Notes – October 7, 2025

Topic: Discussion on Experimental Results of RBGS_GMRES, BGS_GMRES and built-in GMRES

1. Overview of Experiment Setup

Parameter	Description
Solver	RBGS-GMRES, BGS-GMRES, GMRES(built-in)
Runs	1
Matrices Tested	A1, A2, A3, A4
Sketch Sizes	s = 1, 5, 10, 20
Metric	Relative residual, $ \left\lVert A*Qy - b \right\rVert $

2. Experimental Results by Matrix

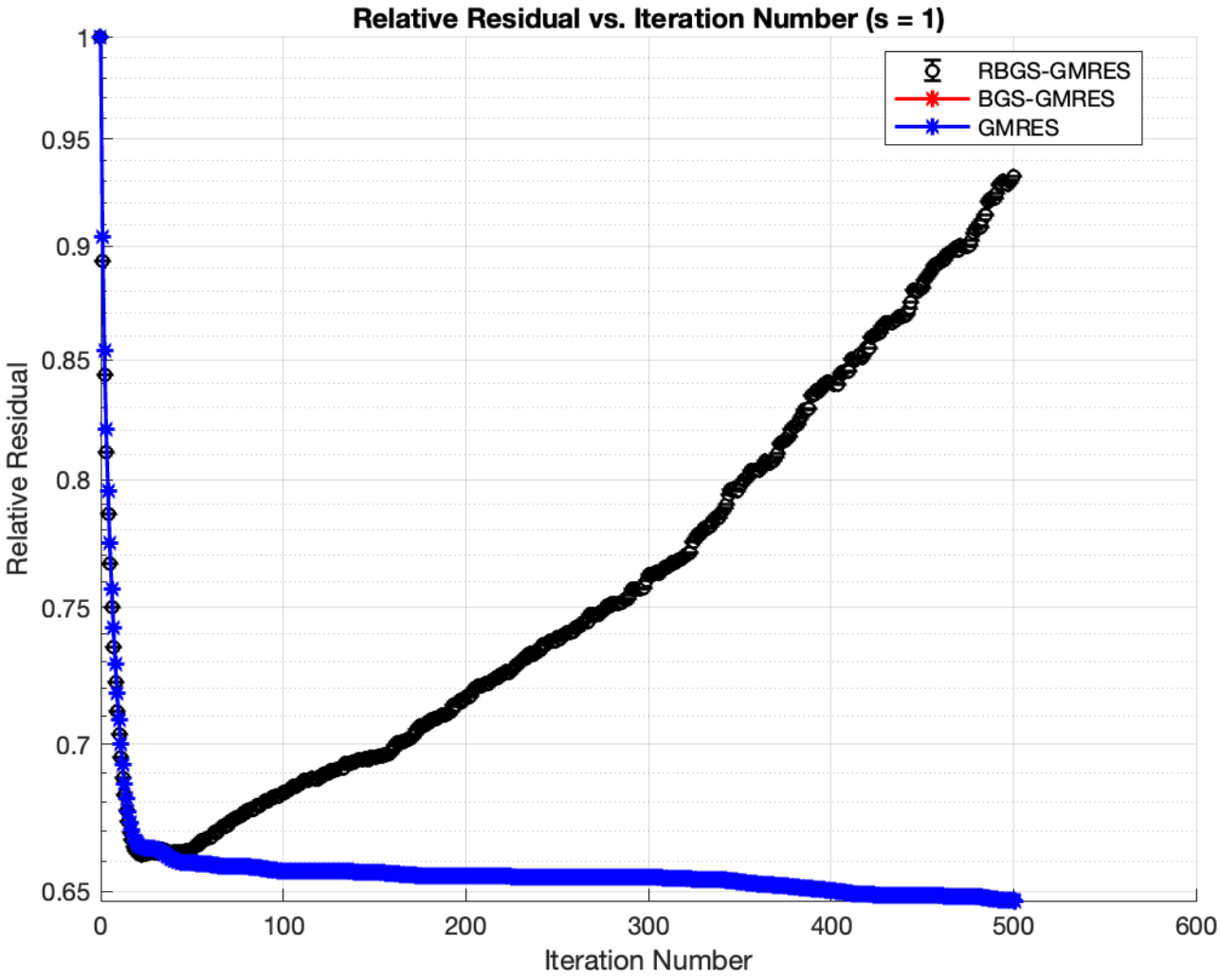
2.1 Matrix A1 – Nonsymmetric matrix

Matrix info: (n = 2500, alpha = 0.5, lambda = logspace(0,-5,n).')
Sketch info: (m = 501, d = 2 * m)
Converge info: (ctol = 1e-8)

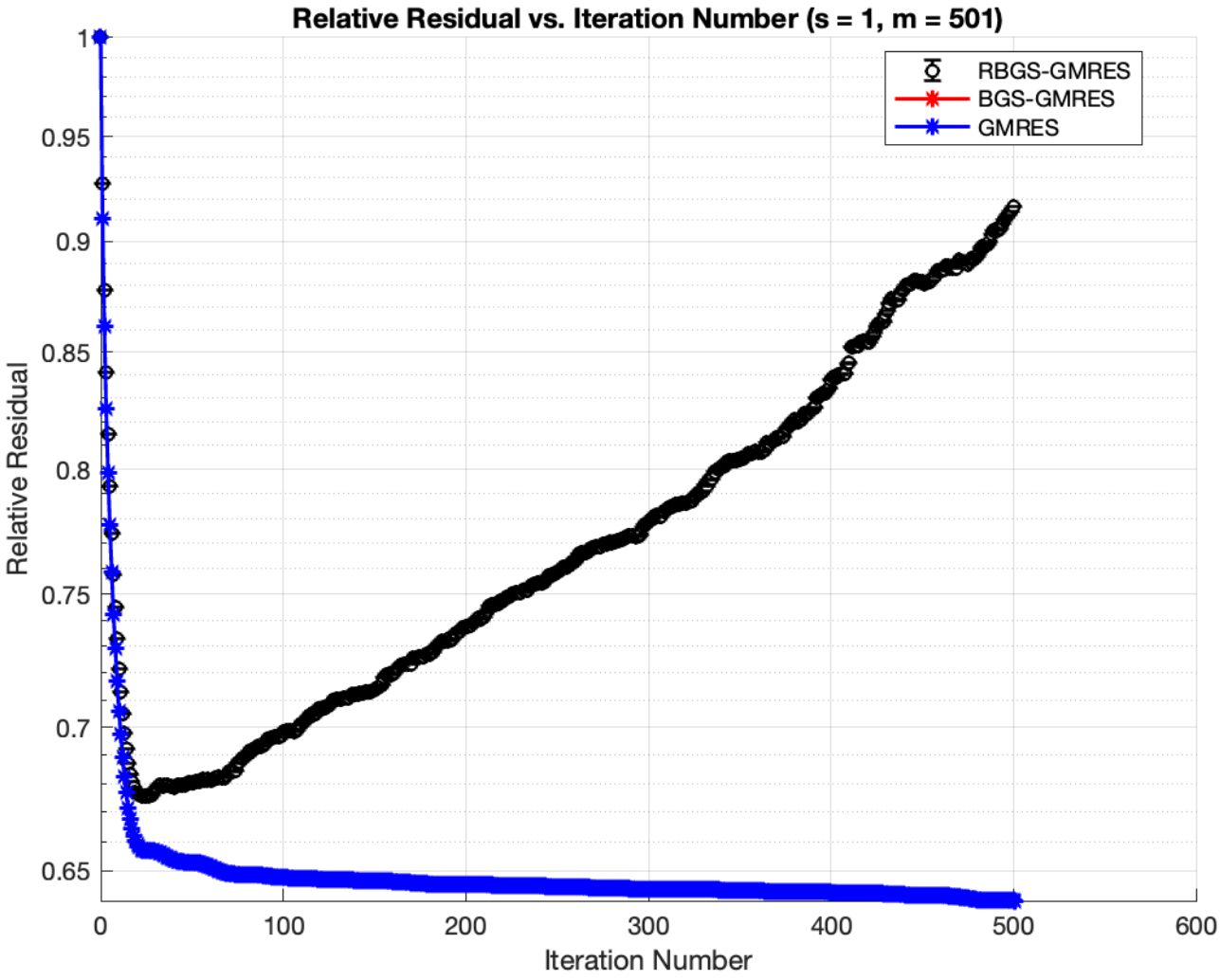
➤ Results for Step Sizes

relative residual

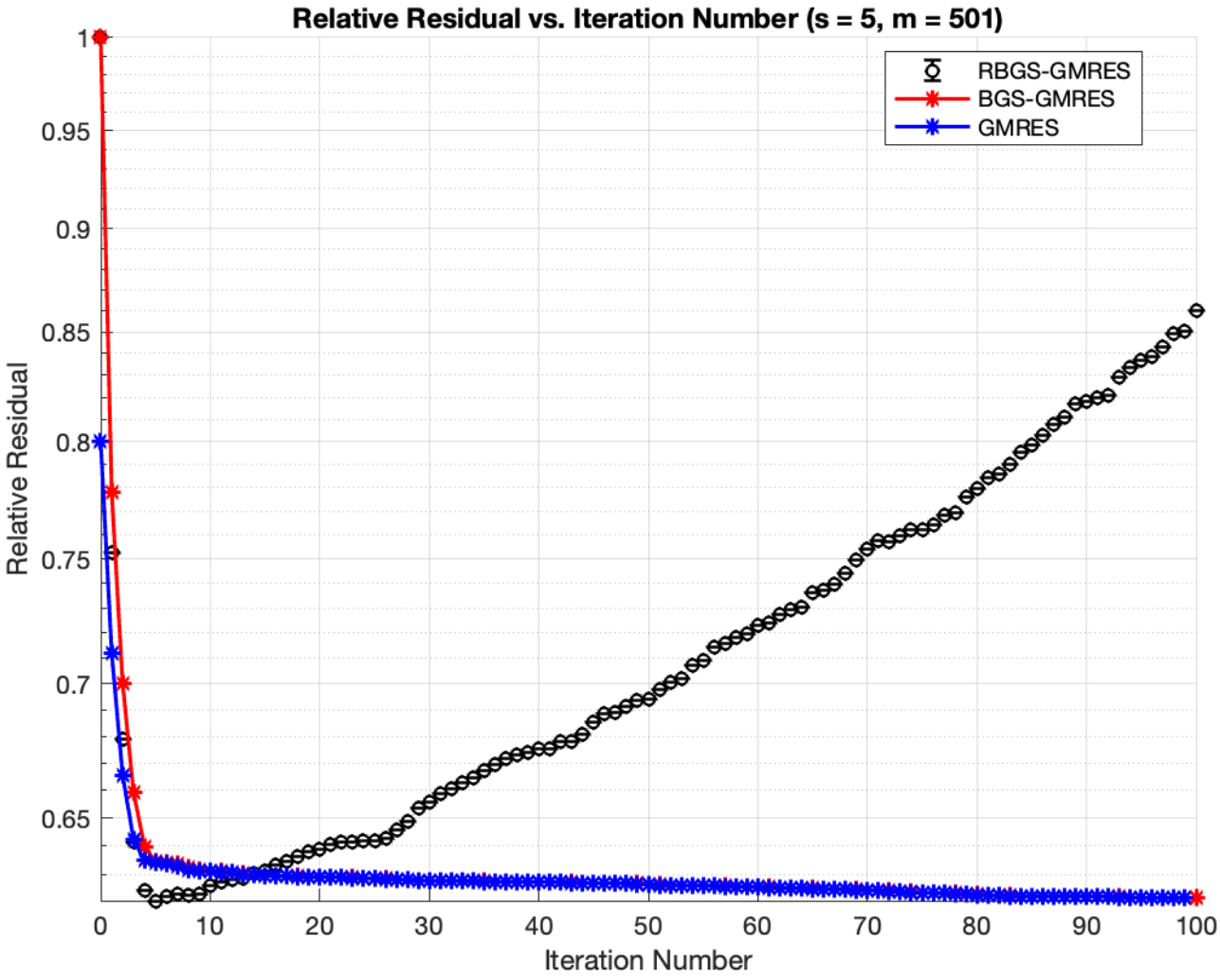
- s=1


Relative Residual vs. Iteration Number (s = 1)
RBGS-GMRES
BGS-GMRES
GMRES
Relative Residual
Iteration Number
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0
100
200
300
400
500
600

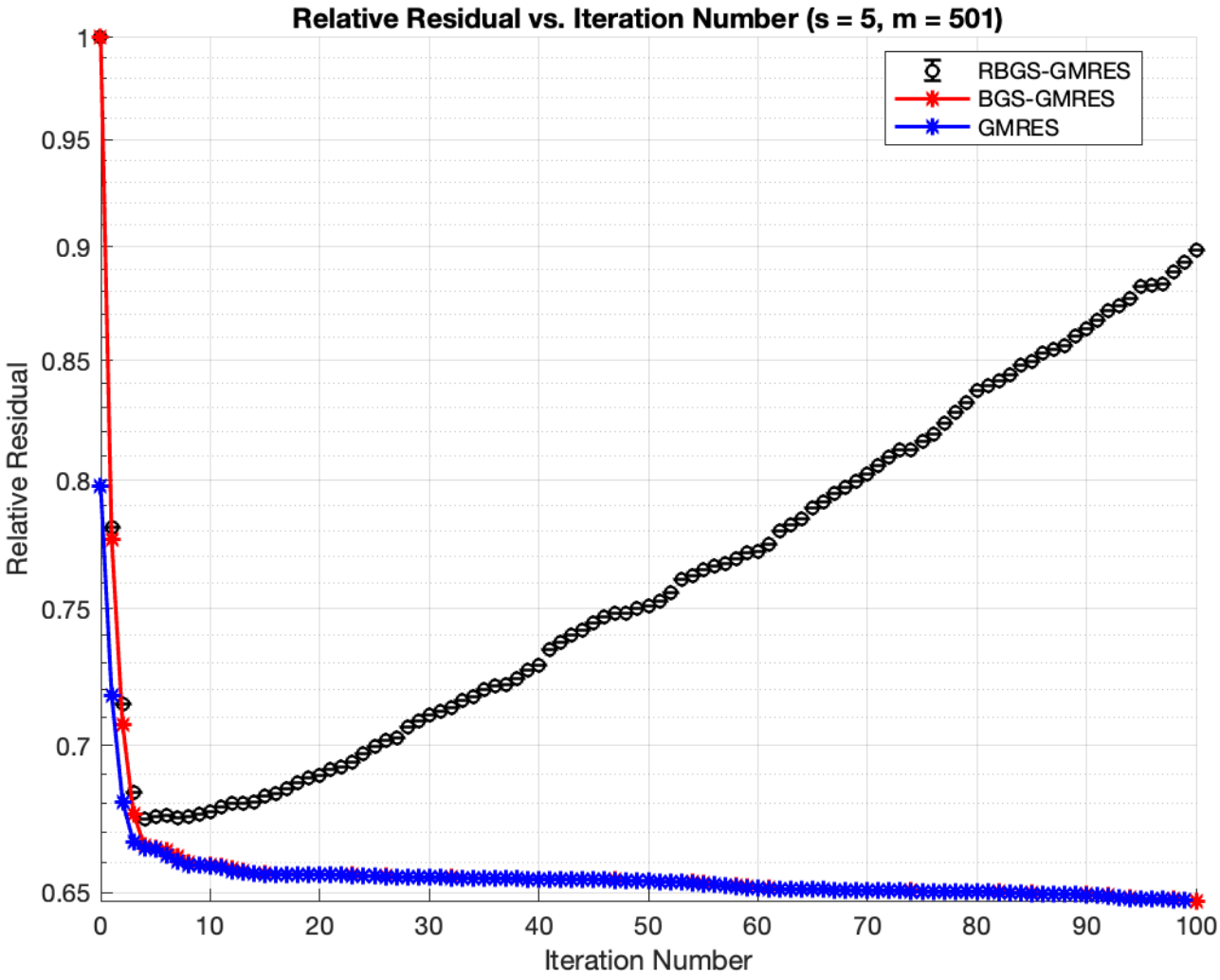
| residual after reorthogonalization



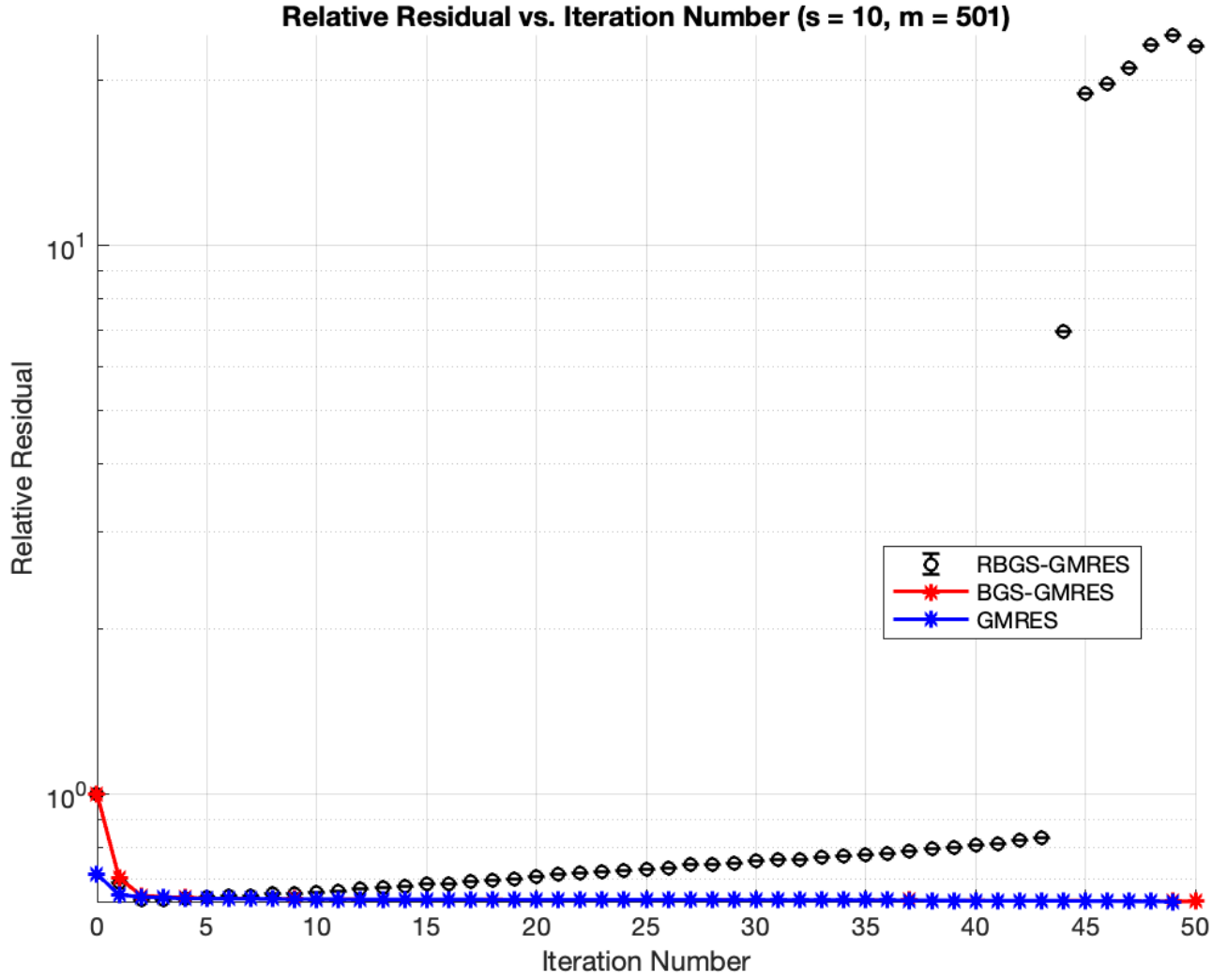
- s=5


Relative Residual vs. Iteration Number (s = 5, m = 501)
RBGS-GMRES
BGS-GMRES
GMRES
Relative Residual
Iteration Number
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0
10
20
30
40
50
60
70
80
90
100

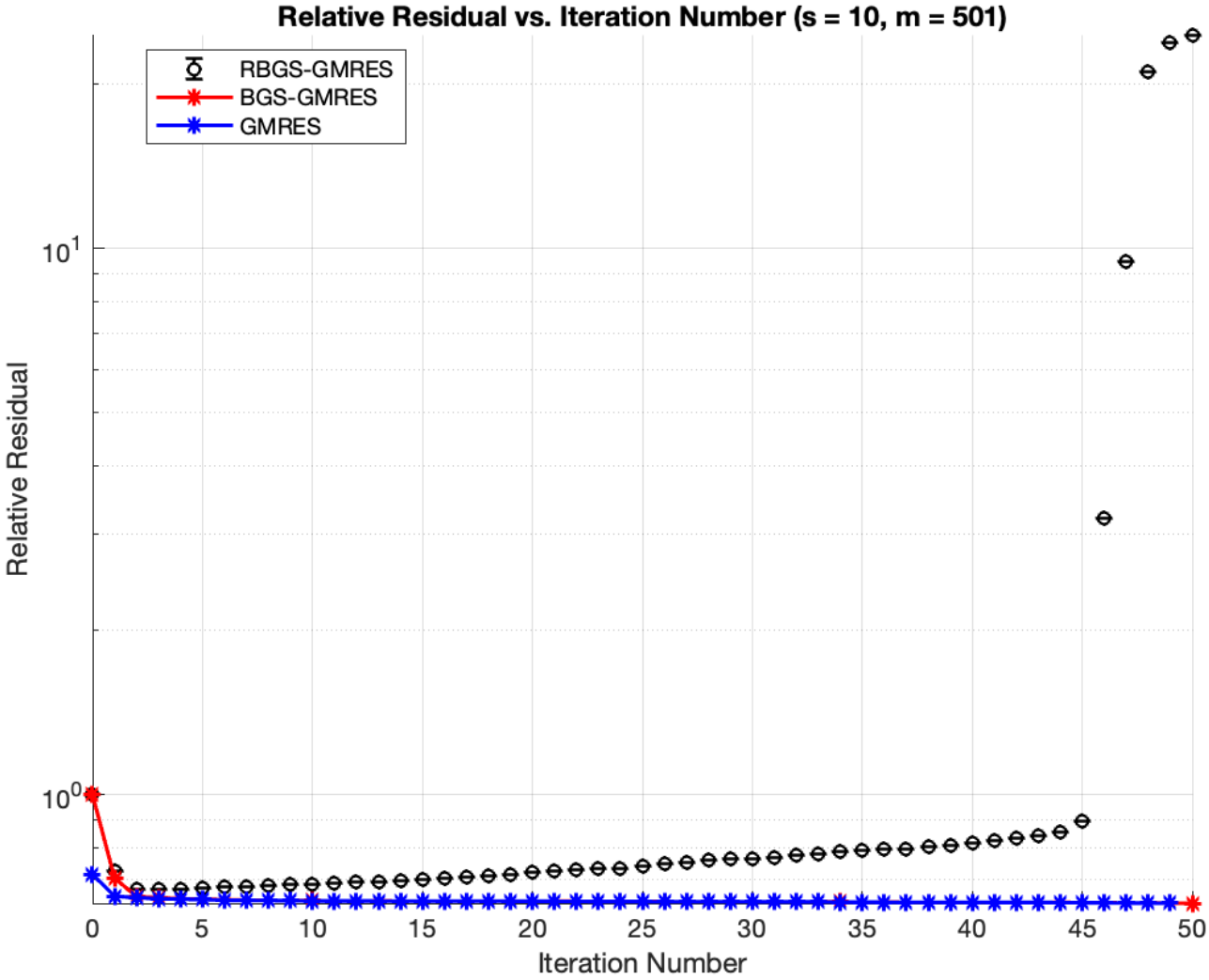
| esidual after reorthogonalization



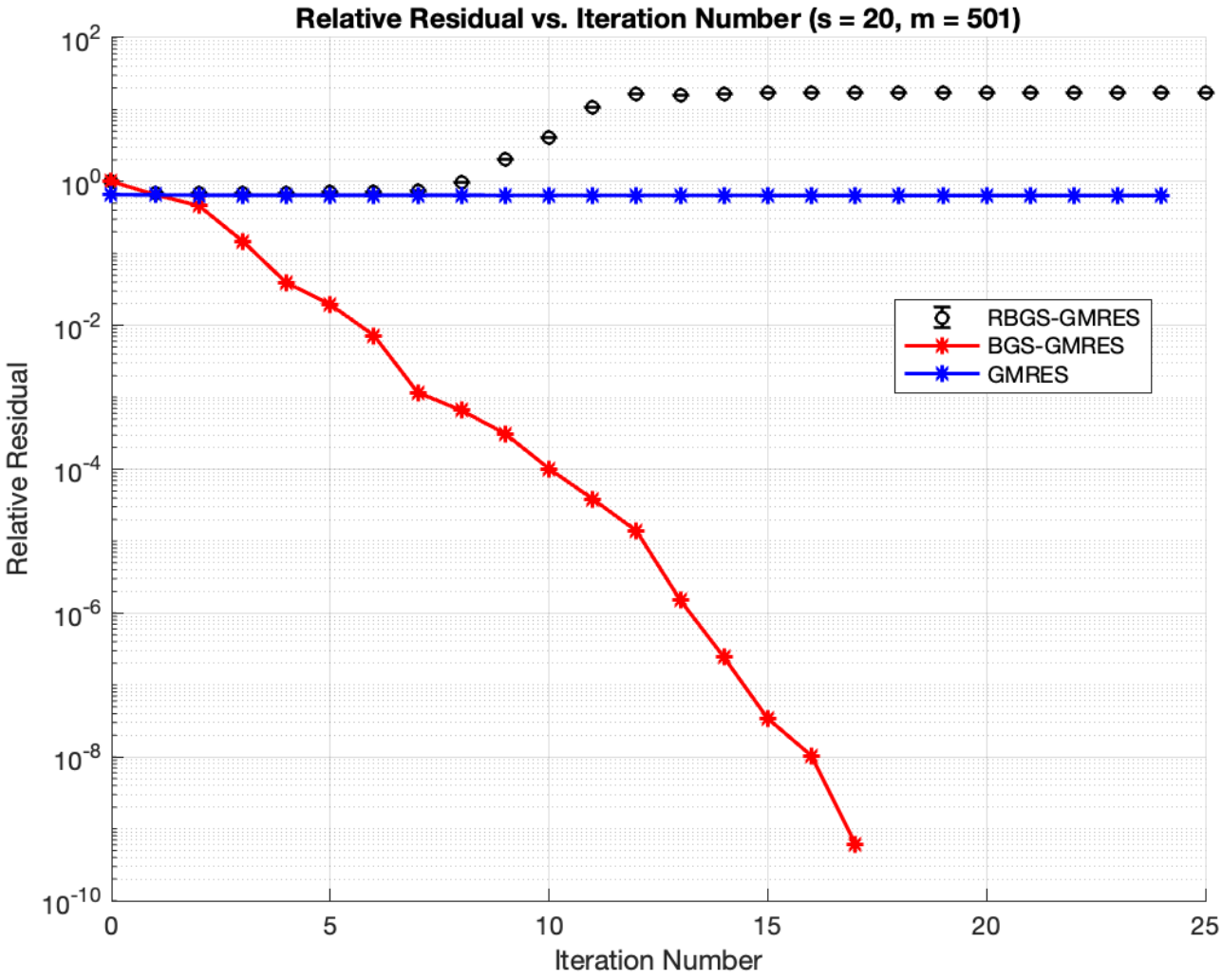
- s=10


Relative Residual vs. Iteration Number (s = 10, m = 501)
Relative Residual
Iteration Number
RBGS-GMRES
BGS-GMRES
GMRES
10^1
10^0
0
5
10
15
20
25
30
35
40
45
50

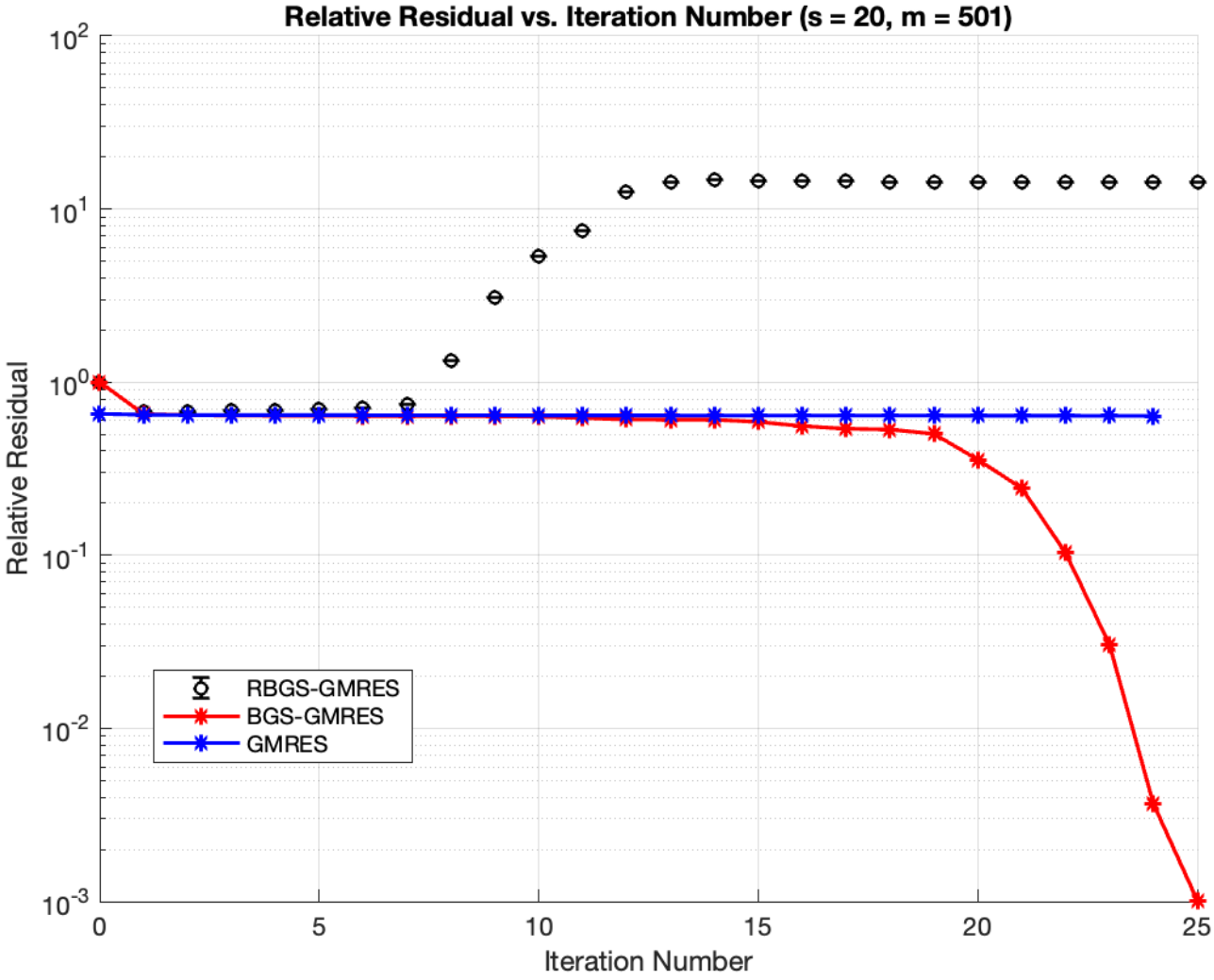
| residual after reorthogonalization



- s=20


Relative Residual vs. Iteration Number (s = 20, m = 501)
Relative Residual
Iteration Number
RBGS-GMRES
BGS-GMRES
GMRES
10^2
10^0
10^-2
10^-4
10^-6
10^-8
10^-10
0
5
10
15
20
25

|residual after reorthogonalization

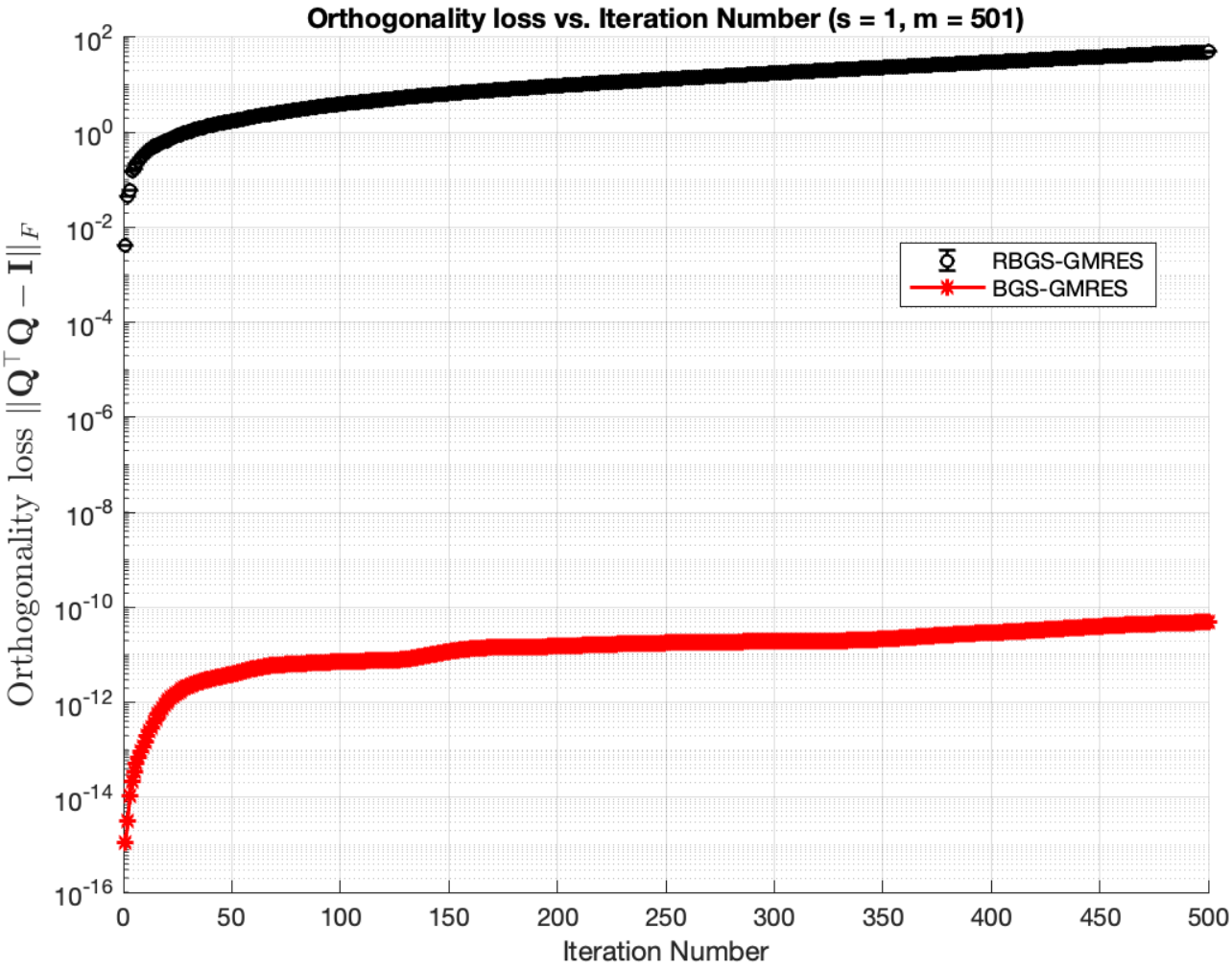


Discussion:

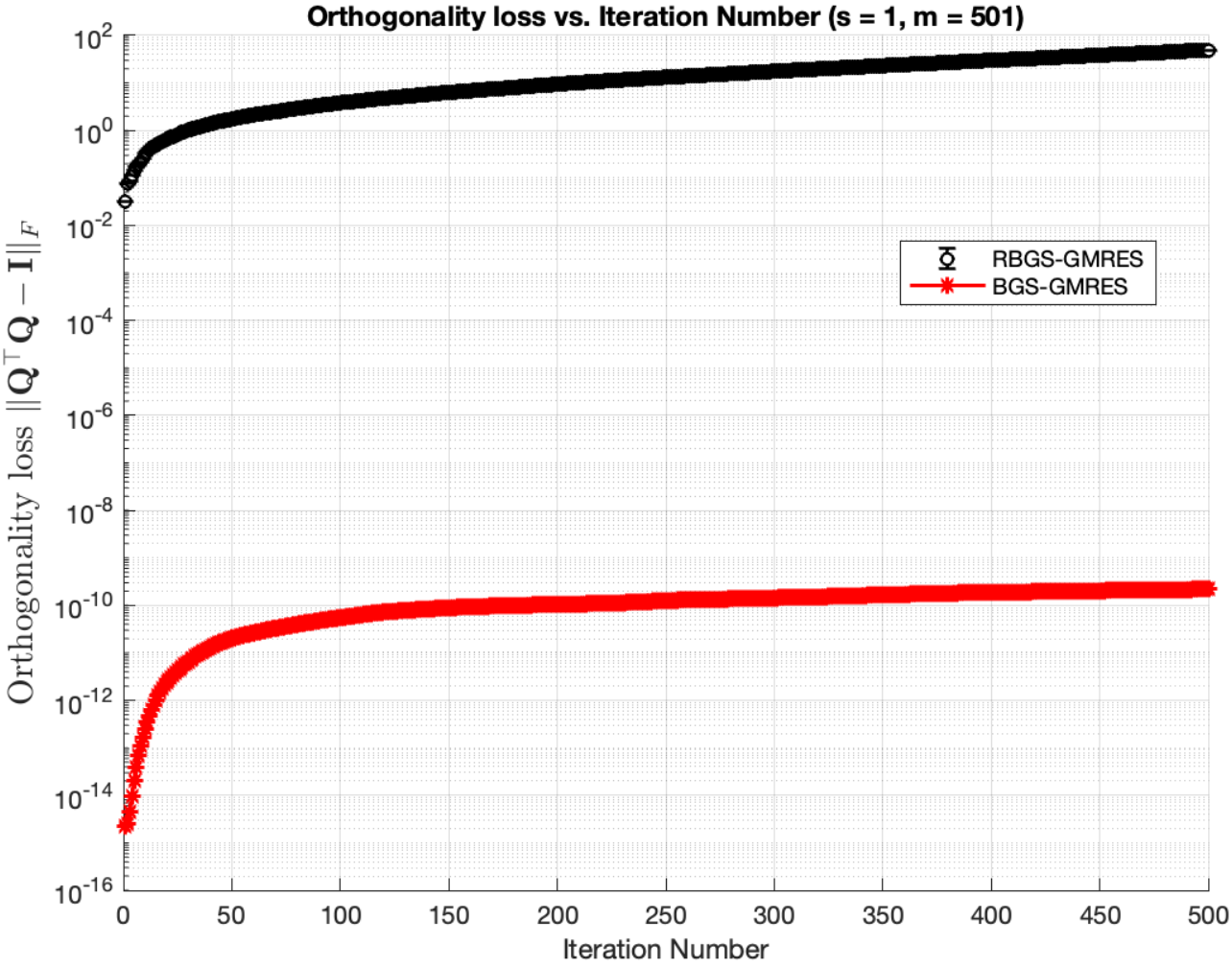
- In the implementation, m = 501 is fixed
- When s = 1, BGS_GMRES and built-in GMRES coincide, showing that BGS_GMRES is fine in implementation
- RBGS_GMRES divergences in each case
- When s increases, BGS_GMRES begins to converge, and when s = 20, BGS even detects convergence after 340 steps
- It seems that when s=10, the residual does not conform to the theoretical condition $ \left\lVert b-Ax_\star \right\rVert \le \left\lVert b-A\tilde{x} \right\rVert \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\xi} \left\lVert b-Ax_\star \right\rVert,\quad \xi=\tfrac{1}{2}. $

Check Orthogonality

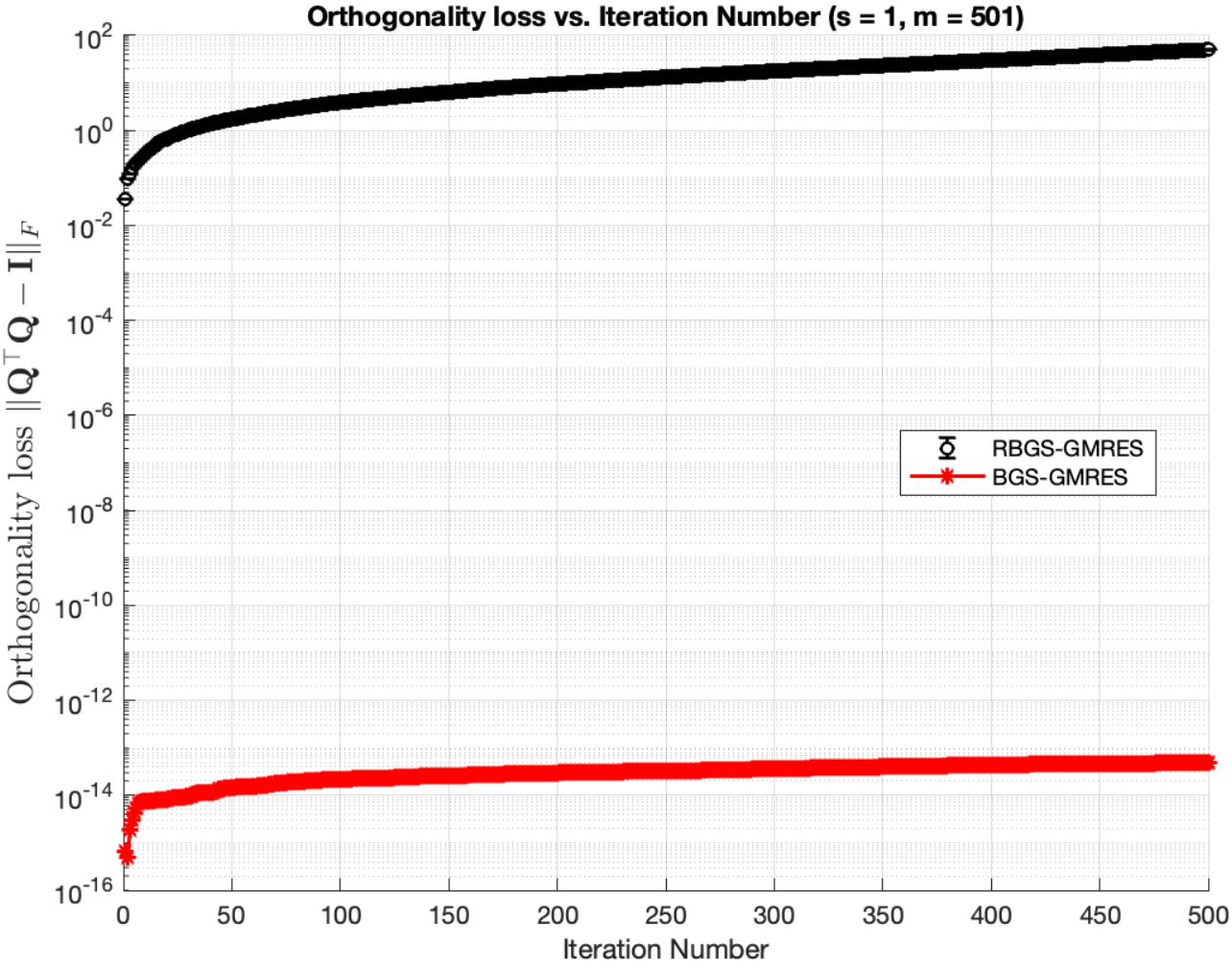
- When s=1,



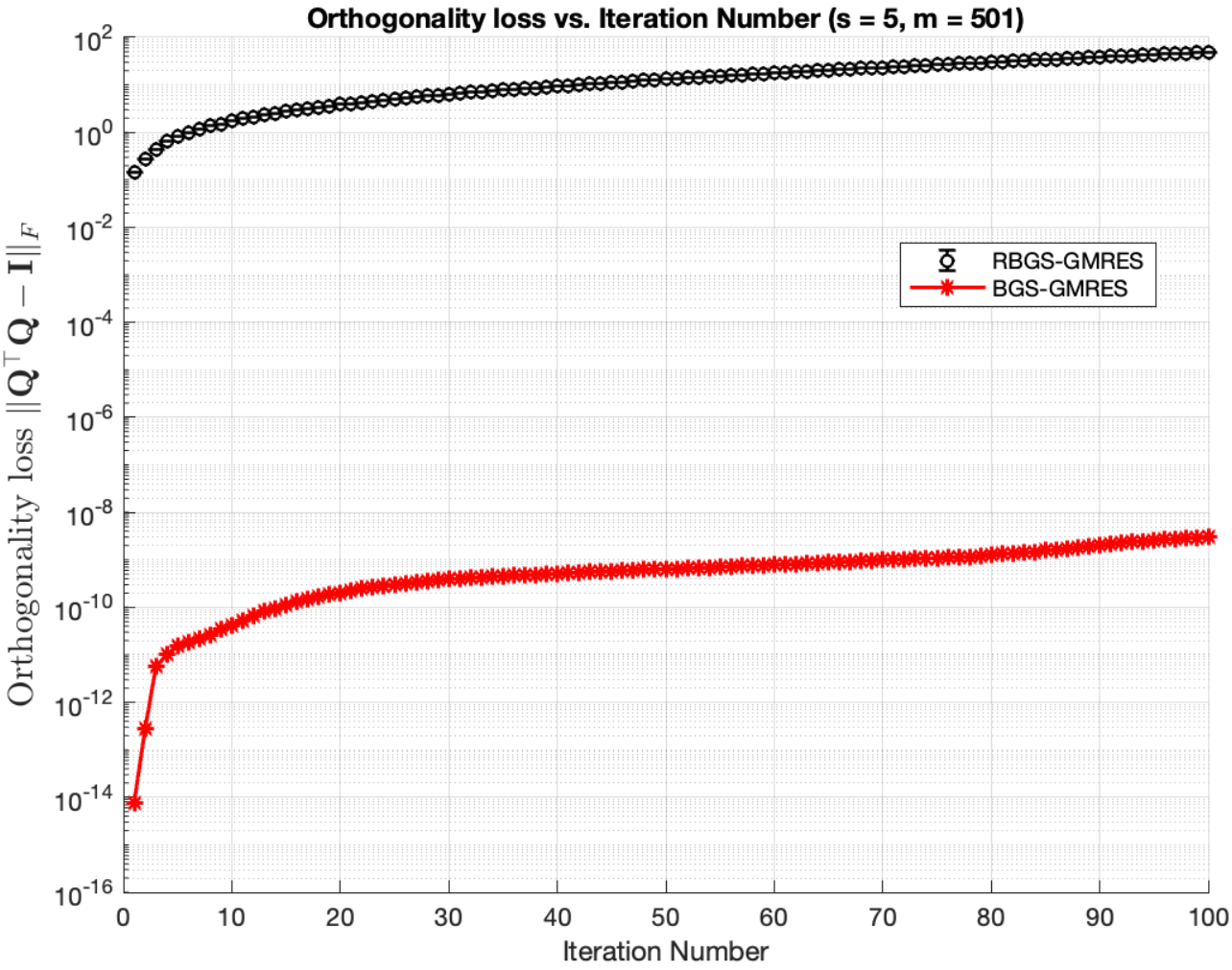
|With RBG reorthogonalization



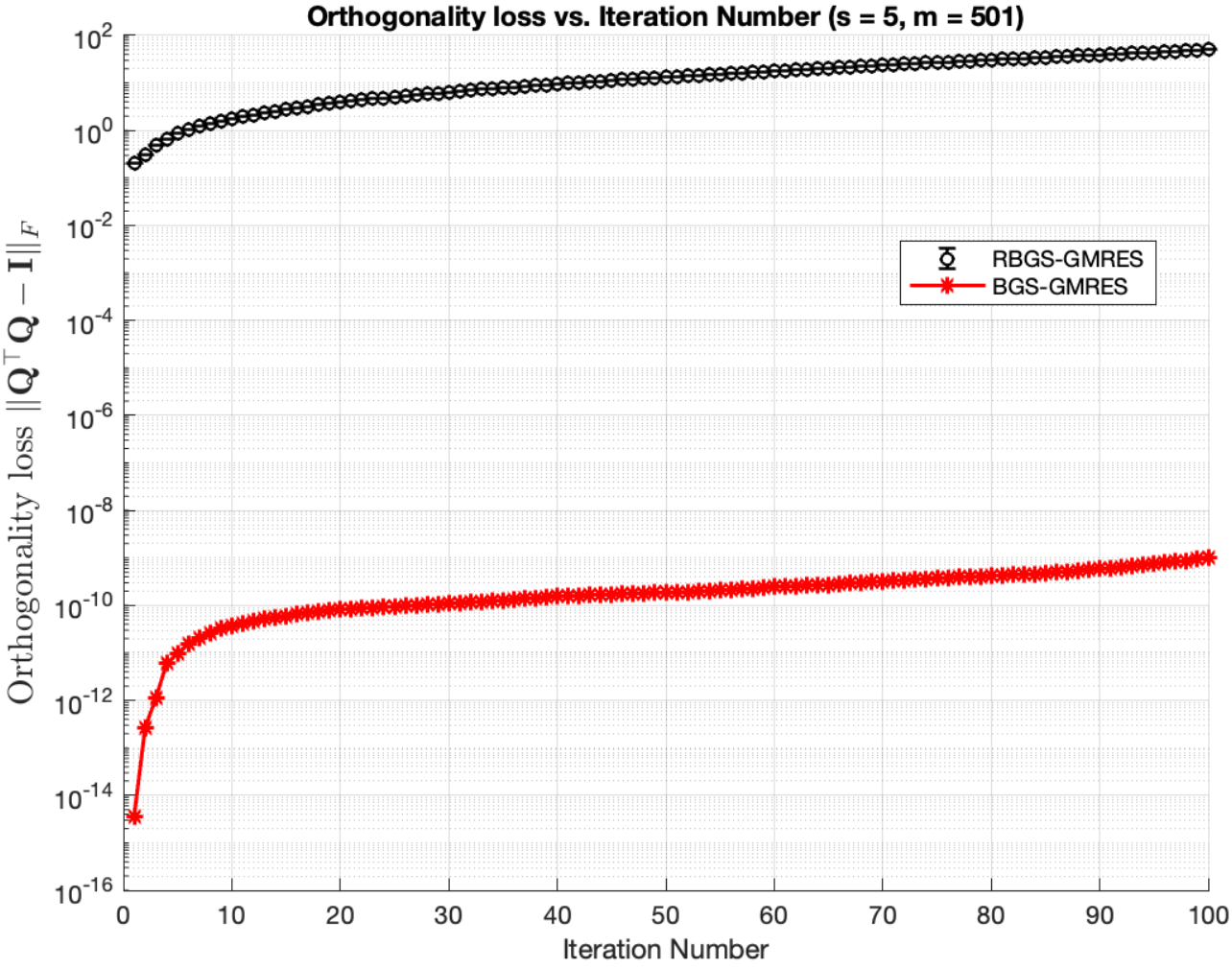
|With both reorthogonalization


Orthogonality loss vs. Iteration Number (s = 1, m = 501)
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
RBGS-GMRES
BGS-GMRES
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
0
50
100
150
200
250
300
350
400
450
500

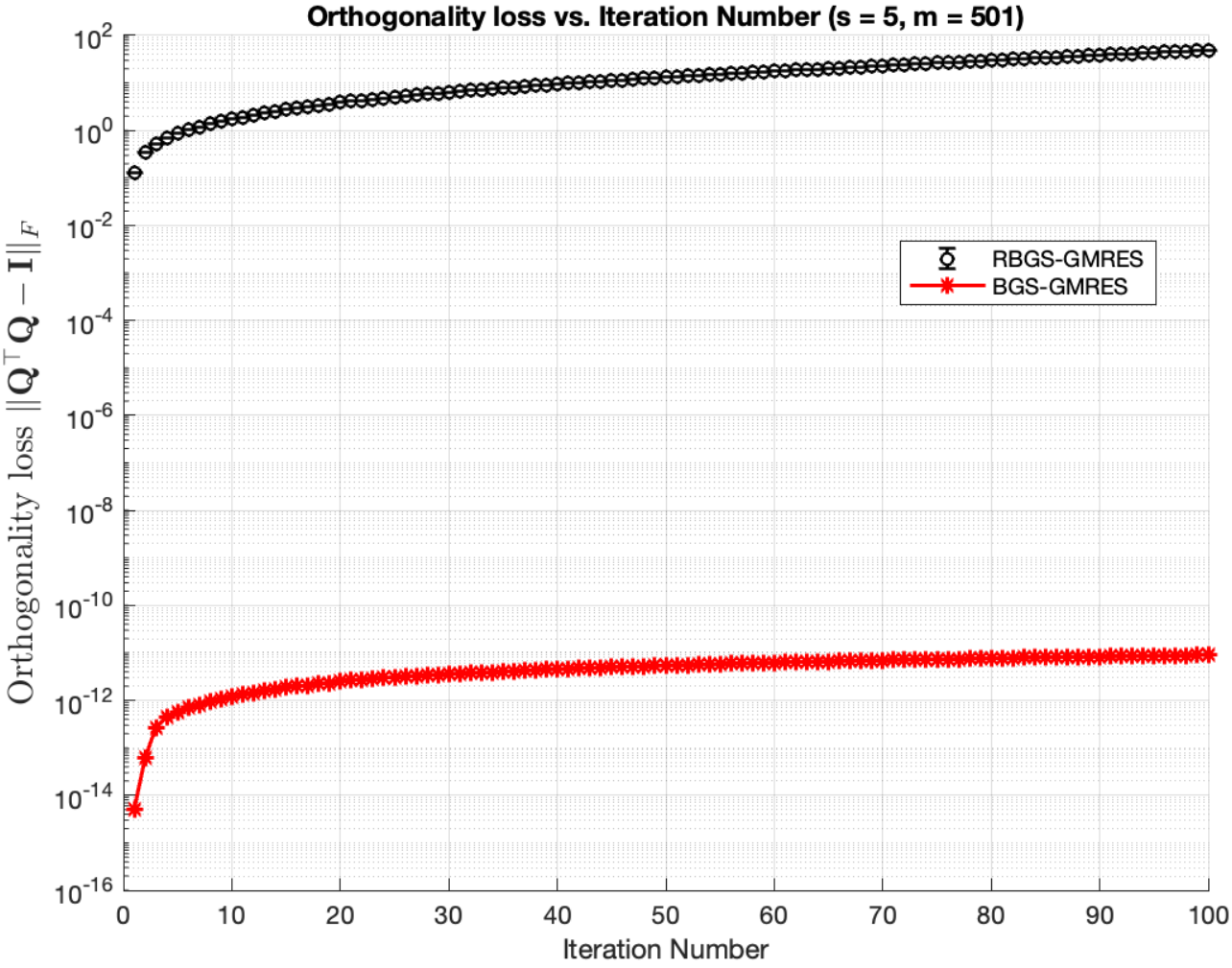
- When s=5,



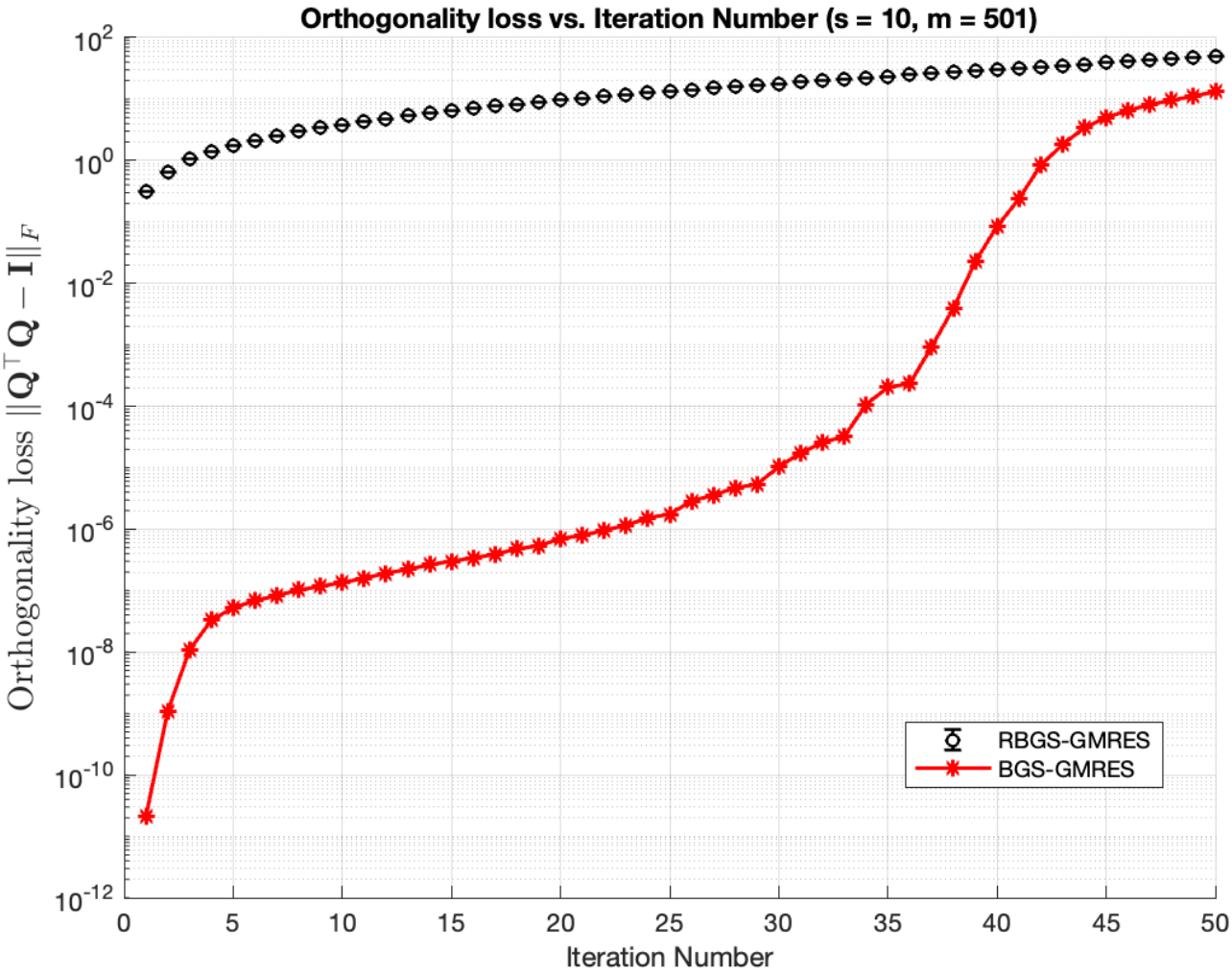
|With RBG reorthogonalization



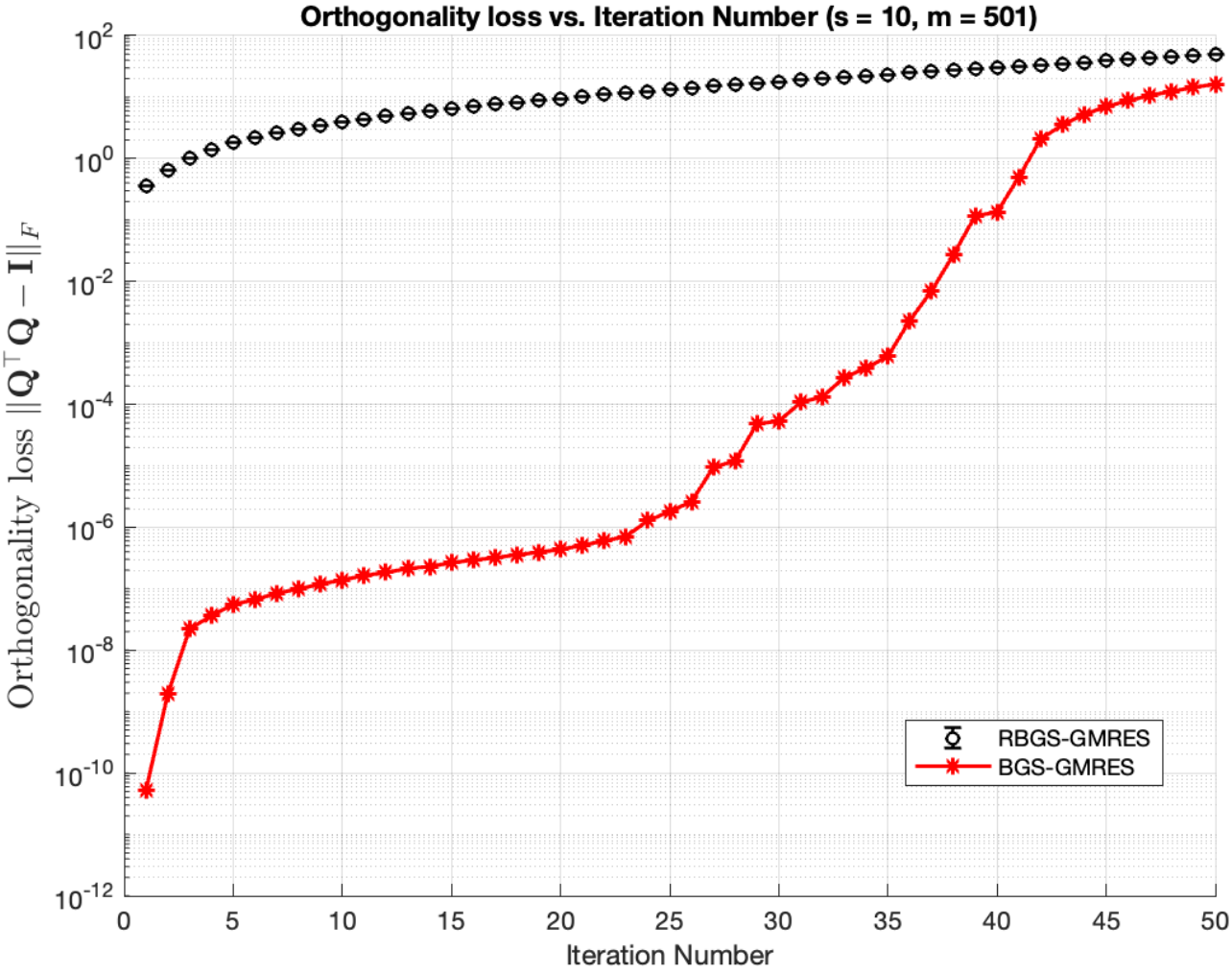
|With both reorthogonalization


Orthogonality loss vs. Iteration Number (s = 5, m = 501)
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
RBGS-GMRES
BGS-GMRES
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
0
10
20
30
40
50
60
70
80
90
100

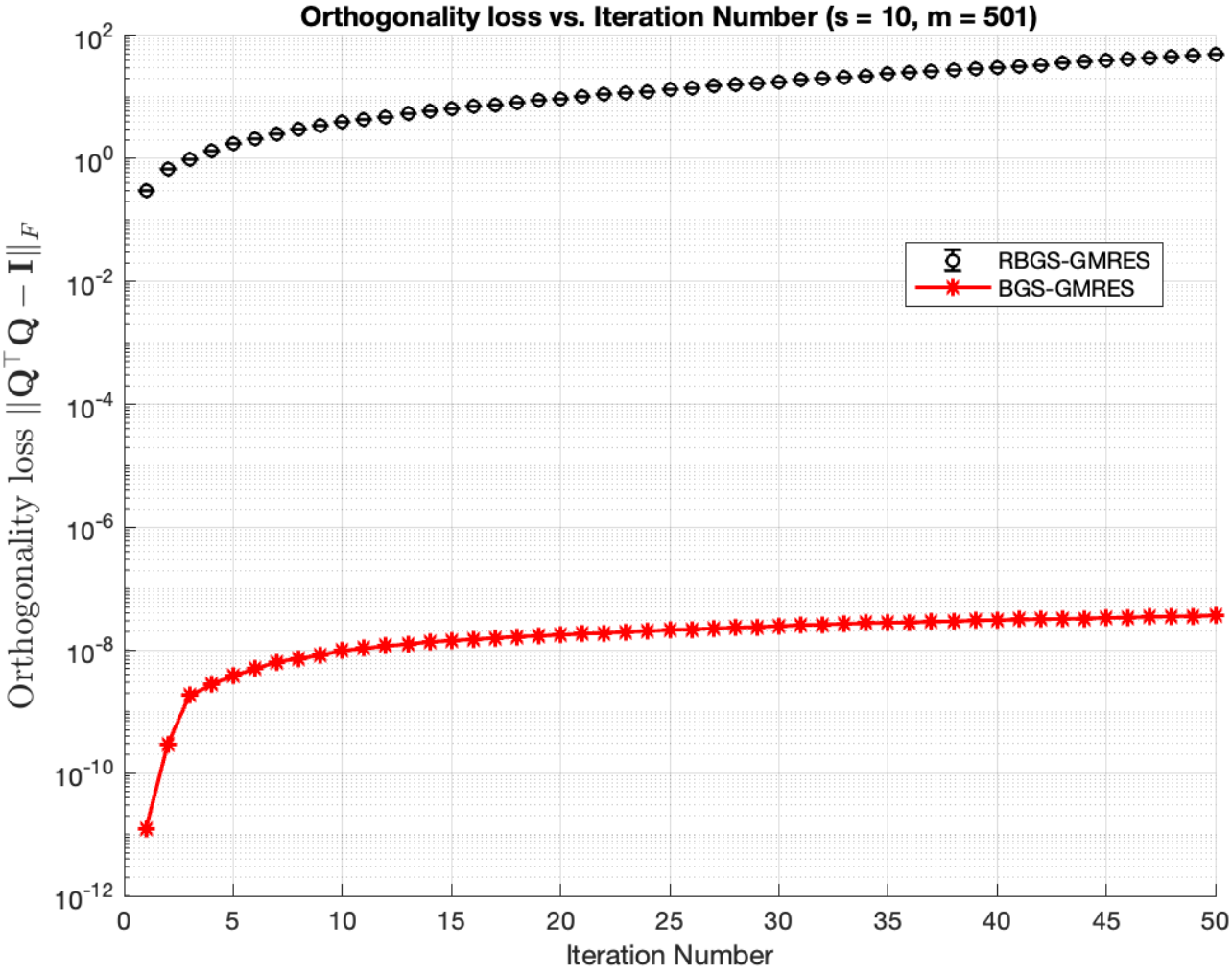
- When s=10,



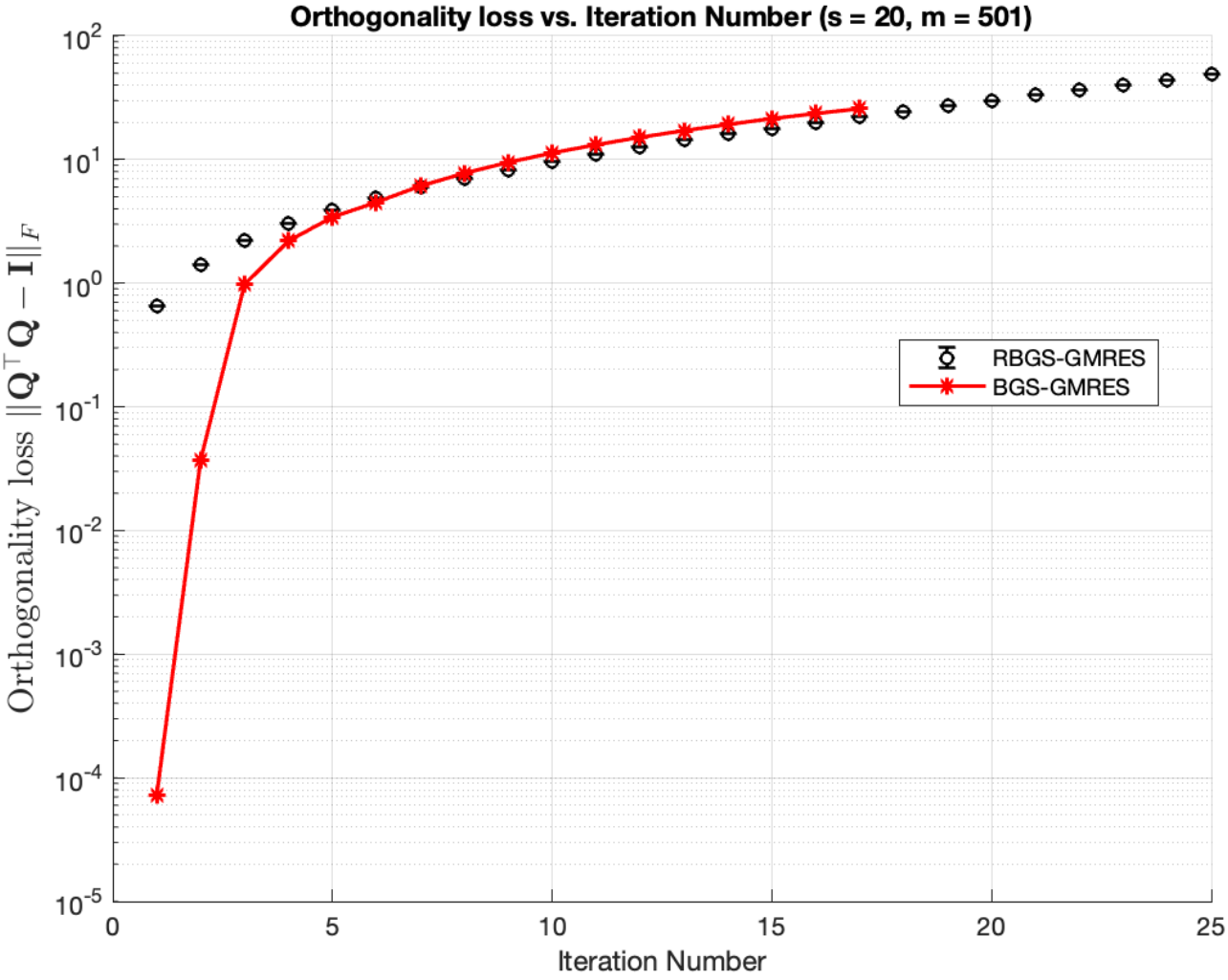
|With RBG reorthogonalization



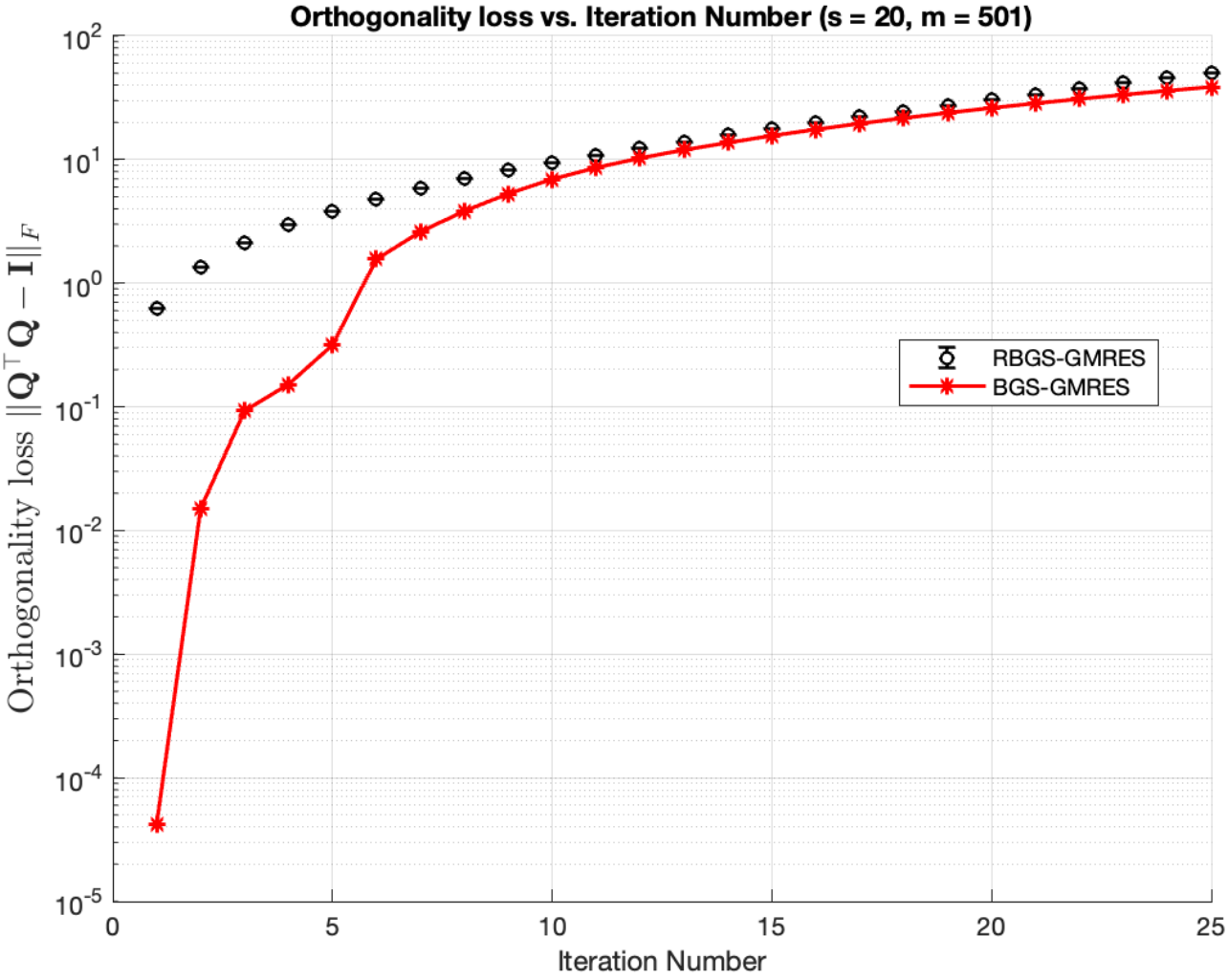
|With both reorthogonalization


Orthogonality loss vs. Iteration Number (s = 10, m = 501)
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
RBGS-GMRES
BGS-GMRES
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
0
5
10
15
20
25
30
35
40
45
50

- When s=20,


Orthogonality loss vs. Iteration Number (s = 20, m = 501)
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
RBGS-GMRES
BGS-GMRES
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0
5
10
15
20
25

|With both reorthogonalization

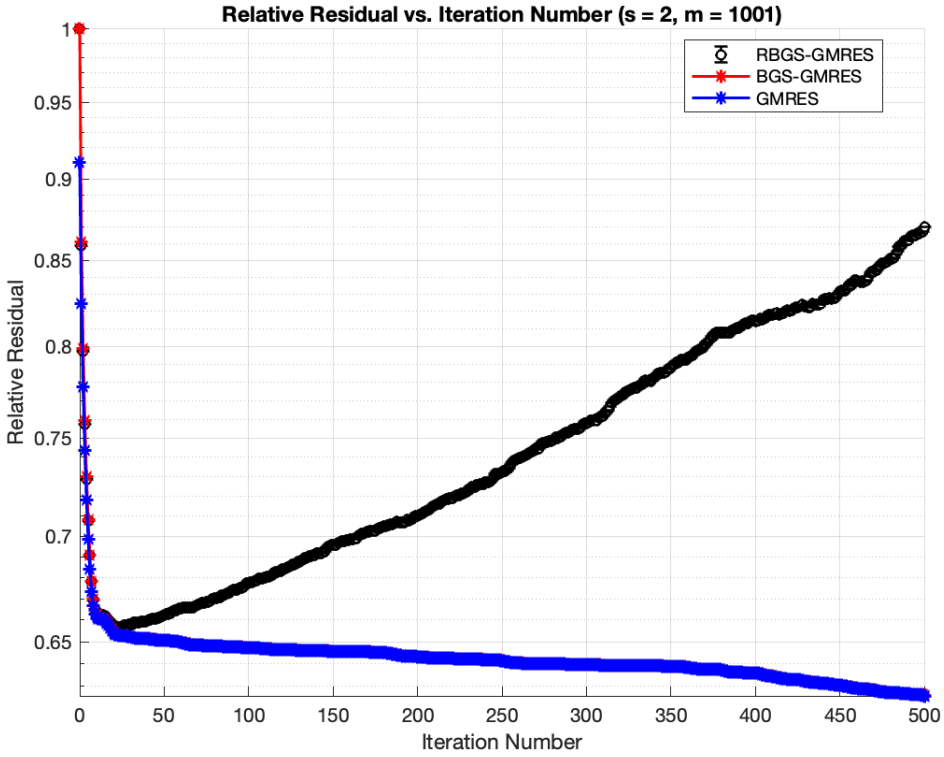
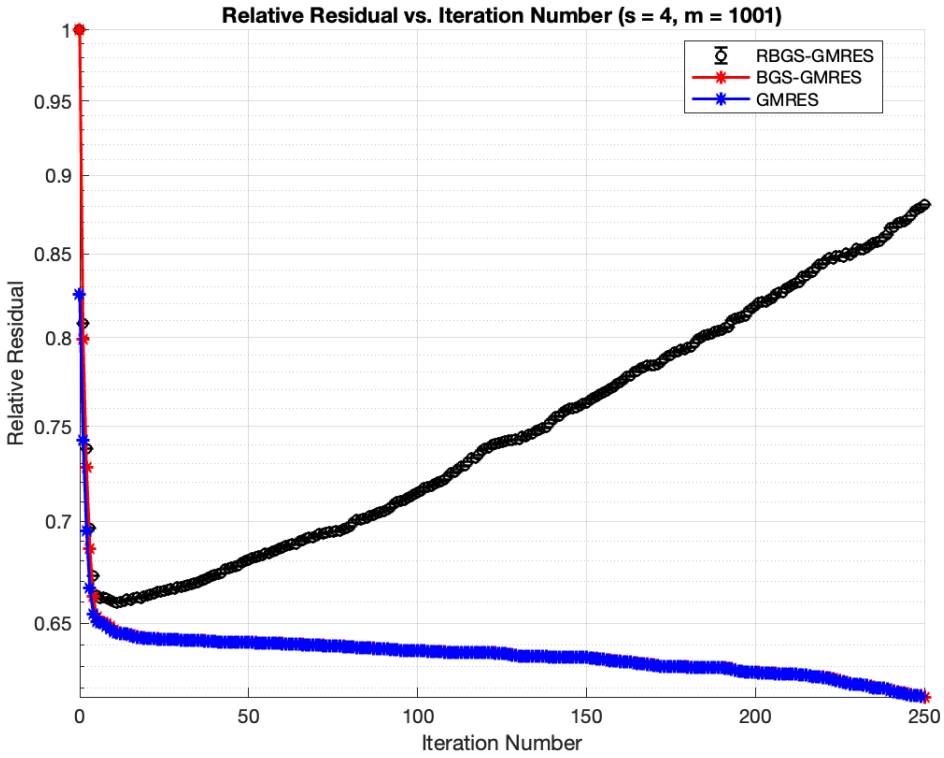


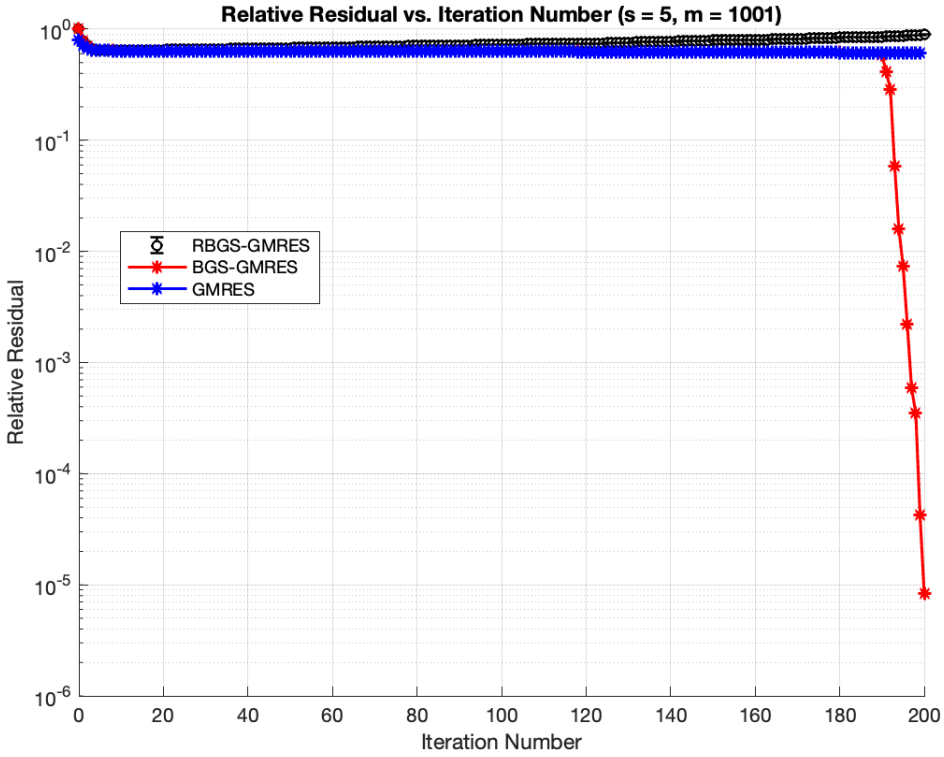
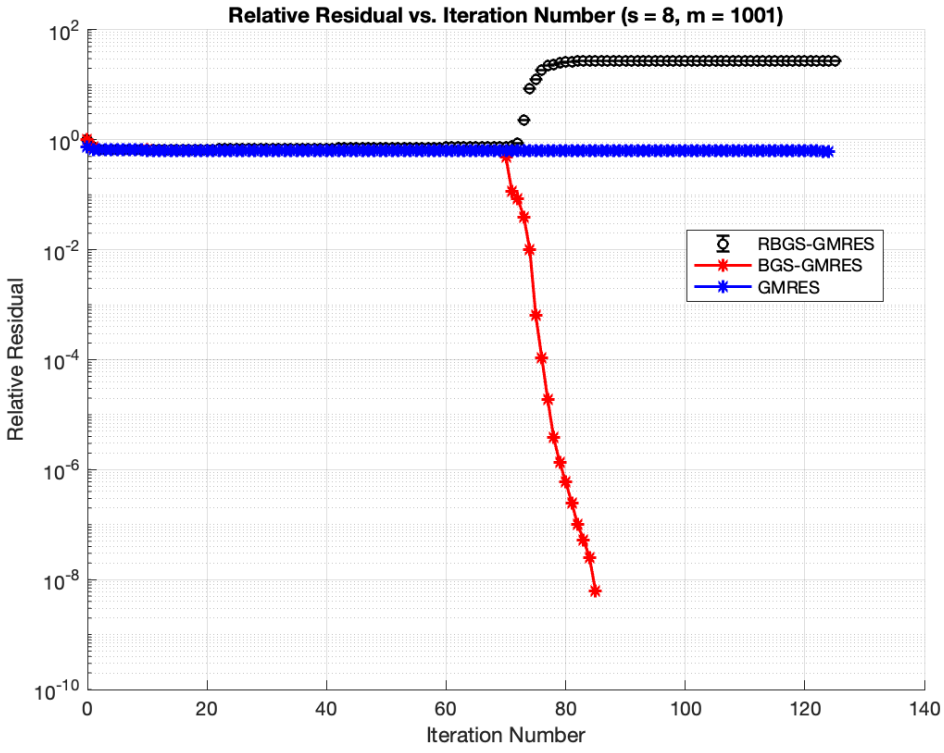
Discussion:

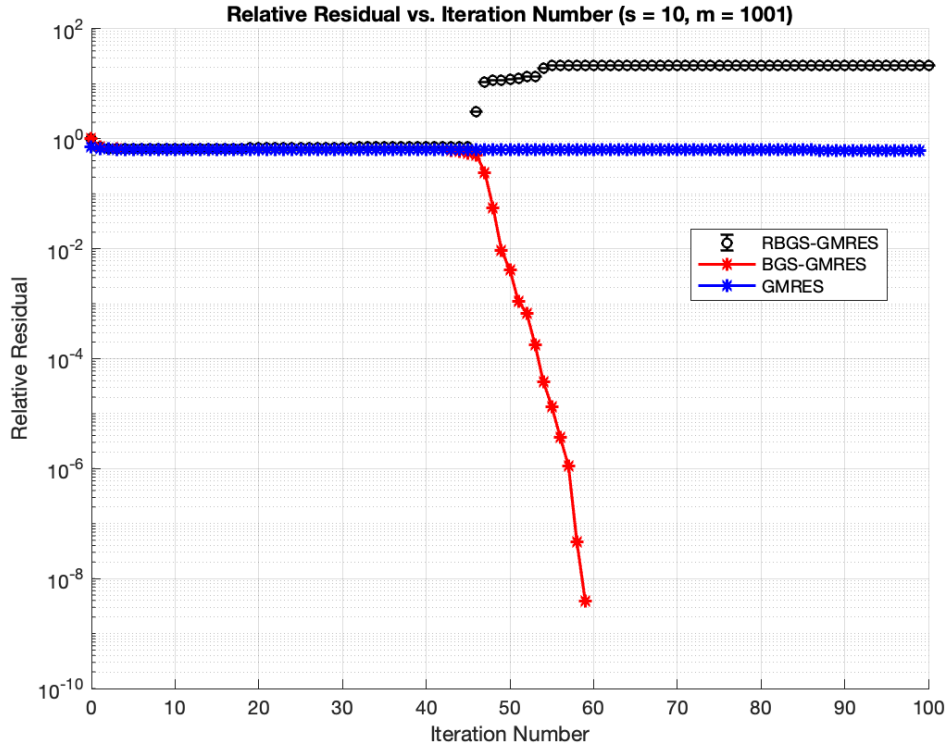
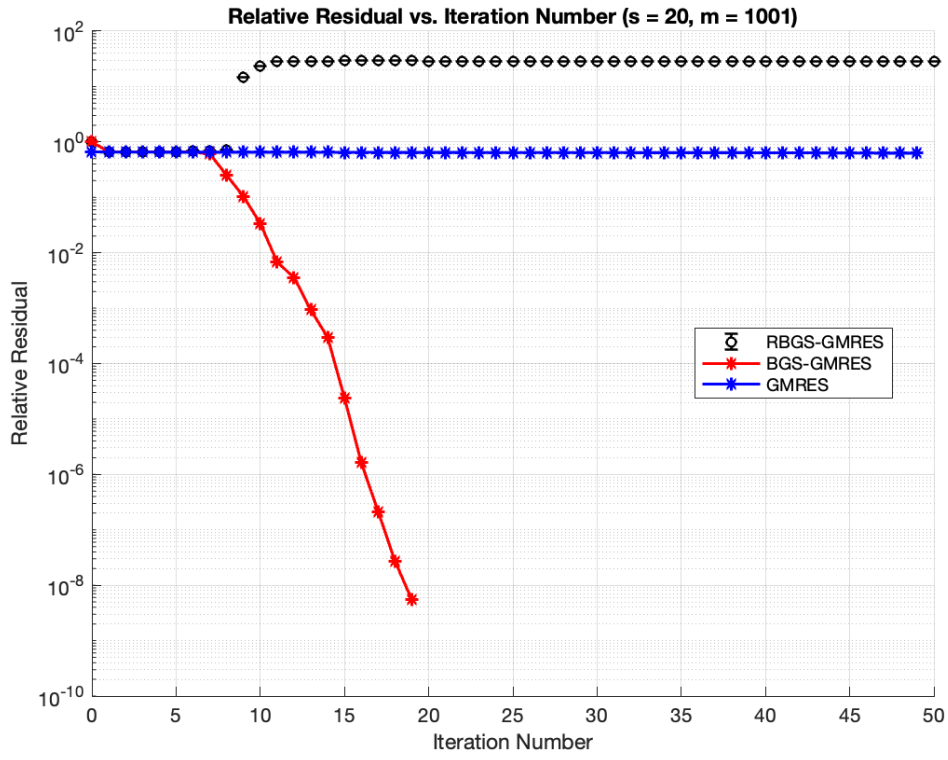
- After reorthogonalization, the θ-orthonormal Q generated by RBGS shows little improvement in orthogonality, although a slight enhancement can be observed in the relative residual. In contrast, the orthogonality improvement of BGS is more obvious

When m = 1000

s	Relative Residual Plot	Iteration Count	Runtime (s)

s	Relative Residual Plot	Iteration Count	Runtime (s)
2			
4			

s	Relative Residual Plot	Iteration Count	Runtime (s)
5			
8			

s	Relative Residual Plot	Iteration Count	Runtime (s)
10			
20			

Discussion:

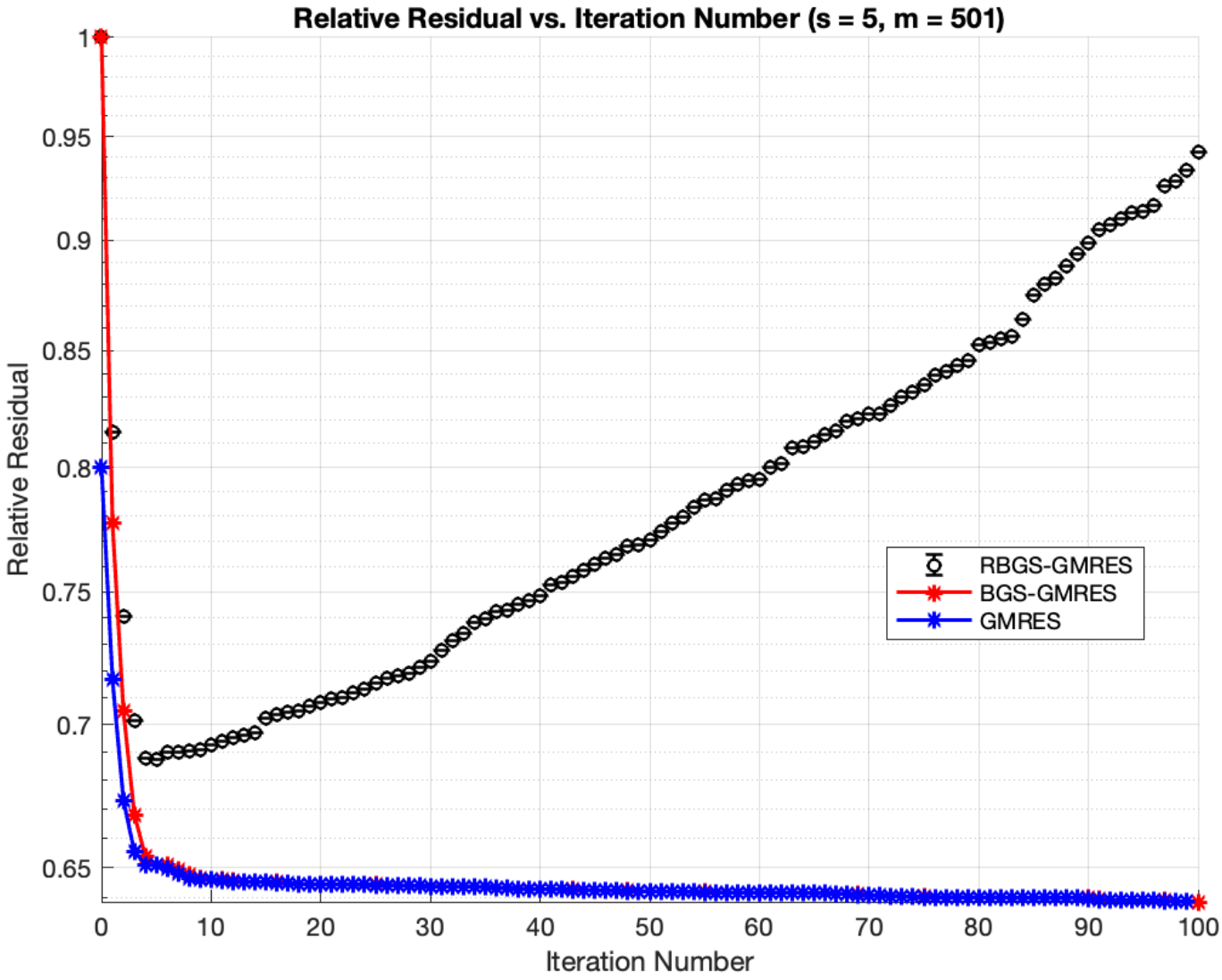
- When s=8, BGS converged after 680 steps
- When s=10, BGS converged after 590 steps
- When s=20, BGS converged after 380 steps

2.2 Matrix A2 – Hermitian matrix

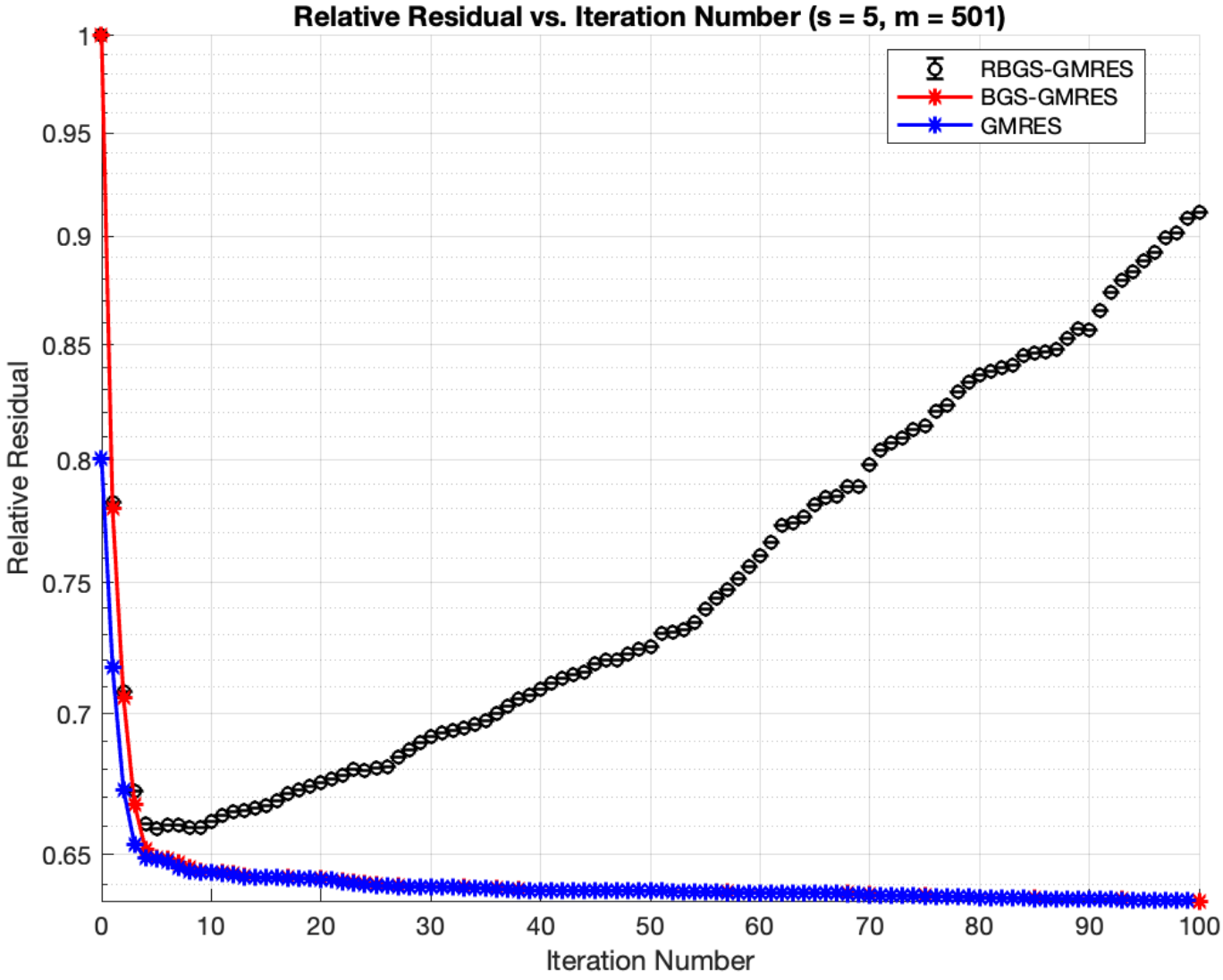
Matrix info: (n = 2500, alpha = 0, lambda = logspace(0,-5,n).')
Sketch info: (m = 501, d = 2 * m)\
Converge info: (ctol = 1e-8)

➤ Results for Step Sizes

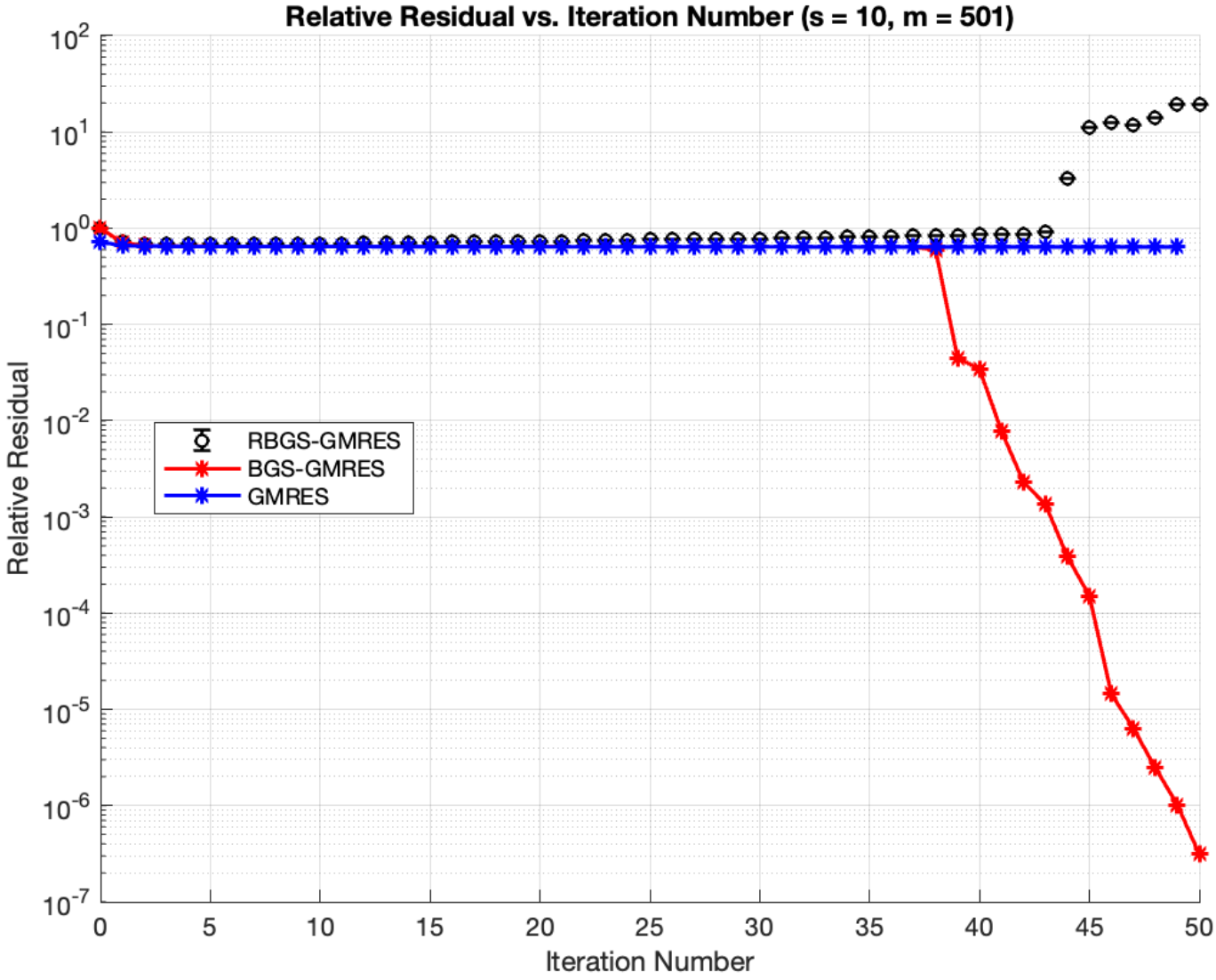
relative residual
-s=5,



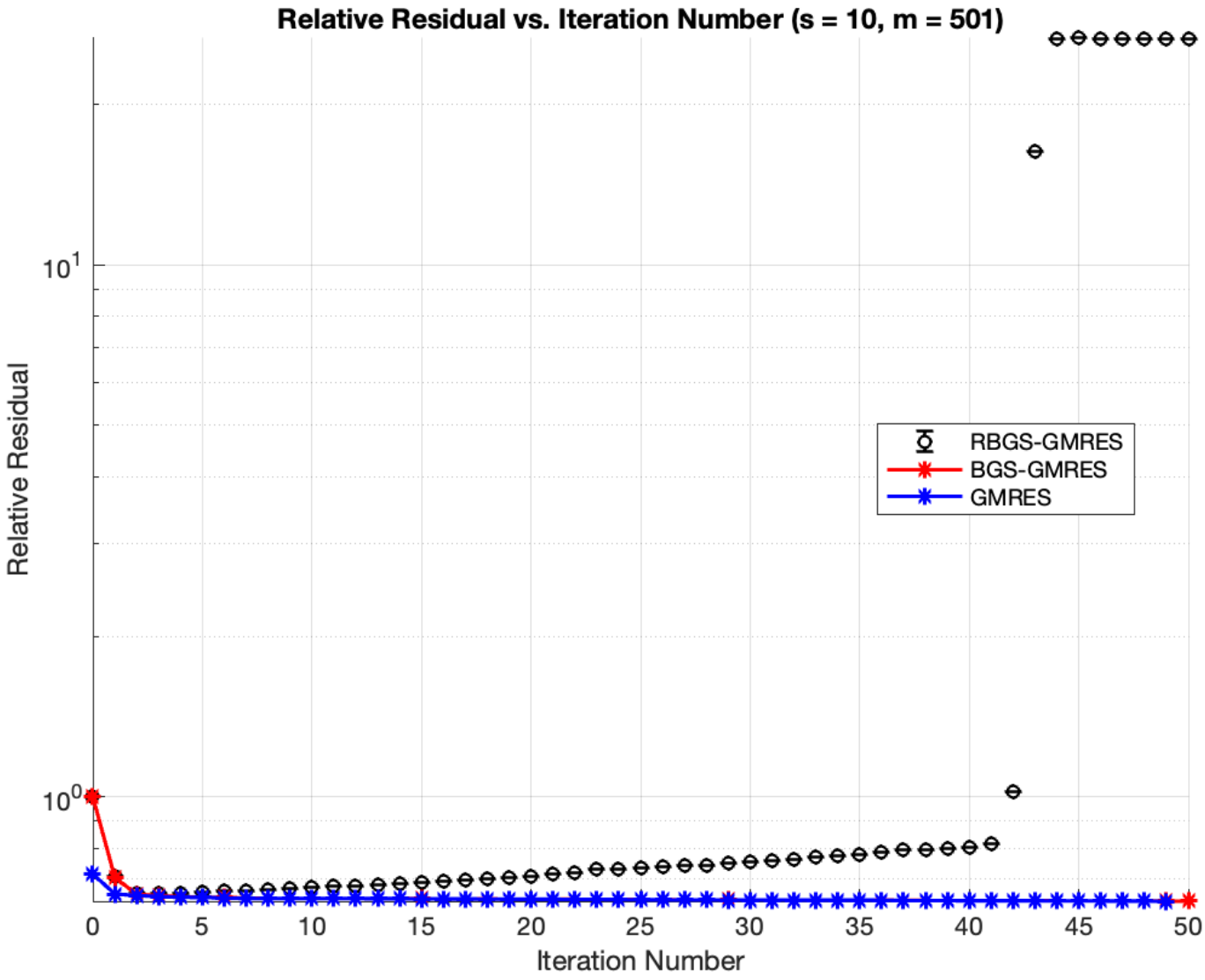
| residual after reorthogonalization


Relative Residual vs. Iteration Number (s = 5, m = 501)
RBGS-GMRES
BGS-GMRES
GMRES
Relative Residual
Iteration Number
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0
10
20
30
40
50
60
70
80
90
100

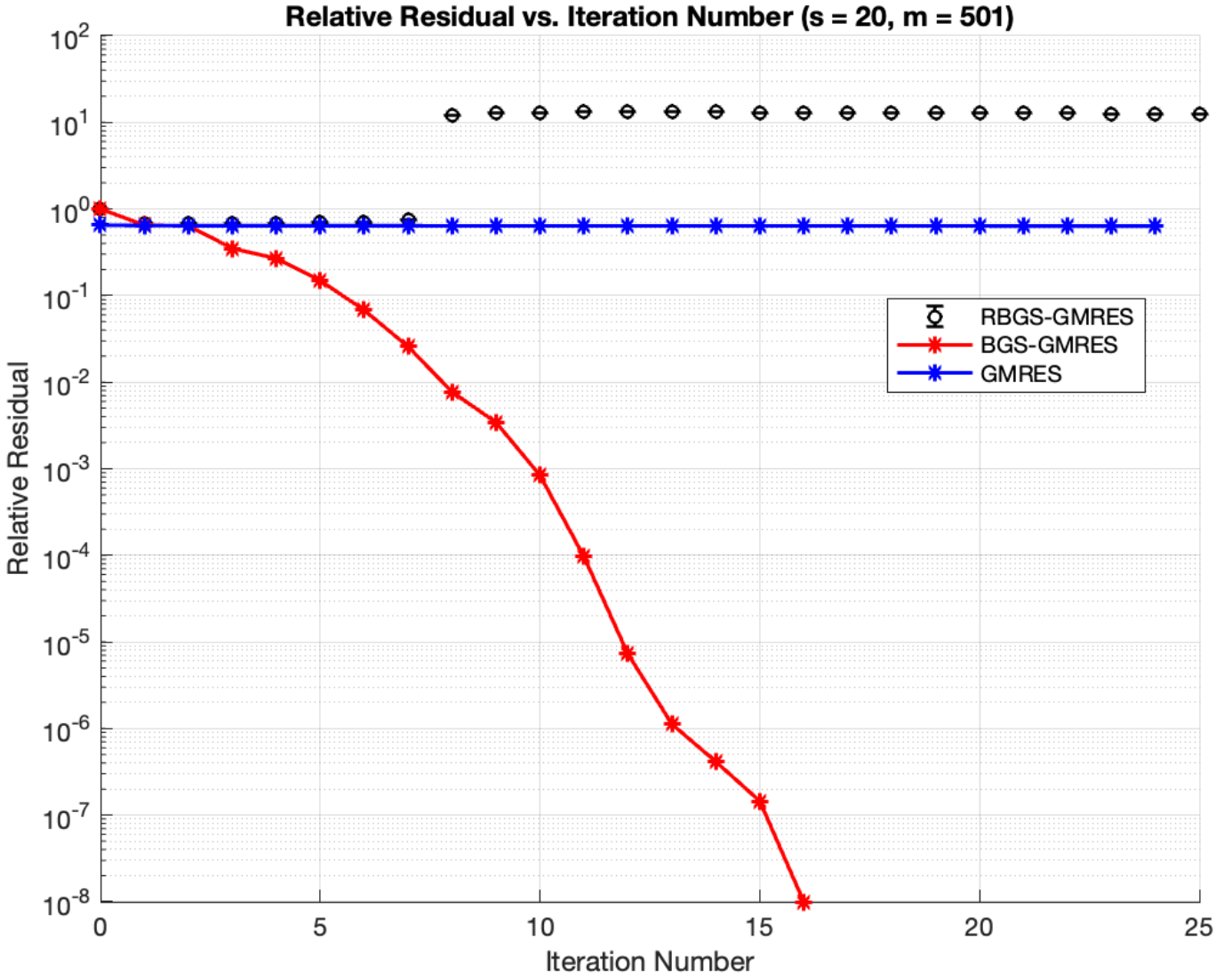
-s=10,



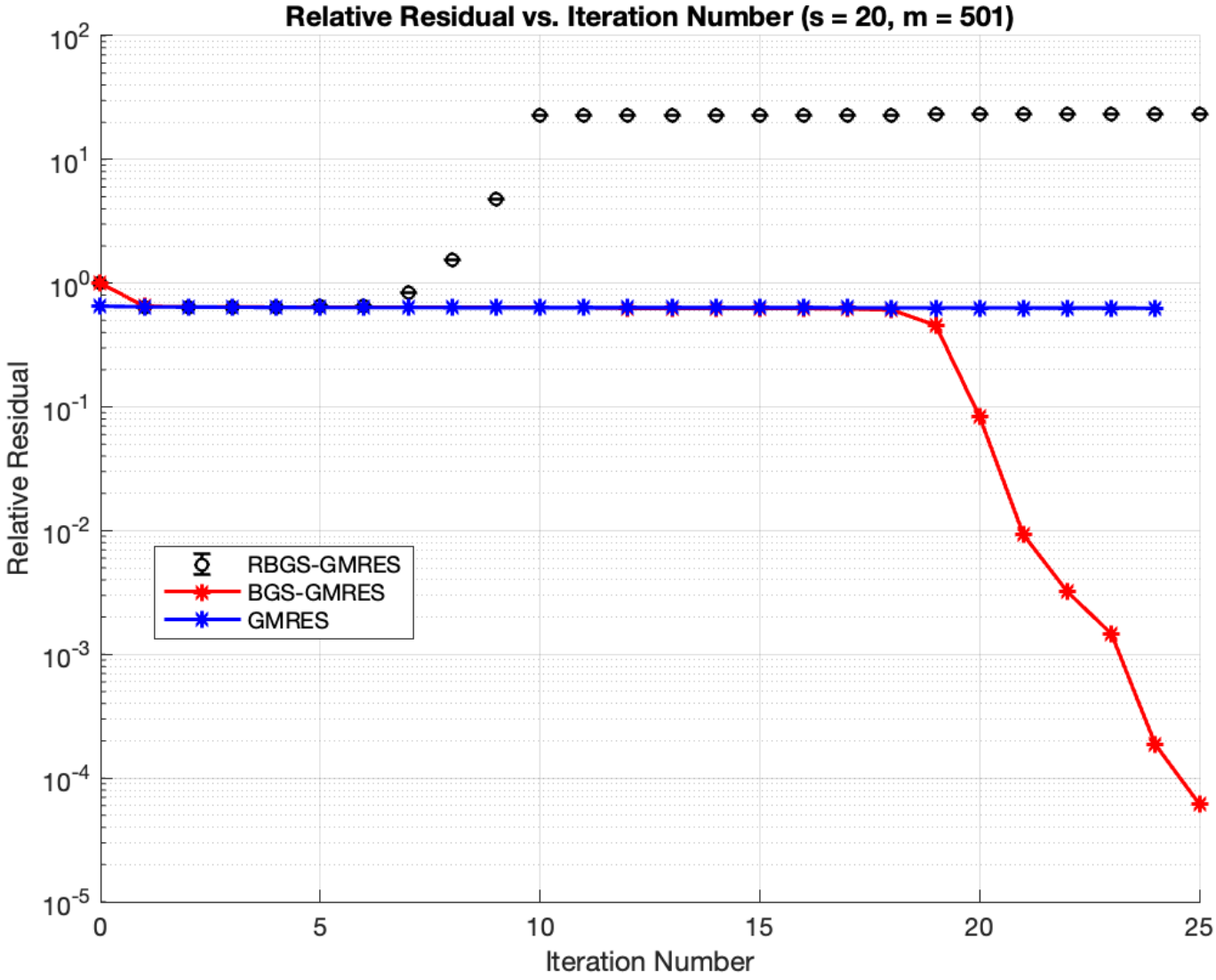
| residual after reorthogonalization


Relative Residual vs. Iteration Number (s = 10, m = 501)
Relative Residual
Iteration Number
RBGS-GMRES
BGS-GMRES
GMRES
10^1
10^0
0
5
10
15
20
25
30
35
40
45
50

-s=20,



| residual after reorthogonalization

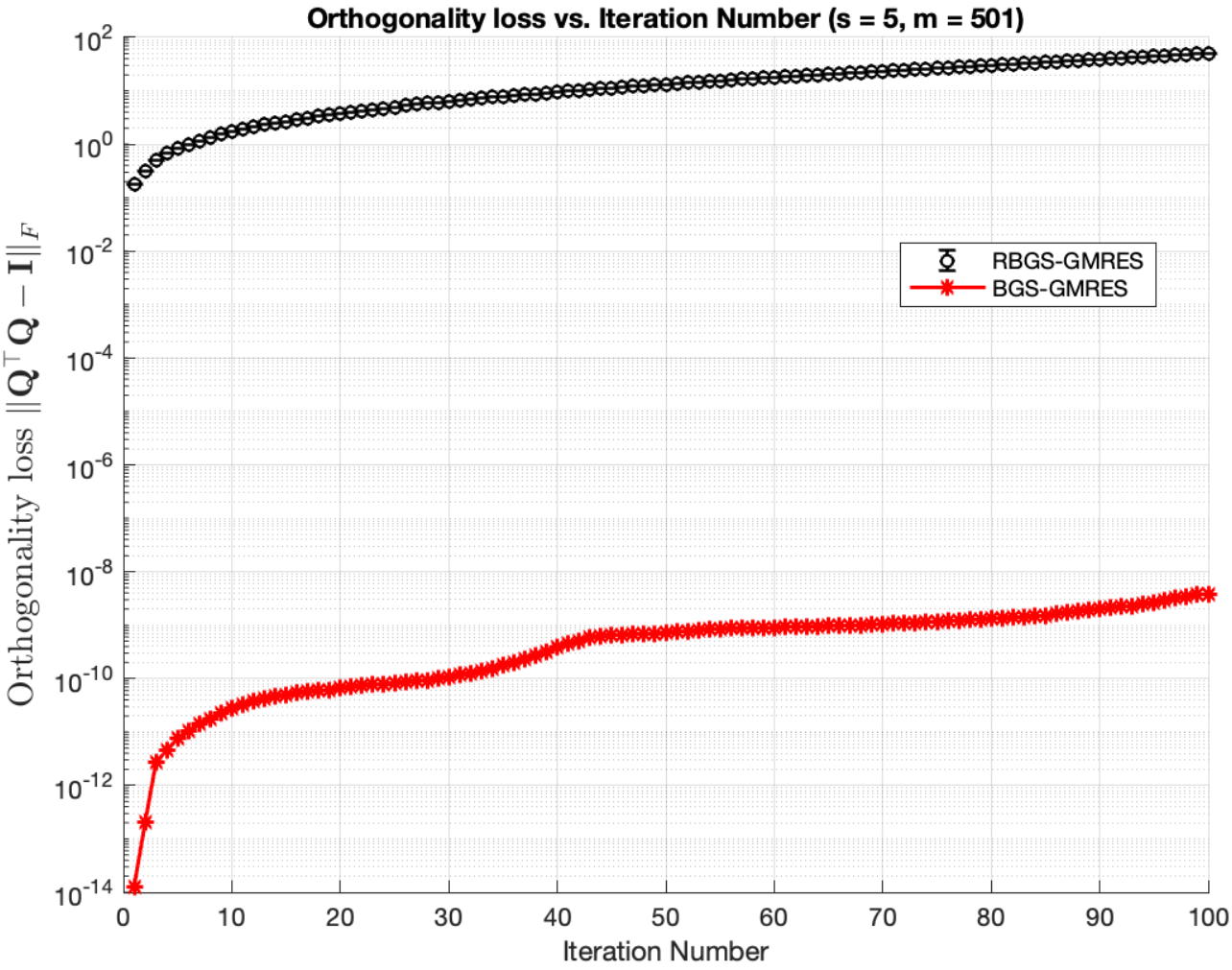


Discussion:

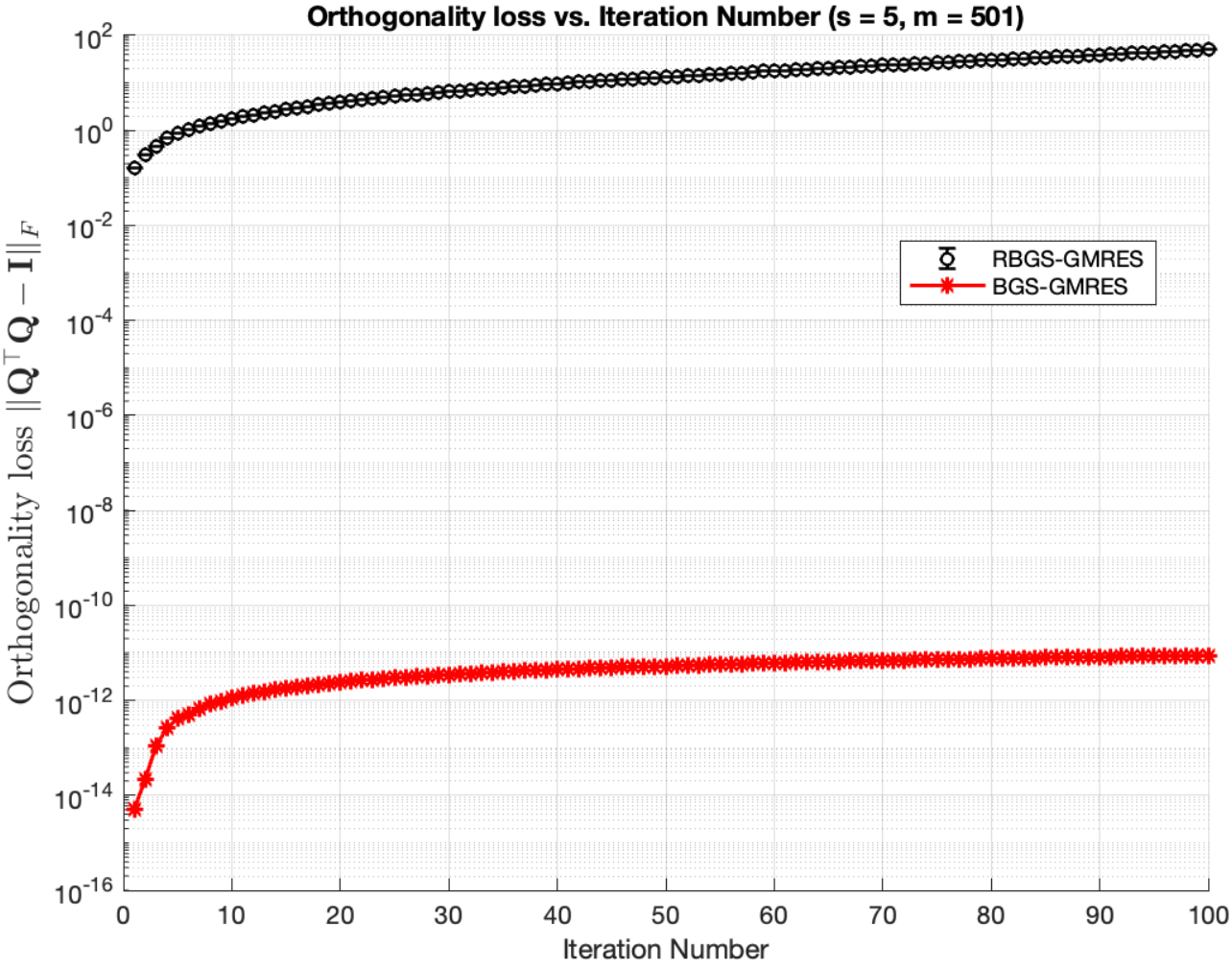
- When s=20, BGS converged after 320 steps

Check Orthogonality

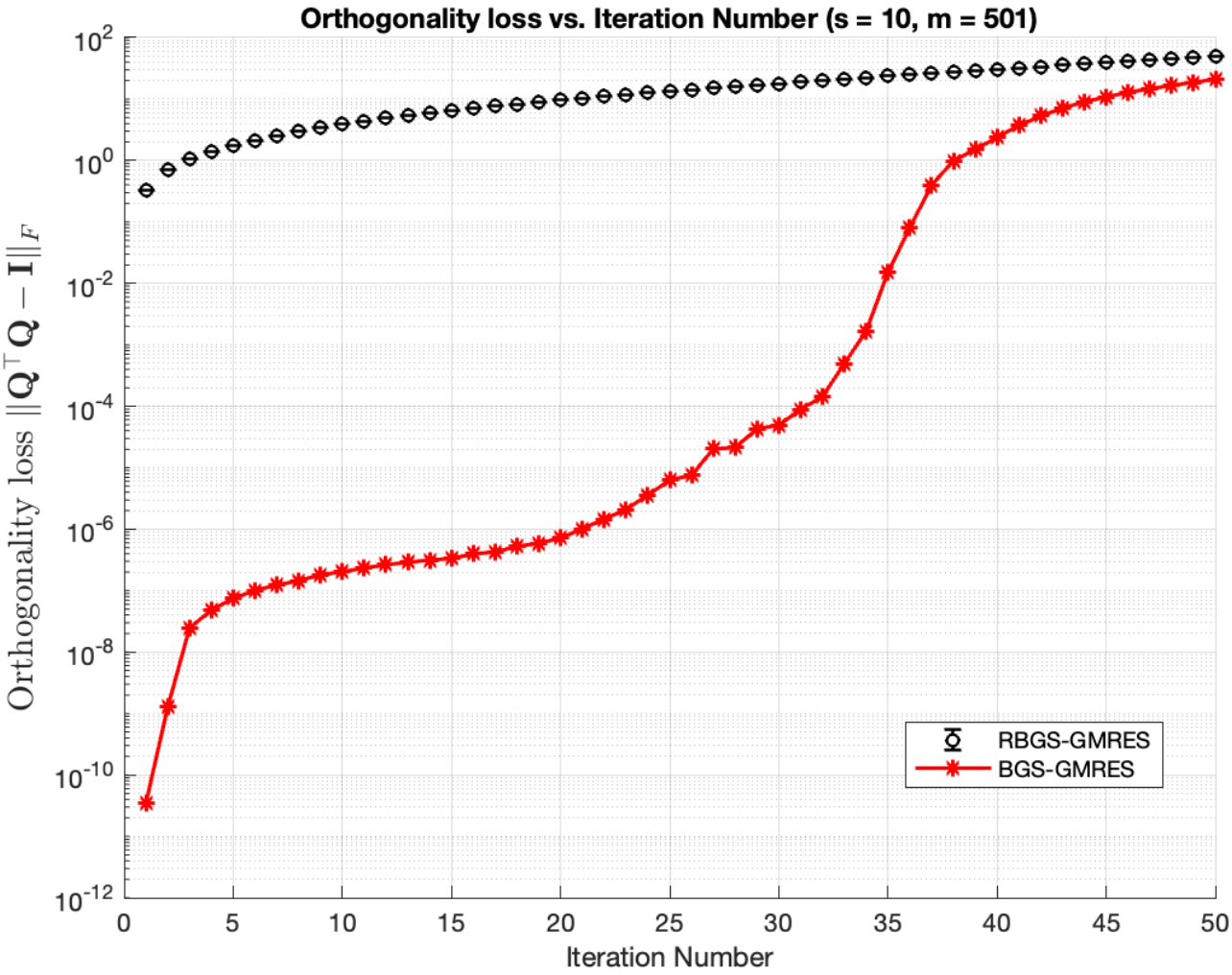
- When s=5,


Orthogonality loss vs. Iteration Number (s = 5, m = 501)
RBGS-GMRES
BGS-GMRES
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
0
10
20
30
40
50
60
70
80
90
100

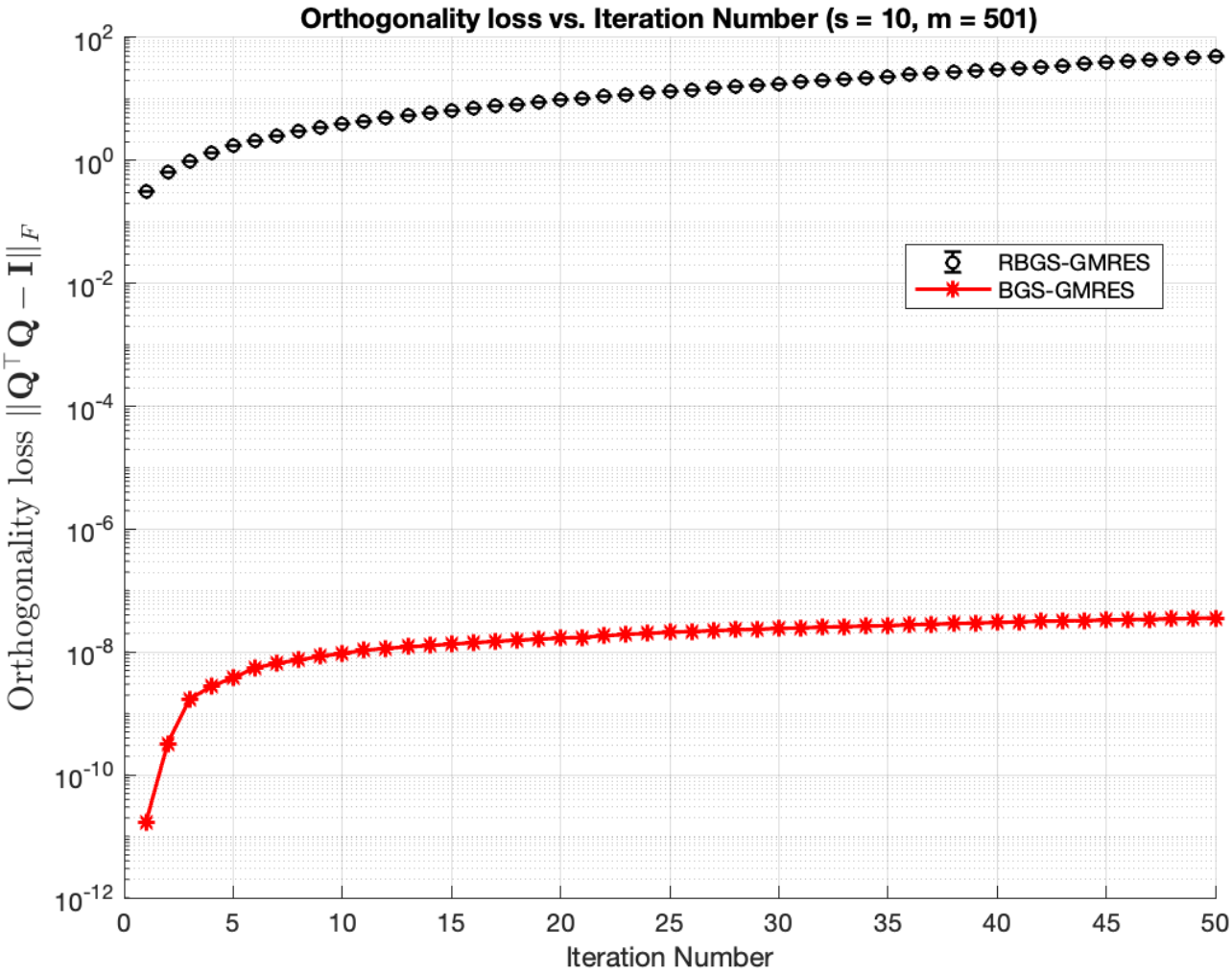
|With both reorthogonalization



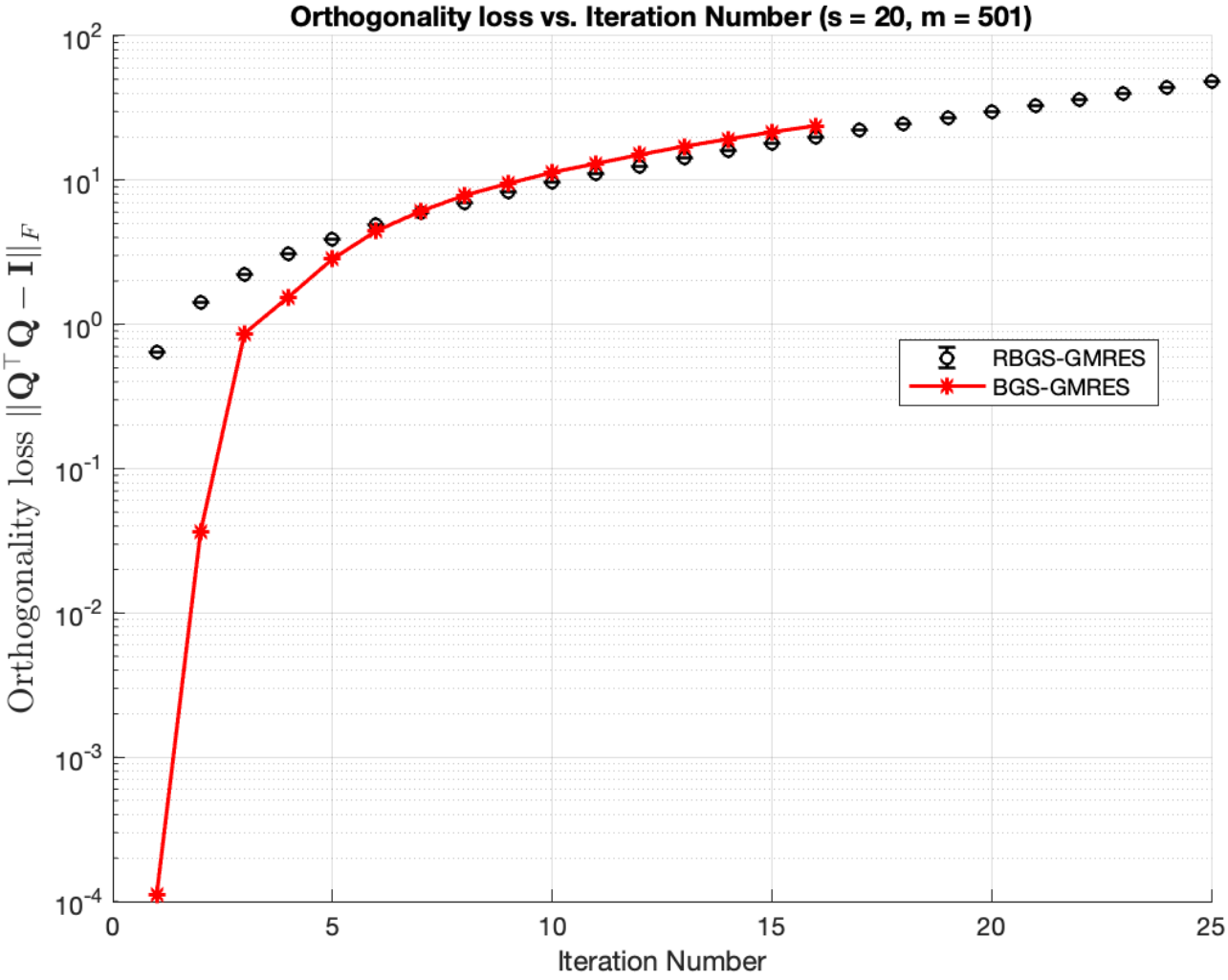
- When s=10,


Orthogonality loss vs. Iteration Number (s = 10, m = 501)
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
RBGS-GMRES
BGS-GMRES

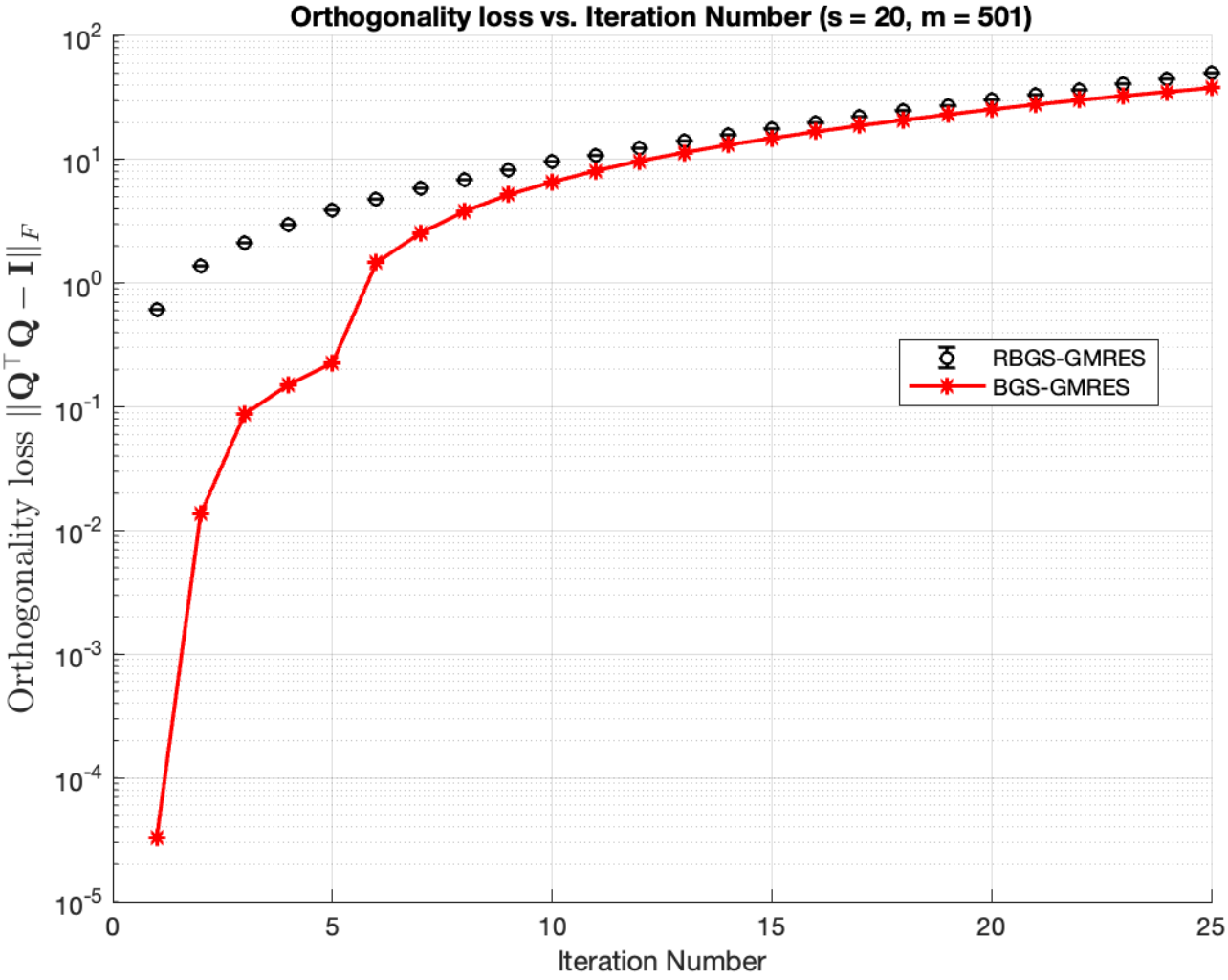
|With both reorthogonalization



- When s=20,


Orthogonality loss vs. Iteration Number (s = 20, m = 501)
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
RBGS-GMRES
BGS-GMRES
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
5
10
15
20
25

|With both reorthogonalization

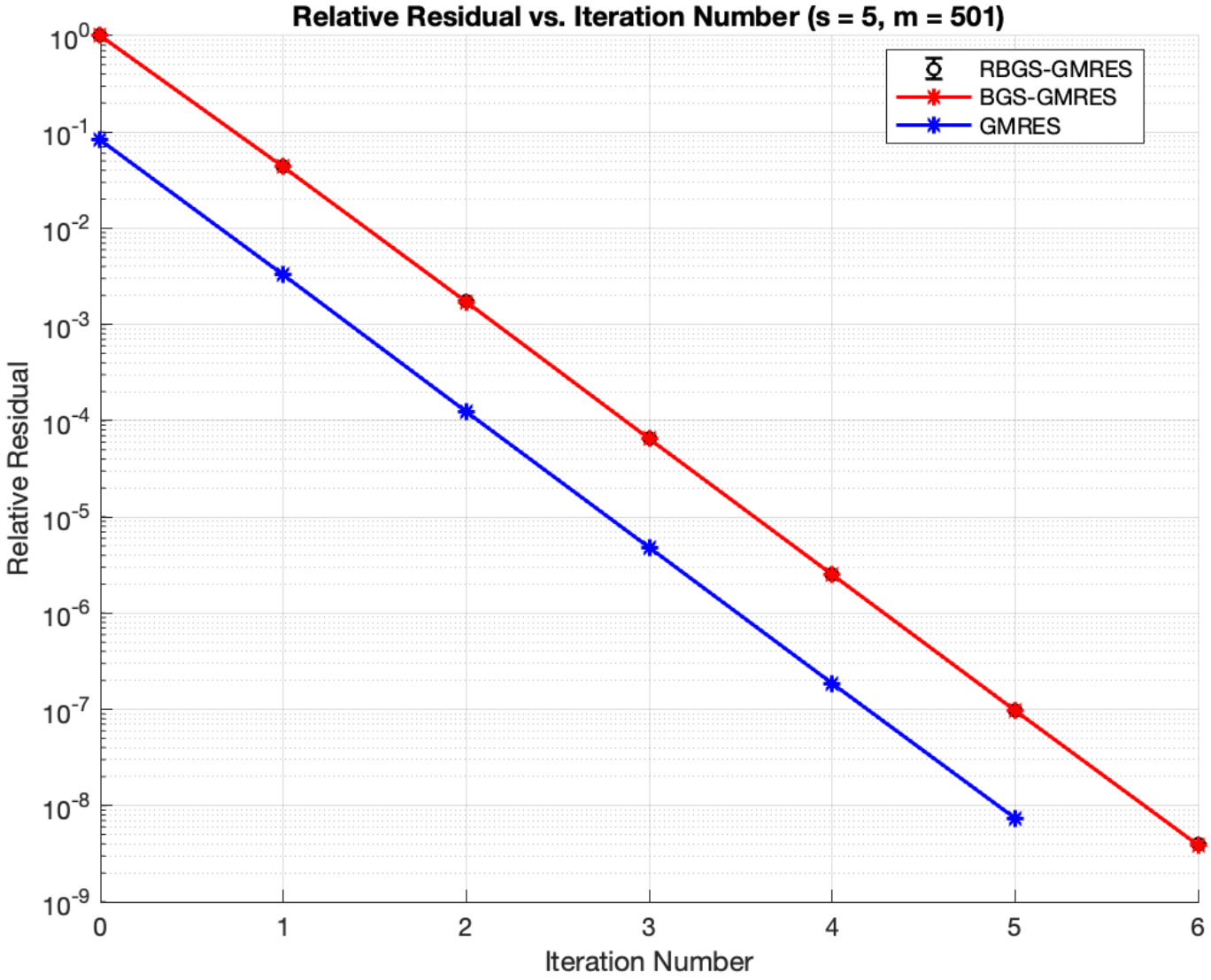


Discussion:

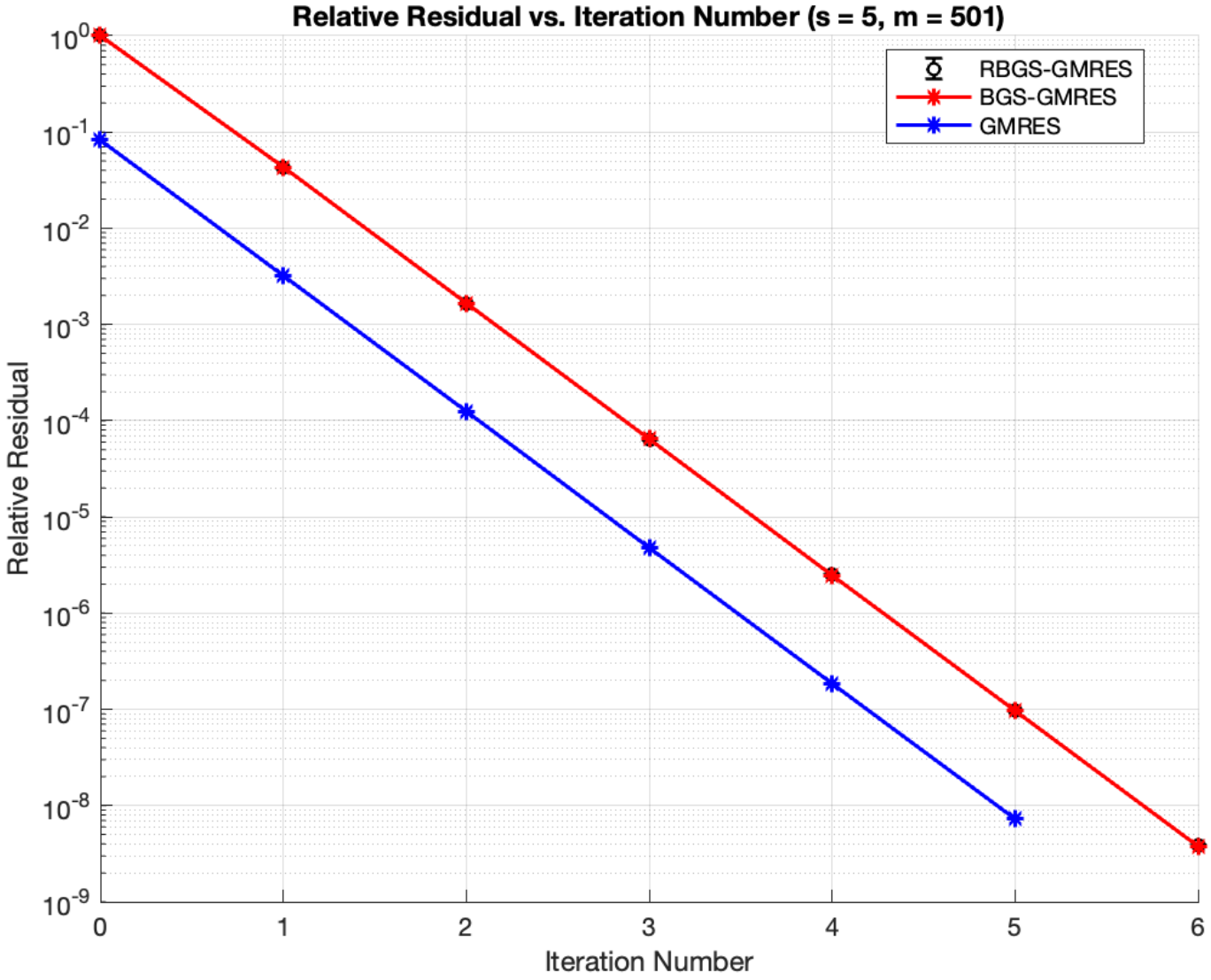
2.3 Matrix A3 – matrix with smaller condition number

Matrix info: (n = 2500, alpha = 0.5, lambda = logspace(0,-1,n).')
Sketch info: (m = 501, d = 2 * m)\
Converge info: (ctol = 1e-8)

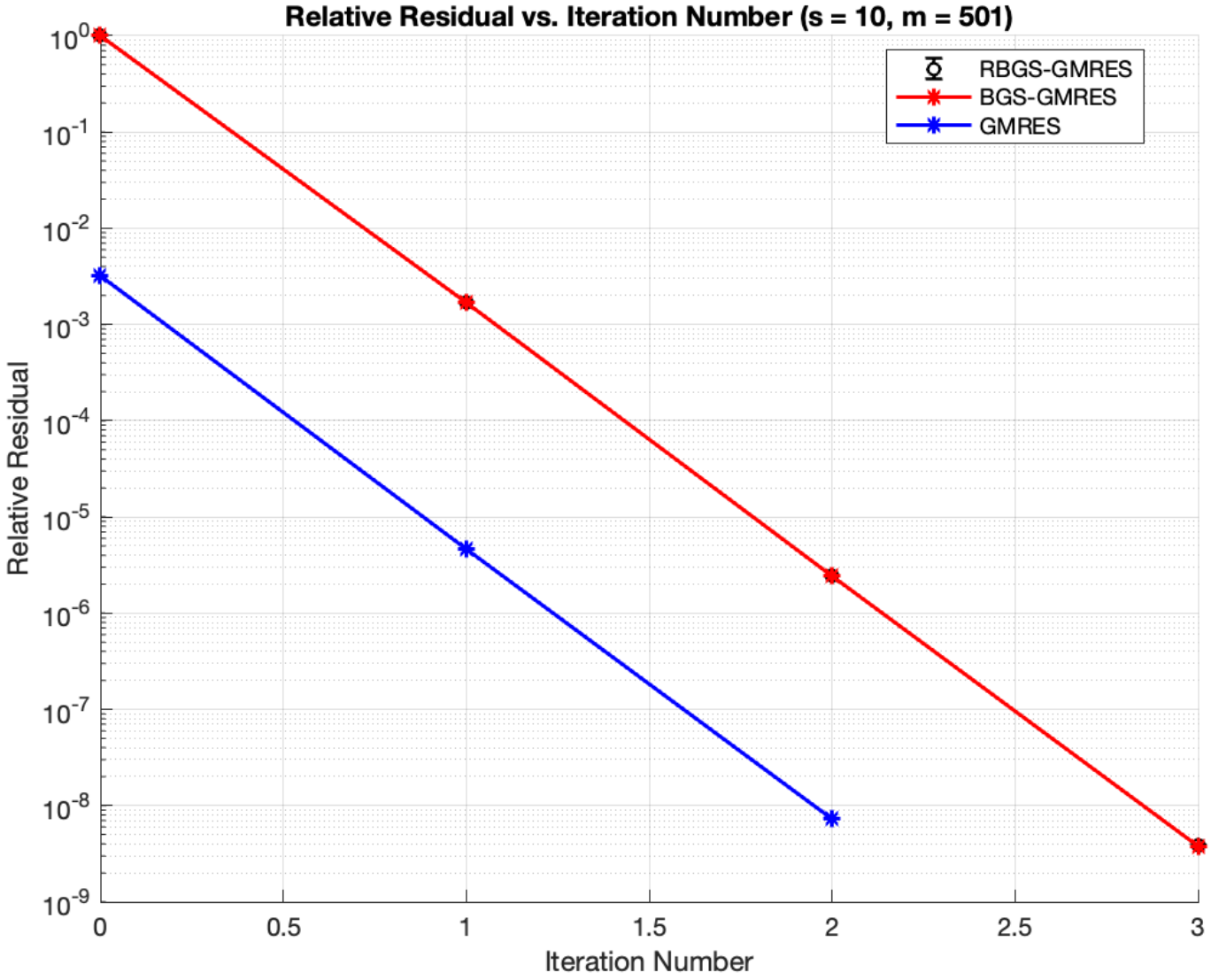
relative residual
-s=5,



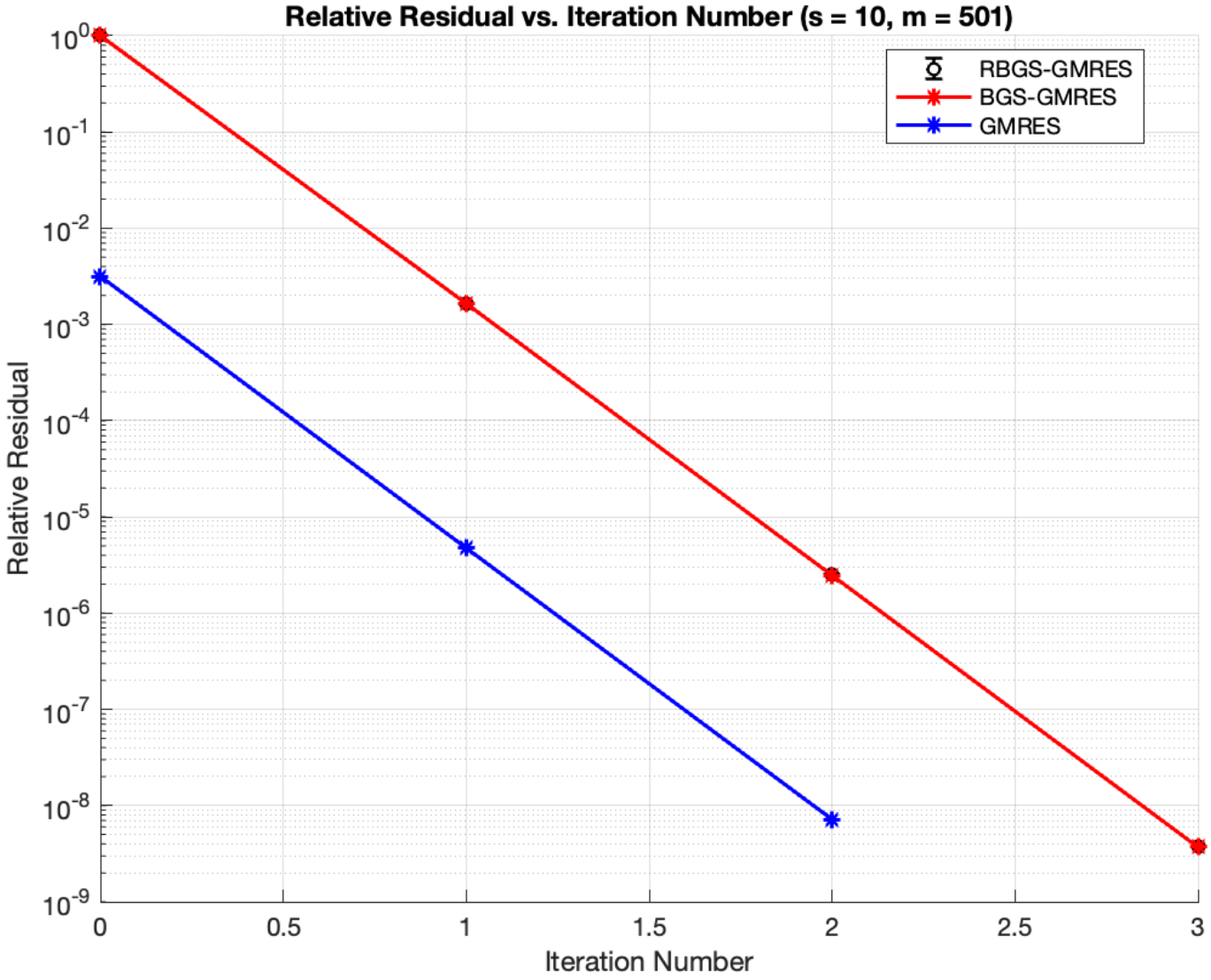
| residual after reorthogonalization


Relative Residual vs. Iteration Number (s = 5, m = 501)
RBGS-GMRES
BGS-GMRES
GMRES
Relative Residual
Iteration Number
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
0
1
2
3
4
5
6

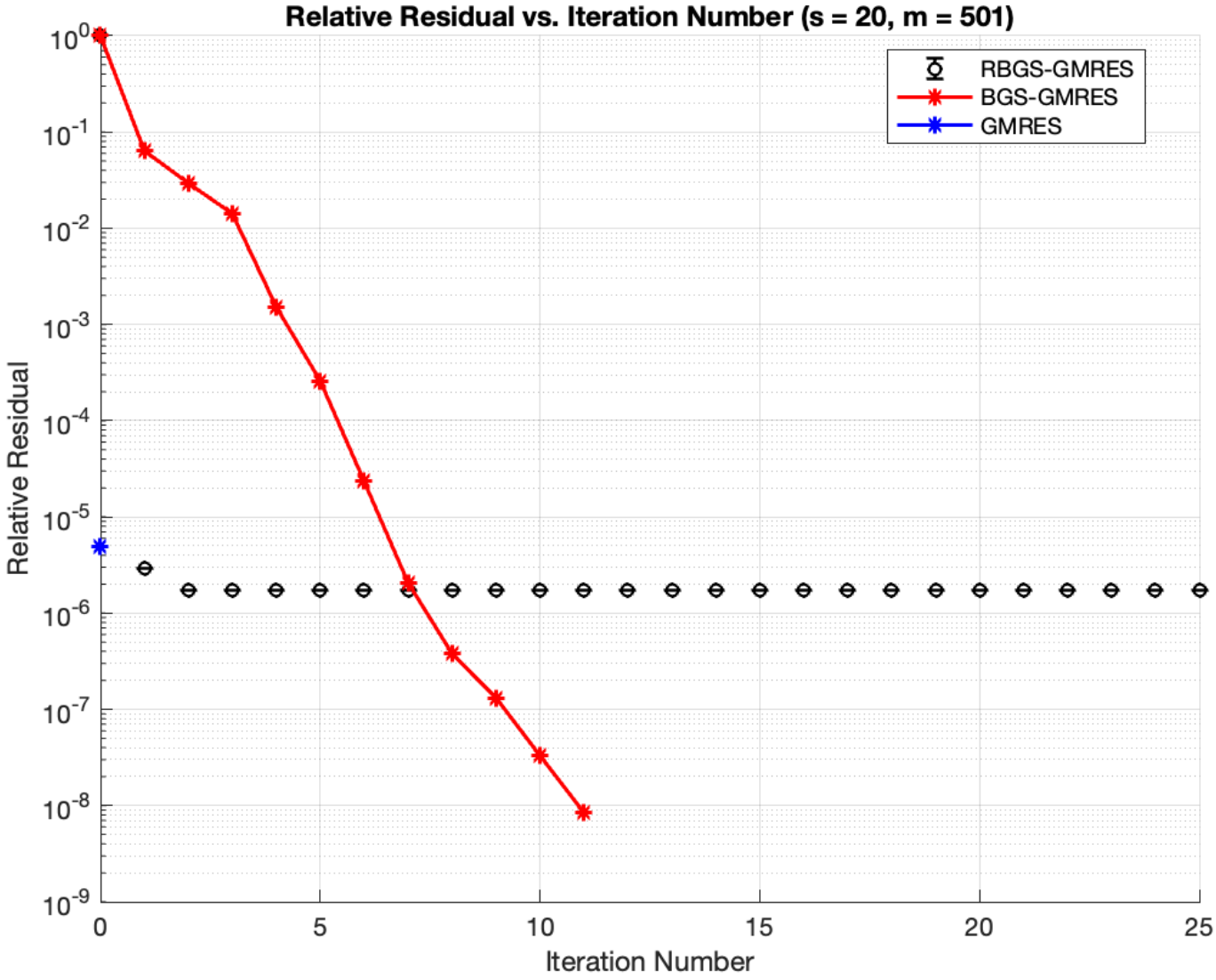
-s=10,



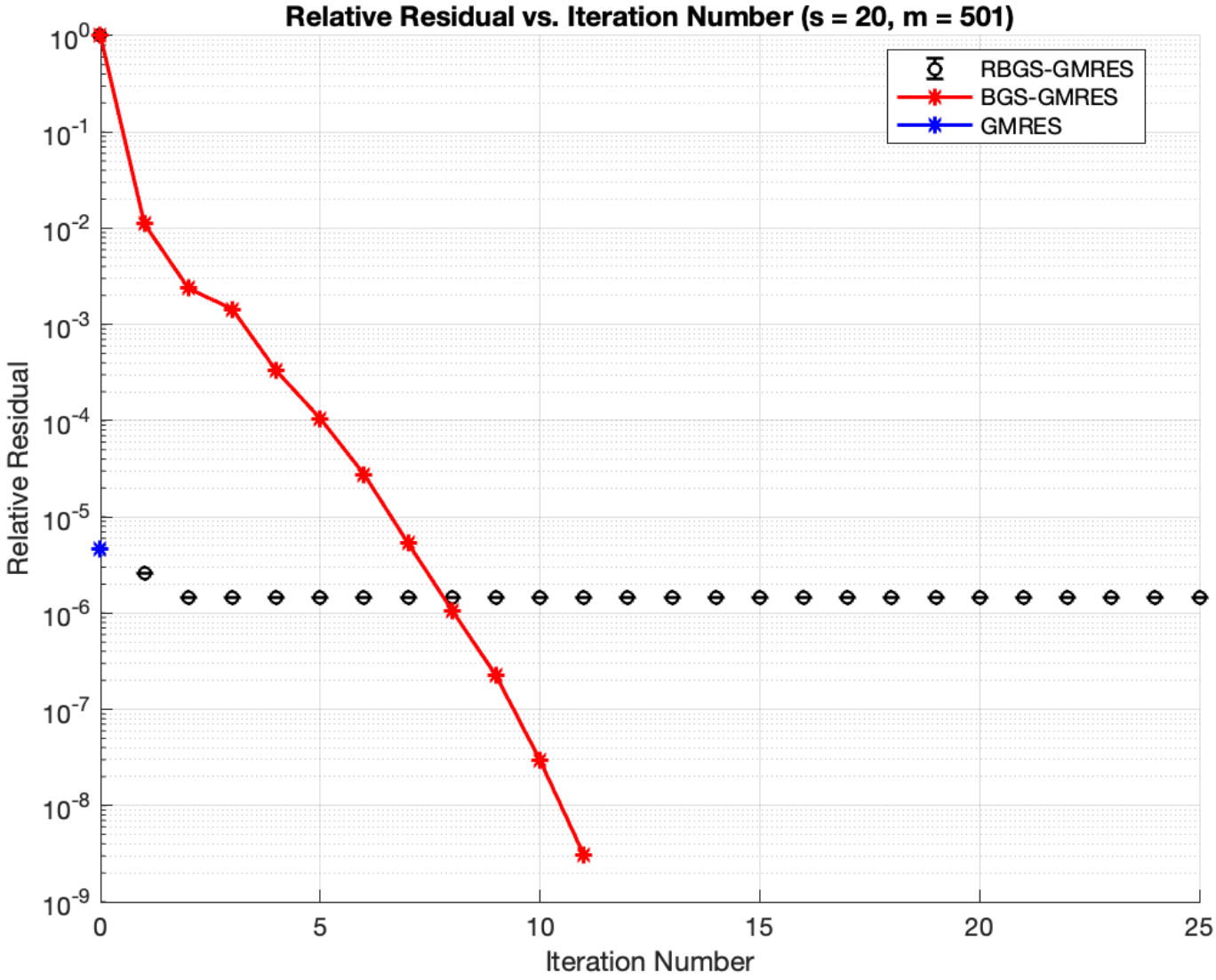
| residual after reorthogonalization


Relative Residual vs. Iteration Number (s = 10, m = 501)
RBGS-GMRES
BGS-GMRES
GMRES
Relative Residual
Iteration Number
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-8
10^-9
0
0.5
1
1.5
2
2.5
3

-s=20,



| residual after reorthogonalization

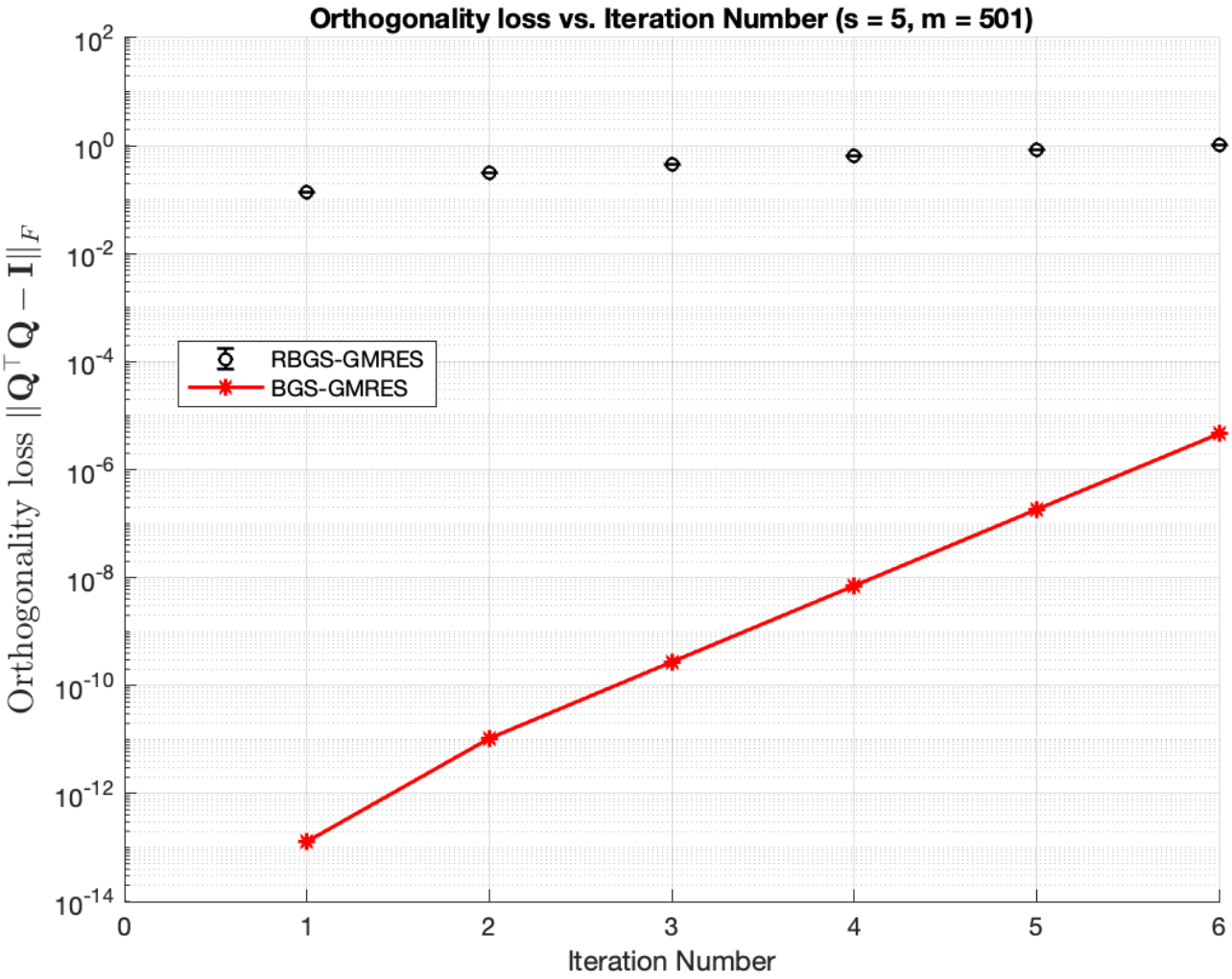


Observation:

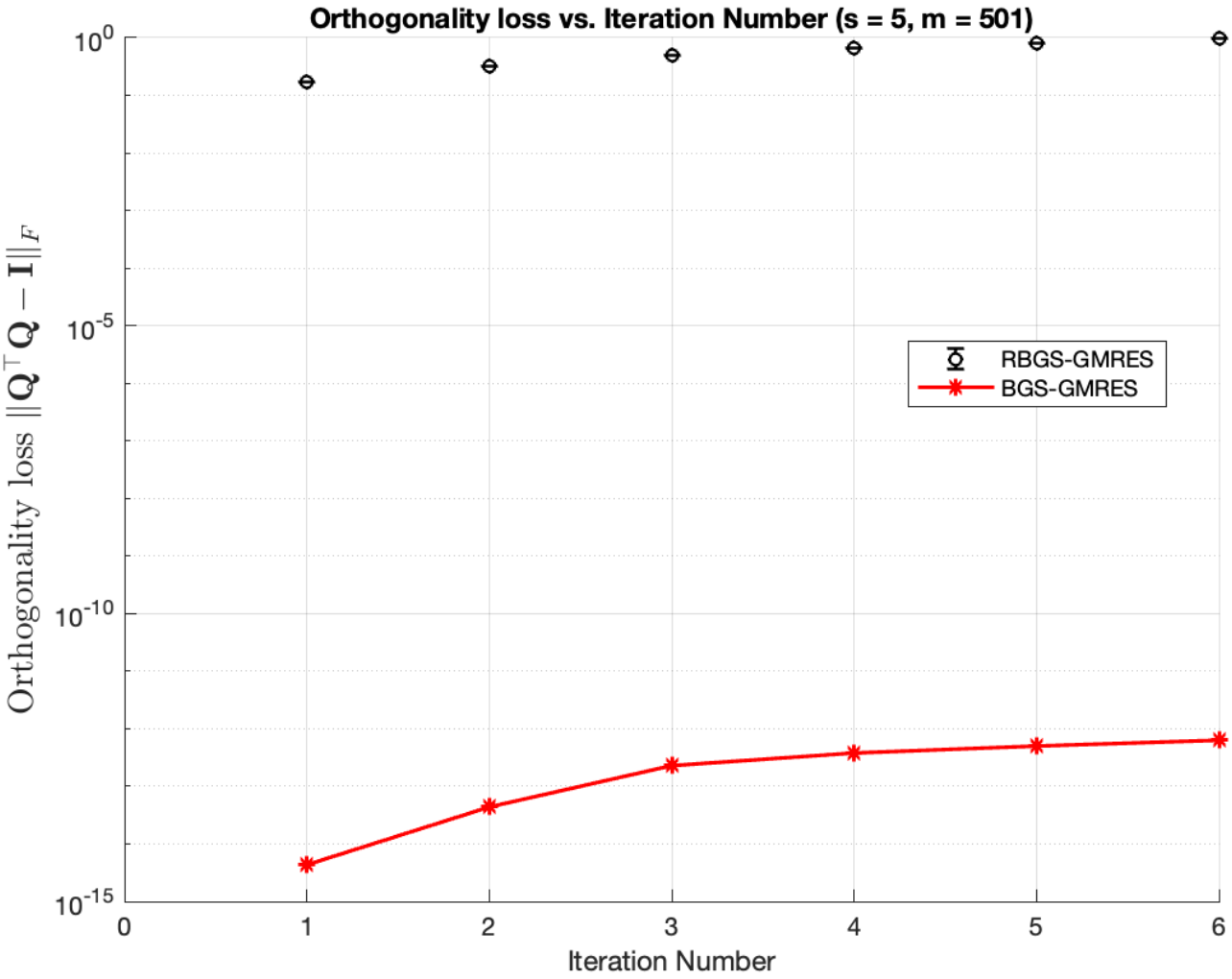
- When s=5/10, all the algorithms converge after 30 steps
- When s=20, BGS converged after 220 steps

Check Orthogonality

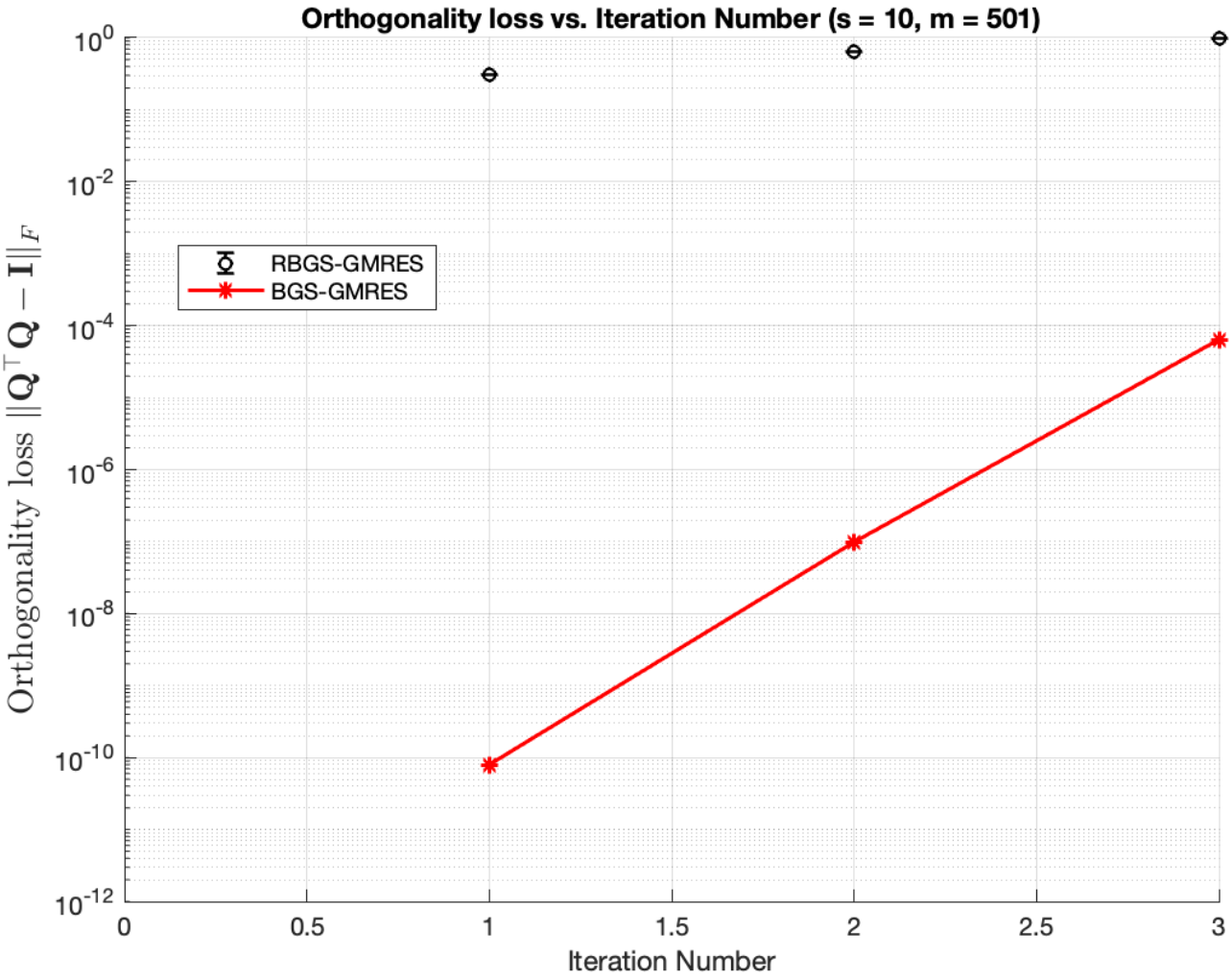
- When s=5,


Orthogonality loss vs. Iteration Number (s = 5, m = 501)
RBGS-GMRES
BGS-GMRES
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
0
1
2
3
4
5
6

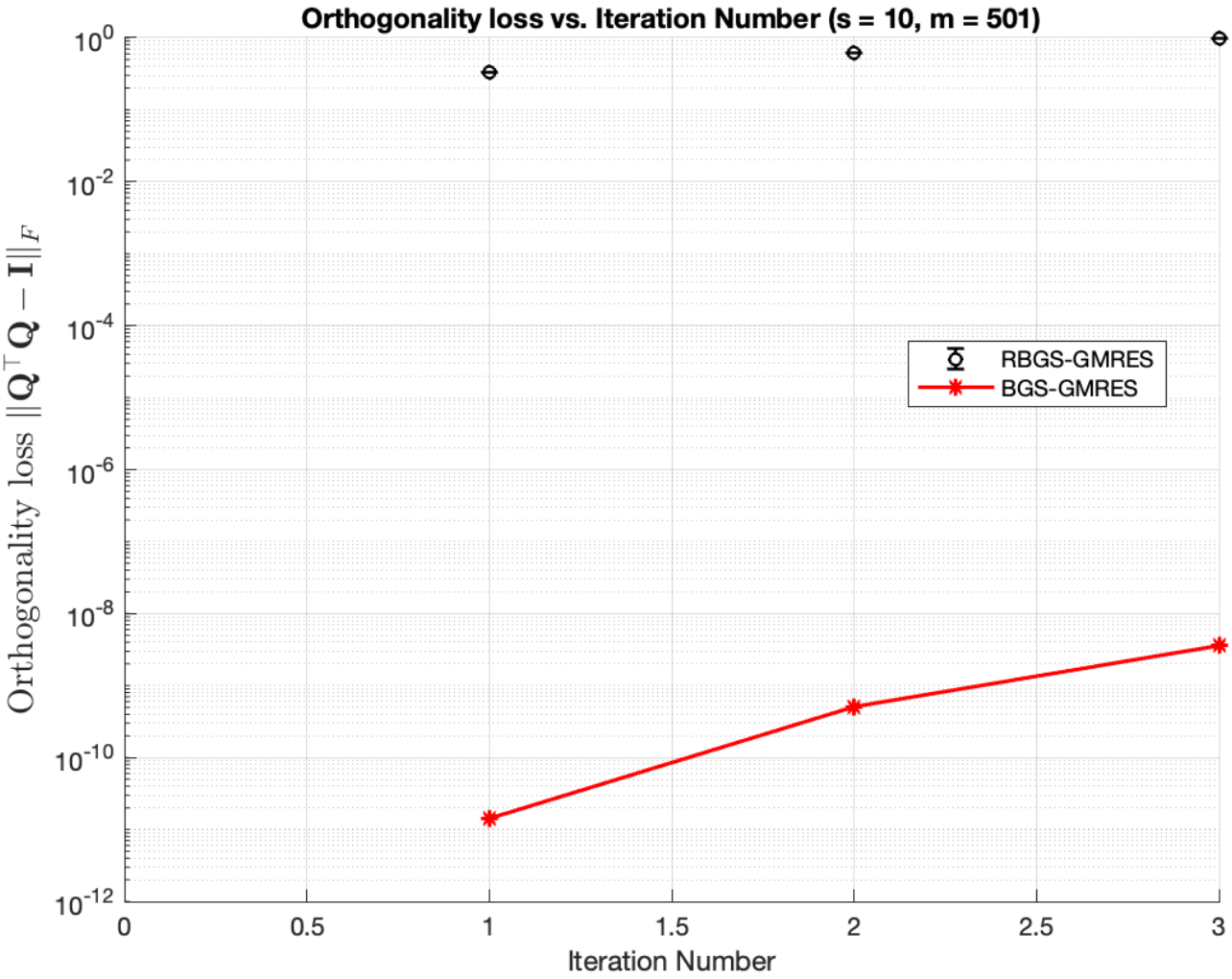
|With both reorthogonalization



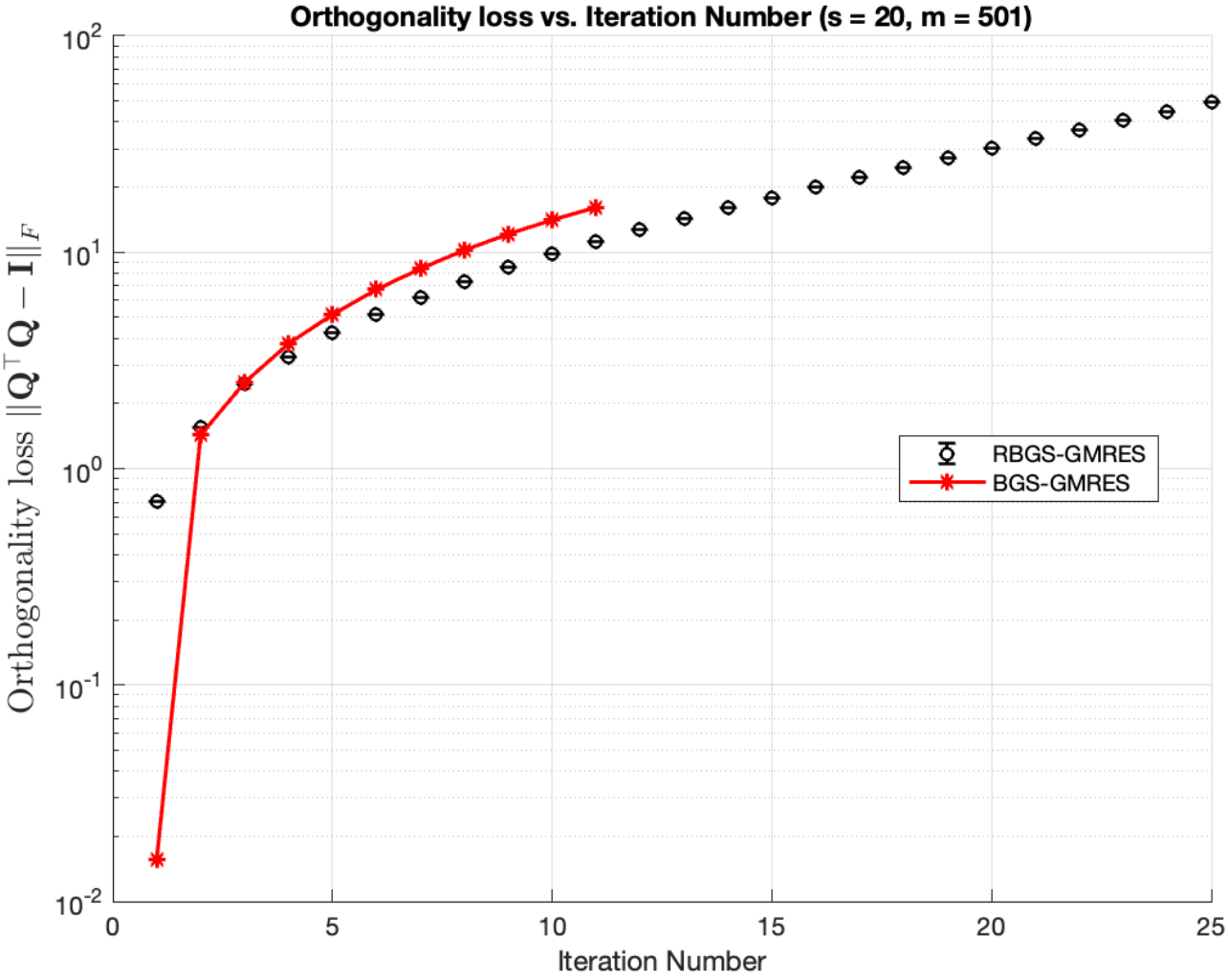
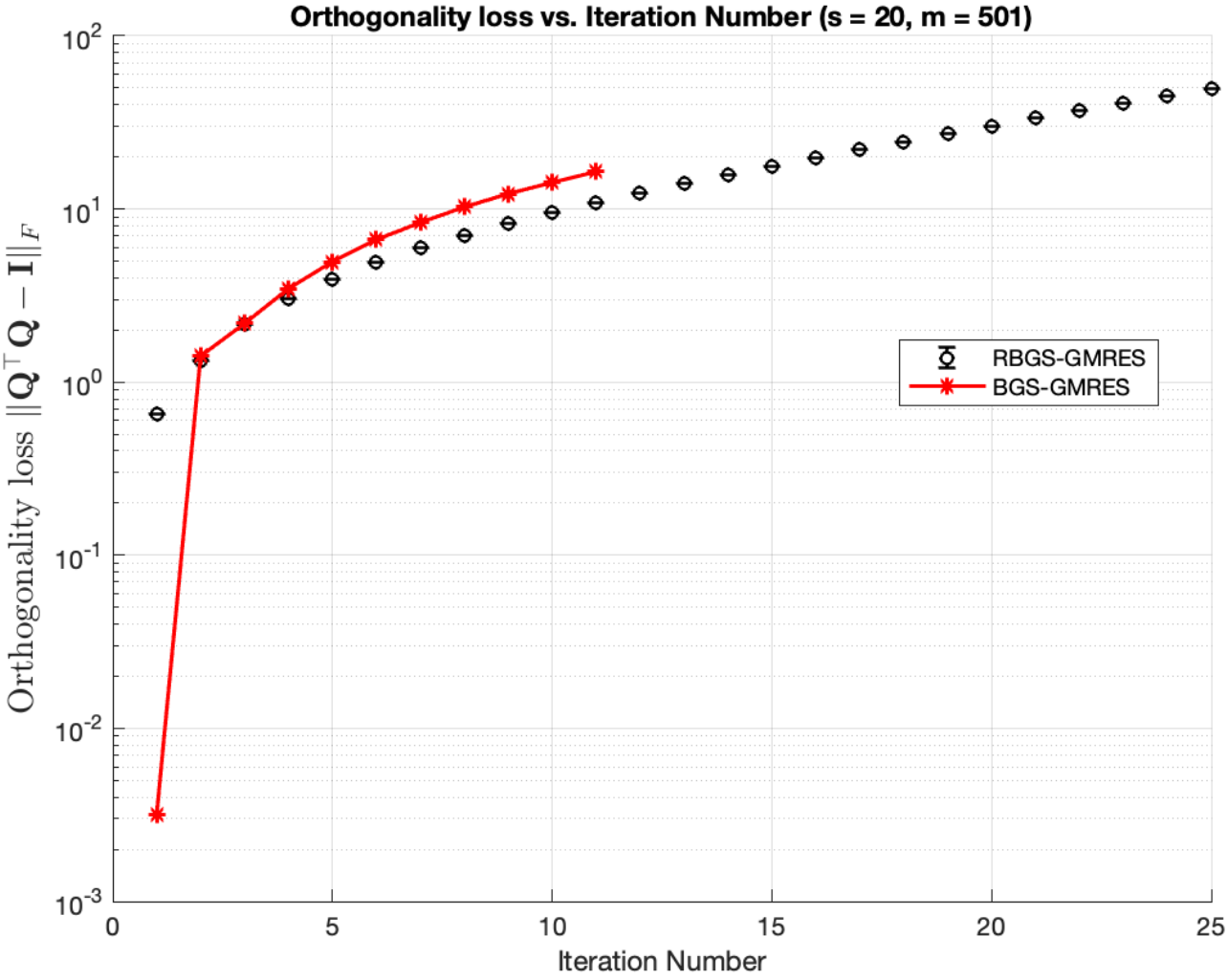
- When s=10,


Orthogonality loss vs. Iteration Number (s = 10, m = 501)
RBGS-GMRES
BGS-GMRES
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
0
0.5
1
1.5
2
2.5
3

|With both reorthogonalization



- When s=20,


Orthogonality loss vs. Iteration Number (s = 20, m = 501)
Orthogonality loss $\|\mathbf{Q}^\top\mathbf{Q}-\mathbf{I}\|_F$
Iteration Number
RBGS-GMRES
BGS-GMRES
10^2
10^1
10^0
10^{-1}
10^{-2}
0
5
10
15
20
25

|With both reorthogonalization