

Quantum error correction III:

Introduction to stabilizers

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What is a stabilizer?

Let's start with some notation, we need signed tensor product of the matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

eg: $-X \otimes I \otimes Z \otimes Z \otimes Y$

or more compactly: $-XIZZY$

A signed tensor product will also simply be called an operator

What is a stabilizer?

The simplest stabilizers are just operators that leave some useful state unchanged, eg:

$$\begin{aligned} Z|0\rangle &= |0\rangle \\ - Z|1\rangle &= |1\rangle \\ X|+\rangle &= |+\rangle \\ - X|-\rangle &= |-\rangle \end{aligned}$$

These states are called eigenstates of their associated stabilizer, and are unique up to global phase given the stabilizer

What is a stabilizer?

The simplest stabilizers are just operators that leave some useful state unchanged, eg:

$Z 0\rangle = 0\rangle$	$ 0\rangle \rightarrow +Z$
$-Z 1\rangle = 1\rangle$	$ 1\rangle \rightarrow -Z$
$X +\rangle = +\rangle$	$ +\rangle \rightarrow +X$
$-X -\rangle = -\rangle$	$ -\rangle \rightarrow -X$

Since the eigenstates are uniquely specified by the stabilizers, we can write them instead of the states.

What is a stabilizer?

Consider $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$

This state is stabilized by 3 independent stabilizers

+XXX

+ZZI

+IZZ

For complex states, the list of independent stabilizers can be far more compact than the state itself

Some algebra

Let's work through one example of showing a stabilizer stabilizes a state:

$$\begin{aligned} ZZI|\Psi\rangle &= \frac{1}{\sqrt{2}} ZZI(|000\rangle + |111\rangle) \\ &= \frac{1}{\sqrt{2}} (ZZI|000\rangle + ZZI|111\rangle) \\ &= \frac{1}{\sqrt{2}} (|000\rangle + (-1)^2|111\rangle) \\ &= \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \\ &= |\Psi\rangle \end{aligned}$$

What about errors?

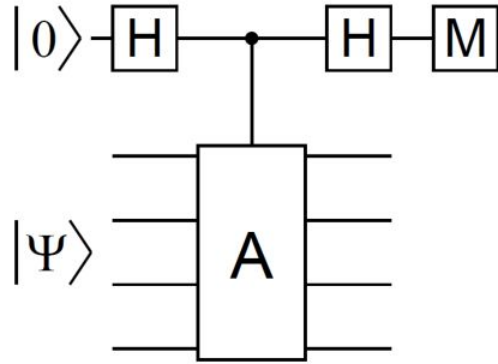
Consider the effect of errors: $X_2|\Psi\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle)$

What does this do to our stabilizers?
$$\begin{aligned} Z_2 Z_1 X_2 |\Psi\rangle &= Z_2 X_2 Z_1 |\Psi\rangle \\ &= -X_2 Z_2 Z_1 |\Psi\rangle \\ &= -X_2 |\Psi\rangle \end{aligned}$$

To say the same thing another way: $(-Z_2 Z_1) X_2 |\Psi\rangle = X_2 |\Psi\rangle$ or:
$$\begin{array}{l} +XXX \\ -ZZI \\ +IZZ \end{array}$$

If we can work out how to measure the sign of a stabilizer, we can detect errors.

General operator measurement

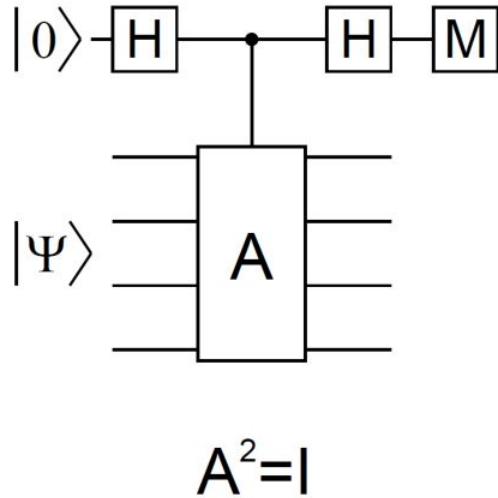


$|0\rangle|\Psi\rangle$

Try to work out the output of this circuit

$$A^2 = I$$

General operator measurement



$$\begin{aligned}
 &|0\rangle|\Psi\rangle \\
 H &\rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\Psi\rangle \\
 &= \frac{1}{\sqrt{2}}(|0\rangle|\Psi\rangle + |1\rangle|\Psi\rangle) \\
 \text{controlled-}A &\rightarrow \frac{1}{\sqrt{2}}(|0\rangle|\Psi\rangle + |1\rangle A|\Psi\rangle) \\
 H &\rightarrow \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\Psi\rangle + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)A|\Psi\rangle \right) \\
 &= \frac{1}{\sqrt{2}} \left(|0\rangle \frac{1}{\sqrt{2}}(|\Psi\rangle + A|\Psi\rangle) + |1\rangle \frac{1}{\sqrt{2}}(|\Psi\rangle - A|\Psi\rangle) \right)
 \end{aligned}$$

A zero measurement means the output is the +1 eigenstate of A, one means -1 eigenstate

Locating an error with stabilizers

$$X_2|\Psi\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle) \longrightarrow \begin{array}{l} +XXX \\ -ZZI \\ +IZZ \end{array}$$

$$X_1|\Psi\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle)$$

$$X_0|\Psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle)$$

What will the signs of the stabilizers be after these errors?

Locating an error with stabilizers

$$X_2|\Psi\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |011\rangle) \longrightarrow \begin{array}{l} +XXX \\ -ZZI \\ +IZZ \end{array}$$

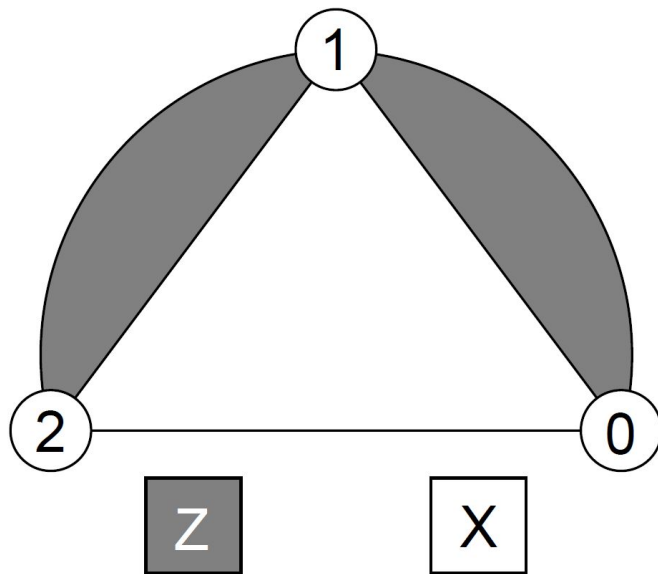
$$X_1|\Psi\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |101\rangle) \longrightarrow \begin{array}{l} +XXX \\ -ZZI \\ -IZZ \end{array}$$

$$X_0|\Psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |110\rangle) \longrightarrow \begin{array}{l} +XXX \\ +ZZI \\ -IZZ \end{array}$$

Representing stabilizers as a picture

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

+XXX
+ZZI
+IZZ

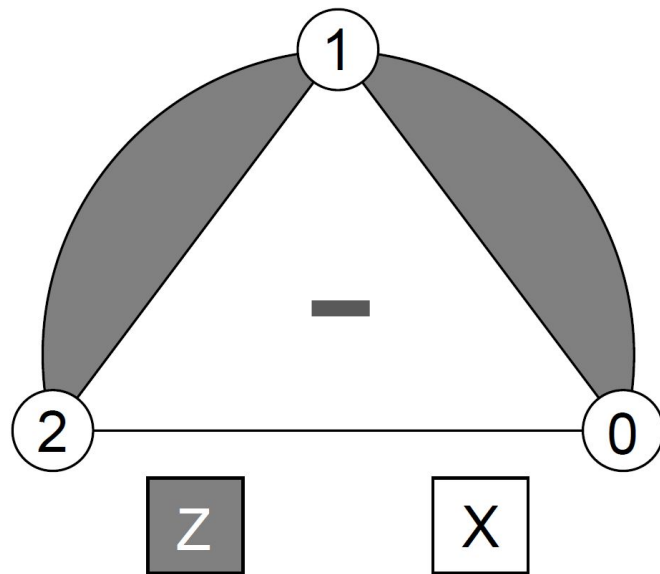


These are all the same state.
What happens after a Z error?

Representing stabilizers as a picture

$$Z|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$$

-XXX
+ZZI
+IZZ



X stabilizer is flipped.
Next time: the surface code