Regular Expressions with Backreferences and Lookaheads Capture NLOG

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CyberAgent, Inc.

Introduction: Modern Regular Expressions

Classical Regular Expressions (REGEX)

Example. $$\begin{aligned} [\![aa\ a^*]\!] &= [\![aa(\epsilon + a + aa + aaa + \cdots)]\!] \\ &= \{a^n : n \geqslant 2\} \end{aligned}$$

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$$(\ aa\,a^*\ )_x \quad \backslash x \quad \backslash x\ ^*$$

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$$(\, aa\ a^* \,)_x \ \backslash x \ \backslash x^*$$

$(\ E\ )_x$: **Capturing**
stores a string
matched with E
to a variable x

$\backslash x$: **Referring**
reads the string
stored in x

$(aa\,a^*)_x \; \backslash x \; \backslash x^*$ means $\{a^n : n$ is composite$\}$.

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n is composite $\iff \exists i, j \geqslant 2.\ n = i \cdot j$

$(aa\,a^*)_x \setminus x \setminus x^*$ means $\{a^n :$ n is composite$\}$.

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The expression behaves like the “Sieve of Eratosthenes”:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
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31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
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81	82	83	84	85	86	87	88	89	90
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The expression behaves like the “Sieve of Eratosthenes”:

$$\overset{x}{\overbracket{aa}} \backslash x \backslash x^*$$

$$(aa\,a^*)_x \;\; \backslash x \;\; \backslash x^* =$$

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The expression behaves like the “Sieve of Eratosthenes”:

$$\overbrace{aa}^{x}\,(aa)\ (aa)^*\ (= \{a^{2\cdot j} : j \geqslant 2\})$$

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$$\begin{aligned} & \overset{x}{\overbracket{aa}}\,(aa)\ (aa)^* \ (=\{a^{2\cdot j} : j \geqslant 2\}) \\ (aa\,a^*)_x \ \ \backslash x \ \ \backslash x^* = \ + \ & \overset{x}{\overbracket{aaa}}\,(aaa)\ (aaa)^* \ (=\{a^{3\cdot j} : j \geqslant 2\}) \end{aligned}$$

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& + \overbracket{aaa}^{x}\,(aaa)\ (aaa)^* \ (= \{a^{3\cdot j} : j \geqslant 2\}) \\
& + \overbracket{aaaa}^{x}\,(aaaa)\ (aaaa)^* \ (= \left\{a^{4\cdot j} : j \geqslant 2\right\}) \\
& + \cdots
\end{aligned}
$$

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\end{aligned}
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So, $[\![\,(aaa^*)_x \setminus x \setminus x^*\,]\!] = \{a^n : n$ is composite$\}$.

Can we represent the prime numbers $\left\{a^2, a^3, a^5, a^7, \ldots\right\}$??

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$$?(\ aa\ a^*\ \$\ ) \quad !(\ \underbrace{(aa\ a^*)_x\ \backslash x\ \backslash x^*}_{\text{composite checker}}\ \$\ ) \quad a^*$$

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Classical Regular Expressions (REGEX)

$$\text{Example. } [\![aa\ a^*]\!] = [\![aa(\epsilon + a + aa + aaa + \cdots)]\!]$$
$$= \{a^n : n \geqslant 2\}$$

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Example. $$[\![aa\ a^*]\!] = [\![aa(\epsilon + a + aa + aaa + \cdots)]\!] = \{a^n : n \geqslant 2\}$$

Mode … Lookaheads

Positive Lookahead

$?(\ E\ )$: if the input matches with E, then we continue.
Else: Fail

$\backslash x\ ^*$

$$?(\ aa\ a^*\ \$\ )\ ?(\ \underbrace{(aa\ a^*)_x\ \backslash x\ \backslash x^*}_{\text{composite checker}}\ \$\ )\ a^*$$

Introduction: Modern Regular Expressions

Classical Reg

Example

a^7: $a \quad a \quad a \quad a \quad a \quad a \quad a$

$$= \{a^n : n \geqslant 2\}$$

Mode Lookaheads

Positive Lookahead

?(E) : if the input matches with E, then we continue.

Else: Fail

$\backslash x$ *

. ($aa\,a$ $) .($\underbrace{(aa\,a\ )_x\ \backslash x\ \backslash x^*}_{\text{composite checker}}$ $) a^*

Introduction: Modern Regular Expressions

Classical Reg

Example

a^7: $a\ a\ a\ a\ a\ a\ a$

$$= \{a^n : n \geqslant 2\}$$

Mode Lookaheads

$?(\ E\ )$:

Positive Lookahead

if the input matches with E,

then we continue.

Else: Fail

$x\ ^*$

$$?(\ aaa^*\ \$\ )\ \ !(\ \underbrace{(aaa^*)_x\ \backslash x\ \backslash x^*}_{\text{composite checker}}\ \$\ )\ a^*$$

Introduction: Modern Regular Expressions

Classical Reg

Example

a^7: $a \; a \; a \; a \; a \; a \; a$

Mode

$?($ E $)$:

Posi

if th

then we continue.

Else: Fail

$!($ E $)$:

Negative Lookahead

if the input **never** matches with E,

then we continue.

Else: Fail

$!($ $\underbrace{(aa\,a^*)_x \; \backslash x \; \backslash x^*}_{\text{composite checker}}$ $\$$ $)$ a^*

Introduction: Modern Regular Expressions

Classical Reg

Example

a^7: $a \; a \; a \; a \; a \; a \; a$ *not match!*

Mode

Negative Lookahead

$!(\; E \;)$: if the input **never** matches with E, then we continue.
Else: Fail

Posi

$?(\; E \;)$: if th

then we continue.

Else: Fail

$x \backslash x^* \; \$ \;) \; a^*$

composite checker

Introduction: Modern Regular Expressions

Classical Reg

Example $\cdot\,)]\!]$

$\{a^n : n \geq 2\}$

a^7: $a\ a\ a\ a\ a\ a\ a$ *accept!*

Mode

?(E) :

Posi

if th

then we continue.

Else: Fail

!(E) :

Negative Lookahead

if the input **never** matches with E,

then we continue.

Else: Fail

$?(\ aa\ a\ \$\ )\ \ !(\ \underbrace{(aa\ a\ )_x\ \backslash x\ \backslash x^*}_{\text{composite checker}}\ \$\ )\ a^*$

Introduction: Modern Regular Expressions

Classical Reg

Example

$\{a^n : n \geq 2\}$

a^{10}: $a \; a \; a \; a \; a \; a \; a \; a \; a \; a$

Mode

$?(\; E \;)$:

Posi

if th

then we continue.

Else: Fail

$!(\; E \;)$:

Negative Lookahead

if the input **never** matches with E,

then we continue.

Else: Fail

$$?(\; aa\,a^* \;\$\;)\; !(\; \underbrace{(aa\,a^*)_x \setminus x \setminus x^*}_{\text{composite checker}} \;\$\;)\; a^*$$

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Classical Reg

Example

a^{10}: $a\ a\ a\ a\ a\ a\ a\ a\ a\ a$

$\{a^n : n \geq 2\}$

Mode

$?(\ E\ )$:

Posi

if th

then we continue.

Else: Fail

$!(\ E\ )$:

Negative Lookahead

if the input **never** matches with E, then we continue.

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$$?(\ aa\ a\ \$\ )\ !(\ \underbrace{(aa\ a\ )_x\ \backslash x\ \backslash x^*}_{\text{composite checker}}\ \$\ )\ a^*$$

Introduction: Modern Regular Expressions

Classical Reg

Example

a^{10}: a a a a a a a a a a *Fail!*

$\{a^n : n \geq 2\}$

Mode

?(E) :

Posi

if th

then we continue.

Else: Fail

!(E) :

Negative Lookahead

if the input **never** matches with E,

then we continue.

Else: Fail

$$?(\ aa\,a^*\ \$\ )\ !(\ \underbrace{(aa\,a^*)_x\ \backslash x\ \backslash x^*}_{\text{composite checker}}\ \$\ )\ a^*$$

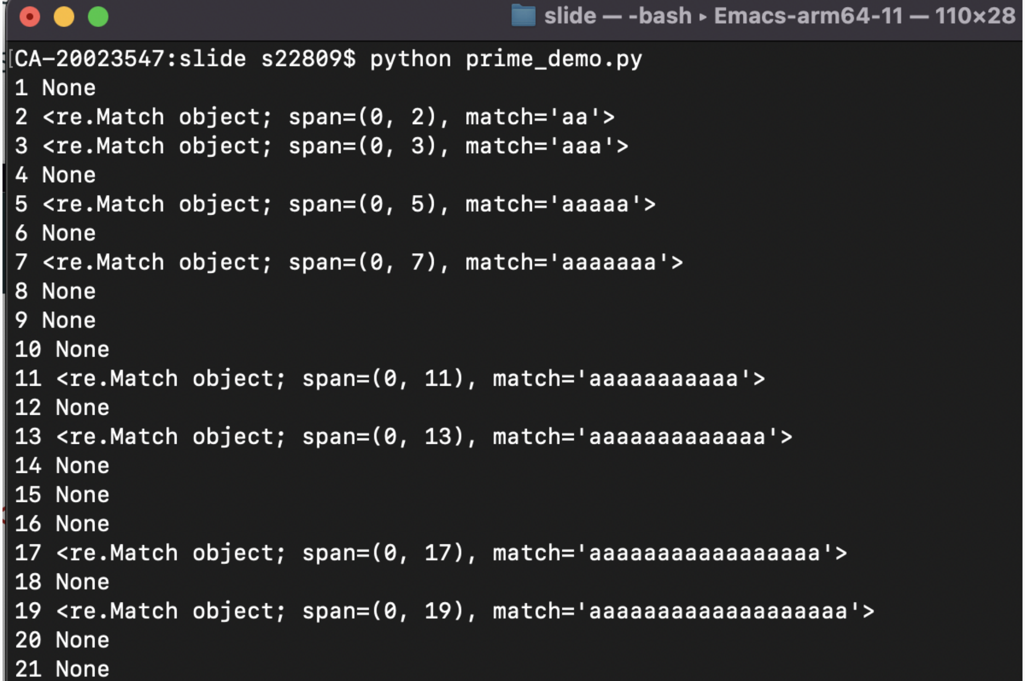
Introduction

Coding and experimenting in Python

```python
import re

prime = r'(?!((?P<X>(aaa*))(?P=X)(?P=X)*$))aa(a*)'

for i in range(1, 50):
    w = 'a' * i
    result = re.fullmatch(prime, w)
    print(i, result)
```



```
CA-20023547:slide s22809$ python prime_demo.py
1 None
2 <re.Match object; span=(0, 2), match='aa'>
3 <re.Match object; span=(0, 3), match='aaa'>
4 None
5 <re.Match object; span=(0, 5), match='aaaaa'>
6 None
7 <re.Match object; span=(0, 7), match='aaaaaaa'>
8 None
9 None
10 None
11 <re.Match object; span=(0, 11), match='aaaaaaaaaaa'>
12 None
13 <re.Match object; span=(0, 13), match='aaaaaaaaaaaaa'>
14 None
15 None
16 None
17 <re.Match object; span=(0, 17), match='aaaaaaaaaaaaaaaaa'>
18 None
19 <re.Match object; span=(0, 19), match='aaaaaaaaaaaaaaaaaaa'>
20 None
21 None
```

Main Results on REWBLk

Question: *How expressive are modern REGEXs?*

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Answer: We have two new results.

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In the paper, modern REGEXs are called as REWBLk.
(***R***egular ***E***xpressions ***W***ith ***B***ackreferences and ***L***oo***k***aheads).

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1. The language class of REWBLK $=$ **NL** (nondeterministic log-space).

   We have the following translation:

$$E : \text{REWBLK} \iff M : \text{nondeterministic log-space TM } s.t.\ [\![E]\!] = L(M)$$

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   $$E : \text{REWBLk} \Longleftrightarrow M : \text{nondeterministic log-space TM } s.t.\ [\![E]\!] = L(M)$$

2. The complexity of REWBLk-membership problem is **PSPACE**-complete.
   - **Input**: a REWBLk expression E & a string w
   - **Output**: True if E accepts w. False otherwise.

Main Results on REWBLk

Question: *How expressive are modern REGEXs?*

In this talk,
we only focus on 1 REWBLk $=$ **NL**

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T

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1 The language class of REWBLk $=$ **NL** (nondeterministic log-space).

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2 The complexity of REWBLk-membership problem is **PSPACE**-complete.

- **Input**: a REWBLk expression E & a string w
- **Output**: True if E accepts w. False otherwise.

Outline of this talk

1. Informal overview of REWBLk and results. ✅
2. Formal semantics of REWBLk
3. Idea of the proof of **NL** $\subseteq$ REWBLk
4. Idea of the proof of **NL** $\supseteq$ REWBLk

Syntax and Semantics of REWBLκ

Syntax and Semantics of REWBLk

① *Syntax.* Expressions are inductively defined via the following grammar:

$$\begin{array}{llll} E & ::= & \epsilon \mid \sigma \mid E + E \mid E\,E \mid E^* & \text{(REGEX part)} \\ & \mid & (E)_x \mid \backslash x & \text{(Backreferences part)} \\ & \mid & ?(E) \mid !(E) & \text{(Lookaheads part)} \end{array}$$

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② ***Configurations*** $\langle \underset{\textit{index}}{\underline{p}} , \underset{\textit{variables assign.}}{\underline{\Lambda}} \rangle$. Pairs of an index and an assign.:

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$$\begin{array}{lcccccccc} & & & & & \Lambda & & & \\ & & & & & \triangledown & & & \\ \text{input } w & : & a & b & \ldots & a & \ldots & a & \\ \text{index} & : & 0 & 1 & & p & & |w|-1 & |w| \end{array}$$

Syntax and Semantics of REWBLk

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$$
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$$
\begin{array}{lccccccc}
 & & & & & \overset{\Lambda}{\triangledown} & & \\
\text{input } w & : & a & b & \ldots & a & \ldots & a & \\
\text{index} & : & 0 & 1 & & p & & |w|-1 & |w|
\end{array}
$$

For a variable x, $\Lambda(x)$ means the stored string in x.

Semantics Function & Language

③ ***Semantics function.*** Let w be an input string.

$$[\![E,\ \langle\ \underset{\text{index}}{\underline{p}}\ ,\ \underset{\text{variables assign.}}{\underline{\Lambda}}\ \rangle]\!]_w = \{\ \langle q_1, \Lambda_1\rangle,\ \langle q_2, \Lambda_2\rangle,\ \ldots\ \}.$$

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It means that:

1. From the current configuration $\langle p, \Lambda\rangle$, we execute E;

$$\begin{array}{lcl}
 & & \qquad\qquad\qquad\ \Lambda \\
 & & \qquad\qquad\qquad\ \triangledown \\
\text{input string } w & : & a\ b\ c\ b\ \dots\ a\ \dots\dots\dots\dots\ b\ \dots\ c\ \dots\ b\ a\ a \\
\text{index} & : & 0\ 1\ 2\ 3\qquad\ p
\end{array}$$

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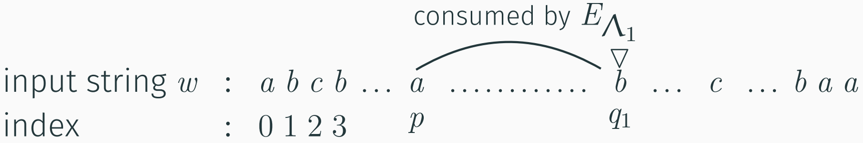
2. then obtain new configurations $\langle q_1, \Lambda_1\rangle$



consumed by E_{Λ_1}

input string w : $a\ b\ c\ b\ \dots\ a\ \dots\dots\dots\dots\ \overset{\triangledown}{b}\ \dots\ c\ \dots\ b\ a\ a$

index : $0\ 1\ 2\ 3$ p q_1

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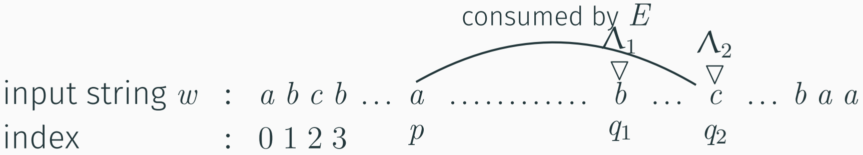
It means that:

1. From the current configuration $\langle p, \Lambda\rangle$, we execute E;
2. then obtain new configurations $\langle q_1, \Lambda_1\rangle$,$\langle q_2, \Lambda_2\rangle$, and so on.



consumed by E

Λ_1 $\triangledown$ Λ_2 $\triangledown$

input string w : $a\ b\ c\ b\ \dots\ a\ \dots\dots\dots\dots\ b\ \dots\ c\ \dots\ b\ a\ a$

index : $0\ 1\ 2\ 3\ \quad p \quad q_1 \quad q_2$

Semantics Function & Language

③ ***Semantics function.*** Let w be an input string.

$$[\![E, \langle \underbrace{p}_{\text{index}}, \underbrace{\Lambda}_{\text{variables assign.}} \rangle]\!]_w = \{\, \langle q_1, \Lambda_1 \rangle, \langle q_2, \Lambda_2 \rangle, \ldots \,\}.$$

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		Λ_ϵ $\triangledown$					
input string w	:	a	b	c	....	a	
index	:	0	1	2		$\lvert w\rvert - 1$	$\lvert w\rvert$

Semantics Function & Language

③ ***Semantics function.*** Let w be an input string.

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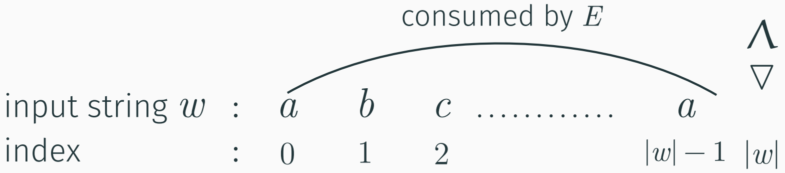

consumed by E

Λ

$\triangledown$

input string w : a b c .... a

index : 0 1 2 $|w|-1$ $|w|$

Semantics Function & Language

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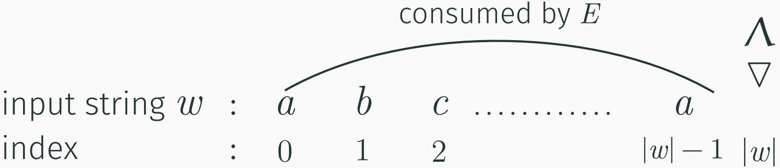

input string w	:	a	b	c	....	a	
index	:	0	1	2		$\lvert w\rvert - 1$	$\lvert w\rvert$

Def. The language of E: $[\![E]\!] = \{w : E$ accepts $w\}$.

Semantics of REWBLᴋ: REGEX part

Let w be an input string.

$$[\![\epsilon,\ \langle p, \Lambda\rangle]\!] \;=\; \{\,\langle p, \Lambda\rangle\,\},$$

$$[\![\sigma,\ \langle p, \Lambda\rangle]\!] \;=\; \begin{cases} \{\,\langle p+1, \Lambda\rangle\,\} & \text{if } w[p] \text{ is defined } \& \ w[p] = \sigma, \\ \emptyset & \text{otherwise} \end{cases}$$

Examples

Semantics of REWBLK: REGEX part

Let w be an input string.

$$\llbracket \epsilon, \langle p, \Lambda \rangle \rrbracket = \{ \langle p, \Lambda \rangle \},$$

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Semantics of REWBLk: REGEX part

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\end{aligned}$$

Examples

$$[\![a,\ a \quad b \quad b \quad \overset{\Lambda}{\overset{\triangledown}{a}}\ ]\!] = a \quad b \quad b \quad a \quad \overset{\Lambda}{\triangledown}$$

Semantics of REWBLK: REGEX part

Let w be an input string.

$$
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[\![\epsilon,\ \langle p, \Lambda\rangle]\!] &= \{\, \langle p, \Lambda\rangle \,\}, \\
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\end{aligned}
$$

Examples

$$[\![b,\ a \quad b \quad b \quad \overset{\Lambda}{\overset{\triangledown}{a}}]\!] = \emptyset$$

Semantics of REWBLK: REGEX part

$$[\![E_1 + E_2,\ \langle p, \Lambda\rangle]\!] \ = \ [\![E_1,\ \langle p, \Lambda\rangle]\!] \cup [\![E_2,\ \langle p, \Lambda\rangle]\!]$$

Semantics of REWBLᴋ: REGEX part

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Semantics of REWBLK: REGEX part

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[\![E_1 + E_2,\ \langle p, \Lambda\rangle]\!] &= [\![E_1,\ \langle p, \Lambda\rangle]\!] \cup [\![E_2,\ \langle p, \Lambda\rangle]\!] \\
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[\![E^*,\ \langle p, \Lambda\rangle]\!] &= \bigcup_{i=0}^{\infty} [\![E^i,\ \langle p, \Lambda\rangle]\!] \\
&=
\end{aligned}$$

Semantics of REWBLK: REGEX part

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&\cup\ \llbracket E\ E,\ \langle p, \Lambda \rangle \rrbracket \cup \llbracket E\ E\ E,\ \langle p, \Lambda \rangle \rrbracket \cup \cdots
\end{aligned}$$

Syntax and Semantics of Backreferences

$(E)_x$: We save a string consumed by applying E to the variable x.

$$[\![(E)_x,\ \langle p, \Lambda\rangle]\!] = \Big\{\langle\, p,\ \Lambda'[x := w[p..q)\,]\,\rangle\ :\ \langle q, \Lambda'\rangle \in [\![E,\ \langle p, \Lambda\rangle]\!]\Big\}$$

$\backslash x$: We use the stored string in x.

$$[\![\backslash x,\ \langle p, \Lambda\rangle]\!] = [\![\Lambda(x), \langle p, \Lambda\rangle]\!]$$

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Example.

$$((a+b)^*)_x \,\#\, \backslash x$$

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Example.

$$(\,(a + b)^*\,)_x \ \#\ \backslash x$$

$$\text{input}:\ \overset{\overset{\Lambda_\epsilon}{\triangledown}}{a}\quad b\quad a\quad a\quad \#\quad a\quad b\quad a\quad a$$

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$$((a+b)^*)_x \ \#\ \backslash x$$

$$x \mapsto abaa$$

input : $\overbrace{a\ \ b\ \ a\ \ a}^{\text{consumed}}\ \ \overset{\triangledown}{\#}\ \ a\ \ b\ \ a\ \ a$

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So, $((a+b)^*)_x \,\#\, \backslash x$ accepts a non-CFL $\{w \,\#\, w : w \in (a+b)^*\}$.

Syntax and Semantics of Lookaheads

Negative Lookaheads. We continue if the computation of E fails:

$$[\![!(E),\ \langle p, \Lambda\rangle]\!] = \begin{cases} \{\langle p,\ \Lambda\rangle\} & \text{if } [\![E,\ \langle p, \Lambda\rangle]\!] = \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

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Example: Implementing End-of-String checker \$

In REGEX libraries, the special symbol \$ checks if we are in the EOS pos.

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Proposition: Closed under complementation

Let E be an expression. $!(E\,\$)\Sigma^*$ accepts the complement of $[\![E]\!]$.

Syntax and Semantics of Lookaheads

Positive Lookaheads

$$[\![?(E),\ \langle p,\Lambda\rangle]\!] = \Big\{\langle p,\Lambda'\rangle : \langle q,\Lambda'\rangle \in [\![E,\ \langle p,\Lambda\rangle]\!]_w\Big\}.$$

Example

$$?\big((a^*)_x\big)$$

$$\begin{array}{llccccccccccc} & & & & & & & \Lambda & & & & & \\ & & & & & & & \triangledown & & & & & \\ \text{input string } w & : & a & b & c & b & \ldots & a & a & a & a & b & \ldots\ldots \\ \text{index} & : & 0 & 1 & 2 & 3 & & p & & & & & \end{array}$$

Syntax and Semantics of Lookaheads

Positive Lookaheads

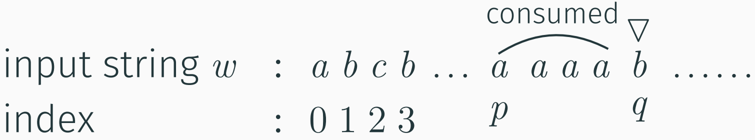
$$[\![?(E),\ \langle p,\Lambda\rangle]\!] = \left\{ \langle p,\Lambda'\rangle : \langle q,\Lambda'\rangle \in [\![E,\ \langle p,\Lambda\rangle]\!]_w \right\}.$$

1. We apply E from the current position;

Example

$$?\big({\color{red}(a^*)_x}\big)$$

$$x \mapsto aaaa$$



consumed $\triangledown$

input string w : $a\ b\ c\ b\ \ldots\ a\ a\ a\ a\ b\ \ldots\ldots$

index : $0\ 1\ 2\ 3$ p q

Syntax and Semantics of Lookaheads

Positive Lookaheads

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1. We apply E from the current position;
2. and go back to the invocation point with *inheriting* assignments.

Example

$$?\big((a^*)_x\big)$$

$$x \mapsto aaaa$$

input string w : $a\ b\ c\ b\ \ldots\ \overset{\triangledown}{a}\ a\ a\ a\ b\ \ldots\ldots$

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Proving「NL ⊆ REWBLκ」: Translating NL-machines to REWBLκ

Proving NL $\subseteq$ REWBLK

Our transformation consists of two steps:

M : nondeterministic log-space machine

↓

$\mathcal{A}$: two-way multihead finite automaton

↓

E : REWBLK

Proving NL ⊆ REWBLk

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This translation comes from the following result:

> *The language class of two-way multihead automata equals to* ***NL****.*

📘 *On Non-Determinancy in Simple Computing Devices*, J. Hartmanis, 1971.

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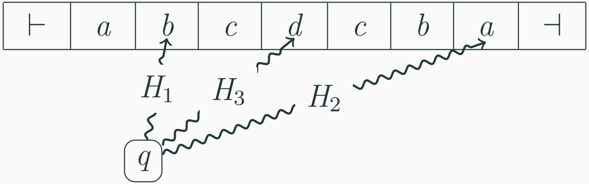
We only need to focus on this translation.

Simulating *heads* is easier than simulating log-space tapes.

E : REWBLk

Quick and informal review of two-way multihead finite automata

The following is a configuration of a **3-head** two-way finite automaton:



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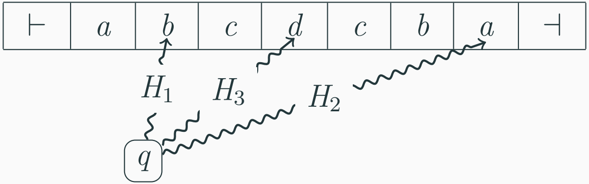

Generally, on every k-head two-way finite automaton,

- the input word is surrounded by end-markers $\vdash \cdots \dashv$.

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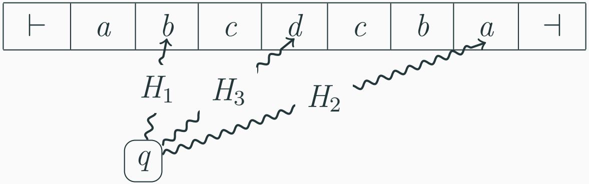

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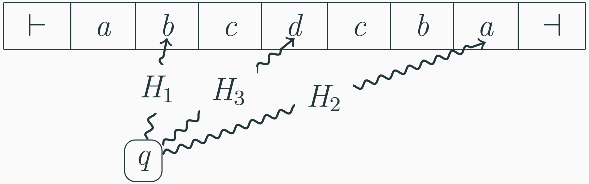

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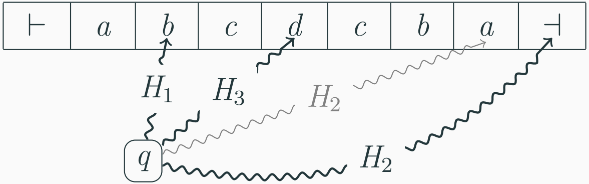

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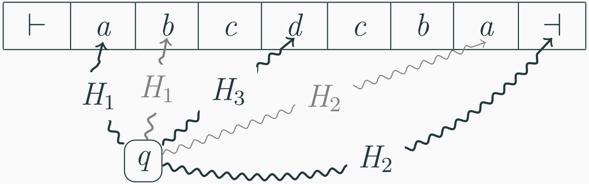

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- the input word is surrounded by end-markers $\vdash \cdots \dashv$.
- there are finitely many states;
- there are k-heads H_1, H_2, … H_k;
- as usual automata, each head can move right; and
- each head can also move left.

Proving NL $\subseteq$ REWBLK

As example, we try to accept the following typical CFL with REWBLK:

$$L_{\text{reverse}} = \left\{ w\#\, w^{R} : w \in \Sigma^{*} \right\}$$

w^R is the reverse of w.

For example, $a\,b\,c\#\,c\,b\,a \in L_{\text{reverse}}$.

Proving NL ⊆ REWBLk

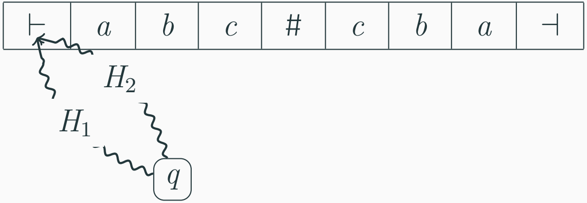
As example, we try to accept the following typical CFL with REWBLk:

$$L_{\text{reverse}} = \left\{ w\#\, w^R : w \in \Sigma^* \right\}$$

w^R is the reverse of w.

For example, $a\, b\, c\#\, c\, b\, a \in L_{\text{reverse}}$.

We use a 2-head two-way finite automaton as follows:



initial configuration

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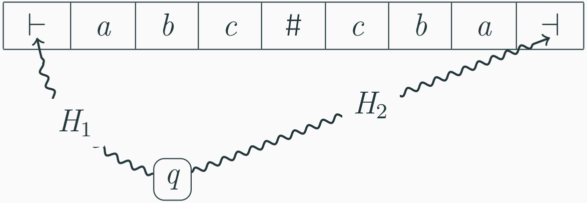

move H_2 to ⊣ while checking that # appears only once.

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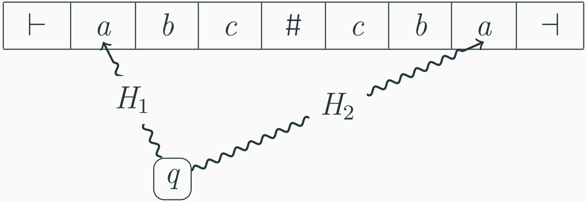

move heads toward the center while they read the same symbol.

Proving NL $\subseteq$ REWBLK

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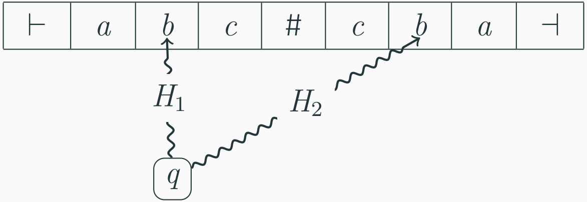

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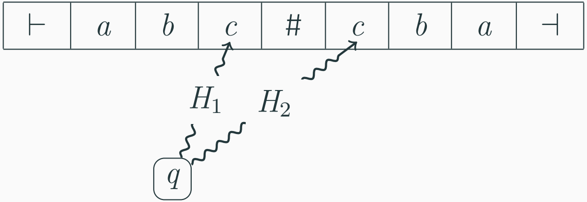

move heads toward the center while they read the same symbol.

Proving NL ⊆ REWBLk

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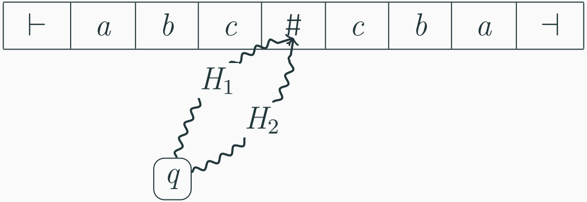


if the heads reach # at the same time, we accept the input.

Proving NL $\subseteq$ REWBLK

$$L_{\text{reverse}} = \left\{ w\#\, w^R : w \in \Sigma^* \right\}$$

w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

Proving NL $\subseteq$ REWBLk

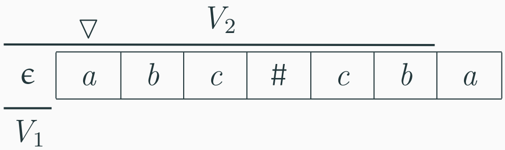
$$L_{\text{reverse}} = \{w\# w^R : w \in \Sigma^*\}$$

w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

① We initialize V_1 and V_2 by:

$$?((\epsilon)_{V_1}) \qquad ?(\,(\Sigma^*)_{V_2}\ \Sigma\ \$\,)$$



Proving NL $\subseteq$ REWBLk

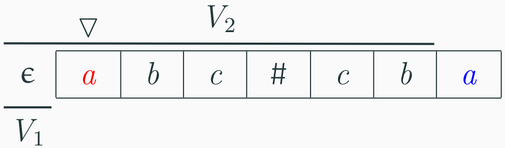
$$L_{\text{reverse}} = \left\{ w\#\, w^R : w \in \Sigma^* \right\}$$

w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

② Check V_1 and V_2 scan the same symbol by

$$\bigcup_{\sigma \in \Sigma} ?(\backslash V_1\ \sigma)\ ?(\backslash V_2\ \sigma)$$



Proving NL $\subseteq$ REWBLk

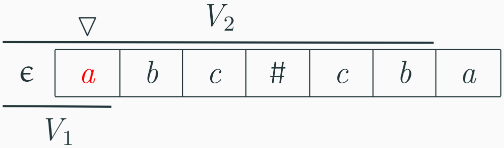
$$L_{\text{reverse}} = \{ w\# w^R : w \in \Sigma^* \}$$

w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

③ Change/Move V_1 and V_2. First, we expand V_1. It is easy:

$$?\big((\backslash V_1\ \Sigma)_{V_1}\big)$$



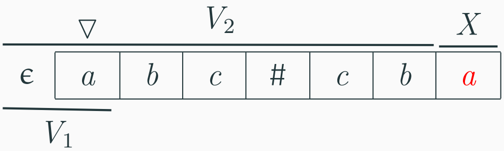
Proving NL ⊆ REWBLk

$$L_{\text{reverse}} = \{ w \# w^R : w \in \Sigma^* \}$$ w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

④ We shrink V_2 in two steps. First, find the complement string X:

$$?(\backslash V_2\ (\Sigma^*)_X\ \$)$$



Proving NL $\subseteq$ REWBLk

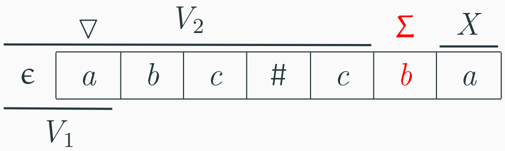
$$L_{\text{reverse}} = \left\{ w\#\, w^R : w \in \Sigma^* \right\}$$

w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

④ Using the complement string X, we shrink V_2:

$$?\big((\Sigma^*)_{V_2}\ (\Sigma \setminus X)\ \$\big)$$



Proving NL ⊆ REWBLκ

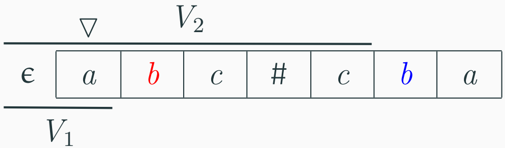
$$L_{\text{reverse}} = \left\{ w \# w^R : w \in \Sigma^* \right\}$$

w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

⑤ We repeat the same. Check both the variables scan the same symbol:

$$\bigcup_{\sigma \in \Sigma} ?(\backslash V_1 \sigma)\ ?(\backslash V_2 \sigma)$$



ϵ	a	b	c	#	c	b	a

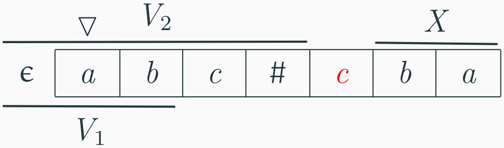
Proving NL $\subseteq$ REWBLk

$L_{\text{reverse}} = \{ w \# w^R : w \in \Sigma^* \}$ w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

⑥ We again expand V_1 and shrink V_2 by

$$?((\backslash V_1\ \Sigma)_{V_1})\ \ ?(\backslash V_2\ (\Sigma^*)_X\ \$)\ \ ?((\Sigma^*)_{V_2}\ (\Sigma\ \backslash X)\ \$)$$



Proving NL $\subseteq$ REWBLk

$$L_{\text{reverse}} = \left\{ w\#\, w^R : w \in \Sigma^* \right\}$$

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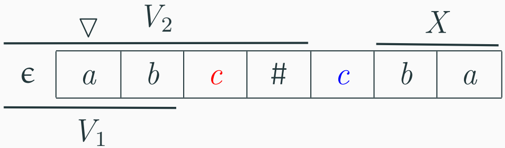

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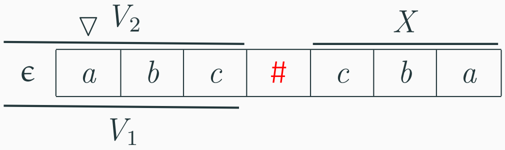
$$?\big((\backslash V_1\ \Sigma)_{V_1}\big)\ \ ?\big(\backslash V_2\ (\Sigma^*)_X\ \$\big)\ \ ?\big((\Sigma^*)_{V_2}\ (\Sigma\ \backslash X)\ \$\big)$$



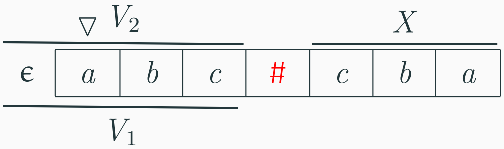
Proving NL $\subseteq$ REWBLκ

$$L_{\text{reverse}} = \{w\# w^R : w \in \Sigma^*\}$$ w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

⑧ We again expand V_1 and shrink V_2.

$$?((\backslash V_1\ \Sigma)_{V_1})\ \ ?(\backslash V_2\ (\Sigma^*)_X\ \$)\ \ ?((\Sigma^*)_{V_2}\ (\Sigma\ \backslash X)\ \$)$$



ϵ	a	b	c	#	c	b	a

On the basis of this simulation idea, we have「**Thm. NL** $\subseteq$ REWBLκ.」

Proving REWBLᴋ $\subseteq$ NL: Translating REWBLᴋ to NL machines

Proving REWBLk ⊆ NL

Basically, we generalize $\underbrace{\text{the classical regex-to-automaton translation}}_{\text{McNaughton–Yamada–Thompson construction}}$.

Proving REWBLk $\subseteq$ NL

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For example, from the following expression,

$$E_{\mathsf{STconn}} = (V^*)_C\ ?(\#)\ \Big(?\big(\Sigma^*\#\backslash C \to (\Sigma^*)_C\#\big)\Big)^*\ \Sigma^*\ \#\ \backslash C$$

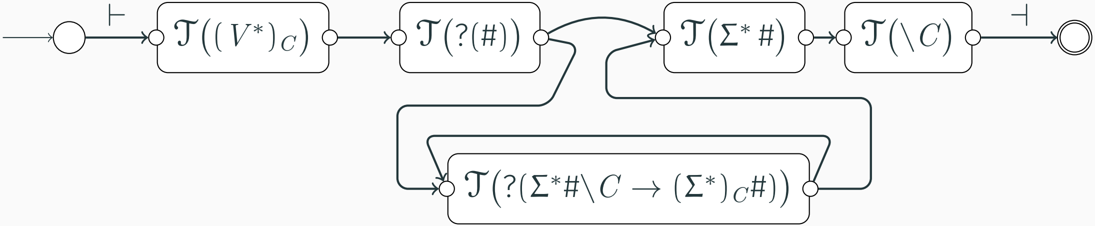
Proving REWBLк ⊆ NL

Basically, we generalize $\underbrace{\text{the classical regex-to-automaton translation}}_{\text{McNaughton–Yamada–Thompson construction}}$.

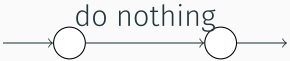
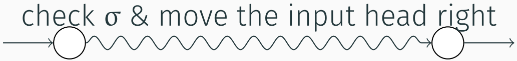
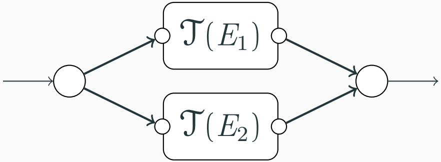
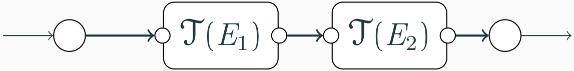
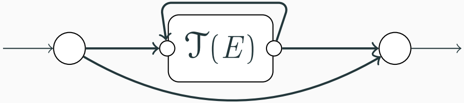
For example, from the following expression,

$$E_{\mathsf{STconn}} = (\,V^*)_C\;?(\#)\;\Big(?(\Sigma^*\#\backslash C \to (\Sigma^*)_C\#)\Big)^*\;\Sigma^*\;\#\;\backslash C$$

we obtain the following nlog-space machine $\mathcal{T}(E_{\mathsf{STconn}})$:
(note: inputs of machines are surrounded by ⊢ and ⊣)

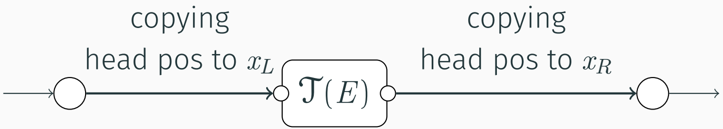

⊢
$\mathcal{T}((\,V^*)_C)$
$\mathcal{T}(?(\#))$
$\mathcal{T}(\Sigma^*\#)$
$\mathcal{T}(\backslash C)$
⊣
$\mathcal{T}(?(\Sigma^*\#\backslash C \to (\Sigma^*)_C\#))$

Translation rules for REGEX part

REWBLK	:	nlog-space Turing machines
$\mathcal{T}(\epsilon)$	$=$	
$\mathcal{T}(\sigma)$	$=$	
$\mathcal{T}(E_1 + E_2)$	$=$	
$\mathcal{T}(E_1\ E_2)$	$\Mapsto$	
$(E)^*$	$\Mapsto$	

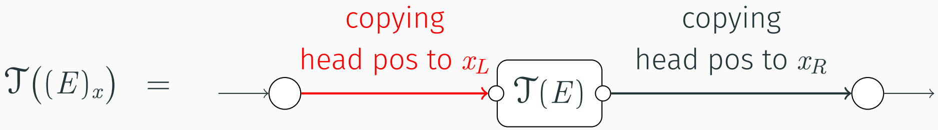
Translation rules for $(E)_x$ and $\backslash x$

$\mathcal{T}((E)_x) \;=$



$$\begin{array}{lcl} & & \qquad\qquad\qquad\quad \triangledown \\ \text{input tape} & : & \vdash\ a\ b\ c\ b\ \ldots\ a\ \ldots\ldots\ldots\ldots\ b\ \ldots\ c\ b\ a\ a\ \dashv \\ \text{pos} & : & \qquad\qquad\qquad\quad N \\ \text{working tape } x_L & : & \\ \text{working tape } x_R & : & \end{array}$$

Translation rules for $(E)_x$ and $\backslash x$



$\mathcal{T}((E)_x) =$

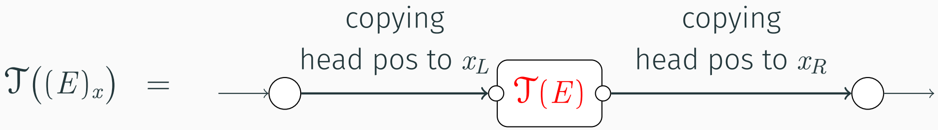
copying head pos to x_L

$\mathcal{T}(E)$

copying head pos to x_R

input tape	:	$\vdash\ a\ b\ c\ b\ \dots\ \overset{\triangledown}{a}\ \dots\dots\dots\dots\ b\ \dots\ c\ b\ a\ a\ \dashv$
pos	:	N
working tape x_L	:	binary encoding of N
working tape x_R	:	

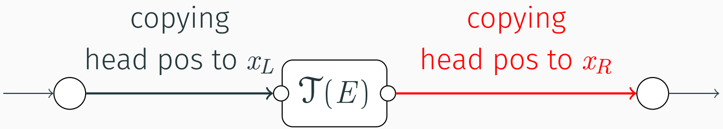
Translation rules for $(E)_x$ and $\backslash x$



		consumed by E ▽
input tape	:	$\vdash\ a\ b\ c\ b\ \ldots\ a\ \overbrace{\ldots\ldots\ldots\ldots}\ b\ \ldots\ c\ b\ a\ a\ \dashv$
pos	:	N M
working tape x_L	:	binary encoding of N
working tape x_R	:	

Translation rules for $(E)_x$ and $\backslash x$

$\mathfrak{T}((E)_x) \quad =$


copying head pos to x_L

$\mathfrak{T}(E)$

copying head pos to x_R

consumed by E

input tape : $\vdash \; a \; b \; c \; b \; \ldots \; a \; \overbrace{\ldots\ldots\ldots\ldots}^{} \; \overset{\triangledown}{b} \; \ldots \; c \; b \; a \; a \; \dashv$

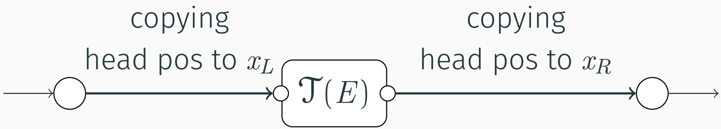
pos : N M

working tape x_L : binary encoding of N

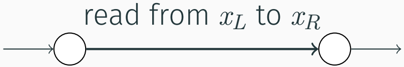
working tape x_R : binary encoding of M

Translation rules for $(E)_x$ and $\backslash x$

$\mathcal{T}((E)_x) \quad =$



$\mathcal{T}(\backslash x) \quad =$



consumed by E

input tape : $\vdash\ a\ b\ c\ b\ \ldots\ a\ \overbrace{\ldots\ldots\ldots\ldots}\ \overset{\triangledown}{b}\ \ldots\ c\ b\ a\ a\ \dashv$

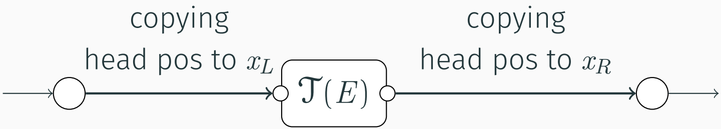
pos : N M

working tape x_L : binary encoding of N

working tape x_R : binary encoding of M

Translation rules for $(E)_x$ and $\backslash x$

$\mathcal{T}((E)_x) \quad =$

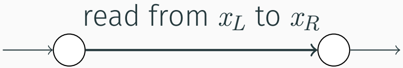


copying head pos to x_L

$\mathcal{T}(E)$

copying head pos to x_R

$\mathcal{T}(\backslash x) \quad =$



read from x_L to x_R

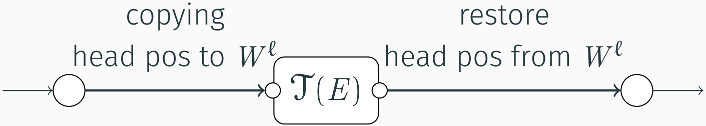
consumed by E

input tape	:	$\vdash\ a\ b\ c\ b\ \ldots\ a\ \overparen{\ldots\ldots\ldots\ldots}\ \overset{\nabla}{b}\ \ldots\ c\ b\ a\ a\ \dashv$
pos	:	$N \qquad M$
working tape x_L	:	binary encoding of N
working tape x_R	:	binary encoding of M

⚠ We need to normalize and remove patterns like $(\cdots x \cdots)_x$, $(\cdots(\cdots)_x \cdots)_x$.

Translation rules for $?(E)$ and $!(E)$

$\mathcal{T}(?(E)^{\ell}) \quad =$



copying

head pos to W^{ℓ}

$\mathcal{T}(E)$

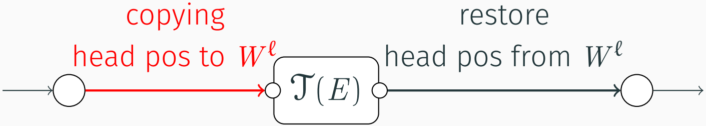
restore

head pos from W^{ℓ}

$$\begin{array}{lcccc}
 & & & \triangledown & \\
\text{input tape} & : & \vdash \ a\ b\ c\ b \ \dots & a & \dots\dots\dots\dots \ b \ \dots \ c\ b\ a\ a \ \dashv \\
\text{pos} & : & & N & \\
\text{working tape } W^{\ell} & : & & &
\end{array}$$

Translation rules for $?(E)$ and $!(E)$

$\mathcal{T}(?(E)^\ell) \quad =$



copying

head pos to W^ℓ

$\mathcal{T}(E)$

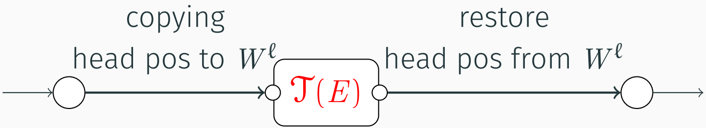
restore

head pos from W^ℓ

input tape	:	$\vdash \ a\ b\ c\ b\ \ldots\ \overset{\triangledown}{a}\ \ldots\ldots\ldots\ldots\ \ b\ \ldots\ c\ b\ a\ a\ \dashv$
pos	:	N
working tape W^ℓ	:	binary encoding of N

Translation rules for $?(E)$ and $!(E)$

$\mathcal{T}(?(E)^{\ell}) \quad =$

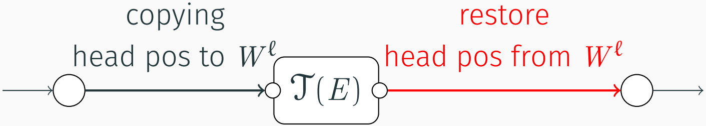


consumed by E $\triangledown$

input tape	:	$\vdash \; a \; b \; c \; b \; \ldots \; a \; \ldots\ldots\ldots\ldots \; b \; \ldots \; c \; b \; a \; a \; \dashv$
pos	:	N $\qquad\qquad$ M
working tape W^{ℓ}	:	binary encoding of N

Translation rules for $?(E)$ and $!(E)$

$\mathcal{T}(?(E)^\ell) \quad =$



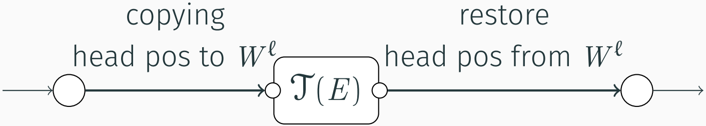
copying head pos to W^ℓ

restore head pos from W^ℓ

input tape	:	$\vdash \; a \; b \; c \; b \; \ldots \; \overset{\triangledown}{a} \; \ldots\ldots\ldots\ldots \; b \; \ldots \; c \; b \; a \; a \; \dashv$
pos	:	N (under a) $\qquad M$ (under b)
working tape W^ℓ	:	binary encoding of N

Translation rules for $?(E)$ and $!(E)$

$\mathcal{T}(?(E)^{\ell}) \quad =$

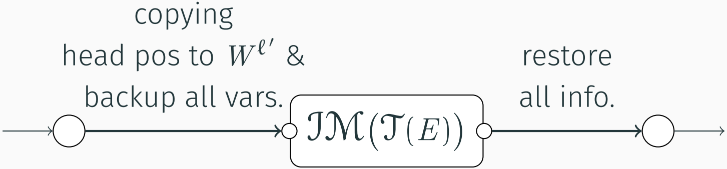


copying head pos to W^{ℓ}

$\mathcal{T}(E)$

restore head pos from W^{ℓ}

$\mathcal{T}(!(E)^{\ell'}) \quad =$



copying head pos to $W^{\ell'}$ & backup all vars.

$\mathcal{JM}(\mathcal{T}(E))$

restore all info.

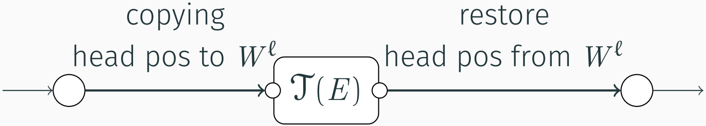
input tape : $\vdash \; a \; b \; c \; b \; \ldots \; \overset{\triangledown}{a} \; \ldots\ldots\ldots\ldots \; b \; \ldots \; c \; b \; a \; a \; \dashv$

pos : N M

working tape W^{ℓ} : binary encoding of N

Translation rules for $?(E)$ and $!(E)$

$$\mathcal{T}(?(E)^{\ell}) \quad =$$

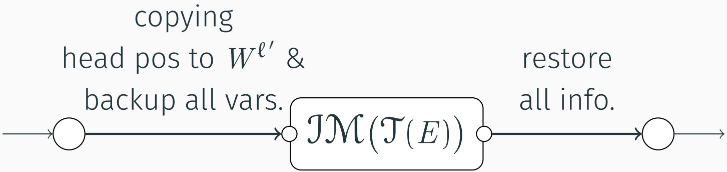


copying head pos to W^{ℓ}

$\mathcal{T}(E)$

restore head pos from W^{ℓ}

$$\mathcal{T}(!(E)^{\ell'}) \quad =$$



copying head pos to $W^{\ell'}$ & backup all vars.

$\mathcal{IM}(\mathcal{T}(E))$

restore all info.

input tape	:	$\vdash\ a\ b\ c\ b\ \ldots\ \overset{\triangledown}{a}\ \ldots\ldots\ldots\ldots\ \ b\ \ldots\ c\ b\ a\ a\ \dashv$
pos	:	N $\qquad\qquad$ M
working tape W^{ℓ}	:	binary encoding of N

$\mathcal{IM}(M)$ is the complement version of M computed by Immerman's construction. Please recall **NL** $=$ **co-NL** by Immerman–Szelepcsényi theorem.

Conclusion

	Language class	Comp. of Membership Problem
REGEX	$=$ Dspace($O(1)$)	**NL**-complete (1)
REGEX + *Backreferences*	$\subseteq$ **NL** [1] $\subseteq$ **INDEX** [2]	**NP**-complete (2)
REGEX + *Lookaheads*	$=$ REGEX [3]	**P**-complete (3)
REWBLk	$=$ **NL** NEW	**PSPACE**-complete NEW

[1] *Inside the class of regex languages*, M.L. Schmid, 2013

[2] *On the expressive power of regular expressions with backreferences*, T.Nogami & T. Terauchi, 2023

[3] *Alternation*, A. Chandra, D. Kozen, & L. Stockmeyer, 1981

(1) is equivalent to the st-connectivity (directed-graph connectivity) problem

(2) *Algorithms for finding patterns in strings*, A. Aho, 1990

(3) *A note on the space complexity of some decision problems for finite automata*, T. Jiang & B. Ravikumar, 1991

Bonus Slides

Normalizing expressions

Let's consider an expression:

$$(\backslash x\ (a\ \backslash x)_x\ \backslash x)_x,$$

which violates conditions $(\cdots \backslash x \cdots)_x$ and $(\cdots (\cdots)_x \cdots)_x$.

We can normalize it as follows:

$$\begin{array}{rl} & (\backslash x\ (a\ \backslash x)_x\ \backslash x)_x \\ \Rrightarrow & (\backslash x\ (a\ \backslash x)_x\ \backslash x)_y\ ?((\backslash y)_x) \\ \Rrightarrow & (\backslash x\ (a\ \backslash x)_z\ ?((\backslash z)_x)\ \backslash x)_y\ ?((\backslash y)_x) \end{array}$$

From outside to inside, we replace labels $(\cdots)_x$ with a frash one $(\cdots)_\alpha$ and then add $?((\backslash \alpha)_x)$ for copying the content:

$$(\cdots)_x \Rrightarrow (\cdots)_\alpha ?((\backslash \alpha)_x)$$