Common Statistical Models

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Anbin Rhee

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Contents

What is a "Model"?

- Analogy: A toy airplane

  - A toy airplane is not a real plane, but it helps

    us understand how a real plane flies

- Statistical Models are similar

  - They are simplified mathematical

    representations of complex real-world data

  - They help us understand relationships

    between variables and make predictions

Key Concepts: Variables

- What is a variable?

  - Anything that can be measured or observed and can vary across observations

- Independent Variable (IV) : Predictor / Cause

  - The variable we change or observe to see if it has an effect

    (The amount of fertilizer you give a plant)

- Dependent Variable (DV) : Outcome / Effect

  - The variable we measure to see if it changes in response to the independent variable

    (The height of the plant)

Key Concepts: Variable Types

- Numerical

  - Can be any value within a range

    - Continuous : can take any values withing a range

    - Discrete : can only take specific, separate values

  - e.g., temperature, sales, study hours, the number of children

- Categorical

  - Falls into distinct categories

  - e.g., gender, product type, teaching method (A/B/C)

Simple Linear Regression (SLR)

Example: Can we predict a student's **Exam Score** based on their **Study Hours**?

- We want to predict **a continuous numerical outcome** using **one continuous numerical predictor**
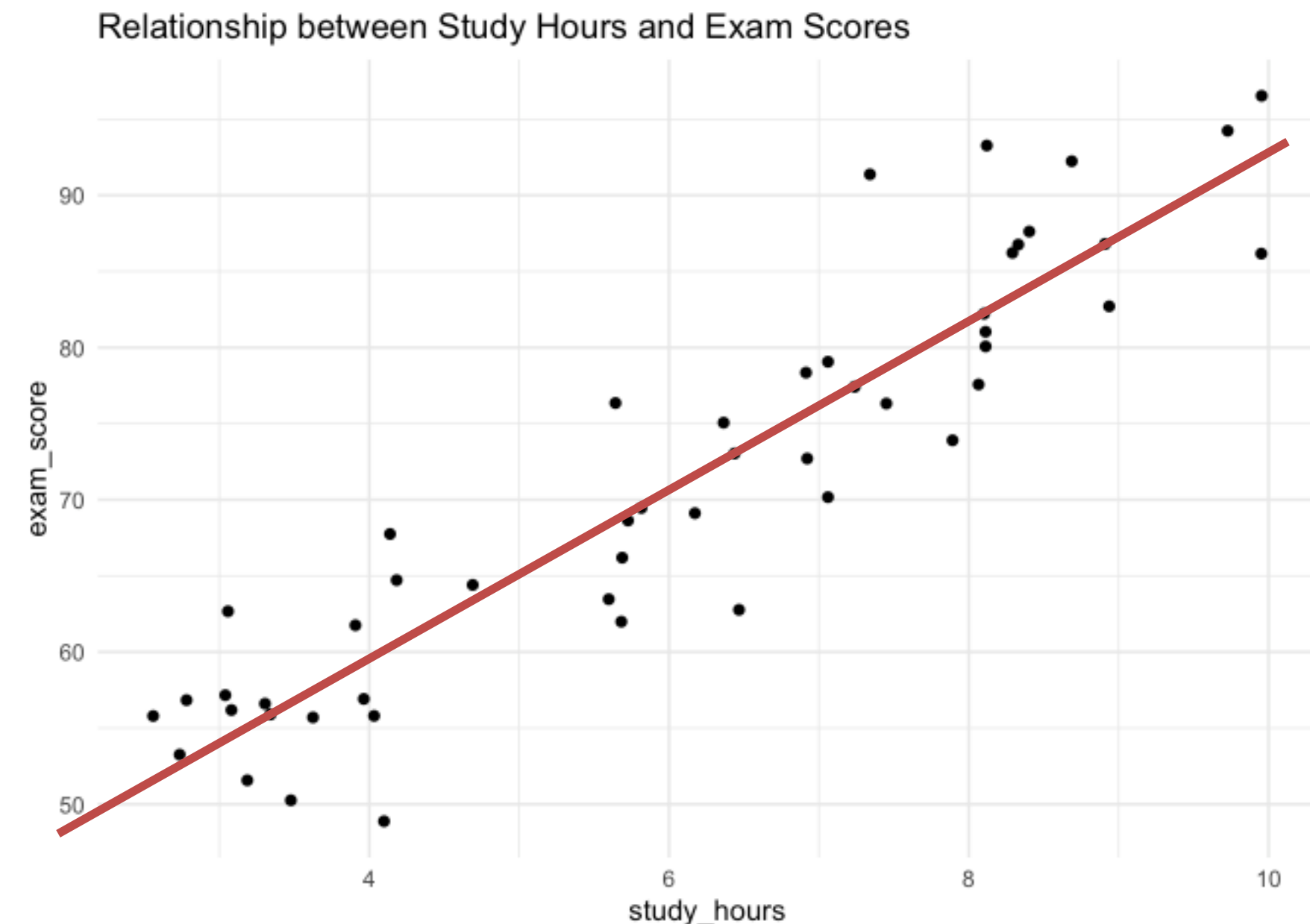
Relationship between Study Hours and Exam Scores

Simple Linear Regression

- The Conceptual Model (The 'True' Relationship in the population) :

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

- Y_i : The true outcome (Dependent variable) for individual i (e.g., Exam Score)

- X_i : The predictor (Independent variable) for individual i (e.g., Study Hours)

- β_0 : The true average value of Y when X is 0 (Intercept)

- β_1 : The true average change in Y for every one-unit increase in X (Slope)

- ϵ_i : The part of Y_i that the model cannot explain for individual i (Error term)

Simple Linear Regression

- The Estimated Model (From Our Data) :

$$\hat{Y}_i = \widehat{\beta_0} + \widehat{\beta_1} X_i$$

  - $\hat{Y}_i$ (Y-hat) : The predicted outcome for individual i (e.g., Exam Score)

  - $\widehat{\beta_0}$ (Estimated Intercept) : Our best guess for β_0 based on our data

  - $\widehat{\beta_1}$ (Estimated Slope) : Our best guess for β_1 based on our data

- Error vs. Residual

  - Error (ϵ_i): Unobserved difference between the true Y_i and the true model's prediction

  - Residual (e_i) : Observed difference between the actual Y_i and our model's predicted $\hat{Y}_i$

$$e_i = Y_i - \hat{Y}_i : \text{We want small residuals!}$$

Simple Linear Regression: Assumptions

1. Independence of Errors

   - Errors (ϵ_i) are independent of each other

2. Linearity

   - The relationship between X and Y is approximately linear

3. Constant Variance of Errors (Homoscedasticity)

   - The variability of the residuals should be roughly constant across all levels

     of the predictor X

4. Normality

   - The residuals (e_i) are approximately normally distributed

Simple Linear Regression: Standard Error

- Standard Error (SE) of Coefficients:

  - How much our estimates would likely vary if we took many different samples from the same population

  - A smaller SE means our estimate is more precise and reliable (less-sample to sample variation)

Simple Linear Regression: T-Test

- Hypotheses

$$H_0: \beta_1 = 0 \text{ vs. } H_1: \beta_1 \neq 0$$

- We need statistical test used to get the p-value for individual coefficients in regression

- It calculates a t-statistics for each coefficient:

$$t = \frac{Coefficient\ Estimate\ (\beta_1) - Hypothesized\ Value\ (\beta_1 = 0\ under\ H_0)}{Standard\ Error\ of\ Coefficient\ (\beta_1)}$$

  - A smaller SE means our estimate is more precise and reliable

- It measures how many standard errors away our estimated coefficient is from the hypothesized value

- A **large** |t| means our estimate is further from zero, making it **less** likely to be due to chance

SLR: Confidence Interval (CI)

- Instead of just a single point estimate, a CI provides a range of plausible values for the true population parameters

- Example: A 95% CI for the slope might be (a, b). This means, "We are 95% confident that the true average change in Y for a one-unit change in X lies somewhere between a and b"

- It gives us a sense of the precision of our estimate and the range of values we'd expect the true parameters to fall in if we repeated the experiment many times

Simple Linear Regression: Interpretation

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 37.6210     2.1316   17.65   <2e-16 ***
study_hours  5.5231     0.3314   16.66   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.13 on 48 degrees of freedom
Multiple R-squared:  0.8526,    Adjusted R-squared:  0.8495
F-statistic: 277.7 on 1 and 48 DF,  p-value: < 2.2e-16
```

- Intercept(37.6210): The predicted exam score for a student who studied 0 hours

- Study_hours (5.5231): For every additional hour of study, the exam score increases on average by 5.52 points

Multiple Linear Regression

Example: Can we predict an individual's Salary based on their Year of Experience, Education Level, and Hours worked Per Week?

- Dependent variable : Salary   *Continuous*

- Independent variables : Year of Experience,   *Continuous*

  Education Level   *Categorical*

  Hours Worked per Week   *Continuous*

- We want to predict/explain a **continuous numerical outcome** using two or more independent variables

- Extends simple linear regression by adding more predictors

Multiple Linear Regression

- The Conceptual Model (The 'True' Relationship in the population) :

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \epsilon_i$$

- It is adding more X terms to our simple linear regression equation

- Y_i : Salary for individual i

- X_{1i} : Years of Experience for individual i

- X_{2i} : Education Level for individual i (High school, Bachelor, Master, PhD)

- X_{3i} : Hours Worked per Week for individual i

- Each β_j (slope) now represents the average change in Y for a one-unit change in that specific X, while holding all other X variables constant.

Handling Categorical Predictors

- For handling categorical predictors, one category becomes the reference group

- Statistical software automatically convert categorical variables into '**dummy variables**' (0s and 1s) to indicate each category

- The coefficients for other categories show the predicted difference compared to this reference group

- For the salary example, we can set High School as our reference group for Education Level

Multiple Linear Regression: Interpretation

```
Coefficients:
                          Estimate Std. Error t value Pr(>|t|)
(Intercept)               35480.39    3987.61   8.898 4.05e-14 ***
experience                 1907.66      85.39  22.341  < 2e-16 ***
education_levelBachelor    5375.48    1441.44   3.729 0.000328 ***
education_levelMaster     10936.71    1410.09   7.756 1.03e-11 ***
education_levelPhD        15381.78    1411.88  10.895  < 2e-16 ***
hours_per_week              115.86      93.38   1.241 0.217811
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4658 on 94 degrees of freedom
Multiple R-squared:  0.8648,    Adjusted R-squared:  0.8576
F-statistic: 120.3 on 5 and 94 DF,  p-value: < 2.2e-16
```

- Experience (1907.66): Each additional year of experience increases predicted salary by about $1907, adjusting for education level and hours worked (very strong effect)

- Education level – Bachelor: $5375 more than High School, on average

Comparing Multiple Groups: ANOVA

Example: The company wants to compare the average purchase amounts among customers exposed to three different marketing campaigns (A, B, and C) to see which campaign performs best.

- Comparing the means of a **continuous Dependent Variable (purchase amount)** among **three marketing groups** (**Campaign A, B, C**)
- Analysis Of Variance (ANOVA) helps us:
  - Determine if there is a statistically significant difference among the group means

Comparing Multiple Groups: ANOVA

- The Conceptual Model:

$$Y_i = \mu + \alpha_j + \epsilon_i$$

- Y_i : The outcome (Dependent variable) for individual i (Purchase Amount)

- μ : The overall average of Y across all groups (Grand Mean)

- α_j : The effect of being in group j (Campaign A, B, or C). It represents how much the mean of group j deviates from the grand mean

- ϵ_i : The part of Y_i that the model cannot explain

Comparing Multiple Groups: ANOVA

- Hypothesis:

$$H_0: \alpha_1 = \alpha_2 = \alpha_3 \text{ vs. } H_1: \text{at least one group differ}$$

- ANOVA looks at two types of variation in the data:

  1. Between-Group Variation: The variability among the group means themselves (effect)

  2. Within-Group Variation: The variability within each group (noise)

- F-statistics:

$$F = \frac{Variability\ between\ Group\ Means}{Variability\ within\ Groups}$$

  - A larger 'F' value suggest the group differences are large compared to the random differences withing each group, indicating significant group effects

ANOVA: Interpretation

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   49.064      1.310  37.455  < 2e-16 ***
campaignB     13.121      1.853   7.083 1.03e-11 ***
campaignC      6.383      1.853   3.446 0.000652 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 13.1 on 297 degrees of freedom
Multiple R-squared:  0.1445,    Adjusted R-squared:  0.1388
F-statistic: 25.09 on 2 and 297 DF,  p-value: 8.556e-11
```

- CampaignB (13.121): Customers in Campaign B had an average purchase amount that was $13.121 higher than customers in Campaign A. This difference is highly statistically significant

Logistic Regression

Example: The medical company has developed a new treatment to improve treatment success compared to the standard treatment, after accounting for patient age and gender

- Outcome (Y) is either success or failure   *Binary*

- Predictors (X): Treatment group (New vs. Standard)   *Categorical*

  Age   *Continuous*

  Gender   *Categorical*

- Linear regression predicts continuous values. It could predict "success" as -0.5 or 1.2, which don't make sense for a Yes/No outcome

Logistic Regression

- Instead of predicting the outcome (0 or 1) directly, Logistic Regression predicts the **probability of the 'Yes/Success' outcome** occurring (e.g., the probability of treatment success)

- The Model:

$$log(Odds\ of\ success) = \beta_0 + \beta_1 * Treatment + \beta_2 * Age + \beta_3 * Gender$$

- $Odds = \dfrac{Probability\ of\ Event}{1-Probability\ of\ Event} = \dfrac{Probability\ of\ Success}{Probability\ of\ Failure}$

- Purpose: To predict the probability of a binary outcome and understand which factors increase or decrease the odds of that outcome

Logistic Regression

- The Model:

$$log(Odds\ of\ success) = \beta_0 + \beta_1 * Treatment + \beta_2 * Age + \beta_3 * Gender$$

- $Odds = \dfrac{Probability\ of\ Event}{1 - Probability\ of\ Event}$

- β_1 : change in log-odds of success for new treatment compared to standard

- Exp(β_1): **odds ratio of success** for new treatment

  - How many times higher or lower the odds of success are under the new treatment compared to the standard treatment

- Logistic regression models the **probability of success**, allowing adjustment for other factors (age, gender) while comparing treatment effects

Logistic Regression: Interpretation

```
Coefficients:
                  Estimate Std. Error z value Pr(>|z|)
(Intercept)        1.01130    0.95185   1.062   0.2880
treatmentStandard -0.74231    0.35994  -2.062   0.0392 *
age               -0.03402    0.01900  -1.790   0.0735 .
genderMale        -0.55005    0.35980  -1.529   0.1263
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- TreatmentStandard (Estimate: -0.74231): Patients in the Standard treatment group have exp(-0.74231)=0.476 times the odds (or are about 52.4% less likely) of achieving treatment success compared to patients in the New treatment group, holding age and gender constant. This difference is statistically significant.

Quick Quiz

- What statistical model would you use to compare the average productivity across these three training programs?

  Example: A company tests three types of training programs to improve employee productivity. After the training , they measure each employee's number of tasks completed per day.

- **ANOVA model** to compare the mean productivity across three training program

Important Considerations

- Garbage in, garbage out

  - Bad data -> bad insight

- Correlation is NOT causation

  - Just because two variables are related does not mean one causes the other

- Models are Simplifications

  - They are mathematical abstractions of reality. They do not capture every nuance

- Overfitting

  - A model that is too complex for your data might fit your current data perfectly but perform poorly on new, unseen data

Summary

- **Simple Linear Regression**: Predicting a continuous outcome from one continuous predictor (e.g., Exam Score from Study Hours)

- **Multiple Linear Regression**: Predicting a continuous outcome from multiple predictors (continuous or categorical) (e.g., Salary from Experience, Education, Hours)

- **ANOVA**: Comparing the means of a continuous outcome across three or more groups (e.g., Purchase Amount from Campaign A, B, and C)

- **Logistic Regression**: Predicting a binary (Yes/No) outcome from one or more predictors (e.g., Treatment Success from Treatment Group, Age, Gender)