ECON 204: Microeconomics III — **Spring 2017**

Lecture 8 — May 22nd

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8.1 Mechanism Design

- $A,\ N = \{1, ..., n\}$
- $\Theta_i \rightarrow$ "types"
- $M = M_1 \times ... \times M_n$
- $g : M \rightarrow A$

Example 1. voting setting: simplify mechanism to: ask for type.

Note 1. This example introduces us revelation principle: fancy mechanism $\rightarrow$ direct mechanism.

$$(M, g): \quad \textit{fancy mechanism}$$
$$(\Theta, f): \quad \textit{direct mechanism}$$

8.2 Revelation Principle

Revelation Principle: $u_i(a, \theta_i)$ private values, i.e. u_i is a function of θ_i and doesn't depend on other θ_j's. Consider mechanism s.t. $\forall i, \theta_i \in \Theta_i$, $\exists m(\theta_i)$ which is a dominant strategy for i at $u_i(\cdot, \theta_i)$, then $\exists$ social choice function $f : \Theta \rightarrow A$, s.t. θ_i is dominant strategy at $u_i(\cdot, \theta_i)$ $\forall i,\ \theta_i \in \Theta_i$, and $f(\theta) = g(m(\theta)) = g(m(\theta_1), ..., m(\theta_n))$.

Proof.

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)$$
$$u_i(g(m(\theta_i), m(\theta_{-i})), \theta_i) \geq u_i(f(m(\hat{\theta}_i), m(\theta_{-i})), \theta_i), \quad \forall \theta_i, \theta_{-i}$$

$\square$

Note 2. weak dominance is good; strong dominance works as well it's just that it's rarely seen.

8.3 BE version of Revelation Principle

Theorem 1. Consider mechanism s.t. $\exists m : \cup\Theta_i \to \cup M_i$ which is a Bayesian equilibrium, i.e.

$$m(\theta_i) \in M_i, \forall \theta_i \in \Theta_i$$

$$E_{\theta_i}[u_i(g(m(\theta_i), m(\theta_{-i}))] \geq E_{\theta_i}[u_i(g(\hat{m}_i, m(\theta_{-i}))], \quad \forall \theta_i \in \Theta_i, \hat{m}_i \in M_i$$

where E_{θ_i} is expectation (over θ_{-i}) conditioning on θ_i being realized $E_{\theta_i}[u_i(g(m(\theta_i), m(\theta_{-i}))] = \int u_i(g(m(\theta_i), m(x)) f_{\theta_{-i}|\theta_i}(x)dx$

then $\exists$ social choice function $f : \Theta \to A$
s.t. $\sigma(\theta_i) = \theta_i$ is Bayesian equilbirum at $u_i(\cdot, \theta_i)$ $\forall i,\ \theta_i \in \Theta_i$

$$E_{\theta_i}[u_i(f(\theta_i, \theta_{-i}))] \geq E_{\theta_i}[u_i(f(\hat{\theta}_i, \theta_{-i})], \quad \forall \theta_i \in \Theta_i, \hat{\theta}_i \in \Theta_i$$

and $f(\theta) = g(m(\theta)) = g(m(\theta_1), ..., m(\theta_n))$.

Note 3. The construction in revelation principle doesn't maintain "full" equilibrium correspondence, i.e. can lose or pick up more equilibria; it only makes sure that it preserves the particular one.

Example 2. Plurality Rule:
1. $\exists$ equilibrium where everyone sends most preferred candidate.
2. everyone $\to a$ $\quad (n \geq 3)$
3. only vote for a/b $\quad (n \geq 3)$

Example 3. Extensive Form:
Person 1 observes person 2's type. Cannot directly apply theorem, but can indirectly apply.

8.4 Gibbard-Satterthwaite Theorem

- $A = \{a, ...\}$ finite $\quad N = \{1, ..., n\}$
- $\Theta = (\Theta_1, ..., \Theta_n) \quad f : \Theta \to A$
- $\Theta_i \to$ rich: $\forall \succ \in L(A),\ \exists \theta_i \in \Theta_i\ s.t.$

$$u_i(a, \theta_i) > u_i(b, \theta_i) \Leftrightarrow a \succ b$$

- $\Theta_i \to$ no indifference: $\forall \theta_i \in \Theta_i$

$$u_i(c, \theta_i) = u_i(\theta_i) \Rightarrow a = b$$

1. We will only look at direct mechanisms from now on since Revelation Theorem shows that equilibriums are preserved? $\Rightarrow$ G-S theorem.

Definition 1. s.c.f. $f : \Theta \to A$ is strategy-proof if truth is dominant strategy.

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i) \quad \forall i, \theta_i, \theta_{-i}, \hat{\theta}_i$$

Theorem 2. $\forall i$, Θ_i satisfies (richness, no indifference), $A \geq 3$. If f has full range, i.e. $range(f) = A$, and is strategy proof, then it's dictatorial.

Restate the theorem: With richness, and range(f) ≥ 3, strategy proof $\Leftrightarrow$ dictatorial on range(f).

Note 4. 1. no indifference condition is minor; for the sake of simplicity.
2. Single-peaked preferences will violate richness. $1 \succ 4 \succ 3$ impossible
3. dictatorial $\Rightarrow$ strategy-proof naturally.
4. By theorem, a lot of the mechanisms we saw before work with rich preferences and are are not dictatorial, thus not strategy-proof.

Example 4. Borda rule note dictatorial $\Rightarrow$ not strategy-proof

8.5 Vickrey Auction

Example 5. Auction:

- 1 good to allocate
- n people to buy
- $A = \{\{1, ..., n\}\} \times R^n$, $a = (d, t_1, ..., t_n)$, where d is assignment, t is cost.

Utility $u_i(a, \theta_i) = \theta_i I\{d = i\} - t_i$, violating richness. i always rank a s.t. $d = i$ higher than a with $d \neq i$

$$(i, t_1, ...t_n) \succ_i (j, t_1, ..., t_n)$$

Example 6. Vickrey Auction: (mostly refers to 2nd price auction. A similar example: US selling treasury bonds, top 1000 bidders get it paying the price equal to 1001-st person's bid)

- $\Theta_i = R_+$ "possible real values"
- $\hat{\theta} = (\hat{\theta}_1, ..., \hat{\theta}_n)$ "bids"
- $s_1(\hat{\theta}) = \max_i \hat{\theta}$ $s_2(\hat{\theta}) =$ 2nd highest $\hat{\theta}_i$ "order statistics"

- $d(\hat{\theta}) = i$ s.t. $\hat{\theta}_i = s_1(\hat{\theta})$ break ties $\sim$ lowest index
- $t_i = 0$ if $d(\hat{\theta}_i) \neq i$ $\quad t_i = s_2(\hat{\theta})$ otherwise

Now utility becomes $\theta_i I\{d=i\} - t_i = \begin{cases} \theta_i - \max_{j\neq i}(\hat{\theta}_j) & \hat{\theta}_i > \max_{j\neq i}(\hat{\theta}_j) \\ 0 & \hat{\theta}_i < \max_{j\neq i}(\hat{\theta}_j) \end{cases}$

Dominant Strategy: $\hat{\theta}_i = \theta_i \rightarrow \begin{cases} win\ when\ \theta_i > \max_{j\neq i}(\hat{\theta}_j) \\ lose\ when\ \theta_i < \max_{j\neq i}(\hat{\theta}_j) \end{cases}$

Note 5. G-S Theorem's result is bad. We want something that is (Vickrey auction is) strategy proof, but not dictatorial (by violating richness assumption.)

8.6 Build Public Good

- $x \in R^+ \rightarrow$ amount of public good
- $c(x) \rightarrow$ cost of public good cost divided equally $\frac{c(x)}{n}$
- $A = \{x \in R^+\}$
- utility $\theta_i x - \frac{c(x)}{n}$

For the rest of the problem, we set $c(x) = x^2$ Then optimal level of public good for person i satisfy

$$FOC : \theta_i - \frac{2x}{n} = 0$$
$$\Rightarrow x^* = \frac{n\theta_i}{2}$$

Rule: $\hat{\theta} \rightarrow$

$$f(\hat{\theta}) = \frac{n}{2} med(\hat{\theta}_i)$$

This is strategy-proof: telling the truth $\rightarrow$ is median, no lie makes better.
$\rightarrow$ is lower than median, will only change the result if it changes to higher than median, and that increases $a = f(\hat{\theta})$ and makes it worse.

Note 6. 1. Preferences are single-peaked, thus not rich.
2. Other order statistics (e.g. $\frac{n}{2}\hat{\theta}_{[2]}$) works equally well.
3. "Phantom Voter Theorem"

Example 7. Unefficiency: The above mechanism is not efficient.
$n = 3$
$\theta_1 = 1 \quad x_1^* = \frac{3}{2}$
$\theta_2 = 2 \quad x_2^* = 3$
$\theta_3 = 4 \quad x_3^* = 6$

total:

$$7x - x^2$$
$$\Rightarrow x^* = 3.5$$

Note 7. If $\theta_3 = 10$, person 3 will want high x and willing to pay full cost. But if we change the mechanism so that higher types pay more, people will lie.