18.217 PROBLEM SET (FALL 2019)

Instructions:

- All submissions must be **typed in LaTeX** and submitted as PDF on Stellar (try Overleaf if you are looking for an online LaTeX editor without requiring installations). Please name your file `ps#_Lastname_Firstname.pdf` and remember to include your name in each file.
  Suggested LaTeXtemplate for homework submissions.
- Please acknowledge, **individually for every problem at the beginning of each solution**, a list of all collaborators and sources consulted (people, books, websites, etc.). Write *sources consulted: none* even if no sources are consulted. Failure acknowledge sources will lead to an automatic 10% penalty.
- You may not look up solutions to homework problems online or offline.
- Please turn in the problems marked ps1 and ps1⋆ for problem set 1, etc., by midnight of each due date (see course homepage). Do not submit the other problems—they are for you to practice.
- *Late policy.* Late submissions will be penalized by 20% per each late day. For example, for an assignment due on Sunday, a submission worth x points if turned in on time will be worth $0.6x$ points if submitted on Tuesday.
- *Collaboration policy.* You are strongly encouraged to start early and first work on the problems on your own. Reasonable collaboration is permitted, but everyone must write their solutions individually and acknowledge their collaborators.
- *Bonus problems*, marked by ⋆, are more challenging. A grade of A- may be attained by only solving the non-starred problems. To attain a grade of A or A+, you should solve a substantial number of starred problems. (No hints will be given for bonus problems, e.g., during office hours.)
- Please try to **fit your solution within one page** for each unstarred problem/part (standard 1-inch margins and 11pt font). The spirit of this policy is to encourage you to think first before you write. Distill your ideas, structure your arguments, and eliminate unnecessary steps. If necessary, some details of routine calculations may be skipped provided that you give convincing explanations.
- This file will be updated as the term progresses. Please check back regularly. There will be an announcement whenever each problem set is complete.
- You are encouraged to include figures whenever they are helpful. Here are some recommended ways to produce figures in decreasing order of learning curve difficulty:
  (1) TikZ or other drawing script
  (2) IPE (which supports LaTeX) or other drawing app
  (3) photo/scan (I recommend the Dropbox app on your phone, which has a nice scanning feature that produces clear monochrome scans)

Problems begin on the next page. Last updated: September 12, 2019

A. Introduction

ps1 A1. *Ramsey's theorem*

(a) Let s and r be positive integers. Show that there is some integer $n = n(s,r)$ so that if every edge of the complete graph K_n on n vertices is colored with one of r colors, then there is a monochromatic copy of K_s.

(b) Let $s \geq 3$ be a positive integer. Show that if the edges of the complete graph on $\binom{2s-2}{s-1}$ vertices are colored with 2 colors, then there is a monochromatic copy of K_s.

ps1 A2. Prove that it is possible to color $\mathbb{N}$ using two colors so that there is no infinitely long monochromatic arithmetic progression.

ps1 A3. *Many monochromatic triangles*

(a) True or false: If the edges of K_n are colored using 2 colors, then at least $1/4 - o(1)$ fraction of all triangles are monochromatic. (Note that $1/4$ is the fraction one expects if the edges were colored randomly.)

(b) True or false: if the edges of K_n are colored using 3 colors, then at least $1/9 - o(1)$ fraction of all triangles are monochromatic.

(c) ($\star$ do not submit) True or false: if the edges of K_n are colored using 2 colors, then at least $1/32 - o(1)$ fraction of all triangles are monochromatic.

B. Forbidding subgraphs

ps1 B1. Show that a graph with n vertices and m edges has at least $\frac{4m}{3n}\left(m - \frac{n^2}{4}\right)$ triangles.

ps1$\star$ B2. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains at least $\lfloor n/2 \rfloor$ triangles.

ps1$\star$ B3. Prove that every n-vertex graph with at least $\lfloor n^2/4 \rfloor + 1$ edges contains some edge in at least $(1/6 - o(1))n$ triangles, and that this constant $1/6$ is best possible.

ps1 B4. Let G be an n-vertex triangle free graph with at least $\lfloor n^2/4 \rfloor - k$ edges. Prove that G can be made bipartite by removing at most k edges.

B5. Let G be a K_{r+1}-free graph. Prove that there is another graph H on the same vertex set as G such that $\chi(H) \leq r$ and $d_H(x) \geq d_G(x)$ for every vertex x (here $d_H(x)$ is the degree of x in H, and likewise with $d_G(x)$ for G). Give another proof of Turán's theorem from this fact.

B6. *Turán density.* Let H be a r-uniform hypergraph, let its *Turán number* $\mathrm{ex}^{(r)}(n,H)$ be the maximum number of edges in an r-uniform hypergraph on n vertices that does not contain H as a subgraph. Prove that the fraction $\mathrm{ex}^{(r)}(n,H)/\binom{n}{r}$ is a nonincreasing function of n, so that it has a limit $\rho(H)$ as $n \to \infty$, called the *Turán density* of H.

ps1 B7. *Supersaturation.* Let H be a graph and ρ a constant such that $\limsup_{n\to\infty} \mathrm{ex}(n,H)/\binom{n}{2} \leq \rho$. Prove that for every $\epsilon > 0$ there exists some constant $c = c(H,\epsilon) > 0$ such that for sufficiently large n, every n-vertex graph with at least $(\rho + \epsilon)\binom{n}{2}$ edges contains at least $cn^{v(H)}$ copies of H.