FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

FACULTY OF SCIENCES

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# BOUNDARY CONTROLLABILITY OF NODAL PROFILE FOR HYPERBOLIC SYSTEMS


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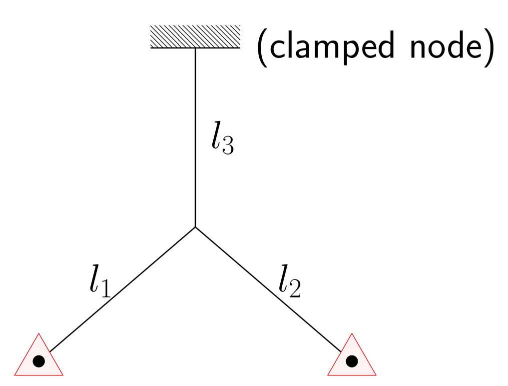
## Motivation and Introduction

**Boundary controllability of nodal profile for a coupled system of linear and quasi-linear hyperbolic systems** is a new class of exact boundary controllability problems proposed in recent years according to practical application requirements. There are already quite mature processing methods and results. Challenges have been raised in the theoretical in-depth treatment, and there are many problems to be studied in depth.
**Aim.**The boundary traces of state to exactly fit any given profile as function of time on a node after a suitable time $t = T$ by means of boundary controls.

## Linear Case - Duality Method

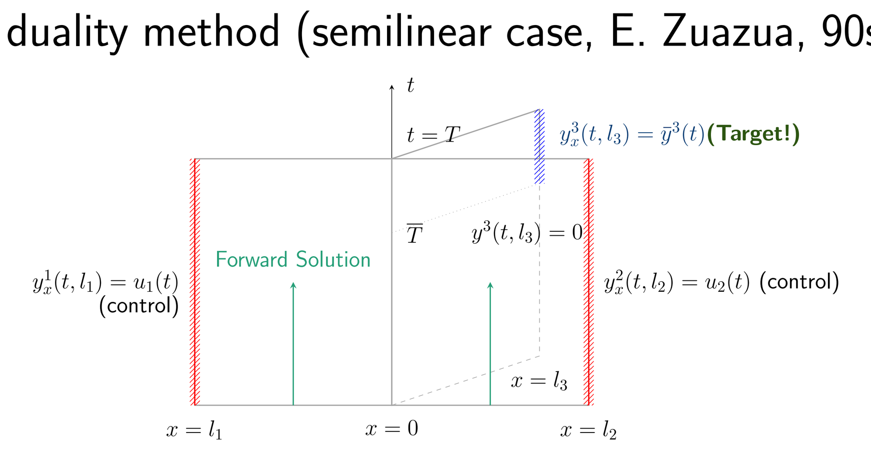



**Toy Example (Three string network).**
The simplest non-trivial network of strings that cannot be reduced to a single string.

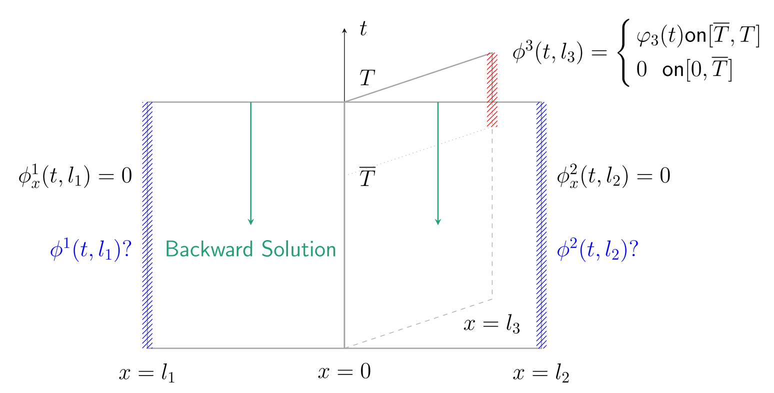
- $y^i = y^i(t,x) : [0, l_i] \to \mathbb{R}, i = 1,2,3,$ the transversal displacements of the strings, governed by linear wave equation.
- Neumann controls $y_x^i(t, l_i) = u^i(t) \in L^2(0,T), i = 1,2.$
- Coupled at one of their extremes at $x = 0$.

**Nodal Control Problem.** If we can find two controls, such that trace data of solution $\mathbf{y} := (y^1, y^2, y^3) \in C([0,T]; V) \cap C^1([0,T]; H)$ at the clamped node meets with the demand that $y_x^3(t, l_3) = \bar{y}_3(t)$ with given target function $\bar{y}_3 \in L^2(\overline{T}, T)$ after certain time $\overline{T}$, where $V = \{\psi \in \prod_{i=1}^3 H^1(0,l_i) | \psi^1(0) = \psi^2(0) = \psi^3(0), \psi^3(l_3) = 0\}$, $H = \prod_{i=1}^3 L^2(0,l_i)$.

**Duality Method.** From the idea of HUM method (linear case, J.L.Lions, 80s) and duality method (semilinear case, E. Zuazua, 90s), we shift this problem to a dual problem.




**Original Problem.** Fulfill given nodal profile by two controls for the original $y$-system.




**Dual Problem.** Observation the trace at $x = l_1, l_2$ for the adjoint $\phi$-system.

Step 1 Controllability of nodal profile $\iff \mathcal{F} : U \longrightarrow L^2(\overline{T}, T)$, $\mathcal{F}(u^1, u^2) = y_x^3(t, l_3)$ is onto from the control space $U := L^2(0,T) \times L^2(0,T)$ to the nodal profile function space.

Step 2 Write down the adjoint system and compute the adjoint $\mathcal{F}^*$ by duality equality between the original system and its adjoint.

Step 3 The surjectivity of $\mathcal{F} \iff$ the observability inequality holds for some $c > 0$:
$$\|\mathcal{F}^*(x_2)\|_U \geq c\|x_2\|_{L_2(\overline{T},T)}, \quad \forall x_2 \in L_2(\overline{T}, T).$$

**Main Result([1]).** The three-string system with two Neumann controls is controllable of nodal profile at $x = l_3$ in time $t \in (\overline{T}, T)$. By proving
**Theorem.**Let $\overline{T} > l_3 + \max\{l_1, l_2\}$ and $T > \overline{T}$. There exists a constant $c > 0$ such that
$$\| \left(\phi^1(\cdot, l_1), \phi^2(\cdot, l_2)\right) \|_{L^2(0,T) \times L^2(0,T)} \geq c\|\varphi_3(\cdot)\|_{L^2(\overline{T},T)}$$
holds for the dual problem.

Key. Side-wise D' Alembert formula.

Remark. The number of controls can be reduced to 1.

## Perspectives

- Design HUM control (Optimal in $L^2$).
- Uncontrollable case.
- General framework for other boundary conditions.
- For constructive method:
  - Extension to entropy solution.
  - Extension to general first order systems.
  - Numerics.

## Nonlinear Case - Constructive Method

**Workplan.**

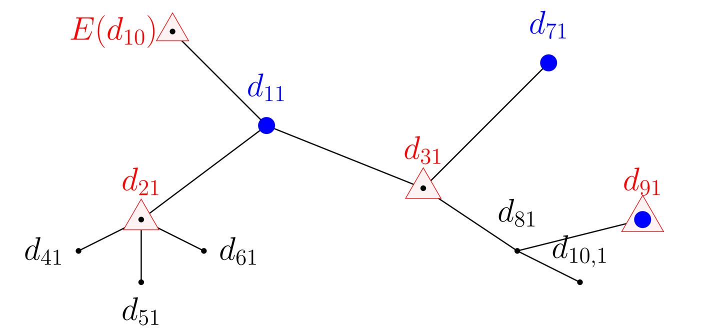
- Modeling: Meaningful physical model. Description of interface conditions.
- Wellposedness: Existence and uniqueness of semi-global classical solution.
- Local exact controllability.
- Solve the desired control by explicit constructive method with modular structure (Li& Rao, 02).

- **Results on theory.** For nodal control problem of hyperbolic models on network, we ask and answer :
  - Q How long? Infimum of controllability time.
  - A Determined by the eigenvalues and length from the control node to the charged node.
  - Q How many? Minimum number of controls.
  - A Sum of freedom degree of charged nodes.
  - Q Where? Design the position of controls.
  - A Ensure that the whole network can be divided into independent controllable sub-networks.
  - Q What is? Calculate the controls.
  - A Take the nodal trace of constructed solution (not unique) as the desired control.

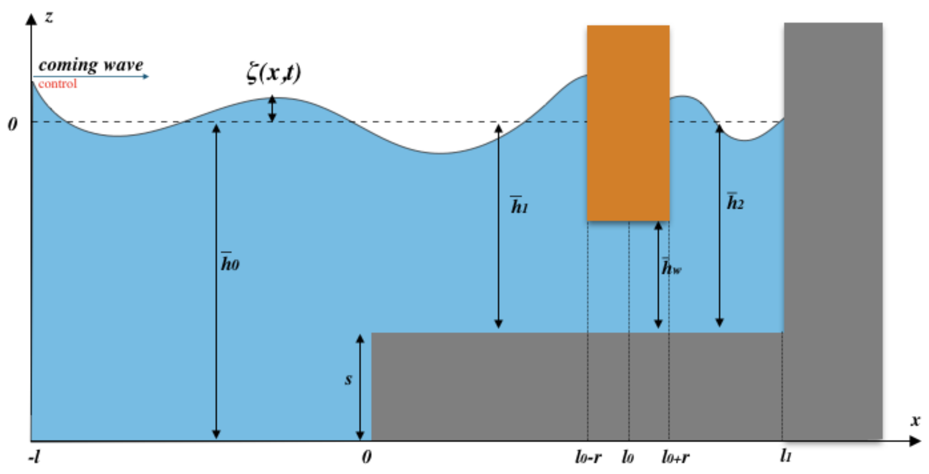
## Results and Applications - 1d Nonlinear Hyperbolic Systems

**Network of Vibrating Strings ([2]).**

- Planar Tree-like Network.
- Each string follows quasilinear wave equation $y_{tt} - (K(y, y_x))_x = F(y, y_x, y_t)$, where $K$ is $C^2$ function, such that $K_v(u,v) > 0$, and $F$ is $C^1$ functions with $F(0,0,0) = 0$.
- Dynamics at multiple nodes: Kirchhoff and continuity conditions.




**Fulfill demand of nodal profiles at $d_{11}, d_{71}$ and $d_{91}$ with 4 controls.**
(Charged node): with given (partial or total) nodal profiles.
(Controlled node): with boundary control to be determined.



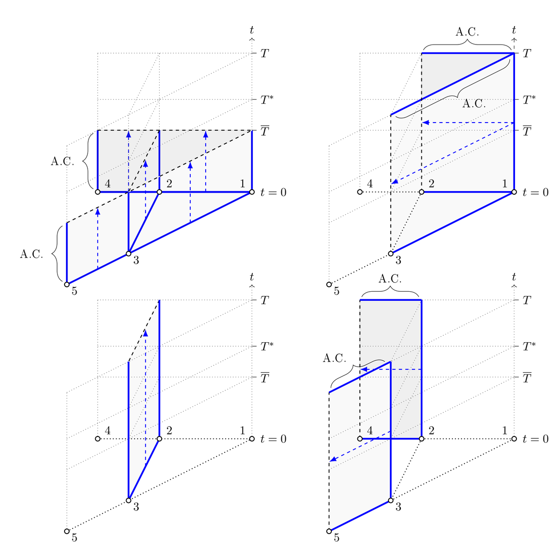

**Shallow Water System with a Partially Immersed Obstacle ([3]).**
One dimensional nonlinear shallow water system, describing the free surface flow of water as well as the flow under a fixed gate structure.

- 1D nonlinear shallow water equation:
$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(q^2/h\right) + gh\partial_x \zeta = 0, \end{cases}$$
where $\zeta(t,x)$ is the free surface elevation, $h(t,x)$ is the fluid height, $q(t,x)$ is the horizontal discharge.
- Transmission conditions:
$$\zeta_0(t,0) = \zeta_1(t,0), \quad q_0(t,0) = q_1(t,0),$$
$$\begin{cases} q_2(t, l_0 + r) = q_1(t, l_0 - r) = q_w(t), \\ \left[\dfrac{q_i^2}{2h_i^2} + g\zeta_i\right]_{i=1, x=l_0-r}^{i=2, x=l_0+r} = -\alpha \dfrac{\mathrm{d}}{\mathrm{d}t} q_w(t), \end{cases}$$
where $\alpha = 2r/h_w$.

- Results.
  - Exactly fit given demand in fluid height and horizontal discharge at the end $x = l_1$ by $(\zeta_2, q_2)(t, l_1) = (\bar{h}_2 + \zeta_B(t), q_B(t)), t \in [T, \overline{T}]$ with $q_B = 0$, $\zeta_B(t)$ chosen as any given $C^1$ function.
  - after a suitable time $T > \overline{T}$ by means of boundary control $f$ acting at the left end $x = -l : \zeta_0 = f(t), t \in [0,T]$, where $\overline{T} = \left(\frac{l}{\sqrt{g\bar{h}_0}} + \frac{l_1 - 2r}{\sqrt{g\bar{h}_1}}\right)$.
  - Numerical algorithm.

**Geometrically Exact Beams on Networks with loops ([4]).**

- Networks of shearable beams that may undergo large motions.
  - GEB Model. Quasilinear.
  - $\Leftrightarrow$(by a nonlinear transformation)
  - IGEB Model. Semilinear. 1st Order.
- Specific Network with a cycle.
  - Lack of null controllability, but possible to realize the controllability of nodal profile.



**Constructive Method Proceeding on "A-shaped network"**

## Selected publications


[1] Y. Wang, G. Leugering, T.T. Li (2021). **HUM Method to the Exact Boundary Controllability of Nodal Profile for Vibrating Strings.** In prepation.

[2] Y. Wang, T.T. Li (2019). **Exact Boundary Controllability of Partial Nodal Profile for Wave Equations.** Submitted.

[3] G. Vergara-Hermosilla, G. Leugering, Y. Wang (2021). **Boundary Controllability Of a System Modeling a Partially Immersed Obstacle.** Submitted.

[4] G. Leugering, C. Rodriguez, Y. Wang (2021). **Nodal Profile Control For Networks Of Geometrically Exact Beams.** https://arxiv.org/abs/2103.13064.



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