Oriented Matroids from Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$

Chi Ho Yuen

Brown University

Joint Work with Marcel Celaya (ETH) and Georg Loho (Kassel)

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Oriented Matroid: An abstraction of linear (in)dependence over $\mathbb{R}$.

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$$\begin{pmatrix} 3 & 1 & 0 & 2 \\ -1 & 1 & 2 & -1 \end{pmatrix}$$

$$\chi(a,b) = \operatorname{sign}\det\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} = +, \chi(a,d) = -, \chi(c,b) = -.$$

A Crash Course in Oriented Matroids

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Definition

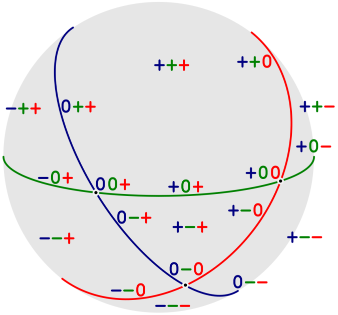
A *chirotope* is a (non-zero) map $\chi : E^d \to \{+,-,0\}$ that is

- alternating;
- Grassmann–Plücker: $\forall\ |X| = d-1, |Y| = d+1$, $(-1)^k\chi(X, y_k)\chi(Y \setminus y_k)$'s either contain both a +ve and a -ve term, or are all zeros.
  Follows from "usual" GP: $\sum_{k=1}^{d+1}(-1)^k \det(A|_{X,y_k}) \det(A|_{Y\setminus y_k}) = 0$.

Pseudosphere Arrangements

Theorem (Folkman-Lawrence 1978)

Oriented Matroids $\Leftrightarrow$ Pseudosphere Arrangements $\approx$ *Real Hyperplane Arrangements.*



Covectors $\Leftrightarrow$ Sign patterns of the regions $\approx$ Sign patterns of the row span.

Oriented Matroids in Mathematics

- Graph Theory: directed graphs, flow networks
- Convex Geometry: real hyperplane arrangements, polytopes
- Optimization: linear programming (simplex method) and beyond
- Algebraic Geometry: strata of real Grassmannians (Mnëv's universality theorem)
- Topology: real vector bundles and their characteristic classes

Matching Fields

What if instead of $\det(A|_\sigma)$'s, we only compute one term per $\det(A|_\sigma)$?

$$\operatorname{sign}\det\begin{pmatrix} +\mathbf{e^{18}} & -e^{6} & -e^{9} \\ -e^{12} & -e^{16} & +\mathbf{e^{22}} \\ +e^{20} & -\mathbf{e^{14}} & +e^{2} \end{pmatrix} = + = (-1)(1 \cdot -1 \cdot 1)$$

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Notation: Entries of $A \Leftrightarrow$ Edges of $K_{R,E}$, with $|R| = d, |E| = n$.

Definition

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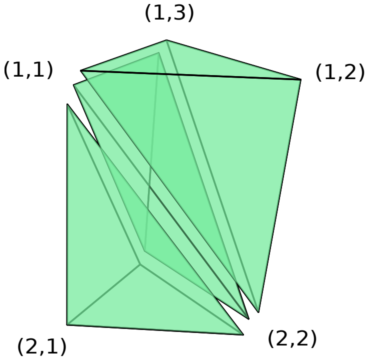
Example: Take the max. perfect matchings w.r.t. generic weights.
Motivation: Tropical geometry & Gröbner theory of maximal minors [Sturmfels–Zelevinsky 93].

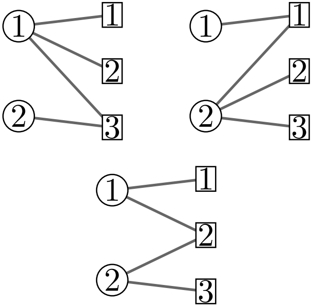
Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$

Notation: Vertices of $\triangle_{d-1} \times \triangle_{n-1} \Leftrightarrow$ Edges of $K_{R,E}$.

Proposition

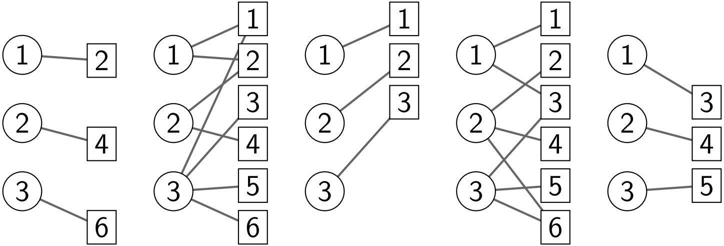
The vertices of any full-dim simplex in $\triangle_{d-1} \times \triangle_{n-1}$ form a spanning tree.





Polyhedral Matching Fields

Fix a triangulation. Take all perfect matchings that are subgraphs of some trees. This gives a *polyhedral matching field*.



Why Triangulations of $\triangle_{d-1} \times \triangle_{n-1}$?

Reason I: They appear in many places as "tropical matrices".

- Algebraic Geometry: toric Hilbert schemes, Schubert calculus
- Tropical Geometry: tropical convexity, Stiefel tropical linear spaces
- Optimization: tropical linear programming, mean payoff game
- Tropical pseudohyperplane arrangements, tropical oriented matroids, trianguloids, etc

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Reason II: Correct direction in view of [Sturmfels–Zelevinsky]

$$\text{Coherent} \subsetneq \text{Polyhedral} \subsetneq \text{Linkage}$$

Main Theorems

Theorem (Celaya–Loho–Y. 2020+)

Polyhedral matching fields induce uniform oriented matroids.

Proof Strategy: Divide-and-conquer with matroid subdivisions.

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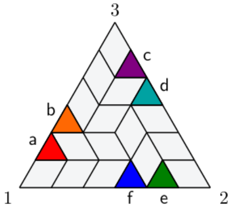
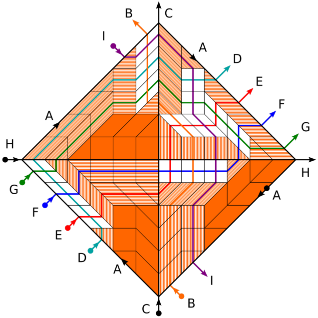
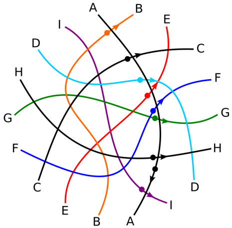
Topological representation directly from the triangulation.







Selected Topics

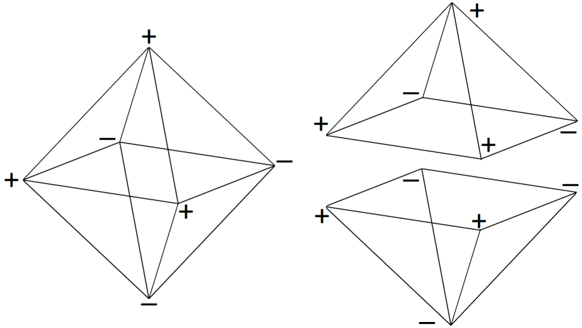
- Proof of the Main Theorem goto
- Topologicial Representation goto
- Connection with Optimization goto
- Extension to Matroids over Hyperfields goto

Proof Sketch: Divide

Divide: The triangulation induces a *matroid subdivision* of the hypersimplex by *transversal matroid polytopes* (of the trees).

Definition

Matroid polytope: $\mathrm{conv}\{\mathbf{e}_B : B \in \mathcal{B}(M)\}$.
Matroid subdivision: Subdivision of a MP by MPs.
Transversal matroid: $\sigma \subset E$ is a basis iff $\exists\ R \equiv \sigma$ perfect matching in T.



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Conquer: Each restriction is a chirotope (realizable by A restricted to the edges of the tree).

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Merge:

Lemma (Celaya–Loho–Y.)

Let $\chi : \mathcal{B}(M) \to \{+,-\}$ and $M_1, \ldots, M_k$ be a matroid subdivision of M. If every χ_{M_i} is a chirotope, then χ is also a chirotope.

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PROOF: Reduce to 3-term GP and analyze the subdivision on 3-dim faces.

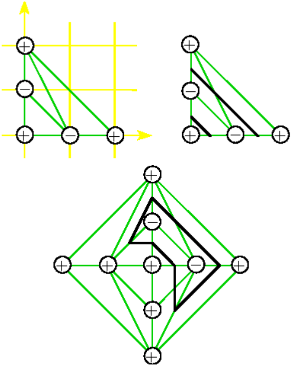
Definition

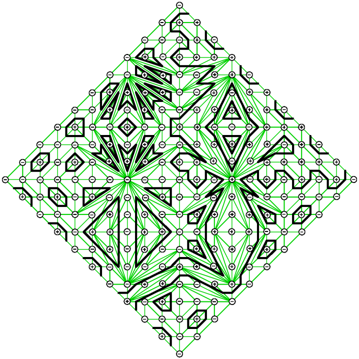
3-term GP relation: $\forall a, b, c, d \in E$,
$\chi(a, b, \_)\chi(c, d, \_), -\chi(a, c, \_)\chi(b, d, \_), \chi(a, d, \_)\chi(b, c, \_)$,
either contain both a $+$ve and a $-$ve term, or are all zeros.

goto B goto C goto D goto E

Viro's Patchworking

A method developed by Viro to construct real hypersurfaces of prescribed topology from signed *regular* triangulation of $n\triangle_{d-1}$.



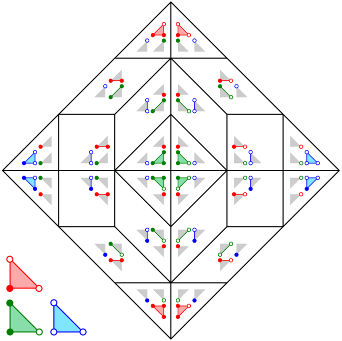


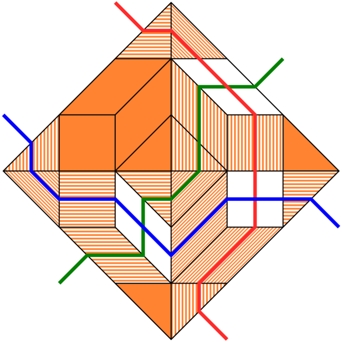
Theorem (Viro 1980's)

The locus is isotopic to some real algebraic hypersurface.

Patchworking Oriented Matroids

Using *Cayley trick*, convert a triangulations of $\triangle_{d-1} \times \triangle_{n-1}$ into a *fine mixed subdivisions* of $n\triangle_{d-1}$.





Theorem (Celaya–Loho–Y. 2020+)

The locus is a pseudosphere arrangement representing χ.

Some Proof Ingredients

Combinatorics: The face poset is the *covector lattice*.

- Faces $\Rightarrow$ Signed Forests $\Rightarrow$ Covectors: Restrict to individual cells.
- Surjectivity: Borsuk–Ulam + Topological Representation Theorem.

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Topology: The CW complex is *regular*.

- View patchworking as a stepwise cell merging.
- Show that every step preserves regularity.

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Tropical Linear Programming

$(+,\times) \mapsto (\min, +)$

$$\text{(Optional)Maximize } \min(x+1, y-5, z-1) \text{ subject to}$$
$$x+0 \le \min(y+0, z+0)$$
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- Equivalent to *mean payoff game*: In a weighted (bipartite) digraph with a starting v_0, A and B take turns and moves to a new vertex $v_i \xrightarrow{w_i} v_{i+1}$. Can B guarantee $\sum w_i$ is bounded from above?
- Applications: Scheduling problems, analysis of resource-constrained systems, string matching etc

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- Applications: Scheduling problems, analysis of resource-constrained systems, string matching etc
- Signed triangulations of $\triangle_{d-1} \times \triangle_{n-1}$ are the correct framework, matchings are the bases in pivot rules of tropical simplex method.

Algorithms and Complexity

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- Candidate: Simplex method with combinatorial pivot rules, e.g., those that can be described using the underlying oriented matroid.
- The construction presented here systematically relates the combinatorics of classical and tropical LP, hence (potentially) relates the two open problems.

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Matroids over Hyperfields

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A **strong** matroid over $\mathbb{H}$ is an alternating $\chi : E^d \to \mathbb{H}$ such that

$$0 \in \boxplus_{k=1}^{d+1} (-1)^k \chi(X, y_k) \otimes \chi(Y \setminus y_k).$$

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EXAMPLE: Oriented matroids = Matroids over $\mathbb{S}$.

Also linear subspaces, matroids, valuated matroids, phase matroids...

Caution: In general, $\{\text{Strong matroids}\} \subsetneq \{\text{Weak matroids}\}$.

Our Theorem for Matroids over Hyperfields

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Suppose $\mathbb{H}$ *has the IP. Then given a polyhedral* $\{M_\sigma\}$ *and a nowhere zero* $\mathbb{H}$*-matrix* A, $\chi(\sigma) := \mathrm{sign}(M_\sigma) \bigotimes_{e \in M_\sigma} A_e$ *is a* **weak** *matroid over* $\mathbb{H}$.

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Question

Is the same statement true for strong matroids?

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Thank you!

Marcel Celaya, Georg Loho, Chi Ho Yuen. *Oriented Matroids from Triangulations of Products of Simplices.* arXiv:2005.01787.

________________. *Patchworking Oriented Matroids.* arXiv:2010.12018.