25392319 Yifei Yue

Question 1. (Calculus Review)

a) $f(x,y) = a_1 x^2 y^2 + a_4 xy + a_5 x + a_7$

1. first order derivatives of f with respect to x:

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}(a_1 x^2 y^2) + \frac{\partial}{\partial x}(a_4 xy) + \frac{\partial}{\partial x}(a_5 x) + \frac{\partial}{\partial x}(a_7)$$

$$= 2a_1 y^2 x + a_4 y + a_5 + 0$$

$$\therefore \frac{\partial f(x,y)}{\partial x} = 2a_1 x y^2 + a_4 y + a_5$$

2. second order derivatives of f with respect to x:

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial}{\partial x}(2a_1 x y^2) + \frac{\partial}{\partial x}(a_4 y) + \frac{\partial}{\partial x}(a_5)$$

$$= 2a_1 y^2 + 0 + 0$$

$$\frac{\partial^2 f(x,y)}{\partial x^2} = 2a_1 y^2$$

3. first order derivatives of f with respect to y:

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}(a_1 x^2 y^2) + \frac{\partial}{\partial y}(a_4 xy) + \frac{\partial}{\partial y}(a_5 x) + \frac{\partial}{\partial y}(a_7)$$

$$= 2a_1 x^2 y + a_4 x$$

4. second order derivatives of f with respect to y.

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial}{\partial y}(2a_1x^2y) + \frac{\partial}{\partial y}(a_4x)$$
$$= 2a_1x^2$$

5. second order with respect to x and y

$\because \frac{\partial f(x,y)}{\partial x} = 2a_1xy^2 + a_4y + a_5$

$$\frac{\partial}{\partial y}(2a_1xy^2 + a_4y + a_5) = 4a_1xy + a_4$$

$\therefore \frac{\partial^2 f(x,y)}{\partial x \partial y} = 4a_1xy + a_4$

b) $f(x,y) = a_1x^2y^2 + a_2x^2y + a_3xy^2 + a_4xy + a_5x + a_6y + a_7$

In the same manner.

1. $\frac{\partial f(x,y)}{\partial x} = 2a_1xy^2 + a_3y^2 + 2a_2xy + a_4y + a_5$

2. $\frac{\partial^2 f(x,y)}{\partial x^2} = 2a_1y^2 + 2a_2y$

3. $\frac{\partial f(x,y)}{\partial y} = a_2x^2 + 2a_1x^2y + a_4x + 2a_3xy + a_6$

4. $\frac{\partial f(x,y)}{\partial y^2} = 2a_1x^2 + 2a_3x$

5. $\frac{\partial^2 f(x,y)}{\partial x \partial y} = 4a_1xy + 2a_2x + 2a_3y + a_4$

c) $\sigma(x) = \frac{1}{1+e^{-x}}$

$$\frac{\partial \sigma(x)}{\partial x} = \frac{\partial}{\partial x}\left[(1+e^{-x})^{-1}\right]$$

By chain rule:

$$= -\frac{1}{(1+e^{-x})^2}(-e^{-x})$$

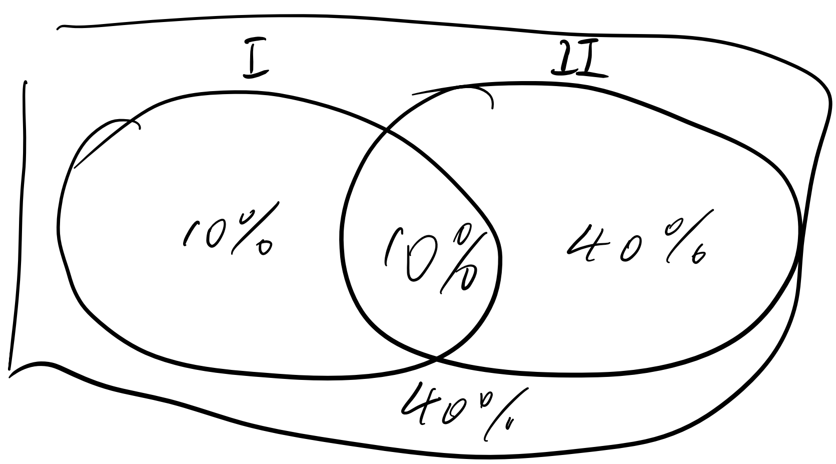
$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{1}{1+e^{-x}} \cdot \frac{e^{-x}}{1+e^{-x}}$$

$$= \sigma(x)(1-\sigma(x))$$

Question 2.

a)



$P(A) = 20\%$

$P(B) = 50\%$

$P(A \vee B) = 60\%$

$P(\bar{A}\bar{B}) = 40\%$

b) i) $r = 1 - \left(\frac{1}{6} + \frac{1}{12} + \frac{1}{12} + \frac{1}{6} + \frac{1}{6}\right)$

$= 1 - \left(\frac{2}{3}\right)$

$= \frac{1}{3}$

ii) $P(X=2, Y=3) = \frac{1}{6}$

iii) $P(X=3) = 0 + \frac{1}{3} + 0 = \frac{1}{3}$

, $P(X=3 \mid Y=2) = 1 - P(X \neq 3 \wedge Y \neq 2)$

$= 1 - \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{6}\right)$

$= \frac{5}{12}$

Question 3

a) By row echelon

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 & 2 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & 0 & -2 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 3 & 4 & \cdots \end{bmatrix}$$

$\therefore \dim(A) = 3$

b) $\dim(B) = 1$

c) $\dim(A^T) = \dim(A) = 3$

b)

i) AB, BA incorrect dimensions

ii) AC

$$= \begin{pmatrix} 1\cdot7+3\cdot2+4\cdot2 & 1\cdot3+3\cdot1+4\cdot2 & 1\cdot3+3\cdot1+4\cdot2 \\ 2\cdot7+2\cdot2+1\cdot2 & 2\cdot3+2\cdot1+1\cdot2 & 2\cdot3+2\cdot1+1\cdot2 \\ 6\cdot7+4\cdot2+3\cdot2 & 6\cdot3+4\cdot1+3\cdot2 & 6\cdot3+4\cdot1+3\cdot2 \end{pmatrix}$$

$$= \begin{pmatrix} 21 & 14 & 14 \\ 20 & 10 & 10 \\ 56 & 28 & 28 \end{pmatrix}$$

$$CA = \begin{pmatrix} 31 & 39 & 40 \\ 10 & 12 & 12 \\ 18 & 18 & 16 \end{pmatrix}$$

iii) AD

$$= \begin{pmatrix} 1\cdot4+3\cdot4+4\cdot1 & 1\cdot2+3\cdot6+4\cdot3 \\ 2\cdot4+2\cdot4+1\cdot1 & 2\cdot2+2\cdot6+1\cdot3 \\ 6\cdot4+4\cdot4+3\cdot1 & 6\cdot2+4\cdot6+3\cdot3 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 32 \\ 17 & 19 \\ 43 & 45 \end{pmatrix}$$

DA incorrect dimensions

iv) DC incorrect dimensions

CD

$$= \begin{pmatrix} 7\cdot 4+3\cdot 4+3\cdot 1 & 7\cdot 2+3\cdot 6+3\cdot 3 \\ 2\cdot 4+1\cdot 4+1\cdot 1 & 2\cdot 2+1\cdot 6+1\cdot 3 \\ 2\cdot 4+2\cdot 4+2\cdot 1 & 2\cdot 2+2\cdot 6+2\cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 43 & 41 \\ 13 & 13 \\ 18 & 22 \end{pmatrix}$$

$D^T C$

$$= \begin{pmatrix} 4 & 4 & 1 \\ 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 32 & 18 & 18 \\ 32 & 18 & 18 \end{pmatrix}$$

v) Bu

$$= \begin{pmatrix} 2\cdot 1+4\cdot 3 \\ 1\cdot 1+1\cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix}14\\4\end{pmatrix}$$

uB incorrect dimensions

vi) Au incorrect dimensions

vii) Av

$$= \begin{pmatrix} 1\cdot2+3\cdot4+4\cdot1 \\ 2\cdot2+2\cdot4+1\cdot1 \\ 6\cdot2+4\cdot4+3\cdot1 \end{pmatrix}$$

$$= \begin{pmatrix}18\\13\\31\end{pmatrix}$$

vA incorrect dimensions

viii) Bv has incorrect dimensions

f) inverse of A

$$\begin{pmatrix}1&3\\4&1\end{pmatrix}^{-1} = \frac{1}{\det(A)}\begin{pmatrix}1&-3\\-4&1\end{pmatrix}$$

$$= -\frac{1}{11}\begin{pmatrix}1&-3\\-4&1\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{11}&\frac{3}{11}\\ \frac{4}{11}&-\frac{1}{11}\end{pmatrix}$$

g) $\begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix}^{-1} = 0$ because matrix is singular

h) Recall that $(AB)^T = B^T A^T$

We have $(X^T X)^T = X^T (X^T)^T$

$= X^T X$

Therefore, $X^T X$ is always symmetric.