Calculus 1 10/17 Note
Module Class 07

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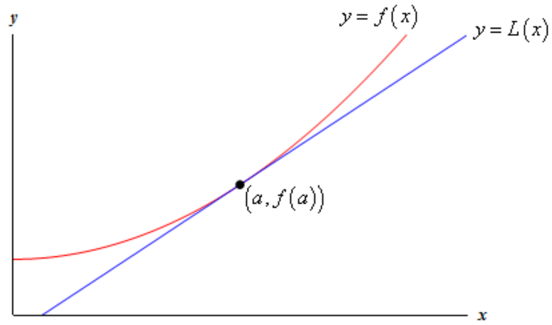
Section 3.10: Linear Approximations and Differentials



Recall (Tangent line)

Give the function $f(x)$ and if $f(x)$ is differentiable at $x = a$, then the tangent line of $f(x)$ at $x = a$ is

$$y = f(a) + f'(a)(x - a)$$

Based on the above tangent line, we will have the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

This is called the **linear approximation** or **tangent line approximation** of f at a.
Hence, the **linearization** of f at a is

$$L(x) = f(a) + f'(a)(x - a)$$

Example:

Determine the linear approximation for $\sin\theta$ at $\theta = 0$.

Sol.

All that we need to do is compute the tangent line to $\sin\theta$ at $\theta = 0$.

$$f(\theta) = \sin\theta,\ f'(\theta) = \cos\theta \quad \text{so} \quad f(0) = 0,\ f'(0) = 1$$

The lineear approximation is,

$$L(\theta) = f(0) + f'(0)(\theta - a) = 0 + 1 \cdot (\theta - 0) = \theta$$

So as long as θ stays small we can say that $\sin\theta = \theta$.

Differentials

The **differential** dy is defined in terms of dx by the equation

$$dy = f'(x)\ dx.$$

Moreover, let $dy = \Delta y$ and $dx = \Delta x$, then the corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x).$$

Example:

Compute dy and Δy if $y = \cos(x^2 + 1) - x$ as x changes from $x = 2$ to $x = 2.03$.

Sol.

First let's compute actual the change in y, Δy.

$$\Delta y = \cos\left((2.03)^2 + 1\right) - 2.03 - (\cos(2^2 + 1) - 2) \approx 0.083581127$$

Now let's get the formula for dy.

$$dy = (-2x\ \sin(x^2 + 1) - 1)\ dx$$

Next, the change in x from $x = 2$ to $x = 2.03$ is $\Delta x = 0.03$ and so we then assume that $dx \approx \Delta x = 0.03$.

This gives an approximate change in y of,

$$dy = (-2 \cdot 2 \cdot \sin(2^2 + 1) - 1)(0.03) \approx 0.085070913$$

We can see that in fact we do have that $\Delta y \approx dy$ provided we keep Δx small.

Exercise:

1. Find the differential dy and evaluate dy for the given values of x and dx.
   (a) $y = e^{x/10},\ x = 0,\ dx = 0.1$
   (b) $y = \dfrac{x+1}{x-1},\ x = 2,\ dx = 0.05$
2. Use a linear approximation (or differentials) to estimate the given number.
   (a) $\sqrt[3]{1001}$
   (b) $\cos 29^o$

Sol.

1. (a) $dy = \dfrac{1}{10}e^{x/10}\,dx$ and $dy = 0.01$ (b) $dy = \dfrac{-2}{(x-1)^2}\,dx$ and $dy = -0.1$
2. (a) $10 + \dfrac{1}{300} \approx 10.003$ (b) $\frac{1}{2}\sqrt{3} + \dfrac{\pi}{360} \approx 0.875$

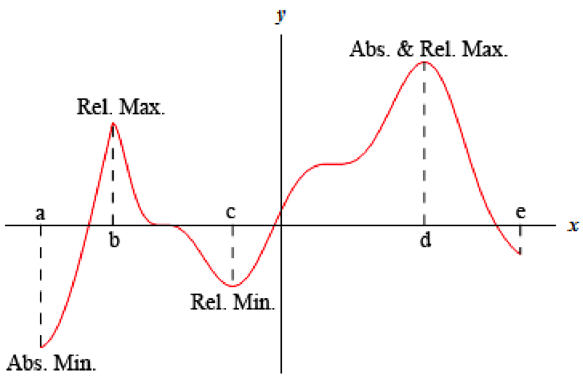
Section 4.1: Maximum and Minimum Values

Definition

Let c be a number in the domain D of a function f. Then $f(c)$ is the

1. **absoulte maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.
2. **absoulte minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.
3. **local maximum** value of f if $f(c) \geq f(x)$ when x is near c.
4. **local minimum** value of f if $f(c) \leq f(x)$ when x is near c.

Example graph:



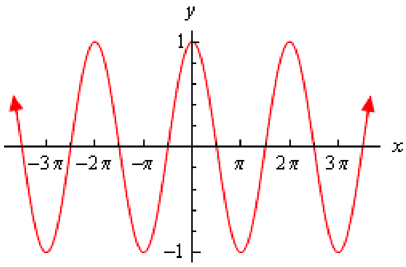
Example:

Identify the absolute extrema and relative extrema for the following function.

$$f(x) = \cos x$$

Sol.

We've not restricted the domain for this function. Here is the graph.



Cosine has extrema (relative and absolute) that occur at many points.

Cosine has both relative and absolute maximums of 1 at

$$x = \ldots,\ -4\pi,\ -2\pi,\ 0,\ 2\pi,\ 4\pi,\ \ldots$$

Cosine also has both relative and absolute minimums of -1 at

$$x = \ldots,\ -3\pi,\ -\pi,\ \pi,\ 3\pi,\ \ldots$$

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absoulte maximum value $f(c)$ and an absoulte minimum value $f(d)$ at some number c and d in $[a, b]$.

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Definition

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

So if f has a local maximum or minimum at c, then c is a critical number of f.

Example:

Determine all the critical points for the function.

$$f(x) = x^2 \ln(3x) + 6$$

Sol.

Before getting the derivative let's notice that since we can't take the log of a negative number or zero we will only be able to look at $x > 0$.

The derivative is then,

$$f'(x) = 2x\ln(3x) + x^2\left(\frac{3}{3x}\right) = 2x\ln(3x) + x = x(2\ln(3x) + 1)$$

First note that, despite appearances, the derivative will not be zero for $x = 0$.

As noted above the derivative doesn't exist at $x = 0$ because of the natural logarithm and so the derivative can't be zero there!

So, the derivative will only be zero if,

$$2\ln(3x) + 1 = 0 \;\Rightarrow\; \ln(3x) = -\frac{1}{2}$$

Recall that we can solve this by exponentiating both sides.

$$e^{\ln(3x)} = e^{-1/2} \;\Rightarrow\; 3x = e^{-1/2} \;\Rightarrow\; x = \frac{1}{3\sqrt{e}}$$

Hence, there is a single critical point for this function.

Exercise:

1. Find the critical numbers of the function.
   (a) $g(x) = 4x - \tan x$
   (b) $h(x) = 3x - \arcsin x$

2. Find the absolute maximum and absolute minimum values of f on the given interval.
   (a) $f(x) = x^3 - 6x^2 + 5$, $[-3, 5]$
   (b) $f(x) = xe^{x/2}$, $[-3, 1]$

Sol.
1. (a) $x = \frac{\pi}{3} + 2n\pi,\ \frac{5\pi}{3} + 2n\pi,\ \frac{2\pi}{3} + 2n\pi$ and $\frac{4\pi}{3} + 2n\pi$
(b) $x = \pm\frac{2}{3}\sqrt{2}$

2. (a) absolute maximum is $f(0) = 5$ and absolute minimum is $f(-3) = 76$
(b) absolute maximum is $f(1) = \sqrt{e}$ and absolute minimum is $f(-2) = \frac{-2}{e}$

Section 4.2: The Mean Value Theorem

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

Example:

Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = 2x^2 - 4x + 5, \quad [-1, 3]$$

Sol.

f is a polynomial, so it's continuous and differentiable on $\mathbb{R}$, and hence, continuous on $[1, 3]$ and differentiable on $(1, 3)$.

Since $f(-1) = 11$ and $f(3) = 11$, f satisfies all the hypotheses of Rolle's Theorem.

$$f'(c) = 4c - 4 \quad \text{and} \quad f'(c) = 0 \;\Leftrightarrow\; 4c - 4 = 0 \;\Leftrightarrow\; c = 1.$$

$c = 1$ is in the interval $(-1, 3)$, so 1 satisfies the conclusion of Rolle's Theorem.

The Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

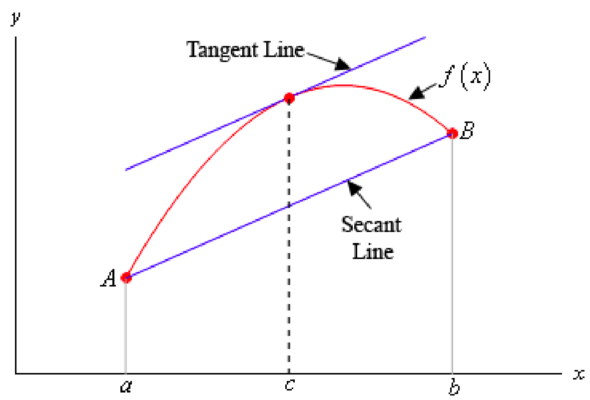
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$



Example:

Verify that the function satisfies the three hypotheses of Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Mean Value Theorem.

$$f(x) = 2x^2 - 3x + 1, \ [0, 2]$$

Sol.

f is continuous on $[0, 2]$ and differentiable on $(0, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$.

$$f'(c) = \frac{f(b) - f(a)}{b - a} \ \Leftrightarrow \ 4c - 3 = \frac{f(2) - f(0)}{2 - 0} = 1 \ \Leftrightarrow \ 4c = 4 \ \Leftrightarrow \ c = 1,$$

which is in $(0, 2)$.

Corollary

If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

Exercise:

1. Show that $f(x) = 4x^5 + x^3 + 7x - 2$ has exactly one real root. [Intermediate Value Theorem and Rolle's Theorem]
2. Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the following function. [Mean Value Theorem]

Sol.
2. $c = \dfrac{-4 + \sqrt{76}}{6}$