The Critical Group of Hypercubes and Beyond

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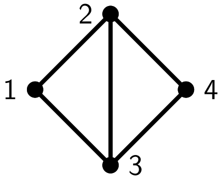
Laplacian, Critical Group, and Matrix–Tree Theorem

Definition

- Laplacian: $L := D - A$
  $D =$ diagonal matrix of vertex degrees, $A =$ adjacency matrix.
- Critical group: $\operatorname{coker} L := \mathbb{Z}^V / \operatorname{row}_{\mathbb{Z}} L = \mathcal{K}(G) \oplus \mathbb{Z}$.

G : 

$$, L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}, \mathcal{K}(G) \cong \mathbb{Z}/8\mathbb{Z}$$

As known as *sandpile group* or *Jacobian*.

Theorem (Kirchhoff's Matrix–Tree Theorem)

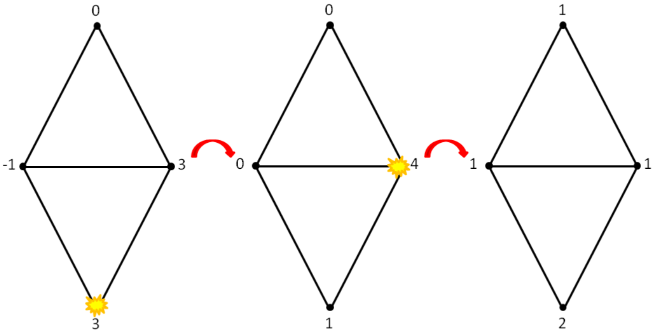
$|\mathcal{K}(G)| = \#$ *of spanning trees of* G.

Examples of Critical Groups

$\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$

- $\mathcal{K}(K_n) = \mathbb{Z}_n^{n-2}$
- $\mathcal{K}(K_{m,n}) = \mathbb{Z}_m^{n-2} \oplus \mathbb{Z}_n^{m-2} \oplus \mathbb{Z}_{mn}$ [Lorenzini '91]
- $\mathcal{K}(W_n) = \begin{cases} \mathbb{Z}_{L_n}^2, & n \text{ odd} \\ \mathbb{Z}_{F_n} \oplus \mathbb{Z}_{5F_n}, & n \text{ even} \end{cases}$ [Biggs '99]
- $\mathcal{K}(K_{n_1,\ldots,n_r}) = \bigoplus \mathbb{Z}_{N_i}^{n_i-2} \oplus \mathbb{Z}_g \oplus \mathbb{Z}_{\sigma_1\sigma_2/g} \oplus \bigoplus_{i\geq 3} \mathbb{Z}_{\sigma_i N}$ [Reiner et al. '03]
  $N = \sum n_i, N_i = N - n_i, g = \gcd(r-1, n_1, \ldots, n_r), \bigoplus \mathbb{Z}_{\sigma_i} \cong \bigoplus \mathbb{Z}_{N_i}, \sigma_1 \mid \sigma_2 \ldots$
- $\mathcal{K}(K_{n+1}^{(a,b)}) = \mathbb{Z}_{a+bn}^{n-2} \oplus \mathbb{Z}_{\gcd(a,b)} \oplus \mathbb{Z}_{a(a+bn)/\gcd(a,b)}$ [Eu–Fu–Lai '10]
- $\mathcal{K}(Paley(q)) = \mathbb{Z}_{(q-1)/4}^{(q-1)/2} \oplus \bigoplus \mathbb{Z}_{p^\lambda}^{f(\lambda)}$ [Chandler et al. '15]
  $Paley(q)$: $V = \mathbb{F}_q, q = p^t; a \sim b \Leftrightarrow \exists c \in \mathbb{F}_q, a - b = c^2$
  $f(\lambda) = \sum_{i=0}^{\min\{\lambda, t-\lambda\}} \frac{t}{t-i}\binom{t-i}{i}\binom{t-2i}{\lambda-i}(-p)^i\left(\frac{p+1}{2}\right)^{t-2i}$

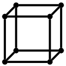
Chip-firing and Abelian Sandpile Model



The critical group is to a graph as

- The Picard group is to a Riemann surface.
  Riemann–Roch and Abel–Jacobi theory of graphs [Baker–Norine '07]
- The ideal class group is to a number field.
  Iwasawa theory of graphs [Gonet '22], Cohen–Lenstra heuristics of random graphs [Clancy et al. '15]

The Critical Group of Hypercubes



Q: How about the critical group of the hypercube graphs Q_N's?
A: The full description is still unknown! But a lot has been worked out.

Theorem (Bai 2003, Ducey–Jalil 2014)

The odd component of $\mathcal{K}(Q_N)$ is isomorphic to that of $\bigoplus_{k=1}^{N} \mathbb{Z}_{2k}^{\binom{N}{k}}$.

Proposition (Bai 2003)

$\mathcal{K}(Q_N)$ has exactly $2^{N-1}-1$ many 2-elementary factors; exactly $2^{N-2}-2^{\lfloor \frac{N-2}{2} \rfloor}$ of them are $\mathbb{Z}_2$.

Proposition (Anzis–Prasad 2016)

The 2-elementary factors of $\mathcal{K}(Q_N)$ are at most $2^{N+\lfloor \log_2 N \rfloor}$.

New Results

- $\mathcal{K}(Q_N)$ has exactly $2^{N-1}-1$ many 2-elementary factors;
- The 2-elementary factors of $\mathcal{K}(Q_N)$ are at most $2^{N+\lfloor \log_2 N \rfloor}$.

General setup: Q_N as the Cayley graph of $\mathbb{F}_2^N$ with generators $M = \{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$.

Theorem (Gao–Marx-Kuo–McDonald–Y. 2024)

1. *$\mathcal{K}(Cayley(\mathbb{F}_2^N; M))$ has at least $2^{N-1}-1$ many 2-elementary factors, with equality if and only if $\sum_{\mathbf{v}\in M} \mathbf{v} \neq \mathbf{0}$;*
2. *The 2-elementary factors of $\mathcal{K}(Cayley(\mathbb{F}_2^N; M))$ are at most $2^{|M|+\lfloor \log_2 N \rfloor - 1}$, which can be tight;*
3. *The largest 2-elementary factor of $\mathcal{K}(Q_N)$ is 2^ν, where $\nu = \max\{\max_{x<N}\{x+\nu_2(x)\}, N+\nu_2(N)-1\}$.*

$\nu_2(x)$ is the largest integer k such that $2^k \mid x$.

Proof Ingredient

Theorem (Benkart–Klivans–Reiner 2018)

$\mathcal{K}(Cayley(\mathbb{F}_2^N; M)) \oplus \mathbb{Z}$ is isomorphic as an abelian group to $\mathbb{Z}[x_1, x_2, \ldots, x_N]/\langle x_1^2 - 1, \ldots, x_N^2 - 1, |M| - \sum_{i=1}^{|M|} \prod_{j=1}^{N} x_j^{M_{ji}} \rangle$.

Elements are the (virtual) rep. of $\mathbb{F}_2^N$, $+$ is direct sum, $\cdot$ is tensor product.

SKETCH OF PROOF OF (1), GENERIC CASE:

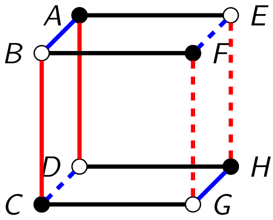
- Work over $\mathbb{Z}_2$ and construct a ring isomorphism to $\mathbb{Z}_2[u_1, \ldots, u_{N-1}]/(u_1^2, \ldots, u_{N-1}^2)$
- $\#$ of 2-elementary factors (plus 1)
- $=$ the dimension of the rings
- $= \#$ of standard monomials of the new ring
- $= \#$ of square-free monomials in $u_1, \ldots, u_{N-1} = 2^{N-1}$.

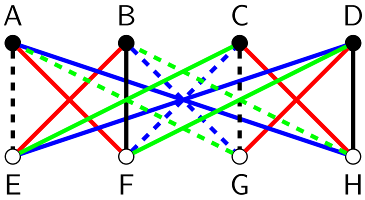
Adinkras

Definition (Faux–Gates 2004)

An *Adinkra/Cliffordinkra* is a (connected, simple) graph with each edge colored by one of N colors and is either solid or dashed, such that:

1. the graph is bipartite;
2. every vertex is incident to exactly one edge of each color;
3. for every pair of distinct colors, the graph restricted to these edges is a disjoint union of 4-cycles;
4. each bi-color 4-cycle contains an odd number of dashed edges.





Some Physics (that I don't really know)

Supersymmetry (SUSY): Every boson ϕ has an associated fermion ψ and vice versa.

Physicists are interested in *SUSY superalgebras*, some particularly interesting/useful ones satisfy:

- the algebra is generated by $Q_1, \ldots, Q_N, H := \sqrt{-1}\partial_t$;
- each Q_i takes some ϕ to some ψ up to signs and H, vice versa;
- $Q_i Q_j + Q_j Q_i = 2\delta_{ij} H$ and $Q_i H = H Q_i$.

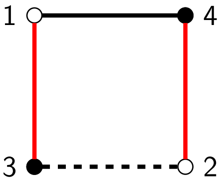
If we pretend H does nothing, we get a *Clifford algebra* $Cl(0, N)$: $Q_i^2 = I, Q_i Q_j = -Q_j Q_i$.

Signed Graphs

Definition

- Signed Graph: A graph with a signing $E \to \{+, -\}$ of the edges.
- Laplacian: $L := D - A$, but an entry of A is -1 if the edge is $-ve$.
- Critical group: $\mathcal{K}(G) := \operatorname{coker} L$.

G :



$$, L = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & \mathbf{1} & -1 \\ -1 & \mathbf{1} & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}, \mathcal{K}(G) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

$|\mathcal{K}(G)|$: A combinatorial interpretation via Matrix–Tree Theorem for signed graphs [Zaslavsky 1982], also as the volume of tropical Prym varieties [Len–Zakharov 2022].

Basic properties of Adinkras

Theorem (DFGHILM 2008)

The underlying graph of an Adinkra is a Cayley graph of $\mathbb{F}_2^n$. (Complete classification exists but omitted here.)

Proposition

The eigenvalues of L are $N \pm \sqrt{N}$, each has multiplicity $\#V/2$. Hence $\det L = (N^2 - N)^{\#V/2}$.

The Critical Group of an Adinkra

Theorem (Iga–Klivans–Kostiuk–Y. 2023)

The odd component of $\mathcal{K}(A)$ is isomorphic to that of $\mathbb{Z}_{N^2-N}^{\#V/2}$.

Theorem (Y. 2024)

Let $\#V/2+m-1$ be the number of 2-elementary factors of the critical group of the underlying Cayley graph of A. Then

$$\mathcal{K}(A) \cong \mathbb{Z}_2^m \oplus \mathbb{Z}_{(N^2-N)/2}^m \oplus \mathbb{Z}_{N^2-N}^{\#V/2-m}.$$

Monodromy Pairing

Every critical group $\mathcal{K}$ has a canonical perfect pairing $\langle \cdot, \cdot \rangle : \mathcal{K} \times \mathcal{K} \to \mathbb{Q}/\mathbb{Z}$.

Def: For $[\mathbf{x}] \in \mathcal{K}$, pick $m \in \mathbb{N}$ and $\mathbf{f} \in \mathbb{Z}^v$ such that $L\mathbf{f} = m\mathbf{x}$. $\langle [\mathbf{x}], [\mathbf{y}] \rangle := \mathbf{f}^T \mathbf{y}/m$.

This pairing is to a graph as the Weil pairing is to an elliptic curve.

Proposition (Y. 2024)

Fix a color c. $\mathbf{e}_u - \mathbf{e}_v$'s of edges uv of color c form an "orthonormal set" with $\langle \mathbf{e}_u - \mathbf{e}_v, \mathbf{e}_u - \mathbf{e}_v \rangle = 2/N$ (up to vertex switching).

Proof Ingredient: A concrete description of $\langle \cdot, \cdot \rangle$ for (signed) graphs with two non-zero Laplacian eigenvalues. [Hung–Y. 2022]

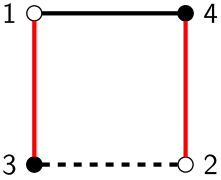
Corollary

$\mathcal{K}(A)$ contains a subgroup isomorphic to $\mathbb{Z}_{N/\gcd(2,N)}^{\#V/2}$.

Colored Laplacian

For the odd component of $\mathcal{K}$, the most difficult part is to determine # of p-elementary factors of L for odd $p \mid N$.

Fix a color c and replace each "1" in L from an edge of color c by x.



$$\hat{L} = \begin{pmatrix} x+1 & 0 & -x & -1 \\ 0 & x+1 & 1 & -x \\ -x & 1 & x+1 & 0 \\ -1 & -x & 0 & x+1 \end{pmatrix}$$

Proposition

$\det \hat{L} = [2(N-1)x + (N-1)(N-2)]^{\#V/2}$.

Some Algebraic Setup

We can further modulo the entries by p and/or setting $x = 1$.

$$\begin{tikzcd} \mathbb{Z}[x] \arrow[r, "x\mapsto 1"] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{F}_p[x] \arrow[r, "x\mapsto 1"] & \mathbb{F}_p \end{tikzcd}, \quad \begin{tikzcd} \hat{L} \arrow[r] \arrow[d] & L \arrow[d] \\ \tilde{L} \arrow[r] & \overline{L} \end{tikzcd}$$

$\det L = (N^2 - N)^{\#V/2}, \det \tilde{L} = (-2(x-1))^{\#V/2}$.

p-rank for odd $p|N$

- $\det L = (N^2 - N)^{\#V/2}$. By [Lorenzini 2008], each elementary factor divides $N^2 - N$, so L has $\geq \#V/2$ many p-elementary factors.
- $\det \tilde{L} = (-2(x-1))^{\#V/2}$. So $\tilde{L}$ has $\leq \#V/2$ many $(x-1)$-elementary factors.

Lemma

$\#$ of p-elementary factors of L = corank of $\overline{L}$
$=$ $\#$ of $(x-1)$-elementary factors of $\tilde{L}$.

$\Rightarrow$ All numbers are exactly $\#V/2$.

Proposition

Let $M \in \mathbb{Z}^{n\times n}$. The $\#$ of p-elementary factors of M equals $\min\{\mathrm{ord}_{x-1} \det \hat{M} \in \mathbb{F}_p[x] : \hat{M} \in \mathbb{Z}[x]^{n\times n}, \hat{M}|_{x=1} = M\}$.

Other interesting instances of this proposition?

Thank you!

N	$\mathrm{Syl}_2(\mathcal{K}(Q_N))$
2	$\mathbb{Z}_4$
3	$\mathbb{Z}_2\mathbb{Z}_8^2$
4	$\mathbb{Z}_2^2\mathbb{Z}_8^4\mathbb{Z}_{32}$
5	$\mathbb{Z}_2^6\mathbb{Z}_8^4\mathbb{Z}_{16}\mathbb{Z}_{64}^4$
6	$\mathbb{Z}_2^{12}\mathbb{Z}_4^4\mathbb{Z}_8\mathbb{Z}_{32}^4\mathbb{Z}_{64}^{10}$
7	$\mathbb{Z}_2^{28}\mathbb{Z}_4\mathbb{Z}_{16}^8\mathbb{Z}_{32}^6\mathbb{Z}_{64}^{14}\mathbb{Z}_{128}^6$
8	$\mathbb{Z}_2^{56}\mathbb{Z}_4^2\mathbb{Z}_{16}^{16}\mathbb{Z}_{32}^{12}\mathbb{Z}_{64}^{28}\mathbb{Z}_{128}^{12}\mathbb{Z}_{1024}$
9	$\mathbb{Z}_2^{120}\mathbb{Z}_4^{10}\mathbb{Z}_{16}^{16}\mathbb{Z}_{32}^{26}\mathbb{Z}_{64}^{48}\mathbb{Z}_{128}^{26}\mathbb{Z}_{512}\mathbb{Z}_{2048}^8$
10	$\mathbb{Z}_2^{240}\mathbb{Z}_4^{36}\mathbb{Z}_8^{26}\mathbb{Z}_{32}^{16}\mathbb{Z}_{64}^{148}\mathbb{Z}_{256}\mathbb{Z}_{1024}^{26}\mathbb{Z}_{2048}^{18}$
11	$\mathbb{Z}_2^{496}\mathbb{Z}_4^{66}\mathbb{Z}_8^{32}\mathbb{Z}_{16}^{100}\mathbb{Z}_{64}^{164}\mathbb{Z}_{128}\mathbb{Z}_{512}^{100}\mathbb{Z}_{2048}^{64}$