Ch 5: Amortization Schedules and Sinking Funds

Please read Ch 5.1 to 5.6 from the Theory of Interest.

5.1 Introduction

– there are two methods for paying off a loan:

(i) Amortization Method - borrower makes installment payments at periodic intervals

(ii) Sinking Fund Method - borrower makes installment payments as the annual interest comes due and pays back the original loan as a lump-sum at the end. The lump-sum is built up with periodic payments going into a fund called a "sinking fund".

5.2 Finding the Outstanding Loan Balance (outstanding principal/unpaid balance/remaining loan indebtedness)

Amortization method

- Borrower repays the lender by means of installments at periodic intervals
- Installments form an annuity
- PV of all the installments = original amount
- There are two methods for determining the outstanding loan once the payment process commences:

  (i) Prospective Method

  (ii) Retrospective Method

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Retrospective Method

- calculates outstanding loan balance by looking into the past
- Let P be the level repayments that are payable at the end of each year
- the outstanding loan at time t, B_t, is equal to the accumulated value of the loan *less* the accumulated value of the payments made to date

$$B_t^r = L(1+i)^t - Ps_{\overline{t|}i}$$

Finding the Outstanding Balance at different times:

$t = 1$: $L(1 + i) - P$

$t = 2$: $[L(1 + i) - P](1+ i) - P$

$t = 3$: $\{[L(1 + i) - P](1+ i) - P\}(1+ i) - P$

$t = t$: $\{[L(1 + i) - P](1+ i) - P\}(1+ i) - P$ … t times

$= L(1 + i)^t - P((1 + i)^{t-1} + (1 + i)^{t-2} + \ldots + (1 + i) + 1)$

$= L(1 + i)^t - Ps_{\overline{t|}}$

Example:

Say we have a loan of \$10,000. If it takes 6 years to repay the loan with annual payments at $i = 12\%$. Find the annual payment.

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How much we owe after 1 payment (without interest)?

How much we owe after 1 payment (taken into account of interest)?

NOTE:

The amount that we paid to reduce the loan amount:

The amount that we paid for interest:

How much we owe after 2^{nd} payment (taken into account of interest) or balance at year 2?

NOTE:

The amount that we paid to reduce the loan amount:

The amount that we paid for interest:

What is the outstanding loan balance at year 3, 4, 5, and 6?

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Prospective Method

- calculates outstanding loan balance by looking into the future
- the original loan at time 0 represents the present value of future repayments.
- The original loan can be represented as follows:

$$\text{Loan} = B_0^p = Pa_{\overline{n}|i}$$

- the outstanding loan at time *t*, B_t, represents the present value of the remaining future repayments

$$B_t^p = Pa_{\overline{n-t}|i}$$

- this assumes that the repayment schedule determined at time 0 has been adhered to; otherwise, the prospective method will not work

Outstanding balance at different time:

$t = 0$: $B_0^p = Pa_{\overline{n}|}$

$t = 1$:

$$\begin{aligned} B_1^p &= Pa_{\overline{n}|}(1+i) - P \\ &= P\left(\frac{1-v^n}{i}\right)(1+i) - P = \frac{P(1+i) - Pv^n(1+i) - Pi}{i} \\ &= \frac{P(1+i-v^{n-1}-i)}{i} = \frac{P(1-v^{n-1})}{i} \\ &= Pa_{\overline{n-1}|} \end{aligned}$$

$t = t$: $B_t^p = Pa_{\overline{n-t}|}$

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Example 5.1 (TOI, pg 155)

A loan of \$1000 is being repaid with 10 payments of \$2000 followed by 10 payments of \$1000 the end of each half-year. If the nominal rate of interest convertible semiannually is 10%, find the outstanding loan balance immediately after 5 payments have been made by:

1) the prospective method
2) the retrospective method

Example:

L = 15000, i = 0.08. Annual payments of \$2000 starting in 1 year. What is the outstanding balance after 3 years?

Example:

Jim gets a loan for \$5000 at $i^{(4)} = 0.08$.

He plans on making quarterly payments at the end of each quarter for 3 years. What is the payment amount?

Jim misses the 5th and 6th payments. What does Jim owe at the time of the 7th payment to catch up?

3 options:

1) Make 3 payments at once
2) Payoff the whole loan
3) Redo the payments

Time	0	1	2	…	4	5	6	7	8	…	12
Original		P	P	…	P	P	P	P	P	…	P
Opt 1		P	P	…	P	0	0	A	P	…	P
Opt 2		P	P	…	P	0	0	B	0	…	0
Opt 3		P	P	…	P	0	0	C	C	…	C

Remark:

1) the prospective method is preferable when the size of each level payment and the number of remaining payments is known
2) the retrospective method is preferable when the number of remaining payments or a final irregular payment is unknown.

5.3 Amortization Schedules

– let a loan be repaid with end-of-year payments of 1 over the next n years

– the loan at time 0 is $1a_{\overline{n}|}$

– an annual end-of-year payment of 1 using the amortization method will contain an interest payment, I_t, and a principal repayment, P_t

– in other words, $1 = I_t + P_t$

Interest Payment (I_t)

– I_t is intended to cover the interest obligation that is payable at the end of year t. The interest is based on the outstanding loan balance at the beginning of year t.

$$I_t = i\left(1a_{\overline{n-(t-1)}|i}\right) = i\left(\frac{1-v^{n-(t-1)}}{i}\right) = 1-v^{n-(t-1)}$$

Principal Repayment

Once the interest owed for the year is paid off, then the remaining portion of the amortization payment goes towards paying back the principal:

$$P_t = 1 - I_t$$

$$= 1 - [1 - v^{n-(t-1)}]$$

$$P_t = v^{n-(t-1)}$$

Outstanding Loan Balance

– outstanding loan at the end of year t can also be viewed as the outstanding loan at the beginning of year t *less* the principal repayment that has just occurred

$$\begin{aligned} B_t &= B_{t-1} - P_t \\ &= 1a_{\overline{n-(t-1)}|i} - v^{n-(t-1)} \\ &= v + v^2 + \cdots + v^{n-t} + v^{n-(t-1)} - v^{n-(t-1)} \\ &= v + v^2 + \cdots + v^{n-t} \\ &= a_{\overline{n-t}|i} \end{aligned}$$

The following amortization schedule illustrates the progression of the loan repayments:

Year (t)	Payment	I_t	P_t	B_t
1	1	$1-v_i^n$	v_i^n	$a_{\overline{n-1}\|i}$
2	1	$1-v_i^{n-1}$	v_i^{n-1}	$a_{\overline{n-2}\|i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	1	$1-v_i^{n-(t-1)}$	$v_i^{n-(t-1)}$	$a_{\overline{n-t}\|i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	1	$1-v_i^2$	v_i^2	$a_{\overline{1}\|i}$
n	1	$1-v_i$	v_i	0
Total	n	$n-a_{\overline{n}\|i}$	$a_{\overline{n}\|i}$	

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NOTE: the principal repayments increase geometrically by (1+ i)

i.e. $P_{t+n} = P_t\,(1 + i)^n$. This is to be expected since the outstanding loan gets smaller with each principal repayment and as a result, there is less interest accruing which leaves of the amortization payment left to pay off principal.

Example (TOI, pg 158, 159):

Consider the construction of an amortization schedule for a 1000 loan repaid in 4 annual payments (R) if i = 8%.

Year	Payment	I_t	P_t	B_t
0				
1				
2				
3				
4				

Example:

A $125000 mortgage is being repaid with monthly installments at the end of each month for 20 years. If $i^{(2)} = 0.0917045$, find the outstanding loan balance at the end of 1 year. What is the amount of principal paid and what is the total amount of interest paid by the end of 1 year?

Example 5.3 (TOI, pg 161)

A $1000 loan is being repaid by payments of $100 at the end of each quarter for as long as necessary, plus a smaller final payment. If the nominal rate of interest convertible quarterly is 16%, find the amount of principal and interest in the 4^{th} payment.

Example 5.4 (TOI, pg 161)

A borrows $10,000 from B and agrees to repay it with equal quarterly installments of principal and interest at 8% convertible quarterly over 6 years. At the end of 2 years, B sells the right to receive future payments to C at a price that will yield C 10% convertible quarterly. Find the total amount of interest received by:

a) C

b) B

5.4 Sinking Funds

– let a loan of $1a_{\overline{n}|i}$ be repaid with single lump-sum payment at time *n*.

– If annual end-of-year interest payments of $ia_{\overline{n}|i}$ are being met each year, then the lump-sum required at $t = n$ is the original loan amount. (i.e. the interest on the loan never gets to grow with interest)

– let the lump-sum that is to be built up in a "sinking fund" be credited with interest rate *i*

Sinking Fund Payment

– if the lump-sum is to be built up with annual end-of-year payments for the next *n* years, then the sinking fund payment or deposit is calculated as:

- $$P = deposit = \frac{Loan}{s_{\overline{n}|i}} = \frac{a_{\overline{n}|i}}{s_{\overline{n}|i}}$$

– the total annual payment for year *t* made by the borrower is the annual interest due on the loan *plus* the sinking fund payment:

$$\text{Payment}_t = ia_{\overline{n}|i} + \frac{a_{\overline{n}|i}}{s_{\overline{n}|i}} = a_{\overline{n}|i}\left(i + \frac{1}{s_{\overline{n}|i}}\right) = a_{\overline{n}|i}\left(\frac{1}{a_{\overline{n}|i}}\right) = 1$$

– in other words, the annual payment under the sinking fund method is the same annual payment under the amortization method

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Net Amount of Loan

– the accumulated value of the sinking fund at time *t* is the accumulated value of the sinking fund payments made to date:

$$SF_t = \left(\frac{a_{\overline{n}|i}}{s_{\overline{n}|i}} \right) s_{\overline{t}|i} = \left(\frac{v^n s_{\overline{n}|i}}{s_{\overline{n}|i}} \right) s_{\overline{t}|i} = v^n s_{\overline{t}|i}$$

– the loan itself will never grow as long as the annual interest growth, $ia_{\overline{n}|i}$, is paid off at the end of each year

– "net" amount of loan outstanding = the loan amount that is not covered by the balance in the sinking fund = Loan - SF_t

$$\begin{aligned} Net\ Loan_t &= Loan - SF_t \\ &= a_{\overline{n}|i} - v^n s_{\overline{t}|i} \\ &= \frac{1-v^n}{i} - v^n \frac{(1+i)^t - 1}{i} \\ &= \frac{1 - v^n - v^n (1+i)^t + v^n}{i} = \frac{1-v^{n-t}}{i} \\ &= a_{\overline{n-t}|i} \end{aligned}$$

Net Amount of Interest

– the actual interest cost to the borrower for year *t* is referred to as the "net" amount of interest

– "net" amount of interest is the difference between what amount of interest has been paid and what amount of interest has been earned

– the borrower pays interest to the lender in the amount of $ia_{\overline{n}|i}$ each year

– the borrower also earns interest in the sinking fund of iSF_{t-1} each year

$$\begin{aligned} ia_{\overline{n}|i} - i \cdot SF_{t-1} &= ia_{\overline{n}|i} - i \cdot \left(\frac{a_{\overline{n}|i}}{s_{\overline{n}|i}} \right) s_{\overline{t-1}|i} \\ &= i\frac{1-v^n}{i} - iv^n \frac{(1+i)^t - 1}{i} \\ &= 1 - v^n - v^n(1+i)^{t-1} + v^n \\ &= 1 - v^{n-(t-1)} \end{aligned}$$

Sinking Fund Increase

– the sinking fund grows each year by the amount of interest that it earns and by the end-of-year contribution that it receives

– the increase in the sinking fund, $SF_t - SF_{t-1}$:

$$SF_t = SF_{t-1}(1+i) + \left(\frac{a_{\overline{n}|i}}{s_{\overline{n}|i}} \right)$$

$$\begin{aligned} SF_t - SF_{t-1} &= iSF_{t-1} + \left(\frac{a_{\overline{n}|i}}{s_{\overline{n}|i}} \right) = iv^n s_{\overline{t-1}|i} + \frac{v^n s_{\overline{n}|i}}{s_{\overline{n}|i}} \\ &= v^n \left(i\frac{(1+i)^{t-1} - 1}{i} + 1 \right) = v^n (1+i)^{t-1} \\ &= v^{n-(t-1)} \end{aligned}$$

Example: Consider a Sinking Fund Schedule for a Loan of $1000 repaid Over 4 Years at 8%

Year	Interest Paid	SF Deposit	Interest Earned on SF	Amt in SF	O/B
0					1000
1	80	221.92	0	221.92	778.08
2	80	221.92	17.75	461.59	538.41
3	80	221.92	36.93	720.44	279.56
4	80	221.92	57.64	1000	0

Remark:

SF Method	Amortization
total payment (Interest paid on the loan + sinking fund deposits)	payment amount
net interest paid (Interest paid on the loan – interest earned on this SF)	interest paid
annual increment (SF deposit + interest earned on the SF)	principal repaid
net amount of the loan (original amount of loan – the amount in the SF)	Outstanding loan balance

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What Happens When The Sinking Fund Earns Rate *j*, not *i*

– usually, the interest rate on borrowing, *i*, is greater than the interest rate offered by investing in a fund, *j*

– the total payment under the sinking fund approach is then

$$i \cdot Loan + \frac{Loan}{s_{\overline{n}|j}}$$

Example (TOI, pg 198, 24)

A borrower is repaying a loan with 10 annual payments of $1000. Half of the loan is repaid by the amortization method at 5% effective. The other half of the loan is repaid by the sinking fund method in which the lender receives 5% effective on the investment and the sinking fund accumulates at 4% effective. Find the amount of the loan.

Example:

Payment of \$36000 is made at the end of each year to payoff \$400,000 in 31 years. If the borrower adopted the sinking fund method with the SF paying $j = 0.03$ effective. What is i?

5.5 Differing Payment Periods and Interest Conversion Periods

Example 5.8 (TOI, pg. 171)

A debt is being amortized by means of monthly payments at an annual effective rate of interest of 11%. If the amount of principal in the 3rd payment is $1000, find the amount of principal in the 33rd payment.

Example 5.9 (TOI, pg. 171, 172)

A borrower takes out a loan of $2000 for 2 years. Construct a sinking fund schedule if the lender receives 10% effective on the loan and if the borrower replaces the amount of loan with semiannual deposits in a sinking fund earning 8% convertible quarterly.

Year	Interest Paid	SF Deposit	Interest Earned on SF	Amount In SF	Net Amount of Loan
0					
¼	0	0	0	0	2000
½	0	470.7	0	470.7	1529.3
¾	0	0	9.41	480.11	1519.89
1	200	470.7	9.60	960.41	1039.59
1 ¼	0	0	19.21	979.62	1020.38
1 ½	0	470.7	19.59	1469.91	530.09
1 ¾	0	0	29.4	1499.31	500.69
2	200	470.7	29.99	2000	0

5.6. Varying Payments for Amortization Schedules and Sinking Funds

- what happens when the payments on a loan are not level?
- must resort to using general principals in order to evaluate the interest payment, principal payment, the outstanding loan and the amortization schedules
- consider the following 4 scenarios:

(i) payments increase/decrease arithmetically

(ii) payments increase/decrease geometrically

(iii) equal amounts of principal are paid each period

(iv) payments randomly vary

Payments Increase/Decrease Arithmetically

Example 5.10 (TOI, pg 174)

A borrower is repaying a loan at 5% effective with payments at the end of each year for 10 years, such that the payment the first year is $200, the 2^{nd} year $190, and so forth, until the 10^{th} year it is $110.

a) Find the amount of the loan.

b) Find the principal and interest paid in the 5^{th} period.

Payments Increase/Decrease Geometrically

Example 5.13: (TOI, pg 176)

A borrows \$10000 from B and agrees to repay it with a series of 10 installments at the end of each year such that each installment is 20% greater than the preceding installment. The rate of interest on the loan is 10% effective. Find the amount of principal repaid in the first 3 installments.

	Payment	Interest Paid	Principal Repaid	OSB
0				10000
1	720.89	1000	-279.11	10279.11
2	865.07	1027.911	-162.84	10441.95
3	1038.08	1044.1953	-6.11	10448.07
4	1245.70	1044.80667	200.89	10247.18
5	1494.84	1024.71755	470.12	9777.06
6	1793.81	977.705549	816.10	8960.96
7	2152.57	896.095604	1256.47	7704.49
8	2583.08	770.448563	1812.63	5891.85
9	3099.70	589.185499	2510.51	3381.35
10	3719.63	338.134544	3381.50	-0.15

Equal Amounts of Principal Paid Each Period

Example:

A 20,000 loan is repaid with 20 equal end-of-year principal payments where the principal payments are $P_1 = P_2 = \cdots = P_{20} = 20{,}000/20 =$ 1000 and with 20 interest payments at 3%,

$I_t = 3\% \times [20{,}000 - 1{,}000(t-1)]$.

(a) What is accumulated value of the 11*th* to 20*th* payments if 5% can be earned for the next 5 years and 4% can be earned, thereafter?

(b) What is the value at $t = 10$?