Flavor wave model homework problems

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A. In this part, we will develop a method solving the ground state without diagonalizing the Hamiltonian.

(1) Assume a particle can be discribed by spin only, with $S = 1$. and the Hamiltonian is

$$H = S_z.$$

Write down the matrix form for H, the eigenstates of H and their energies. We will write down the ground state to be $|a_0\rangle$

(2) Assume a initial state to be $|\psi_0\rangle = \frac{1}{\sqrt{3}}(|S_z = -1\rangle + |S_z = 0\rangle + |S_z = 1\rangle)$. Let $\varepsilon = 10^{-2}$. Calculate

$$|\phi_1\rangle = (1 - \varepsilon H)\,|\psi_0\rangle$$

$$|\psi_1\rangle = \frac{|\phi_1\rangle}{\langle\phi_1|\phi_1\rangle}$$

Look at the $|\psi_1\rangle$, and show that there are more weights of ground state in $|\psi_1\rangle$ than $|\psi_0\rangle$.

(3) Define

$$|\phi_{n+1}\rangle = (1 - \varepsilon H)\,|\psi_n\rangle$$

$$|\psi_{n+1}\rangle = \frac{|\phi_{n+1}\rangle}{\langle\phi_{n+1}|\phi_{n+1}\rangle}.$$

Draw (i)$|\langle a_0|\psi_n\rangle|^2$ as a function of n. and (ii) $\langle\psi_n|H|\psi_n\rangle$ as a function of n. Also try with differenct positive values of ε and see how the value will change the speed of converging.

(4) repeat (2) and (3) with $|\psi_0\rangle = |S_z = 0\rangle$. (Some weird thing could happen, depending on the computer.)

(5) For a general Hamiltonian

$$H = \sum_{i=0}^{d-1} E_i\,|a_i\rangle\,\langle a_i|$$

with the eigenenergies $E_0 < E_1 \leqslant E_2 \leqslant \ldots \leqslant E_{d-1}$, prove that $\forall\,|\psi_0\rangle$ that $\langle a_0|\psi_0\rangle \neq 0$, $\exists\varepsilon > 0$, the series $\{|\psi_n\rangle\}$ generatred in (2) satisfies

$$\lim_{n\to\infty} |\langle a_0|\psi_n\rangle|^2 = 1.$$

Hint: Consider an ε that $0 < \varepsilon < 1/|E_{d-1}|$ and find a q that $0 < q < 1$ and $\left(1 - \sqrt{|\langle a_0|\psi_{n+1}\rangle|^2}\right) < q\left(1 - \sqrt{|\langle a_0|\psi_n\rangle|^2}\right)$.

(6) Now we know that this algorithrm always converges. Let's figure out how to make it fast. Consider a Hamiltonian in (5) with conditions $d > 2$ and $E_0 < E_1 < E_{d-1}$. Define:

$$|\phi_{n+1}\rangle = (-H + z)\,|\psi_n\rangle$$

$$|\psi_{n+1}\rangle = \frac{|\phi_{n+1}\rangle}{\langle\phi_{n+1}|\phi_{n+1}\rangle}.$$

where z is a real number and define $x_n = 1 - |\langle a_0|\psi_n\rangle|$. Prove that (i) $\forall z > \frac{-E_0 - E_{d-1}}{2}$, $\exists q$, $0 < q < 1$ and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = q,$$

and (ii) q is minimized when

$$z = -\frac{1}{2}\left(E_1 + E_{d-1}\right)$$

B. This part is to use the method in A to solve the ground state with mean-field approximation.