ECON 204: Microeconomics III **Spring 2017**

Lecture 6 — May 15th

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6.1 Arrow's Theorem

- $N = \{1, ..., n\}$
- $A =$ finite, $\#A \geq 3$
- $L_i(A) \rightarrow$ complete, transitive, asymmetric
- $R(\succ) \rightarrow$ complete, transitive on A.

Definition 1. Unanimity: if $a \succ_i b$, $\forall i$, then

$$a\ R(\succ)\ b,\ \textit{not}\ b\ R(\succ)\ a$$

Definition 2. Arrow's Independence of Irrelevant Alternatives (AIIA):
$\succ, \succ'$ *s.t.* $a \succ_i b$ iff $a \succ_i' b$, then

$$a\ R(\succ) b \Longleftrightarrow a\ R(\succ')\ b$$

Note 1. AIIA is inconsistent with Borda Rule.

Arrow's Theorem. If $\#A \geq 3$, R complete, transitive, unanimous, and AIIA, $\Leftrightarrow$ dictorial, i.e.

$$\exists\ i \in N$$
$$s.t.\ R(\succ) = \succ_i,\ \forall \succ$$

Note 2. "Sudoku Version": with $n = 2$, $\#A = 3$

Definition 3.

$$b = top(\succ_i) \qquad if\ b \succ_i a\ \forall a \neq b$$
$$b = bottom(\succ_i) \qquad if\ a \succ_i b \forall a \neq b$$

Observe that top and bottom exists and is unique for each individual preference.

Definition 4. We use $a\ P(\succ)\ b$ to denote $a\ R(\succ)\ b$ and not $b\ R(\succ)\ a$.

Lemma 1. If $\succ$ $s.t.$ $\forall i$ either $b = bottom(\succ_i)$ or $b = top(\succ_i)$, then

$$either\ b\ P(\succ)\ a\ \ \forall a \neq b$$

$$or\ a\ P(\succ)\ b\ \ \forall a \neq b$$

unique top and unique bottom

Proof. Suppose on the contrary, $\exists c, a, s.t.$

$$a\ R(\succ)\ b\ R(\succ)\ c$$

we construct $\succ'$ $s.t.$

$$b = top(\succ_i')\ \ if\ b = top(\succ_i)$$

$$b = bottom(\succ_i')\ \ if\ b = bottom(\succ_i)$$

$$c \succ_i' a, \quad \forall i$$

then by AIIA

$$a\ R(\succ')\ b\ R(\succ')\ c$$

and applying transitivity,

$$a\ R(\succ')\ c$$

On the other hand, by unanimity,

$$c\ P(\succ')\ a\ \ \Rightarrow contradiction!$$

(We used the fact that $\#A \geq 3$.)

□

Proof. Step 2. It can be checked easily that if let $\succ^{bottom}$ denote arbitrary preference with $b = bottom(\succ_i^{bottom})$, and $\succ^{top}$ denote preference with b on top.

$$\succ^{(k)}\ is\ s.t.\ \succ_i^{(k)} = \begin{cases} \succ_i^{bottom} & \forall i > k \\ \succ_i^{top} & \forall i \leq k \end{cases}$$

By unanimity, we know

$$b = top(R(\succ^{top}))$$

$$b = bottom(R(\succ^{bottom}))$$

Define $i^*(b)$ to be the first k $s.t.$ b is unique top

$$b\ P(\succ^{(k)})\ a, \quad \forall a$$

$\rightarrow$ "$i^*(b, \succ) = i^*(b)$"

Step 3. Person $i^*(b)$ dictates on all $c \neq b, a \neq b$, i.e. $\forall \succ$, $\forall a, c \neq b$

$$a \succ_{i^*(b)} c \Rightarrow aP(\succ)c$$

$$c \succ_{i^*(b)} a \Rightarrow cP(\succ)a$$

We define $\succ'$ as follows:

$$\succ' = \succ \ \ ignoring\ b$$

$$b = top(\succ') \ \ \forall j < i^*(b) \quad b = bottom(\succ') \ \ \forall j > i^*(b)$$

$$a \ \succ'_{i^*(b)} \ b \ \succ'_{i^*(b)} \ c$$

Step 4. consistency of dictatorship.

□

6.2 Voting Systems

Definition 5. Plurality: "A plurality vote (in North America) or relative majority (in the United Kingdom) describes the circumstance when a candidate or proposition polls more votes than any other, but does not receive a majority.

For example, if 100 votes were cast, including 45 for Candidate A, 30 for Candidate B and 25 for Candidate C, then Candidate A received a plurality of votes but not a majority. " – Wiki
(vote for the most favorite; candidate with the most votes win.)

Definition 6. Borda count: "The Borda count is a single-winner election method in which voters rank options or candidates in order of preference. The Borda count determines the outcome of a debate or the winner of an election by giving each candidate, for each ballot, a number of points corresponding to the number of candidates ranked lower. Once all votes have been counted the option or candidate with the most points is the winner.

Because it sometimes elects broadly acceptable options or candidates, rather than those preferred by a majority, the Borda count is often described as a consensus-based voting system rather than a majoritarian one."– Wiki
(assign points to rankings, and sum; candidate with the most points win.)

Note 3. In plurality or Borda count, irrelevant alternatives do matter.

Example 1. # ppl with preferences:
34 $a \succ c \succ b$
33 $b \succ a \succ c$
32 $c \succ b \succ a$
Plurality gives a; after knocking c out, plurality gives b.

Definition 7. Plurality Elimination: "Plurality with Elimination is carried out in rounds. After each round of voting the candidate (or alternative) with the fewest first place votes is eliminated and a new round of voting is done with the remaining candidates. When only two candidates remain in a round, the candidate with the most votes wins the election."– Web

Definition 8. Condorcet consistent: "The Condorcet candidate (a.k.a. Condorcet winner) is the person who would win a two-candidate election against each of the other candidates in a plurality vote. For a set of candidates, the Condorcet winner is always the same regardless of the voting system in question.

A voting system satisfies the Condorcet criterion if it always chooses the Condorcet winner when one exists."– Wiki (Condorcet winner beats everything by majority.)

Note 4. There may not always be a Condorcet winner. Say if there's cycle.

Definition 9. Copeland Rule: "Copeland's method or Copeland's pairwise aggregation method is a Condorcet method in which candidates are ordered by the number of pairwise victories, minus the number of pairwise defeats."– Wiki

Note 5. "When there is no Condorcet winner, this method often leads to ties. For example, if there is a three-candidate majority rule cycle, each candidate will have exactly one loss, and there will be an unresolved tie between the three."– Wiki

Example 2. People with preferences:

$$\begin{array}{rl} 48 & c \succ b \succ a \\ 2 & b \succ c \succ a \\ 49 & a \succ b \succ c \end{array}$$

plurality $\rightarrow a$
plurality elimination $\rightarrow c$ (b is eliminated first)
copeland $\rightarrow b$

6.3 Nash's Independence of Irrelevant Alternatives & Single-Peaked Preferences

Definition 10. A choice rule $C(\succ, B) \subseteq B$, is a function that takes preference $\succ$, subset of alternatives $B \subseteq A$ and outputs choice set.

Definition 11. Nash's Independence of Irrelevant Alternatives: $B \subseteq B' \subseteq A$, if

$$C(\succ, B') \cap B \neq \emptyset$$

then

$$C(\succ, B) = B \cap C(\succ, B')$$

Theorem 1. Arrow's 1959: C satisfies $NIIA$, then
$\exists R \; s.t. \; C$ is rationalized by R:

$$C(\succ, B) = \{a \in B | \; a \; R(\succ) \; b \;\; \forall b \in B\}$$

Definition 12. "Single-peaked" preferences: Say utility functions u_i. Preferences are single peaked if $\exists b(i) \in A$ s.t.

$$b \leq c < a \Rightarrow u_i(b) \leq u_i(c) > u_i(a)$$

$$b \geq c > a \Rightarrow u_i(b) \leq u_i(c) > u_i(a)$$

Note 6. The definition works when A is discrete as well.

Claim 1.

- Majority rule is now transitive.
- Condorcet winner (n odd) is median peak. "Median voter theorem" (easy to show in graph)