18.217 PROBLEM SET (FALL 2019)

Instructions:

- All submissions must be **typed in LaTeX** and submitted as PDF on Stellar (try Overleaf if you are looking for an online LaTeX editor without requiring installations). Please name your file `ps#_Lastname_Firstname.pdf` and remember to include your name in each file. Suggested LaTeX template for homework submissions.
- Please acknowledge, **individually for every problem at the beginning of each solution**, a list of all collaborators and sources consulted (people, books, websites, etc.). Write *sources consulted: none* even if no sources are consulted. Failure acknowledge sources will lead to an automatic 10% penalty.
- You may not look up solutions to homework problems online or offline.
- Please turn in the problems marked ps1 and ps1⋆ for problem set 1, etc., by midnight of each due date (see course homepage). Do not submit the other problems—they are for you to practice.
- *Late policy.* Late submissions will be penalized by 20% per each late day. For example, for an assignment due on Sunday, a submission worth x points if turned in on time will be worth $0.6x$ points if submitted on Tuesday.
- *Collaboration policy.* You are strongly encouraged to start early and first work on the problems on your own. Reasonable collaboration is permitted, but everyone must write their solutions individually and acknowledge their collaborators.
- *Bonus problems*, marked by $\star$, are more challenging. A grade of A- may be attained by only solving the non-starred problems. To attain a grade of A or A+, you should solve a substantial number of starred problems. (No hints will be given for bonus problems, e.g., during office hours.)
- Please try to **fit your solution within one page** for each unstarred problem/part (standard 1-inch margins and 11pt font). The spirit of this policy is to encourage you to think first before you write. Distill your ideas, structure your arguments, and eliminate unnecessary steps. If necessary, some details of routine calculations may be skipped provided that you give convincing explanations.
- This file will be updated as the term progresses. Please check back regularly. There will be an announcement whenever each problem set is complete.
- You are encouraged to include figures whenever they are helpful. Here are some recommended ways to produce figures in decreasing order of learning curve difficulty:
  (1) TikZ or other drawing script
  (2) IPE (which supports LaTeX) or other drawing app
  (3) photo/scan (I recommend the Dropbox app on your phone, which has a nice scanning feature that produces clear monochrome scans)

Problems begin on the next page. Last updated: November 27, 2019

A. Introduction

ps1 A1. *Ramsey's theorem*

(a) Let s and r be positive integers. Show that there is some integer $n = n(s, r)$ so that if every edge of the complete graph K_n on n vertices is colored with one of r colors, then there is a monochromatic copy of K_s.

(b) Let $s \geq 3$ be a positive integer. Show that if the edges of the complete graph on $\binom{2s-2}{s-1}$ vertices are colored with 2 colors, then there is a monochromatic copy of K_s.

ps1 A2. Prove that it is possible to color $\mathbb{N}$ using two colors so that there is no infinitely long monochromatic arithmetic progression.

ps1 A3. *Many monochromatic triangles*

(a) True or false: If the edges of K_n are colored using 2 colors, then at least $1/4 - o(1)$ fraction of all triangles are monochromatic. (Note that $1/4$ is the fraction one expects if the edges were colored uniformly at random.)

(b) True or false: if the edges of K_n are colored using 3 colors, then at least $1/9 - o(1)$ fraction of all triangles are monochromatic.

(c) ($\star$ do not submit) True or false: if the edges of K_n are colored using 2 colors, then at least $1/32 - o(1)$ fraction of all copies of K_4's are monochromatic.

(d) (do not submit) Prove that for every s and r, there is some constant $c > 0$ so that for every sufficiently large n, if the edges of K_n are colored using r colors, then at least c fraction of all copies of K_s are monochromatic.

B. Forbidding subgraphs

ps1 B1. Show that a graph with n vertices and m edges has at least $\frac{4m}{3n}\left(m - \frac{n^2}{4}\right)$ triangles.

ps1$\star$ B2. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles.

ps1$\star$ B3. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains some edge in at least $(1/6 - o(1))n$ triangles, and that this constant $1/6$ is best possible.

B4. *K_{r+1}-free graphs close to the Turán bound are nearly r-partite*

ps1 (a) Let G be an n-vertex triangle-free graph with at least $\lfloor n^2/4 \rfloor - k$ edges. Prove that G can be made bipartite by removing at most k edges.

ps1$\star$ (b) Let G be an n-vertex K_{r+1}-free graph with at least $e(T_{n,r}) - k$ edges, where $T_{n,r}$ is the Turán graph. Prove that G can be made r-partite by removing at most k edges.

B5. Let G be a K_{r+1}-free graph. Prove that there is another graph H on the same vertex set as G such that $\chi(H) \leq r$ and $d_H(x) \geq d_G(x)$ for every vertex x (here $d_H(x)$ is the degree of x in H, and likewise with $d_G(x)$ for G). Give another proof of Turán's theorem from this fact.

B6. *Turán density.* Let H be a r-uniform hypergraph, let its *Turán number* $\mathrm{ex}^{(r)}(n, H)$ be the maximum number of edges in an r-uniform hypergraph on n vertices that does not contain H as a subgraph. Prove that the fraction $\mathrm{ex}^{(r)}(n, H)/\binom{n}{r}$ is a nonincreasing function of n, so that it has a limit $\pi(H)$ as $n \to \infty$, called the *Turán density* of H.

ps1 B7. *Supersaturation.* Let H be a graph and ρ a constant such that $\limsup_{n\to\infty} \mathrm{ex}(n,H)/\binom{n}{2} \le \rho$. Prove that for every $\epsilon > 0$ there exists some constant $c = c(H,\epsilon) > 0$ such that for sufficiently large n, every n-vertex graph with at least $(\rho+\epsilon)\binom{n}{2}$ edges contains at least $cn^{v(H)}$ copies of H.

ps1 B8. Let S be a set of n points in the plane, with the property that no two points are at distance greater than 1. Show that S has at most $\lfloor n^2/3 \rfloor$ pairs of points at distance greater than $1/\sqrt{2}$. Also, show that the bound $\lfloor n^2/3 \rfloor$ is tight (i.e., cannot be improved).

ps1 B9. (How *not* to define density in a product set) Let $S \subset \mathbb{Z}^2$. Define

$$d_k(S) = \max_{\substack{A,B\subset\mathbb{Z}\\|A|=|B|=k}} \frac{|S\cap(A\times B)|}{|A||B|}.$$

Show that $\lim_{k\to\infty} d_k(S)$ exists and is always either 0 or 1.

B10. Show that, for every $\epsilon > 0$, there exists $\delta > 0$ such that every graph with n vertices and at least ϵn^2 edges contains a copy of $K_{s,t}$ where $s \ge \delta \log n$ and $t \ge n^{0.99}$.

ps2 B11. *Density version of Kővári–Sós–Turán.* Prove that for every positive integers $s \le t$, there are constants $C, c > 0$ such that every n-vertex graph with $p\binom{n}{2}$ edges contains at least $cp^{st}n^{s+t}$ copies of $K_{s,t}$, provided that $p \ge Cn^{-1/s}$.

ps2⋆ B12. *Hypergraph Kővári–Sós–Turán and a proof of Erdős–Stone–Simonovits*

(a) Prove that for every positive integer t there is some C so that every 3-uniform hypergraph on n vertices and at least $Cn^{3-t^{-2}}$ edges (i.e., triples) contains a copy of $K^{(3)}_{t,t,t}$, the complete tripartite 3-uniform hypergraph with t vertices in each part.

(b) Deduce that $\mathrm{ex}(n,H) \le (\frac{1}{4}+o(1))n^2$ for every graph H with $\chi(H) \le 3$.

(c) Explain how to generalize the above strategy to prove the Erdős–Stone–Simonovits theorem for every H (sketch the key steps).

ps2 B13. Find a graph H with $\chi(H) = 3$ and $\mathrm{ex}(n,H) > \frac{1}{4}n^2 + n^{1.99}$ for all sufficiently large n.

ps2⋆ B14. *Construction of a C_6-free graph.* Let q be an odd prime power. Let S denote the quadratic surface in the 4-dimensional projective space over $\mathbb{F}_q$ (whose points are nonzero points of $\mathbb{F}_q^5$ modulo the equivalence relation $(x_0,x_1,x_2,x_3,x_4) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4)$ for $\lambda \in \mathbb{F}_q^\times$) given by the equation (you may use another quadratic form if you wish)

$$x_0^2 + 2x_1x_2 + 2x_3x_4 = 0.$$

Let $\mathcal{L}$ be the set of lines contained in S.

(a) Prove that no three lines of $\mathcal{L}$ lie in the same plane.

(b) Show that the point-line incidence bipartite graph between S and $\mathcal{L}$ is a $(q+1)$-regular graph on $2(q^3+q^2+q+1)$ vertices with no cycles of length at most 6. Conclude that $\mathrm{ex}(n,C_6) \ge cn^{4/3}$ for some constant $c > 0$.

The next two problems concern the dependent random choice technique.

ps2 B15. Let $\epsilon > 0$. Show that, for sufficiently large n, every K_4-free graph with n vertices and at least ϵn^2 edges contains an independent set of size at least $n^{1-\epsilon}$.

ps2⋆ B16. *Extremal numbers of degenerate graphs*

(a) Prove that there is some absolute constant $c > 0$ so that for every positive integer r, every n-vertex graph with at least $n^{2-c/r}$ edges contains disjoint vertex subsets A and B such that every subset of r vertices in A has at least n^c neighbors in B and every subset of r vertices in B has at least n^c neighbors in A.

(b) We say that a graph H is *r-degenerate* if its vertices can be ordered so that every vertex has at most r neighbors that appear before it in the ordering. Show that for every r-degenerate bipartite graph H there is some constant $C > 0$ so that $\mathrm{ex}(n, H) \le Cn^{2-c/r}$, where c is the same absolute constant from part (a) (c should not depend on H or r).

ps2 B17. Let T be a tree with k edges. Show that $\mathrm{ex}(n, T) \le kn$.

ps2⋆ B18. Show that every n-vertex triangle-free graph with minimum degree greater than $2n/5$ is bipartite.

C. Szemerédi's regularity lemma and applications

For simplicity, you are welcome to apply the equitable version of Szemerédi's regularity lemma.

C1. Let G be a graph and $X, Y \subset V(G)$. If (X, Y) is an $\epsilon\eta$-regular pair, then (X', Y') is ϵ-regular for all $X' \subset X$ with $|X'| \ge \eta|X|$ and $Y' \subset Y$ with $|Y'| \ge \eta|Y|$.

C2. Let G be a graph and $X, Y \subset V(G)$. Say that (X, Y) is *ϵ-homogeneous* if for all $A \subset X$ and $B \subset Y$, one has

$$|e(A, B) - |A|\,|B|\,d(X, Y)| \le \epsilon\,|X|\,|Y|\,.$$

Show that if (X, Y) is ϵ-regular, then it is ϵ-homogeneous. Also, show that if (X, Y) is ϵ^3-homogeneous, then it is ϵ-regular.

ps2 C3. *Unavoidability of irregular pairs.* Let the *half-graph* H_n be the bipartite graph on $2n$ vertices $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ with edges $\{a_i b_j : i \le j\}$.

(a) For every $\epsilon > 0$, explicitly construct an ϵ-regular partition of H_n into $O(1/\epsilon)$ parts.

(b) Show that there is some $c > 0$ such that for every $\epsilon \in (0, c)$, every integer k and sufficiently large multiple n of k, every partition of the vertices of H_n into k equal-sized parts contains at least ck pairs of parts which are not ϵ-regular.

ps2 C4. Show that there is some absolute constant $C > 0$ such that for every $0 < \epsilon < 1/2$, every graph on n vertices contains an ϵ-regular pair of vertex subsets each with size at least δn, where $\delta = 2^{-\epsilon^{-C}}$.

C5. *Existence of a regular set.* Given a graph G, we say that $X \subset V(G)$ is *ϵ-regular* if the pair (X, X) is ϵ-regular, i.e., for all $A, B \subset X$ with $|A|, |B| \ge \epsilon|X|$, one has $|d(A, B) - d(X, X)| \le \epsilon$. This problem asks for two different proofs of the claim: for every $\epsilon > 0$, there exists $\delta > 0$ such that every graph contains an ϵ-regular subset of vertices of size at least δ fraction of the vertex set.

ps3 (a) Prove the claim using Szemerédi's regularity lemma, showing that one can obtain the ϵ-regular subset by combining a suitable sub-collection of parts from a regular partition.

ps3⋆ (b) Give an alternative proof of the claim showing that one can take $\delta = \exp(-\exp(\epsilon^{-C}))$ for some constant C.

ps3 C6. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex K_4-free graph with at least $(\frac{1}{8} + \epsilon)n^2$ edges contains an independent set of size at least δn.

C7. Show that for ever $\epsilon > 0$, there exists $\delta > 0$ such that every n-vertex K_4-free graph with at least $(\frac{1}{8} - \delta)n^2$ edges and independence number at most δn can be made bipartite by removing at most ϵn^2 edges.

ps3 C8. Show that the number of non-isomorphic n-vertex triangle-free graphs is $2^{(1/4+o(1))n^2}$.

C9. Show that for every H there exists some $\delta > 0$ such that for all sufficiently large n, if G is an n-vertex graph with average degree at least $(1 - \delta)n$ and the edges of G are colored using 2 colors, then there is a monochromatic copy of H.

ps3 C10. Show that for every H and $\epsilon > 0$ there exists $\delta > 0$ such that every graph on n vertices without an induced copy of H contains an induced subgraph on at least δn vertices whose edge density is at most ϵ or at least $1 - \epsilon$.

C11. *Random graphs are ϵ-regular.* Let G be a random bipartite graph between disjoint sets of vertices X and Y with $|X| = |Y| = n$, such that every pair in $X \times Y$ appears as an edge of G independently with the same probability. Show that there is some absolute constant $c > 0$ such that with probability at least $1 - e^{-n^{1+c}}$ for sufficiently large n, the pair (X, Y) is ϵ-regular in G with $\epsilon = n^{-c}$.

(You may use the following special case of the Azuma–Hoeffding inequality: if $X_1, \dots, X_N$ are independent random variables taking values in $[-1, 1]$, and $S = X_1 + \cdots + X_N$, then $\mathbb{P}(S \geq \mathbb{E}S + t) \leq e^{-t^2/(2N)}$.)

ps3⋆ C12. Show that for every graph H there is some graph G such that if the edges of G are colored with two colors, then some induced subgraph of G is a monochromatic copy of H.

ps3⋆ C13. Show that for every $c > 0$, there exists $c' > 0$ such that every graph on n vertices with at least cn^2 edges contains a d-regular subgraph with $d \geq c'n$ (here *d-regular* refers to every vertex having degree d).

ps4 C14. Show that there is a constant $c > 0$ so that for every sufficiently small $\epsilon > 0$ and sufficiently large $n > n_0(\epsilon)$ there exists an n-vertex graph with at most $\epsilon^{c\log(1/\epsilon)} n^3$ triangles that cannot be made triangle-free by removing fewer than ϵn^2 edges. (In particular, this shows that one cannot take $\delta = \epsilon^C$ for some constant $C > 0$ in the triangle removal lemma.)

C15. *Removal lemma for bipartite graphs with polynomial bounds.* Prove that for every bipartite graph H, there is a constant C such that for every $\epsilon > 0$, every n-vertex graph with fewer than $\epsilon^C n^{v(H)}$ copies of H can be made H-free by removing at most ϵn^2 edges.

ps4 C16. Let H be a n-vertex 3-uniform hypergraph such that every 6 vertices contain strictly fewer than 3 triples. Prove that H has $o(n^2)$ edges.

(Hint in white:)

ps4 C17. Assuming the tetrahedron removal lemma for 3-uniform hypergraphs, deduce that if $A \subset [N]^2$ contains no axes-aligned squares (i.e., four points of the form $(x, y), (x + d, y), (x, y + d), (x + d, y + d)$, where $d \neq 0$), then $|A| = o(N^2)$.

ps4⋆ C18. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subset [n]$ has fewer than δn^2 many triples $(x, y, z) \in A^3$ with $x + y = z$, then there is some $B \subset A$ with $|A \setminus B| \leq \epsilon n$ such that B is sum-free, i.e., there do not exist $x, y, z \in B$ with $x + y = z$.

D. Spectral graph theory and pseudorandom graphs

ps4 D1. Let G be an n-vertex graph. The *Laplacian* of G is defined to be $L_G = D_G - A_G$, where A_G is the adjacency matrix of G and D_G a diagonal matrix whose entry corresponding to the vertex $v \in V(G)$ is the degree of v in G (so that L_G is a symmetric matrix with all row sums zero). Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of L_G, with $\lambda_1 = 0$ corresponding to the all-1 vector. Prove that for every $S \subset V(G)$ with $|S| \le n/2$, one has (writing $\overline{S} := V(G) \setminus S$)

$$e(S, \overline{S}) \ge \frac{1}{2}\lambda_2 |S|$$

ps4⋆ D2. Let p be an odd prime and $A, B \subset \mathbb{Z}/p\mathbb{Z}$. Show that

$$\left| \sum_{a \in A} \sum_{b \in B} \left(\frac{a+b}{p} \right) \right| \le p\sqrt{p}$$

where (a/p) is the Legendre symbol defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod p \\ 1 & \text{if } a \text{ is a nonzero quadratic residue mod } p \\ -1 & \text{if } a \text{ is a quadratic nonresidue mod } p \end{cases}$$

D3. *Quasirandom transitive graphs.* Prove that if an n-vertex d-regular vertex-transitive graph G satisfies

$$\left| e(X,Y) - \tfrac{d}{n}|X||Y| \right| \le \epsilon dn \qquad \text{for all } X, Y \subseteq V(G),$$

then all the eigenvalues of the adjacency matrix of G, other than the largest one, are at most $8\epsilon d$ in absolute value.

ps4 D4. Prove that the diameter of an (n, d, λ)–graph is at most $\lceil \log n / \log(d/\lambda) \rceil$. (The *diameter* of a graph is the maximum distance between a pair of vertices.)

D5. Let G be an n-vertex d-regular graph. Suppose n is divisible by k. Color the vertices of G with k colors (not necessarily a proper coloring) such that each color appears exactly n/k times. Suppose that all eigenvalues, except the top one, of the adjacency matrix of G are at most d/k in absolute value. Show that there is a vertex of G whose neighborhood contains all k colors.

ps4 D6. Prove that for every positive integer d and real $\epsilon > 0$, there is some constant $c > 0$ so that if G is an n-vertex d-regular graph with adjacency matrix A_G, then at least cn of the eigenvalues of A_G are greater than $2\sqrt{d-1} - \epsilon$.

ps4⋆ D7. Show that for every d and r, there is some $\epsilon > 0$ such that if G is a d-regular graph, and $S \subset V(G)$ is such that every vertex of G is within distance r of S, then the top eigenvalue of the adjacency matrix of $G - S$ (i.e., remove S and its incident edges from G) is at most $d - \epsilon$.

ps4⋆ D8. Prove or disprove: there exists an absolute constant C such that the adjacency matrix of every n-vertex Cayley graph has an eigenbasis in $\mathbb{C}^n$ (consisting of n orthonormal unit eigenvectors) all of whose coordinates are each at most $C/\sqrt{n}$ in absolute value.

E. Graph limits and homomorphism density inequalities

Note: A "graphon" is a symmetric measurable function $W\colon [0,1]^2 \to [0,1]$.

ps5

E1. *Weak regularity decomposition (instead of partition).*

(a) Let $\epsilon > 0$. Show that for every graphon W, there exist measurable $S_1, \dots, S_k, T_1, \dots, T_k \subseteq [0,1]$ and reals $a_1, \dots, a_k \in \mathbb{R}$, with $k < \epsilon^{-2}$, such that
$$\left\| W - \sum_{i=1}^{k} a_i \mathbf{1}_{S_i \times T_i} \right\|_\square \le \epsilon.$$
The above conclusion allows one to approximate an arbitrary graph(on) as a sum of most ϵ^{-2} components. In the next following parts, you will show how to recover a regularity partition from the approximation above.

(b) Show that the stepping operator $\mathcal{P}$ is contractive with respect to the cut norm, in the sense that if $W\colon [0,1]^2 \to \mathbb{R}$ is a measurable symmetric function, then $\|W_\mathcal{P}\|_\square \le \|W\|_\square$.

(c) Let $\mathcal{P}$ be a partition of $[0,1]$ into measurable sets. Let U be a graphon that is constant on $S \times T$ for each $S, T \in \mathcal{P}$. Show that for every graphon W, one has
$$\|W - W_\mathcal{P}\|_\square \le 2\|W - U\|_\square.$$

(d) Use (a) and (c) to give a different proof of the weak regularity lemma (with slightly worse bounds than the one given in class): show that for every $\epsilon > 0$ and every graphon W, there exists partition $\mathcal{P}$ of $[0,1]$ into $2^{O(1/\epsilon^2)}$ measurable sets such that $\|W - W_\mathcal{P}\|_\square \le \epsilon$.

E2. Define $W\colon [0,1]^2 \to \mathbb{R}$ by $W(x,y) = 2\cos(2\pi(x-y))$. Let G be a graph. Show that $t(G,W)$ is the number of ways to orient all edges of G so that every vertex has the same number of incoming edges as outgoing edges.

E3. Show that for every $\epsilon > 0$ there is some $C > 0$ such that if W is a graphon, and $S \subset [0,1]$ is a set of such that, writing $W \circ W(x,z) = \int_{[0,1]} W(x,y)W(y,z)\,dy$,
$$\int_{[0,1]} |W \circ W(s,z) - W \circ W(t,z)|\,dz > \epsilon$$
for all distinct $s, t \in S$, then $|S| \le C$.

E4. Let W be a $\{0,1\}$-valued graphon. Suppose graphons W_n satisfy $\|W_n - W\|_\square \to 0$ as $n \to \infty$. Show that $\|W_n - W\|_1 \to 0$ as $n \to \infty$.

ps5

E5. *"Regularity lemma" for bounded degree graphs.* The r-*local sample* of a graph G is defined to be the random rooted graph induced by all vertices within distance r from a uniform random vertex v of G, and setting v to be the root.

Show that for every $\epsilon > 0$ and $r, \Delta \in \mathbb{N}$ there exists $M = M(\epsilon, r, \Delta)$ such that if G is a graph with maximum degree at most Δ, then there exists a graph H on at most M vertices such that the r-local samples of G and H differ by at most ϵ in total variation distance.

E6. *Strong regularity lemma.* In this problem, you will give an alternate proof of the strong regularity lemma with explicit bounds.

Let $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \dots)$ be a sequence of positive reals. By repeatedly applying the weak regularity lemma, show that there is some $M = M(\boldsymbol{\epsilon})$ such that for every graphon W, there is a pair of partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[0,1]$ into measurable sets, such that $\mathcal{Q}$ refines $\mathcal{P}$, $|\mathcal{Q}| \le M$ (here $|\mathcal{Q}|$ denotes the number of parts of $\mathcal{Q}$),

$$\|W - W_{\mathcal{Q}}\|_\square \le \epsilon_{|\mathcal{P}|} \qquad \text{and} \qquad \|W_{\mathcal{Q}}\|_2^2 \le \|W_{\mathcal{P}}\|_2^2 + \epsilon_1^2.$$

Furthermore, deduce the strong regularity lemma in the following form: one can write

$$W = W_{\mathrm{str}} + W_{\mathrm{psr}} + W_{\mathrm{sml}}$$

where W_{str} is a k-step-graphon with $k \le M$, $\|W_{\mathrm{psr}}\|_\square \le \epsilon_k$, and $\|W_{\mathrm{sml}}\|_1 \le \epsilon_1$. State your bounds on M explicitly in terms of $\boldsymbol{\epsilon}$. (Note: the parameter choice $\epsilon_k = \epsilon/k^2$ roughly corresponds to Szemerédi's regularity lemma, in which case your bound on M should be an exponential tower of 2's of height $\epsilon^{-O(1)}$; if not then you are doing something wrong.)

ps5 E7. *Inverse counting lemma.* Using the moments lemma ($t(F,U) = t(F,W)$ for all F implies $\delta_\square(U,W) = 0$) and compactness of the space of graphons, deduce that for every $\epsilon > 0$, there exist $k \in \mathbb{N}$ and $\eta > 0$ such that if U and W are graphons such that $|t(F,U) - t(F,W)| \le \eta$ for all graphs F on k vertices, then $\delta_\square(U,W) \le \epsilon$.

ps5⋆ E8. *Generalized maximum cut.* For symmetric measurable functions $W, U \colon [0,1]^2 \to \mathbb{R}$, define

$$\mathcal{C}(W,U) := \sup_{\varphi} \langle W, U^{\varphi} \rangle = \sup_{\varphi} \int W(x,y) U(\varphi(x), \varphi(y))\, dx dy,$$

where φ ranges over all measure-preserving bijections on $[0,1]$. Extend the definition of $\mathcal{C}(\cdot,\cdot)$ to graphs by $\mathcal{C}(G,\cdot) := \mathcal{C}(W_G,\cdot)$, etc.

(a) Is $\mathcal{C}(U,W)$ continuous jointly in (U,W) with respect to the cut norm? Is it continuous in U if W is held fixed?

(b) Show that if W_1 and W_2 are graphons such that $\mathcal{C}(W_1,U) = \mathcal{C}(W_2,U)$ for all graphons U, then $\delta_\square(W_1,W_2) = 0$.

(c) Let $G_1, G_2, \ldots$ be a sequence of graphs such that $\mathcal{C}(G_n,U)$ converges as $n \to \infty$ for every graphon U. Show that $G_1, G_2, \ldots$ is convergent.

(d) Can the hypothesis in (c) be replaced by "$\mathcal{C}(G_n,H)$ converges as $n \to \infty$ for every graph H"?

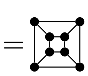
E9. (a) Let G_1 and G_2 be two graphs such that $\hom(F,G_1) = \hom(F,G_2)$ for every graph F. Show that G_1 and G_2 are isomorphic.

(b) Let G_1 and G_2 be two graphs such that $\hom(G_1,H) = \hom(G_2,H)$ for every graph H. Show that G_1 and G_2 are isomorphic.

E10. Fix $0 < p < 1$. Let G be a graph on n vertices with average degree at least pn. Prove:

(a) The number of labeled copies of $K_{3,3}$ in G is at least $(p^9 - o(1))n^6$.

ps5 (b) The number of labeled 6-cycles in G is at least $(p^6 - o(1))n^6$. (You may not use part (d) for part (b))

ps5 (c) The number of labeled copies of $Q_3 =$  in G is at least $(p^{12} - o(1))n^8$.

ps5⋆ (d) The number of labeled paths on 4 vertices in G is at least $(p^3 - o(1))n^4$.

ps5⋆ E11. Let $\mathcal{F}_m$ denote the set of all m-edge graphs without isolated vertices (up to isomorphism). Suppose $p \in [0,1]$ is a constant, and G_n is a sequence of graphs such that

$$\lim_{n\to\infty} \sum_{F \in \mathcal{F}_m} t(F, G_n) = \sum_{F \in \mathcal{F}_m} p^{|E(F)|}$$

for every positive integer m. Prove that G_n converges to the constant graphon p.

ps5⋆ E12. Prove there is a function $f\colon [0,1] \to [0,1]$ with $f(x) \geq x^2$ and $\lim_{x\to 0} f(x)/x^2 = \infty$ such that

$$t(K_4^-, W) \geq f(t(K_3, W))$$

for all graphons W. Here K_4^- is K_4 with one edge removed.

F. Fourier analysis and linear patterns

Some conventions: for $f\colon \mathbb{F}_p^n \to \mathbb{C}$ with prime p,

- $\widehat{f}(r) = \mathbb{E}_{x \in \mathbb{F}_p^n} f(x)\omega^{-r\cdot x}$ where $\omega = e^{2\pi i/p}$
- $\|f\|_s := (\mathbb{E}[|f|^s])^{1/s}$
- $\|\widehat{f}\|_\infty = \max_{r \in \mathbb{F}_p^n} |\widehat{f}(r)|$

ps5 F1. *Fourier does not control 4-AP counts.* Let $A = \{x \in \mathbb{F}_5^n : x \cdot x = 0\}$. Write $N = 5^n$.

(a) Show that $|A| = (1/5 + o(1))N$ and $|\widehat{1_A}(r)| = o(1)$ for all $r \neq 0$.

(b) Show that $|\{(x,y) \in \mathbb{F}_5^n : x, x+y, x+2y, x+3y \in A\}| \neq (5^{-4} + o(1))N^2$.

ps6 F2. *Linearity testing.* Show that for every prime p and real $\epsilon > 0$, there exists $\delta > 0$ such that if $f\colon \mathbb{F}_p^n \to \mathbb{F}_p$ is a function such that

$$\mathbb{P}_{x,y \in \mathbb{F}_p^n}(f(x) + f(y) = f(x+y)) \geq 1 - \delta$$

then there exists some $a \in \mathbb{F}_p^n$ such that

$$\mathbb{P}_{x \in \mathbb{F}_p^n}(f(x) = a_1x_1 + \cdots + a_nx_n) \geq 1 - \epsilon,$$

where in the above $\mathbb{P}$ expressions x and y are chosen i.i.d. uniform from $\mathbb{F}_p^n$.

ps6 F3. *Counting solutions to a single linear equation.*

(a) Given a function $f\colon \mathbb{Z} \to \mathbb{C}$ with finite support, define $\widehat{f}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by

$$\widehat{f}(t) = \sum_{n\in\mathbb{Z}} f(n) e^{-2\pi i n t}.$$

Let $c_1, \dots, c_k \in \mathbb{Z}$. Let $A \subset \mathbb{Z}$ be a finite set. Show that

$$|\{(a_1, \dots, a_k) \in A^k : c_1a_1 + \cdots + c_ka_k = 0\}| = \int_0^1 \widehat{1_A}(c_1t)\widehat{1_A}(c_2t)\cdots\widehat{1_A}(c_kt)\, dt.$$

(b) Show that if a finite set A of integers contains $\beta|A|^2$ solutions $(a,b,c) \in A^3$ to $a+2b = 3c$, then it contains at least $\beta^2|A|^3$ solutions $(a,b,c,d) \in A^4$ to $a+b = c+d$.

ps6 F4. Let $a_1, \dots, a_m, b_1, \dots, b_m, c_1, \dots, c_m \in \mathbb{F}_2^n$. Suppose that the equation $a_i + b_j + c_k = 0$ holds if and only if $i = j = k$. Show that there is some constant $\epsilon > 0$ such that $m \leq (2-\epsilon)^n$ for all sufficiently large n.

F5. *Strong arithmetic regularity lemma.* Show that for every $\boldsymbol{\epsilon} = (\epsilon_0, \epsilon_1, \dots)$ with $1 \ge \epsilon_0 \ge \epsilon_1 \ge \cdots$ there exists $m = m(\boldsymbol{\epsilon})$ such that for every $f\colon \mathbb{F}_3^n \to [0,1]$ there exist a pair of subspaces $W \le U$ of $\mathbb{F}_3^n$ with codimW $\le m$ and a decomposition

$$f = f_{\mathrm{str}} + f_{\mathrm{psr}} + f_{\mathrm{sml}}$$

such that

- $f_{\mathrm{str}} = f_U$ and $f_{\mathrm{str}} + f_{\mathrm{sml}} = f_W$,
- $\|\widehat{f_{\mathrm{psr}}}\|_\infty \le \epsilon_{\operatorname{codim} U}$
- $\|f_{\mathrm{sml}}\|_2 \le \epsilon_0$

F6. *Counting lemma for 3-APs with restricted differences.* Let $f\colon \mathbb{F}_3^n \to [0,1]$ be written as $f = f_{\mathrm{str}} + f_{\mathrm{psr}} + f_{\mathrm{sml}}$ where

- f_{str} and $f_{\mathrm{str}} + f_{\mathrm{sml}}$ take values in $[0,1]$,
- $\|\widehat{f_{\mathrm{psr}}}\|_\infty \le \eta$, and
- $\|f_{\mathrm{sml}}\|_2 \le \epsilon$.

Let U be a subspace of $\mathbb{F}_3^n$. Show that there is some absolute constant C so that

$$\left|\mathbb{E}_{x\in\mathbb{F}_3^n, y\in U}(f(x)f(x+y)f(x+2y) - f_{\mathrm{str}}(x)f_{\mathrm{str}}(x+y)f_{\mathrm{str}}(x+2y)\right|) \le C(|U^\perp|\eta + \epsilon)$$

F7. *Gowers U^2 uniformity norm.* Let Γ be a finite abelian group. For $f\colon \Gamma \to \mathbb{C}$, define

$$\|f\|_{U^2} := \left(\mathbb{E}_{x,h,h'\in\Gamma} f(x)\overline{f(x+h)f(x+h')}f(x+h+h')\right)^{1/4}.$$

(a) Show that the expectation above is always a nonnegative real number, so that the above expression is well defined. Also, show that $\|f\|_{U^2} \ge |\mathbb{E} f|$.

(b) For $f_1, f_2, f_3, f_4\colon \Gamma \to \mathbb{C}$, let

$$\langle f_1, f_2, f_3, f_4\rangle = \mathbb{E}_{x,h,h'\in\Gamma} f_1(x)\overline{f_2(x+h)f_3(x+h')}f_4(x+h+h').$$

Prove that

$$|\langle f_1, f_2, f_3, f_4\rangle| \le \|f_1\|_{U^2}\,\|f_2\|_{U^2}\,\|f_3\|_{U^2}\,\|f_4\|_{U^2}$$

(c) By noting that $\langle f_1, f_2, f_3, f_4\rangle$ is multilinear, and using part (b), show that

$$\|f+g\|_{U^2} \le \|f\|_{U^2} + \|g\|_{U^2}.$$

Conclude that $\|\ \|_{U^2}$ is a norm.

(d) Show that $\|f\|_{U^2} = \|\widehat{f}\|_{\ell^4}$, i.e.,

$$\|f\|_{U^2}^4 = \sum_{\gamma\in\widehat{\Gamma}} |\widehat{f}(\gamma)|^4.$$

Furthermore, deduce that if $\|f\|_\infty \le 1$, then

$$\|\widehat{f}\|_\infty \le \|f\|_{U^2} \le \|\widehat{f}\|_\infty^{1/2}.$$

(This gives a so-called "inverse theorem" for the U^2 norm: if $\|f\|_{U^2} \ge \delta$ then $|f(\gamma)| \ge \delta^2$ for some $\gamma \in \widehat{\Gamma}$, i.e., if f is not U^2-uniform, then it must correlate with some character.)

G. Structure of set addition

ps6 G1. Show that for every real $K \geq 1$ there is some C_K such that for every finite set A of an abelian group with $|A+A| \leq K|A|$, one as $|nA| \leq n^{C_K}|A|$ for every positive integer n.

ps6⋆ G2. Show that there is some constant C so that if S is a finite subset of an abelian group, and k is a positive integer, then $|2kS| \leq C^{|S|}\,|kS|$.

ps6⋆ G3. Show that for every sufficiently large K there is there some finite set $A \subset \mathbb{Z}$ such that $|A+A| \leq K|A|$ and $|A-A| \geq K^{1.99}|A|$.

ps6⋆ G4. Show that for every finite subsets A, B, C in an abelian group, one has

$$|A+B+C|^2 \leq |A+B|\,|A+C|\,|B+C|\,.$$

ps6 G5. Let $A \subset \mathbb{Z}$ with $|A| = n$.

(a) Let p be a prime. Show that there is some integer t relatively prime to p such that $\|at/p\|_{\mathbb{R}/\mathbb{Z}} \leq p^{-1/n}$ for all $a \in A$.

(b) Show that A is Freiman 2-isomorphic to a subset of $[N]$ for some $N = (4+o(1))^n$.

(c) Show that (b) cannot be improved to $N = 2^{n-2}$.

(You may use the fact that the smallest prime larger than m has size $m + o(m)$.)

G6. Let $r_3(N)$ denote the size of the largest subset of $[N]$ without a 3-AP. Show that there is some constant $c > 0$ so that if A is 3-AP-free, then $|A+A| \geq c|A|^{1+c} r_3(|A|)^{-c}$.

ps6 G7. Let $A \subset \mathbb{F}_2^n$ with $|A| = \alpha 2^n$.

(a) Show that if $|A+A| < 0.99 \cdot 2^n$, then there is some $r \in \mathbb{F}_2^n \setminus \{0\}$ such that $|\widehat{1_A}(r)| > c\alpha^{3/2}$ for some constant $c > 0$.

(b) By iterating (a), show that $A+A$ contains 99% of a subspace of codimension $O(\alpha^{-1/2})$.

(c) Deduce that $4A$ contains a subspace of codimension $O(\alpha^{-1/2})$ (i.e., Bogolyubov's lemma with better bounds than the one shown in class)