

## 18.217 PROBLEM SET (FALL 2019)

### Instructions:

- All submissions must be **typed in L<sup>A</sup>T<sub>E</sub>X** and submitted as PDF on Stellar (try Overleaf if you are looking for an online L<sup>A</sup>T<sub>E</sub>X editor without requiring installations). Please name your file `ps#_Lastname_Firstname.pdf` and remember to include your name in each file.  
Suggested L<sup>A</sup>T<sub>E</sub>X template for homework submissions.
- Please acknowledge, **individually for every problem at the beginning of each solution**, a list of all collaborators and sources consulted (people, books, websites, etc.). Write `sources consulted: none` even if no sources are consulted. Failure to acknowledge sources will lead to an automatic 10% penalty.
- You may not look up solutions to homework problems online or offline.
- Please turn in the problems marked `ps1` and `ps1★` for problem set 1, etc., by midnight of each due date (see course homepage). Do not submit the other problems—they are for you to practice.
- *Late policy.* Late submissions will be penalized by 20% per each late day. For example, for an assignment due on Sunday, a submission worth  $x$  points if turned in on time will be worth  $0.6x$  points if submitted on Tuesday.
- *Collaboration policy.* You are strongly encouraged to start early and first work on the problems on your own. Reasonable collaboration is permitted, but everyone must write their solutions individually and acknowledge their collaborators.
- *Bonus problems*, marked by ★, are more challenging. A grade of A- may be attained by only solving the non-starred problems. To attain a grade of A or A+, you should solve a substantial number of starred problems. (No hints will be given for bonus problems, e.g., during office hours.)
- Please try to **fit your solution within one page** for each unstarred problem/part (standard 1-inch margins and 11pt font). The spirit of this policy is to encourage you to think first before you write. Distill your ideas, structure your arguments, and eliminate unnecessary steps. If necessary, some details of routine calculations may be skipped provided that you give convincing explanations.
- This file will be updated as the term progresses. Please check back regularly. There will be an announcement whenever each problem set is complete.
- You are encouraged to include figures whenever they are helpful. Here are some recommended ways to produce figures in decreasing order of learning curve difficulty:
  - (1) TikZ or other drawing script
  - (2) IPE (which supports LaTeX) or other drawing app
  - (3) photo/scan (I recommend the Dropbox app on your phone, which has a nice scanning feature that produces clear monochrome scans)

Problems begin on the next page.

Last updated: September 24, 2019

## A. INTRODUCTION

- ps1 A1. *Ramsey's theorem*
- (a) Let  $s$  and  $r$  be positive integers. Show that there is some integer  $n = n(s, r)$  so that if every edge of the complete graph  $K_n$  on  $n$  vertices is colored with one of  $r$  colors, then there is a monochromatic copy of  $K_s$ .
  - (b) Let  $s \geq 3$  be a positive integer. Show that if the edges of the complete graph on  $\binom{2s-2}{s-1}$  vertices are colored with 2 colors, then there is a monochromatic copy of  $K_s$ .
- ps1 A2. Prove that it is possible to color  $\mathbb{N}$  using two colors so that there is no infinitely long monochromatic arithmetic progression.
- ps1 A3. *Many monochromatic triangles*
- (a) True or false: If the edges of  $K_n$  are colored using 2 colors, then at least  $1/4 - o(1)$  fraction of all triangles are monochromatic. (Note that  $1/4$  is the fraction one expects if the edges were colored uniformly at random.)
  - (b) True or false: if the edges of  $K_n$  are colored using 3 colors, then at least  $1/9 - o(1)$  fraction of all triangles are monochromatic.
  - (c) ( $\star$  do not submit) True or false: if the edges of  $K_n$  are colored using 2 colors, then at least  $1/32 - o(1)$  fraction of all copies of  $K_4$ 's are monochromatic.

## B. FORBIDDING SUBGRAPHS

- ps1 B1. Show that a graph with  $n$  vertices and  $m$  edges has at least  $\frac{4m}{3n} \left(m - \frac{n^2}{4}\right)$  triangles.
- ps1 $\star$  B2. Prove that every  $n$ -vertex graph with at least  $\lfloor n^2/4 \rfloor + 1$  edges contains at least  $\lfloor n/2 \rfloor$  triangles.
- ps1 $\star$  B3. Prove that every  $n$ -vertex graph with at least  $\lfloor n^2/4 \rfloor + 1$  edges contains some edge in at least  $(1/6 - o(1))n$  triangles, and that this constant  $1/6$  is best possible.
- B4.  *$K_{r+1}$ -free graphs close to the Turán bound are nearly  $r$ -partite*
- ps1 (a) Let  $G$  be an  $n$ -vertex triangle-free graph with at least  $\lfloor n^2/4 \rfloor - k$  edges. Prove that  $G$  can be made bipartite by removing at most  $k$  edges.
- ps1 $\star$  (b) Let  $G$  be an  $n$ -vertex  $K_{r+1}$ -free graph with at least  $e(T_{n,r}) - k$  edges, where  $T_{n,r}$  is the Turán graph. Prove that  $G$  can be made  $r$ -partite by removing at most  $k$  edges.
- B5. Let  $G$  be a  $K_{r+1}$ -free graph. Prove that there is another graph  $H$  on the same vertex set as  $G$  such that  $\chi(H) \leq r$  and  $d_H(x) \geq d_G(x)$  for every vertex  $x$  (here  $d_H(x)$  is the degree of  $x$  in  $H$ , and likewise with  $d_G(x)$  for  $G$ ). Give another proof of Turán's theorem from this fact.
- B6. *Turán density*. Let  $H$  be a  $r$ -uniform hypergraph, let its *Turán number*  $\text{ex}^{(r)}(n, H)$  be the maximum number of edges in an  $r$ -uniform hypergraph on  $n$  vertices that does not contain  $H$  as a subgraph. Prove that the fraction  $\text{ex}^{(r)}(n, H)/\binom{n}{r}$  is a nonincreasing function of  $n$ , so that it has a limit  $\rho(H)$  as  $n \rightarrow \infty$ , called the *Turán density* of  $H$ .
- ps1 B7. *Supersaturation*. Let  $H$  be a graph and  $\rho$  a constant such that  $\limsup_{n \rightarrow \infty} \text{ex}(n, H)/\binom{n}{2} \leq \rho$ . Prove that for every  $\epsilon > 0$  there exists some constant  $c = c(H, \epsilon) > 0$  such that for sufficiently large  $n$ , every  $n$ -vertex graph with at least  $(\rho + \epsilon)\binom{n}{2}$  edges contains at least  $cn^{v(H)}$  copies of  $H$ .

- ps1** B8. Let  $S$  be a set of  $n$  points in the plane, with the property that no two points are at distance greater than 1. Show that  $S$  has at most  $\lfloor n^2/3 \rfloor$  pairs of points at distance greater than  $1/\sqrt{2}$ . Also, show that the bound  $\lfloor n^2/3 \rfloor$  is tight (i.e., cannot be improved).

- ps1** B9. (How *not* to define density in a product set) Let  $S \subset \mathbb{Z}^2$ . Define

$$d_k(S) = \max_{\substack{A, B \subset \mathbb{Z} \\ |A|=|B|=k}} \frac{|S \cap (A \times B)|}{|A||B|}.$$

Show that  $\lim_{k \rightarrow \infty} d_k(S)$  exists and is always either 0 or 1.

- B10. Show that, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that every graph with  $n$  vertices and at least  $\epsilon n^2$  edges contains a copy of  $K_{s,t}$  where  $s \geq \delta \log n$  and  $t \geq n^{0.99}$ .

- ps2** B11. *Density version of Kővári–Sós–Turán.* Prove that for every positive integers  $s \leq t$ , there are constants  $C, c > 0$  such that every  $n$ -vertex graph with  $p \binom{n}{2}$  edges contains at least  $cp^{st}n^{s+t}$  copies of  $K_{s,t}$ , provided that  $p \geq Cn^{-1/s}$ .

- ps2** B12. *Hypergraph Kővári–Sós–Turán and a proof of Erdős–Stone–Simonovits*

- (a) Prove that for every positive integer  $t$  there is some  $C$  so that every 3-uniform hypergraph on  $n$  vertices and at least  $Cn^{3-t-2}$  edges (i.e., triples) contains a copy of  $K_{t,t,t}^{(3)}$ , the complete tripartite 3-uniform hypergraph with  $t$  vertices in each part.
- (b) Deduce that  $\text{ex}(n, H) \leq (\frac{1}{4} + o(1))n^2$  for every graph  $H$  with  $\chi(H) \leq 3$ .
- (c) Explain how to generalize the above strategy to prove the Erdős–Stone–Simonovits theorem for every  $H$  (sketch the key steps).

- ps2** B13. Find a graph  $H$  with  $\chi(H) = 3$  and  $\text{ex}(n, H) > \frac{1}{4}n^2 + n^{1.99}$  for all sufficiently large  $n$ .

- ps2\*** B14. *Construction of a  $C_6$ -free graph.* Let  $q$  be an odd prime power. Let  $S$  denote the quadratic surface in the 4-dimensional projective space over  $\mathbb{F}_q$  (whose points are nonzero points of  $\mathbb{F}_q^5$  modulo the equivalence relation  $(x_0, x_1, x_2, x_3, x_4) \sim (\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4)$  for  $\lambda \in \mathbb{F}_q^\times$ ) given by the equation (you may use another quadratic form if you wish)

$$x_0^2 + 2x_1x_2 + 2x_3x_4 = 0.$$

Let  $\mathcal{L}$  be the set of lines contained in  $S$ .

- (a) Prove that no three lines of  $\mathcal{L}$  lie in the same plane.
- (b) Show that the point-line incidence bipartite graph between  $S$  and  $\mathcal{L}$  is a  $(q+1)$ -regular graph on  $2(q^3 + q^2 + q + 1)$  vertices with no cycles of length at most 6. Conclude that  $\text{ex}(n, C_6) \geq cn^{4/3}$  for some constant  $c > 0$ .

*The next two problems concern the dependent random choice technique.*

- ps2** B15. Let  $\epsilon > 0$ . Show that, for sufficiently large  $n$ , every  $K_4$ -free graph with  $n$  vertices and at least  $\epsilon n^2$  edges contains an independent set of size at least  $n^{1-\epsilon}$ .

- ps2\*** B16. *Extremal numbers of degenerate graphs*

- (a) Prove that there is some absolute constant  $c > 0$  so that for every positive integer  $r$ , every  $n$ -vertex graph with at least  $n^{2-c/r}$  edges contains disjoint vertex subsets  $A$  and  $B$  such that every subset of  $r$  vertices in  $A$  has at least  $n^c$  neighbors in  $B$  and every subset of  $r$  vertices in  $B$  has at least  $n^c$  neighbors in  $A$ .

- (b) We say that a graph  $H$  is  $r$ -degenerate if its vertices can be ordered so that every vertex has at most  $r$  neighbors that appear before it in the ordering. Show that for every  $r$ -degenerate graph  $H$  there is some constant  $C > 0$  so that  $\text{ex}(n, H) \leq Cn^{2-c/r}$ , where  $c$  is the same absolute constant from part (a) ( $c$  should not depend on  $H$  or  $r$ ).

ps2
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- B17. Let  $T$  be a tree with  $k$  edges. Show that  $\text{ex}(n, T) \leq kn$ .