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Testing and improving the robustness of amortized Bayesian inference for cognitive models

Yufei Wu, Stefan T. Radev, Francis Tuerlinckx

March 27th 2025

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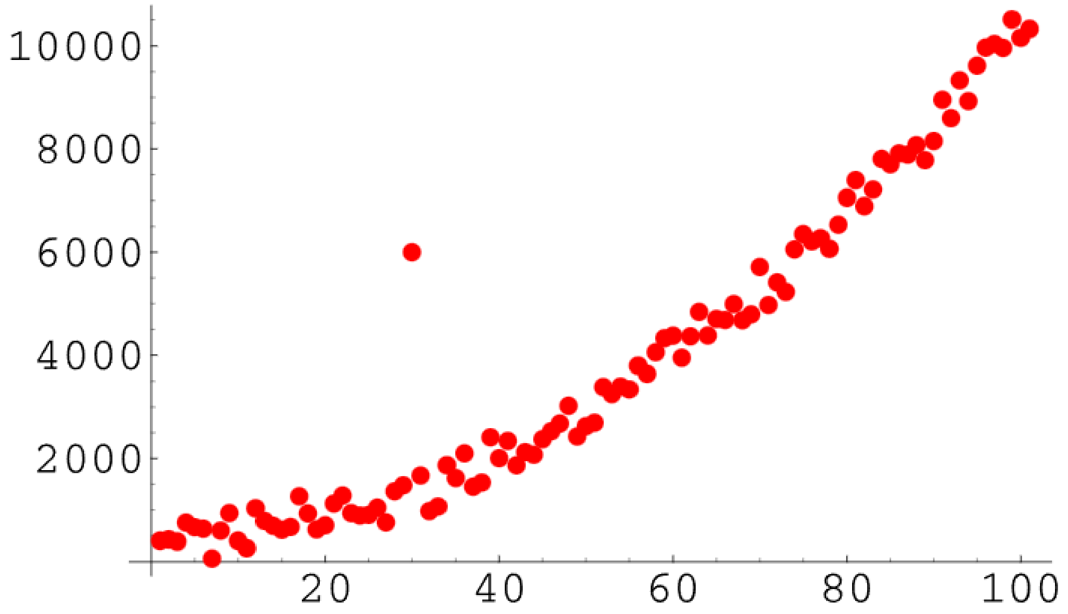
The presence of outliers

- Robustness refers to the resilience of an estimator to outliers.



Cognitive models

- Mathematical expressions of cognitive processes with interpretable parameters
- Some models are sensitive to outliers due to the nature of its assumptions
- E.g., the drift diffusion model (DDM)

Research question

- How to study the influence of outliers of complex stochastic cognitive models?
- How to robustify the inference of complex stochastic cognitive models?

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Testing and improving the robustness of amortized Bayesian inference for Cognitive Models

Origin of robust statistics

ROBUST ESTIMATION OF A LOCATION PARAMETER[1]

BY PETER J. HUBER[2]

University of California, Berkeley

Which is more robust?

$$\boldsymbol{x} \sim N(\mu, 1^2)$$

Original Data	Contaminated Data
1.66	1.66
1.82	1.82
0.43	0.43
1.37	1.37
0.82	0.82
0.11	0.11
1.27	1.27
3.10	3.10
0.95	0.95
1.20	-100

$$Mean_1 = 1.27$$
$$Median_1 = 1.24$$

$$Mean_2 = -8.85$$
$$Median_2 = 1.24$$

Tools to assess robustness

- 1. Empirical influence function (Cook & Weisberg, 1982)
  - Add outlier x^c with different values
- 2. Breakdown point (Donoho & Huber, 1982)
  - Add outlier x^c with different fractions

Empirical influence function

Original Data
1.66
1.82
0.43
1.37
0.82
0.11
1.27
3.10
0.95
1.20

$Mean_1 = 1.27$

Contaminated Data
1.66
1.82
0.43
1.37
0.82
0.11
1.27
3.10
0.95
-100

x^c

$Mean_2 = -8.85$

$$\begin{aligned}
&\mathrm{EIF}_{-100} \\
&= Mean_2 - Mean_1 \\
&= -8.85 - 1.27 \\
&= -10.12
\end{aligned}$$

Empirical influence function

Original Data
…
…
…
…
…
…
…
…
…
…

$Mean_1$

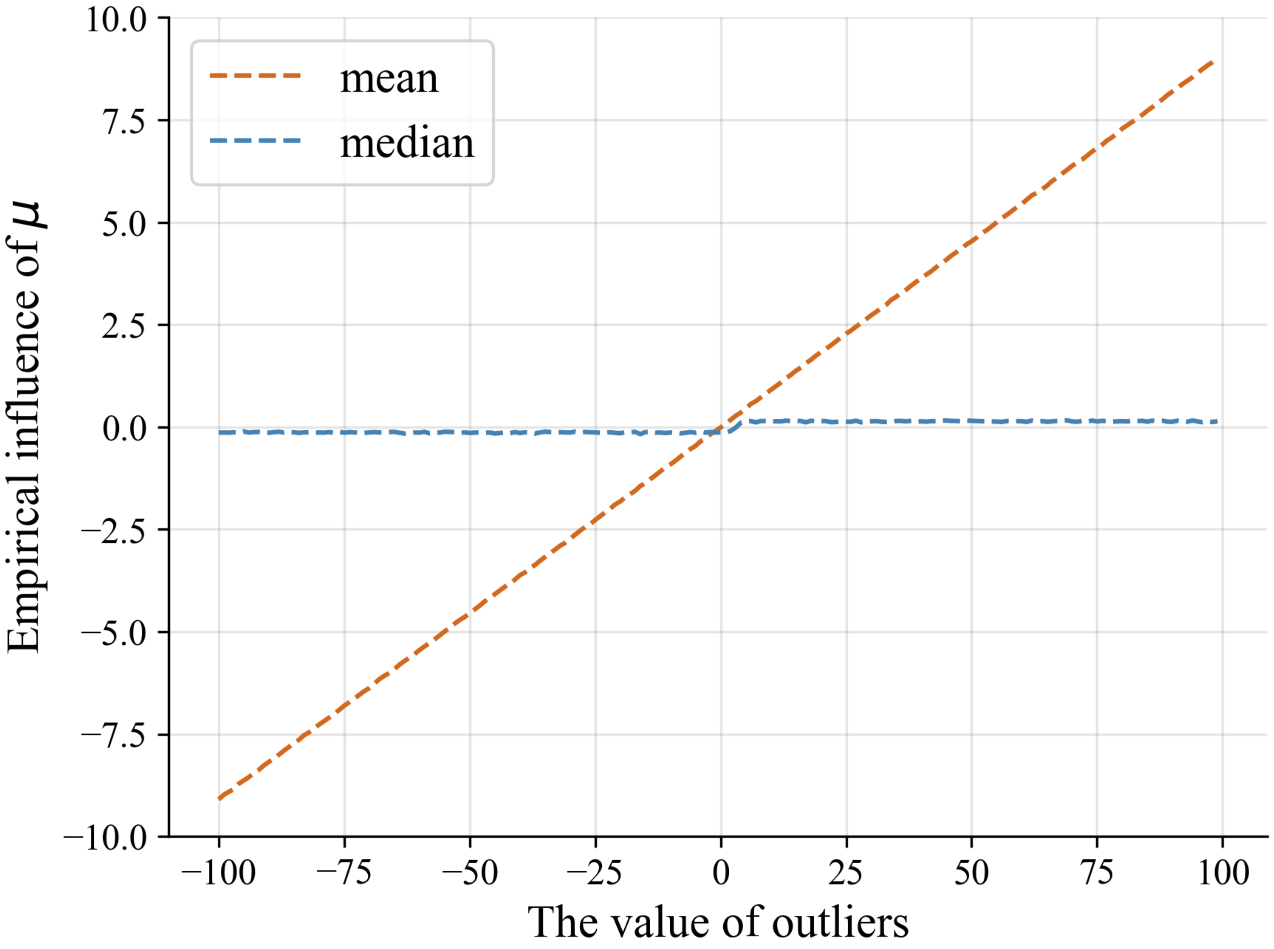
Contaminated Data
…
…
…
…
…
…
…
…
…
-100

x^c

$Mean_2$

$$\overline{\text{EIF}}_{-100} = \overline{Mean_1 - Mean_2}$$

Empirical influence function



For each $x^c = k$,
200 datasets are simulated.

Tools to estimate robustness

- 1. Empirical influence function (Cook & Weisberg, 1982)
  - Add outlier x^c with different values
- 2. Breakdown point (Donoho & Huber, 1982)
  - Add outlier x^c with different fractions

Breakdown point

Contaminated Data
1.66
1.82
0.43
1.37
0.82
0.11
1.27
3.10
-100
-100

x^c

$Mean = -18.9$

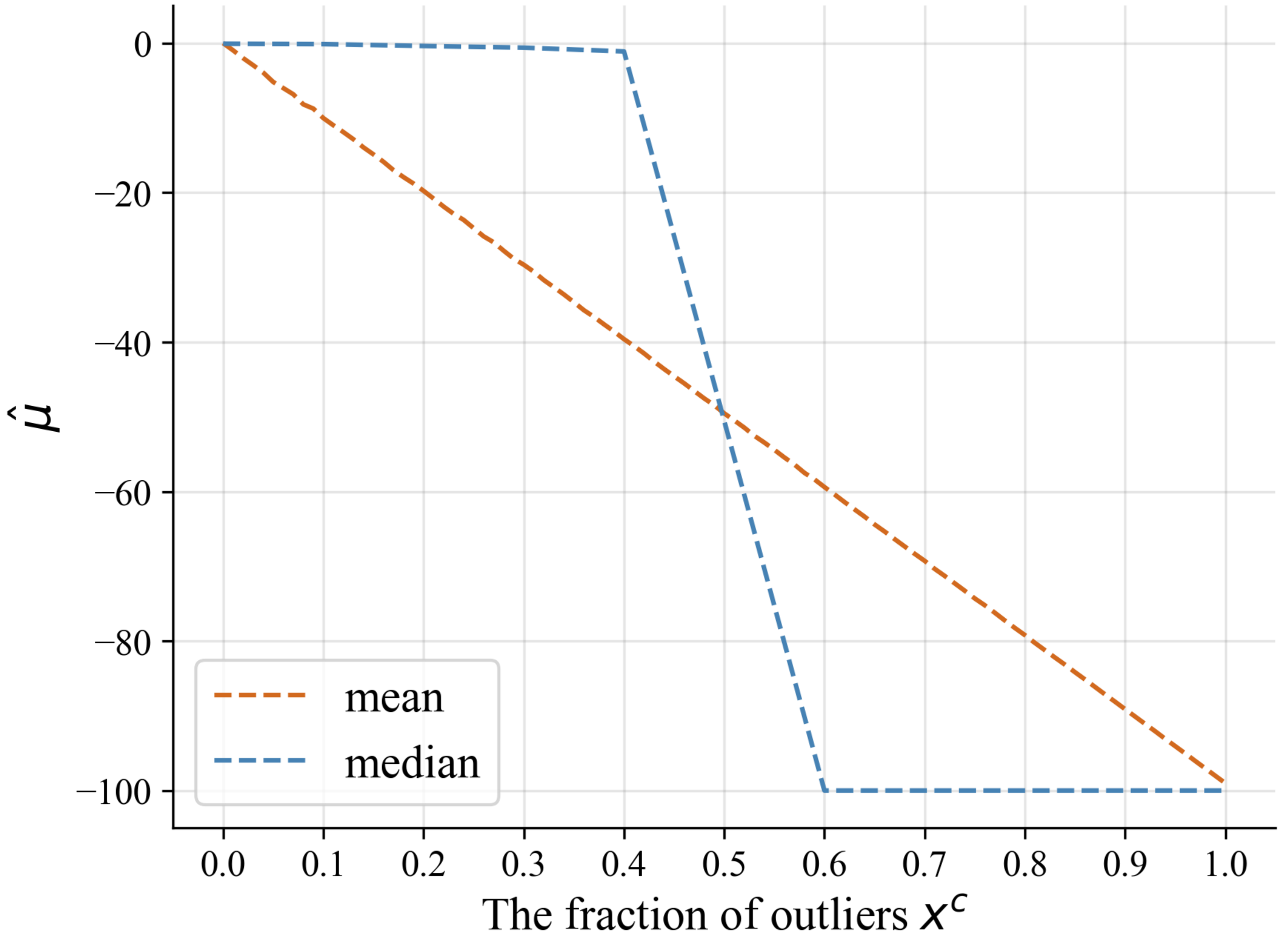
Breakdown point

Contaminated Data
…
…
…
…
…
…
…
…
-100
-100

x^c

$\overline{Mean}$

Breakdown point ($x^c = -100$)

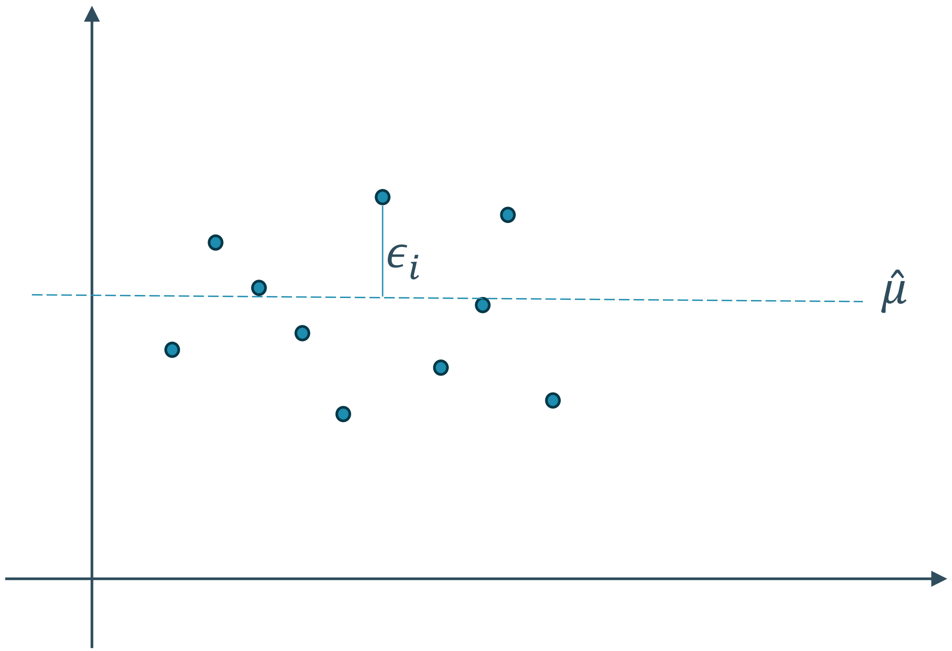


Tools to estimate robustness

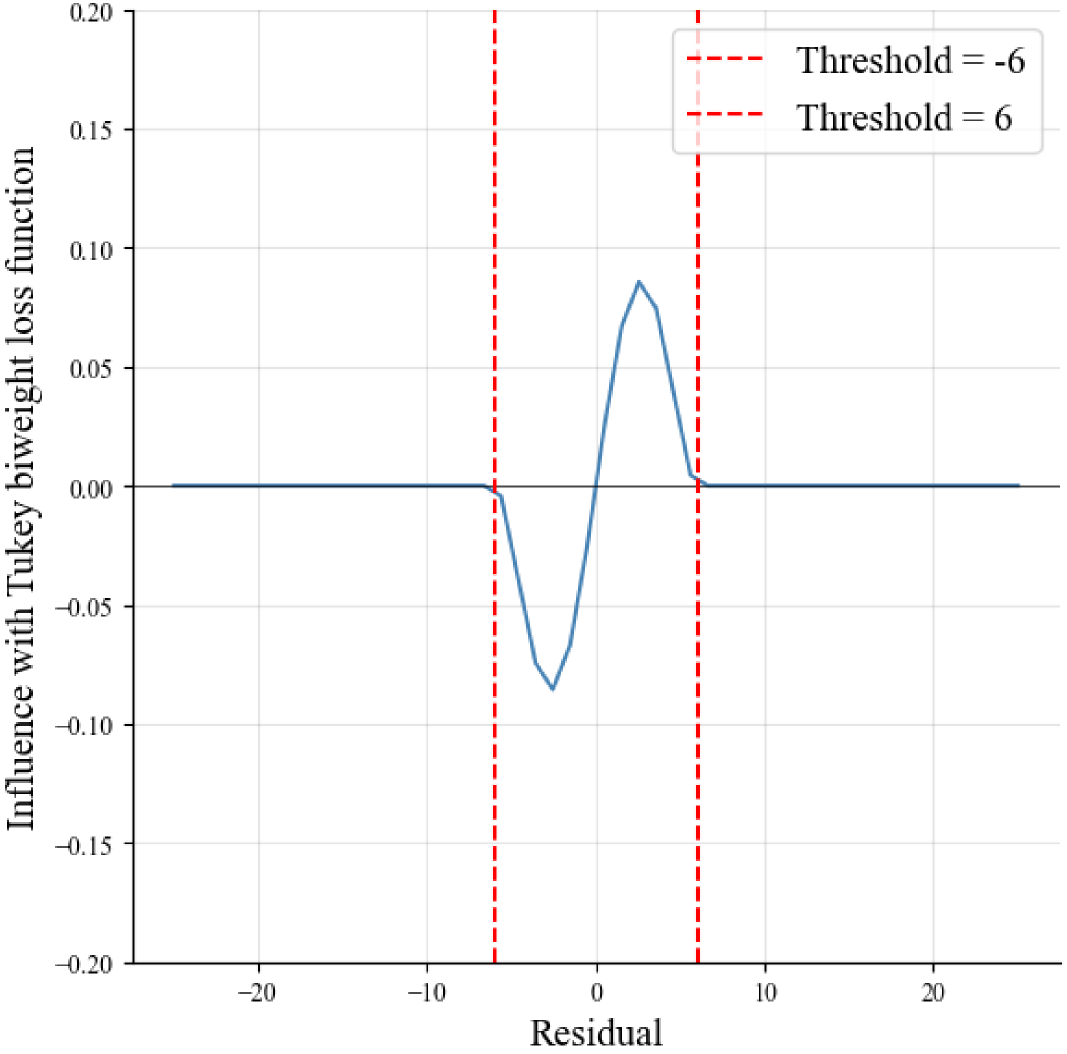
- 1. Empirical influence function (Cook & Weisberg, 1982)
    - Add outlier x^c with different values
- 2. Breakdown point (Donoho & Huber, 1982)
    - Add outlier x^c with different fractions

How to make $\hat{\mu}$ more robust?

- **Minimizing the loss function:** $\sum_{i=1}^{n} \rho(\epsilon_i)$
    - **Mean:** $\rho(\epsilon_i) = \epsilon_i^2$
    - **Median:** $\rho(\epsilon_i) = |\epsilon_i|$
    - **Tukey's biweight function** (Tukey, 1979):
    - $\rho(\epsilon_i) = \begin{cases} \left(1 - \left(\frac{\epsilon_i}{k}\right)^2\right)^2, & if\ |\epsilon_i| \leq k \\ 0, & if\ |\epsilon_i| > k \end{cases}$



Tukey’s biweight function



But…

- 1. Modifying loss function can be difficult

Can we bypass the likehood-based framework?

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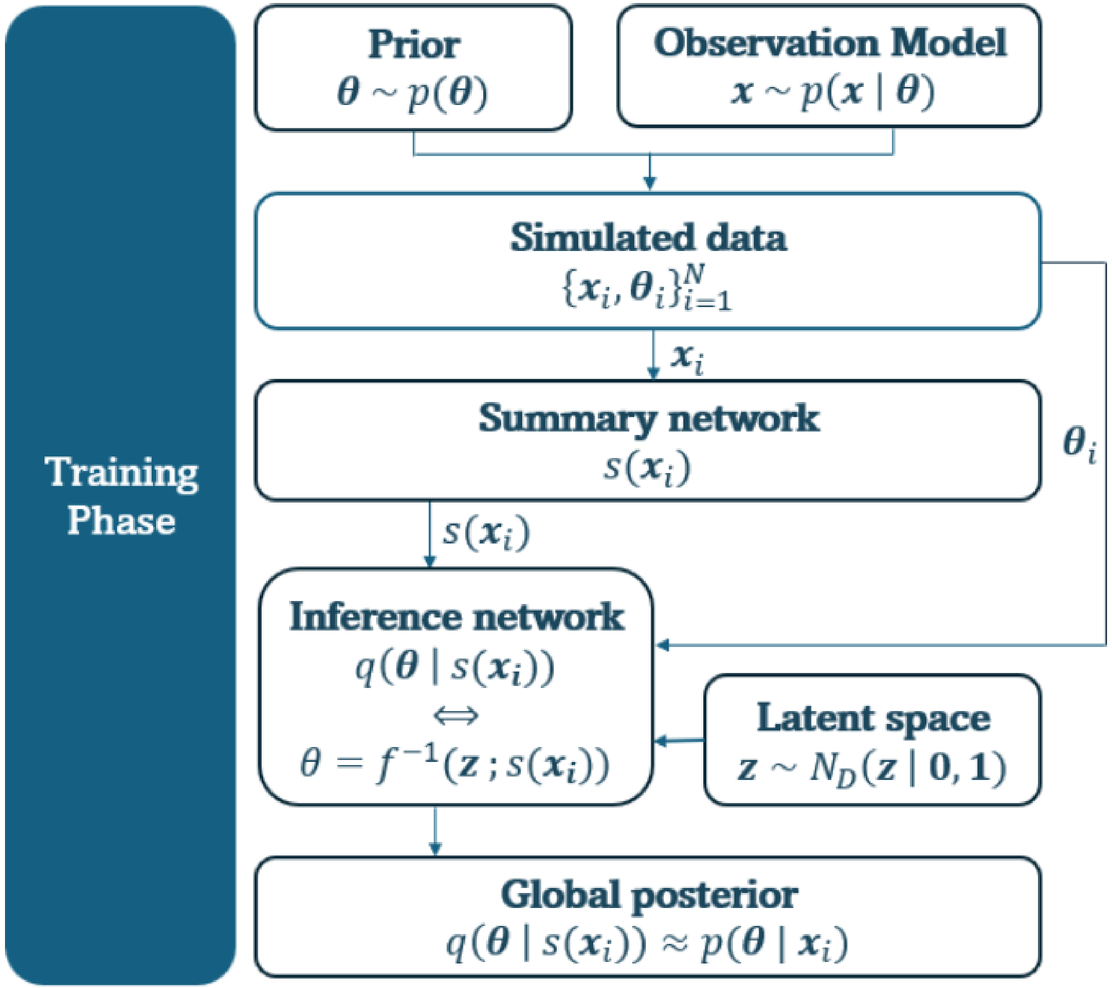
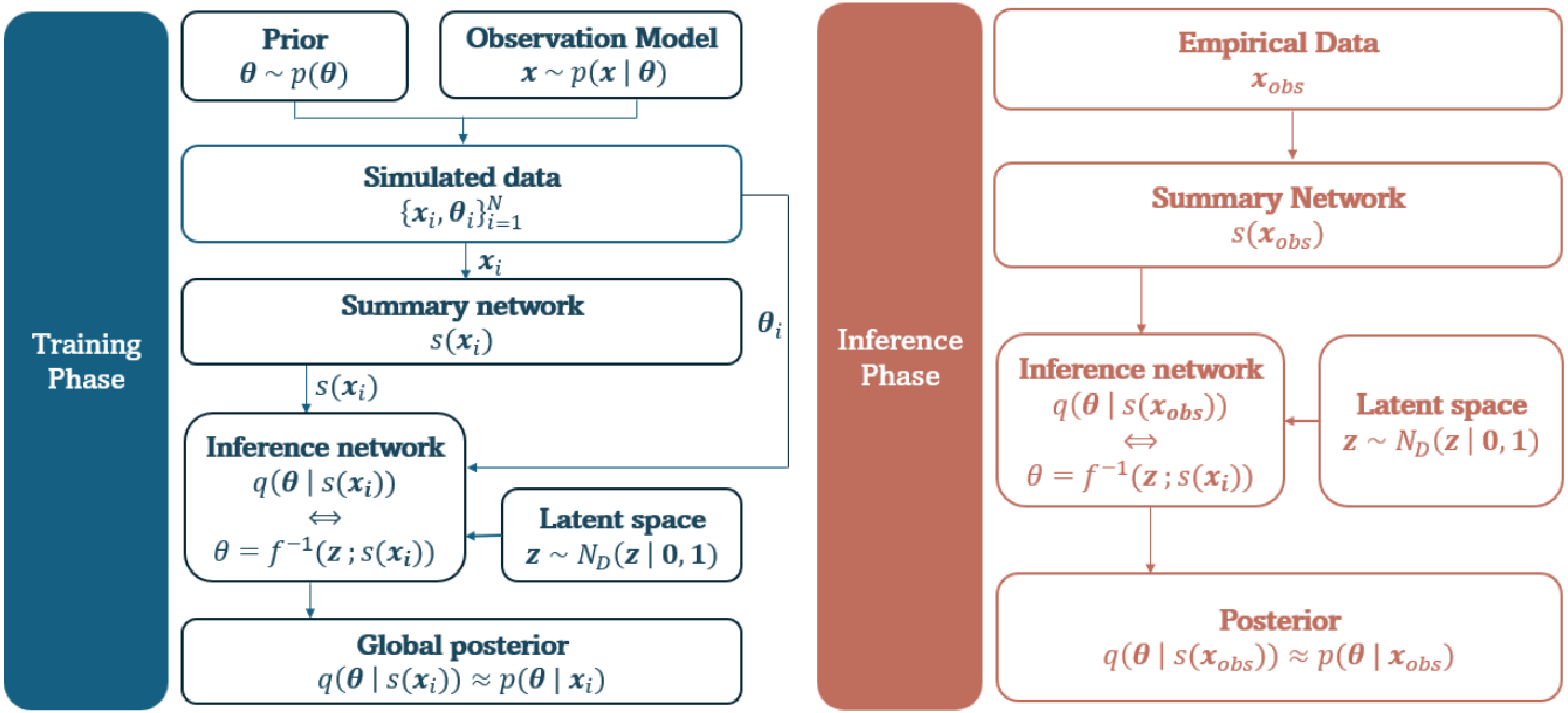
Testing and improving the robustness of amortized Bayesian inference for Cognitive Models

Training


Training Phase
Prior
$\boldsymbol{\theta} \sim p(\boldsymbol{\theta})$
Observation Model
$\boldsymbol{x} \sim p(\boldsymbol{x} \mid \boldsymbol{\theta})$
Simulated data
$\{\boldsymbol{x}_i, \boldsymbol{\theta}_i\}_{i=1}^N$
$\boldsymbol{x}_i$
Summary network
$s(\boldsymbol{x}_i)$
$\boldsymbol{\theta}_i$
$s(\boldsymbol{x}_i)$
Inference network
$q(\boldsymbol{\theta} \mid s(\boldsymbol{x}_i))$
$\Leftrightarrow$
$\theta = f^{-1}(\boldsymbol{z}\,; s(\boldsymbol{x}_i))$
Latent space
$\boldsymbol{z} \sim N_D(\boldsymbol{z} \mid \boldsymbol{0}, \boldsymbol{1})$
Global posterior
$q(\boldsymbol{\theta} \mid s(\boldsymbol{x}_i)) \approx p(\boldsymbol{\theta} \mid \boldsymbol{x}_i)$

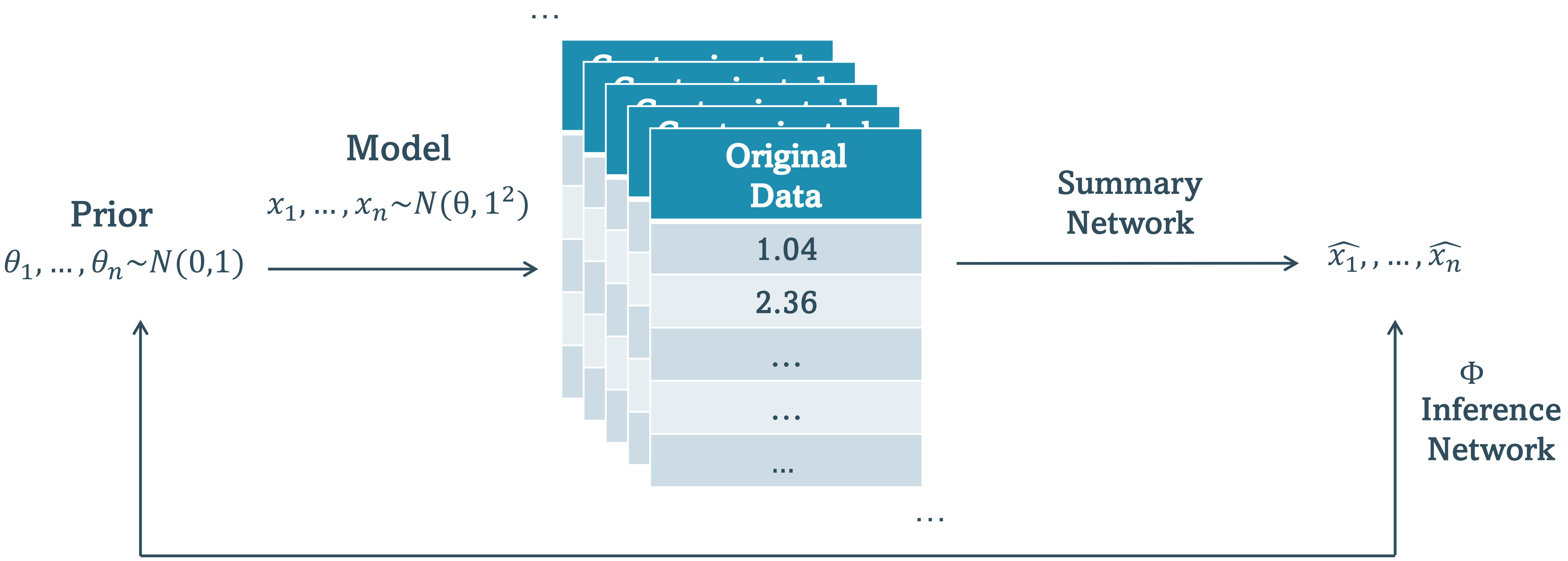
Inference


Training Phase
Prior
$\boldsymbol{\theta} \sim p(\boldsymbol{\theta})$
Observation Model
$\boldsymbol{x} \sim p(\boldsymbol{x} \mid \boldsymbol{\theta})$
Simulated data
$\{\boldsymbol{x}_i, \boldsymbol{\theta}_i\}_{i=1}^{N}$
$\boldsymbol{x}_i$
$\boldsymbol{\theta}_i$
Summary network
$s(\boldsymbol{x}_i)$
$s(\boldsymbol{x}_i)$
Inference network
$q(\boldsymbol{\theta} \mid s(\boldsymbol{x}_i))$
$\Leftrightarrow$
$\theta = f^{-1}(\boldsymbol{z}\,; s(\boldsymbol{x}_i))$
Latent space
$\boldsymbol{z} \sim N_D(\boldsymbol{z} \mid \boldsymbol{0}, \boldsymbol{1})$
Global posterior
$q(\boldsymbol{\theta} \mid s(\boldsymbol{x}_i)) \approx p(\boldsymbol{\theta} \mid \boldsymbol{x}_i)$
Inference Phase
Empirical Data
$\boldsymbol{x}_{obs}$
Summary Network
$s(\boldsymbol{x}_{obs})$
Inference network
$q(\boldsymbol{\theta} \mid s(\boldsymbol{x}_{obs}))$
$\Leftrightarrow$
$\theta = f^{-1}(\boldsymbol{z}\,; s(\boldsymbol{x}_i))$
Latent space
$\boldsymbol{z} \sim N_D(\boldsymbol{z} \mid \boldsymbol{0}, \boldsymbol{1})$
Posterior
$q(\boldsymbol{\theta} \mid s(\boldsymbol{x}_{obs})) \approx p(\boldsymbol{\theta} \mid \boldsymbol{x}_{obs})$

Advantages

- Expensive training, cheap inference
- Likelihood-free

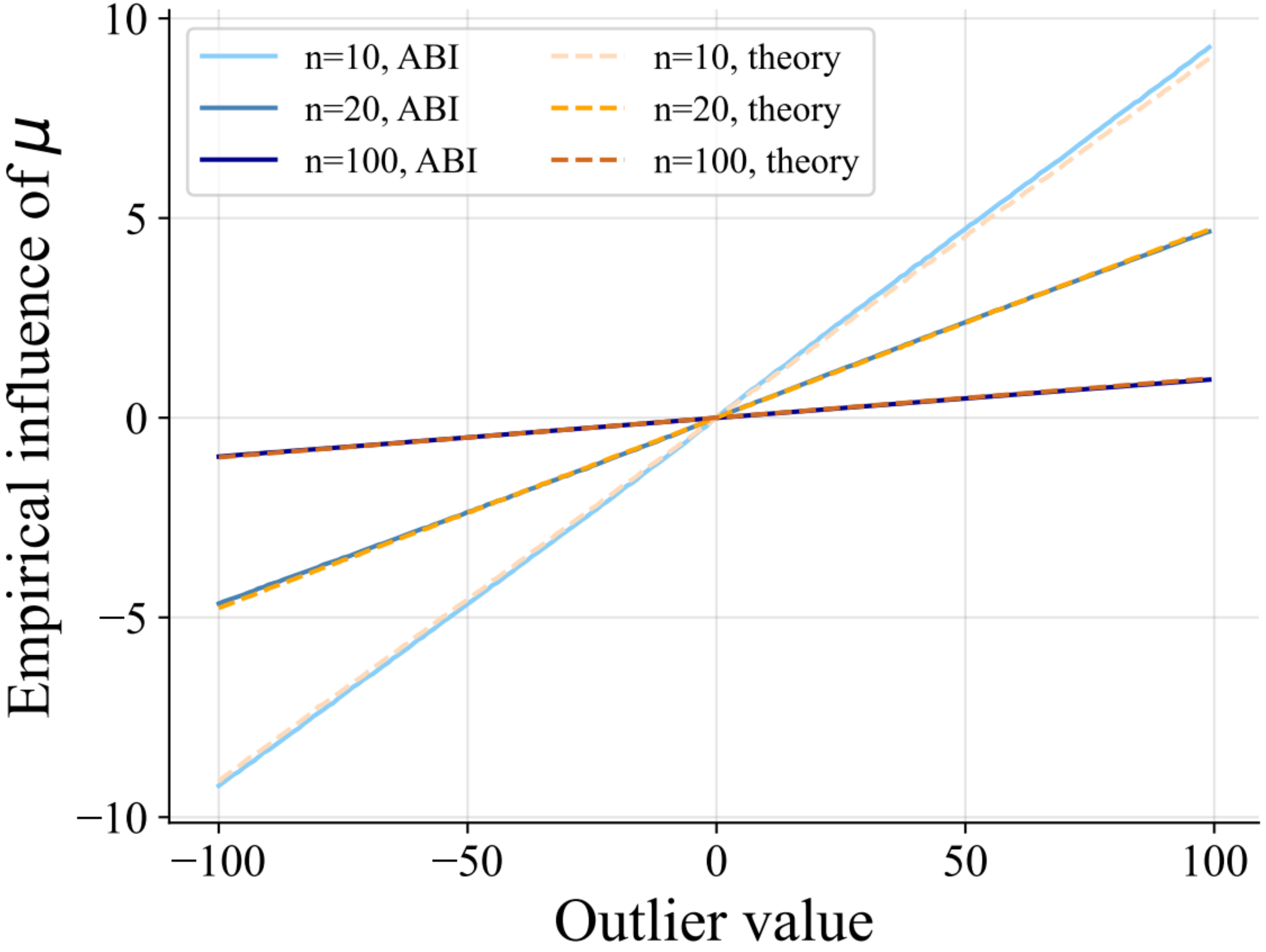
Standard estimator training



Standard estimator training

- Summary network: DeepSet(summary_dim=2) (other settings stay the same as in bayesflow 1.1.4/ bayesflow stable-legacy branch)
- Inference network: Normalizingflow (Couplingflow)

Empirical influence function



Assuming outliers in simulation

$\boldsymbol{x} \sim N(\mu, 1^2)$

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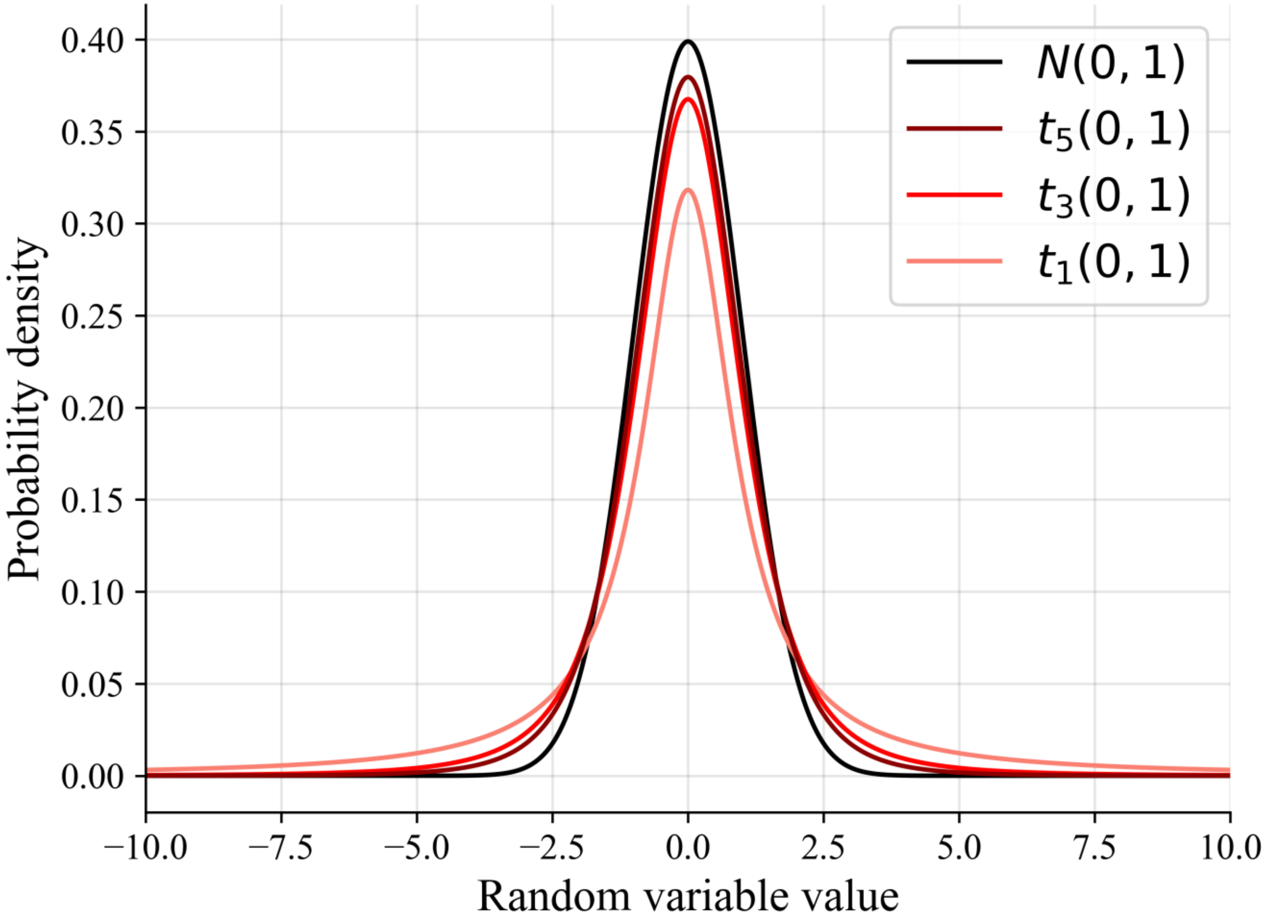
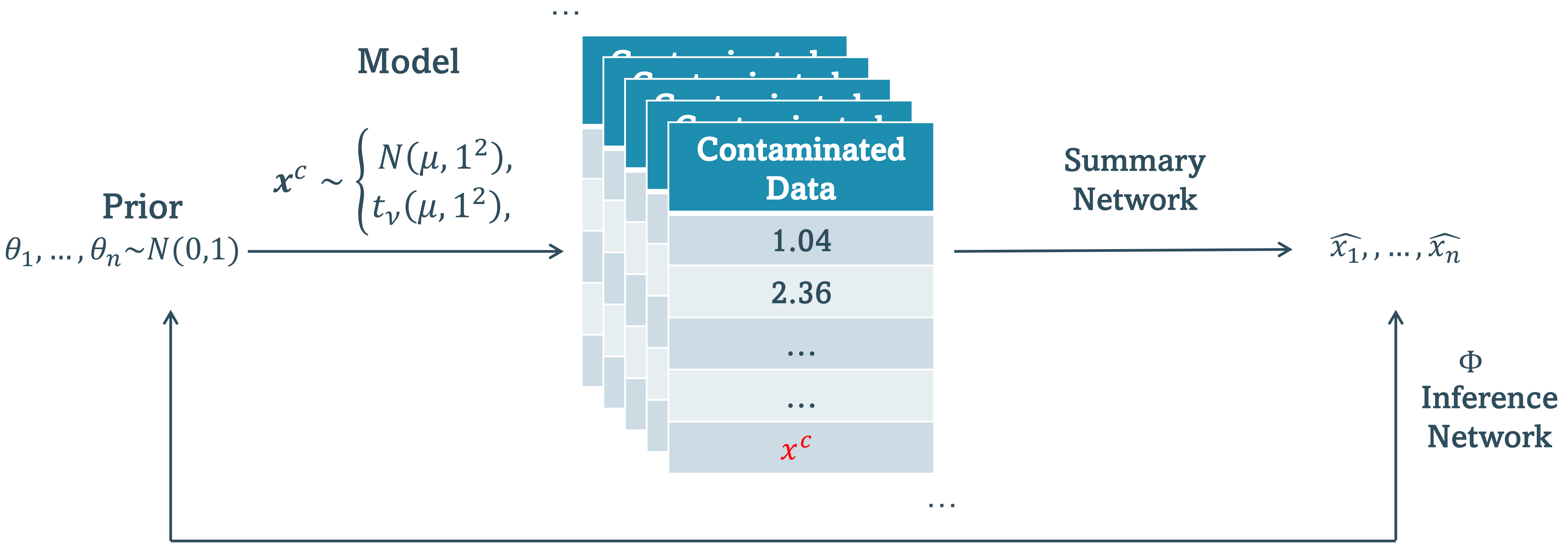
Assuming outliers in simulation

$x \sim N(\mu, 1^2)$

$$x^c \sim \begin{cases} N(\mu, 1^2), & with\ probablity\ 1 - \pi = 0.9 \\ t_\nu(\mu, 1^2), & with\ probablity\ \pi = 0.1 \end{cases}$$



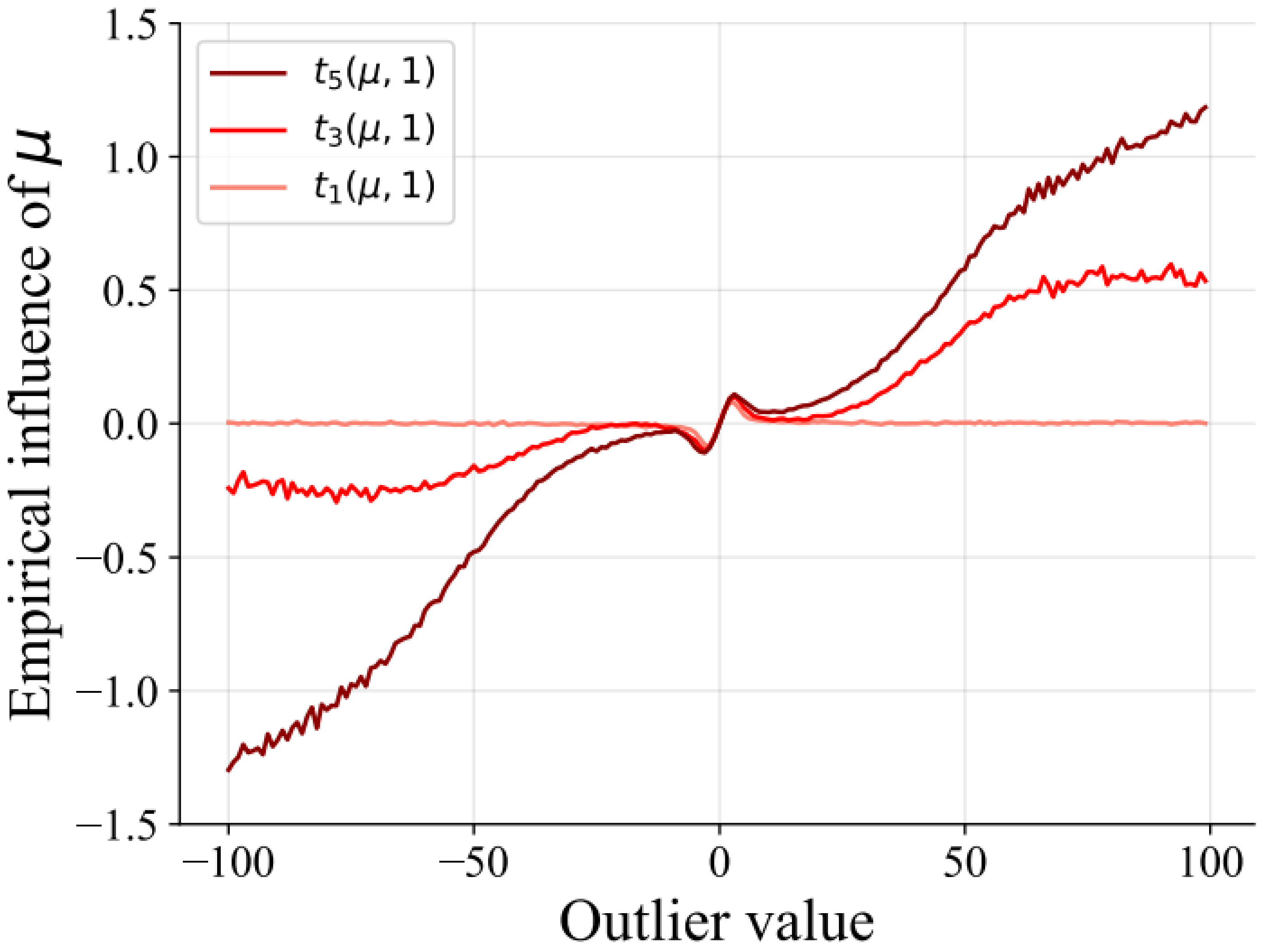
Robust estimator training



Empirical influence function

The smaller the ν,

the more robust the estimator.

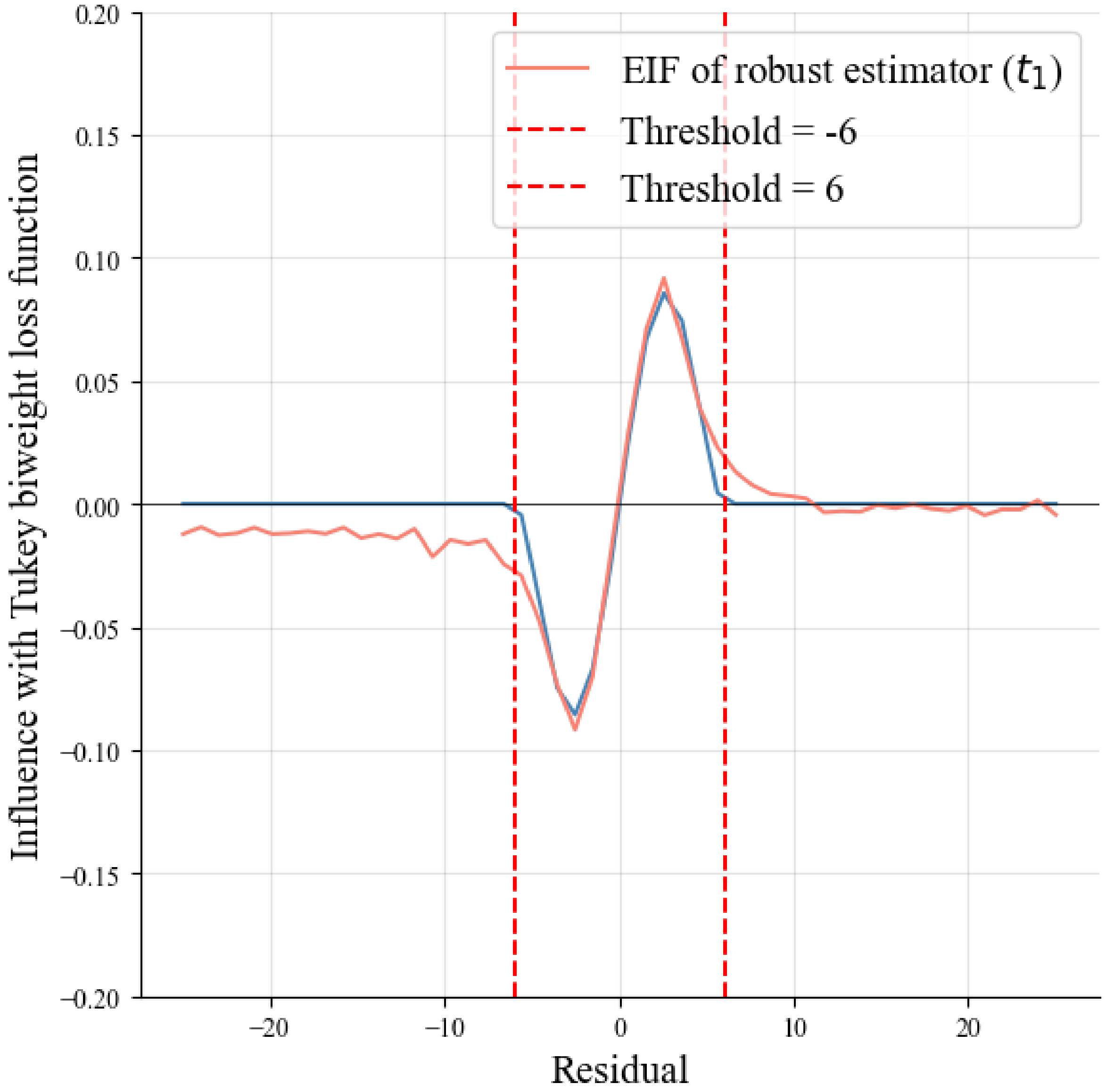


How many outliers during training?

Estimator		Regular outliers $Q_3 + 1.5 \cdot IQR$	%	Far outliers $Q_3 + 3 \cdot IQR$	%
Robust	normal + 10% t_1	2.828	2.585	4.949	1.269
	normal + 10% t_3	2.734	1.280	4.785	0.174
	normal + 10% t_5	2.719	1.008	4.758	0.051
Standard	normal	2.698	0.698	4.721	0.0002

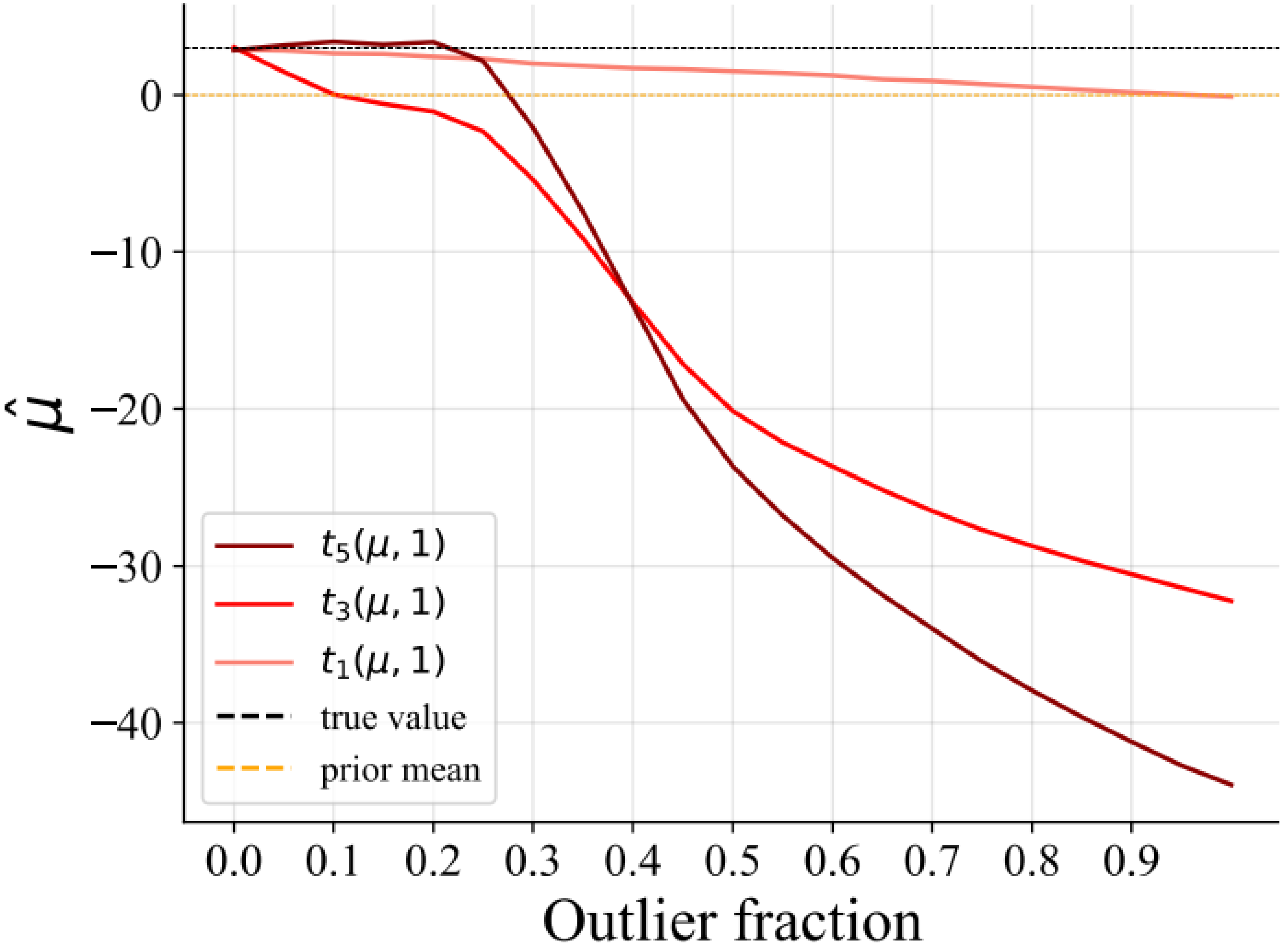
Coincidence

- Those (specific) neural networks works in a similar way as an M-estimator in traditional robust statistics.



Breakdown point ($x^c = -100$)

When there is no information, estimation goes to the 0, the prior mean.



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Robust Amortized Bayesian Inference of Cognitive Model Parameters

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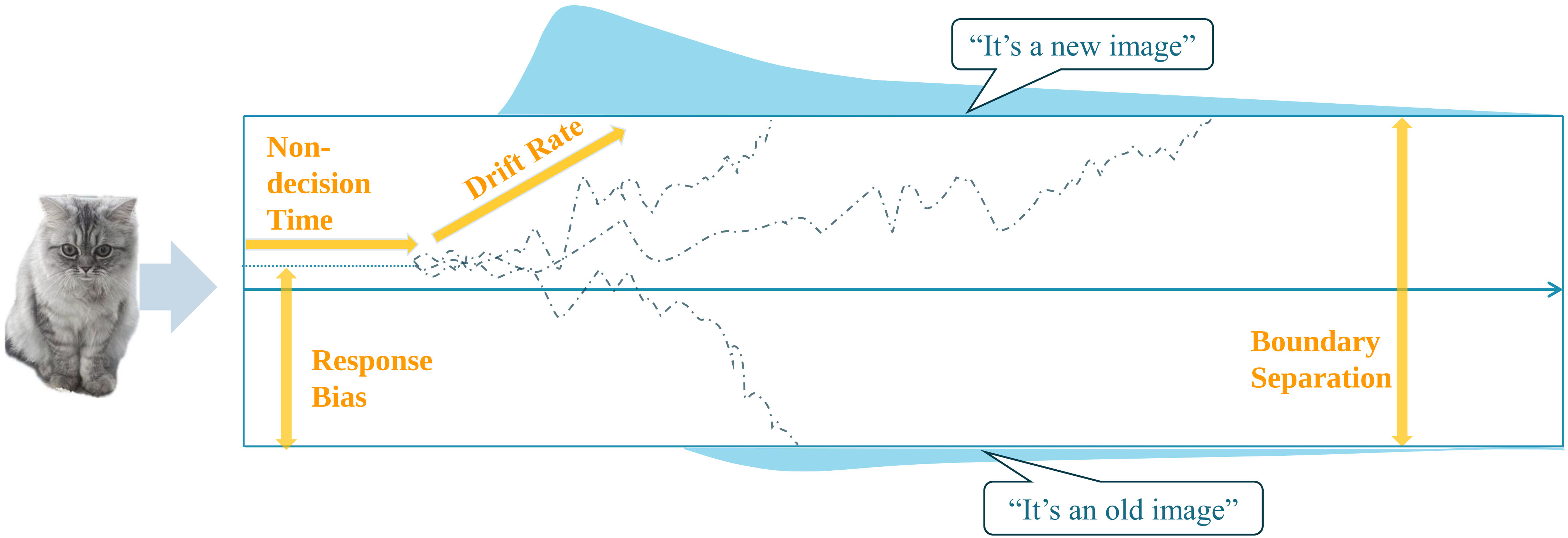
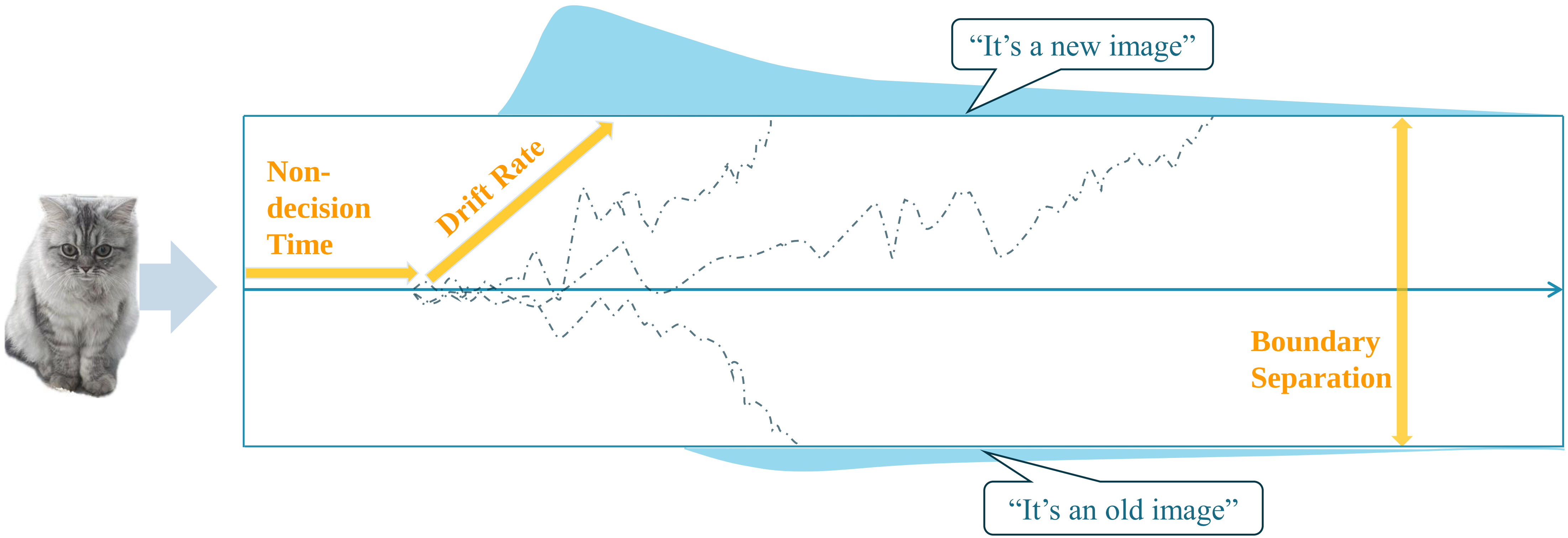
Drift diffusion model (DDM)

Condition	Response	Reaction Time
0	0	0.63
1	1	0.89
0	1	0.23
…	…	…



Drift diffusion model (EZ version)



Sufficient summary statistics

$$P_c = \frac{1}{1+\exp(h)}$$

$$M_{RT} = \left(\frac{a}{2v}\right)\frac{1-\exp(h)}{1+\exp(h)} + T_{er}$$

$$V_{RT} = \left(\frac{as^2}{2v^3}\right)\frac{2h\exp(h)-\exp(2h)+1}{[\exp(h)+1]^2}$$

$$h = -va/s^2$$

Sufficient summary statistics

- P_c, M_{RT}, V_{RT}

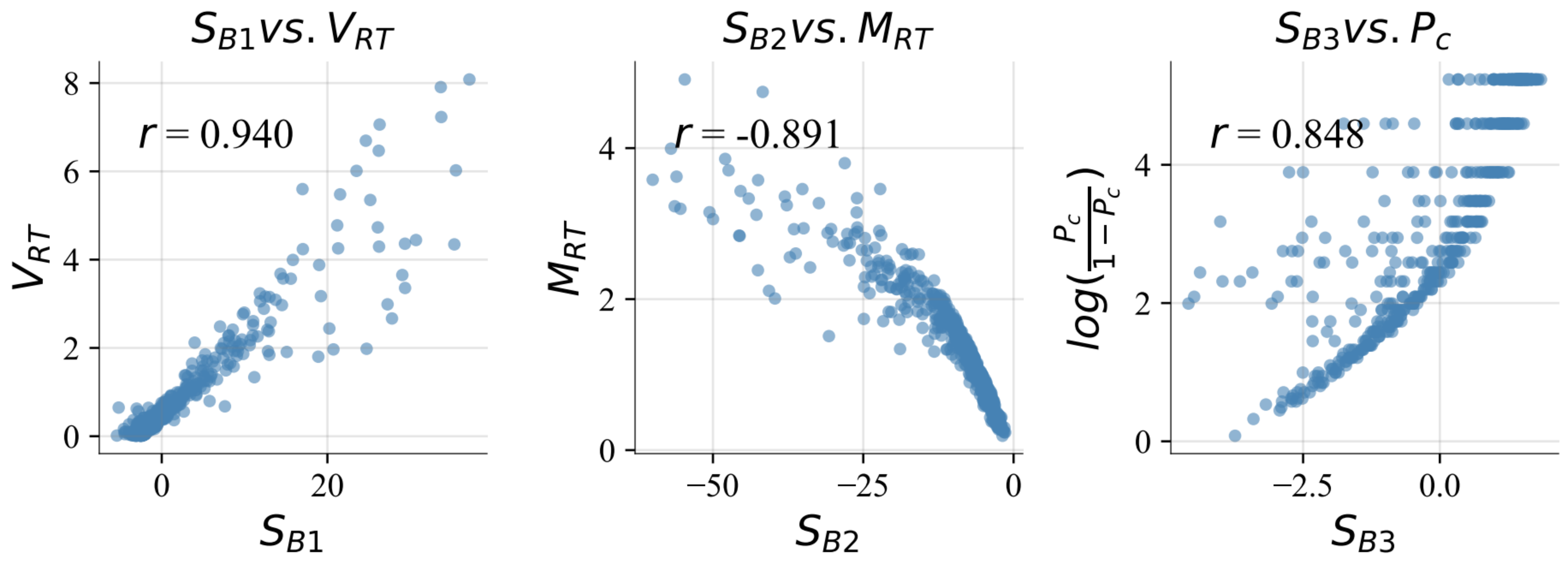
EZ version DDM estimator training

- Summary network: SetTransformer(summary_dim=3) (other settings stay the same as in bayesflow 1.1.4/ bayesflow stable-legacy branch)
- Inference network: Normalizingflow (Couplingflow)

Learned summary statistics

- S_{B1}, S_{B2}, S_{B3}

Sufficient vs. learned summary statistics



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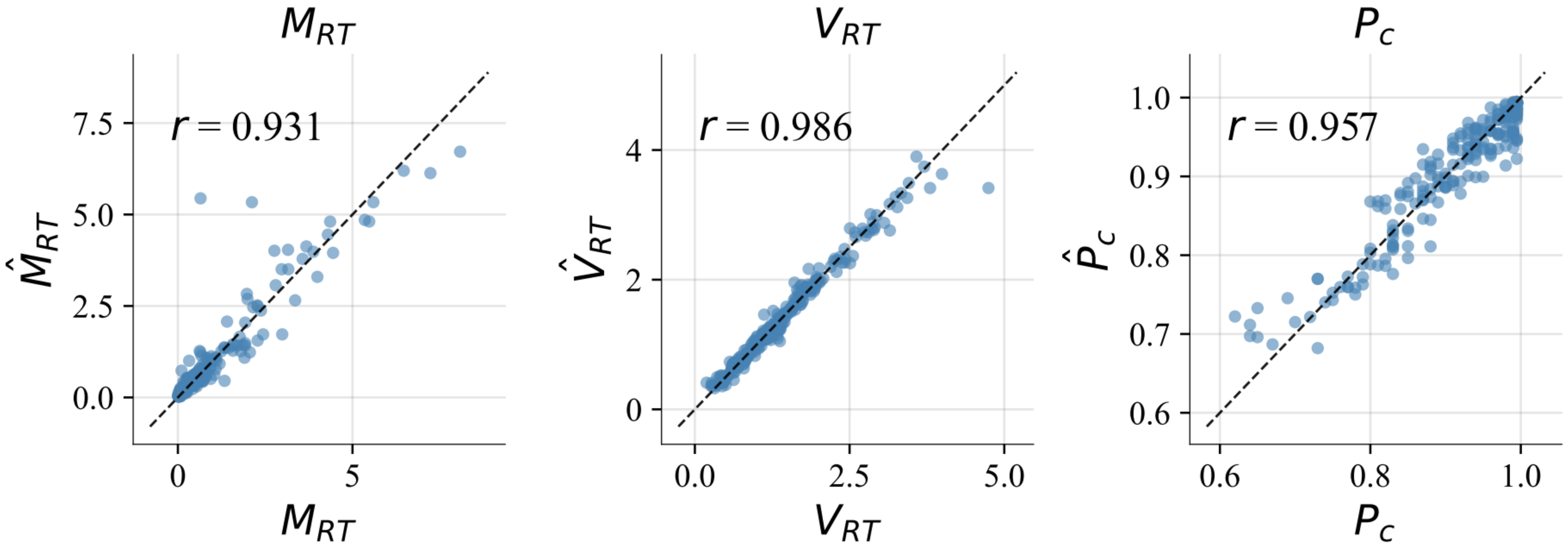
Sufficient vs. learned summary statistics

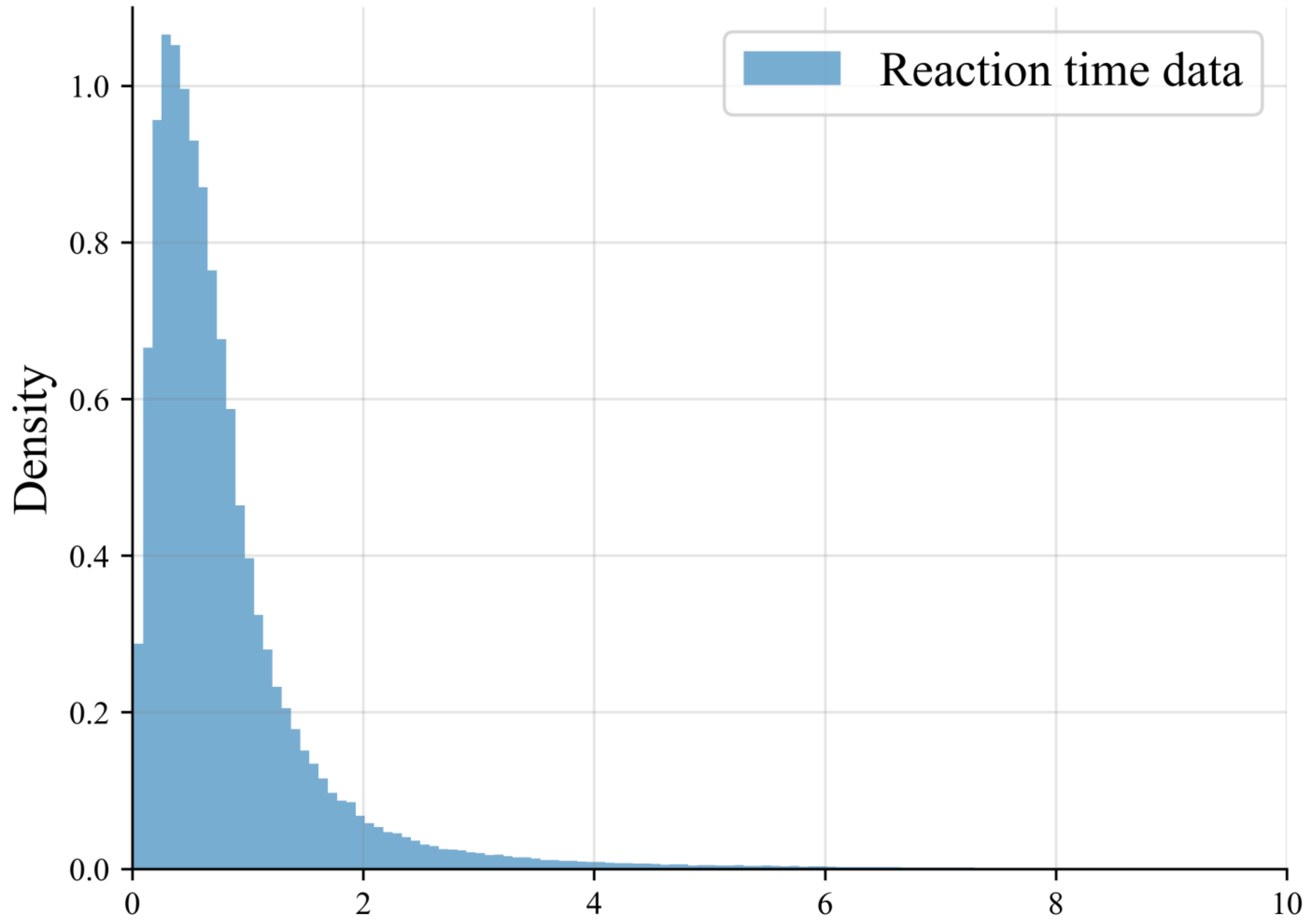


Sensitive to outliers by nature

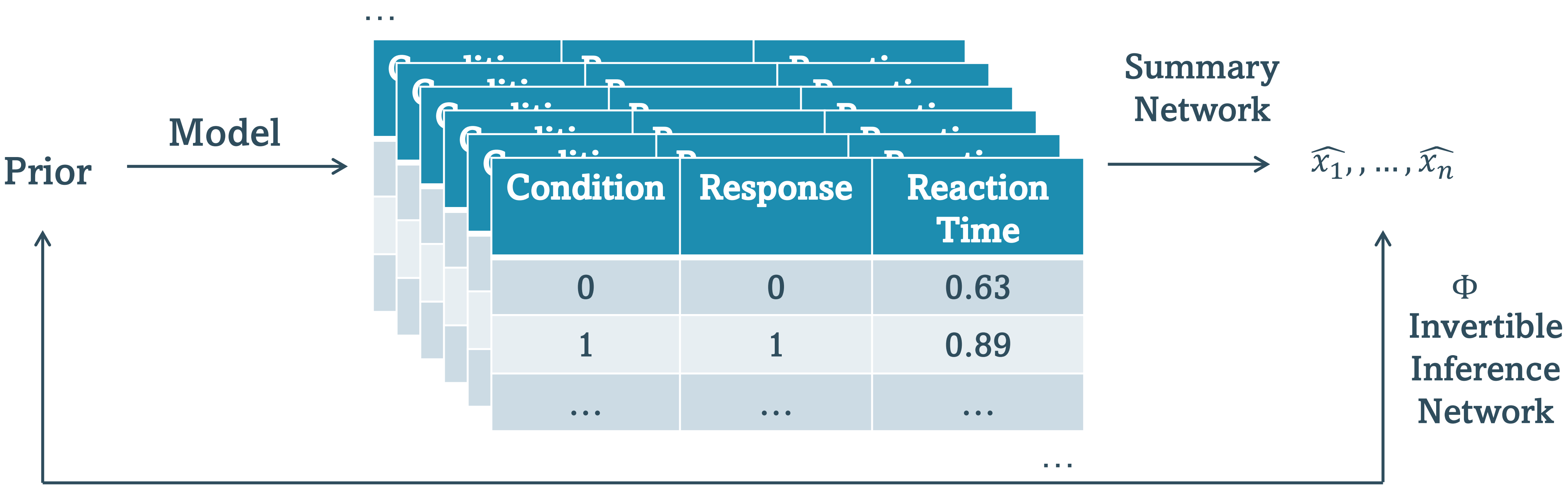
- non-decision time is shorter than the shortest reaction time

Simulation

$$(rt, r) \sim Wiener(v, a, z, T_{er})$$



Standard estimator training



Condition	Response	Reaction Time
0	0	0.63
1	1	0.89
…	…	…

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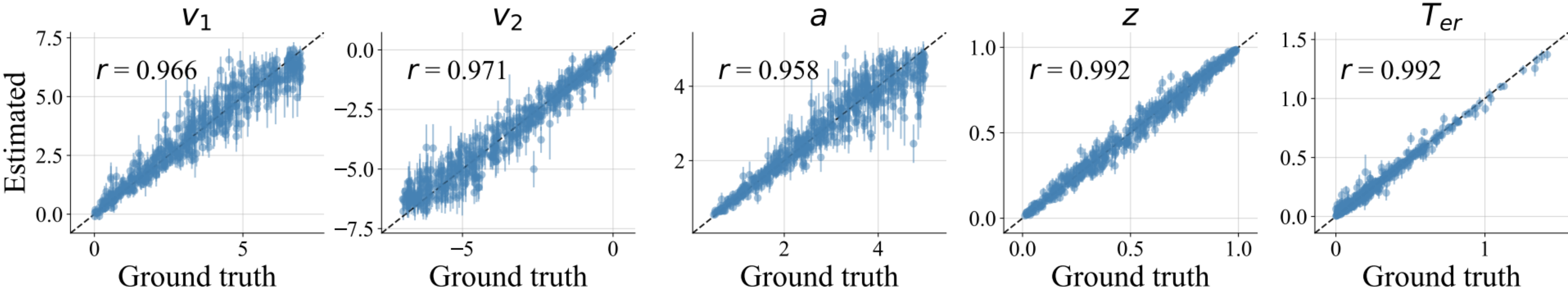
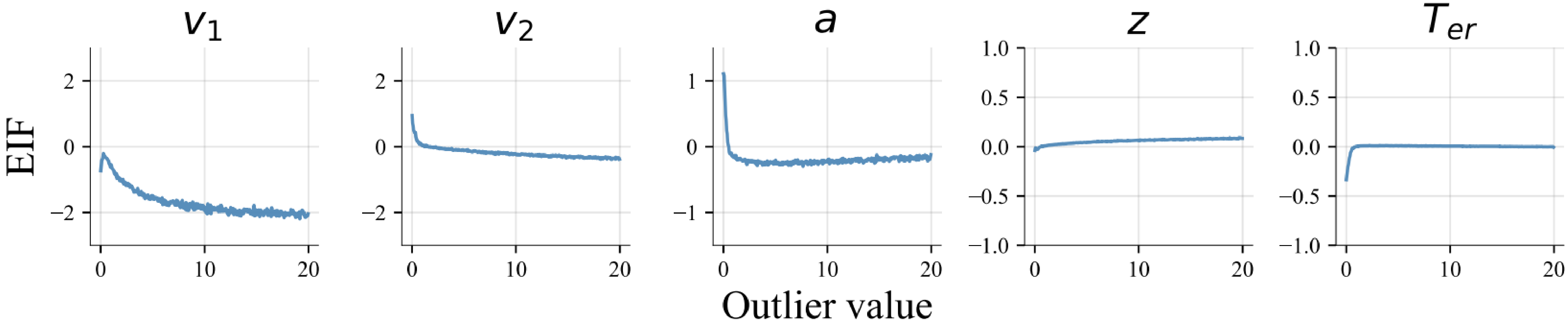
Parameter recovery



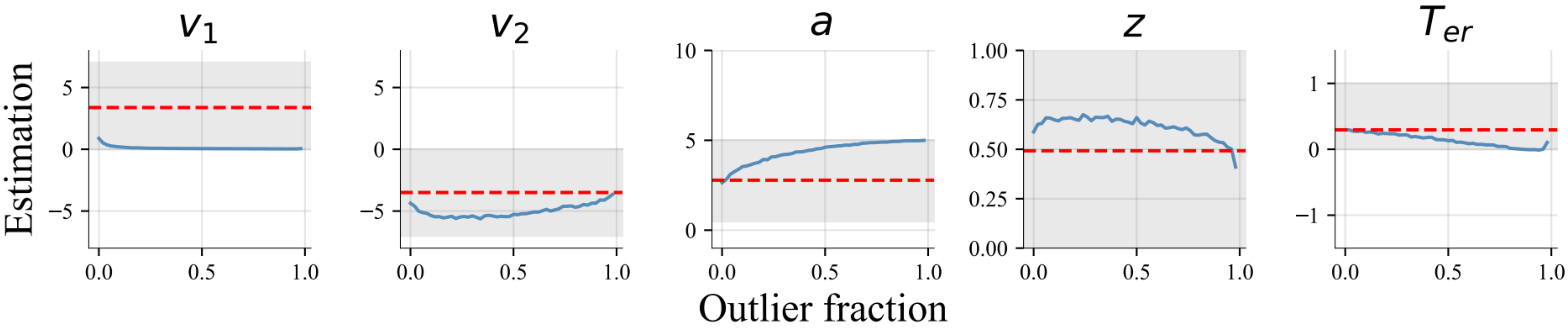
Empirical influence function



Empirical influence function

- Short outlier:
  - Underestimated non-decision time
  - Overestimated time for evidence accumulation process in both conditions
  - Reflected on the rising boundary separation
- Long outlier:
  - Overestimated time for evidence accumulation process in condition 1
  - Drift rate in condition 1 is underestimated

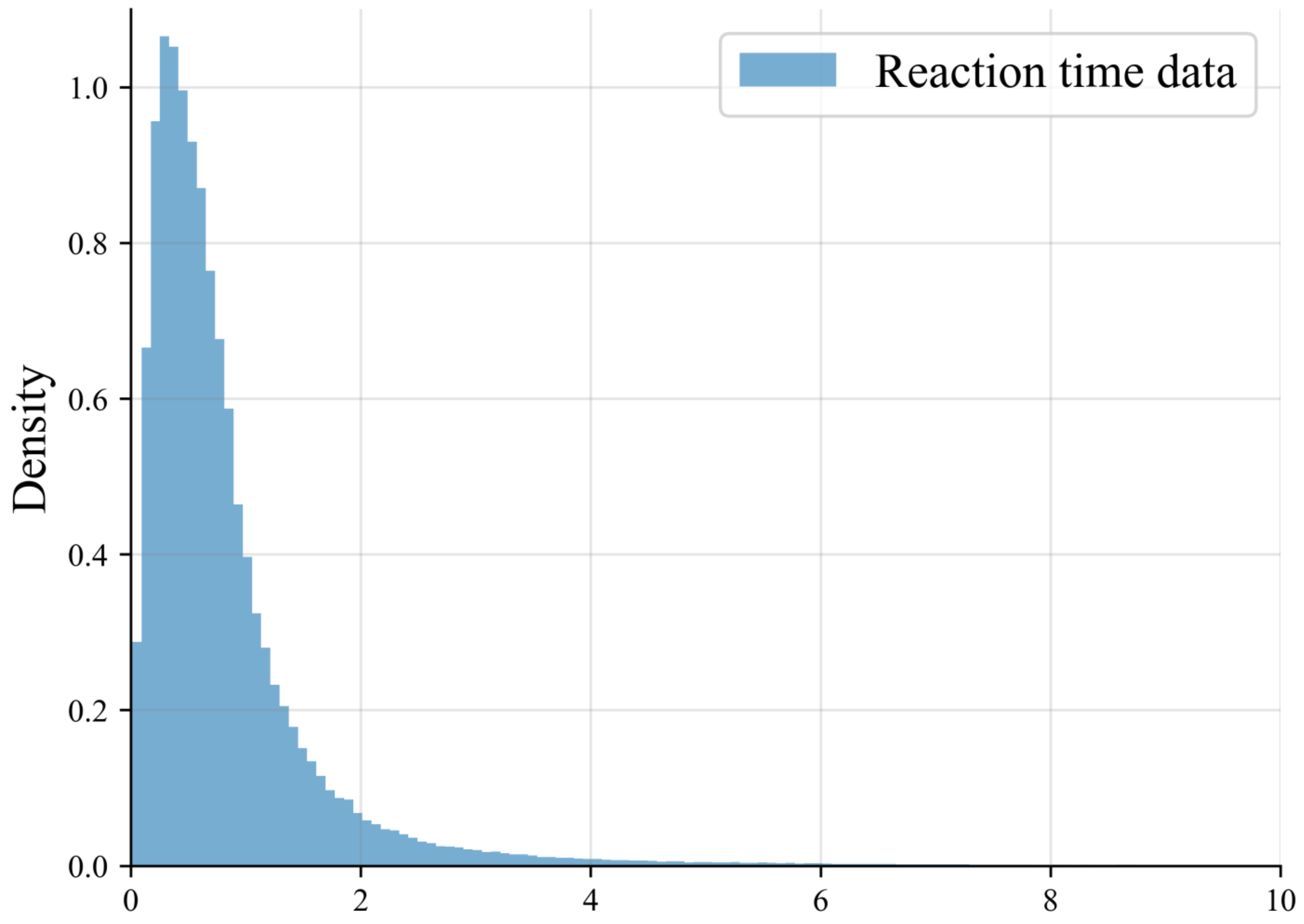
Breakdown point ($x^c = 20s$)



Simulation

$$(rt, r) \sim Wiener(v, a, z, T_{er})$$

$$(rt^c, r^c) \sim \begin{cases} Wiener(v, a, z, T_{er}), & 1 - \pi = 0.9 \\ contamination\ distribution & \pi = 0.1 \end{cases}$$



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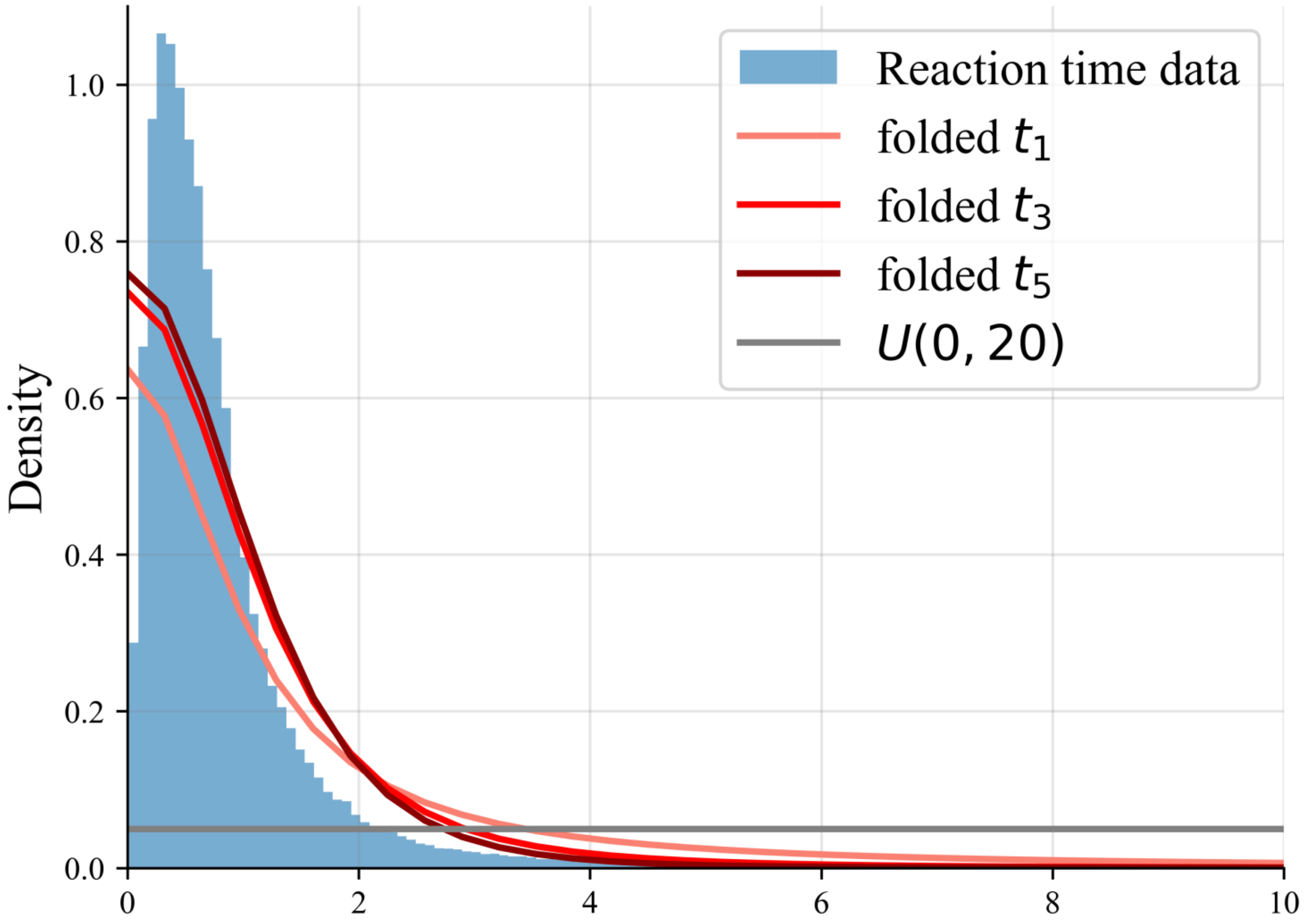
Assuming outliers in simulation

$$rt^c \sim \begin{cases} folded\ t_1 \\ folded\ t_3 \\ folded\ t_5 \\ U(0,20) \end{cases}$$

$$r^c \sim Bern(0.5)$$



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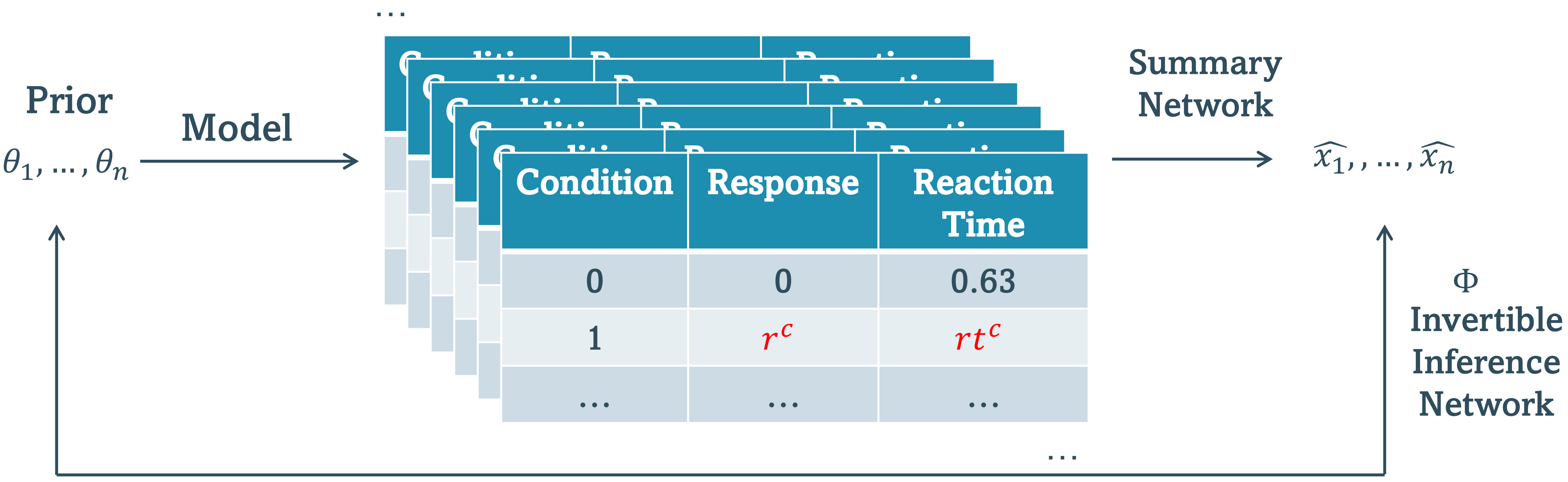
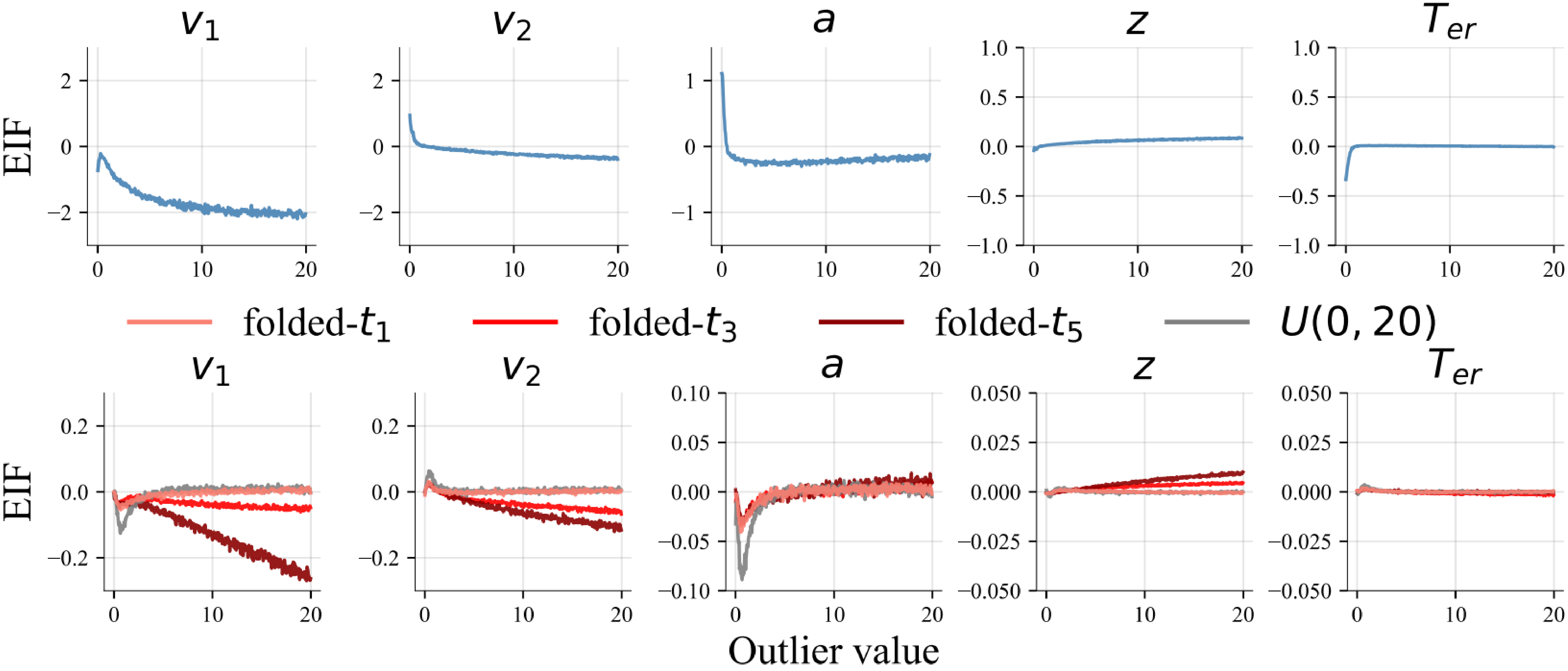
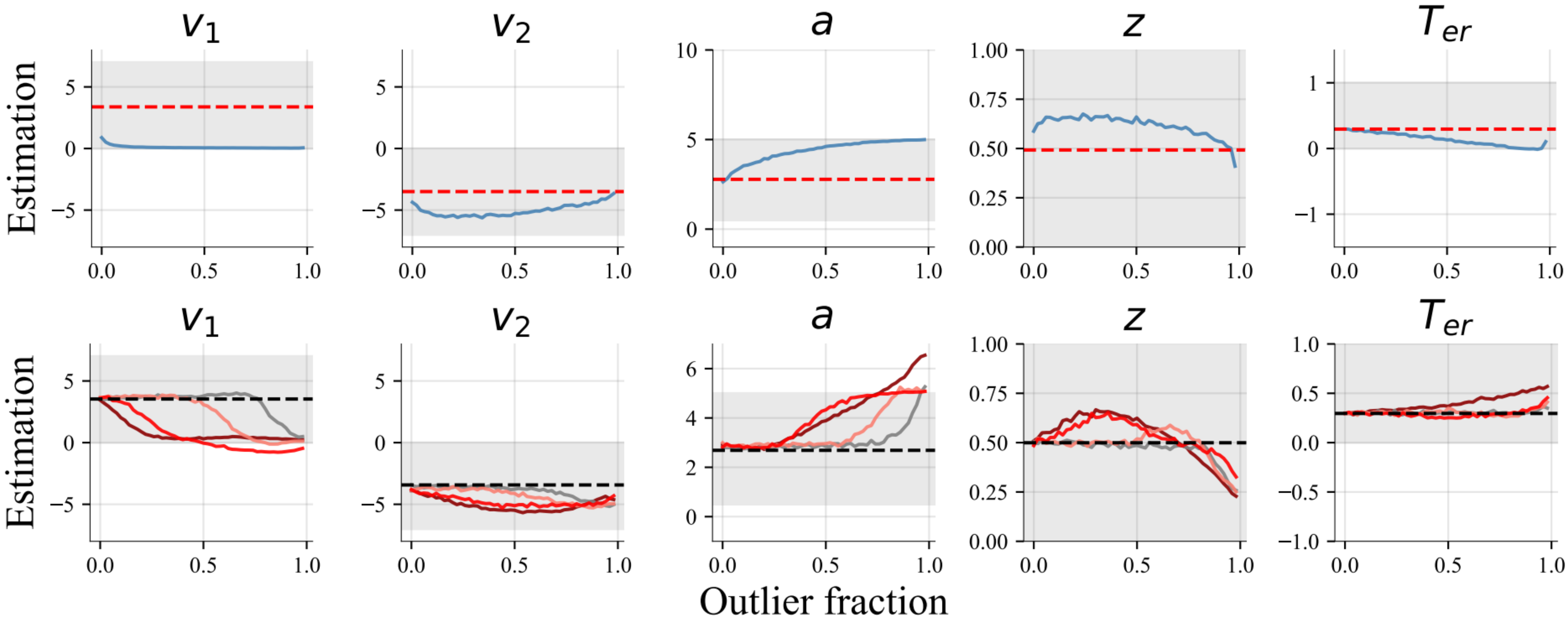
Robust estimator training



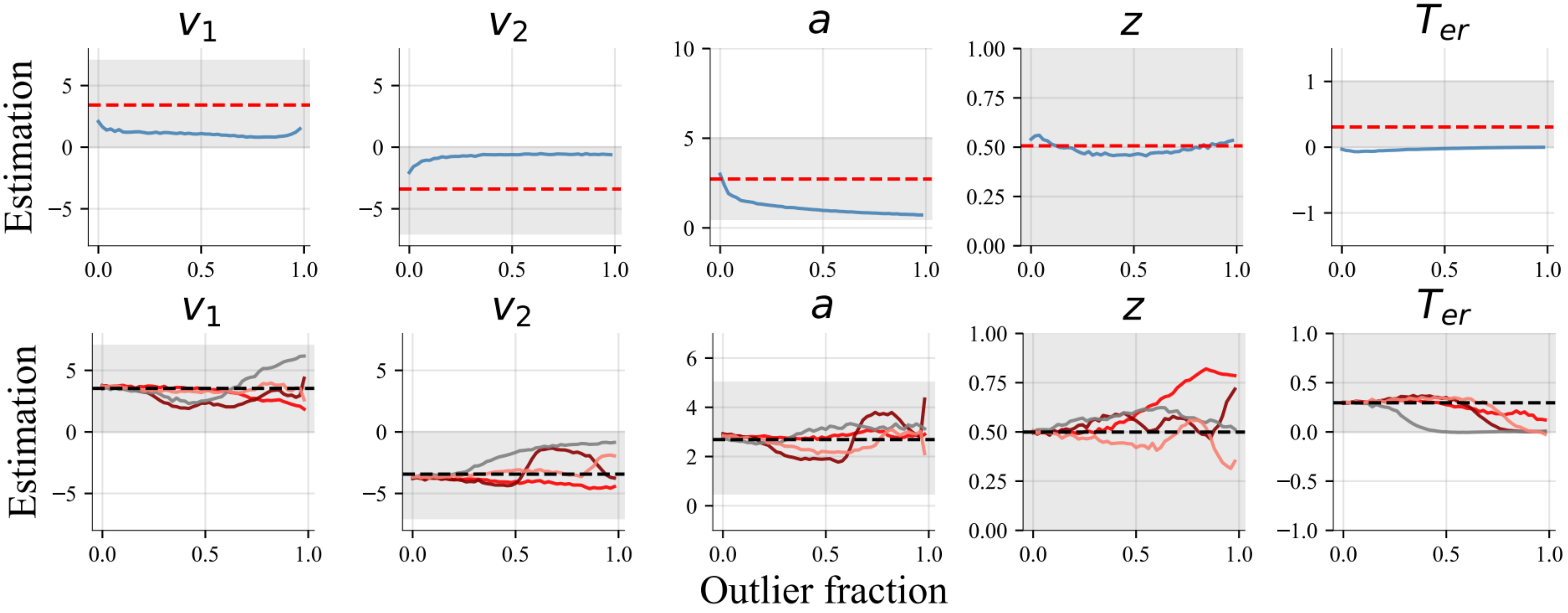
Empirical influence function



Breakdown point ($x^c = 20s$)



Breakdown point ($x^c = 0.01s$)



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The cost of robustness

$$\text{MAE Ratio}_j^{S,R} = \frac{1}{B}\sum_{b=1}^{B}\frac{|\hat{\theta}_j^S(\boldsymbol{x}_b) - \theta_{j(b)}|}{|\hat{\theta}_j^R(\boldsymbol{x}_b) - \theta_{j(b)}|},$$

Robust estimator	v_1	v_2	a	z	T_{er}
t_1	0.76	0.79	0.769	0.88	0.792
t_3	0.731	0.722	0.728	0.865	0.764
t_5	0.728	0.74	0.734	0.862	0.733
$U(0,20)$	0.876	0.894	0.907	0.936	0.848

$$\text{Posterior Variance Ratio}_j^{S,R} = \frac{1}{B}\sum_{b=1}^{B}\frac{\text{var}(\hat{\theta}_j^S \mid \boldsymbol{x}_b)}{\text{var}(\hat{\theta}_j^R \mid \boldsymbol{x}_b)},$$

Robust estimator	v_1	v_2	a	z	T_{er}
t_1	0.613	0.596	0.621	0.732	0.571
t_3	0.637	0.569	0.634	0.694	0.581
t_5	0.618	0.569	0.672	0.713	0.568
$U(0,20)$	0.723	0.672	0.723	0.756	0.636

Summary

- 1. Amortized Bayesian inference works:
  - It makes accurate inference through learning the mapping between data and $P(\theta|x)$
- 2. Amortized Bayesian inference can be robust:
  - Assuming the presence of outliers in simulation process increases the robustness of amortized Bayesian inference
- 3. The robustness depends on the contamination distribution:
  - Different contamination distributions have different effect on robustness
- 4. In our given example, the estimator with t_1 performs the best.

Thanks!