Nonconvex Methods for Phase Retrieval

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Paper Choice

Main reference paper:
"Solving Random Quadratic Systems of Equations Is Nearly as Easy as Solving Linear Systems", by Yuxin Chen and Emmanuel Candès (2015)

Outline

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Problem Setup

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  - Let $x \in \mathbb{C}^n$ or $\in \mathbb{R}^n$ be our signal of interest.
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  - Our sensing vectors $\{a_i\}_{i=1}^m$ are known a priori.
    - Typically, a_i are a standard Gaussian ensemble (real or complex).

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  - Our sensing vectors $\{a_i\}_{i=1}^m$ are known a priori.
    - Typically, a_i are a standard Gaussian ensemble (real or complex).
- We can additionally consider the case where $y_i \approx |\langle a_i, x \rangle|^2$ i.e. we have noisy measurements.

A Convex Approach

- Consider the equivalent problem where phase retrieval is cast as a matrix completion problem.
- Create the lifted problem where $x \Rightarrow X = xx^*; y = \mathcal{A}(X)$.

$$\begin{aligned} \text{minimize} \quad & \operatorname{rank}(X) \\ \text{subject to} \quad & y = \mathcal{A}(X) \\ & X \succeq 0 \end{aligned} \tag{1}$$

- However, Problem (1) is NP-hard in general. However, the following relaxation is tractable:

$$\begin{aligned} \text{minimize} \quad & \operatorname{Tr}(X) \\ \text{subject to} \quad & y = \mathcal{A}(X) \\ & X \succeq 0 \end{aligned} \tag{2}$$

Positive: Excellent Statistical Results

- Problem (2) is referred to as *PhaseLift* (Candès, 2011), which comes with desirable statistical results.

Theorem

Consider an arbitrary signal x in $\mathbb{R}^n$ or $\mathbb{C}^n$ and suppose that the number of measurements obeys $m \geq c_0 n \log n$, where c_0 is a sufficiently large constant. Then in both the real and complex cases, the solution to the trace-minimization program is exact with high probability in the sense that (2) has a unique solution obeying

$$\hat{X} = xx^*.$$

This holds with probability $1 - 3e^{-\gamma \frac{m}{n}}$, where γ is a positive absolute constant.

Negative: Computation Time

- However, PhaseLift's accuracy comes with a high computation cost...

```
Running PhaseLift algorithm
Estimating signal of length 100 using the optimal spectral initializer with 800 measurements...
Initialization finished.
Iteration = 1 | IterationTime = 0.063222 | Residual = 9.728678e-01
Iteration = 2 | IterationTime = 0.089038 | Residual = 6.612825e-01
Iteration = 3 | IterationTime = 0.113212 | Residual = 5.667797e-01
Iteration = 4 | IterationTime = 0.136444 | Residual = 3.524499e-01
```

```
Iteration = 494 | IterationTime = 12.635114 | Residual = 1.277772e-03
Iteration = 495 | IterationTime = 12.659512 | Residual = 9.223048e-04
Signal recovery required 495 iterations (12.659512 secs)
```

```
Running PhaseLift algorithm
Estimating signal of length 1024 using the optimal spectral initializer with 8192 measurements..
Initialization finished.
Iteration = 1 | IterationTime = 21.572238 | Residual = 9.540755e-01
```

```
Iteration = 39 | IterationTime = 293.624844 | Residual = 1.466370e-01
Iteration = 40 | IterationTime = 299.433933 | Residual = 8.710826e-02
Iteration = 41 | IterationTime = 305.922098 | Residual = 4.571505e-02
Signal recovery required 41 iterations (305.922098 secs)
```

- Using nonconvex optimization, we can solve the same problem much quicker

Same Scenario with a Nonconvex Regime...

Much more reasonable!

```
Running TWF algorithm
Estimating signal of length 100 using the optimal spectra
Initialization finished.
Iter = 1 | IterationTime = 0.013 | Resid = 1.000e+00 | St
Iter = 2 | IterationTime = 0.018 | Resid = 3.525e-01 | St
Iter = 3 | IterationTime = 0.019 | Resid = 3.691e-01 | St
Iter = 4 | IterationTime = 0.021 | Resid = 5.944e-01 | St
Iter = 5 | IterationTime = 0.023 | Resid = 3.232e-01 | St
Iter = 6 | IterationTime = 0.024 | Resid = 3.187e-01 | St
Iter = 7 | IterationTime = 0.026 | Resid = 1.522e-01 | St
Iter = 8 | IterationTime = 0.027 | Resid = 6.019e-02 | St
Iter = 9 | IterationTime = 0.029 | Resid = 7.550e-02 | St
Iter = 10 | IterationTime = 0.032 | Resid = 6.070e-02 | S
Iter = 11 | IterationTime = 0.035 | Resid = 6.115e-03 | S
Iter = 12 | IterationTime = 0.036 | Resid = 7.215e-03 | S
Iter = 13 | IterationTime = 0.036 | Resid = 5.871e-03 | S
Iter = 14 | IterationTime = 0.040 | Resid = 1.574e-02 | S
Iter = 15 | IterationTime = 0.040 | Resid = 1.846e-03 | S
Iter = 16 | IterationTime = 0.041 | Resid = 3.107e-03 | S
Iter = 17 | IterationTime = 0.041 | Resid = 3.921e-03 | S
Iter = 18 | IterationTime = 0.042 | Resid = 7.544e-04 | S
Iter = 19 | IterationTime = 0.053 | Resid = 4.316e-04 | S
Iter = 20 | IterationTime = 0.057 | Resid = 2.209e-04 | S
Signal recovery required 20 iterations (0.000000 secs)
```

```
Running TWF algorithm
Estimating signal of length 1024 using the optimal spectral init
Initialization finished.
Iter = 1 | IterationTime = 0.126 | Resid = 1.000e+00 | Stepsize
Iter = 2 | IterationTime = 0.175 | Resid = 5.601e-01 | Stepsize
Iter = 3 | IterationTime = 0.224 | Resid = 4.051e-01 | Stepsize
Iter = 4 | IterationTime = 0.273 | Resid = 5.303e-01 | Stepsize
Iter = 5 | IterationTime = 0.320 | Resid = 8.231e-01 | Stepsize
Iter = 6 | IterationTime = 0.367 | Resid = 3.968e-01 | Stepsize
Iter = 7 | IterationTime = 0.415 | Resid = 1.698e-01 | Stepsize
Iter = 8 | IterationTime = 0.462 | Resid = 1.019e-01 | Stepsize
Iter = 9 | IterationTime = 0.510 | Resid = 1.662e-01 | Stepsize
Iter = 10 | IterationTime = 0.558 | Resid = 1.060e-01 | Stepsize
Iter = 11 | IterationTime = 0.607 | Resid = 5.515e-02 | Stepsize
Iter = 12 | IterationTime = 0.656 | Resid = 2.802e-02 | Stepsize
Iter = 13 | IterationTime = 0.705 | Resid = 2.041e-02 | Stepsize
Iter = 14 | IterationTime = 0.753 | Resid = 2.784e-02 | Stepsize
Iter = 15 | IterationTime = 0.800 | Resid = 2.618e-02 | Stepsize
Iter = 16 | IterationTime = 0.874 | Resid = 4.689e-03 | Stepsize
Iter = 17 | IterationTime = 0.926 | Resid = 3.575e-03 | Stepsize
Iter = 18 | IterationTime = 0.980 | Resid = 2.351e-03 | Stepsize
Iter = 19 | IterationTime = 1.025 | Resid = 9.933e-03 | Stepsize
Iter = 20 | IterationTime = 1.071 | Resid = 4.090e-03 | Stepsize
Iter = 21 | IterationTime = 1.207 | Resid = 8.782e-04 | Stepsize
Iter = 22 | IterationTime = 1.342 | Resid = 1.550e-04 | Stepsize
Iter = 23 | IterationTime = 1.467 | Resid = 2.310e-04 | Stepsize
Signal recovery required 23 iterations (0.000000 secs)
```

Outline

Nonconvex Optimization Setup

- As we know, nonconvex optimization in general is NP-hard.
- However, with proper problem formulation, we can create a tractable problem. We break it down into the two following parts:
  1. a careful initialization (spectral initialization)
  2. a series of updates refining this initial estimate by iteratively applying an update rule, (a gradient descent based scheme).
- The combination of standard spectral initalization with a gradient descent scheme is known as *Wirtinger Flow* (Candès, 2014).

Wirtinger Flow

- First, we initialize with *Spectral Initialization*
- Then, we use a gradient descent scheme
- Let $f(z) := \frac{1}{2m} \sum_{i=1}^{m} \left(a_i^\top z - y_i\right)^2$

Algorithm 1 Wirtinger flow for phase retrieval

Input: $\{\boldsymbol{a}_j\}_{1\le j\le m}$ and $\{y_j\}_{1\le j\le m}$.
Spectral initialization: Let $\lambda_1(\boldsymbol{Y})$ and $\widetilde{\boldsymbol{x}}^0$ be the leading eigenvalue and eigenvector of

$$\boldsymbol{Y} = \frac{1}{m}\sum_{j=1}^{m} y_j \boldsymbol{a}_j \boldsymbol{a}_j^\top,$$

respectively, and set $\boldsymbol{x}^0 = \sqrt{\lambda_1(\boldsymbol{Y})/3}\,\widetilde{\boldsymbol{x}}^0$.
Gradient updates: **for** $t = 0, 1, 2, \ldots, T-1$ **do**

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \eta_t \nabla f\left(\boldsymbol{x}^t\right).$$

Wirtinger Flow Setup: Exact Recovery Results

Theorem 3.3 *Let $\boldsymbol{x}$ be an arbitrary vector in $\mathbb{C}^n$ and $\boldsymbol{y} = |\boldsymbol{A}\boldsymbol{x}|^2 \in \mathbb{R}^m$ be m quadratic samples with $m \geq c_0 \cdot n \log n$, where c_0 is a sufficiently large numerical constant. Then the Wirtinger flow initial estimate $\boldsymbol{z}_0$ normalized to have squared Euclidean norm equal to $m^{-1} \sum_r y_r$,*[3] *obeys*

$$\operatorname{dist}(\boldsymbol{z}_0, \boldsymbol{x}) \leq \frac{1}{8} \|\boldsymbol{x}\| \tag{3.1}$$

with probability at least $1 - 10e^{-\gamma n} - 8/n^2$ (γ is a fixed positive numerical constant). Further, take a constant learning parameter sequence, $\mu_\tau = \mu$ for all $\tau = 1, 2, \ldots$ and assume $\mu \leq c_1/n$ for some fixed numerical constant c_1. Then there is an event of probability at least $1 - 13e^{-\gamma n} - me^{-1.5m} - 8/n^2$, such that on this event, starting from any initial solution $\boldsymbol{z}_0$ obeying (3.1), we have

$$\operatorname{dist}(\boldsymbol{z}_\tau, \boldsymbol{x}) \leq \frac{1}{8} \left(1 - \frac{\mu}{4}\right)^{\tau/2} \cdot \|\boldsymbol{x}\|.$$

Outline

Truncated Wirtinger Flow (TWF)

- Truncated Wirtinger Flow (TWF) introduces some novel techniques to improve upon Wirtinger Flow
    - A distinct objective function
    - An adaptive iterative algorithm
    - Removes outliers at each step to mitigate high impact measurements
- These improvements result in a tighter initial guess, with improved descent directions
- TWF also extends the theory to handle noisy systems, i.e. $y_i = |\langle a_i, x \rangle|^2 + \eta$ or $y \sim \text{Poisson}(|\langle a_i, x \rangle|^2)$
- TWF is capable of solving a quadratic system of equations only about 4 times slower than solving a least squares problem of the same size

TWF Setup (Informal)

1. **Initialization:** compute an initial guess $\boldsymbol{z}^{(0)}$ by means of a spectral method applied to a subset $\mathcal{T}_0$ of the observations $\{y_i\}$;

2. **Loop:** for $0 \le t < T$,

$$\boldsymbol{z}^{(t+1)} = \boldsymbol{z}^{(t)} + \frac{\mu_t}{m} \sum_{i \in \mathcal{T}_{t+1}} \nabla \ell(\boldsymbol{z}^{(t)}; y_i) \tag{7}$$

   for some index subset $\mathcal{T}_{t+1} \subseteq \{1, \cdots, m\}$ determined by $\boldsymbol{z}^{(t)}$.

Remarks:

- Both the initialization and the gradient flow are regularized in a data dependent manner
- The step size can either be taken to be a constant ($\mu_t = 0.3$ is recommended), or can be calculated using backtracking line search

TWF Setup: (Formal)

Algorithm 1 Truncated Wirtinger Flow.

Input: Measurements $\{y_i \mid 1 \le i \le m\}$ and sampling vectors $\{\boldsymbol{a}_i \mid 1 \le i \le m\}$; trimming thresholds α_z^{lb}, α_z^{ub}, α_h, and α_y (see default values in Table 1).

Initialize $\boldsymbol{z}^{(0)}$ to be $\sqrt{\frac{mn}{\sum_{i=1}^m \|\boldsymbol{a}_i\|^2}}\lambda_0\tilde{\boldsymbol{z}}$, where $\lambda_0 = \sqrt{\frac{1}{m}\sum_{i=1}^m y_i}$ and $\tilde{\boldsymbol{z}}$ is the leading eigenvector of

$$\boldsymbol{Y} = \frac{1}{m}\sum_{i=1}^m y_i \boldsymbol{a}_i \boldsymbol{a}_i^* \mathbf{1}_{\{|y_i| \le \alpha_y^2 \lambda_0^2\}}. \tag{26}$$

Loop: for $t = 0 : T$ **do**

$$\boldsymbol{z}^{(t+1)} = \boldsymbol{z}^{(t)} + \frac{2\mu_t}{m}\sum_{i=1}^m \frac{y_i - \left|\boldsymbol{a}_i^* \boldsymbol{z}^{(t)}\right|^2}{\boldsymbol{z}^{(t)*}\boldsymbol{a}_i} \boldsymbol{a}_i \mathbf{1}_{\mathcal{E}_1^i \cap \mathcal{E}_2^i}, \tag{27}$$

where

$$\mathcal{E}_1^i := \left\{ \alpha_z^{\mathrm{lb}} \le \frac{\sqrt{n}}{\|\boldsymbol{a}_i\|}\frac{|\boldsymbol{a}_i^*\boldsymbol{z}^{(t)}|}{\|\boldsymbol{z}^{(t)}\|} \le \alpha_z^{\mathrm{ub}} \right\}, \qquad \mathcal{E}_2^i := \left\{ |y_i - |\boldsymbol{a}_i^*\boldsymbol{z}^{(t)}|^2| \le \alpha_h K_t \frac{\sqrt{n}}{\|\boldsymbol{a}_i\|}\frac{|\boldsymbol{a}_i^*\boldsymbol{z}^{(t)}|}{\|\boldsymbol{z}^{(t)}\|} \right\}, \tag{28}$$

$$\text{and} \quad K_t := \frac{1}{m}\sum_{l=1}^m \left| y_l - |\boldsymbol{a}_l^* \boldsymbol{z}^{(t)}|^2 \right|.$$

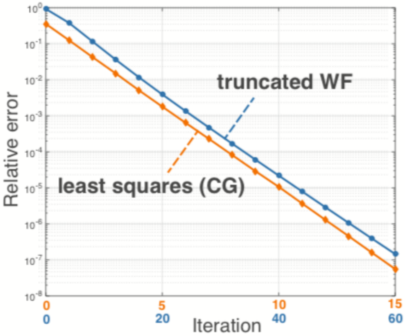
Output $\boldsymbol{z}_T$.

Outline

"Solving TWF is Nearly as Easy as Solving LS"

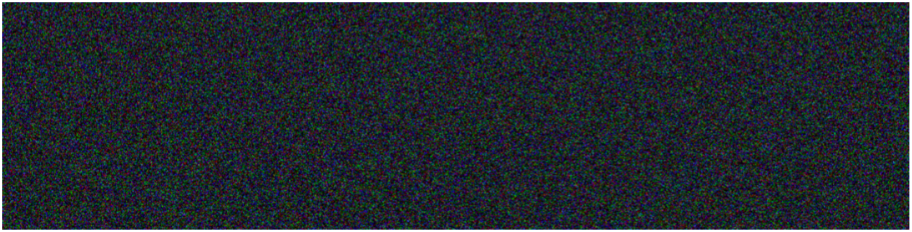
Comparing the relative error of an iterate recovered from TWF vs. the relative error of an iterate from least squares at each iteration.

- Least squares is given complete phase information, i.e. recover $x \in \mathbb{R}^n$ from $y_i = a_i^\top x$.



Impact of Truncated Spectral Initialization

The original image (top) vs. the initialization given by standard spectral initialization.



Impact of Truncated Spectral Initialization

The original image (top) vs. the initialization given by truncated spectral initialization.

Exact Recovery Results

Theorem 1 (Exact recovery). *Consider the noiseless case (1) with an arbitrary signal $\boldsymbol{x} \in \mathbb{R}^n$. Suppose that the step size μ_t is either taken to be a positive constant $\mu_t \equiv \mu$ or chosen via a backtracking line search. Then there exist some universal constants $0 < \rho, \nu < 1$ and $\mu_0, c_0, c_1, c_2 > 0$ such that with probability exceeding $1 - c_1 \exp(-c_2 m)$, the truncated Wirtinger Flow estimates (Algorithm 1 with parameters specified in Table 1) obey*

$$\operatorname{dist}(\boldsymbol{z}^{(t)}, \boldsymbol{x}) \quad \leq \quad \nu(1-\rho)^t \|\boldsymbol{x}\|, \quad \forall t \in \mathbb{N}, \tag{13}$$

provided that

$$m \geq c_0 n \quad \text{and} \quad 0 < \mu \leq \mu_0.$$

As explained below, we can often take $\mu_0 \approx 0.3$.

Improved Stability

Theorem 2 (Stability). *Consider the noisy case (14). Suppose that the step size μ_t is either taken to be a positive constant $\mu_t \equiv \mu$ or chosen via a backtracking line search. If*

$$m \geq c_0 n, \quad \mu \leq \mu_0, \quad \textit{and} \quad \|\boldsymbol{\eta}\|_\infty \leq c_1 \|\boldsymbol{x}\|^2, \tag{15}$$

then with probability at least $1 - c_2 \exp(-c_3 m)$, the truncated Wirtinger Flow estimates (Algorithm 1 with parameters specified in Table 1) satisfy

$$\mathrm{dist}(\boldsymbol{z}^{(t)}, \boldsymbol{x}) \;\lesssim\; \frac{\|\boldsymbol{\eta}\|}{\sqrt{m}\|\boldsymbol{x}\|} + (1-\rho)^t \|\boldsymbol{x}\|, \quad \forall t \in \mathbb{N} \tag{16}$$

simultanesouly for all $\boldsymbol{x} \in \mathbb{R}^n$. Here, $0 < \rho < 1$ and $\mu_0, c_0, c_1, c_2, c_3 > 0$ are some universal constants.

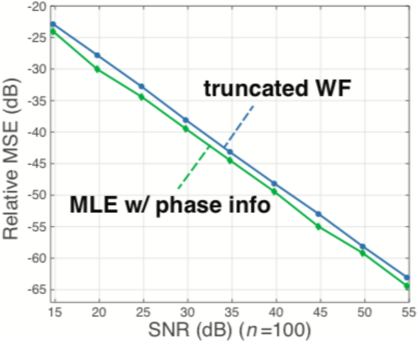
Under the Poisson noise model (4), one has

$$\mathrm{dist}(\boldsymbol{z}^{(t)}, \boldsymbol{x}) \;\lesssim\; 1 + (1-\rho)^t \|\boldsymbol{x}\|, \quad \forall t \in \mathbb{N} \tag{17}$$

with probability approaching one, provided that $\|\boldsymbol{x}\| \geq \log^{1.5} m$.

MSE vs. SNR

We compare the MSE of the output from TWF with the MSE of a Maximum Likelihood Estimate of x where the phase information is known.



≈ 1.5 dB difference in MSE independent of the SNR!

Outline

Truncated Spectral Initialization is "Good Enough"

Proposition 3. *Fix $\delta > 0$ and $\boldsymbol{x} \in \mathbb{R}^n$. Consider the model where $y_i = |\boldsymbol{a}_i^\top \boldsymbol{x}|^2 + \eta_i$ and $\boldsymbol{a}_i \overset{\text{ind.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$. Suppose that*

$$|\eta_i| \leq \varepsilon \max\{\|\boldsymbol{x}\|^2,\ |\boldsymbol{a}_i^\top \boldsymbol{x}|^2\}, \quad 1 \leq i \leq m \tag{145}$$

for some sufficiently small constant $\varepsilon > 0$. With probability exceeding $1 - \exp(-\Omega(m))$, the solution $\boldsymbol{z}^{(0)}$ returned by the truncated spectral method obeys

$$\operatorname{dist}(\boldsymbol{z}^{(0)}, \boldsymbol{x}) \leq \delta \|\boldsymbol{x}\|, \tag{146}$$

provided that $m > c_0 n$ for some constant $c_0 > 0$.

Truncated Spectral Initialization is "Good Enough"

Recall: $Y = \frac{1}{m}\sum_{i=1}^{m} y_i a_i a_i^\top$

- If we were considering the not-truncated spectral method we would...
  1. Analyze the difference between Y and $\mathbb{E}[Y]$.
  2. Apply random matrix theory to bound the difference of Y and $\mathbb{E}[Y]$ by δ.
  3. Use Davis-Kahan $\sin\Theta$ theorem to bound the difference in their top eigenvectors

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    3. Use Davis-Kahan sin Θ theorem to bound the difference in their top eigenvectors
- In the case of truncated spectral initialization, bound Y by Y_1 and Y_2, through precise theory we can create similar bounds

Regularity Condition

We would like to show that our loss function has a "convex enough" region about the global optima.

$$\left\langle \boldsymbol{h}, -\frac{1}{m}\nabla\ell_{\mathrm{tr}}(\boldsymbol{z})\right\rangle \;\geq\; \frac{\mu}{2}\left\|\frac{1}{m}\nabla\ell_{\mathrm{tr}}(\boldsymbol{z})\right\|^2 + \frac{\lambda}{2}\|\boldsymbol{h}\|^2$$

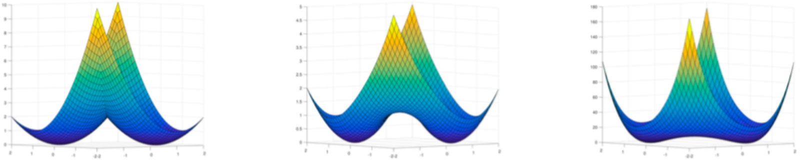
If a local regularity condition holds, we can prove that we have linear convergence.

$$\begin{aligned}
\mathrm{dist}^2\left(\boldsymbol{z} + \frac{\mu}{m}\nabla\ell_{\mathrm{tr}}(\boldsymbol{z}), \boldsymbol{x}\right) &\leq \left\|\boldsymbol{z} + \frac{\mu}{m}\nabla\ell_{\mathrm{tr}}(\boldsymbol{z}) - \boldsymbol{x}\right\|^2 \\
&= \|\boldsymbol{h}\|^2 + \left\|\frac{\mu}{m}\nabla\ell_{\mathrm{tr}}(\boldsymbol{z})\right\|^2 + 2\mu\left\langle \boldsymbol{h}, \frac{1}{m}\nabla\ell_{\mathrm{tr}}(\boldsymbol{z})\right\rangle \\
&\leq \|\boldsymbol{h}\|^2 + \left\|\frac{\mu}{m}\nabla\ell_{\mathrm{tr}}(\boldsymbol{z})\right\|^2 - \mu^2\left\|\frac{1}{m}\nabla\ell_{\mathrm{tr}}(\boldsymbol{z})\right\|^2 - \mu\lambda\|\boldsymbol{h}\|^2 \\
&= (1-\mu\lambda)\,\mathrm{dist}^2(\boldsymbol{z}, \boldsymbol{x})
\end{aligned}$$

Outline

Improvements through Reshaped Wirtinger Flow

- Recall: $y_i = |\langle a_i, x\rangle|^2$ and $\ell_{WF}(z) = \frac{1}{2m}\sum_{i=1}^{m}(|a_i^\top z|^2 - y_i)^2$.
- In Reshaped Wirtinger Flow (RWF), we adopt the loss function $\ell_{RWF}(z) = \frac{1}{2m}\sum_{i=1}^{m}(|a_i^\top z| - \sqrt{y_i})^2$
- RWF loss function has *lower order terms*

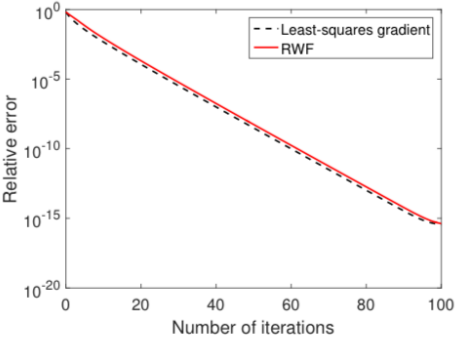


(a) Quadratic surface (b) Expected loss of RWF (c) Expected loss of WF

Figure 3: (a) Surface of quadratic function $f(z) = \min\{(\boldsymbol{z}-\boldsymbol{x})^T(\boldsymbol{z}-\boldsymbol{x}), (\boldsymbol{z}+\boldsymbol{x})^T(\boldsymbol{z}+\boldsymbol{x})\}$ with $\boldsymbol{x} = [1 -1]^T$. (b) Expected loss function of RWF for $\boldsymbol{x} = [1 -1]^T$. (c) Expected loss function of WF for $\boldsymbol{x} = [1 -1]^T$.

Convergence Improvements with RWF

- RWF takes almost the *exact same* number of interations as least-squares!



(a) Convergence behavior

RWF Algorithm

Algorithm 1 Reshaped Wirtinger Flow

Input: $\boldsymbol{y} = \{y_i\}_{i=1}^m$, $\{\boldsymbol{a}_i\}_{i=1}^m$;

Parameters: Lower and upper thresholds α_l, α_u for truncation in initialization, stepsize μ;

Initialization: Let $\boldsymbol{z}^{(0)} = \lambda_0 \tilde{\boldsymbol{z}}$, where $\lambda_0 = \frac{mn}{\sum_{i=1}^m \|\boldsymbol{a}_i\|_1} \cdot \left(\frac{1}{m}\sum_{i=1}^m y_i\right)$ and $\tilde{\boldsymbol{z}}$ is the leading eigenvector of

$$\boldsymbol{Y} := \frac{1}{m}\sum_{i=1}^m y_i \boldsymbol{a}_i \boldsymbol{a}_i^* \mathbf{1}_{\{\alpha_l \lambda_0 < y_i < \alpha_u \lambda_0\}}. \tag{6}$$

Gradient loop: for $t = 0 : T-1$ do

$$\boldsymbol{z}^{(t+1)} = \boldsymbol{z}^{(t)} - \frac{\mu}{m}\sum_{i=1}^m \left(\boldsymbol{a}_i^* \boldsymbol{z}^{(t)} - y_i \cdot \frac{\boldsymbol{a}_i^* \boldsymbol{z}^{(t)}}{|\boldsymbol{a}_i^* \boldsymbol{z}^{(t)}|}\right) \boldsymbol{a}_i. \tag{7}$$

Output $\boldsymbol{z}^{(T)}$.

Why is RWF so fast?

- Intuitively speaking, the gradient of the loss function for RWF is very similar to the gradient of a least-squares loss function (in the case where phase is known)
  - $\nabla \ell_{ls}(z) = \frac{1}{m} \sum_{i=1}^{m} (a_i^\top z - a_i^\top x) a_i$

Why is RWF so fast?

- Intuitively speaking, the gradient of the loss function for RWF is very similar to the gradient of a least-squares loss function (in the case where phase is known)
  - $\nabla \ell_{ls}(z) = \frac{1}{m}\sum_{i=1}^{m}(a_i^\top z - a_i^\top x)a_i$
- Additionally, we can prove that if we initialize close enough to the true signal, then $a_i^\top z$ has the same sign as $a_i^\top x$ for large $|a_i^\top z|$

Lemma 1. *Let $\boldsymbol{a}_i \sim \mathcal{N}(0, \boldsymbol{I}_{n\times n})$. For any given $\boldsymbol{x}$ and $\boldsymbol{z}$, independent from $\{\boldsymbol{a}_i\}_{i=1}^{m}$, satisfying $\|\boldsymbol{x}-\boldsymbol{z}\| < \frac{\sqrt{2}-1}{\sqrt{2}}\|\boldsymbol{x}\|$, we have*

$$\mathbb{P}\{(\boldsymbol{a}_i^T\boldsymbol{x})(\boldsymbol{a}_i^T\boldsymbol{z}) < 0 \big| (\boldsymbol{a}_i^T\boldsymbol{x})^2 = t\|\boldsymbol{x}\|^2\} \le erfc\left(\frac{\sqrt{t}\|\boldsymbol{x}\|}{2\|\boldsymbol{z}-\boldsymbol{x}\|}\right), \tag{11}$$

where $erfc(u) := \frac{2}{\sqrt{\pi}}\int_u^\infty \exp(-\tau^2)d\tau$.

Implicit Regularization

- Do we need regularization, as in the case of TWF?
- Recent theory shows that in the case of gradient descent, we observe a phenomenon called "implicit regularization".
- This implicit regularization feature allows gradient descent to proceed in a far more aggressive $\Rightarrow$ substantial computational savings.

Implicit Regularization

- If our iterate lies in what is referred to as a *region of incoherence and contraction*, we enjoy a much nicer geometry about the true signal

$$\|\boldsymbol{x} - \boldsymbol{x}^\natural\|_2 \le \delta \|\boldsymbol{x}^\natural\|_2 \qquad \text{and}$$
$$\max_{1 \le j \le m} \left|\boldsymbol{a}_j^\top (\boldsymbol{x} - \boldsymbol{x}^\natural)\right| \lesssim \sqrt{\log n}\, \|\boldsymbol{x}^\natural\|_2,$$

- Within this region, we know that the Hessian satisfies

$$(1/2) \cdot I_n \preceq \nabla^2 f(x) \preceq O(\log n) \cdot I_n$$

One can safely adopt a far more aggressive step size (as large as $\eta_t = O(1/\log n)$) to achieve acceleration, as long as the iterates stay within the RIC.

Implicit Regularization

- Standard gradient descent theory does not ensure that we remain within the RIC at each iteration
- However, we use a new tricked called "leave-one-out"
- The leave-one-out trick enables analysis to ensure that each iteration is within the RIC as long as it is incoherent enough with our measurement matrix

Leave one out trick

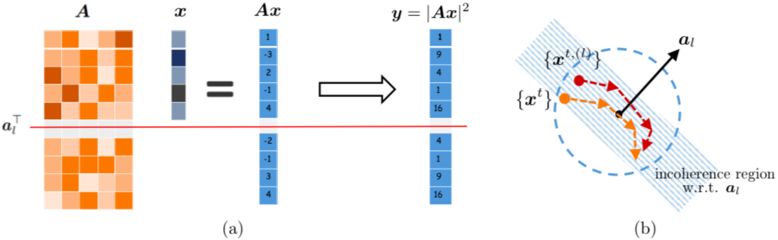


Figure 4: Illustration of the leave-one-out sequence w.r.t. a_l. (a) The sequence $\{x^{t,(l)}\}_{t\geq 0}$ is constructed without using the lth sample. (b) Since the auxiliary sequence $\{x^{t,(l)}\}$ is constructed without using a_l, the leave-one-out iterates stay within the incoherence region w.r.t. a_l with high probability. Meanwhile, $\{x^t\}$ and $\{x^{t,(l)}\}$ are expected to remain close as their construction differ only in a single sample.

Other Forms of Initialization

- As we have emphasized, we would like to initialize very carefully in nonconvex optimization.
- New findings have found success in the use of deterministic *pre-processing* functions.
  - let $z_i = \mathcal{T}(y_i)$
  - Initialize by take the top eigenvector of $D = \frac{1}{m}\sum_{i=1}^{m} z_i a_i a_i^\top$
- Utilizing such pre-processing functions can allow us to decrease the number of measurements required to guarentee good initialization.

...Or do we need to be careful?

- It would great it we could initialize randomly
  - That way, our initialization can be independent of our model, therefore being robust to model mismatch.
- In fact, a lot of people use random initialization, even though the theory is poorly understood.
- Very new theory provides intuition as to why random initialization is often fine

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However, if I know that I can initialize very well with a spectral method... why wouldn't I?

Outline

Conclusion

- Nonconvex optimization for phase retrieval is a very new and vibrant field of study
- Through careful choice of loss function and initialization, nonconvex optimization problems are highly tractable and recover very accurate solutions, fast
- Truncated Wirtinger Flow introduces data-dependent regularization on both initialization and the loss function to initialization, improve convergence speed, as well as stability results.
- More recent research points towards additional open problems: can we pick better loss functions? Do we need regularization? Can we initialize better?

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