Calculus 2 11/14 Note
Module Class 07

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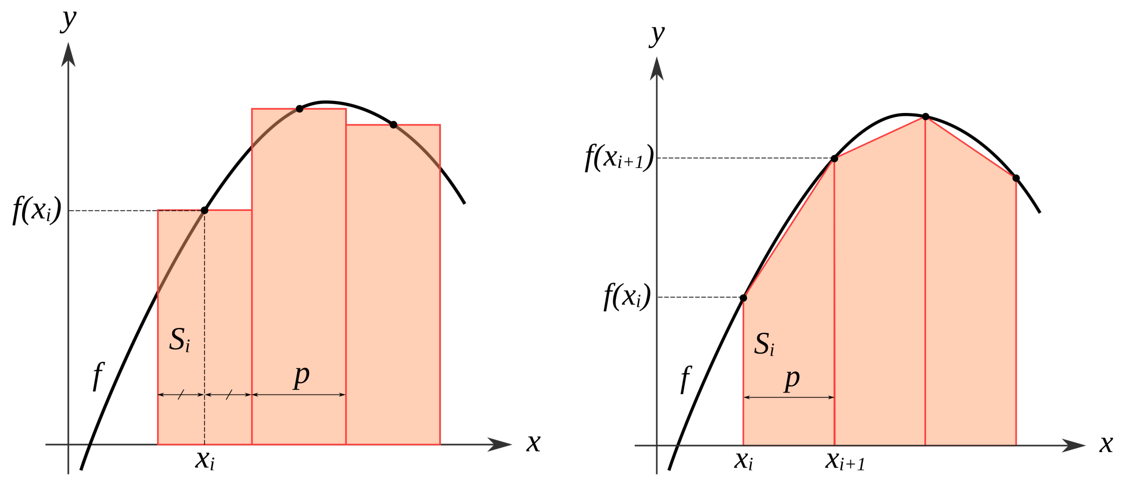
Section 5.1: Areas and Distances

Definition

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

Graph:



Example:
Determine a region whose area is equal to the given limit.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$$

Sol.
$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y = \tan x$ on the interval $[0, \frac{\pi}{4}]$, since for $y = \tan x$ on $[0, \frac{\pi}{4}]$ with $\Delta x = \frac{\pi/4 - 0}{n} = \frac{\pi}{4n}$, $x_i = 0 + i\Delta x = \frac{i\pi}{4n}$ and $x_i^* = x_i$.

The expression for the area is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\frac{i\pi}{4n}\right) \frac{\pi}{4n}.$$

Section 5.2: The Definite Integral

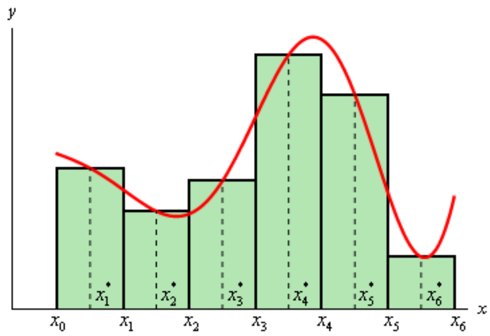
Definition of a Definite Integral

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n sub-intervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1}, x_i]$. Then the **definite Integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Graph:



Theorem

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuties, then f in integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

Theorem

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\,\Delta x$$

Example:
Express the limit as a definite integral on the given interval.

1. $\lim\limits_{n\to\infty} \sum_{i=1}^{n} x_i \sqrt{1 + x_i^3}\, \Delta x, \quad [2,\, 5]$

   Sol.

$$\int_2^5 x\sqrt{1 + x^3}\, dx$$

2. $\lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i}\, \Delta x, \quad [\pi,\, 2\pi]$

   Sol.

$$\int_{\pi}^{2\pi} \frac{\cos x}{x}\, dx$$

Midpoint Rule

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\, \Delta x = \Delta x[f(\overline{x}_1) + \cdots + f(\overline{x}_n)]$$

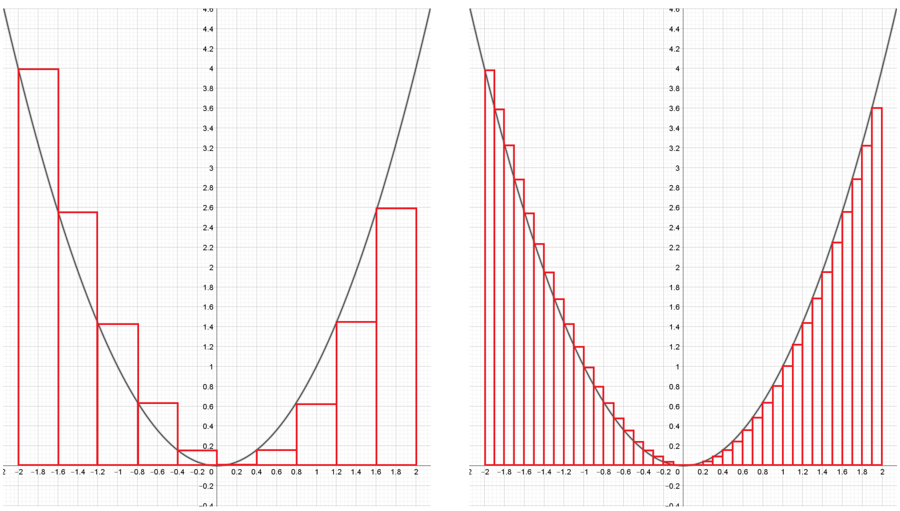
where

$$\Delta x = \frac{b-a}{n}$$

and

$$\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1},\, x_i]$$

Graph:



Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
6. If $f(x) \geq 0$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\,dx \geq 0$$
7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$
8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

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Section 5.3: The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

Suppose f is con

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$

Example:
Find the derivative of the function.

1. $g(x) = \int_1^x \ln(1+t^2)\,dt$

   Sol.

   By Fundamental Theorem of Calculus, then
$$g'(x) = \frac{1}{x^3+1}$$

2. $f(x) = \int_x^1 \cos\sqrt{t}\,dt$

Sol.

By Fundamental Theorem of Calculus, since

$$f(x) = \int_x^1 \cos\sqrt{t}\,dt = -\int_1^x \cos\sqrt{t}\,dt,$$

then

$$f'(x) = -\cos\sqrt{x}$$

3. $h(x) = \int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt$

Sol.

Let $u = \tan x$. Then $\frac{du}{dx} = \sec^2 x$.

Also $h'(x) = \frac{h(x)}{dx} = \frac{h(x)}{du} \cdot \frac{du}{dx}$, so

$$f'(x) = \frac{d}{du}\int_0^u \sqrt{t+\sqrt{t}}\,dt \cdot \frac{du}{dx} = \sqrt{u+\sqrt{u}} \cdot \sec^2 x = \sqrt{\tan x + \sqrt{\tan x}} \cdot \sec^2 x$$

4. $F(x) = \int_{1-2x}^{1+2x} t\sin t\,dt$

Sol.

$$\begin{aligned}
F(x) &= \int_{1-2x}^{1+2x} t\sin t\,dt = \int_0^{1+2x} t\sin t\,dt + \int_{1-2x}^0 t\sin t\,dt \\
&= \int_0^{1+2x} t\sin t\,dt - \int_0^{1-2x} t\sin t\,dt \\
&= [(1+2x)\sin(1+2x)]\,(2) - [(1-2x)\sin(1-2x)]\,(-2) \\
&= (2+4x)\sin(1+2x) + (2-4x)\sin(1-2x)
\end{aligned}$$

Section 5.4: Indefinite Integrals and the Net Change Theorem

Formula

$$\int c\,f(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Example:
Evaluate the integral.

1. $\int_0^1 \frac{4}{1+x^2}\,dx$

   Sol.

$$\int_0^1 \frac{4}{1+x^2}\,dx = [4\arctan x]_{x=0}^1 = 4\arctan 1 - 4\arctan 0 = 4\cdot\frac{\pi}{4} - 4\cdot 0 = \pi$$

2. $\int_{-10}^{10} \frac{2e^x}{\sinh x+\cosh x}\,dx$

   Sol.

$$\int_{-10}^{10} \frac{2e^x}{\sinh x+\cosh x}\,dx = \int_{-10}^{10} \frac{2e^x}{\frac{e^x-e^{-x}}{2}+\frac{e^x+e^{-x}}{2}}\,dx = \int_{-10}^{10} 2\,dx = [2x]_{x=-10}^{10} = 20$$

Exercise:
Evaluate the integral.

1. $\int_0^{\pi/3} \frac{\sin\theta+\sin\theta\,\tan^2\theta}{\sec^2\theta}\,d\theta$

2. $\int_0^2 |2x-1|\,dx$

Sol.
1. $\frac{1}{2}$ 2. $\frac{5}{2}$

Section 5.5: The Substitution Rule

The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

The Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$

Example: Evaluate the integral.

1. $\int \frac{(\arctan x)^2}{x^2+1}\,dx$

   Sol.

   Let $u = \arctan x$. Then

   $$du = \frac{1}{x^2+1}\,dx,$$

   so

   $$\int \frac{(\arctan x)^2}{x^2+1}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\arctan x)^3 + C,$$

   where C is the constant.

2. $\int \frac{\cos(\ln x)}{x}\,dx$

   Sol.

   Let $u = \ln x$. Then

   $$du = \frac{1}{x}\,dx,$$

   so

   $$\int \frac{\cos(\ln x)}{x}\,dx = \int \cos u\,du = \sin u + C = \sin(\ln x) + C,$$

   where C is the constant.

3. $\int_0^a x\sqrt{a^2 - x^2}\, dx$

Sol.

Assume $a > 0$. Let $u = a^2 - x^2$, so $du = -2x\, dx$.

When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Thus,

$$\int_0^a x\sqrt{a^2 - x^2}\, dx = \int_{a^2}^0 u^{1/2} \left(-\frac{1}{2}\, du\right) = \frac{1}{2}\int_0^{a^2} u^{1/2}\, du = \frac{a^3}{3}$$

4. $\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$

Sol.

Let $u = \sin^{-1} x$, so $du = \frac{dx}{\sqrt{1-x^2}}$.

When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1 - x^2}}\, dx = \int_0^{\pi/6} u\, du = \frac{\pi^2}{72}$$

Exercise: Evaluate the integral.

1. $\int x^2\sqrt{2 + x}\, dx$

2. $\int_0^{T/2} \sin\left(\frac{2\pi x}{T - \alpha}\right)\, dx$

Sol.

1. $\frac{2}{7}(2 + x)^{7/2} - \frac{8}{5}(2 + x)^{5/2} + \frac{8}{3}(2 + x)^{3/2} + C$, where C is the constant.

2. $\frac{T}{\pi}\cos\alpha$