Real Analysis Homework
Chapter 4. Product Measures
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Exercise 1

Let $A \subseteq X$ and let B be a ν-measurable subset of Y. If $A \times B$ is measurable with respect to the product measure $\mu \times \nu$, is A necessarily measurable with respect to μ?

Proof.
No, A is NOT necessarily measurable with respect to μ.

If $X = Y =$ Borel σ algebra of $\mathbb{R}$, $\mu = \nu =$ Lebesgue measue and A is a Lebesgue measuable set in $\mathbb{R}$ which is not Borel set, then $A \times \mathbb{R}$ is measurable with respect to the product measure but $A \notin X$, that is A may not be measurable with respect to μ.

Exercise 2

Let $\mathbb{N}$ be the set of natural numbers, $\mathcal{M} = 2^{\mathbb{N}}$, and c the counting measure defined by setting $c(E)$ equal to the number of points in E if E is finite and ∞ if E is an infinite set. Prove that every function $f : \mathbb{N} \to \mathbb{R}$ is measurable with respect to c and that f is integrable over $\mathbb{N}$ with respect to c if and only if the series $\sum_{k=1}^{\infty} f(k)$ is absolutely convergent in which case

$$\int_{\mathbb{N}} f\, dc = \sum_{k=1}^{\infty} f(k).$$

Proof.

1. Use the fact that f is integrable iff $|f|$ is integrable.

   For nonnegative, measurable function $|f|$ in measure space $(\mathbb{N}, \mathcal{M}, c)$, we have

   $$\int_{\mathbb{N}} |f|\, dc = \sum_{k=1}^{\infty} |f(k)|.$$

   Moreover, we also know that $|f|$ is integrable iff $\int_{\mathbb{N}} |f|\, dc = \sum_{k=1}^{\infty} |f(k)| < +\infty$.

   Thus, every function $f : \mathbb{N} \to \mathbb{R}$ is measurable with respect to c and that f is integrable over $\mathbb{N}$ with respect to c if and only if the series $\sum_{k=1}^{\infty} f(k)$ is absolutely convergent.

2. Following, we will show that

$$\int_{\mathbb{N}} f\, dc = \sum_{k=1}^{\infty} f(k).$$

Use the fact that for nonnegative, measurable function g in measure space $(\mathbb{N}, \mathcal{M}, c)$, we have

$$\int_{\mathbb{N}} g\, dc = \sum_{k=1}^{\infty} g(k).$$

Since we can write f as $f = f^+ - f^-$ where $f^+,\ f^- \geq 0$, then

$$\begin{aligned}
&\int_{\mathbb{N}} |f|\, dc = \sum_{k=1}^{\infty} |f(k)| \\
\Rightarrow &\int_{\mathbb{N}} |f^+ - f^-|\, dc = \sum_{k=1}^{\infty} |f^+(k) - f^-(k)| \\
\Rightarrow &\int_{\mathbb{N}} f^+\, dc + \int_{\mathbb{N}} f^-\, dc = \sum_{k=1}^{\infty} f^+(k) + \sum_{k=1}^{\infty} f^-(k) \\
\Rightarrow &\int_{\mathbb{N}} f^+\, dc - \sum_{k=1}^{\infty} f^-(k) = \sum_{k=1}^{\infty} f^+(k) - \int_{\mathbb{N}} f^-\, dc \\
\Rightarrow &\int_{\mathbb{N}} f^+\, dc - \int_{\mathbb{N}} f^-\, dc = \sum_{k=1}^{\infty} f^+(k) - \sum_{k=1}^{\infty} f^-(k) \\
\Rightarrow &\int_{\mathbb{N}} f\, dc = \sum_{k=1}^{\infty} f(k).
\end{aligned}$$

Exercise 3

Let $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu) = (\mathbb{N}, \mathcal{M}, c)$, the measure space defined in the preceding problem. State the Fubini and Tonelli Theorems explicitly for this case.

Proof.

Suppose every function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is measurable with respect to $\mu \times \nu$ and f is integrable over $\mathbb{N} \times \mathbb{N}$ with respect to $\mu \times \nu$.

The Tonelli's Theorems for counting measure on $\mathbb{N} \times \mathbb{N}$, if f is $\mathcal{M} \times \mathcal{M}$ and $f \geq 0$, then $\int_{\mathbb{N}} f\, d\mu$ is measurable with respect to c, $\int_{\mathbb{N}} f\, dc$ is measurable with respect to μ, and

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f\, d\mu \right] d\nu = \int_{\mathbb{N}\times\mathbb{N}} f\, d(\mu \times \nu) = \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f\, d\nu \right] d\mu.$$

The Fubini's Theorems for counting measure on $\mathbb{N} \times \mathbb{N}$, if f is integrable over $\mathbb{N} \times X$ with repect to $c \times \mu$, that is

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f\, d\nu \right] d\mu < \infty,$$

then

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f\, d\mu \right] d\nu = \int_{\mathbb{N}\times\mathbb{N}} f\, d(\mu \times \nu) = \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f\, d\nu \right] d\mu.$$

Exercise 4(i)

Let $(\mathbb{N}, \mathcal{M}, c)$ be the measure space defined in **Exercise 2** and $(X, \mathcal{A}, \mu)$ a general measure space. Consider $\mathbb{N} \times X$ with the product measure $c \times \mu$.

Show that a subset E of $\mathbb{N} \times X$ is measurable with respect to $c \times \mu$ if and only if for each natural number k, $E_k = \{x \in X | (k, x) \in E\}$ is measurable with respect to μ.

Proof.
($\Rightarrow$):
We can get from **Lemma 4.8.2** in Real Analysis by Fon-Che Liu.

($\Leftarrow$):
We can find some $n \in \mathbb{N}$ such that $E = \cup_n(\{n\} \times E_n)$, since E_n is measurable with respect to μ and $\{n\} \subset \mathbb{N}$ is measurable with respect to c, then E is measurable with respect to $c \times \mu$.

Exercise 4(ii)

Let $(\mathbb{N}, \mathcal{M}, c)$ be the measure space defined in **Exercise 2** and $(X, \mathcal{A}, \mu)$ a general measure space. Consider $\mathbb{N} \times X$ with the product measure $c \times \mu$.
Show that a function $f : \mathbb{N} \times X \to \mathbb{R}$ is measurable with respect to $c \times \mu$ if and only if for each natural number k, $f(k, \cdot) : X \to \mathbb{R}$ is measurable with respect to μ.

Proof.
($\Rightarrow$):
We can get from **Corollary 4.8.2** in Real Analysis by Fon-Che Liu.

($\Leftarrow$):
Let $f_k = f(k, \cdot) : X \to \mathbb{R}$. By **Exercise 4(i)**, we know that if E_k is measurable, then $\{k\} \times E_k$ is measurable. Thus,

$$f^{-1}((a, \infty]) = \bigcup_k \{k\} \times f_k^{-1}((a, \infty])$$

is a countable union of measurable sets, so f is measurable.

Exercise 4(iii)

Let $(\mathbb{N}, \mathcal{M}, c)$ be the measure space defined in **Exercise 2** and $(X, \mathcal{A}, \mu)$ a general measure space. Consider $\mathbb{N} \times X$ with the product measure $c \times \mu$.
Show that a function $f : \mathbb{N} \times X \to \mathbb{R}$ is integrable over $\mathbb{N} \times X$ with repect to $c \times \mu$ if and only if for each natural number k, $f(k, \cdot) : X \to \mathbb{R}$ is integrable over X with respect to μ and

$$\sum_{k=1}^{\infty} \int_X |f(k, x)| \, d\mu(x) < \infty.$$

Proof.
Use the fact that f is integrable over $\mathbb{N} \times X$ with repect to $c \times \mu$ iff $|f|$ is integrable over $\mathbb{N} \times X$ with repect to $c \times \mu$.

Also, by **Exercise 4(ii)**, we have f is measurable with respect to $c \times \mu$, then $|f|$ is measurable and nonnegative, hence, we have

$$\int_{\mathbb{N}} |f| \, dc = \sum_{k=1}^{\infty} |f(k, x)|.$$

Moreover, $|f|$ is integrable over $\mathbb{N} \times X$ with repect to $c \times \mu$ iff

$$\int_{\mathbb{N}\times X} |f|\, d(c \times \mu) = \int_X \left(\int_{\mathbb{N}} |f|\, dc \right) d\mu = \int_X \left(\sum_{k=1}^{\infty} |f(k,x)| \right) d\mu = \sum_{k=1}^{\infty} \int_X |f(k,x)|\, d\mu < +\infty.$$

Since $\int_X |f(k,x)|\, d\mu < \sum_{n=1}^{\infty} \int_X |f(n,x)|\, d\mu < +\infty$, then $f(k,\cdot) : X \to \mathbb{R}$ is integrable over X with respect to μ.

Exercise 4(iv)

Let $(\mathbb{N}, \mathcal{M}, c)$ be the measure space defined in **Exercise 2** and $(X, \mathcal{A}, \mu)$ a general measure space. Consider $\mathbb{N} \times X$ with the product measure $c \times \mu$.
Show that if the function $f : \mathbb{N} \times X \to \mathbb{R}$ is integrable over $\mathbb{N} \times X$ with repect to $c \times \mu$ then

$$\int_{\mathbb{N}\times X} f\, d(c \times \mu) = \sum_{k=1}^{\infty} \int_X f(k,x)\, d\mu(x) < \infty.$$

Proof.

Since the function $f : \mathbb{N} \times X \to \mathbb{R}$ is integrable over $\mathbb{N} \times X$ with repect to $c \times \mu$, by **Exercise 4(iii)**, we then have

$$\begin{aligned}
\int_{\mathbb{N}\times X} f\, d(c \times \mu) &= \int_X \left(\int_{\mathbb{N}} f\, dc \right) d\mu \\
&= \int_X \left(\sum_{k=1}^{\infty} f(k,x) \right) d\mu \\
&= \sum_{k=1}^{\infty} \int_X f(k,x)\, d\mu(x) \\
&\leq \sum_{k=1}^{\infty} \int_X |f(k,x)|\, d\mu(x) < \infty.
\end{aligned}$$

Exercise 5

Let $(X, \mathcal{A}, \mu) = (Y, \mathcal{B}, \nu) = (\mathbb{N}, \mathcal{M}, c)$, the measure space defined in **Exercise 2**. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by setting

$$f(x,y) = \begin{cases} 2 - 2^{-x} & \text{if } x = y \\ -2 + 2^{-x} & \text{if } x = y + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that f is measurable with respect to the product measure $c \times c$. Also show that

$$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n)\, dc(m) \right] dc(n) \neq \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n)\, dc(n) \right] dc(m).$$

Is this a contradiction either of Fubini's Theorem or Tonelli's Theorem?

Proof.

1. Show that f is measurable with respect to the product measure $c \times c$.

   Since $\mathbb{N}$ is countable and the only set for which $c \times c$ measure is 0 is the emptyset, the $c \times c$ measurable sets of $\mathbb{N} \times \mathbb{N}$ are all subsets, hence f is measurable.

2. Show that $\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n)\, dc(m)\right] dc(n) \neq \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n)\, dc(n)\right] dc(m)$.

   Since
   $$\sum_{m=1, m\in\mathbb{N}}^{\infty} \sum_{n=1, n\in\mathbb{N}}^{\infty} f(m,n) = f(1,1) = 1.5,$$
   but
   $$\begin{aligned} \sum_{n=1, n\in\mathbb{N}}^{\infty} \sum_{m=1, m\in\mathbb{N}}^{\infty} f(m,n) &= \sum_{n=1, n\in\mathbb{N}}^{\infty} (2 - 2^{-n}) + (-2 + 2^{-n-1}) \\ &= \sum_{n=1, n\in\mathbb{N}}^{\infty} 2^{-n-1} - 2^{-n} \\ &= - \sum_{n=1, n\in\mathbb{N}}^{\infty} 2^{-n-1} \\ &= -0.5 \neq 1.5. \end{aligned}$$
   So
   $$\int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n)\, dc(m)\right] dc(n) \neq \int_{\mathbb{N}} \left[\int_{\mathbb{N}} f(m,n)\, dc(n)\right] dc(m).$$

3. Is this a contradiction either of Fubini's Theorem or Tonelli's Theorem?

   No, it doesn't contradict Fubini's theorem or Tonelli's Theorem either.

   Since the measure c is σ-finite, hence it show we can NOT remove the assumption of non-negativeness or integrability.

Exercise 10

Let h and g be integrable functions on X and Y, and define $f(x,y) = h(x)g(y)$. Show that f is integrable on $X \times Y$ with respect to the product measure, then

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_X h\, d\mu \int_Y g\, d\nu.$$

Proof.

1. **Measurability:**

   Use the fact that a function is measurable if and only if it is the pointwise limit of a sequence of simple functions.

   So take simple a_n, b_n such that $h(x) = \lim_{n\to\infty} a_n(x)$ for every x and $g(y) = \lim_{n\to\infty} b_n(y)$ for every y.

Then

$$f(x,y) = h(x)g(y) = \lim_{n\to\infty} a_n(x)b_n(y)$$

for every (x,y).

It only remains to observe that each $(x,y) \mapsto a_n(x)b_n(y)$ is a simple function to conclude that f is measurable.

This follows from

$$1_X(x)1_Y(y) = 1_{X\times Y}(x,y).$$

2. **Integrability:**

   Take two nondecreasing sequences of nonnegative simple functions $a_n(x), b_n(y)$ converging pointwise to $|h(x)|$ and $|g(y)|$ respectively. Then $a_n(x)b_n(y)$ is a nondecreasing sequence of nonnegative simple functions converging pointwise to $|f|$.

   By the monotone convergence theorem

   $$\int_{X\times Y} |f|\, d(\mu\times\nu) = \lim_{n\to\infty} \int_{X\times Y} a_n(x)b_n(y)\, d(\mu\times\nu)(x,y).$$

   By definition of the product measure

   $$\int_{X\times Y} 1_{X\times Y}(x,y)\, d(\mu\times\nu) = (\mu\times\nu)(X\times Y) = \mu(X)\nu(Y) = \int_X 1_X(x)\, d\mu \int_Y 1_Y(y)\, d\nu.$$

   By linearity, this extends to simple functions. Hence, by monotone convergence again,

   $$\int_{X\times Y} a_n(x)b_n(y)\, d(\mu\times\nu)(x,y) = \int_X a_n(x)\, d\mu \int_Y b_n(y)\, d\nu \;\longrightarrow\; \int_X |h|\, d\mu \int_Y |g|\, d\nu.$$

   So f is integrable with

   $$\int_{X\times Y} |f|\, d(\mu\times\nu) = \int_X |h|\, d\mu \int_Y |g|\, d\nu.$$

   Note that applying the above to $h^\pm$ and $g^\pm$, we can deduce

   $$\int_{X\times Y} f\, d(\mu\times\nu) = \int_X h\, d\mu \int_Y g\, d\nu.$$