Quadrotor Dynamics Model

Two reference frames and their corresponding coordinate systems are used to describe the attitude and position of the quadrotor. One is the inertial frame expressed by $I\{x, y, z\}$, and the other is the body-fixed frame expressed by $B\{x, y, z\}$. The structure of the quadrotor is shown in Fig. 1. The construction of the mathematical model is based on the following assumptions: (a) the quadrotor's structure is symmetrical; (b) the quadrotor's body and propellers are rigid; (c) free stream air velocity is zero; (d) the motors' dynamics are relatively fast and can be neglected; (e) the flexibility of the blade is relatively small and can be neglected; (f) drag is supposed to be linear [2] [3].

The mass of the quadrotor is denoted by $m \in \mathbb{R}$ and the inertial tensor $\boldsymbol{I} = \operatorname{diag}\left(I_{xx}, I_{yy}, I_{zz}\right) \in \mathbb{R}^{3\times3}$ is defined in the body frame, which is symmetric by assumption (a). Let vector $\boldsymbol{\xi} = [x, y, z]^T \in \mathbb{R}^3$ and $\boldsymbol{v} = [v_x, v_y, v_z]^T \in \mathbb{R}^3$ represent the position and velocity along x, y and z axis in the inertial frame, respectively. The elements of $\boldsymbol{\eta} = [\phi\ \theta\ \psi]^T \in \mathbb{R}^3$ are Tait-Bryan Euler angles. Assume that the Euler angles are bounded as follows: $\phi \in (-\pi/2, \pi/2),\ \theta \in (-\pi/2, \pi/2),\ \psi \in (-\pi, \pi]$.

The angular velocities around x, y and z axis are denoted by the elements of $\boldsymbol{\omega} = [p\ q\ r]^T \in \mathbb{R}^3$ in the body frame, respectively. The angular velocity vector $\boldsymbol{\zeta}$ is related to the Euler angular velocity vector $\boldsymbol{\omega}$ in the body fixed frame, and the relation can be expressed by $\dot{\boldsymbol{\eta}} = \boldsymbol{R}_a^o \boldsymbol{\omega}$, where $\boldsymbol{R}_a^o$ is a transformation matrix from angular velocity around the axis to euler angular velocity given by

$$\boldsymbol{R}_a^o = \begin{bmatrix} 1 & \boldsymbol{s\phi t\theta} & \boldsymbol{c\phi t\theta} \\ 0 & \boldsymbol{c\phi} & -\boldsymbol{s\phi} \\ 0 & \boldsymbol{s\phi/c\theta} & \boldsymbol{c\phi/c\theta} \end{bmatrix}. \tag{1}$$

In the above equation, instead of $\boldsymbol{sinx}\ (\boldsymbol{cosx}, \boldsymbol{tanx}),\ \boldsymbol{sx}\ (\boldsymbol{cx}, \boldsymbol{tx})$ is used for simplicity, with similar abbreviations employed subsequently.
Besides, the linear transformation of a vector from the body-fixed frame to the inertial frame is represented by rotation matrix $\boldsymbol{R} \in \mathbb{R}^3 \times \mathbb{R}^3$ as follows:

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{c\theta c\psi} & \boldsymbol{s\phi s\theta c\psi} - \boldsymbol{c\phi s\psi} & \boldsymbol{c\phi s\theta c\psi} + \boldsymbol{s\phi s\psi} \\ \boldsymbol{c\theta s\psi} & \boldsymbol{s\phi s\theta s\psi} + \boldsymbol{c\phi c\psi} & \boldsymbol{c\phi s\theta s\psi} - \boldsymbol{s\phi c\psi} \\ -\boldsymbol{s\theta} & \boldsymbol{s\phi c\theta} & \boldsymbol{c\phi c\theta} \end{bmatrix}. \tag{2}$$

The translational force is F_l in the body frame, which is described by $F_l = \sum_{i=1}^4 F_i$, where F_i is the thrust moment generated by each motor. Furthermore, F_i is defined as $F_i = c_T \varpi_i^2$, where the positive lumped constant parameter c_T denotes the thrust factor of the propeller and ϖ_i is the angular velocity of the motor i. Then, the input torque $\boldsymbol{\tau_p}$ are given by:

$$\boldsymbol{\tau_p} = \begin{bmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_1 - F_3) \\ T_1 - T_2 + T_3 - T_4 \end{bmatrix}, \tag{3}$$

where l is the distance from the motor to the center of gravity and $T_i = c_Q \varpi_i^2 (i = 1 \sim 4)$ is anti-torque generated by motor i with the torque coefficient c_Q.

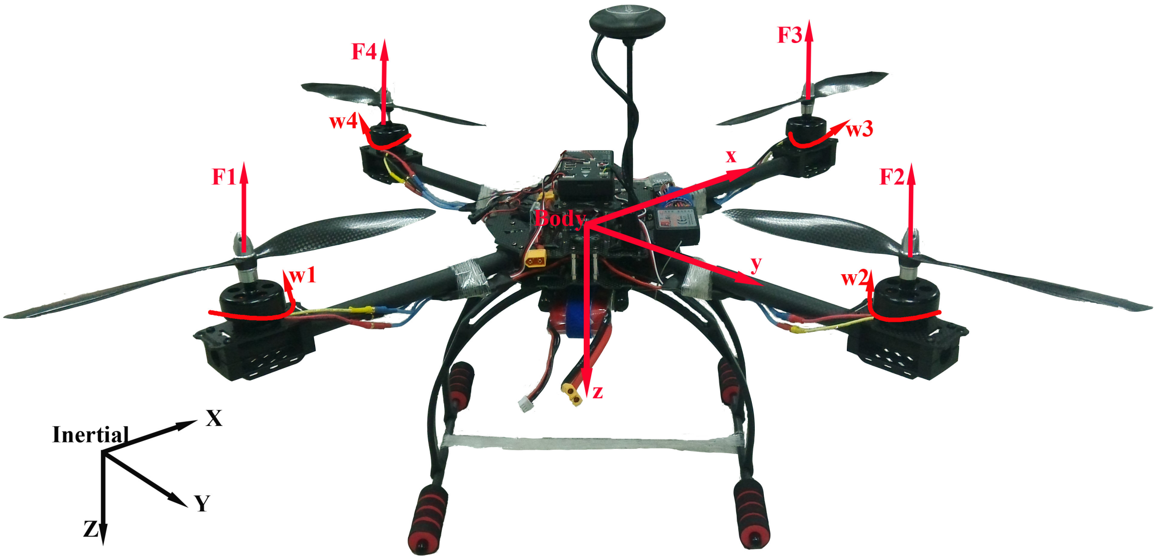

Fig 1. Schematic of a quadrotor with the body frame and inertial frame

Using the well-known rigid-body equations[4], the complete mathematical model of a quadrotor can be expressed as

$$\dot{\boldsymbol{\xi}} = \boldsymbol{v}, \tag{4}$$

$$\dot{\boldsymbol{v}} = \frac{1}{m}(mg\boldsymbol{e_3} - \boldsymbol{R}F_l\boldsymbol{e_3} - \boldsymbol{D}_v\boldsymbol{v}), \tag{5}$$

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}_a^o\boldsymbol{\omega}, \tag{6}$$

$$\dot{\boldsymbol{\omega}} = \boldsymbol{I}^{-1}(\boldsymbol{\tau_p} - \hat{\boldsymbol{\omega}}\boldsymbol{I}\boldsymbol{\omega} - \boldsymbol{D}_\omega\boldsymbol{\omega}), \tag{7}$$

where $\boldsymbol{D_v}$ is the lumped drag force coefficient defined by $\boldsymbol{D_v} = \mathrm{diag}(D_x^i,\ D_y^i,\ D_z^i)$[3], $\boldsymbol{D_\omega}$ is the lumped drag torque coefficient defined by $\boldsymbol{D_\omega} = \mathrm{diag}(D_\phi,\ D_\theta,\ D_\psi)$\cite{novel:modirrousta}, the $hat\ map\ \hat{\cdot} : \mathbb{R} \to so(3)$ is defined on the condition that $\hat{x}y = x \times y$ for all $x,\ y \in \mathbb{R}^3$, and $\boldsymbol{e_3} = [0\ 0\ 1]^T$.

This mathematical model can be divided into two subsystems: a fully-actuated subsystem (6,7), and an under-actuated subsystem (4,5). Thus, the whole model of the quadrotor is an under-actuated system.

References

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