mathcal{R}_j$ the set of elements removed.

Recall that

$$\hat{\mathcal{C}} \ = \ \cap_{j=1}^{\infty} \mathcal{C}_j$$

Next, we note that each of the $\mathcal{C}_j$ is the disjoint union of 2^j intervals which we denote $C_{j,\,k}$.

Because the intervals are centrally situated we know that after n iterations x lies no further than 2^{-n} froom an element of $\mathcal{R}_n$.

For each n, choose such an element, x'_n, of $\mathcal{R}_j$. This yeilds a convergent sequence $x'_n \to x$ because for any $\epsilon > 0$ we simply choose n large enough that $2^{-n} < \epsilon$.

Next, note that the subsequence x_n of odd numbered terms in x'_n must also converge and each $x_n \in \mathcal{I}$. Then $x_n \to x$ and so x is a limit point of a sequence in $\mathcal{I}$. $\square$

Now we observe that $\mathcal{I}$ and $\hat{\mathcal{C}}$ are disjoint so we apply the claim to get that $\hat{\mathcal{C}} \subseteq \partial \bar{\mathcal{I}}$. Then by monotonicity of the measure and the results of **Exercise 1.4**, we have

$$m(\bar{\mathcal{I}}) \geq m(\hat{\mathcal{C}}) > 0.$$

Verifying that $\mathcal{I}$ satisfies the requirements.

Exercise 1.13

The following deals with G_δ and F_σ sets.

(a) Show that a. closed set is a G_δ and an open set is an F_σ.

[Hint: If F is closed, consider $\mathcal{O}_n = \left\{ x : d(x, F) < \frac{1}{n} \right\}$.]

(b) Give an example of an F_σ which is not a G_δ.

[Hint: This is more difficult; let F be a denumerable set that is dense.]

(c) Give an example of a Borel set which is not a G_δ nor an F_σ.

Proof.

(a) Let F be the closed set and consider $\mathcal{O}_n = \left\{ x : d(x, F) < \frac{1}{n} \right\}$.

Since F is closed, $d(x, F) = 0$ if and only if $x \in F$ therefore

$$F = \cap_{n \in \mathbb{N}} \mathcal{O}_n.$$

Thus, $F \in F_\sigma$.

Let G be an open set. Then G^c is closed and hence is an F_σ set. Therefore G is a G_δ set by de Morgan.

(b) ***Definition.*** A topological space X is called a **Baire space** if for any countable collection $\{A_n\}$ of closed sets of X each of which has empty interior in X, their union $\cup A_n$ also has empty interior in X.

Theorem. (Baire Category Theroem) Complete metric spaces are Baire spaces.

We can definitely choose an F as in the hint for any $\mathbb{R}^d$ because $\mathbb{R}^d$ is a separable topological space (points with rational coordinates form a dense set).

Since F is a countable union of singletons, it is an F_σ. To show that it is not G_δ, we will refer to the Baire category theorem.

Now suppose F is G_δ to get a contadiction. Then $F = \cap_{n\in\mathbb{N}} U_n$ where U_n's are a countable collection of open sets.

Since F is dense, each U_n is dense.

Write $C_n = \mathbb{R}^d \setminus U_n$. Then since $Int(C_n)$ is disjoint from the dense set U_n, we have $Int(C_n) = \emptyset$. So

$$\mathbb{R}^d \setminus F = \cup_{n\in\mathbb{N}} C_n$$

is a countable union of closed sets with empty interiors.

But F is also a countable union of closed sets with empty interiors, namely singletons. Thus $\mathbb{R}^d = (\mathbb{R}^d \setminus F) \cup F$ is a countable union of closed sets with empty interiors.

But by Baire Category Theorem, $\mathbb{R}^d$ is a Baire space, hence $\mathbb{R}^d$ has empty interior, nonsense. Therefore, F is not G_δ.

(c) ***Lemma.*** Let X be any topological space and A, $B \subseteq X$.

(1) If A and B are F_σ sets so are $A \cap B$ and $A \cup B$.

(2) If A and B are G_δ stes so are $A \cap B$ and $A \cup B$.

Proof.

(2) follows from (1) by taking complements. To prove (1), write $A = \cup_{n\in\mathbb{N}} F_n$ and $B = \cup_{n\in\mathbb{N}} L_n$ where F_n, L_n are closed. Then

$$A \cup B = \cup_{n\in\mathbb{N}}(F_n \cup L_n),\ A \cap B = \cup_{n,\,m\in\mathbb{N}}(F_n \cap L_m)$$

are F_σ sets. □

Let $F = [0, \infty) \cap \mathbb{Q}$ and $G = (-\infty, 0] \cap (\mathbb{R} \setminus \mathbb{Q})$ in $\mathbb{R}$. Note that closed sets in $\mathbb{R}$ are trivially F_σ and also G_δ by part (a). Then by ***Lemma.*** the set $E \cap [0, \infty) = F$ is G_δ.

But then the set $F' = (-\infty, 0] \cap \mathbb{Q}$ is also G_δ because F' is the image of F under the homeomorphism $x \mapsto -x$ of $\mathbb{R}$. Thus $F \cup F' = \mathbb{Q}$ is G_δ. This contradicts what we've shown in part (b). Thus, E is not G_δ.

Now suppose that E is F_σ. The $E \cap (-\infty, 0] = G$ is F_σ. Similar to above, from here it follows that $\mathbb{R} \setminus \mathbb{Q}$ is F_σ which then implies that $\mathbb{Q}$ is G_δ, again a contradiction. Thus, E is neither G_δ nor F_σ.

Exercise 1.16

The Borel-Cantelli lemma. Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^d : x \in E_k,\ \text{for infinitely many } k\} \\ &= \limsup_{k\to\infty}(E_k). \end{aligned}$$

(a) Show that E is measurable.

(b) Prove $m(E) = 0$.

[Hint: Write $E = \cap_{n=1}^{\infty} \cup_{k \geq n} E_k$.]

Proof.

Verify the hint:

Assume $x \in E_k$ for infinitely many k. This implies that for every $n \in \mathbb{N}$, there exists $k \geq n$ such that $x \in E_k$. That is, for every $n \in \mathbb{N}$, $x \in \cup_{k \geq n} E_k$ and hence $x \in \cap_{n \in \mathbb{N}} \cup_{k \geq n} E_k$.

(a) By definition of the limilt superior, we have

$$E \ = \ \cap_{n=1}^{\infty} \ \cup_{k \geq n} E_k$$

Each $\cup_{k \geq n} E_k$ is a countable union of measurable sets, and hence measurable. Then E is a countable intersection of measurable sets, and so it is also measurable.

(b) Assume to the contary that $m(E) = \delta > 0$.

Notice that if we define

$$X_n \ = \ \cap_{k=1}^{N} \cup_{n \geq k} \ E_n = \ \cup_{n \geq N} E_n$$

Then $X_n \searrow E$, and $\forall N$,

$$\delta \leq m(X_n) = m\left(\cup_{n \geq N} E_n\right) \leq \sum_{n=N}^{\infty} m(E_n)$$

which contradicts the precondition $\sum_{k=1}^{\infty} m(E_k) < \infty$.

Exercise 1.19

Here are some observations regarding the set operation $A + B$.

(a) Show that if either A and B is open, then $A + B$ is open.

(b) Show that if A and B are closed, then $A + B$ is measurable.

(c) Show, however, that $A + B$ might not be closed even though A and B are closed.

[Hint: For (b) show that $A + B$ is an F_σ set.]

Proof.

(a) Without loss of generality, suppose that A is open and choose $x \in A + B$. This means that there is an $a \in A$ and a $b \in B$ such that $a + b = x$.

Because A is open, we can choose a $\delta > 0$ such that $B_\delta(a) \subset A$.

Claim that $B_\delta(x) \subset A + B$.

We can see this because any $y \in B_\delta(x)$ satisfies $y = x + \varepsilon$, where $|\varepsilon| < \delta$. Then $y = a + b + \varepsilon = (a + \varepsilon) + b$.

We note that $(a + \varepsilon) \in A$ so $y \in A + B$. Hence, $B_\delta(x) \subset A + B$ and so $A + B$ and so $A + B$ is open because x was arbitrary.

(b) Suppose that A, $B \subset \mathbb{R}^d$ are closed. Note that it will suffice to prove the special case of A, B compact because we can write

$$A_k = \cup_{k=1}^{\infty} A \cap B_k(0) \text{ and } B_j = \cup_{j=1}^{\infty} B \cap B_j(0)$$

where $B_i(0)$ is the ball of radius i centered at the origin. Then A_k, B_j are compact for every k, j and therefore can then write

$$A + B = \cup_{j=1}^{\infty} \cup_{k=1}^{\infty} A_k + B_j$$

Hence, if each of the $A_k + B_j$ are measurable, then $A + B$ will. So we have reduced the prblem to the follwing claim.

Claim. If X and Y are compact subsets of $\mathbb{R}^d$, then the set $X + Y$ is compact.

Proof.

Recall that a set in $\mathbb{R}^d$ is compact if and only if every sequence has a convergent subsequence.

Consider any sequence $\{z_n\}_{n=1}^{\infty}$ in $X + Y$. Then by the definition of $X + Y$, we have that each of the z_n can be written $z_n = x_n + y_n$ where $x_n \in X$ and $y_n \in Y$.

Becasue X, Y are compact, we can find convergent subsequences $x_{n_k} \to x$ in X and $y_{n_k} \to y$ in Y. Because X and Y are closed, we know that $x \in X$ and $y \in Y$ so we can see that for any $\varepsilon > 0$ and sufficiently large n, k

$$|(x + y) - (x_{n_k} + y_{n_k})| = |(x - x_{n_k}) + (y - y_{n_k})| < |x - x_{n_k}| + |y - y_{n_k}| < \varepsilon$$

Hence, $z_n = x_n + y_n$ has a convergent subsequence in $X + Y$ which shows that $X + Y$ is compact. $\square$

Going back to the original problem, we have that $A + B$ can be written as a countable union of compact sets, and hence closed. This means that $A+B$ is not only measurable, but actually $\mathcal{F}_\sigma$.

(c) Define

$$A = \{(x, y) \in \mathbb{R}^2 \,|\, y \geq b - mx,\, b > 0,\, m > 0\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 \,|\, y \geq -b + mx,\, b > 0,\, m > 0\}$$

Then we have that A^c and B^c are open, so A and B are closed. But

$$A + B = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}$$

which is open.

Exercise 1.20

Show that there exist closed sets A and B with $m(A) = m(B) = 0$, but $m(A+B) > 0$:

(a) In $\mathbb{R}$, let $A = \mathcal{C}$ (the Cantor set), $B = \mathcal{C}/2$. Note that $A + B \supseteq [0,1]$.

(b) In $\mathbb{R}^2$, observe that if $A = I \times \{0\}$ and $B = \{0\} \times I$ (where $I = [0,1]$), then $A + B = I \times I$.

Proof.

(a) Let $x \in [0,1]$. We know that x has a ternary expansion

$$x = \sum_{n \in \mathbb{N}} a_n 3^{-n}.$$

Then if we define

$$b_n = \begin{cases} a_n & \text{if } a_n \neq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$c_n = \begin{cases} 2 & \text{if } a_n = 1 \\ 0 & \text{otherwise} \end{cases}$$

then we have

$$\begin{aligned} x &= \sum_{n \in \mathbb{N}} b_n 3^{-n} + \sum_{n \in \mathbb{N}} \frac{c_n}{2} 3^{-n} \\ &= \sum_{n \in \mathbb{N}} b_n 3^{-n} + \frac{1}{2} \sum_{n \in \mathbb{N}} c_n 3^{-n} \in \mathcal{C} + \mathcal{C}/2 \end{aligned}$$

since $\{b_n\}$ and $\{c_n\}$ are sequences of 0's and 2's.

As x was arbitrary above, we obtain $[0,1] \subseteq A + B$. Hence $m(A+B) \geq 1$, but A and B are closed sets of measure zero.

(b) Given $(x,y) \in I \times I$, $(x,y) = (x,0) + (0,y) \in A + B$. $I \times I \subseteq A + B$ and the reverse containment is similar so we have $A + B = I \times I$.

Therefore $m(A+B) = 1$, however both A and B can be covered by rectangles of area ε for any given $\varepsilon > 0$, hence, $m(A) = m(B) = 0$.

Exercise 1.26

Suppose $A \subset E \subset B$, where A and B are measurable sets of finite measure.
Prove that if $m(A) = m(B)$, then E is measurable.

Proof.
Suppose that A, B are measurable sets, and that $A \subset E \subset B$.
We want to prove that E is measurable.

Fisrt, we note that because $A \subset E$ that we can write

$$E \;=\; A \cup (E - A)$$

Because A is measurable, it will suffice to show that $E - A$ is measurable, then E will be a union of two measurable sets, therefore, E is measurable.

Observe that because $A \subset E \subset B$ that $(E - A) \subset (B - A)$. We use monotonicity of the measure to get

$$m_*(E - A) \leq m_*(B - A) = m(B - A) = m(B) - m(A) = 0$$

So $E - A$ is a subset of a set with measure 0, and is measurable as a result. This immediately gives that E is measurable.