A Deterministic Distributed Algorithm for Weighted All Pairs Shortest Paths Through Pipelining

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Abstract

We present a new pipelined approach to compute all pairs shortest paths (APSP) in a directed graph with nonnegative integer edge weights (including zero weights) in the CONGEST model in the distributed setting. Our deterministic distributed algorithm computes shortest paths of distance at most Δ for all pairs of vertices in at most $2n\sqrt{\Delta}+2n$ rounds, and more generally, it computes h-hop shortest paths for k sources in $2\sqrt{nkh}+n+k$ rounds. The algorithm is simple, and it has some novel features and a nontrivial analysis. It uses only the directed edges in the graph for communication. This algorithm can be used as a base within asymptotically faster algorithms that match or improve on the current best deterministic bound of $\tilde{O}(n^{3/2})$ rounds for this problem when edge weights are O(n) or shortest path distances are $\tilde{O}(n^{3/2})$. These latter results are presented in a companion paper [1].

1 Introduction

Designing distributed algorithms for various network and graph problems such as shortest paths [2, 8, 11, 13, 9] is a extensively studied area of research. The Congest model (described in Sec 1.2) is a widely-used model for these algorithms, see [2, 4, 8, 11]. In this paper we consider distributed algorithms for the computing all pairs shortest paths (APSP) and related problems in a graph with non-negative edge weights in the Congest model.

In sequential computation, shortest paths can be computed much faster in graphs with non-negative edge-weights (including zero weights) using the classic Dijkstra's algorithm [3] than in graphs with negative edge-weights. Additionally, negative edge-weights raise the possibility of negative weight cycles in the graph, which usually do not occur in practice, and hence are not modeled by real-world weighted graphs. Thus, in the distributed setting, it is of importance to design fast shortest path algorithms that can handle non-negative edge-weights, including edges of weight zero.

The presence of zero weight edges creates challenges in the design of distributed algorithms as observed in [8]. (We review related work in Section 1.3.) One approach used for positive integer edge weights is to replace an edge of weight d with d unweighted edges and then run an unweighted APSP algorithm such as [11, 13] on this modified graph. This approach is used in approximate APSP

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algorithms [12, 10]. However such an approach fails when zero weight edges may be present. There are a few known algorithms that can handle zero weights, such as the $\tilde{O}(n^{5/4})$ -round randomized APSP algorithm of Huang et al. [8] (for polynomially bounded non-negative integer edge weights) and the $\tilde{O}(n^{3/2})$ -round deterministic APSP algorithm of Agarwal et al. [2] (for graphs with arbitrary edge weights).

1.1 Our Results

We present a new pipelined approach for computing APSP and related problems on an n-node graph G = (V, E) with non-negative edge weights $w(e), e \in E$, (including zero weights). Our results hold for both directed and undirected graphs and we will assume w.l.o.g. that G is directed. Our distributed algorithm uses only the directed edges for communication, while the algorithms in [8, 2] need to use the underlying undirected graph as the communication network.

Our Pipelined APSP Algorithm for Weighted Graphs. An h-hop shortest path from u to v in G is a path from u to v of minimum weight among all paths with at most h edges (or hops). The central algorithm we present is for computing h-hop APSP, or more generally, the h-hop k-sources shortest path problem ((h,k)-SSP), with an additional constraint that the shortest paths have distance at most Δ in G. We also compute an (h,k)-SSP tree for each source, which contains an h-hop shortest path from the source to every other vertex to which there exists a path with weight at most Δ . In the case of multiple h-hop shortest paths from a source s to a vertex v, this tree contains the path with the smallest number of hops, breaking any further ties by choosing the predecessor vertex with smallest ID. Our algorithm (Algorithm 1 in Section 3) is compact and easy to implement, and has no large hidden constant factors in its bound on the number of rounds. It can be viewed as a (substantial) generalization of the pipelined method for unweighted APSP given in [13], which is a refinement of [11]. Our algorithm uses key values that depend on both the weighted distance and the hop length of a path, and it can store multiple distance values for a source at a given node, with the guarantee that the shortest path distance will be identified. This algorithm (Algorithm 1) achieves the bounds in the following theorem.

Theorem 1.1. Let G = (V, E) be a directed or undirected edge-weighted graph, where all edge weights are non-negative integers (with zero-weight edges allowed). The following deterministic bounds can be obtained in the Congest model for shortest path distances at most Δ .

- (i) (h, k)-SSP in $2\sqrt{\Delta kh} + k + h$ rounds.
- (ii) APSP in $2n\sqrt{\triangle} + 2n$ rounds.
- (iii) k-SSP in $2\sqrt{\Delta kn} + n + k$ rounds.

Follow-up Results. In a companion paper [1], we build on this pipelined algorithm to present several improved results, which we summarize here. We improve on the bounds given in (ii) and (iii) of Theorem 1.1 by combining our pipelined Algorithm 1 with a modified version of the APSP algorithm in [2] to obtain the bounds stated in the following Theorems 1.2 and 1.3.

Theorem 1.2. [1] Let G = (V, E) be a directed or undirected edge-weighted graph, where all edge weights are non-negative integers bounded by λ (with zero-weight edges allowed). The following deterministic bounds can be obtained in the CONGEST model.

- (i) APSP in $O(\lambda^{1/4} \cdot n^{5/4} \log^{1/2} n)$ rounds.
- (ii) k-SSP in $O(\lambda^{1/4} \cdot nk^{1/4} \log^{1/2} n)$ rounds.

Theorem 1.3. [1] Let G = (V, E) be a directed or undirected edge-weighted graph, where all edge weights are non-negative integers (with zero edge-weights allowed), and the shortest path distances are bounded by \triangle . The following deterministic bounds can be obtained in the Congest model.

- (i) APSP in $O(n(\triangle \log^2 n)^{1/3})$ rounds.
- (ii) k-SSP in $O((\triangle kn^2 \log^2 n)^{1/3})$ rounds.

The results in Theorem 1.2 and 1.3 improve on the $\tilde{O}(n^{3/2})$ deterministic APSP bound of Agarwal et al. [2] for significant ranges of values for both λ and Δ , as stated below.

Corollary 1.4. [1] Let G = (V, E) be a directed or undirected edge-weighted graph with non-negative edge weights (and zero-weight edges allowed). The following deterministic bounds hold for the Congest model for $1 \ge \epsilon \ge 0$.

- (i) If the edge weights are bounded by $\lambda = n^{1-\epsilon}$, then APSP can be computed in $O(n^{3/2-\epsilon/4} \log^{1/2} n)$ rounds.
- (ii) For shortest path distances bounded by $\Delta = n^{3/2-\epsilon}$, APSP can be computed in $O(n^{3/2-\epsilon/3}\log^{2/3}n)$ rounds.

The corresponding bounds for the weighted k-SSP problem are: $O(n^{5/4-\epsilon/4}k^{1/4}\log^{1/2}n)$ (when $\lambda = n^{1-\epsilon}$) and $O(n^{7/6-\epsilon/3}k^{1/3}\log^{2/3}n)$ (when $\Delta = n^{3/2-\epsilon}$). Note that the result in (i) is independent of the value of Δ (depends only on λ) and the result in (ii) is independent of the value of λ (depends only on Δ).

Our pipelined technique can be adapted to give simpler methods for some procedures in the randomized distributed weighted APSP algorithms in Huang et al. [8]. In [1] we present simple deterministic algorithms that match the congest and dilation bounds in [8] for two of the three procedures used there: the *short-range* and *short-range-extension* algorithms. Those simplified algorithms are both obtained using a streamlined single-source version of our pipelined APSP algorithm (Algorithm 1). Several other results for distributed computation of shortest paths are presented in [1].

After discussing the Congest model and related reults, the rest of this paper will present our new pipelined algorithm.

1.2 Congest Model

In the CONGEST model, there are n independent processors interconnected in a network by bounded-bandwidth links. We refer to these processors as nodes and the links as edges. This network is modeled by graph G = (V, E) where V refers to the set of processors and E refers to the set of links between the processors. Here |V| = n and |E| = m.

Each node is assigned a unique ID between 1 and poly(n) and has infinite computational power. Each node has limited topological knowledge and only knows about its incident edges. For the weighted APSP problem we consider, each edge has a positive or zero integer weight that can be represented with $B = O(\log n)$ bits. Also if the edges are directed, the corresponding communication channels are bidirectional and hence the communication network can be represented by the underlying undirected graph U_G of G (this is also considered in [8, 13, 7]). It turns out that our basic pipelined algorithm does not need this feature though our faster algorithm does.

The computation proceeds in rounds. In each round each processor can send a message of size $O(\log n)$ along edges incident to it, and it receives the messages sent to it in the previous round. The model allows a node to send different message along different edges though we do not need this

feature in our algorithm. The performance of an algorithm in the Congest model is measured by its round complexity, which is the worst-case number of rounds of distributed communication.

1.3 Related Work

Weighted APSP. The current best bound for the weighted APSP problem is due to the randomized algorithm of Huang et al. [8] that runs in $\tilde{O}(n^{5/4})$ rounds. This algorithm works for graphs with polynomially bounded integer edge weights (including zero-weight edges), and the result holds with w.h.p. in n. For graphs with arbitrary edge weights, the recent result of Agarwal et al. [2] gives a deterministic APSP algorithm that runs in $\tilde{O}(n^{3/2})$ rounds. This is the current best bound (both deterministic and randomized) for graphs with arbitrary edge weights as well as the best deterministic bound for graphs with integer edge weights.

In a companion paper [1] we build on the pipelined algorithm we present here to obtain an algorithm for non-negative integer edge-weights (including zero-weighted edges) that runs in $\tilde{O}(n\Delta^{1/3})$ rounds where the shortest path distances are at most Δ and in $\tilde{O}(n^{5/4}\lambda^{1/4})$ rounds when the edge weights are bounded by λ . This result improves on the $\tilde{O}(n^{3/2})$ deterministic APSP bound of Agarwal et al. [2] when either edge weights are at most $n^{1-\epsilon}$ or shortest path distances are at most $n^{3/2-\epsilon}$, for any $\epsilon > 0$. In [1] we also give an improved randomized algorithm for APSP in graphs with arbitrary edge weights that runs in $\tilde{O}(n^{4/3})$ rounds, w.h.p. in n.

Weighted k-SSP. The current best bound for the weighted k-SSP problem is due to the Huang et al's [8] randomized algorithm that runs in $\tilde{O}(n^{3/4} \cdot k^{1/2} + n)$ rounds. This algorithm is also randomized and only works for graphs with integer edge weights. The deterministic APSP algorithm in [2] can be shown to give an $O(n \cdot \sqrt{k \log n})$ round deterministic algorithm for k-SSP. In this paper, we present a deterministic algorithm for non-negative (including zero) integer edge-weighted graphs that runs in $\tilde{O}((\triangle \cdot n^2 \cdot k)^{1/3})$ rounds where the shortest path distances are at most \triangle and in $\tilde{O}((\lambda k)^{1/4}n)$ rounds when the edge weights are bounded by λ .

2 Overview of the $O(n\sqrt{\triangle})$ Pipelined Algorithm for APSP

The starting point for our weighted APSP algorithm is the distributed algorithm for unweighted APSP in [13], which is a streamlined variant of an earlier APSP algorithm [11]. This unweighted APSP algorithm is very simple: each source initiates its distributed BFS in round 1. Each node v retains the best (i.e., shortest) distance estimate it has received for each source, and stores these estimates in sorted order (breaking ties by source id). Let d(s) (or $d_v(s)$) denote the shortest distance estimate for source s at v and let pos(s) be its position in sorted order $(pos(s) \ge 1)$. In a general round r, node v sends out a shortest distance estimate d(s) if r = d(s) + pos(s). Since d(s) is nondecreasing and pos(s) is increasing, there will be at most one d(s) at v that can satisfy this condition. It is shown in [13] that if the current best distance estimate d(s) for a source s reaches v in round r then r < d(s) + pos(s). Since d(s) < n for any source s and s and s are all sources arrive at any given node s in less than s rounds.

In the class of graphs that we deal with, d(s) is at most \triangle for all s, v, and it appears plausible that the above pipelining method would apply here as well. Unfortunately, this does not hold since we allow zero weight edges in the graph. The key to the guarantee that a d(s) value arrives at v before round d(s) + pos(s) in the unweighted case in [13] is that the predecessor y that sent its $d_y(s)$ value to v must have had $d_y(s) = d_v(s) - 1$. (Recall that in the unweighted case, $d_y(s)$ is

simply the hop-length of the path taken from s to y.) If we have zero-weight edges this guarantee no longer holds for the weighted path length, and it appears that the key property of the unweighted pipelining methodogy no longer applies. Since edge weights larger than 1 are also possible (as long as no shortest path distance exceeds \triangle) we also have the property that the hop length of a path can be either greater than or less than its weighted distance.

2.1 Our (h, k)-SSP algorithm

Algorithm 1 in the next section is our pipelined algorithm for a directed graph G = (V, E) with non-negative edge-weights. The input is G, together with the subset S of k vertices for which we need to compute h-hop shortest path trees. An innovative feature of this algorithm is that the key κ it uses for a path is not its weighted distance, but a function of both its hop length l and its weighted distance d. More specifically, $\kappa = d \cdot \gamma + l$, where $\gamma = \sqrt{kh/\Delta}$. This allows the key to inherit some of the properties from the algorithms in [11, 13] through the fact that the hop length is part of κ 's value, while also retaining the weighted distance which is the actual value that needs to be computed.

The new key κ by itself is not sufficient to adapt the algorithm for unweighted APSP in [13] to the weighted case. In fact, the use of κ can complicate the computation since one can have two paths from s to v, with weighted distances $d_1 < d_2$, and yet for the associated keys one could have $\kappa_1 > \kappa_2$ (because the path with the smaller weight can have a larger hop-length). Our algorithm handles this with another unusual feature: it may maintain several (though not all) of the key values it receives, and may also send out several key values, even some that it knows cannot correspond to a shortest distance. These features are incorporated into a carefully tailored algorithm that terminates in $O(\sqrt{\Delta kh})$ rounds with all h-hop shortest path distances from the k sources computed.

It is not difficult to show that eventually every shortest path distance key arrives at v for each source from which v is reachable when Algorithm 1 is executed. In order to establish the bound on the number of rounds, we show that our pipelined algorithm maintains two important invariants:

Invariant 1: If an entry Z is added to $list_v$ in round r, then $r < \lceil Z.\kappa + pos(Z) \rceil$, where $Z.\kappa$ is Z's key value.

Invariant 2: The number of entries for a given source s at $list_v$ is at most $\sqrt{\Delta h/k} + 1$.

Invariant 1 is the natural generalization of the unweighted algorithms [11, 13] for the key κ that we use. On the other hand, to the best of our knowledge, Invariant 2 has not been used before, nor has the notion of storing multiple paths or entries for the same source at a given node. By Invariant 2, the number of entries in any list is at most $\sqrt{\Delta kh} + k$, so $pos(Z) \leq \sqrt{\Delta kh} + k$ for every list at every round. Since the value of any κ is at most $\Delta \cdot \gamma + h$, by Invariant 1 every entry is received by round $2\sqrt{\Delta kh} + k + h$. We give the details in the next section, starting with a step-by-step description of Algorithm 1 followed by its analysis.

3 The Pipelined (h, k)-SSP Algorithm

We now describe Algorithm 1. Recall that the key value we use for a path π is $\kappa = d \cdot \gamma + l$, where $\gamma = \sqrt{kh/\Delta}$, d is the weighted path length, and l is the hop-length of π . At each node v our

Table 1: Notations

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Global Parameters:		
S	set of sources	
k	number of sources, or $ S $	
h	maximum number of hops in a shortest path	
Δ	maximum weighted distance of a shortest path	
n	number of nodes	
γ	parameter equal to $\sqrt{hk/\Delta}$	
Local Variables at node v :		
d_x^*	current shortest path distance from x to v; same as $d_{x,v}^*$	
$list_v$	list at v for storing the SP and non-SP entries	

Variables/Parameters for entry $Z = (\kappa, d, l, x)$ in $list_v$:	
κ	key for Z ; $\kappa = d \cdot \gamma + h$
d	weight (distance) of the path associated with this entry
l	hop-length of the path associated with this entry
x	start node (i.e. source) of the path associated with this entry
p	parent node of v on the path associated with this entry
ν	number of entries for source x at or below Z in $list_v$ (not stored explicitly)
$flag-d^*$	flag to indicate if Z is the current SP entry for source x
pos	position of Z in $list_v$ in a round r; same as pos^r , pos_v^r
SP	shortest path

algorithm maintains a list, $list_v$, of the entries and associated data it has retained. Each element Z on $list_v$ is of the form $Z = (\kappa, d, l, x)$, where x is the source vertex for the path corresponding to κ , d, and l. The elements on $list_v$ are ordered by key value κ , with ties first resolved by the value of d, and then by the label of the source vertex. We use $Z.\nu$ to denote the number of keys for source x stored on $list_v$ at or below Z. The position of an element Z in $list_v$ is given by pos(Z), which gives the number of elements at or below Z on $list_v$. If the vertex v and the round r are relevant to the discussion we will use the notation $pos_v^r(Z)$, but we will remove either the subscript or the superscript (or both) if they are clear from the context. We also have a flag $Z.flag-d^*$ which is set if Z has the smallest (d, κ) value among all entries for source x (so d is the shortest weighted distance from s to v among all keys for x on $list_v$). A summary of our notation is in Table 1.

Initially, when round r = 0, $list_v$ is empty unless v is in the source set S. Each source vertex $x \in S$ places an element (0,0,0,x) on its $list_x$ to indicate a path of weight 0 and hop length 0 from x to x, and $Z.flag-d^*$ is set to true. In Step 1 of the Initialization round 0, node v initializes the distance from every source to ∞ . In Step 2 every source vertex initializes the distance from itself to 0 and adds the corresponding entry in its list. There are no Sends in round 0.

In a general round r, in Step 1 of Algorithm 1, v checks if $list_v$ contains an entry Z with $\lceil Z.\kappa + pos_v(Z) \rceil = r$. If there is such an entry Z then v sends Z to its neighbors, along with $Z.\nu$ and $Z.flag-d^*$ in Step 2. Steps 3-13 describe the steps taken at v after receiving a set of incoming messages I from its neighbors. In Step 7 an entry Z is created from an incoming message M, updated to reflect the d and l values at v. Step 9 checks if Z has a shorter distance than the current shortest path entry, Z^* , at v, or a shorter hop-length (if the distance is the same), or a parent with smaller ID (if both distance and hop-length are same). And if so, then Z is marked as SP in Step 10 and is then inserted in $list_v$ in Step 11. Otherwise, if Z is a non-SP it is inserted into $list_v$ in Step 13 only if the number of entries on $list_v$ for source x with key $\langle Z.\kappa$ in $list_v$ is less than $Z^-.\nu$. This is the rule that decides if a received entry that is not the SP entry is inserted into $list_v$.

Steps 1-4 of procedure INSERT perform the addition of a new entry Z to $list_v$. In Step 1 Z is inserted in $list_v$ in the sorted order of (κ, d, x) . The algorithm then moves on to remove an existing entry for source x on $list_v$ if the condition in Step 2 holds. This condition checks if there is a non-SP entry above Z in $list_v$. If so then the closest non-SP entry above Z is removed in Steps 3-4.

```
Initialization: Initialization procedure for Algorithm 1 at node v
Input: set of sources S
1: for each x \in S do d_x^* \leftarrow \infty
2: if v \in S then d_v^* \leftarrow 0; add an entry Z = (0, 0, 0, v) to list_v; Z.flag-d^* \leftarrow true
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Algorithm 1 Pipelined (h,k)-SSP algorithm at node v for round r

```
1: send [Steps 1-2]: if there is an entry Z with [Z.\kappa + pos_n^r(Z)] = r
          then compute Z.\nu and form the message M = \langle Z, Z.flag-d^*, Z.\nu \rangle and send M to all neighbors
 3: receive [Steps 3-13]: let I be the set of incoming messages
 4: for each M \in I do
           let M = (Z^- = (\kappa^-, d^-, l^-, x), Z^-.flag-d^*, Z^-.\nu) and let the sender be y.
           \kappa \leftarrow \kappa^- + w(y,v) \cdot \gamma + 1; \ d \leftarrow d^- + w(y,v); \ l \leftarrow l^- + 1
6:
           Z \leftarrow (\kappa, d, l, x); Z. \mathit{flag-d}^* \leftarrow \mathit{false}; Z. p \leftarrow y \quad (Z \text{ may be added to } \mathit{list}_v \text{ in Step } 11 \text{ or } 13) let Z^* be the entry for x in \mathit{list}_v such that Z^*. \mathit{flag-d}^* = \mathit{true}, if such an entry exists (otherwise d_x^* = \infty)
 7:
           \textbf{if } Z^-. \textit{flag-d}^* = true \text{ and } l \leq h \text{ and } ((d < d_x^*) \text{ or } (d = d_x^* \text{ and } Z.\kappa < Z^*.\kappa) \text{ or } (d = d_x^* \text{ and } Z.\kappa = Z^*.\kappa \text{ and } Z.p < Z^*.p))
10:
                 d_x^* \leftarrow d; Z.flag-d^* \leftarrow true; Z^*.flag-d^* \leftarrow false (if Z^* exists)
11:
12:
            else
13:
                 if there are less than Z^-.\nu entries for x with key \leq Z.\kappa then INSERT(Z)
```

```
Insert(Z): Procedure for adding Z to list_v
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```
1: insert Z in list_v in sorted order of (\kappa, d, x)

2: if \exists an entry Z' for x in list_v such that Z'.flag-d^* = false and pos(Z') > pos(Z) then

3: find Z' with smallest pos(Z') such that pos(Z') > pos(Z) and Z'.flag-d^* = false

4: remove Z' from list_v
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Algorithm 1 performs these steps in successive rounds. We next analyze it for correctness and we also show that it terminates with all shortest distances computed before round $r = \lceil 2\sqrt{\Delta kh} + k + h \rceil$.

3.1 Correctness of Algorithm 1

We now provide a sketch for correctness of Algorithm 1. The complete proofs are in Appendix A. The initial Observations and Lemmas given below establish useful properties of an entry Z in a $list_v$ and of $pos_v^r(Z)$ and its relation to $pos_y^r(Z^-)$. We then present the key lemmas. In Lemma 3.9, we show that the collection of entries for a given source x in $list_v$ can be mapped into (d,l) pairs with non-negative l values such that $d=d^*$ for the shortest path entry, and the d values for all other entries are distinct and larger than d^* . (It turns out that we cannot simply use the d values already present in Z's entries for this mapping since we could have two different entries for source x on $list_v$, Z_1 and Z_2 , that have the same d value.) Once we have Lemma 3.9 we are able to bound the number of entries for a given source at $list_v$ by $\frac{h}{\gamma} + 1$ in Lemma 3.11, and this establishes Invariant 2 (which is stated in Section 2). Lemma 3.12 establishes Invariant 1. In Lemma 3.13 we establish that all shortest path values reach node v. With these results in hand, the final Lemma 3.14 for the round bound for computing (h, k)-SSP with shortest path distances at most \triangle is readily established, which then gives Theorem 1.1.

Observation 3.1. Let Z be an entry for a source $x \in S$ added to list_v in round r. Then if Z is removed from list_v in a round $r' \geq r$, it was replaced by another entry for x, Z', such that $pos_v^{r'}(Z) > pos_v^{r'}(Z')$ and $Z.\kappa \geq Z'.\kappa$.

Lemma 3.2. Let Z be an entry in $list_v$. Then $pos_v^{r'}(Z) \ge pos_v^r(Z)$ for all rounds r' > r, for which Z exists in v's list.

Observation 3.3. Let Z be an entry for source x that was added to list_v. If there exists a non-SP entry for x above Z in list_v, then the closest non-SP entry above Z will be removed.

Observation 3.4. Let Z^- be an entry for source x sent from y to v in round r, and let Z be the corresponding entry created for possible addition to list_v in Step 7 of Algorithm 1. If Z is not added to list_v, then there is an entry $Z' \neq Z$ for source x in list_v with Z'-flag-d*= true, and there are at least Z^- . ν entries for x with $key \leq Z$. κ at the end of round r.

Lemma 3.5. Let Z be an entry for source x that is present on list_v in round r. Let r' > r, and let c and c' be the number of entries for source x on list_v that have key value less than Z's key value in rounds r and r' respectively. Then $c' \geq c$.

Lemma 3.5 holds for every round greater than r, even if Z is removed from $list_v$.

Lemma 3.6. Let Z^- be an entry for source x sent from y to v and suppose the corresponding entry Z (Step 7 of Algorithm 2) is added to list_v in round r. Then there are at least Z^- . ν entries at or below Z in list_v for source x.

Proof. Assume inductively that this result holds for all entries on $list_v$ and $list_y$ with key value at most $Z.\kappa$ at all previous rounds and at y in round r as well. (It trivially holds initially.)

Let Z_1^- be the $(Z^-.\nu-1)$ -th entry for source x in $list_y$. Since Z_1^- has a key value smaller than Z^- it was sent to v in an earlier round r'. If the corresponding entry Z_1 created for possible addition to $list_v$ in Step 7 of Algorithm 1, was inserted in $list_v$ then by inductive assumption there were at least $Z_1^-.\nu = Z^-.\nu - 1$ entries for x at or below Z_1 in $list_v$. And by Lemma 3.5 this holds for round r as well and hence the result follows since Z is present above Z_1 in $list_v$.

And if Z_1 was not added to $list_v$ in round r', then by Observation 3.4 there were already $Z^-.\nu - 1$ entries for x with key $\leq Z_1.\kappa$ and by Lemma 3.5 there are at least $Z^-.\nu - 1$ entries for x with key $\leq Z_1.\kappa \leq Z.\kappa$ on $list_v$ at round r and hence the result follows.

Lemma 3.7. Let Z^- be an entry sent from y to v in round r and let Z be the corresponding entry created for possible addition to list_v in Step 7 of Algorithm 1. For each source $x_i \in S$, let there be exactly c_i entries for x_i at or below Z^- in list_y. If Z is added to list_v, then for each $x_i \in S$, there are at least c_i entries for x_i at or below Z in list_v.

Corollary 3.8. Let Z^- be an entry sent from y to v in round r and let Z be the corresponding entry created for possible addition to list_v in Step 7 of Algorithm 1. If Z is added to list_v, then $pos_y^r(Z^-) \leq pos_v^r(Z)$.

Lemma 3.9. Let C be the entries for a source $x \in S$ in list_v in round r. Then the entries in C can be mapped to (d,l) pairs such that each $l \geq 0$ and each $Z \in C$ is mapped to a distinct d value with $Z.\kappa = d \cdot \gamma + l$. Also $d = d_x^*$ if Z is a current shortest path entry, otherwise $d > d_x^*$.

Proof. We will establish this result by induction on j, the number of entries in \mathcal{C} . For the base case, when j=1, we can map d and l to the pair in the single entry Z since $Z.\kappa = d \cdot \gamma + l$. Assume inductively that the result holds at $list_u$ for all nodes u when the number of entries for x is at most j-1. Consider the first time $|\mathcal{C}|$ becomes j at $list_v$, and let this occur when node y sends Z^- to v and this is updated and inserted as Z in $list_v$ in round r.

If Z is inserted as a new shortest path entry with distance value d^* , then the distinct d values currently assigned to the j-1 entries for source x in $list_v$ must all be larger than d^* hence we can simply assign the d and l values in Z as its (d,l) mapping.

If Z is inserted as a non-SP entry then it is possible that the d value in Z has already been assigned to one of the j-1 entries for source x on $list_v$. If this is the case, consider the entries for source x with key value at most Z^- . κ in $list_y$ (at node y). By the check in Step 13 of Algorithm 1 we know that there are j such values. Inductively these j entries have j distinct d^- values assigned to them, and we transform these into j distinct values for $list_v$ by adding $w(y,v) \cdot \gamma + 1$ to each of them. For at least one of these d^- values in y, call it d_1^- , it must be the case that $d' = d_1^- + w(y,v) \cdot \gamma + 1$ is not assigned to any of the j-1 entries for source x below Z in $list_v$. Let Z_1^- be the entry in y's list that is associated with distance d_1^- . It is readily seen that the associated l value for d' in Z on $list_v$ must be greater than 0 and the distance value $d' + w(y,v) \neq d_x^*$ (see proof in Appendix A). So we can assign this d value to Z. We do not need to consider non-SP entries above Z since if there were one, the closest one above Z would have been deleted and j-1 would not have increased to j.

In the general case when the number of entries remains at j after the insert we do need to consider the possibility of the new value assigned to Z being duplicated at an entry Z' above Z. But here we can assign to Z' the d value previously given to the removed entry (and the l needed for Z'. κ will be non-negative because the removed entry must have been below Z' on $list_v$).

Lemma 3.10. Let Z be the current shortest path distance entry for a source $x \in S$ in v's list. Then the number of entries for x below Z in list_v is at most h/γ .

Proof. By Lemma 3.9, we know that the keys of all the entries for x can be mapped to (d, l) pairs such that each entry is mapped to a distinct d value and l > 0.

We have $Z.\kappa = d_x^* \cdot \gamma + l_x^*$, where l_x^* is the hop-length of the shortest path from x to v. Let Z'' be an entry for x below Z in v's list. Then, $Z''.\kappa \leq Z.\kappa$. It implies that $d'' \cdot \gamma \leq d_x^* \cdot \gamma + (l_x^* - l'') < d_x^* \cdot \gamma + h$ which gives $d'' < d_x^* + h/\gamma$. Since $d'' \geq d_x^*$, there can be at most h/γ entries for x below Z in $list_v$. \square

Using Lemma 3.10 we can show that there are at most $h/\gamma + 1$ entries for source x in $list_v$ in the case when the entry for the shortest distance for x is not the topmost entry in $list_v$.

Lemma 3.11. For each source $x \in S$, v's list has at most $h/\gamma + 1$ entries for x.

In Lemmas 3.12-3.13 we establish an upper bound on the round r by which v receives a shortest path entry Z^* .

Lemma 3.12. If an entry Z is added to list_v in round r then $r < \lceil Z.\kappa + pos_v^r(Z) \rceil$.

Proof. The lemma holds in the first round since all entries have non-negative κ , any received entry has hop length at least 1, and the lowest position is 1 so for any entry Z received by v in round 1, $[Z.\kappa + pos_v^1(Z)] \ge 1 + 1 > 1$.

Let r be the first round (if any) in which the lemma is violated, and let it occur when entry Z is added to $list_v$. So $r \ge \lceil Z.\kappa + pos_v^r(Z) \rceil$. Let $r_1 = \lceil Z.\kappa + pos_v^r(Z) \rceil$ (so $r_1 \le r$ by assumption).

Since Z was added to $list_v$ in round r, Z^- was sent to v by a node y in round r. So by Step 1 $r = \lceil Z^-.\kappa + pos_y^r(Z^-) \rceil$. But $Z.\kappa > Z^-.\kappa$ and $pos_v^r(Z) \ge pos_y^r(Z^-)$, hence r must be less than $\lceil Z.\kappa + pos_v^r(Z) \rceil$.

Lemma 3.13. Let $\pi_{x,v}^*$ be a shortest path from source x to v with the minimum number of hops among h-hop shortest paths from x to v. Let $\pi_{x,v}^*$ have l^* hops and shortest path distance $d_{x,v}^*$. Then v receives an entry $Z^* = (\kappa, d_{x,v}^*, l^*, x)$ by round $r < \lceil Z^*, \kappa + pos_v^r(Z^*) \rceil$.

Proof. If an entry $Z^* = (\kappa, d_{x,v}^*, l^*, x)$ is placed on $list_v$ by v then by Lemma 3.12 it is received before round $\lceil Z^*.\kappa + pos_v^r(Z^*) \rceil$ and hence it will be sent in round $r = \lceil Z^*.\kappa + pos_v^r(Z^*) \rceil$ in Step 2. It remains to show that an entry for path $\pi_{x,v}^*$ is received by v. We establish this for all pairs x, v by induction on key value κ .

If $\kappa = 0$, then it implies that the shortest path is the vertex x itself and thus the statement holds for $\kappa = 0$. Let us assume that the statement holds for all keys $< \kappa$ and consider the path $\pi_{x,v}^*$ with key $\kappa = d_{x,v}^* \cdot \gamma + l^*$.

Let (y,v) be the last edge on the path $\pi_{x,v}^*$ and let $\pi_{x,y}^*$ be the subpath of $\pi_{x,v}^*$ from x to y. By construction the path $\pi_{x,y}^*$ is a shortest path from x to y and its hop length l^*-1 is the smallest among all shortest paths from x to y. Hence by the inductive assumption an entry Z^- with $Z^-.\kappa = d_{x,y}^* \cdot \gamma + l^* - 1$ (which is strictly less than $Z^*.\kappa$) is received by y before round $\lceil Z^-.\kappa + pos_y'(Z^-) \rceil$ (by Lemma 3.12) and is then sent to v in round $r' = \lceil Z^-.\kappa + pos_y^{r'}(Z^-) \rceil$ in Step 2. Thus v adds the shortest path entry for x, Z^* , to $list_v$ by end of round r'.

In Lemma 3.14 we establish an upper bound on the round r by which Algorithm 1 terminates.

Lemma 3.14. Let Δ be the maximum shortest path distance in the h-hop paths. Algorithm 1 correctly computes the h-hop shortest path distances from each source $x \in S$ to each node $v \in V$ by round $\lceil \triangle \gamma + h + \triangle \cdot \gamma + k \rceil$.

Proof. An h-hop shortest path has hop-length at most h and weight at most Δ , hence a key corresponding to a shortest path entry will have value at most $\Delta \gamma + h$. Thus by Lemma 3.13, for every source $x \in S$ every node $v \in V$ should have received the shortest path distance entry, Z^* , for source x by round $r = \lceil \Delta \gamma + h + pos_v^r(Z^*) \rceil$.

Now we need to bound the value of $pos_v^r(Z^*)$. By Lemma 3.11, we know that there are at most $h/\gamma + 1$ entries for each source $x \in S$ in a node v's list. Now as there are k sources, v's list has at most $(h/\gamma+1)\cdot k \leq \gamma \cdot \triangle + k$ entries, thus $pos_v^r(Z^*) \leq \gamma \cdot \triangle + k$ and hence $r \leq \lceil \Delta \gamma + h + \gamma \cdot \Delta + k \rceil$. \square

Since $\gamma = \sqrt{hk/\Delta}$, Lemma 3.14 establishes Theorem 1.1.

4 Conclusion

We have presented a new approach to the distributed computation of shortest paths in a graph with non-negative integer weights (including zero weights). Our deterministic pipelined distributed algorithms for weighted shortest paths (both APSP, and for k sources) is novel and very simple. Its asymptotic performance improves on $\tilde{O}(n^{3/2})$ rounds, the current best deterministic distributed algorithm for this problem [2], only in very special cases when shortest path distances are smaller than n. But the algorithm may be relevent even for larger shortest path lengths since it is very simple, and has the very small constant factor 2 in the leading term. As noted in the introduction, we have built on this algorithm to achieve several new results, including improved deterministic APSP for moderately large non-negative integer weights (including zero weights) [1].

A major open problem left by our work is whether we can come up with a similar pipelining strategy when working with the scaled graph in Gabow's scaling technique [5]. Our current pipelined algorithm assumes that all sources see the same weight on each edge, while in the scaling algorithm each source sees a different edge weight on a given edge. We could obtain a deterministic $\tilde{O}(n^{4/3})$ -round APSP algorithm with non-negative polynomially bounded integer weights if our pipelined

strategy can be made to work with Gabow's scaling technique [5]. While this can be handled with n different SSSP computations in conjunction with the randomized scheduling result of Ghaffari [6], it will be very interesting to see if a deterministic pipelined strategy could achieve the same result.

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A Appendix: Correctness of Algorithm 1

<u>Observations and Lemmas 3.1-3.6:</u> In the following Observations and Lemmas we point out the key facts about an entry Z in $list_v$ in our Algorithm 1. We use these in our proofs in this section.

Observation. 3.1. Let Z be an entry for a source $x \in S$ added to list_v in round r. Then if Z is removed from list_v in a round $r' \geq r$, it was replaced by another entry for x, Z', such that $pos_v^{r'}(Z) > pos_v^{r'}(Z')$ and $Z.\kappa \geq Z'.\kappa$.

Proof. An entry is removed from $list_v$ only in Step 4 of INSERT, and this occurs at most once in round r' (through a call from either Step 11 or Step 13 of Algorithm 1). But immediately before that removal an entry Z' with a smaller value was inserted in $list_v$ in Step 1 of INSERT.

Lemma. 3.2. Let Z be an entry in list_v. Then $pos_v^{r'}(Z) \ge pos_v^r(Z)$ for all rounds r' > r, for which Z exists in v's list.

Proof. If not, then it implies that there exists Z' such that Z' was below Z in v's list in round r and was replaced by another entry Z'' that was above Z in a round r'' such that $r' \geq r'' > r$ and hence $pos_v^{r''}(Z'') > pos_v^{r''}(Z')$. But by Observation 3.1 this cannot happen and thus resulting in a contradiction.

Observation. 3.3. Let Z be an entry for source x that was added to list_v. If there exists a non-SP entry for x above Z in list_v, then the closest non-SP entry above Z will be removed from list_v.

Proof. This is immediate from Steps 1-4 of the procedure INSERT.

Observation. 3.4. Let Z^- be an entry for a source x sent from y to v in round r, and let Z be the corresponding entry created for possible addition to list_v in Step 7 of Algorithm 1. If Z is not added to list_v, then there is an entry $Z' \neq Z$ in list_v with Z'-flag-d*= true, and there are at least Z^- -v entries for x with $key \leq Z$. κ at the end of round r.

Proof. This is immediate from Steps 8 and 9 of Algorithm 1 where the current entry Z^* with $Z^*.flag-d^*=true$ is verified to have a shorter distance (or a smaller key if Z and Z^* have the same distance), and by the check in Step 13.

Observation A.1. Let Z^* be a current SP entry for a source $x \in S$ present in list_v. Then $Z^*.l \le h$.

Proof. This is immediate from the check in Step 9.

The above Observation should be contrasted with the fact that $list_v$ could contain entries Z with Z.l > h, but only if $flag-d^*(Z) = false$. In fact it is possible that $list_v$ contains an entry $Z' \neq Z^*$ with $Z'.d = d^*$ and l > h since such an entry would fail the check in Step 9 but could then be inserted in Step 13 of Algorithm 1.

Lemma. 3.5. Let Z be an entry for source x that is present on list_v in round r. Let r' > r, and let c and c' be the number of entries for source x on list_v that have key value less than Z's key value in rounds r and r' respectively. Then $c' \ge c$.

Proof. If c' < c then an entry for x that was present below Z in round r must have been removed without having another entry for x being inserted below Z. But by Observation 3.1 this is not possible since any time an entry for source x is removed from $list_v$ another entry for source x with smaller key value is inserted in $list_v$.

Lemma 3.5 holds for every round greater than r, even if Z is removed from $list_v$. The following stronger lemma holds for rounds greater than r when Z remains on $list_v$.

Lemma A.2. Let Z be a non-SP entry for source x that is present on list_v in round r. Let r' > r, and let c and c' be the number of entries for source x on list_v that have key value less than Z's key value in rounds r and r' respectively. Then c' = c.

Proof. If a new entry Z' with key $< Z.\kappa$ for x is added, then by Observation 3.3 the closest non-SP entry for x with key $> Z'.\kappa$ must be removed from $list_v$ and thus $c' \le c$. Then using Lemma 3.5 we have c' = c.

Lemma. 3.6. Let Z^- be an entry for source x sent from y to v and suppose the corresponding entry Z (Step 7 of Algorithm 2) is added to list_v in round r. Then there are at least Z^- . ν entries at or below Z in list_v for source x.

Proof. Let us assume inductively that this result holds for all entries on $list_v$ and $list_y$ with key value at most $Z.\kappa$ at all previous rounds and at y in round r as well. (It trivially holds initially.)

Let Z_1^- be the $(Z^-.\nu-1)$ -th entry for source x in $list_y$. Since Z_1^- has a key value smaller than Z^- it was sent to v in an earlier round r'. If the corresponding entry Z_1 created for possible addition to $list_v$ in Step 7 of Algorithm 1, was inserted in $list_v$ then by inductive assumption there were at least $Z_1^-.\nu = Z^-.\nu - 1$ entries for x at or below Z_1 in $list_v$. And by Lemma 3.5 this holds for round r as well and hence the result follows since Z is present above Z_1 in $list_v$.

And if Z_1 was not added to $list_v$ in round r', then by Observation 3.4 there were already $Z^-.\nu-1$ entries for x with key $\leq Z_1.\kappa$ and by Lemma 3.5 there are at least $Z^-.\nu-1$ entries for x with key $\leq Z_1.\kappa \leq Z.\kappa$ on $list_v$ at round r and hence the result follows.

Establishing $\operatorname{pos}_y^r(\mathbf{Z}^-) \leq \operatorname{pos}_v^r(\mathbf{Z})$: For an entry Z^- sent from y to v such that Z is the corresponding entry created for possible addition to $list_v$ in Step 7 of Algorithm 1, in Lemma 3.7 and Corollary 3.8 we establish that if Z is added to $list_v$ then $pos_y^r(Z^-) \leq pos_v^r(Z)$, which is an important property of pos.

Lemma. 3.7. Let Z^- be an entry sent from y to v in round r and let Z be the corresponding entry created for possible addition to list_v in Step 7 of Algorithm 1. For each source $x_i \in S$, let there be exactly c_i entries for x_i at or below Z^- in list_y. If Z is added to list_v, then for each $x_i \in S$, there are at least c_i entries for x_i at or below Z in list_v.

Proof. If not there exists an $x_i \in S$ with strictly less than c_i entries for x_i at or below Z in $list_v$.

Let Z_1^- be the c_i -th entry for x_i in $list_y$ (if x_i is Z's source, then Z_1^- is Z^-). If Z_1^- is not Z^- , it is below Z^- in $list_y$ and so was sent in a round r' < r; if $Z_1^- = Z^-$ then r' = r. Let Z_1 be the corresponding entry created for possible addition to $list_v$ in Step 7 of Algorithm 1.

If Z_1 was added to $list_v$ and is also present in $list_v$ in round r, then by Lemma 3.6 and 3.5, there will be at least c_i entries for x_i at or below Z_1 , resulting in a contradiction. And if Z_1 was removed

from $list_v$ in a round r'' < r, then by Lemma 3.5, the number of entries for x_i with key $\leq Z_1.\kappa$ should be at least c_i .

Now if Z_1 was not added to $list_v$ in round r', then by Observation 3.4, we must already have at least c_i entries for x_i with key $\leq Z_1.\kappa$ in round r' and by Lemma 3.5, this must hold for all rounds r'' > r as well.

Corollary. 3.8. Let Z^- be an entry sent from y to v in round r and let Z be the corresponding entry created for possible addition to list_v in Step 7 of Algorithm 1. If Z is added to list_v, then $pos_v^r(Z^-) \leq pos_v^r(Z)$.

A.1 Establishing an Upper bound on $Z.\nu$

In this section (Lemmas 3.9-3.11) we establish an upper bound on the value of $Z.\nu$. This upper bound on $Z.\nu$ immediately gives a bound on the maximum number of entries that can be present in $list_v$ for a source $x \in S$.

Lemma. 3.9. Let C be the entries for a source $x \in S$ in $list_v$ in round r. Then the entries in C can be mapped to (d,l) pairs such that each $l \geq 0$ and each $Z \in C$ is mapped to a distinct d value with $Z.\kappa = d \cdot \gamma + l$. Also $d = d_x^*$ if Z is a current shortest path entry, otherwise $d > d_x^*$.

Proof. We will establish this result by induction on j, the number of entries in \mathcal{C} . For the base case, when j=1, we can map d and l to the pair in the single entry Z since $Z.\kappa = d \cdot \gamma + l$. Assume inductively that the result holds at $list_u$ for all nodes u when the number of entries for x is at most j-1. Consider the first time $|\mathcal{C}|$ becomes j at $list_v$, and let this occur when node y sends Z^- to v and this is updated and inserted as Z in $list_v$ in round r.

If Z is inserted as a new shortest path entry with distance value d^* , then the distinct d values currently assigned to the j-1 entries for source x in $list_v$ must all be larger than d^* hence we can simply assign the d and l values in Z as its (d,l) mapping.

If Z is inserted as a non-SP entry then it is possible that the d value in Z has already been assigned to one of the j-1 entries for source x on $list_v$. If this is the case, consider the entries for source x with key value at most Z^- . κ in $list_y$ (at node y). By the check in Step 13 of Algorithm 1 we know that there are j such values. Inductively these j entries have j distinct d^- values assigned to them, and we transform these into j distinct values for $list_v$ by adding $w'_x(y,v) \cdot \gamma + 1$ to each of them. For at least one of these d^- values in y, call it d_1^- , it must be the case that $d' = d_1^- + w'_x(y,v) \cdot \gamma + 1$ is not assigned to any of the j-1 entries for source x below z in z in z in z be the entry in z is list that is associated with distance z we show that the associated z value for z on z on z in z in z in z on z in z

$$\begin{split} (d'+w_x'(y,v)) \cdot \gamma + l &= Z.\kappa \\ &= Z^-.\kappa + w_x'(y,v) \cdot \gamma + 1 \\ &\geq Z_1^-.\kappa + w_x'(y,v) \cdot \gamma + 1 \\ &= d' \cdot \gamma + l_1^- + w_x'(y,v) \cdot \gamma + 1 \\ &= (d'+w_x'(y,v)) \cdot \gamma + l_1^- + 1 \end{split} \qquad \text{(since } (pos_y(Z^-) > pos_y(Z_1^-)))$$

Hence $l \ge l_1^- + 1 > 0$.

Since Z is a non-SP entry we also need to argue that $d' + w'_x(y, v) \neq d_x^*$. If not then by induction, it implies that the entry Z_1^- for x in y's list correspond to the current shortest path entry for x in $list_y$. Since Z_1^- gives the shortest path distance from x to y, the corresponding shortest path entry for x must be below Z in v's list and by induction, it must have d_x^* associated with it. This results in a contradiction since we chose the distance value, $d' + w'_x(y, v)$, such that it was different from the distances associated with the other (j-1) entries for x in v's list.

We have shown that the lemma holds the first time a j-th entry is added to $list_v$ for source x. To complete the proof we now show that the lemma continues to hold if a new entry Z for source x is added to $list_v$ while keeping the number of entries at j. The argument is the same as the case of having j entries for source x for the first time except that we also need to consider duplication of a d value at an entry above the newly inserted Z. For this we proceed as in the previous case. Let Z be inserted in position $p \leq j$. We assign a d value to Z as in the previous case, taking care that the d value assigned to Z is different from that for the p-1 entries below Z. Suppose Z's d value has been assigned to another entry Z'' in $list_v$ above Z. Then, we consider Z', the entry that was removed (in Step 5 of INSERT) in order to keep the total number of entries for source x at j. We assign to Z'' the value d' that was assigned to Z'. Since Z'' has a larger key value than Z' we will need to use an l'' at least as large as that used for Z' (call it l') in order satisfy the requirement that $Z''.\kappa = d' \cdot \kappa + l''$. Since l' must have been non-negative, l'' will also be non-negative as required, and all d values assigned to the entries for x will be distinct.

Lemma. 3.10. Let Z be the current shortest path distance entry for a source $x \in S$ in v's list. Then the number of entries for x below Z in list_v is at most $\gamma \cdot \frac{n}{k}$.

Proof. By Lemma 3.9, we know that the keys of all the entries for x can be mapped to (d, l) pairs such that each entry is mapped to a distinct d value and l > 0.

We have $Z.\kappa = d_x^* \cdot \gamma + l_x^*$, where l_x^* is the hop-length of the shortest path from x to v. Let Z'' be an entry for x below Z in v's list. Then, $Z''.\kappa \leq Z.\kappa$. It implies

$$d'' \cdot \gamma + l'' \le d_x^* \cdot \gamma + l_x^*$$

$$d'' \cdot \gamma \le d_x^* \cdot \gamma + (l_x^* - l'')$$

$$d'' \le d_x^* + \frac{(l_x^* - l'')}{\gamma}$$

$$\le d_x^* + \frac{(h-1)}{\gamma}$$

$$< d_x^* + \frac{h}{\gamma}$$

$$= d_x^* + \frac{h}{\gamma^2} \cdot \gamma$$

$$= d_x^* + \frac{n}{k} \cdot \gamma$$

Thus $d'' < d_x^* + \frac{n}{k} \cdot \gamma$. Since $d'' \ge d_x^*$, there can be at most $\frac{n}{k} \cdot \gamma$ entries for x below Z in $list_v$.

Lemma. 3.11. For each source $x \in S$, v's list has at most $\frac{n}{k} \cdot \gamma + 1$ entries for x.

Proof. On the contrary, let Z be an entry for source $x \in S$ with the smallest key such that Z is the $(\gamma \cdot \frac{n}{k} + 2)$ -th entry for x in $list_v$. Let y be the sender of Z to v and let the corresponding entry in y's list be Z^- .

If Z was added as a non-SP entry, then by Lemma 3.6 there are at least $\gamma \cdot \frac{n}{k} + 2$ entries for x at or below Z^- in $list_y$, resulting in a contradiction as Z is the entry with the smallest key that have this ν value.

Otherwise if Z was added as a current shortest path entry, then by Lemma 3.10, Z can have at most $\gamma \cdot \frac{n}{k}$ entries below it in any round and hence there are at most $\gamma \cdot \frac{n}{k} + 1$ at or below Z in $list_v$ in all rounds (and if Z is later marked as non-SP then by Lemma A.2 $Z.\nu$ will stay fixed at that value), again resulting in a contradiction.

A.2 Establishing an Upper Bound on the round r by which an entry Z is sent

Lemma. 3.12. If an entry Z is added to list_v in round r then $r < Z.\kappa + pos_v^r(Z)$.

Proof. The lemma holds in the first round since all entries have non-negative κ , any received entry has hop length at least 1, and the lowest position is 1 so for any entry Z received by v in round 1, $Z.\kappa + pos_v^1(Z) \ge 1 + 1 > 1$.

Let r be the first round (if any) in which the lemma is violated, and let it occur when entry Z is added to $list_v$. So $r \ge Z.\kappa + pos_v^r(Z)$. Let $r_1 = Z.\kappa + pos_v^r(Z)$ (so $r_1 < r$ by assumption).

Since Z was added to $list_v$ in round r, Z^- was sent to v by a node y in round r. So $r = Z^-.\kappa + pos_y^r(Z^-)$. But $Z.\kappa > Z^-.\kappa$ and $pos_v^r(Z) \ge pos_y^r(Z^-)$, hence r must be less than $Z.\kappa + pos_v^r(Z)$. \square

Lemma. 3.13. Let $\pi_{x,v}^*$ be a shortest path from source x to v with the minimum number of hops among h-hop shortest paths from x to v. Let $\pi_{x,v}^*$ have l^* hops and shortest path distance $d_{x,v}^*$. Then v receives an entry $Z^* = (\kappa, d_{x,v}^*, l^*, x)$ by round $r < Z^*.\kappa + pos_v^r(Z^*)$.

Proof. If an entry $Z^* = (\kappa, d_{x,v}^*, l^*, x)$ is placed on $list_v$ by v then by Lemma 3.12 it is received before round $Z^*.\kappa + pos_v^r(Z^*)$ and hence it will be sent in round $r = Z^*.\kappa + pos_v^r(Z^*)$ in Step 1. It remains to show that an entry for path $\pi_{x,v}^*$ is received by v. We establish this for all pairs x, v by induction on key value κ .

If $\kappa = 0$, then it implies that the shortest path is the vertex x itself and thus the statement holds for $\kappa = 0$. Let us assume that the statement holds for all keys $< \kappa$ and consider the path $\pi_{x,v}^*$ with key $\kappa = d_{x,v}^* \cdot \gamma + l^*$.

Let (y,v) be the last edge on the path $\pi_{x,v}^*$ and let $\pi_{x,y}^*$ be the subpath of $\pi_{x,v}^*$ from x to y. By construction the path $\pi_{x,y}^*$ is a shortest path from x to y and its hop length l^*-1 is the smallest among all shortest paths from x to y. Hence by the inductive assumption an entry Z^- with $Z^-.\kappa = d_{x,y}^* \cdot \gamma + l^* - 1$ (which is strictly less than $Z^*.\kappa$) is received by y before round $Z^-.\kappa + pos_y^{'}(Z^-)$ (by Lemma 3.12) and is then sent to v in round $v' = Z^-.\kappa + pos_y^{'}(Z^-)$ in Step 1. Thus v adds the shortest path entry for x, Z^* , to $list_v$ by the end of round v'.

A.3 Establishing an Upper Bound on the round r by which Algorithm 1 terminates

Lemma. 3.14. Algorithm 1 correctly computes the h-hop shortest path distances from each source $x \in S$ to each node $v \in V$ by round $(n-1)\gamma + h + n \cdot \gamma + k$.

Proof. An h-hop shortest path has hop-length at most h and weight at most n-1, hence a key corresponding to a shortest path entry will have value at most $(n-1)\gamma + h$. Thus by Lemma 3.13, for every source $x \in S$ every node $v \in V$ should have received the shortest path distance entry, Z^* , for source x by round $r = (n-1)\gamma + h + pos_r^r(Z^*)$.

Now we need to bound the value of $pos_v^r(Z^*)$. By Lemma 3.11, we know that there are at most $\gamma \cdot \frac{n}{k} + 1$ entries for each source $x \in S$ in a node v's list. Now as there are k sources, v's list has at most $(\gamma \cdot \frac{n}{k} + 1) \cdot k \leq \gamma \cdot n + k$ entries, thus $pos_v^r(Z^*) \leq \gamma \cdot n + k$ and hence $r \leq (n-1)\gamma + h + \gamma \cdot n + k$. \square

Since $\gamma = \sqrt{\frac{hk}{n}}$, Lemma 3.14 establishes the bounds given in Theorem 1.1.