Finding the most optimal factor to focus on for the largest improvement in my smash effectiveness for badminton.

Math IA: HL AA

1. Introduction

The most popular shot in badminton, and the shot that wins you the most points is the smash. The smash is what gives badminton the title of the fastest sport in the world, with a record setting speed of 565 km/h or around 157 m/s. At elite levels, the smash is responsible for 54% of unconditional winners, or forced errors (McErlain-Naylor et al., 2020).

As the smash is arguably the most important shot and by far the most popular shot in badminton, improving your smash is a big part of training. Personally I have been playing badminton for over 7 years, yet my smash has always been a lacking aspect in my game that I want to improve.

The main objective of a smash is to reach the opponent in as little time as possible, thus allowing your opponent the least time to react and attempt to hit a return. The 2 main factors of the smash that affects the time taken, is initial velocity, as well as launch angle.

Therefore the aim for this exploration is finding the most optimal factor (initial velocity or launch angle) to focus on training in order to have the biggest improvement to my smash by decreasing the time taken to reach my opponent.

To achieve my aim, I will be collecting data for my smash. Then using equations of motion to solve for a mathematical relationship between time taken, initial velocity and angle. Using that relationship, I will be plotting its graph and computing the slope of the tangent line at my current values for both areas (instantaneous rate of change) to quantify the short term impact of improving each area. The secant line slope between my current and profession values would also be calculated to quantify long term impact. Doing so I can mathematically calculate which area is the most optimal to focus on training for the biggest improvement to my smash in both the short and long term.

To create a focused and mathematically manageable exploration, I will examine and optimize one specific type of badminton smash under controlled conditions. The smash I will analyze is the straight smash because it is the most commonly used smash in badminton and a crucial part of competitive play. Specifically, I will focus on a jump straight smash executed from the rear court, where the goal is to minimize the time it takes for the shuttle to reach the opponent. By narrowing the scope in this way, I can reduce external variables and ensure that the mathematical modeling remains targeted, consistent, and relevant.

2. Calculations

2.1 Finding force of drag:

As drag is a significant force in projectile motion equations, and especially badminton, finding an accurate equation for the force of drag is essential for accurate results.
To find the force of drag acting on the shuttle we use the drag formula:

$$F_D = \frac{1}{2}Cpv^2A$$

F_D = Force of drag
C = Drag Coefficient
p = Air Density
v = Velocity
A = Reference area (cross section area)

This is the formula for quadratic drag, it is used due to the badminton shuttle's drag greatly affecting the flight path of the shuttle. As linear drag where ($F_D \propto v$) deals with slow speeds, quadratic drag is instead used, where the object travels at higher speeds and inertial forces dominate.

To ensure accuracy, the shuttle model used during calculations is the LingMei 90Pro due to it being the shuttle I used when collecting data. Reference area of the shuttle can be found in Figure 1. The dimensions of the shuttle are found in Figure 2.
This data will not be fully accurate due to each shuttle having natural imperfections and differences. Also the shape and fluff of the shuttle can change shape when hit. However this would only create minor uncertainties that wouldn't hugely impact results.

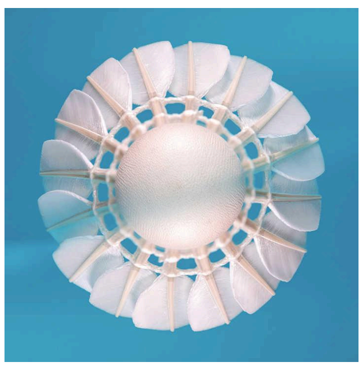

Figure 1. Reference Area for LingMei 90Pro

Diameter	About 6.5cm
Weight	About 5.0g

Figure 2. Dimensions of the Lingmei 90 pro (Anhui Lingmei Sporting Goods Co., Ltd., 2024)

Calculations for Reference Area(A):
Due to the rather simple shape, this shape would be calculated as a circle. The gaps in the shuttle and other imperfections would be ignored and shouldn't have a significant impact on the final results.

$$A = (\frac{d}{2})^2 \pi$$

$$A = (\frac{6.5}{2})^2 \pi\, cm^2$$

$$A = 10.5625\pi\, cm^2$$

$$A = 0.00105625\pi\, m^2$$

Drag Coefficient(C) is set at 0.5. This is taken from the findings of an experiment conducted by the Washington State University (Paudel, 2023). This measurement will not be fully accurate due to the tests not using the same shuttle as the one I am using. However, the Drag Coefficient for my shuttle would only have a slight difference and would also round to 0.5, effectively having no impact on results. I also have no way of collecting this data myself, so using outside research is the only feasible method.

Air Density(p) is set at $1.225\ kg/m^3$. This is the average air density at 15 degrees celsius. 15 degrees was taken directly from the thermostat at the gym where I collected my data.

Velocity(v) is dependent on each individual smash and can not be generalized. As my velocity can not be directly measured with the equipment that I have access to, it would be left as an unknown variable during calculations.

When dealing with only horizontal and vertical components of drag:

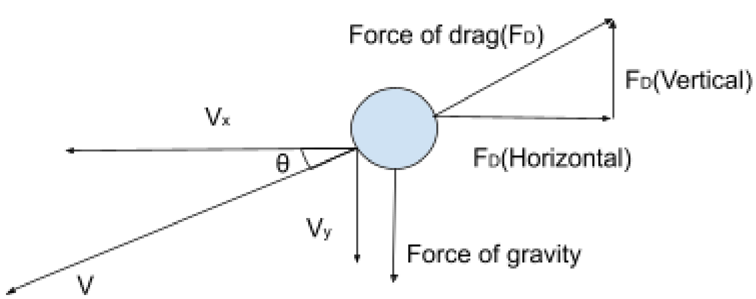

Figure 3. Free body diagram of shuttle in motion

In equations below, θ is not constant as angle changes as the force of gravity and drag acts on the shuttle. However, we assume θ to be constant at the value of the launch angle in order to keep the math within the scope of this exploration. This will make our equation differ from real collected values, however due to badminton smashes taking less than a second to reach the ground, while traveling at high speeds, the difference in θ will not be drastic enough to see significant changes in any result we find. From this point forward, all uses of θ would represent launch angle, despite not being fully accurate.

$F_D(Horizontal) = \frac{1}{2}Cpv_x^2Acos\theta$, where $v_x = vcos\theta$ Seen in Figure 3.

The horizontal drag force depends on the effective cross-sectional area of the shuttlecock facing the airflow. When the shuttlecock is struck at a **launch angle(θ)**, its feathers tilt relative to the launch angle, reducing the reference area of in horizontal calculations. To calculate for this reduction, reference area is simplified to a plane, and therefore the horizontal reference area ($A_{horizontal}$) = $Acos\theta$.

This is not fully accurate for a badminton shuttle as it is a non-spherical 3d object, however fully accounting for an 100% accurate reference area would require advanced modeling beyond the scope of this exploration. Here, simplifying to a plane would give the most accurate result using my current mathematical ability. Due to the launch angle in badminton usually being small, not fully accounting for reference area will not significantly impact any results as estimated uncertainty is less than 1%. The use of v_x is also due to this equation only dealing with horizontal motion.

$$F_D(Vertical) = \frac{1}{2}Cpv_y^{\ 2}Asin\theta,\text{ where } v_y = vsin\theta \quad \text{Seen in Figure 3.}$$

All the same reasoning and effects for horizontal also apply to the vertical component of the force of drag.

2.2 Deriving an equation for initial velocity

Objective: To determine the initial velocity of the shuttlecock using measurable quantities: horizontal displacement and time.

To derive an equation for initial velocity, the situation is simplified to only look at horizontal motion. This creates inaccuracy within the method, as technically spitting the motion into its horizontal and vertical components is not mathematically rigorous. This method simplifies the problem using trigonometry to establish the relationship between the components, which only approximates the real value. However this method is still used despite simplification and inaccuracy due to 3 main reasons:

- **Feasibility of mathematics** - As solving for velocity without splitting into its components would require solving coupled system of nonlinear differential equations. This would need understanding of multivariable calculus, vector dynamics, and numerical integration methods due to these equations having no closed-form solution. These all go out of the scope for this IA and beyond what is feasible at my current level of mathematics.
- **Feasibility of data collection** - Data collection for 2D motion such as displacement would be a lot harder to collect, and any data collected would result in significant uncertainties.
- **Relative accuracy** - Giving the limitations above, using the method of splitting the equation into its horizontal and vertical components, would result in the most accurate method feasible.

With this simplification justified, the next step is to derive an equation horizontal displacement ($x(t)$), which can be simply rearranged to isolate initial velocity (v_0).

To derive for an equation for displacement, the formula for displacement would be used:

$$x(t) = \int v_x(t)dt,\text{ where } v_x(t) \text{ is the horizontal component of velocity with respect to time.}$$

Solving the differential equation for horizontal motion to derive $v_x(t)$

$$\frac{dv_x}{dt} = -\frac{1}{m} \cdot F_D(Horizontal)$$

- m = mass
- $\frac{dv_x}{dt}$ Represents horizontal acceleration (rate of change of horizontal velocity)

Substituting in the equation for $F_D(Horizontal)$ found in section 2.1:

$$\frac{dv_x}{dt} = -\frac{1}{m} \cdot \frac{1}{2}Cpv_x^{\ 2}Acos\theta$$

For easier manipulation and simplification, all constant values from Section 2.1 can be put into one constant, k_x. This allows us to focus on the term $v_x^{\ 2}$ without being distracted by constants.

$$\frac{dv_x}{dt} = -k_x v_x^{\ 2}\text{, Where } k_x = \frac{CpA}{2m} cos\theta$$

Calculations for k:

$$k_x = \frac{0.5(1.225\ kg/m^3)(0.00105625\pi\ m^2)}{2(0.005kg)} cos\theta$$

$$k_x = \frac{0.0020324632\ kg/m}{0.01kg} cos\theta$$

$$k_x = 0.20324632 cos\theta\ m^{-1}$$

k_x serves as the quantifiable effect of the drag force in this equation.

Deriving $v_x(t)$:

$$\frac{dv_x}{dt} = -k_x v_x^{\ 2}$$

Setting up integration by isolating variables:

$$\frac{dv_x}{v_x^{\ 2}} = -k_x dt$$

Integrate to inverse the derivative to derive $v_x(t)$**:**

$$\int \frac{dv_x}{v_x^{\ 2}} = \int -k_x dt$$

$$-\frac{1}{v_x(t)} = -k_x t + C\text{, where } C \text{ is a constant of integration}$$

Finding constant C: As velocity when t = 0 is the initial velocity, $v_x(0) = v_{x_0}$:

$$C = \quad -\frac{1}{v_{x_0}}$$

Substituting C back in:

$$\frac{1}{v_x(t)} = k_x t + \frac{1}{v_{x_0}}$$

$$\boxed{v_x(t) = \frac{1}{k_x t + \frac{1}{v_{x_0}}}}$$

Substituting $v_x(t)$ equation into the function for Horizontal displacement to derive $x(t)$:

$$x(t) = \int_0^t v_x(t)dt$$

$$x(t) = \int_0^t \frac{1}{k_x t + \frac{1}{v_{x_0}}} dt$$

Using the formula: $\int \frac{1}{a+bt} dt = \frac{1}{b} ln\,|a + bt| + C_1$, Where C_1 is a constant of integration

$$x(t) = \frac{1}{k_x} ln\,|k_x t + \frac{1}{v_{x_0}}| + C_1$$

As $x(t)$ is the displacement with the change of time, we know when time is 0, the shuttle is in its starting position and there is no displacement. Applying $x(0) = 0$ when $t = 0$, in order to solve for the value of the constant:

$$0 = \frac{1}{k_x} ln(\frac{1}{v_{x_0}}) + C_1$$

$$C_1 = -\frac{1}{k_x} ln(\frac{1}{v_{x_0}})$$

Substituting C_1 back into the equation to derive $x(t)$:

$$x(t) = \frac{1}{k_x} ln\,|k_x t + \frac{1}{v_{x_0}}| - \frac{1}{k_x} ln(\frac{1}{v_{x_0}})$$

$$x(t) = \frac{1}{k_x}(ln\,|k_x t + \frac{1}{v_{x_0}}| - ln(\frac{1}{v_{x_0}}))$$

All values are positive, so the absolute value doesn't matter

$$x(t) = \frac{1}{k_x} ln\,(\frac{k_x t + \frac{1}{v_{x_0}}}{\frac{1}{v_{x_0}}})$$

$$\boxed{x(t) = \frac{1}{k_x} ln(1 + k_x v_{x_0} t)}$$

The general formula for displacement.

Rearranging the displacement function $x(t)$ to isolate for initial velocity:

$$x(t) = \frac{1}{k_x} ln(1 + k_x v_{x_0} t)$$

$$e^{k_x \cdot x(t)} = 1 + k_x v_{x_0} t$$

$$k_x v_{x_0} t = e^{k_x \cdot x(t)} - 1$$

$$v_{x_0} = \frac{e^{k_x \cdot x(t)} - 1}{k_x t}$$

v_{x_0} = Horizontal component of initial velocity = $v_0 cos\theta$

$$v_0 cos\theta = \frac{e^{k_x \cdot x(t)} - 1}{k_x t}$$

$$\boxed{v_0 = \frac{e^{k_x \cdot x(t)} - 1}{k_x t(cos\theta)}}$$

This is the general equation for horizontal velocity. Launch angle, time, and horizontal displacement would need to be collected to solve for initial velocity.

2.3 Data collection

From section 2.2, the quantities needed to calculate initial velocity were time taken(t), horizontal displacement($x(t)$), and launch angle(θ). Data for these quantities along with others needed in later calculations were collected.

I recorded a video of one of my smashes and analysed that smash. Data for time was collected through the Tracker app using the video, by recording the time when the shuttle makes contact with my racquet, then when it goes over the net, and finally when it reaches my opponent.

Due to privacy concerns among other reasons, the video and process of recording data for my smash will not be shown. However the process will be demonstrated by using a match between 2 professional players, Victor Alexleson Vs. Loh Kean Yew, seen in Figure 4. The data portion that was collected from my smash will be displayed in Figure 5.

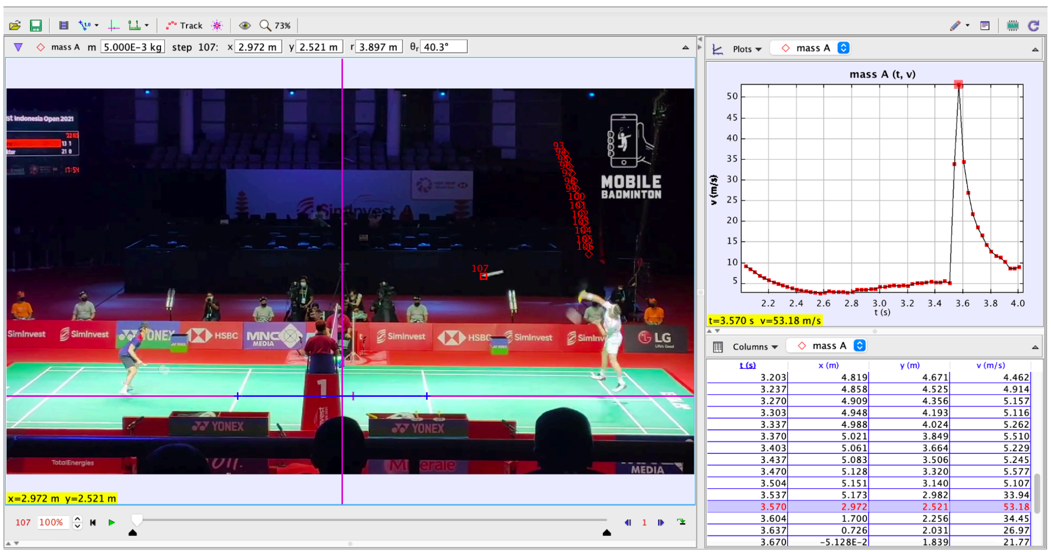

Figure 4. Example of using the tracker app (Displayed data/image are **not** mine and are only for demonstration purposes.)

The graph displayed in the Tracker app is irrelevant to this exploration. The data collected is displayed here:

Frame Position	Time
My contact point for smash	97.464s
Smash crosses net	97.698s
Smash reaches partner	97.988s

Figure 5. Data for time taken, collected from my smash using the Tracker app

Using the time data to calculate for time taken when the smash reaches the net and my opponent:

$$t_{net} = 97.698 - 97.464 = 0.234s$$

$$t_{partner} = 97.988 - 97.464 = 0.524s$$

Launch angle(θ) was calculated by placing a on the protractor video frame and measuring the angle of the shuttle's trajectory in the following frame, as seen in Figure 6. Although during my recording angle, camera warp, and perspective were taken into account, the final measurement still contains some uncertainty and is not fully accurate. Such uncertainties are not quantifiable but we acknowledge that they exist.

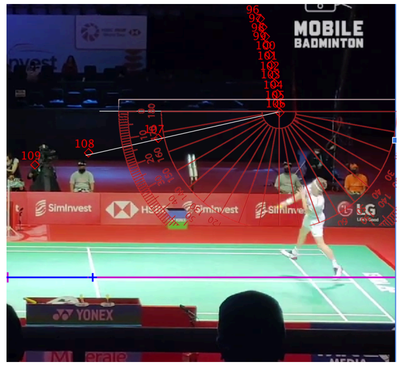

Figure 6. Launch angle calculation demonstration, not my actual smash/data.

Launch angle measured for *my* smash $(\theta) = 8^{\circ}$

Displacements were calculated using scale factor and known distances.

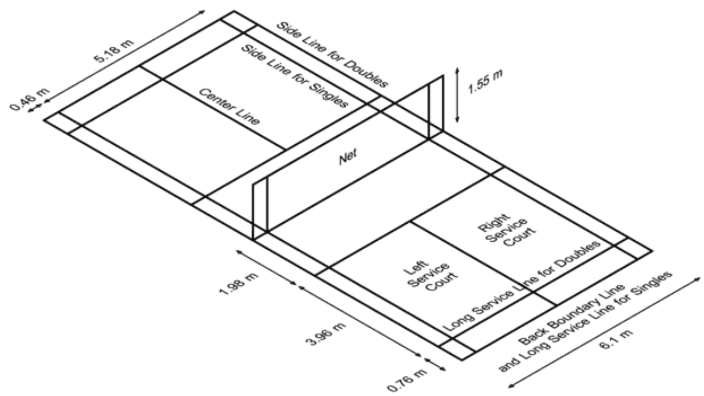

Figure 7. Visualizing dimensions of the badminton court

Since the court section lengths are known, I used them to calculate a scale factor. I measured a known length using a ruler on the video I took, then divided by the real value to get a scale factor. Then I measured an unknown value using a ruler and applied the scale factor to convert the unknown measurement to its real-life length. This does result in minimal uncertainty in the data, however it should not significantly impact any meaningful results.

Example calculation for $y(t)_{net}$, the vertical position of the shuttle when it crosses the net:

Height of net: 1.55m || Height of net in video: 2.7cm || Height of shuttle in video: 3.0cm.

$$\frac{0.027m}{1.55m} = \frac{0.030m}{y(t)_{net}}$$

$$y(t)_{net} = 1.7\overline{2}m$$

Using the same logic, both horizontal displacement when the shuttle reaches the net ($x(t)_{net}$) and horizontal displacement when it reaches my partner ($x(t)_{partner}$) were also calculated.

$$x(t)_{net}: 5.86m$$

$$x(t)_{partner}: 9.82m$$

Contact Height : 3.2m

Note that $x(t)_{net}$, $y(t)_{net}$, contact height, and $t_{partner}$ are not used to solve for initial velocity, but will be used later on in section 2.6.

2.4 Solving for numeric value of initial velocity through data collected:

- $x(t) = 9.82m$ (Parter)
- $Time(t) = 0.524s$ (Partner)
- Launch Angle$(\theta) = 8^{\circ}$
- $k_x = 0.20324632cos\theta$

$$v_0 = \frac{e^{k_x \cdot x(t)} - 1}{k_x t(cos\theta)}$$

$$v_0 \approx 59.53m/s$$

This result falls exactly in line with the average smash initial velocities of players of my skill level (Phomsoupha & Laffaye, 2014). This shows that despite the limitations of my method and data, the end result is still valid and accurate.

2.5 Finding the instantaneous rate of change for my smash.

As stated earlier, the goal of the smash is to reach your opponent in as little time as possible. To quantify the relationship between time(t), initial velocity(v_0), and launch angle(θ), an equation for time is derived using the equation for initial velocity seen in section 2.2:

$$v_0 = \frac{e^{k_x \cdot x(t)} - 1}{k_x t(cos\theta)}$$

Isolate for time:

$$t = \frac{e^{k_x \cdot x(t)} - 1}{k_x v_0 cos\theta}$$

To determine which factor —initial velocity(v_0) or launch angle(θ) — has a greater impact on reducing smash time(t), we analyze the **instantaneous rate of change** of time with respect to both variables to quantify the *short-term impact* of improving both factors. Since both initial velocity and launch angle influence time, **I isolate their effects by keeping one variable constant at my current value while keeping the other as the independent variable.**

Function for time for launch angle (v_0 stays constant at my current v_0, $59.53m/s$):

$$t(\theta) = \frac{e^{k_x \cdot x(t)} - 1}{k_x v_0 cos\theta}$$

Function for time for initial velocity (θ stays constant at my current θ, 8°):

$$t(v_0) = \frac{e^{k_x \cdot x(t)} - 1}{k_x v_0 cos\theta}$$

This approach reflects real-world training:

- A strength-focused training regimen would increase v_0 while keeping θ constant.
- A technique-focused training regimen would increase θ while keeping v_0 constant.

Inorder to find the **instantaneous rate of change** of time, I graphed the time functions, $t(v_0)$ and $t(\theta)$.

The slope of the tangent line on the graphs at my current smash values ($v_0 = 59.93\ m/s$ and $\theta = 8^\circ$ respectively) will be evaluated. As the *slope* of the tangent line represents the sensitivity of the function at that point, it is the instantaneous rate of change of time. Since a tangent line approximates the function locally, it predicts how time would change for an infinitesimally small increase in initial velocity or angle. The implication of the slope value allows us to quantify the short term impact at my current value by seeing how much time decreases per unit increase of both factors. This answers questions such as: if I have a tournament in a week, which factor should I focus on for the most impact?

2.5.1 Finding the effect of launch angle on time:

$$t(\theta) = \frac{e^{k_x \cdot x(t)} - 1}{k_x v_0 cos\theta}$$

$x(t) = 9.82m$
$v_0 = 59.53m/s$
$k_x = 0.20324632cos\theta$

$$t(\theta) = \frac{e^{(0.20324632cos\theta)(9.82)} - 1}{(0.20324632cos\theta)(59.53cos\theta)}$$

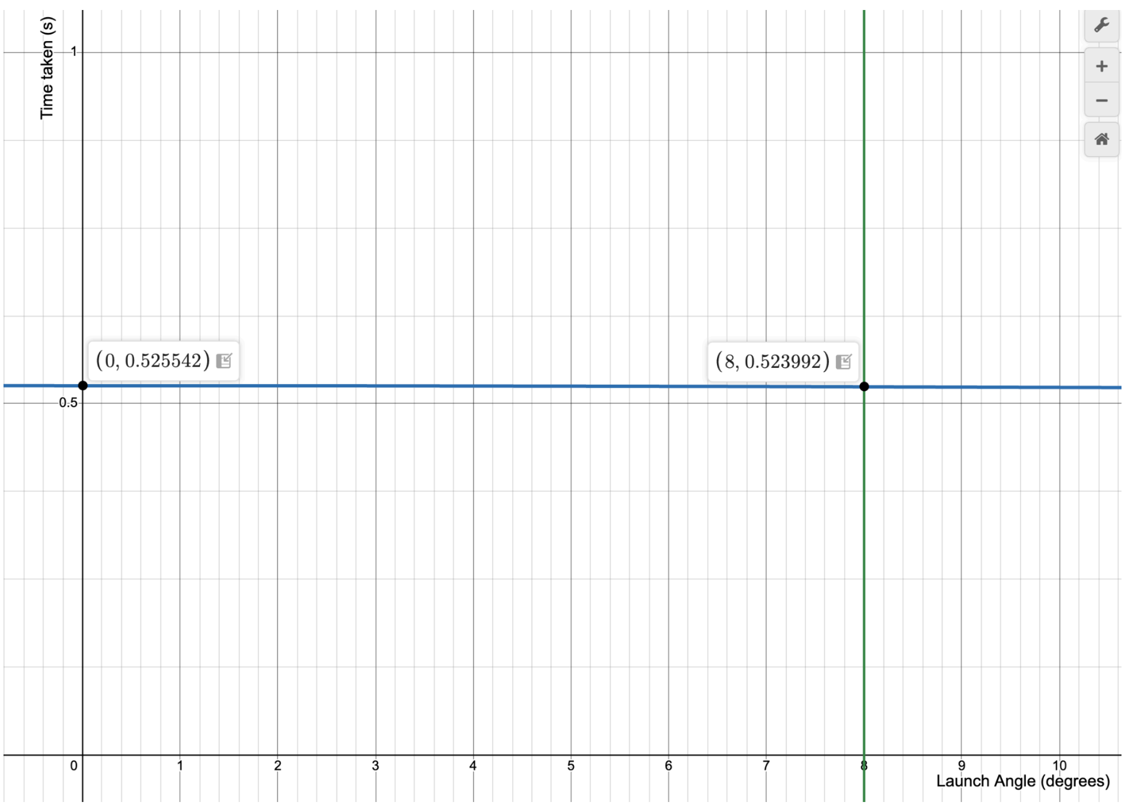

Figure 8. Graphed equation on desmos of launch angle on time.

The graph shows how the change in launch angle affects the time taken of the smash. (8, 0.524)is my current launch angle of 8 degrees with a time of 0.524s. The graph almost seems linear, however time is slightly decreasing as launch angle increases. This may suggest that launch angle is not as important in affecting time taken of the smash, we will see if the final results also follow this logic.

Calculating for the slope of tangent line (instantaneous rate of change) at my current angle (8°):

$$t(\theta) = \frac{e^{(0.20324632cos\theta)(9.82)}-1}{(0.20324632cos\theta)(59.53)(cos\theta)}$$

$$t(\theta) \approx \frac{e^{(1.9959)cos\theta}-1}{12.1(cos^2\theta)}$$

Let $a(\theta) = e^{(1.9959)cos\theta} - 1$ and $b(\theta) = 12.1(cos^2\theta)$

$$a'(\theta) = e^{(1.9959)cos\theta}(-1.9959cos\theta) \parallel b'(\theta) = 12.1(-2cos\theta sin\theta)$$

$$t'(\theta) = \frac{a'(\theta)b(\theta) - b'(\theta)a(\theta)}{[b(\theta)]^2}$$

$$t'(\theta) = \frac{e^{(1.9959)cos\theta}(-1.9959cos\theta)12.1(cos^2\theta) - 12.1(-2cos\theta sin\theta)e^{(1.9959)cos\theta}-1}{(12.1(cos^2\theta))^2}$$

$t'(8) \approx -0.0195$ seconds per radians

However, this is the tangent line assuming the graph is in radians, when in reality the graph is plotted in degrees. This is important as change per radian is an unrealistic measurement with wrong units.

$$t'(8)_{degrees} \approx -0.0195 \cdot \frac{\pi}{180}$$

$t'(8)_{degrees} \approx -0.00034$ seconds per degree

2.5.2 Finding the effect of initial velocity on time:

$$t(v_0) = \frac{e^{k_x \cdot x(t)}-1}{k_x v_0 cos\theta}$$

$x(t) = 9.82m$

$k_x = 0.20324632cos\theta$

$\theta = 8^{\circ}$

$$t(v_0) = \frac{e^{(0.20324632cos8)9.82}-1}{0.20324632cos8v_0(cos8)}$$

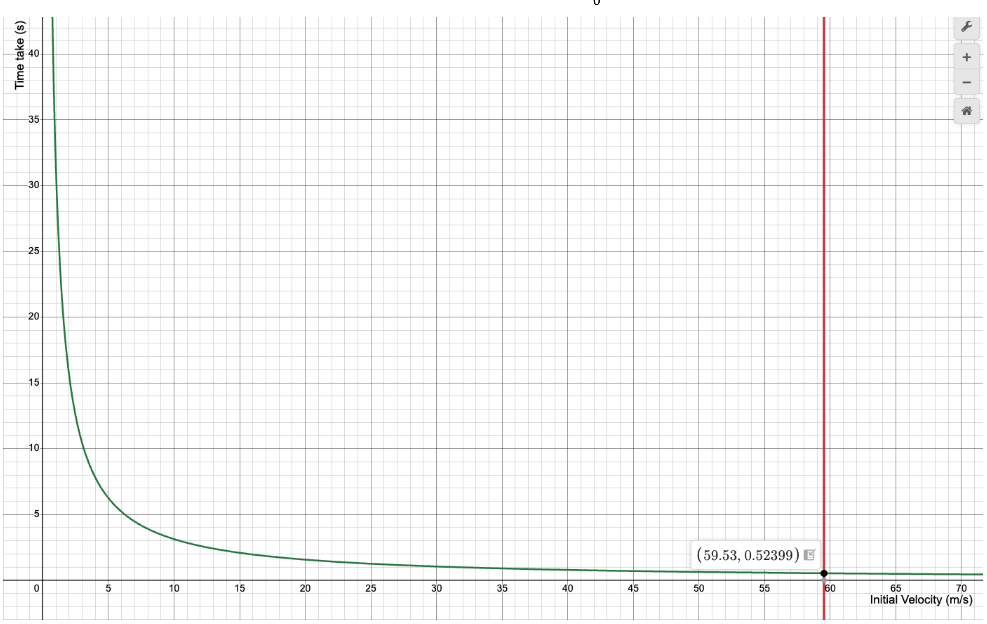

Figure 9. Graphed equation on desmos of initial velocity on time.

The point (59.53, 0.524) represents the current initial velocity and time taken for my smash. The shape of the graph shows an exponential decrease in impact of increasing initial velocity on reducing time taken. The effects of this exponential decrease would be interesting to observe in our final results.

Calculating for slope of tangent line (instantaneous rate of change) at current initial velocity ($59.53m/s$):

$$t(v_0) = \frac{6.22}{0.19931v_0}$$

$$t'(v_0) = -\frac{31.21}{v_0^2}$$

$$t'(59.53) = -\frac{31.0776}{59.53^2}$$

$$t'(59.53) \approx -0.00881 \text{ seconds per 1m/s}$$

2.5.3 Interpretation of results of instantaneous rate of change:

The values calculated represent the change in time per unit increase of initial velocity and launch angle. The instantaneous rate of change for launch angle is -0.00034 seconds decrease per 1 degree increase in launch angle, and -0.00881 seconds decrease per 1 meter/second increase in initial velocity.
A **steeper negative slope** indicates a larger reduction in smash time per unit increase in the variable. This means the more negative the slope, the larger the impact of improving that factor would be on time in the short run. Thus, it would be the better factor to train in the short run. Given the results, we clearly see in the short term improving initial velocity would be much more valuable than improving launch angle. However there are also limitations to the method which can make results not fully reflect real life scenarios.
One major limitation is that the *relative difficulty of improving launch angle versus initial velocity wasn't taken into account*. However, it is near impossible to take into account the difficulty of improving angle versus initial velocity as it is not mathematically quantifiable. Even though full accuracy is impossible to obtain, this method still provides a rigorous mathematical comparison of the 2 factors using per unit change.
The second limitation is the assumption that only 1 factor is going to be changing while the other stays constant. However in reality, both factors could change together and often do. Ignoring this fact during calculations could result in less accurate answers. Although both factors could change together in reality, isolating them allows for a clearer analysis of their individual effects. This simplification is reasonable because training often emphasizes one aspect at a time, and it provides practical guidance on whether strength or technique should be prioritized. Attempting to optimize both factors simultaneously is impractical, as the relationship between changes in initial velocity and launch angle varies across individuals and training regimens. Additionally, accurately modeling their combined effects would require complex mathematical models beyond the scope of this exploration. On top of that, modeling both factors together would not address the aim of this exploration, which is to find the most optimal factor. This means that this method sacrificed some mathematical accuracy for feasible calculations and relevance to the aim, which justifies the use of instantaneous rate of change.

As instantaneous rate of change only measures the short term change, a second method would be used to assess the long term effects of training both factors.

2.6 Finding the average rate of change for both Launch Angle and Initial velocity:

To assess the long term effectiveness of training in either factor, average rate of change, or the slope of the secant line, between my current values for initial velocity and launch angle and the "pro" level for initial velocity and launch angle will be used. Similar to the tangent line, the slope of the secant line also represents the sensitivity of the function, but in this case over a longer period. This simulates long term training, and how time will decrease as I go closer and closer to the pro level over months or years of training.

The pro levels of the secant line will be 100m/s for the initial velocity due to it being around the average initial velocity of pro players (Kwan et Al. 2011).

The pro level for launch angle however is different for each player due to different contact points, techniques, and a variety of other factors. One thing in common is that most pro's smashes go perfectly over the net, meaning the shuttle just barely grazes past the net without hitting it. Therefore I am going to be using the data I have collected for my smash to calculate for my most optimal launch angle that would allow my smash to go perfect over the net, and use that as the launch angle value for the secant line calculations.

2.6.1 Calculations for optimal launch angle using vertical equations of motion:

Using vertical equation of motion to find and equation for $y(t)$:

$$\frac{dv_y}{dt} = -g + \frac{1}{m} \cdot F_D(vertical)$$

$$\frac{dv_y}{dt} = -g + \frac{1}{m} \cdot \frac{1}{2} Cpv_y^2 Asin\theta$$

$$\frac{dv_y}{dt} = -g + k_y v_y^2, \text{ where } k_y = \frac{CpA}{2m} sin\theta$$

Calculations for k_y:

$$k_y = 0.20324632\, sin\theta\, m^{-1}$$

Deriving $v_y(t)$:

In the differential equation, the term $-g$ prevents simple integration like we did for horizontal motion. To get rid of this term, we define a new function $f(t) = v_y(t) - a$ to decouple the differential equation.

Here $f(t)$ is a new function to help simplify the integration process, this is done by eliminating the $-g$ term using a.

As $v_y(t) = f(t) + a$, we can substitute $f(t) + a$ into the vertical equation for motion:

$$\frac{d}{dt}(f + a) = \frac{df}{dt} = k_y(f + a)^2 - g$$

$$\frac{df}{dt} = k_y f^2 + 2k_y af + (k_y a^2 - g)$$

Inorder to eliminate the $-g$ term, we set a so that $k_y a^2 - g = 0$:

$$a = \sqrt{\frac{g}{k_y}}$$

Substituting a int the equation results in:

$$\boxed{\frac{df}{dt} = k_y f^2 + 2\sqrt{gk_y} f}$$

The differential equation is now homogeneous (no constant term), allowing for easier integration. We are able to derive $f(t)$ and substitute it back into $v_y(t) = f(t) + a$ to derive $v_y(t)$.

Integrating $f(t)$:

$$\frac{df}{k_y f^2 + 2\sqrt{gk_y} f} = dt$$

$$\int_{f_0}^{f(t)} \frac{df}{k_y f^2 + 2\sqrt{gk_y} f} = \int_0^t dt$$

The left bounds of this integral is how f evolves from the start of the smash to some later time, while the right bound shows the time elapsed. This ensures the solution aligns with my smash's measured values.

Integrating Left Hand Side:

As this is solving for a general solution, the boundaries are not added.

$$\int \frac{df}{k_y f^2 + 2\sqrt{gk_y} f}$$

$$= \int \frac{df}{k_y (f^2 + \frac{2\sqrt{gk_y}}{k_y} f)}$$

Completing the square to expand the denominator.

Now resembling a difference of squares which has a known integration formula:

$$= \int \frac{df}{k_y ((f + \frac{2\sqrt{gk_y}}{2k_y})^2 - \frac{(2\sqrt{gk_y})^2}{k_y^2})}$$

$$= \int \frac{df}{k_y ((f + \frac{\sqrt{gk_y}}{k_y})^2 - \frac{gk_y}{k_y^2})}$$

To simplify this expression,

$$\text{Let } u = \frac{k_y}{\sqrt{gk_y}} (f + \frac{\sqrt{gk_y}}{k_y})$$

This was chosen to simplify the quadratic term $(f + \frac{\sqrt{gk_y}}{k_y})^2$ as u would match up with the term $\frac{gk_y}{k_y^2}$ as $(f + \frac{\sqrt{gk_y}}{k_y})^2 = (\frac{\sqrt{gk_y}}{k_y} u)^2 = \frac{gk_y}{k_y^2} u^2$.

u can be simplified into:

$$u = \frac{k_y f}{\sqrt{gk_y}} + 1$$

If we differentiate u:

$$\frac{du}{df} = \frac{k_y}{\sqrt{gk_y}}$$

$$df = \frac{\sqrt{gk_y}}{k_y} du$$

We can substitute this back into our original equation, along with $(f + \frac{\sqrt{gk_y}}{k_y})^2$ in terms of u:

$$= \int \frac{1}{k_y(\frac{gk_y}{k_y^2}u^2 - \frac{gk_y}{k_y^2})} \cdot \frac{\sqrt{gk_y}}{k_y} du$$

$$= \int \frac{1}{\frac{gk_y}{k_y}(u^2-1)} \cdot \frac{\sqrt{gk_y}}{k_y} du$$

$$\frac{k_y}{gk_y} \cdot \frac{\sqrt{gk_y}}{k_y} \int \frac{1}{(u^2-1)} du$$

$$\frac{1}{\sqrt{gk_y}} \int \frac{1}{(u^2-1)} du$$

Using a standard result for hyperbolic functions: $\int \frac{1}{(u^2-1)} du = \; - arctanh(u) + C$, where C is a constant of integration:

$$\frac{1}{\sqrt{gk_y}} \int \frac{1}{(u^2-1)} du = \; - \frac{1}{\sqrt{gk_y}} arctanh(u) + C$$

Substituting $u = \frac{k_y f}{\sqrt{gk_y}} + 1$ back in:

$$= \; - \frac{1}{\sqrt{gk_y}} arctanh(\frac{k_y f}{\sqrt{gk_y}} + 1) + C$$

Using LHS result in the original equation:

$$\int_{f_0}^{f(t)} \frac{df}{k_y f^2 + 2\sqrt{gk_y} f} = \int_0^t dt$$

$$- \frac{1}{\sqrt{gk_y}} arctanh(\frac{k_y f}{\sqrt{gk_y}} + 1) + C = t$$

Isolate for $f(t)$

$$arctanh(\frac{k_y f}{\sqrt{gk_y}} + 1) = \; - \sqrt{gk_y}(t - C)$$

$$\frac{k_y f}{\sqrt{gk_y}} = tanh(- \sqrt{gk_y}(t - C)) - 1$$

$$f(t) = \frac{\sqrt{gk_y}}{k_y} (tanh(- \sqrt{gk_y}(t - C)) - 1)$$

Substituting $f(t)$ and a back into $v_y(t)$ equation to derive $v_y(t)$:

$$f(t) = v_y(t) - a$$

$$v_y(t) = \frac{\sqrt{gk_y}}{k_y} (tanh(- \sqrt{gk_y}(t - C)) - 1) + \sqrt{\frac{g}{k_y}}$$

$$v_y(t) = \sqrt{\frac{g}{k_y}} tanh(- \sqrt{gk_y}(t - C)$$

Solving for the constant C by setting $t = 0$, where $v_y(0) = v_{y_0}$

$$v(0) = \sqrt{\frac{g}{k_y}} tanh[- \sqrt{gk_y}(t - C)]$$

$$C = \frac{1}{\sqrt{gk_y}} arctanh(\frac{v_{y_0}}{\sqrt{\frac{g}{k_y}}})$$

Substituting C back into the equation:

$$\boxed{v_y(t) = \sqrt{\frac{g}{k_y}} tanh[-\sqrt{gk_y}(t - \frac{1}{\sqrt{gk_y}} arctanh(\frac{v_{y_0}}{\sqrt{\frac{g}{k_y}}}))]}$$

Here is the final $v_y(t)$ equation. To derive an equation for $y(t)$, we must integrate the $v_y(t)$ equation. For easier integration, I will simplify the equation. This was done so we won't have huge expression that are difficult to manipulate and format:

$$\text{let, } A = \sqrt{\frac{g}{k_y}}, B = -\sqrt{gk_y}, t_0 = \frac{1}{\sqrt{gk_y}} arctanh(\frac{v_{y_0}}{\sqrt{\frac{g}{k_y}}})$$

Then $v_y(t)$ equation becomes:

$$v_y(t) = A\, tanh[B(t - t_0)]$$

Integrate to find vertical position, $y(t)$:

$$\int v_y(t)dt = A \int tanh[B(t - t_0)]dt$$

For easier manipulation:

$$\text{Let } w = B(t - t_0)\text{, therefore } dw = Bdt \Rightarrow dt = \frac{dw}{B}$$

This makes integration straightforward with only a basic term.

$$y(t) = A \int tanh(w)\frac{dw}{B} = \frac{A}{B} \int tanh(w)dw$$

Using the hyperbolic identity $\int tanh(w)dw = ln(cosh(w)) + C_1$:

$$y(t) = \frac{A}{B}\int tanh(w)dw = \frac{A}{B} ln(cosh(w)) + C_1$$

Substituting values for w, A, B, t_0 back in:

$$y(t) = \frac{\sqrt{\frac{g}{k_y}}}{-\sqrt{gk_y}} ln(cosh(-\sqrt{gk_y}(t - \frac{1}{\sqrt{gk_y}} arctanh(\frac{v_{y_0}}{\sqrt{\frac{g}{k_y}}}))) + C_1$$

$$y(t) = -\frac{1}{k_y} ln(cosh(\sqrt{gk_y}(t) - arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}}))) + C_1$$

Using the initial condition that y(0) = 3.2m–contact height from data collection–to find C_1:

$$3.2 = -\frac{1}{k_y} ln(cosh(-\ arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}}))) + C_1$$

$$C_1 = 3.2 + \frac{1}{k_y} ln(cosh(-\ arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}})))$$

substituting C_1 back in:

$$y(t) = -\frac{1}{k_y} ln(cosh(\sqrt{gk_y}(t) - arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}}))) + 3.2 + \frac{1}{k_y} ln(cosh(-arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}})))$$

$$\boxed{y(t) = 3.2 - \frac{1}{k_y} ln(\frac{cosh(-arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}}))}{cosh(\sqrt{gk_y}(t) - arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}}))})}$$

As we are tying the find the launch angle that allows the shuttle perfectly goes over the net, the height of the shuttle $y(t)$ should be equal to 1.55m, the height of the net in badminton.

$$y(t) = 3.2 - \frac{1}{k_y} ln(\frac{cosh(-arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}}))}{cosh(\sqrt{gk_y}(t) - arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}}))}) = 1.55m$$

This needs to hold true when $x(t) = \frac{1}{k_x} ln(1 + k_x v_{0_x} t) = 5.86m$ as that is the horizontal position of the shuttle as it travels over the net.

As we know time(t) in both equations are equal, use the equation for time for the $x(t)$ equation.

$$t = \frac{e^{k_x \cdot x(t)} - 1}{k_x v_0 (cos\theta)}$$

Substituting the equation for time into vertical position equation $y(t)$:

$$y(t) = 3.2 - \frac{1}{k_y} ln(\frac{cosh(-arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}}))}{cosh(\sqrt{gk_y}(\frac{e^{k_x \cdot x(t)} - 1}{k_x v_0 (cos\theta)}) - arctanh(\frac{\sqrt{k_y}v_{y_0}}{\sqrt{g}}))}) = 1.55m$$

Where:

$k_y = 0.20324632 sin\theta \, m^{-1}$

$k_x = 0.20324632 cos\theta \, m^{-1}$

$v_0 = 59.53 m/s$

$g = 9.81 m/s^2$

$x(t) = 5.86m$

Plotting this equation on desmos we get: 9.26 degrees as the max angle in order to clear the net:

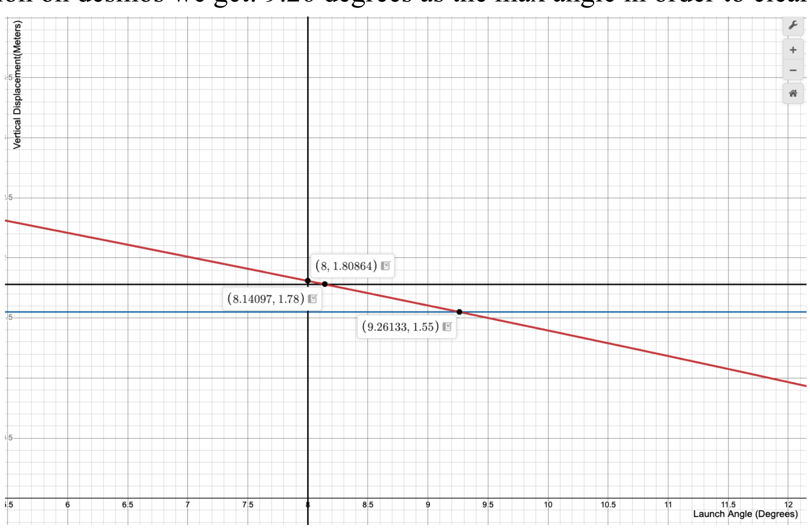

Figure 10. Graphed equation on desmos

Given the data, the point (8.14, 1.78) represents the predicted launch angle of my smash based on my vertical position data, and (8, 1.80) represents the predicted vertical position of my smash based on my launch angle data. We see that the predicted values slightly differ from the actual data collected of 8 degrees, and 1.78 meter vertical position. This is due to the limitations of my method previously discussed as well as uncertainties during data collection. Still, the predicted values are really close to my data, and it shows that assumptions and methodology used during my exploration was reasonable and did not create drastic differences in the final results.

2.6.2 Calculations for secant slope values:

Initial velocity:
Current: (59.32, 0.52401)
Pro Level: (100, 0.31084)
Slope of secant = $\frac{0.31084-0.52401}{100-59.32} = -0.00524$ seconds per 1m/s

Angle:
Current:(8, 0.52399)
Pro Level: (9.28, 0.52347)
Slope of secant = $\frac{0.52347-0.52399}{9.28-8} = -0.00041$ seconds per 1 degree

Similarly to the tangent line, a steeper negative slope indicates a larger impact on reducing time. This means that improving initial velocity is also the best area to focus on in the long run. However, calculations in the long run do have many limitations that are often unavoided due the long run being unpredictable in nature. For example, my calculations assume that all circumstances stay the same, however in the long run I could grow taller, jump higher, which would heavily affect my contact point and therefore also affect what launch angle I am able to hit at. This however is unavoidable and there is no mathematics that could accurately predict the exact circumstances in the long run and therefore this is the best that we can do.

Just like the tangent calculations, the secant calculations also assume that only 1 area changes while the other stays constant. This however is unrealistic in the long run as it is likely that both areas would experience changes. However, as the aim of this exploration is to figure out an area to *focus* on, it is not entirely unreasonable to assume that the area you focus on will experience much greater changes than the other area. Also due to the same reasons mentioned in tangent calculations, solving multivariable calculus is outside the scope of this exploration and would not be effective in fulling the aim of this exploration. Thus this is the most effective method and accurate method of calculating long-term impact at my disposal despite limitations.

3. Reflection/conclusion:

Final Results:
Short-term impact (instantaneous rate of change):
- Initial velocity sensitivity: -0.00881 s per 1 m/s increase
- Launch angle sensitivity: -0.00034 s per 1° increase

Long-term impact (average rate of change):
- Initial velocity sensitivity: -0.00524 s per 1 m/s increase
- Launch angle sensitivity: -0.00041 s per 1° increase

Seeing the results, we can conclude that improving initial velocity is the better factor to focus on training in both the short and long term.
However, the impact of improving initial velocity has had a significant decrease in the long run, while improving angle has increased. While interesting, it does not change any results or findings.

Reflection and implications:
I set out to find the most effective area to focus on to improve my badminton smash. As a result, I found the relative impact of both areas–initial velocity and launch angle–on time taken (the main objective of the smash) in both the short and long run. I was able to quantify the impacts of both variables and determine the most effective area to train. Looking at the results, I can say that I have met my aims for this exploration.
Using the results, I will now look to incorporate more strength sessions into my training. I would also look to incorporate more training on the transfer of power, and how to effectively use power during my on court training as opposed to focusing on hitting steeper shots.

Limitations and justification:
I have already analysed a lot of limitations and justifications in the individual sections. However there are still some limitations with my exploration in general and even my aim that will be discussed.

Firstly, only using time to judge a smash's effectiveness is a huge simplification. There are various other qualities of a smash that were ignored or generalized. Examples of other qualities of an effective smash are, smash placement, smash deception (your opponents can easily receive your smash if they know it's coming), and smash consistency.
However, these qualities are all qualitative. It is really unreasonable to attempt to come up with mathematical representations of these qualities and attempt to calculate improvements in these areas. Also, a lot of these areas come with experience and there is not a specific area to work on inorder to improve them. Therefore measuring the only mathematically quantifiable characteristic of a good smash still allowed for excellent analysis and insight. Therefore I believe that the choice of focusing purely on time was justified and also effective for mathematical analysis.

Another limitation in the method is ignoring the other impacts that both variables have on smash effectiveness outside of time.
- A super hard smash(high initial velocity) results in a less accurate smash and you are more prone to mistakes, However the smash would also be harder for your opponent to control so they are also more likely to make a mistake

- steep angle makes you opponent bend down when getting the smash, allowing for unbalance and a worse quality return, where as if you angle isn't steep, a good player can counter your smash that forces you in trouble

Again, these impacts are not quantifiable and therefore impossible to incorporate into a mathematical model. Therefore not including these effects were justified.

Thirdly, all collected data values can fluctuate. As I am not a robot, the initial velocity, launch angle, contact height, displacement, etc, will be different for each smash. For calculations, I chose a smash where I felt it was "average" but it still limited the results. This means that my calculations only fully reflect the 1 smash's data that was used under those specific conditions. As the exploration is about improving my smash in general, having only data for 1 smash is a limitation. For future improvements, I could have done more trials and recorded the data for more smashes and used an average to have more accurate data.

Evaluation:
My method to achieve my aim of "finding the most optimal factor (initial velocity or launch angle) to focus on training in order to have the biggest improvement to my smash by decreasing the time taken to reach my opponent" was acceptable and the aim was achieved. My final results and all calculated values were close with accepted values. However, we see that some values are not fully accurate such as my calculated angle being 8.15 degrees, while my measured angle was 8 degrees. This can show that both data collection as well as the mathematical model had its limitations, however it is a small enough difference where the accuracy of this exploration as a whole still stands. Overall this exploration maintained a decent level of accuracy and mathematical rigour despite the challenging topic and was successful.

Conclusion:
I have learned a lot from this exploration. For mathematics, I learned many new skills such as integration, equations of motion, hyperbolic trigonometry, and much, much more. I was able to deepen my understanding of existing knowledge on derivatives, tangent and secant lines, functions, and trigonometry just to name a few. This exploration has challenged my mathematical knowledge and abilities much further than I had anticipated. Apart from mathematics I also learned problem solving skills and how to use my limited knowledge of mathematics to attempt to solve more complex problems. When I was first looking into approaches to my problem, all approaches used mathematics that were way above my current level of understanding. This led me to use tangent and secant lines to evaluate for smash effectiveness, or breaking the equation of motion into horizontal and vertical components despite it not being mathematically rigorous or fully accurate.
Overall, I am extremely pleased how my exploration turned out. If revisited in the future, I would want to collect more precise data with better equipment, more trials for data to increase accuracy, and also approach the mathematics in a more rigorous way and use the advanced math that was outside the scope of this exploration.

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