Supplementary File

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***Abstract*—In this supplementary file, the proof of theorems in the manuscript "Robust Team Formation Tracking for Stochastic Multi-Biped Robot System: A Decentralized Control Approach " are given.**

Appendix A
Proof of Theorem 1

Consider the augmented error dynamic models in (20), $u_i(t) \in \mathcal{L}^2_{\mathcal{F}}([0, t_f], \mathbb{R}^{n_u})$ and the Lyapunov functions $V_i(\cdot)$ in (23) for all $i \in \mathbb{N}_{\mathrm{N}}$. By applying Lemma 1, 2, Assumption 4 and letting $\overline{v}_i(t) = 0$, we get:

$$
\begin{aligned}
\mathrm{E}\{dV_i(\overline{e}_i(t))\} &= \mathrm{E}\Bigg\{V^T_{i,\overline{e}_i}\Bigg(\left[\overline{f}_{\mathrm{o},i}(\overline{e}_i(t)) + \Delta\overline{f}_i(\overline{e}_i(t))\right] \\
&\quad + [\overline{g}_{\mathrm{o}} + \Delta\overline{g}_i(\overline{e}_i(t))]u_i(t)\Bigg) + \frac{1}{2}\overline{\sigma}^T_i(\overline{e}_i(t))V_{i,\overline{e}_i\overline{e}_i} \\
&\quad \times \overline{\sigma}_i(\overline{e}_i(t)) + \lambda_i\Big[V_i(\overline{e}_i(t) + \overline{\gamma}_i(\overline{e}_i(t))) - V_i(\overline{e}_i(t))\Big]\Bigg\}dt \\
&\leq \mathrm{E}\Bigg\{V^T_{i,\overline{e}_i}\Big[\overline{f}_{\mathrm{o},i}(\overline{e}_i(t)) + \overline{g}_{\mathrm{o}}u_i(t)\Big] + \frac{1}{2}V^T_{i,\overline{e}_i}V_{i,\overline{e}_i} + \overline{e}^T_i(t)\Delta F^T_i \\
&\quad \times \Delta F_i\overline{e}_i(t) + (\Delta\varepsilon_i)^2u^T_i(t)u_i(t) + \frac{1}{2}\overline{\sigma}^T_i(\overline{e}_i(t))V_{i,\overline{e}_i\overline{e}_i} \\
&\quad \times \overline{\sigma}_i(\overline{e}_i(t)) + \lambda_i\Big[V_i(\overline{e}_i(t) + \overline{\gamma}_i(\overline{e}_i(t))) - V_i(\overline{e}_i(t))\Big]\Bigg\}dt
\end{aligned}
$$

If the HJIs in (24) are satisfied (i.e., $\mathrm{E}\{dV_i(\overline{e}_i(t))\} < 0$), then there exists the positive scalars $c_i > 0$ such that:

$$
\begin{aligned}
\mathrm{E}\{dV_i(\overline{e}_i(t))\} &\leq -c_i\,\mathrm{E}\left\{\|\overline{e}_i(t)\|^2\right\}dt \\
&\leq -\frac{c_i}{a_i}\mathrm{E}\{V_i(\overline{e}_i(t))\}dt
\end{aligned}
$$

By solving for $\mathrm{E}\{V_i(\overline{e}_i(t))\}$ through integration starting with $t = 0$, the following inequalities will hold:

$$
\begin{aligned}
&b_i\,\mathrm{E}\left\{\|\overline{e}_i(t)\|^2\right\} \leq \mathrm{E}\{V_i(\overline{e}_i(t))\} \leq \mathrm{E}\{V_i(0)\}\exp\left\{-\frac{c_i}{a_i}t\right\} \\
&\mathrm{E}\left\{\|\overline{e}_i(t)\|^2\right\} \leq \mathrm{E}\left\{\frac{V_i(0)}{b_i}\right\}\exp\left\{-\frac{c_i}{a_i}t\right\} \qquad \forall i \in \mathbb{N}_{\mathrm{N}}
\end{aligned}
$$

therefore, the exponentially tracking control in the mean-square sense for all the augmented error dynamic models in (20) can be achieved without the consideration of exogenous inputs $\overline{v}_i(t)$. Q.E.D.

Appendix B
Proof of Theorem 2

This will be proved by the fact that all the inequalities in (26) disappear as the optimization results in (25) are achieved. Suppose the optimal solutions ρ^*_i for all $i \in \mathbb{N}_{\mathrm{N}}$ to the indirect optimization problems in (25) are found, but any of the strict inequalities $J_{\infty,i}(u_i(t)) < \rho^*_i$ in (26) still holds. Then, there exists $\rho'_i \in \mathbb{R}_{\geq 0}$ such that $\min\limits_{u_i(t)\in\mathbb{U}_i} J_{\infty,i}(u_i(t)) = \rho'_i < \rho^*_i$, which will violate $\rho^*_i = \min\limits_{u_i(t)\in\mathbb{U}_i} \rho_i$. In other words, if the optimality criteria of the indirect optimization problems in (25) have been reached, then all the inequalities in (26) would reduce to equalities. Therefore, the decentralized robust stochastic H_∞ multi-biped team formation tracking control design in (22) can be equivalently formulated as the indirect optimization problems in (25). Q.E.D.

Appendix C
Proof of Theorem 3

Consider the stochastic H_∞ tracking performances in (21), the augmented error dynamic models in (20) and the Lyapunov functions $V_i(\cdot)$ satisfied with (23) for all $i \in \mathbb{N}_{\mathrm{N}}$, we get:

$$
\begin{aligned}
&\mathrm{E}\left\{\int_0^{t_f}\overline{e}^T_i(t)\overline{Q}_i\overline{e}_i(t) + u^T_i(t)R_iu_i(t)\,dt\right\} \\
&\quad = \mathrm{E}\Bigg\{\int_0^{t_f}\overline{e}^T_i(t)\overline{Q}_i\overline{e}_i(t) + u^T_i(t)R_iu_i(t)\,dt \\
&\quad + \int_0^{t_f}dV_i(\overline{e}_i(t))\Bigg\} + \mathrm{E}\Big\{V_i(\overline{e}_i(0)) - V_i(\overline{e}_i(t_f))\Big\} \\
&\quad \leq \mathrm{E}\{V_i(\overline{e}_i(0))\} + \mathrm{E}\Bigg\{\int_0^{t_f}\overline{e}^T_i(t)\overline{Q}_i\overline{e}_i(t) + u^T_i(t)R_i \\
&\quad\quad \times u_i(t) + V^T_{i,\overline{e}_i}\Bigg\{\Big[\overline{f}_{\mathrm{o},i}(\overline{e}_i(t)) + \Delta\overline{f}_i(\overline{e}_i(t))\Big] + \Big[\overline{g}_{\mathrm{o}} \\
&\quad\quad + \Delta\overline{g}_i(\overline{e}_i(t))\Big]u_i(t) + D\overline{v}_i(t)\Bigg) + \frac{1}{2}\overline{\sigma}^T_i(\overline{e}_i(t))V_{i,\overline{e}_i\overline{e}_i} \\
&+\lambda_i\Big[V_i(\overline{e}_i(t) + \overline{\gamma}_i(\overline{e}_i(t))) - V_i(\overline{e}_i(t))\Big] + \rho_i\overline{v}^T_i(t)\overline{v}_i(t)\Bigg\}dt \\
&\quad \leq \mathrm{E}\{V_i(\overline{e}_i(0))\} + \mathrm{E}\Bigg\{\int_0^{t_f}\overline{e}^T_i(t)\left[\overline{Q}_i + \Delta F^T_i\Delta F_i\right]\overline{e}_i(t) \\
&+u^T_i(t)\Big[R_i + (\Delta\varepsilon_i)^2I_{n_u}\Big]u_i(t) + V^T_{\overline{e}_i}\Big[\overline{f}_{\mathrm{o},i}(\overline{e}_i(t)) + \overline{g}_{\mathrm{o}}u_i(t)\Big] \\
&\quad + \frac{1}{2}V^T_{i,\overline{e}_i}V_{i,\overline{e}_i} + \frac{1}{4\rho_i}V^T_{i,\overline{e}_i}DD^TV_{i,\overline{e}_i} + \frac{1}{2}\overline{\sigma}^T_i(\overline{e}_i(t))V_{\overline{e}_i\overline{e}_i} \\
&\times\overline{\sigma}_i(\overline{e}_i(t)) + \lambda_i\Big[V_i(\overline{e}_i(t) + \overline{\gamma}_i(\overline{e}_i(t))) - V_i(\overline{e}_i(t))\Big]\Bigg\}dt
\end{aligned}
$$

If the HJIs in (28) are satisfied, then the following inequalities hold:

$$
\begin{aligned}
&\mathrm{E}\left\{\int_0^{t_f}\overline{e}^T_i(t)\overline{Q}_i\overline{e}_i(t) + u^T_i(t)R_iu_i(t)\,dt\right\} \\
&\leq \mathrm{E}\{V_i(\overline{e}_i(0))\} + \rho_i\,\mathrm{E}\left\{\int_0^{t_f}\overline{v}^T_i(t)\overline{v}_i(t)\,dt\right\} \qquad \forall i \in \mathbb{N}_{\mathrm{N}}
\end{aligned}
$$

which implies that

$$J_{\infty,i}\left(u_i\left(t\right)\right)=\sup_{\substack{\overline{v}_i(t)\in\\ \mathcal{L}^2_{\mathcal{F}}\left([0,\,t_f],\,\mathbb{R}^{\overline{n_x}}\right)}}\frac{\mathrm{E}\Big\{\int_0^{t_f}\Big[\overline{e}_i^T\left(t\right)\overline{Q}_i\,\overline{e}_i\left(t\right)+u_i^T\left(t\right)\times R_i\,u_i\left(t\right)\Big]dt-V_i\left(\overline{e}_i\left(0\right)\right)\Big\}}{\mathrm{E}\left\{\int_0^{t_f}\overline{v}_i^T\left(t\right)\overline{v}_i\left(t\right)dt\right\}}\leq\rho_i$$

for all $i\in\mathbb{N}_{\mathrm{N}}$. Therefore, if the HJIs in (28) are satisfied, then the upper bounds ρ_i of stochastic H_∞ tracking performance indices $J_{\infty,i}\left(\cdot\right)$ in (21) for all $i\in\mathbb{N}_{\mathrm{N}}$ can be achieved with the corresponding decentralized control laws $u_i\left(t\right)$ and Lyapunov functions $V_i\left(\cdot\right)$. Q.E.D.

APPENDIX D

PROOF OF THEOREM 4

Consider the LPV models in (32), the decentralized state-feedback control laws in (33) with the gain matrices $K_{i,m}$ for $m\in\{1,\,\ldots,\,L_i\}$, and the Lyapunov functions $V_i\left(\overline{e}_i\left(t\right)\right)=\overline{e}_i^T\left(t\right)P_i\,\overline{e}_i\left(t\right)$ with the positive definite matrices $P_i\in\mathbb{R}^{n_e\times n_e}$ for all $i\in\mathbb{N}_{\mathrm{N}}$. Then, we get:

$$\begin{aligned}
&\mathrm{E}\left\{\int_0^{t_f}\overline{e}_i^T\left(t\right)\overline{Q}_i\,\overline{e}_i\left(t\right)+u_i^T\left(t\right)R_i\,u_i\left(t\right)dt\right\}\\
&=\mathrm{E}\bigg\{\int_0^{t_f}\overline{e}_i^T\left(t\right)\overline{Q}_i\,\overline{e}_i\left(t\right)+u_i^T\left(t\right)R_iu_i\left(t\right)dt\\
&\quad+\int_0^{t_f}dV_i\left(\overline{e}_i\left(t\right)\right)\bigg\}+\mathrm{E}\Big\{V_i\left(\overline{e}_i\left(0\right)\right)-V_i\left(\overline{e}_i\left(t_f\right)\right)\Big\}\\
&\leq\mathrm{E}\left\{V_i\left(\overline{e}_i\left(0\right)\right)\right\}+\mathrm{E}\bigg\{\int_0^{t_f}\overline{e}_i^T(t)\,\overline{Q}_i\overline{e}_i\left(t\right)+\bigg(\sum_{m=1}^{L_i}h_{i,m}\left(\overline{e}_i\left(t\right)\right)\\
&\quad\times K_{i,m}\overline{e}_i\left(t\right)\bigg)^TR_i\bigg(\sum_{m=1}^{L_i}h_{i,m}\left(\overline{e}_i\left(t\right)\right)K_{i,m}\overline{e}_i\left(t\right)\bigg)\\
&\quad+\sum_{j=1}^{L_i}h_{i,j}\left(\overline{e}_i\left(t\right)\right)\bigg\{V_{i,\overline{e}_i}^T\Big(\big[\,\overline{A}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{A}_i\left(\overline{e}_i\left(t\right)\right)\\
&\quad+\Delta\overline{f}_i\left(\overline{e}_i\left(t\right)\right)\big]+\sum_{m=1}^{L_i}h_{i,m}\left(\overline{e}_i\left(t\right)\right)\left[\,\overline{g}_{\mathrm{o}}+\Delta\overline{g}_i\left(\overline{e}_i\left(t\right)\right)\right]\\
&\quad\times K_{i,m}\overline{e}_i\left(t\right)+D\overline{v}_i\left(t\right)\Big)+\frac{1}{2}\big[\,\overline{B}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{B}_i\left(\overline{e}_i\left(t\right)\right)\big]^T\\
&\quad\times V_{i,\overline{e}_i\overline{e}_i}\big[\,\overline{B}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{B}_i\left(\overline{e}_i\left(t\right)\right)\big]+\lambda_i\Big[V_i\Big(\overline{e}_i\left(t\right)\\
&\quad+\big[\,\overline{C}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{C}_i\overline{e}_i\left(t\right)\big]\Big)-V_i\left(\overline{e}_i\left(t\right)\right)\Big]\bigg\}\bigg\}dt\\
&\leq\mathrm{E}\left\{V_i\left(\overline{e}_i\left(0\right)\right)\right\}+\sum_{j=1}^{L_i}\sum_{m=1}^{L_i}h_{i,j}\left(\overline{e}_i\left(t\right)\right)h_{i,m}\left(\overline{e}_i\left(t\right)\right)\\
&\quad\times\mathrm{E}\bigg\{\int_0^{t_f}\overline{e}_i^T\left(t\right)\overline{Q}_i\,\overline{e}_i\left(t\right)+\overline{e}_i^T\left(t\right)K_{i,m}^TR_iK_{i,m}\overline{e}_i\left(t\right)\\
&\quad+2\overline{e}_i^T\left(t\right)P_i\Big(\big[\,\overline{A}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{A}_i\left(\overline{e}_i\left(t\right)\right)+\Delta\overline{f}_i\left(\overline{e}_i\left(t\right)\right)\big]\\
&\quad+\Big[\overline{g}_{\mathrm{o}}+\Delta\overline{g}_i\left(\overline{e}_i\left(t\right)\right)\Big]K_{i,m}\overline{e}_i\left(t\right)+D\overline{v}_i\left(t\right)\Big)\\
&\quad+\big[\,\overline{B}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{B}_i\left(\overline{e}_i\left(t\right)\right)\big]^TP_i\big[\,\overline{B}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{B}_i\left(\overline{e}_i\left(t\right)\right)\big]\\
&\quad+\lambda_i\Big(\overline{e}_i^T\left(t\right)P_i\big[\,\overline{C}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{C}_i\left(\overline{e}_i\left(t\right)\right)\big]\\
&\quad+\big[\,\overline{C}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{C}_i\left(\overline{e}_i\left(t\right)\right)\big]^TP_i\overline{e}_i\left(t\right)+\big[\,\overline{C}_{i,j}\,\overline{e}_i\left(t\right)\\
&\quad+\widetilde{C}_i\left(\overline{e}_i\left(t\right)\right)\big]^TP_i\big[\,\overline{C}_{i,j}\,\overline{e}_i\left(t\right)+\widetilde{C}_i\left(\overline{e}_i\left(t\right)\right)\big]\Big)\bigg\}dt\\
&\leq\mathrm{E}\left\{V_i\left(\overline{e}_i\left(0\right)\right)\right\}+\sum_{j=1}^{L_i}\sum_{m=1}^{L_i}h_{i,j}\left(\overline{e}_i\left(t\right)\right)h_{i,m}\left(\overline{e}_i\left(t\right)\right)\\
&\quad\times\mathrm{E}\bigg\{\int_0^{t_f}\overline{e}_i^T\left(t\right)\Big(\overline{Q}_i+K_{i,m}^T\Big[R_i+\left(\Delta\varepsilon_i\right)^2I_{n_u}\Big]K_{i,m}\\
&\quad+P_i\,\overline{A}_{i,j}+\overline{A}_{i,j}^TP_i+P_i\overline{g}_{\mathrm{o}}K_{i,m}+K_{i,m}^T\overline{g}_{\mathrm{o}}^TP_i\\
&\quad+\left(3+\lambda_i\right)P_iP_i+\rho_i^{-1}P_iD^TDP_i+2\overline{B}_{i,j}^TP_i\overline{B}_{i,j}\\
&\quad+\Delta A_i^T\Delta A_i+\left(1+2\overline{\mathrm{eig}}\left(P_i\right)\right)\Delta B_i^T\Delta B_i+\Delta F_i^T\Delta F_i\\
&\quad+\lambda_i\Big[P_i\overline{C}_{i,j}+\overline{C}_{i,j}^TP_i+2\overline{C}_{i,j}^TP_i\overline{C}_{i,j}+2\left(1+\overline{\mathrm{eig}}\left(P_i\right)\right)\\
&\quad\times\Delta C_i^T\Delta C_i\Big]\Big)\overline{e}_i\left(t\right)+\rho_i\overline{v}_i^T\left(t\right)\overline{v}_i\left(t\right)\bigg\}dt
\end{aligned}$$

If the Riccati-like matrix inequalities in (36) are satisfied, then the following inequalities hold:

$$\begin{aligned}
&\mathrm{E}\left\{\int_0^{t_f}\overline{e}_i^T\left(t\right)\overline{Q}_i\,\overline{e}_i\left(t\right)+u_i^T\left(t\right)R_i\,u_i\left(t\right)dt\right\}\\
&\leq\mathrm{E}\left\{V_i\left(\overline{e}_i\left(0\right)\right)\right\}+\rho_i\,\mathrm{E}\left\{\int_0^{t_f}\overline{v}_i^T\left(t\right)\overline{v}_i\left(t\right)dt\right\}\quad\forall i\in\mathbb{N}_{\mathrm{N}}
\end{aligned}$$

which implies that

$$J_{\infty,i}\left(u_i\left(t\right)\right)=\sup_{\substack{\overline{v}_i(t)\in\\ \mathcal{L}^2_{\mathcal{F}}\left([0,\,t_f],\,\mathbb{R}^{\overline{n_x}}\right)}}\frac{\mathrm{E}\Big\{\int_0^{t_f}\Big[\overline{e}_i^T\left(t\right)\overline{Q}_i\,\overline{e}_i\left(t\right)+u_i^T\left(t\right)\times R_i\,u_i\left(t\right)\Big]dt-V_i\left(\overline{e}_i\left(0\right)\right)\Big\}}{\mathrm{E}\left\{\int_0^{t_f}\overline{v}_i^T\left(t\right)\overline{v}_i\left(t\right)dt\right\}}\leq\rho_i$$

Therefore, if the Riccati-like matrix inequalities-constrained optimization problems in (35) and (36) are solved, the decentralized stochastic H_∞ team formation tracking performances for $i\in\mathbb{N}_{\mathrm{N}}$ can be achieved. Q.E.D.

APPENDIX E

PROOF OF COROLLARY 1

Similar to the proof of Theorem 1, with the quadratic Lyapunov functions in the form (34), it is obvious that for the optimal solutions to the optimization problems in (35), the following inequalities hold:

$$\begin{aligned}\overline{\mathrm{eig}}\left(P_i^*\right)\left\|\bar{e}_i\left(t\right)\right\|^2 \geq V_i^*\left(\bar{e}_i\left(t\right)\right) = \bar{e}_i^T\left(t\right)P_i^*\,\bar{e}_i\left(t\right)\\ \geq \underline{\mathrm{eig}}\left(P_i^*\right)\left\|\bar{e}_i\left(t\right)\right\|^2 \quad \forall i \in \mathbb{N}_{\mathrm{N}}\end{aligned}$$

Let the biped robot system exogenous inputs $\bar{v}_i\left(t\right) = 0$ for all $i \in \mathbb{N}_{\mathrm{N}}$. From Lemma 1, the Itô-Lévy formula of $\mathrm{E}\{V_i^*\left(\bar{e}_i\left(t\right)\right)\}$ with the LPV model in (32) is given as follows:

$$\begin{aligned}\mathrm{E}\left\{dV_i^*\left(\bar{e}_i\left(t\right)\right)\right\} = \mathrm{E}\Bigg\{&\sum_{j=1}^{L_i}\sum_{m=1}^{L_i} h_{i,j}\left(\bar{e}_i\left(t\right)\right)h_{i,m}\left(\bar{e}_i\left(t\right)\right)\\ &\times \bar{e}_i^T\left(t\right)\Big(\left(\Delta\varepsilon_i\right)^2 K_{i,m}^{*T}K_{i,m}^* + P_i^*\,\overline{A}_{i,j} + \overline{A}_{i,j}^TP_i^*\\ &+ P_i^*\bar{g}_{\mathrm{o}}K_{i,m}^* + K_{i,m}^{*T}\bar{g}_{\mathrm{o}}^TP_i^* + \left(3+\lambda_i\right)P_i^*P_i^*\\ &+ \rho_i^{*-1}P_i^*DD^TP_i^* + 2\overline{B}_{i,j}^TP_i^*\overline{B}_{i,j}\\ &+ \Delta A_i^T\Delta A_i + \left[1 + 2\overline{\mathrm{eig}}\left(P_i^*\right)\right]\Delta B_i^T\Delta B_i\\ &+ \Delta F_i^T\Delta F_i + \lambda_i\Big[P_i^*\overline{C}_{i,j} + \overline{C}_{i,j}^TP_i^*\\ &+ 2\overline{C}_{i,j}^TP_i^*\overline{C}_{i,j} + 2\left[1 + \overline{\mathrm{eig}}\left(P_i^*\right)\right]\\ &\times \Delta C_i^T\Delta C_i\Big]\Big)\bar{e}_i\left(t\right)\Bigg\}dt \quad \forall i \in \mathbb{N}_{\mathrm{N}}\end{aligned}$$

If the Riccati-like matrix inequalities in (36) are satisfied with the optimal solutions of matrices P_i^* and $K_{i,m}^*$ for all $m \in \{1,\ \ldots,\ L_i\}$, $i \in \mathbb{N}_{\mathrm{N}}$, then there exist the positive scalars c_i for $\mathrm{E}\left\{dV_i^*\left(\bar{e}_i\left(t\right)\right)\right\}$ such that:

$$\begin{aligned}\mathrm{E}\left\{dV_i^*\left(\bar{e}_i\left(t\right)\right)\right\} &\leq -c_i\mathrm{E}\left\{\left\|\bar{e}_i\left(t\right)\right\|\right\}dt\\ &\leq \frac{-c_i}{\overline{\mathrm{eig}}\left(P_i^*\right)}\mathrm{E}\left\{V_i^*\left(\bar{e}_i\left(t\right)\right)\right\}dt\end{aligned}$$

then we get

$$\mathrm{E}\left\{\left\|\bar{e}_i\left(t\right)\right\|^2\right\} \leq \mathrm{E}\left\{\frac{V_i^*\left(0\right)}{\underline{\mathrm{eig}}\left(P_i^*\right)}\right\}\exp\left\{\frac{-c_i}{\overline{\mathrm{eig}}\left(P_i^*\right)}t\right\}$$

which implies that the team formation tracking error vectors and the redundent joint tracking error vectors are exponentially approach to the zero vector in the mean-square sense. Q.E.D.

APPENDIX F

PROOF OF THEOREM 5

Let $W_i = P_i^{-1}$, $Y_{i,m} = K_{i,m}W_i$ for $m \in \{1,\ \ldots,\ L_i\}$ and $\kappa_i \in \mathbb{R}_{\geq 0}$ satisfied with $\kappa_i \leq \underline{\mathrm{eig}}\left(W_i\right)$ for all $i \in \mathbb{N}_{\mathrm{N}}$. After performing multiplication of W_i to the both side of ith Riccati-like matrix inequalities in (36), and replacing the terms of $\overline{\mathrm{eig}}\left(P_i\right)$ in the obtained matrix inequality with their upper bounds κ_i^{-1} for $i \in \mathbb{N}_{\mathrm{N}}$, it is given that:

$$\begin{aligned}&W_i\overline{Q}_iW_i + Y_{i,m}^T\Big[R_i + \left(\Delta\varepsilon_i\right)^2 I_{n_u}\Big]Y_{i,m} + \overline{A}_{i,j}W_i + W_i\overline{A}_{i,j}^T\\ &+ \bar{g}_{\mathrm{o}}Y_{i,m} + Y_{i,m}^T\bar{g}_{\mathrm{o}}^T + 2W_i\overline{B}_{i,j}^TW_i^{-1}\overline{B}_{i,j}W_i + \rho_i^{-1}DD^T\\ &+ \big(3+\lambda_i\big)I_{n_e} + W_i\Delta A_i^T\Delta A_iW_i + W_i\Delta F_i^T\Delta F_iW_i\\ &+ W_i\Delta B_i^T\left[1 + 2\kappa_i^{-1}\right]\Delta B_iW_i + \lambda_i\Big[\overline{C}_{i,j}W_i + W_i\overline{C}_{i,j}^T\\ &+ 2W_i\overline{C}_{i,j}^TW_i^{-1}\overline{C}_{i,j}W_i + W_i\Delta C_i^T + W_i\Delta C_i^T\Big] \prec 0\\ &\forall j,m \in \{1,\ \ldots,\ L_i\} \quad (1)\end{aligned}$$

By Schur complement [39], the following LMIs equivalent to (1) are proposed:

$$\begin{bmatrix} \Xi_{i,j,m} & * & * & * & * & * & * & * & * & * & * \\ Y_{i,m} & -\overline{R}_i^{-1} & * & * & * & * & * & * & * & * & * \\ \overline{B}_{i,j}W_i & 0_{n_e\times n_e} & -\frac{1}{2}W_i & * & * & * & * & * & * & * & * \\ \overline{C}_{i,j}W_i & \vdots & \ddots & -\frac{\lambda_i}{2}W_i & * & * & * & * & * & * & * \\ \widetilde{Q}_i^{\frac{1}{2}}MW_i & & & & -I_{n_e} & * & * & * & * & * & * \\ \Delta A_iW_i & & & & & -I_{n_e} & * & * & * & * & * \\ \Delta F_iW_i & & & & & & -I_{n_e} & * & * & * & * \\ \Delta B_iW_i & & & & & & & -I_{n_e} & * & * & * \\ \Delta B_iW_i & & & & & & & & -\frac{\kappa_i}{2}I_{n_e} & * & * \\ \Delta C_iW_i & \vdots & & & & & & & \ddots & -\frac{\lambda_i}{2}I_{n_e} & * \\ \Delta C_iW_i & 0_{n_e\times n_e} & \cdots & & & & & & \cdots & 0_{n_e\times n_e} & -\frac{\lambda_i\kappa_i}{2}I_{n_e} \end{bmatrix} \prec 0 \quad (2)$$

for all $j,m \in \{1,\ \ldots,\ L_i\}$ and $i \in \mathbb{N}_{\mathrm{N}}$, where $\Xi_{i,j,m} = \overline{A}_{i,j}W_i + W_i\overline{A}_{i,j}^T + \bar{g}_{\mathrm{o}}Y_{i,m} + Y_{i,m}^T\bar{g}_{\mathrm{o}}^T + \left(3+\lambda_i\right)I_{n_e} + \rho_i^{-1}DD^T + \lambda_i\Big[\overline{C}_{i,j}W_i + W_i\overline{C}_{i,j}^T\Big]$ and $\overline{R}_i = \Big[R_i + \left(\Delta\varepsilon_i\right)^2 I_{n_u}\Big]$; $\widetilde{Q}_i^{\frac{1}{2}}$ are the Cholesky decomposition of ith positive definite weighting matrix $\widetilde{Q}_i$ for all $i \in \mathbb{N}_{\mathrm{N}}$ in (21). Because each LMI in (2) is the sufficient condition for the negative definiteness of the corresponding Riccati-like matrix inequalities in (36), the Riccati-like matrix inequalities-constrained optimization problems in (35), (36) can be transformed into the equivalent LMIs-constrained optimization problems in (37), (38).Q.E.D.