Metric-Free Flocking

Motivation

To depict how birds coordinate movement, I simulate a 1D Topological Vicsek model.

On the basic level, I think Vicsek model contains many learning opportunities if I work through the simulation and analysis. Topological distance is interesting because it is more realistic and has scale-free properties compared with the metric distance.

On the interest level, I see that it can look at interesting questions about how order and/or diversity come out as properties of the group, individual, environment.

RESULTS

Here is the cleaned up python code, and all helper functions for plotting its dynamics, confidence intervals, and networks used in the report.

Google Colaboratory

https://colab.research.google.com/drive/1v2qTM28w7Zv9qjCKAk8cvyFZ11A0tvXz#scrollTo=1uQI_O25alFm

Here are the main figures.

- First, I set up some interaction network visualizations to depict clustering and order-disorder transition.

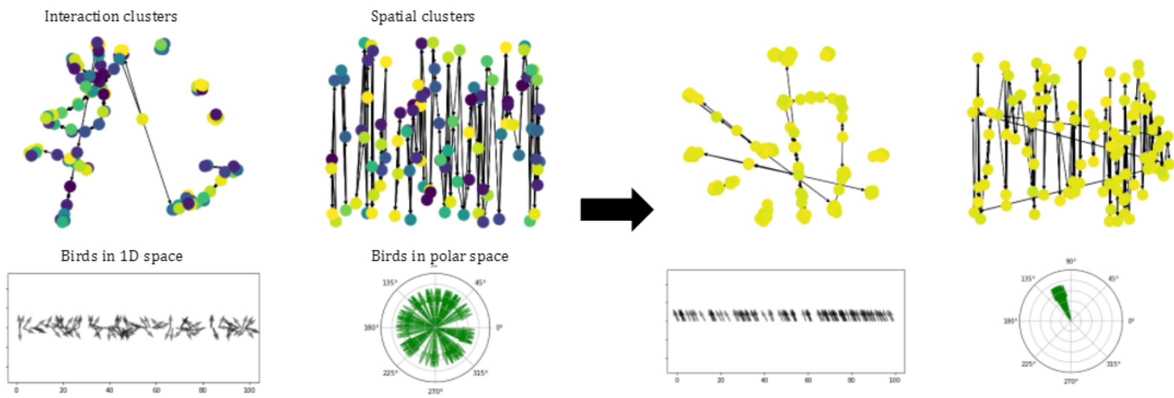
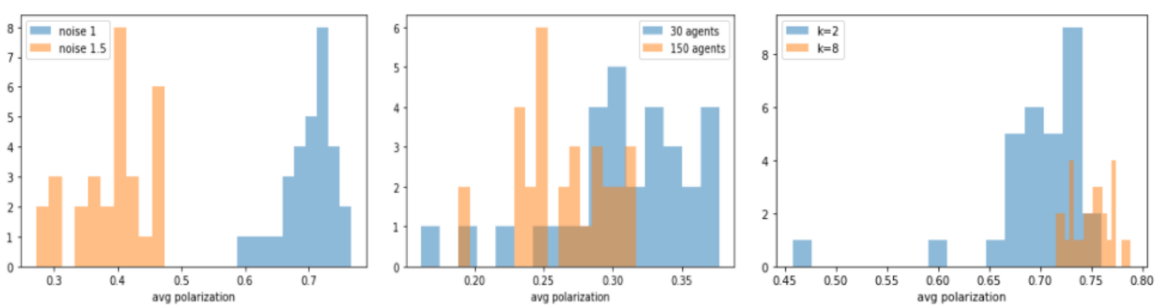


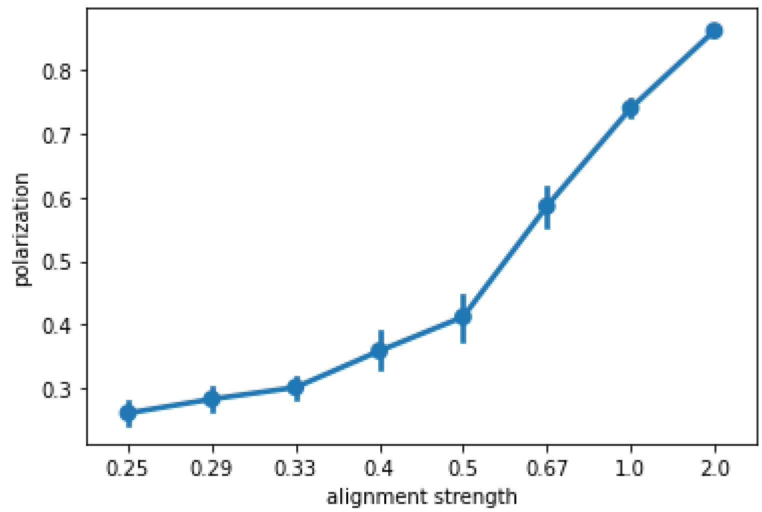
Figure 1. State visualization: from disorder to order

- Second, I tuned single parameters and observed order in polarization.



η=1: [0.686, 0.717]	N=30: [0.252, 0.275]	k=2: [0.695, 0.715]
η=1.5: [0.370, 0.412]	N=150: [0.251, 0.281]	k=8: [0.742, 0.757]
P-value: 5.730e-32	P-value: 2.178e-4	P-value: 3.652e-6

- Alignment strength [0,2]



Params: (N=100,tau=tau,k=3, noise=1,simtime=60)

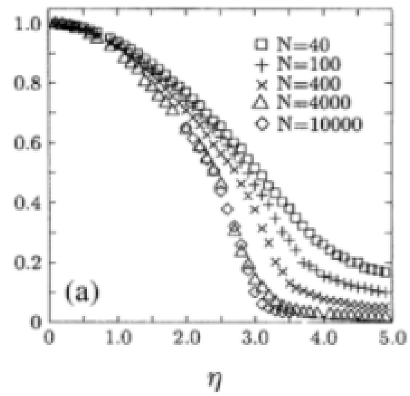
- Density and noise plots are compared to the metric Vicsek (1994) plots.

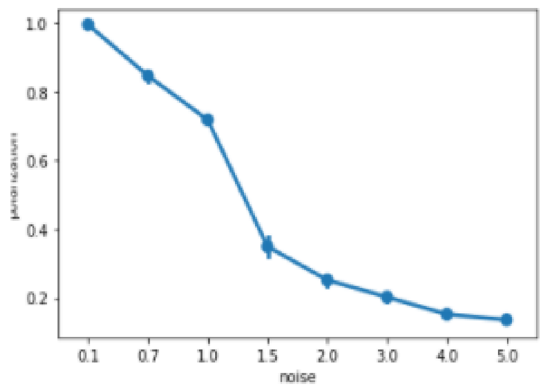
  The density plot looks wonkier. Maybe I should give it more trials or intervals.

Vicsek Metric Vicsek Finding

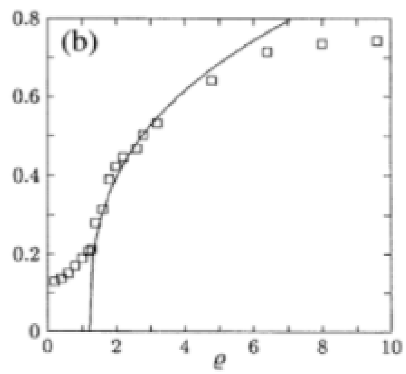
My Topological Viscek Finding

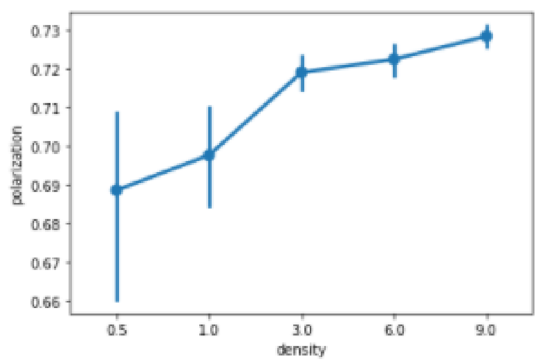
Polarization vs Noise

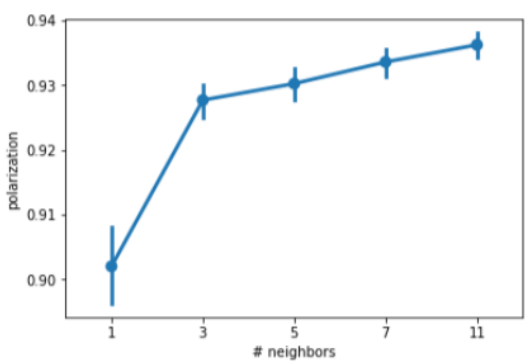


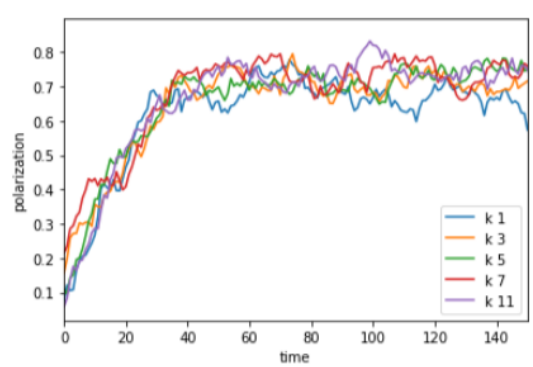


Polarization vs Density



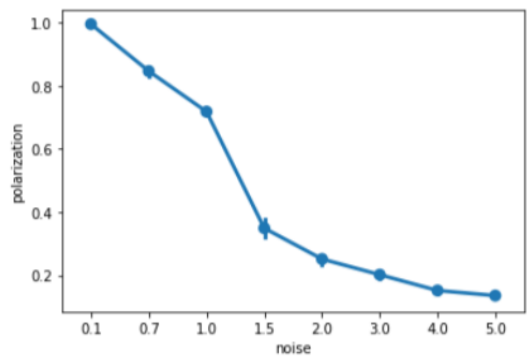


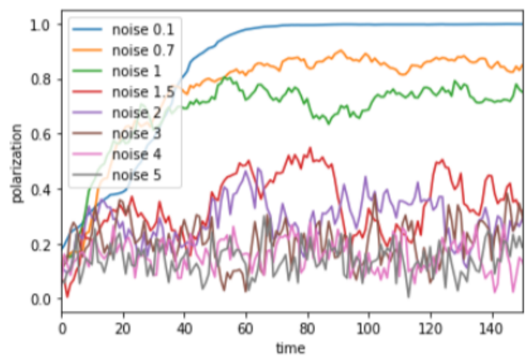




Params: (N=100,tau=tau,k=3, noise=1,simtime=60)

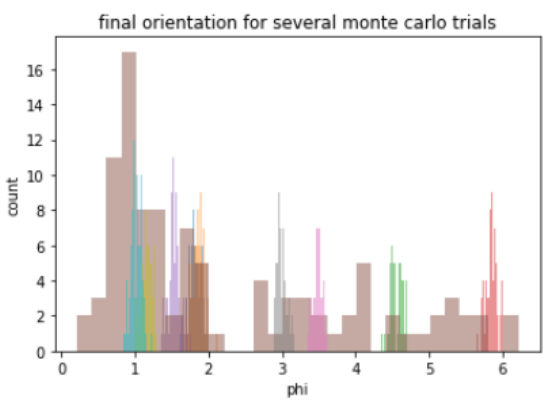
▼ K neighbors plot is new.

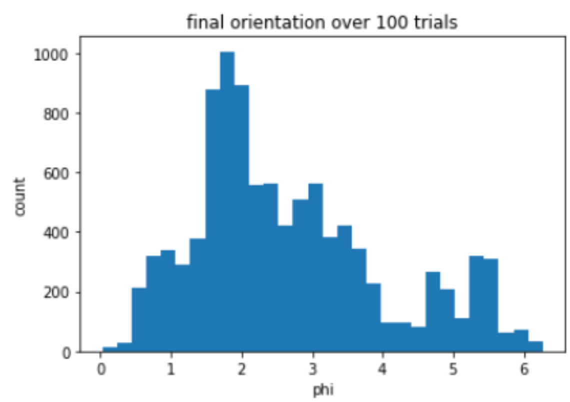




Params: (N=100,tau=1,k=2,simtime=100,noise=n)

▼ I don’t see a preferred direction yet.





▼ The questions that I am interested to address after new years

1. how common are cluster merging? (I only saw it once, in a small system where local effects easily overtake the global order).
2. why are some metric-free networks more likely to bunch than others? what differentiates kNN from vision rules?
3. what is similar and different between the alignment model for birds VS for liquid crystals?
4. I read about someone using PageRank (graph algorithm) to find foodweb hub, and using ODE system parameter tuning to find the equilibrium that is cooperation. What is the equivalent for our system, that incorporates data?

5. when is it appropriate to study with simulation modeling and when some other technique? for example, differential equations, data collection, qualitative descriptive, normative critique of model assumptions.
6. how do you define individuality in flocking? is the flock an individual?

LONG REPORT

Modeling

Rules

While minimal, the Vicsek model can already play out many interesting phenomena to do with flocking, or more generally (because the model's makers seem to like universal patterns), particle-like things that move by absorbing energy and interacting with surroundings.

The model has N particles in a 1D space of size L. The j-th particle moves at a constant speed v0 and an angle ϕ with the x-axis at time t according to the continuous dynamics:

$$\vec{v}_j(t) = v_0 \begin{bmatrix} \cos\phi_j(t) \\ \sin\phi_j(t) \end{bmatrix}$$

At every time increment dt, the angle ϕ updates towards its k-nearest neighbors, with some noise:

$$\frac{d\phi}{dt} = \frac{1}{k\tau} \sum_{i=1}^{k} (\phi_i - \phi_j) + \epsilon\sigma$$

Where ϵ is a Gaussian noise, σ is a scaling factor, and τ is the relaxation time. Relaxation time is how long it takes to reach a stationary state. (the steady state is not stable in this far-from-equilibrium system.) Tau determines the time step-size in the simulation. The bigger tau is, the larger each step in dt, which may lead to larger error and/or faster convergence.

Assumptions

A minimal model means it has many conditions that are sometimes unrealistically simple.

1. Agents have velocity vectors with a angle, but are constrained to move along only the x-axis. One less degree of freedom means there are less possibilities to

analyze, and more people have studied it so I can learn from others who have tools for mean-field, critical value prediction, so on. Though 1D is a very unrealistic, it's supposedly not that limiting on the results. All the interesting clustering still should happen on the projected axis, even if this mechanism is not THE one in reality. I need more evidence: to compare this 1/1.5D with 2D results more to understand if it affects some assumptions about having no preferred direction for alignment.

2. Noise has finite range and is normal. All the information from obstacles in the environment, reaction delay when turning directions, and other things competing for attention are summarized into this random term. This is what makes the model a Monte Carlo simulation, where each trial gives a different result. This additive noise might not fit situations where the current heading interacts with noise like blindspots or a rough storm with many obstacles.
3. An agent takes social cue from its k-nearest neighbors rather than from any agent within a certain radius. This topological neighborhood is closer to nature than the standard metric neighborhood. A study claimed that a typical starling significantly interacts with its 7 or 8 closest neighbors, located at somewhat well-defined angular positions, but irrespective of actual distance. It would be interesting to see how more realistic change to a metric-free constraint on vision impacts the model dynamics.

Parameters

Relaxation time: the time interval for numerically solving the problem which is different from the simulation time because the convergence speed depends on size of system. Held constant at 1 to compare with Vicsek.

Density: the proportion of agent count to space length. Held constant at 1 (100 agents / 100 length) unless it's the tuning knob.

Noise: the size of influence by environment and reaction delay on the orientation update. Held constant at 1 (radian) unless it's the tuning knob.

Speed: the internally propelled agents are only angularly accelerating. Held constant at1 to only change direction for velocity and observe angular change (Imagine a polar plot where r is fixed and theta move).

Number of neighbors: how many neighbors influence this neighbor? Held constant at 3 (in proportion to 100 agents) unless it's the tuning knob.

Measurements

We want to observe Polarization. It describes alignment of agent directions and is the summary statistic over the microscopic variable orientations ϕ. It's calculated as $|u| = \sqrt{\langle\cos\phi\rangle^2 + \langle\sin\phi\rangle^2}$. I didn't understand why we want to look at this instead of average alignment until I saw that when you have two vectors $\pi/4$ and $7\pi/4$, the intuitive mean should be 0 but the average would be π which is reflected. This is a nice metric because it's normalized where |u|=0 means birds are pointing in every direction, and |u|=1 is when all birds fly in single direction. It doesn't matter which exact angle, so long as the direction variance is low. We can observe the transition from disorder to order in this parameter, which means there is a sharp jump from random headings to alignment.

Code

The structure of the code is a simulation class, with a KD tree for the kNN updates and dictionaries to store the N^t arrays of orientation and position data, along with several plotting helper functions.

I make snapshots from various plot-types to check if the code produces the expected behavior. Nothing beats animations so I will make some next time, especially for the dynamics of clusters on the interaction network which is an interesting challenge in 1D.

Periodic kNN:

- At each time-step, each agent finds its neighbors for update. A KD-tree speeds up the brute-force version for searching k neighbors in the N position array from O(kN) to O(k logN). I need to check if the algorithm is finding the true neighbors, especially if the nearest neighbor is wrapped to the other end of the space. I make sure the x-axis is wrapped by padding mirror images during search, and I bound the angle in -2pi,2pi by applying the conditions after each update (I can try a continuous NumPy roll next time for speed.)

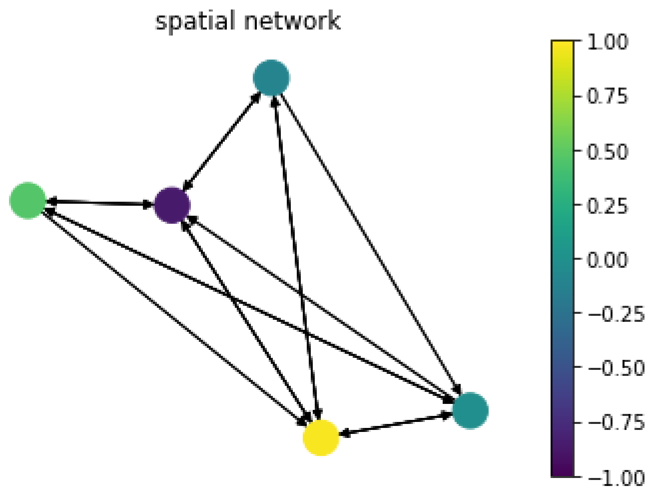


Figure 4. periodic spatial boundary

- The figure (4) network shows neighboring as arrows, orientation as color, with the real x-position and a jittered y-position (in the model, all birds are at the same y-position throughout). I preset the position of a 5 agent system such that the right-most bird has a neighbor on the left-most. Since right-most does point to left-most even though their x-value are far, periodic boundary is working.

Orientation update function:

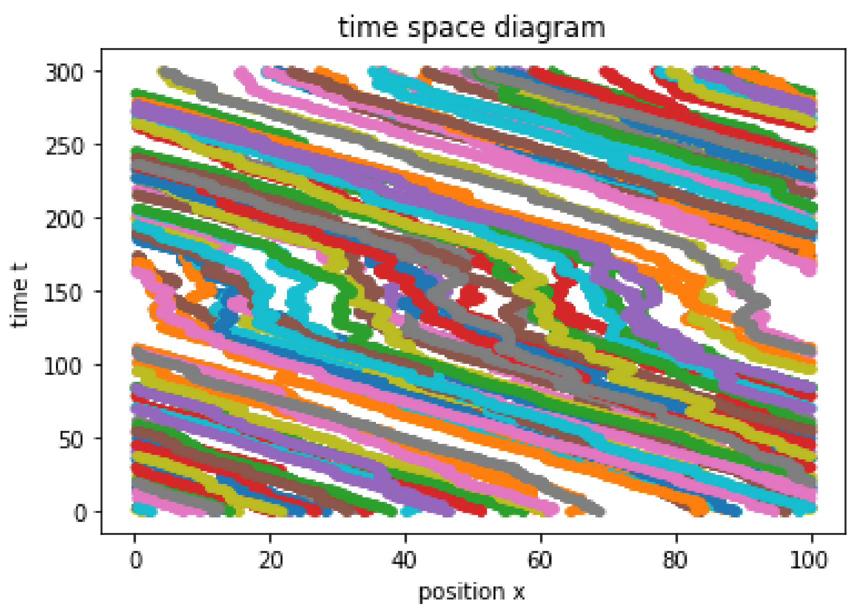
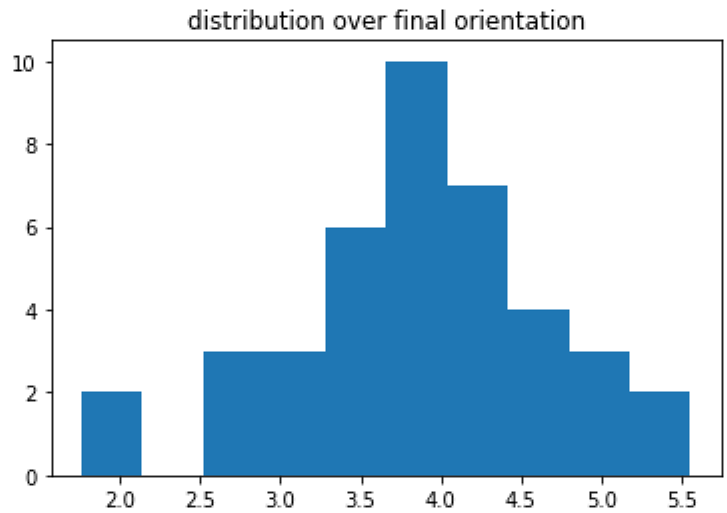
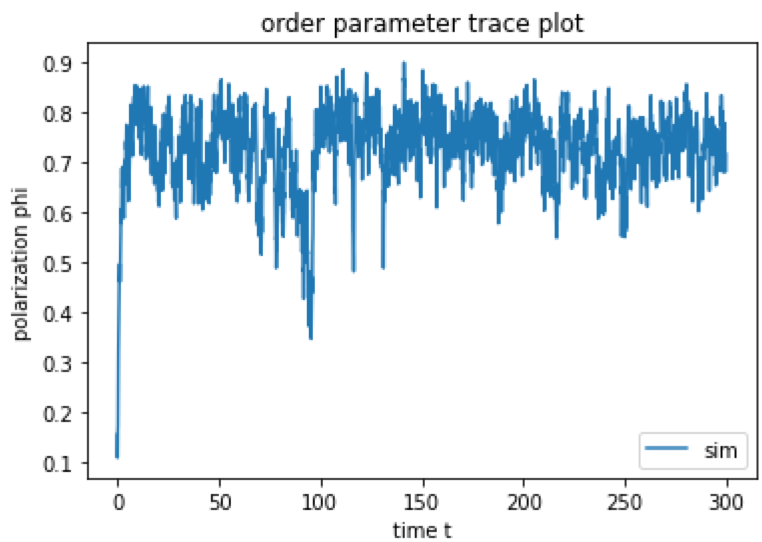


Figure 5 (A)
N=40, k=3, noise=1, L=100, sim time = 100



5(B)



5(C)

- I chose to store the agent data as arrays during simulation, then output a dictionary indexed by time and agent ID. The important thing is to know if the interesting thing — order — happens in our simulation within reasonable computing time. The simplest way to check if social force is working is a time-space diagram (5A). While initial heading was messy, the trajectories become parallel with some wiggling from noise. (5B) shows the distribution over orientations at t=100. (5C) shows the order parameter polarization $|u| = \sqrt{\langle \cos \phi \rangle^2 + \langle \sin \phi \rangle^2}$ which comes from the spread of orientation. The plateau is the stationary state where the flock settles into some structure. Notice that it's oscillating around a value below 1, which means there are some disagreement.

Spatial network and interaction network:

- We haven't studied much about networks in space, where edges represent distance and the metrics for degree and community are complicated.

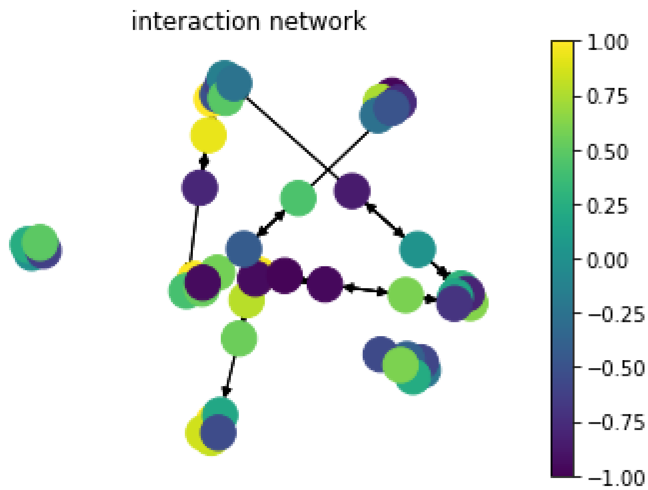
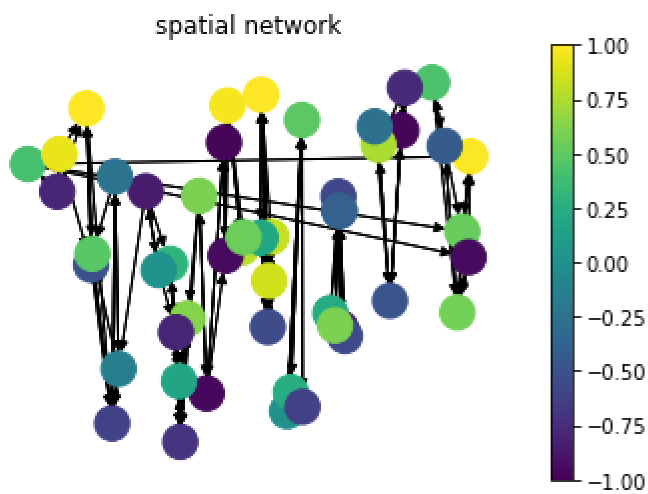
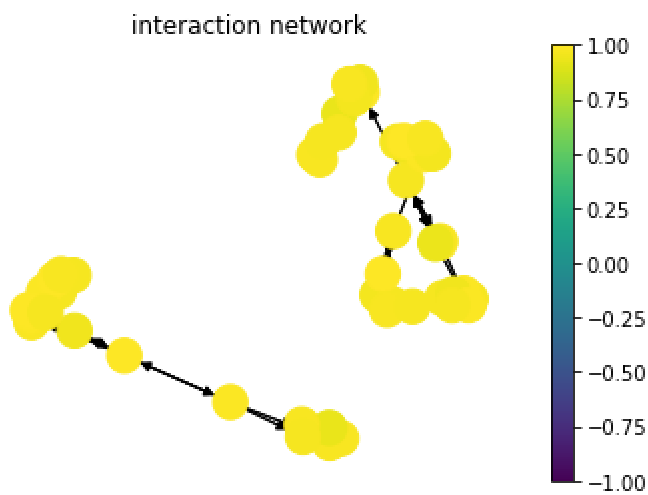


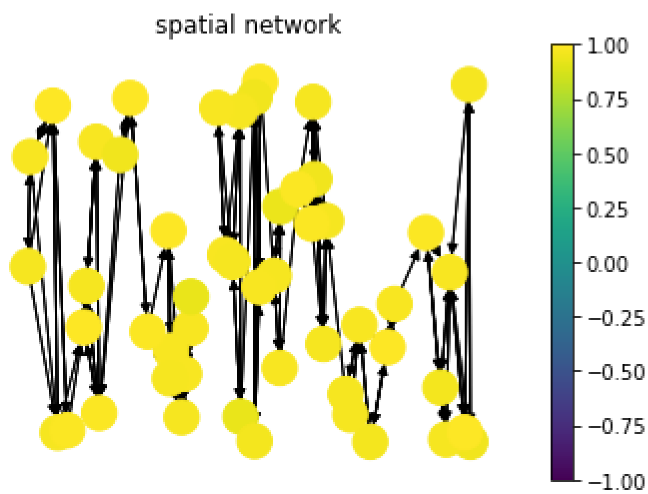
Figure 6 (A) t=0, N=50, k=3, noise=1



6(B)



6(C) t=100



6(D)

- I draw the spatial network by keeping their real x-position, jittering their y-position, then connecting directed neighbors. I draw the interaction network by assigning distance to edges of neighbors and set a spring layout to check for clusters that might look like chains instead of balls in space. Figure 6AB show t=0 and CD show t=100. I see in the interaction network that the orientation evolves to all one angle, and two groups form where they neighbor in-group exclusively. I see in the spatial network that the links between far birds evolve into only edges with close birds. (vertical lines mean they are near each other; y-distance is artificial.)

<u>Initial orientation distribution:</u>

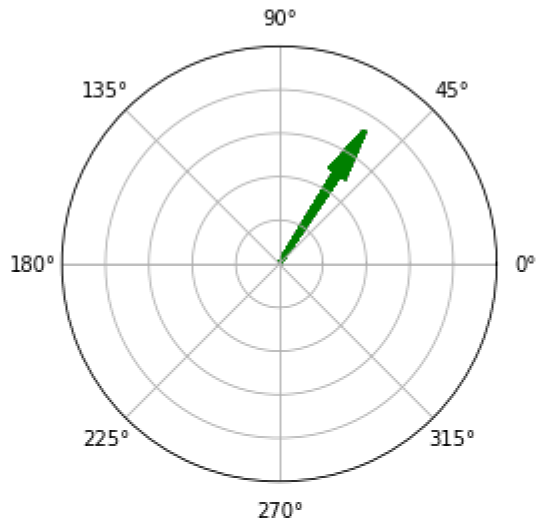
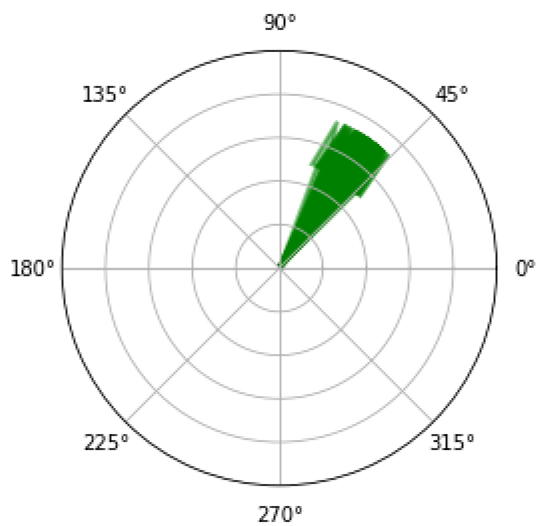


Figure 7 (A) t=0, |u|_{t0} = 1



7(B) t=100, |u|_{tf} < 1

- A confounding factor to time and kind of convergence besides relaxation time & density & noise is what the birds start out as. In figure 7, I initialize a flock that all point to one direction at t=0. At t=100, the polarization actually lowers because of fluctuations, looking like a blur, a dispersion. This is good because it shows some resistance to the initial condition and comes up with perhaps a common pattern.

Empirical analysis

I care about what we can control and how that varies with confounders that we can't change in the real scenario. I will assume that I can control attention but not noise or density. To look into that interaction, I liked the exploration approach from a math problem on the Lorenz equations. I want to apply it to study our model. The steps:

1. Vary an input. Compare the trace and the distribution of the output.
2. Are there attractors (stable states)? At what parameter value does the solution change to one or more different solutions? Does the distance between solutions with perturbed initial conditions diverge over time?

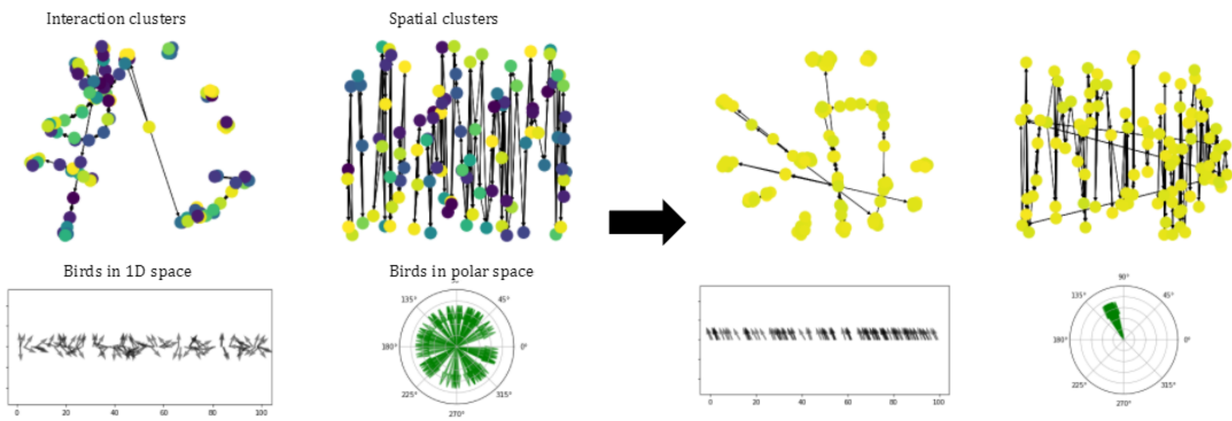


Figure 1. State visualization: from disorder to order

I want to first share the interesting phenomena of the emergence of order. In Figure 1 is the monitor screen for the flock. We can see especially from the polar plot that from t=0 to t=100, a totally random group with very low polarization turned into nearly a unison. There seems to be some attractor that many systems fall into at polarization ~ 1. We might be able to slice with the dynamics in 5(C) and see the transition occur in just over 20 steps, as the neighbors assimilate in color and attract nearby agents to join and mix. Even when we vary parameters like k (in figure 3(A)) the solutions still converge to |u|~1. Perhaps the initial nudges dissipate quickly through some kind of negative feedback mechanism. There are regions in the phase space where it converge to a different alignment though, see finer investigation below (especially figure 10).

Now we can go into the impacts of parameters.

<u>Strategies</u>: I separately varied the parameters for noise, density and number of neighbors. We can assume that the noise represents the messiness of obstacles and the density represents the size of the environment. We want to decide on how many

neighbors should be attended to, because there might be attention trade-offs in focus and sensitivity. I am tuning these background conditions to build up to the control parameter alignment strength $g = \frac{1}{\tau\sigma^2}$ next time.

Outcome: I got the 95% confidence interval and distribution of the expected values of polarization in Figure 9. As a preliminary test on the Monte Carlo property, I checked the variation over trials of the same input. In Figure 8, we can see that orientation (which is the microscopic state that is summarized into the macroscopic state, polarization) settles at different angles in the stationary steps of each trial. When we gather all the angles, the final angle is not distributed evenly as we would expect in a 2D model (because there is no field that the birds prefer, the null result should be uniform final orientation). Maybe there is asymmetry coming from the projection to x-axis. We might increase the trial size and average over longer stationary steps in simulation time to be more sure.

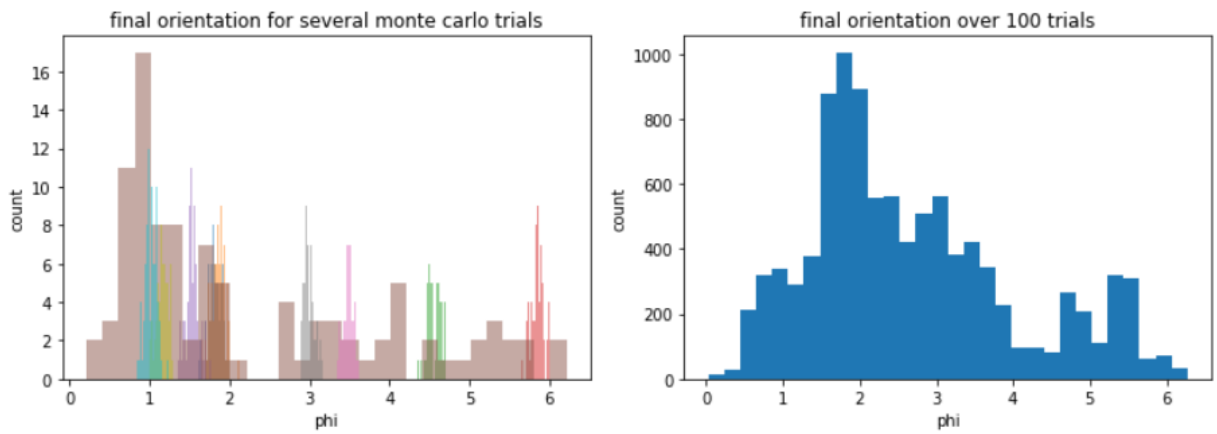
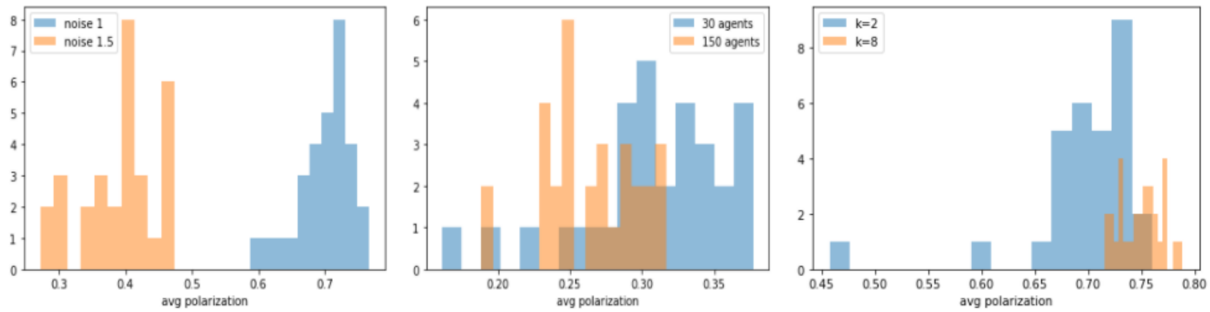


Figure 8. microscopic distribution for orientation over 10 trials

For the treatment combinations, I didn't cap the comparison groups exactly around the critical value for each parameter so this is more a qualitative investigation. The point is to check for coupling when we expect to see some effect. What I expect from reading about the flocking model are that: while holding other variables constant, higher density should have slower and less alignment because it's harder for the particles to clutter before noise or attention limit breaks them apart again. Higher noise would should similarly force particles to separate easier, leader to smaller groups and less consensus. Higher number of neighbors is less clear, but by comparing it to the metric range, averaging over more neighbors should bring more particles together faster.



η=1: [0.686, 0.717]

η=1.5: [0.370, 0.412]

P-value: 5.730e-32

N=30: [0.252, 0.275]

N=150: [0.251, 0.281]

P-value: 2.178e-4

k=2: [0.695, 0.715]

k=8: [0.742, 0.757]

P-value: 3.652e-6

Figure 9. 30 Monte Carlo trials of varying (A) noise, (B) number of agents (density), (C) number of neighbors (attention size).

Looking qualitatively at the difference of means (given the small sample size), noise is obviously very important. A small change in the noise (a difference of max 0.5 radians in the angle update is max 30 degrees which is still not too big in the normal range of noises [0,5].) can cause the polarization to go down to only half the group aligned.

The confidence intervals seem narrow enough for qualitative comparison between extreme conditions because the difference of means is far enough to suggest that the distance is not totally chance. To decrease the variation size, we can increase the number of simulation trials to 50x and/or increase the number of simulation steps to >200 steps so that there are more values in the stationary state to average over. Or we can increase the number of agents and size of space, and lengthen relaxation time and increase noise. These increases should give higher limits of quantification but at the expense of computing power (I haven't done runtime analysis yet, but I am ready for half-day length runs for the more rigorous large system runs that can go beyond the finite size fluctuations and approach central limit).

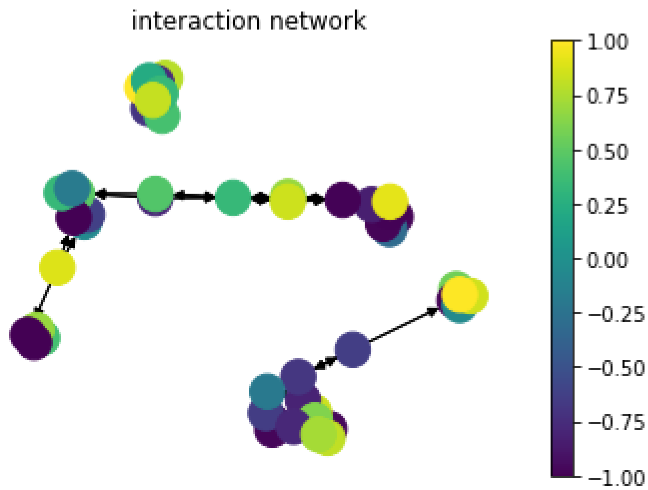
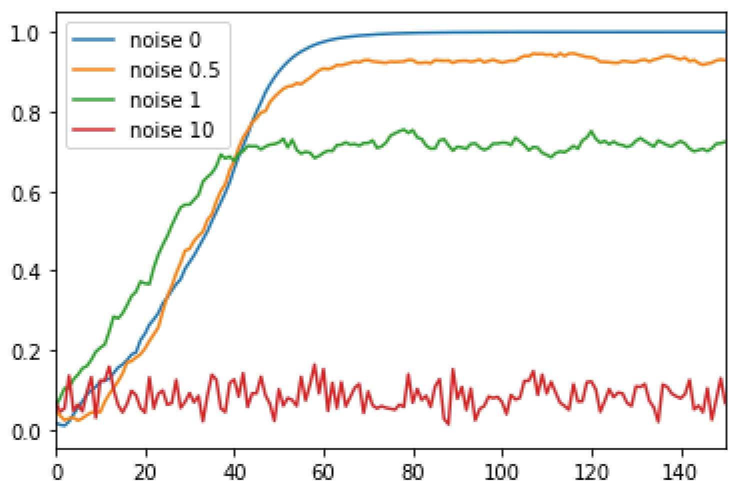


Figure 10. Stuck in clusters even after 300 steps.



10(B). Large noise never percolating.

What it looks like in the state in Figure 10 is that after a very long time (10x the relaxation time for noise=1), there are still two main clusters a a small offshoot, not even aligned with the in-group because the clusters keep switching members. The pattern follows noise size in 10(B) dynamics. It's interesting to see a scenario where order is never reached (average polarization < 1 in a near-stationary state or after a long time). To guess the cause, these are special cases where the cluster structures are so resilient to attraction they keep merging and disbanding, because agents keep swapping neighbors.

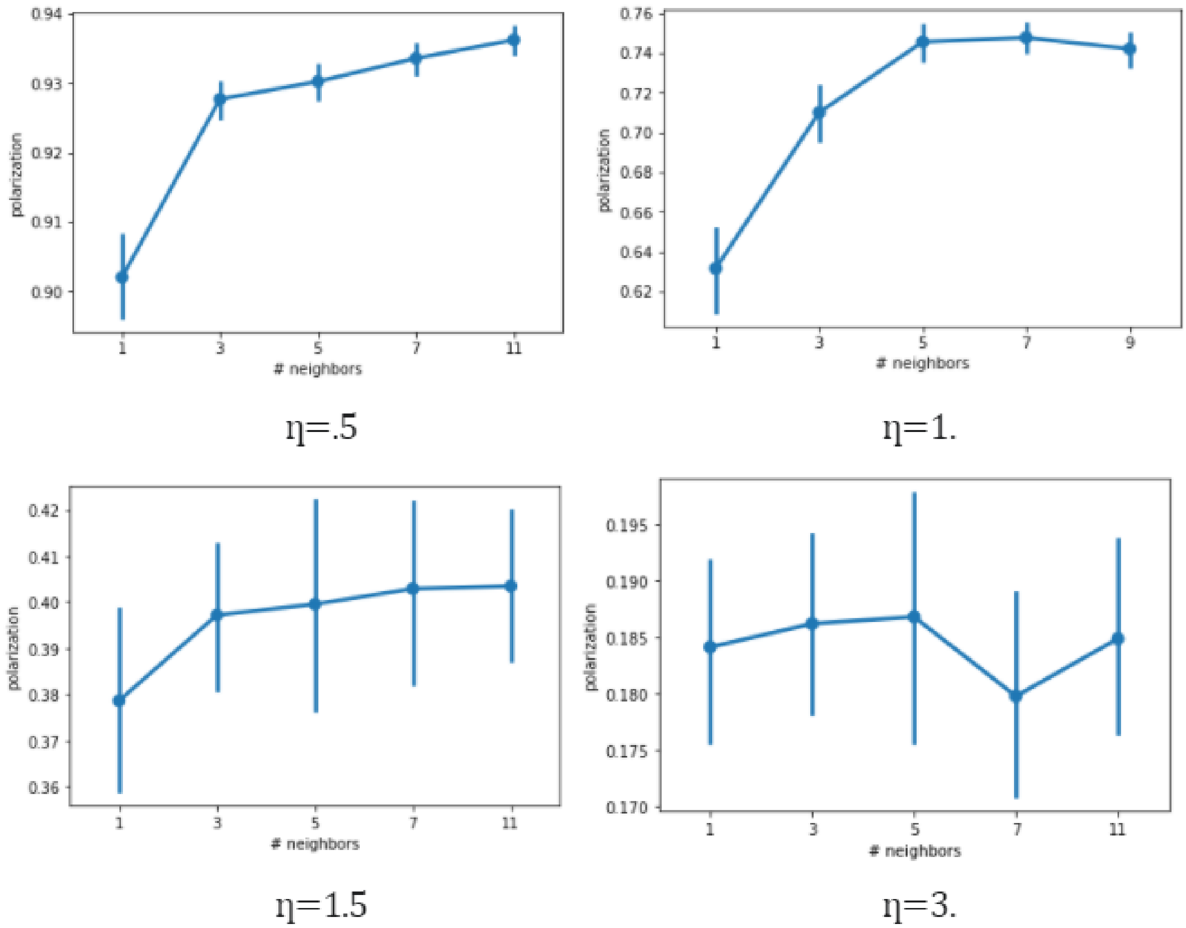


Figure 11. polarization vs neighbor # over 30 trials, for different regimes of noise.

Going back to the original scenario for the model, how does this inform a decision on the programmed vision for the biological robots? We now know that we can inform our decision with the context of the forest. If we are in a clearing, the noise is low from the environmental obstacles. In that region just having a few neighbors is enough to percolate into a very coherent group. Whereas if we are in the midst of a dark wooded swamp, there are so many distractions and it might be good to get as much neighbors as information as possible for a sliver of chance to band together. However, it is not even guaranteed that |u|=1 is possible. Still, sometimes having several clusters is like split teams for exploring. We should study the link between polarization and accuracy in the collective search or decision.

Moreover, this simplistic logic breaks down in the real world, about how birds actually communicate. If I am a bird then I would probably focus on birds who are special to me

— my mom or friend — or birds who attract attention in someway. That might be an additional rule that we can speculate about through agent-based modeling. There are too much irrationality that are complex to describe quantitatively so all of this are simplifications.

Theoretical analysis

I take the reported critical values (noise, density) from the metric 2D model (Vicsek, 1995). Then I observe if the topological 1D model also transitions abruptly (in the outcome, polarization) at similar values or patterns.

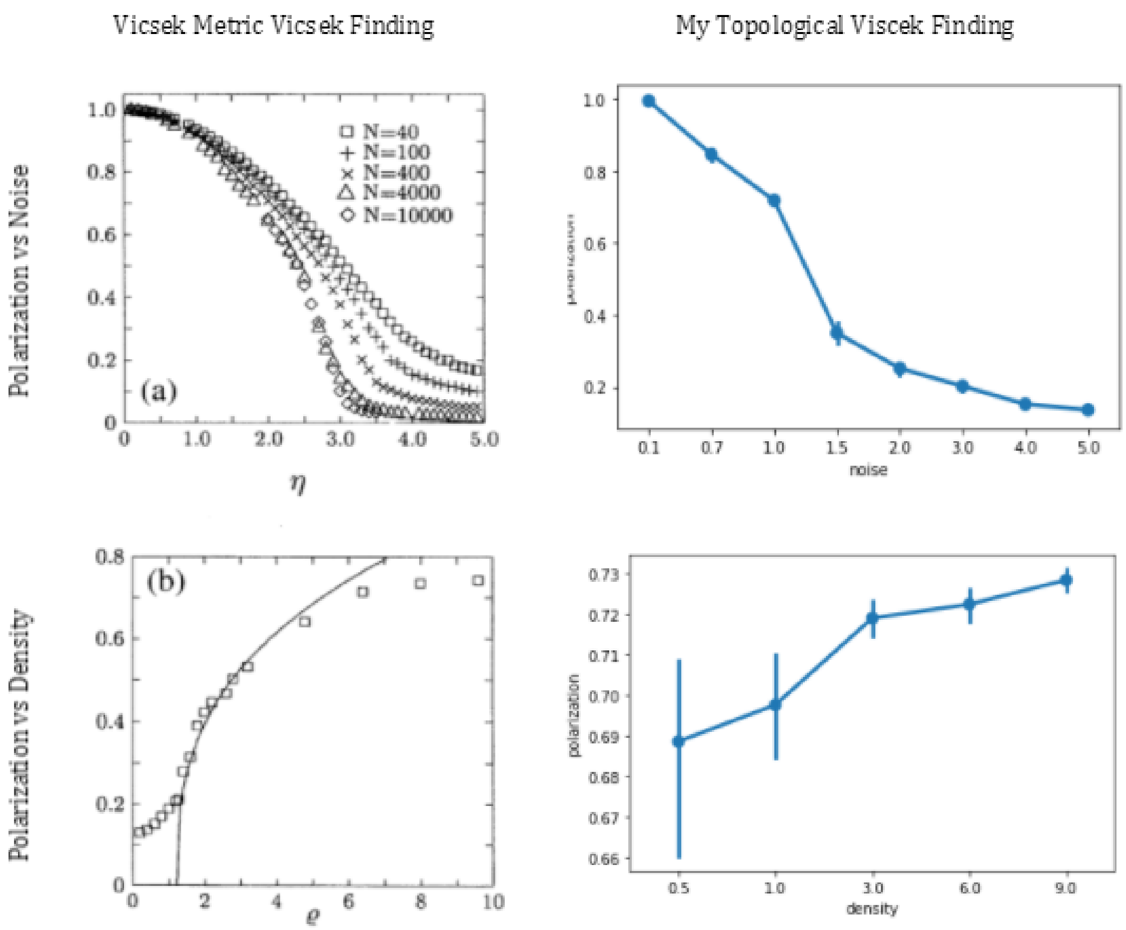


Figure 2. Single-parameter tuning: finding the disorder-order transition critical value. The constants are density=0.4, noise=2 as reported. I held k=3 constant to proxy metric range.

In the comparison above, the y-axis are equivalent but not same order parameter for average velocity and average polarization. For noise regimes, it's surprising that the place where curvature inflects is similar, around 2. for N=40 in metric and around 1.5 for N=40 in mine. The confidence interval wings are pretty small in my 30-trial results but I wouldn't come to certain conclusions. I think the shape is interesting because squinting, I see two regimes for polarization in noise splitting at some value like their reported value 2. I would be interested to see how the fluctuation might have a maximum peak at this critical point. That would be because at both stable states of low and high order, the noise is small and oscillating around some mean, because the structure of messy birds and aligned birds are somewhat stable. But at the critical point, that's when particles manage to cluster from a bit of luck and accumulate but fall apart again — the variation is biggest. On the other hand I'm not sure why the density comparison is so strikingly different. I wonder if this is also fundamental difference from the y-axis. I need to check if my other constants are in the same condition because sometimes a parameter can give very different slope curves with different starting positions. The ideal is to construct a multi-parameter slope field.

The purpose for observing the control of noise and density is to figure out the scaling constant that is the slope to each graph, so that we can construct a partial differential equation-looking analysis where we can slice the prediction by both parameters. Because single-parameter exploration is not enough as we saw in the brute-force version of finding the LCDM constants in cosmology. Perhaps an MCMC approach where we perturb the parameters in smart combinations and numerically approximate the relationship can be an alternative.

But noise and density are known variables. Something cool is taking alignment strength as the control knob. I'm not sure how to do that for this system. I started tuning a new parameter (number of neighbors) to discover transition there and guess a critical value. I couldn't squeeze enough time and paper insight to formalize the relationships into parameter families, check maximum fluctuation of outcome, compare visualizations of the stationary states. But my guess is that we can reformulate the mean-field approximation for critical value of the control parameter (alignment strength) using the topological parameters. Here is a walkthrough of a possible conversion from a discussion. I hope to figure out the exact dimensional relationship and then the critical value constant next.

In the metric model study, they derived a mean field critical density: $\rho_0 > \frac{2D_0}{\gamma\pi\epsilon^2}$ (Vicsek, 1995). The difference in assumption is obviously metric vs topological distance. There are many similarities to the topological model. The density term is the same just for a different dimension $\rho_0 = \langle\rho\rangle = N/L^D$. There is a term for $D_0 \propto \sigma^2$ which is the noise scaling. $\gamma = 1/\tau$ relates to the stepsize in proportion to time required to reach stationary state. $\pi\epsilon^2\rho_0$ is dimensionally how many neighbors per unit area in range. This is key difference between metric range and metric-free counting. I started to make sense of how the EV of particle/unit area is equivalent to EV of network degree (in the kNN interaction network). It agrees with the observation in figure 1, where spatial clusters and interaction network clusters correspond quite well. Then we might convert the $\pi\epsilon^2\rho$ term to relate to the number of neighbors in a constant for $c > 2\sigma^2\tau$. If we have a reasonable enough c(k) then we can find the critical value too at $\tau_* = \frac{c}{2\sigma_*^2}$.

Another approach would be to compare the birds as coupled phase oscillators and compare alignment to synchronization, with the special assumption that the graph is not complete and the natural frequency of the oscillators is 0 and all motion comes from forcing. (Momentum is not conserved in this system.) There are interesting analysis about what social strength is required to get to full synchronization, which is relevant to the question of coordinating many individuals. If we were to transfer the topological attention limit to coupled oscillator, there might be similar local effects there as well.

Next Steps

1. Theoretical analysis: do rescaling analysis by changing one parameter and see how it is similar to changing another parameter. potentially get the set of independent variables (like the g and Peclet).
2. Modeling: look at how kNN differs from vision or Voronoi topological distance.
3. Modeling: compare with data of an alignment system. For example: fish startle. How is it related to the orientational order described for liquid crystals? (I looked at liquid crystal structure → optical polarization property for another assignment)
4. Modeling: Check with natural system data. There was disagreement between starling observations and metric Vicsek predictions which we can check in the topological model. "First, in the relaxation of the spatio-temporal correlation function, non-dissipative modes were found in natural swarms, resulting in an underdamped **correlation function**. This suggests a sort of behavioural inertia in

swarms, not present in the classic Vicsek model. The other difference comes in the value of the **dynamic critical exponent z** resulting from testing the dynamic scaling hypothesis. For natural swarms z assumes the value of approximately 11, while for the Vicsek model z ~ 2." (source)

5. Theoretical: it's feasible to check if maximum outcome variance is at the critical point. that might help with estimating the critical constant.
6. Clustering dynamics: Fission and fusion of clusters. The one time with certain parameters where I saw clusters merging, I would like to see the network before and after the merge. Merging would mean a jump in polarization. Watching animals moving is also just very cute. Maybe there are more biological rules inside the kNN class that are more plausible or specific.