Regular Expressions with Backreferences and Lookaheads Capture NLOG

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Introduction: Modern Regular Expressions

Classical Regular Expressions (REGEX)

Example. $$\begin{aligned} [\![aa\ a^*]\!] &= [\![aa(\epsilon + a + aa + aaa + \cdots)]\!] \\ &= \{a^n : n \geqslant 2\} \end{aligned}$$

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Modern REGEX = REGEX with Backreferences

$$(\ aa\,a^*\ )_x\quad \backslash x\quad \backslash x\ ^*$$

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$$\boxed{(}\,aa\,a^*\boxed{)_x}\ \boxed{\backslash x}\ \boxed{\backslash x}^*$$

$\boxed{(}\ E\ \boxed{)_x}$: **Capturing** stores a string matched with E to a variable x

$\boxed{\backslash x}$: **Referring** reads the string stored in x

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$$\overbrace{aa}^{x}\,(aa)\,(aa)^* \;(= \{a^{2\cdot j} : j \geqslant 2\})$$

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$$
\begin{aligned}
& \ \overset{x}{\overline{aa}}\,(aa)\,(aa)^*\ (= \{a^{2\cdot j} : j \geqslant 2\}) \\
(aa\,a^*)_x \setminus x \setminus x^* = & +\ \overset{x}{\overline{aaa}}\,(aaa)\,(aaa)^*\ (= \{a^{3\cdot j} : j \geqslant 2\})
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$(aa\,a^*)_x \setminus x \setminus x^*$ means $\{a^n : n \text{ is composite}\}$.

$$n \text{ is composite} \iff \exists i, j \geqslant 2.\ n = i \cdot j$$

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So, $[\![\, (aaa^*)_x \setminus x \setminus x^* \,]\!] = \{a^n : n \text{ is composite}\}$.

Can we represent the prime numbers $\left\{a^2, a^3, a^5, a^7, \ldots\right\}$??

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Mode… Lookaheads

Positive Lookahead

$?(\ E\ )$: if the input matches with E, then we continue.

$$?(\ aa\ a^*\ \$\ )\ \ !(\ \underbrace{(aa\ a^*)_x\ \backslash x\ \backslash x^*}_{\text{composite checker}}\ \$\ )\ a^*$$

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Example. $[\![aa\ a^*]\!] = [\![aa(\epsilon + a + aa + aaa + \cdots)]\!]$

$\{a^n : n \geq 2\}$

Mode

Negative Lookahead

$!(\ E\ )$: if the input **never** matches with E, then we continue.

Posi

$?(\ E\ )$: if the input matches with E, then we continue.

$x\ ^*$

$$?(\ aa\ a^*\ \$\ )\ !(\ \underbrace{(aa\ a^*)_x\ \backslash x\ \backslash x^*}_{\text{composite checker}}\ \$\ )\ a^*$$

Introduction

Coding and experimenting in Python

```python
import re

prime = r'(?!((?P<X>(aaa*))(?P=X)(?P=X)*$))aa(a*)'

for i in range(1, 50):
    w = 'a' * i
    result = re.fullmatch(prime, w)
    print(i, result)
```

slide — -bash ▸ Emacs-arm64-11 — 110×28

```
CA-20023547:slide s22809$ python prime_demo.py
1 None
2 <re.Match object; span=(0, 2), match='aa'>
3 <re.Match object; span=(0, 3), match='aaa'>
4 None
5 <re.Match object; span=(0, 5), match='aaaaa'>
6 None
7 <re.Match object; span=(0, 7), match='aaaaaaa'>
8 None
9 None
10 None
11 <re.Match object; span=(0, 11), match='aaaaaaaaaaa'>
12 None
13 <re.Match object; span=(0, 13), match='aaaaaaaaaaaaa'>
14 None
15 None
16 None
17 <re.Match object; span=(0, 17), match='aaaaaaaaaaaaaaaaa'>
18 None
19 <re.Match object; span=(0, 19), match='aaaaaaaaaaaaaaaaaaa'>
20 None
21 None
```

Main Results on REWBLk

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1. The language class of REWBLK $=$ **NL** (nondeterministic log-space).
   We have the following translation:

$$E : \text{REWBLK} \iff M : \text{nondeterministic log-space TM } s.t.\ [\![E]\!] = L(M)$$

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2. The complexity of REWBLk-membership problem is **PSPACE**-complete.
   - **Input**: a REWBLk expression E & a string w
   - **Output**: True if E accepts w. False otherwise.

Main Results on REWBLk

Question: *How expressive are modern REGEXs?*

An

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In this talk,
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Outline of this talk

1. Informal overview of REWBLk and results. ✅
2. Formal semantics of REWBLk
3. Idea of the proof of $\mathbf{NL} \subseteq$ REWBLk
4. Idea of the proof of $\mathbf{NL} \supseteq$ REWBLk

Syntax and Semantics of REWBLk

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① *Syntax.* Expressions are inductively defined via the following grammar:

$$
\begin{array}{rcll}
E & ::= & \epsilon \mid \sigma \mid E + E \mid E\,E \mid E^* & \text{(REGEX part)} \\
& \mid & (E)_x \mid \backslash x & \text{(Backreferences part)} \\
& \mid & ?(E) \mid !(E) & \text{(Lookaheads part)}
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② *Configurations* $\langle \underbrace{p}_{\textit{index}}, \underbrace{\Lambda}_{\textit{variables assign.}} \rangle$. Pairs of an index and an assign.:

$$\begin{array}{lcccccccc} & & & & & \Lambda & & & \\ & & & & & \triangledown & & & \\ \text{input } w & : & a & b & \ldots & a & \ldots & a & \\ \text{index} & : & 0 & 1 & & p & & |w|-1 & |w| \end{array}$$

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 & & & & \triangledown & & & \\
\text{input } w & : & a & b & \dots\ a & \dots & a & \\
\text{index} & : & 0 & 1 & p & & |w|-1 & |w|
\end{array}
$$

For a variable x, $\Lambda(x)$ means the stored string in x.

Semantic Function & Language

③ ***Semantics function.*** Let w be an input string.

$$[\![E, \langle \underbrace{p}_{\text{index}}, \underbrace{\Lambda}_{\text{variables assign.}} \rangle]\!]_w = \{ \langle q_1, \Lambda_1 \rangle, \langle q_2, \Lambda_2 \rangle, \ldots \}.$$

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It means that:

1. From the current configuration $\langle p, \Lambda \rangle$, we execute E;

$$\begin{array}{lcccccccccccccc} & & & & & & \Lambda & & & & & & & & \\ & & & & & & \triangledown & & & & & & & & \\ \text{input string } w & : & a & b & c & b \ldots & a & \ldots\ldots\ldots\ldots & b & \ldots & c & \ldots & b & a & a \\ \text{index} & : & 0 & 1 & 2 & 3 & p & & & & & & & & \end{array}$$

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It means that:

1. From the current configuration $\langle p, \Lambda \rangle$, we execute E;
2. then obtain new configurations $\langle q_1, \Lambda_1 \rangle$

$$\begin{array}{lccccccccccccccc}
 & & & & & & & & \text{consumed by } E_{\Lambda_1} & & & & & & & \\
 & & & & & & & & & \triangledown & & & & & & \\
\text{input string } w & : & a & b & c & b & \ldots & a & \ldots\ldots\ldots & b & \ldots & c & \ldots & b & a & a \\
\text{index} & : & 0 & 1 & 2 & 3 & & p & & q_1 & & & & & &
\end{array}$$

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It means that:

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consumed by E

$$\begin{array}{lcllllllll}
 & & & & & & \overset{\Lambda_1}{\triangledown} & & \overset{\Lambda_2}{\triangledown} & \\
\text{input string } w & : & a\ b\ c\ b & \dots & a & \dots\dots\dots\dots & b & \dots & c & \dots\ b\ a\ a \\
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④ *Acceptance.* E **accepts** w if w can be perfectly consumed by E.

$$\exists \langle p, \Lambda \rangle \in [\![E, \langle 0, \Lambda_\epsilon \rangle]\!]_w.\ p = |w| \qquad \forall x.\, \Lambda_\epsilon(x) = \epsilon$$

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		Λ_ϵ $\triangledown$					
input string w	:	a	b	c	…………	a	
index	:	0	1	2		$\lvert w\rvert - 1$	$\lvert w\rvert$

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Λ

$\triangledown$

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Def. The language of E: $[\![E]\!] = \{w : E \text{ accepts } w\}$.

Semantics of REWBLk: REGEX part

Let w be an input string.

$$[\![\epsilon, \langle p, \Lambda \rangle]\!] = \{ \langle p, \Lambda \rangle \},$$

$$[\![\sigma, \langle p, \Lambda \rangle]\!] = \begin{cases} \{ \langle p+1, \Lambda \rangle \} & \text{if } w[p] = \sigma \text{ and } p+1 \leqslant |w| \\ \emptyset & \text{otherwise} \end{cases}$$

Examples

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\end{aligned}
$$

Examples

$$
[\![a,\ a \quad b \quad b \quad \overset{\Lambda}{\overset{\triangledown}{a}}\ ]\!] = a \quad b \quad b \quad a \quad \overset{\Lambda}{\triangledown}
$$

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Examples

$$\llbracket b, \; a \quad b \quad b \quad \overset{\Lambda}{\overset{\triangledown}{a}} \; \rrbracket = \emptyset$$

Semantics of REWBLκ: REGEX part

$$[\![E_1 + E_2,\ \langle p, \Lambda\rangle]\!] \ = \ [\![E_1,\ \langle p, \Lambda\rangle]\!] \cup [\![E_2,\ \langle p, \Lambda\rangle]\!]$$

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$$[\![E_1\ E_2,\ \ \langle p, \Lambda\rangle]\!] \ = \ \bigcup_{\langle q, \Lambda'\rangle \in [\![E_1, \langle p, \Lambda\rangle]\!]} [\![E_2,\ \langle q, \Lambda'\rangle]\!]$$

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[\![E^*,\ \langle p, \Lambda\rangle]\!] &= \bigcup_{i=0}^{\infty} [\![E^i,\ \langle p, \Lambda\rangle]\!] \\
&=
\end{aligned}
$$

Semantics of REWBLK: REGEX part

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[\![E^*,\ \langle p, \Lambda\rangle]\!] &= \bigcup_{i=0}^{\infty} [\![E^i,\ \langle p, \Lambda\rangle]\!] \\
&= [\![\epsilon,\ \langle p, \Lambda\rangle]\!] \cup [\![E,\ \langle p, \Lambda\rangle]\!] \\
&\cup\ [\![E\ E,\ \langle p, \Lambda\rangle]\!] \cup [\![E\ E\ E,\ \langle p, \Lambda\rangle]\!] \cup \cdots
\end{aligned}
$$

Syntax and Semantics of Backreferences

$(E)_x$: We save a string consumed by applying E to the variable x.

$$[\![(E)_x,\ \langle p, \Lambda\rangle]\!] = \Big\{\langle\, p,\ \Lambda'[x := w[p..q)\,]\,\rangle\ :\ \langle q, \Lambda'\rangle \in [\![E,\ \langle p, \Lambda\rangle]\!]\Big\}$$

$\backslash x$: We use the stored string in x.

$$[\![\backslash x,\ \langle p, \Lambda\rangle]\!] = [\![\Lambda(x), \langle p, \Lambda\rangle]\!]$$

Example.

$$((a+b)^*)_x \,\#\, \backslash x$$

$$\begin{array}{l}\quad\ \Lambda_\epsilon \\ \quad\ \ \triangledown \\ \text{input}: \ a\ \ b\ \ a\ \ a\ \ \#\ \ a\ \ b\ \ a\ \ a\end{array}$$

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Example.

$$((a+b)^*)_x \,\#\, \backslash x$$

$$x \mapsto abaa$$

$$\text{input}: \quad \overbrace{a \quad b \quad a \quad a}^{\text{consumed}} \quad \overset{\triangledown}{\#} \quad a \quad b \quad a \quad a$$

Syntax and Semantics of Backreferences

$(E)_x$: We save a string consumed by applying E to the variable x.

$$[\![(E)_x,\ \langle p, \Lambda\rangle]\!] = \left\{\langle\, p,\ \Lambda'[x := w[p..q)\,]\,\rangle \ :\ \langle q, \Lambda'\rangle \in [\![E,\ \langle p, \Lambda\rangle]\!]\right\}$$

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So, $((a+b)^*)_x \,\#\, \backslash x$ accepts a non-CFL $\{w \,\#\, w : w \in (a+b)^*\}$.

Syntax and Semantics of Lookaheads

Negative Lookaheads. We continue if the computation of E fails:

$$[\![!(E),\ \langle p, \Lambda\rangle]\!] = \begin{cases} \langle p,\ \Lambda\rangle & \text{if } [\![E,\ \langle p, \Lambda\rangle]\!] = \emptyset, \\ \emptyset & \text{otherwise} \end{cases}$$

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Proposition: Closed under complementation

Let E be an expression. $!(E\,\$)\Sigma^*$ accepts the complement of $[\![E]\!]$.

Syntax and Semantics of Lookaheads

Positive Lookaheads

$$[\![?(E),\ \langle p,\Lambda\rangle]\!] = \left\{\langle p,\Lambda'\rangle : \langle q,\Lambda'\rangle \in [\![E,\ \langle p,\Lambda\rangle]\!]_w\right\}.$$

Example

$$?\big((a^*)_x\big)$$

$$\begin{array}{lcl} & & \qquad\qquad\quad\ \ \Lambda \\ & & \qquad\qquad\quad\ \ \triangledown \\ \text{input string } w & : & a\ b\ c\ b\ \ldots\ a\ \ a\ a\ a\ \ b\ \ \ldots\ldots \\ \text{index} & : & 0\ 1\ 2\ 3\ \ \ \ \ \ \ \ p \end{array}$$

Syntax and Semantics of Lookaheads

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1. We apply E from the current position;

Example

$$?\big((a^*)_x\big)$$

$$x \mapsto aaaa$$

consumed

$$\begin{array}{lllllll} \text{input string } w & : & a\ b\ c\ b & \ldots & \overbrace{a\ \ a\ a\ a}^{} & \overset{\triangledown}{b} & \ldots\ldots \\ \text{index} & : & 0\ 1\ 2\ 3 & & p & q & \end{array}$$

Syntax and Semantics of Lookaheads

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1. We apply E from the current position;
2. and go back to the invocation point with *inheriting* assignments.

Example

$$?\big((a^*)_x\big)$$

$$x \mapsto aaaa$$

input string w	:	$a\ b\ c\ b\ \ldots\ \overset{\triangledown}{a}\ \ a\ a\ a\ \ b\ \ \ldots\ldots$
index	:	$0\ 1\ 2\ 3\qquad\quad p$

Proving「NL $\subseteq$ REWBLᴋ」: Translating NL-machines to REWBLᴋ

Proving NL $\subseteq$ REWBLK

Our transformation consists of two steps:

M : nondeterministic log-space machine

$\downarrow$

$\mathcal{A}$: two-way multihead finite automaton

$\downarrow$

E : REWBLK

Proving NL $\subseteq$ REWBLK

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This translation comes from the following result:

*The language class of two-way multihead automata equals to **NL**.*

📘 *On Non-Determinancy in Simple Computing Devices*, J. Hartmanis, 1971.

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We only need to focus on this translation.

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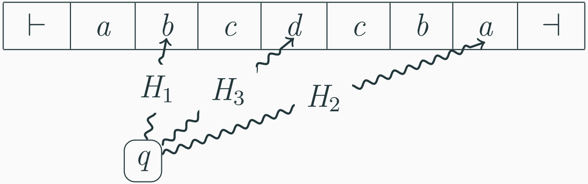
We only need to focus on this translation.

Simulating *heads* is easier than simulating log-space tapes.

E : REWBLk

Quick and informal review of two-way multihead finite automata

The following is a configuration of a **3-head** two-way finite automaton:



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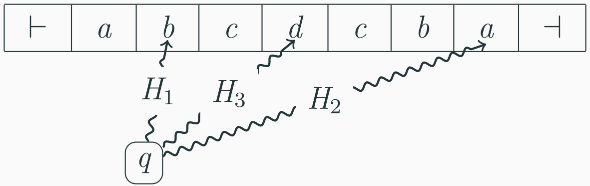

Generally, on every k-head two-way finite automaton,

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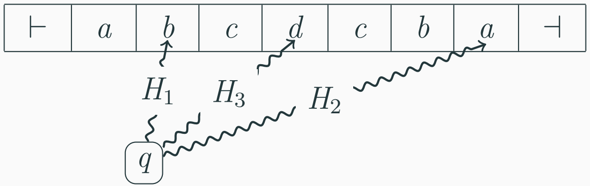

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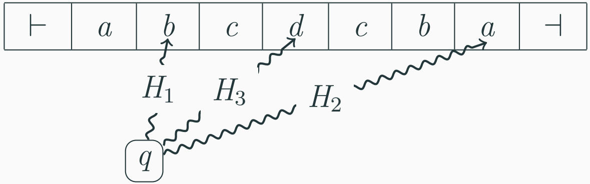

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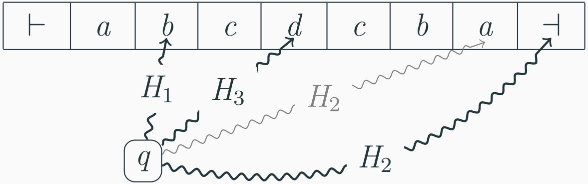

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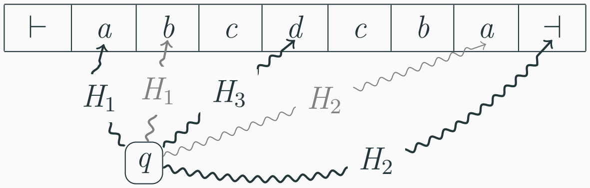

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- there are finitely many states;
- there are k-heads H_1, H_2, … H_k;
- as usual automata, each head can move right; and
- each head can also move left.

Proving NL $\subseteq$ REWBLK

As example, we try to accept the following typical CFL with REWBLK:

$$L_{\text{reverse}} = \left\{ w\#\, w^R : w \in \Sigma^* \right\}$$

w^R is the reverse of w.

For example, $a\, b\, c\# c\, b\, a \in L_{\text{reverse}}$.

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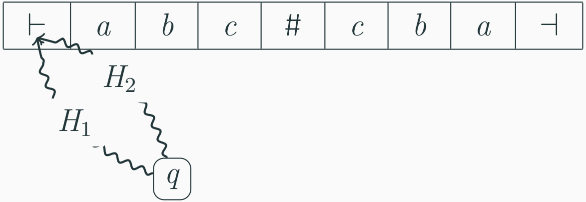

initial configuration

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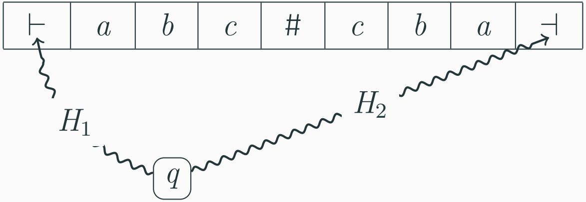

$\boxed{\text{move } H_2 \text{ to } \dashv \text{ while checking that } \# \text{ appears only once.}}$

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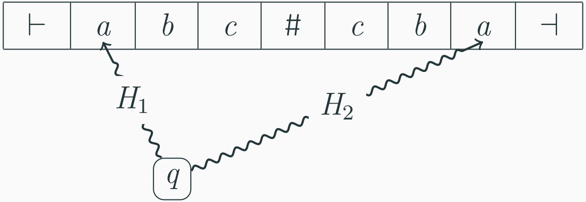

move heads toward the center while they read the same symbol.

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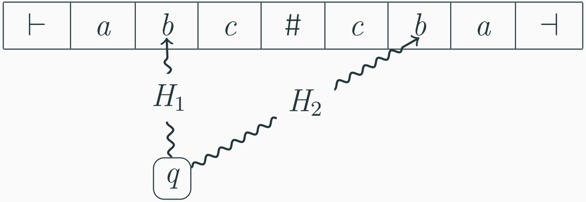

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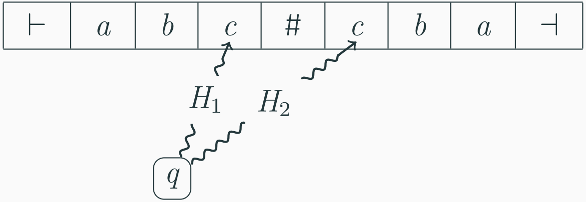

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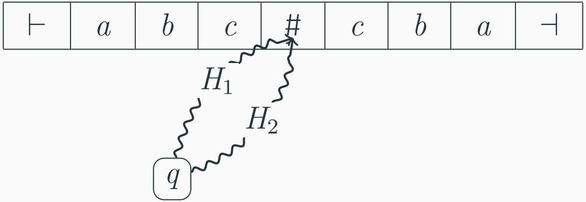

if the heads reach # at the same time, we accept the input.

Proving NL $\subseteq$ REWBLK

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To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

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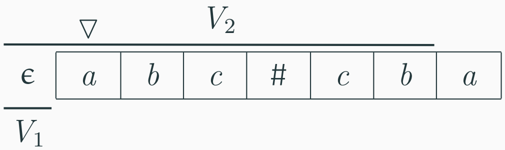
① We initialize V_1 and V_2 by:

$$?\big((\epsilon)_{V_1}\big) \qquad ?\big((\Sigma^*)_{V_2}\ \Sigma\ \$\big)$$



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To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

② Check V_1 and V_2 scan the same symbol by

$$\bigcup_{\sigma \in \Sigma} ?(\backslash V_1\ \color{red}{\sigma}\color{black})\ ?(\backslash V_2\ \color{blue}{\sigma}\color{black})$$

$\triangledown$ V_2

ϵ	$\color{red}{a}$	b	c	$\#$	c	b	$\color{blue}{a}$

V_1

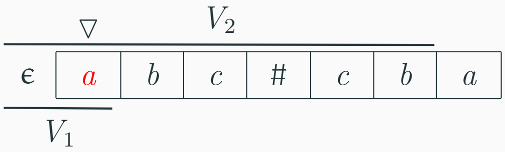
Proving NL $\subseteq$ REWBLk

$$L_{\text{reverse}} = \{ w\#w^R : w \in \Sigma^* \} \qquad w^R \text{ is the reverse of } w.$$

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

③ Change/Move V_1 and V_2. First, we expand V_1. It is easy:

$$?\big((\backslash V_1\ \Sigma)_{V_1}\big)$$



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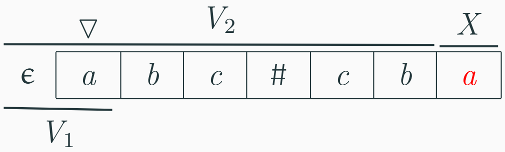
To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

④ We shrink V_2 in two steps. First, find the complement string X:

$$?\left(\backslash V_2\ (\Sigma^*)_X\ \$\right)$$



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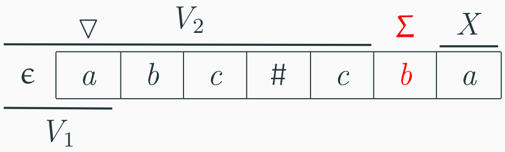
④ Using the complement string X, we shrink V_2:

$$?\Big((\Sigma^*)_{V_2} \ ({\color{red}\Sigma} \setminus X) \ \$ \Big)$$



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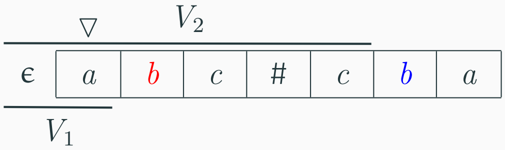
⑤ We repeat the same. Check both the variables scan the same symbol:

$$\bigcup_{\sigma \in \Sigma} ?(\backslash V_1 \sigma)\ ?(\backslash V_2 \sigma)$$



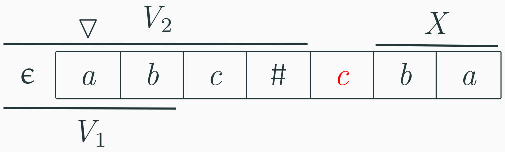
$$L_{\text{reverse}} = \{ w\#\, w^R : w \in \Sigma^* \}$$

w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

⑥ We again expand V_1 and shrink V_2 by

$$?((\backslash V_1\ \Sigma)_{V_1})\ \ ?(\backslash V_2\ (\Sigma^*)_X\ \$)\ \ ?((\Sigma^*)_{V_2}\ (\Sigma \setminus X)\ \$)$$



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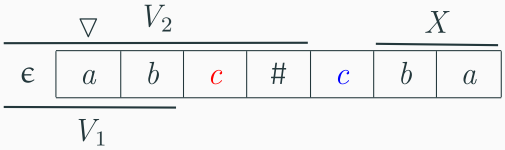

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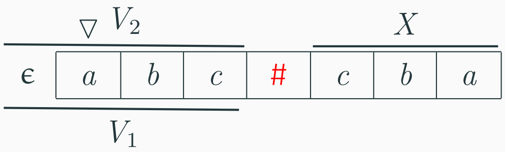
To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

⑧ We again expand V_1 and shrink V_2.

$$?\big((\backslash V_1\ \Sigma)_{V_1}\big)\ \ ?\big(\backslash V_2\ (\Sigma^*)_X\ \$\big)\ \ ?\big((\Sigma^*)_{V_2}\ (\Sigma\ \backslash X)\ \$\big)$$



Proving NL $\subseteq$ REWBLK

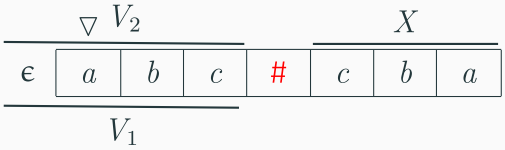
$$L_{\text{reverse}} = \{ w \# w^R : w \in \Sigma^* \}$$

w^R is the reverse of w.

To simulate two heads H_1 and H_2, we use two variables V_1 and V_2.

⑧ We again expand V_1 and shrink V_2.

$$?((\backslash V_1\ \Sigma)_{V_1})\ \ ?(\backslash V_2\ (\Sigma^*)_X\ \$)\ \ ?((\Sigma^*)_{V_2}\ (\Sigma\ \backslash X)\ \$)$$



On the basis of this simulation idea, we have「**Thm. NL** $\subseteq$ REWBLK.」

Proving REWBLk $\subseteq$ NL: Translating REWBLk to NL machines

Proving REWBLk $\subseteq$ NL

Basically, we generalize $\underbrace{\text{the classical regex-to-automaton translation}}_{\text{McNaughton–Yamada–Thompson construction}}$.

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For example, from the following expression,

$$E_{\mathsf{STconn}} = (\,V^*)_C\ ?(\#)\ \Big(?\big(\Sigma^*\,\#\,\backslash C \to (\Sigma^*)_C\,\#\big)\Big)^*\ \Sigma^*\ \#\ \backslash C$$

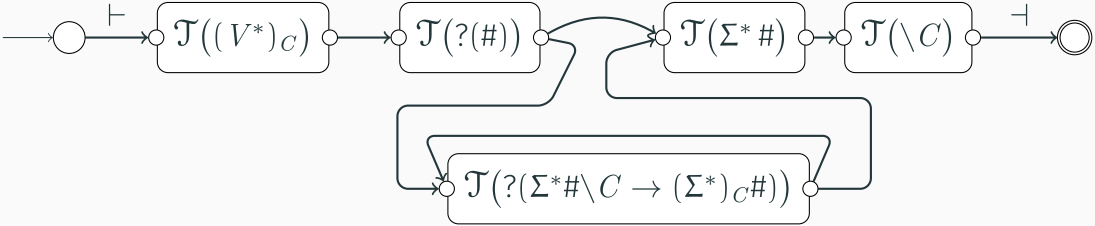
Proving REWBLK $\subseteq$ NL

Basically, we generalize $\underbrace{\text{the classical regex-to-automaton translation}}_{\text{McNaughton–Yamada–Thompson construction}}$.

For example, from the following expression,

$$E_{\mathsf{STconn}} = (V^*)_C\ ?(\#)\ \left(?(\Sigma^*\#\backslash C \to (\Sigma^*)_C\#)\right)^*\ \Sigma^*\ \#\ \backslash C$$

we obtain the following nlog-space machine $\mathcal{T}(E_{\mathsf{STconn}})$:
(note: inputs of machines are surrounded by $\vdash$ and $\dashv$)



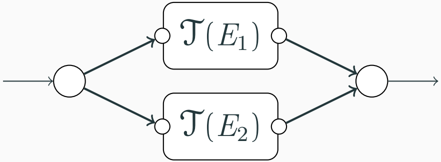
Translation rules for REGEX part

REWBLK : nlog-space Turing machines

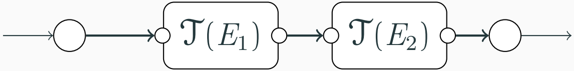
$\mathcal{T}(\epsilon) =$

$\mathcal{T}(\sigma) =$

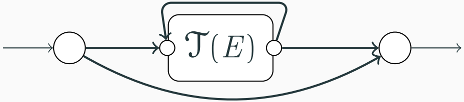
$\mathcal{T}(E_1 + E_2) =$



$\mathcal{T}(E_1\ E_2) \Rrightarrow$

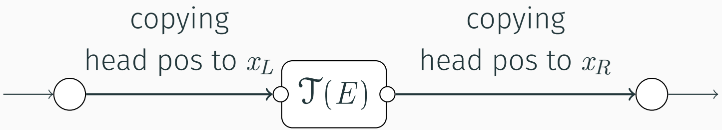


$(E)^* \Rrightarrow$



Translation rules for $(E)_x$ and $\backslash x$

$\mathcal{T}((E)_x) \quad =$

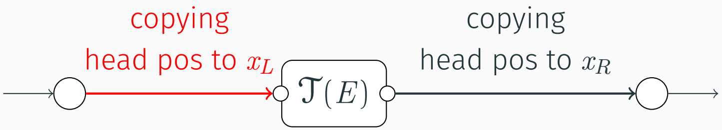


input tape	:	$\vdash \; a \; b \; c \; b \; \ldots \; \overset{\triangledown}{a} \; \ldots\ldots\ldots\ldots \; b \; \ldots \; c \; b \; a \; a \; \dashv$
pos	:	N
working tape x_L	:	
working tape x_R	:	

⚠️ We need to normalize and remove patterns like $(\cdots x \cdots)_x$, $(\cdots(\cdots)_x \cdots)_x$.

Translation rules for $(E)_x$ and $\backslash x$

$\mathcal{T}((E)_x) \quad =$

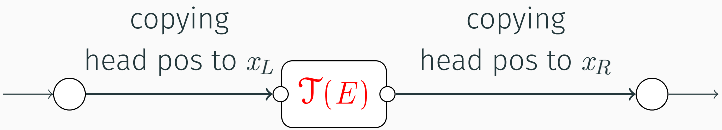


input tape	:	$\vdash \; a\; b\; c\; b\; \ldots\; \overset{\triangledown}{a}\; \ldots\ldots\ldots\ldots\; b\; \ldots\; c\; b\; a\; a \dashv$
pos	:	N
working tape x_L	:	binary encoding of N
working tape x_R	:	

⚠️ We need to normalize and remove patterns like $(\cdots x \cdots)_x$, $(\cdots (\cdots)_x \cdots)_x$.

Translation rules for $(E)_x$ and $\backslash x$

$\mathcal{T}((E)_x) \quad =$



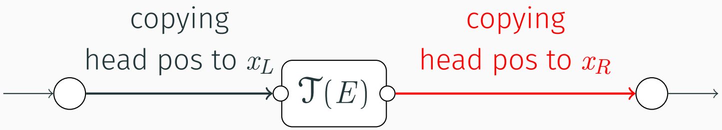
consumed by E

input tape	:	$\vdash \; a \; b \; c \; b \; \ldots \; \overbrace{a \; \ldots\ldots\ldots\ldots \; b}^{}$ $\; \ldots \; c \; b \; a \; a \; \dashv$
pos	:	N ... M
working tape x_L	:	binary encoding of N
working tape x_R	:	

⚠️ We need to normalize and remove patterns like $(\cdots x \cdots)_x$, $(\cdots(\cdots)_x \cdots)_x$.

Translation rules for $(E)_x$ and $\backslash x$

$\mathcal{T}((E)_x) \quad =$



consumed by E

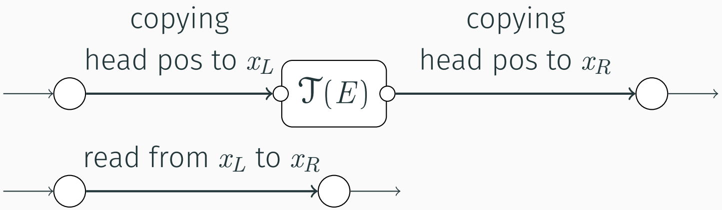
input tape	:	$\vdash \; a \; b \; c \; b \; \ldots \; \overbrace{a \; \ldots\ldots\ldots\ldots \; b}^{} \; \ldots \; c \; b \; a \; a \; \dashv$
pos	:	$N \qquad M$
working tape x_L	:	binary encoding of N
working tape x_R	:	binary encoding of M

⚠️ We need to normalize and remove patterns like $(\cdots x \cdots)_x$, $(\cdots(\cdots)_x \cdots)_x$.

Translation rules for $(E)_x$ and $\backslash x$

$\mathcal{T}((E)_x) \quad = $

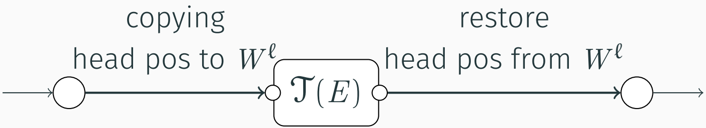
$\mathcal{T}(\backslash x) \quad = $



input tape	:	$\vdash \; a \; b \; c \; b \; \ldots \; a \; \overbrace{\ldots\ldots\ldots\ldots}^{\text{consumed by } E} \; \overset{\triangledown}{b} \; \ldots \; c \; b \; a \; a \; \dashv$
pos	:	$N \qquad M$
working tape x_L	:	binary encoding of N
working tape x_R	:	binary encoding of M

⚠️ We need to normalize and remove patterns like $(\cdots x \cdots)_x$, $(\cdots(\cdots)_x \cdots)_x$.

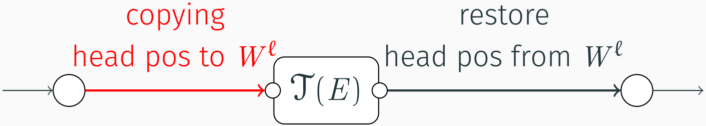
Translation rules for $?(E)$ and $!(E)$

$\mathcal{T}(?(E)^\ell) \quad =$ 

		$\triangledown$
input tape	:	$\vdash \; a \; b \; c \; b \; \ldots \; a \; \ldots\ldots\ldots\ldots \; b \; \ldots \; c \; b \; a \; a \; \dashv$
pos	:	N
working tape W^ℓ	:	

Translation rules for $?(E)$ and $!(E)$

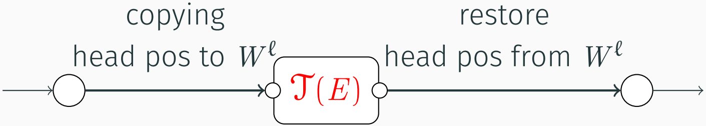
$\mathcal{T}(?(E)^{\ell}) \;=$



		$\triangledown$
input tape	:	$\vdash\ a\ b\ c\ b\ \ldots\ a\ \ldots\ldots\ldots\ldots\ b\ \ldots\ c\ b\ a\ a\ \dashv$
pos	:	N
working tape W^{ℓ}	:	binary encoding of N

Translation rules for $?(E)$ and $!(E)$

$$\mathcal{T}\left(?(E)^{\ell}\right) \quad =$$

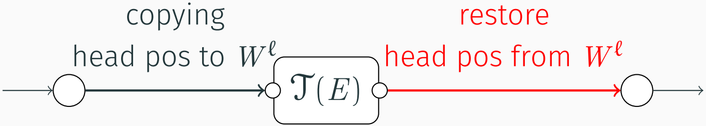


consumed by E

input tape	:	$\vdash \; a \; b \; c \; b \; \ldots \; a \; \ldots\ldots\ldots\ldots \; b \; \ldots \; c \; b \; a \; a \; \dashv$
pos	:	$N \qquad\qquad M$
working tape W^{ℓ}	:	binary encoding of N

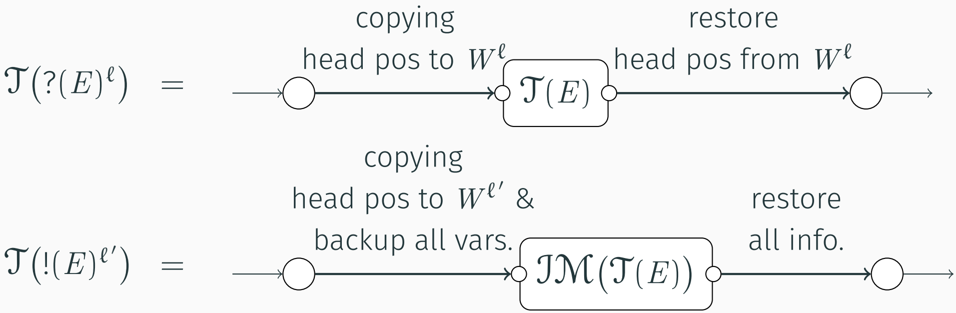
Translation rules for $?(E)$ and $!(E)$

$\mathcal{T}(?(E)^\ell) \quad =$



input tape	:	$\vdash \; a\; b\; c\; b \;\ldots\; \overset{\triangledown}{a} \;\ldots\ldots\ldots\ldots\; b \;\ldots\; c\; b\; a\; a \;\dashv$
pos	:	N (at a), M (at b)
working tape W^ℓ	:	binary encoding of N

Translation rules for $?(E)$ and $!(E)$



input tape	:	$\vdash\ a\ b\ c\ b\ \ldots\ \overset{\triangledown}{a}\ \ldots\ldots\ldots\ldots\ b\ \ldots\ c\ b\ a\ a\ \dashv$
pos	:	N (at a), M (at b)
working tape W^{ℓ}	:	binary encoding of N

Translation rules for $?(E)$ and $!(E)$

$$\mathcal{T}(?(E)^{\ell}) \quad = \quad \longrightarrow \bigcirc \xrightarrow[]{\substack{\text{copying} \\ \text{head pos to } W^{\ell}}} \boxed{\mathcal{T}(E)} \xrightarrow[]{\substack{\text{restore} \\ \text{head pos from } W^{\ell}}} \bigcirc \longrightarrow$$

$$\mathcal{T}(!(E)^{\ell'}) \quad = \quad \longrightarrow \bigcirc \xrightarrow[]{\substack{\text{copying} \\ \text{head pos to } W^{\ell'} \text{ \&} \\ \text{backup all vars.}}} \boxed{\mathcal{IM}(\mathcal{T}(E))} \xrightarrow[]{\substack{\text{restore} \\ \text{all info.}}} \bigcirc \longrightarrow$$

input tape	:	$\vdash \ a \ b \ c \ b \ \ldots \ \overset{\triangledown}{a} \ \ldots\ldots\ldots\ldots \ b \ \ldots \ c \ b \ a \ a \ \dashv$
pos	:	N (at a), M (at b)
working tape W^{ℓ}	:	binary encoding of N

$\mathcal{IM}(M)$ is the complement version of M computed by Immerman's construction. Please recall **NL** $=$ **co-NL** by Immerman–Szelepcsényi theorem.

Conclusion

	Language class	Comp. of Membership Problem
REGEX	$= \text{Dspace}(O(1))$	**NL**-complete (1)
REGEX + *Backreferences*	$\subseteq$ **NL** $[1]$ $\subseteq$ **INDEX** $[2]$	**NP**-complete (2)
REGEX + *Lookaheads*	$=$ REGEX $[3]$	**P**-complete (3)
REWBLk	$=$ **NL** NEW	**PSPACE**-complete NEW

[1] *Inside the class of regex languages*, M.L. Schmid, 2013

[2] *On the expressive power of regular expressions with backreferences*, T.Nogami & T. Terauchi, 2023

[3] *Alternation*, A. Chandra, D. Kozen, & L. Stockmeyer, 1981

(1) is equivalent to the st-connectivity (directed-graph connectivity) problem

(2) *Algorithms for finding patterns in strings*, A. Aho, 1990

(3) *A note on the space complexity of some decision problems for finite automata*, T. Jiang & B. Ravikumar, 1991

Bonus Slides

Normalizing expressions

Let's consider an expression:

$$(\backslash x\ (a\ \backslash x)_x\ \backslash x)_x,$$

which violates conditions $(\cdots \backslash x \cdots)_x$ and $(\cdots (\cdots)_x \cdots)_x$.

We can normalize it as follows:

$$\begin{aligned} & (\backslash x\ (a\ \backslash x)_x\ \backslash x)_x \\ \Rightarrow\quad & (\backslash x\ (a\ \backslash x)_x\ \backslash x)_y\ ?((\backslash y)_x) \\ \Rightarrow\quad & (\backslash x\ (a\ \backslash x)_z\ ?((\backslash z)_x)\ \backslash x)_y\ ?((\backslash y)_x) \end{aligned}$$

From outside to inside, we replace labels $(\cdots)_x$ with a frash one $(\cdots)_\alpha$ and then add $?((\backslash\alpha)_x)$ for copying the content:

$$(\cdots)_x \Rightarrow (\cdots)_\alpha ?((\backslash\alpha)_x)$$