18.S997 (FALL 2017) PROBLEM SET 2

1. Let the *half-graph* H_n be the bipartite graph on $2n$ vertices $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ with edges $\{a_i b_j : i \le j\}$.
   (a) For every $\epsilon > 0$, explicitly construct an ϵ-regular partition of H_n into $O(1/\epsilon)$ parts.
   (b) Show that there is some $c > 0$ such that for every sufficiently small $\epsilon > 0$, every integer k and sufficiently large multiple n of k, every partition of the vertices of H_n into k equal-sized parts contains at least ck pairs of parts which are not ϵ-regular.
2. Show that there is some absolute constant $C > 0$ such that for every $\epsilon > 0$, every graph on n vertices contains an ϵ-regular pair of vertex subsets each with size at least δn, where $\delta = 2^{-\epsilon^{-C}}$.
3. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that every graph on n vertices contains an ϵ-regular subset of vertices of size at least δn. (Here a vertex subset X is called an *ϵ-regular set* if the pair (X, X) is ϵ-regular, i.e., for all $A, B \subset X$ with $|A|, |B| \ge \epsilon |X|$, one has $|d(A, B) - d(X, X)| \le \epsilon$.)
4. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subset [n]$ has fewer than δn^2 many triples $(x, y, z) \in A^3$ with $x + y = z$, then there is some $B \subset A$ with $|A \setminus B| \le \epsilon n$ such that B is sum-free, i.e., there do not exist $x, y, z \in B$ with $x + y = z$.
5. Show that the number of triangle-free graphs on n labeled vertices is $2^{(1/4+o(1))n^2}$.
6. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every K_4-free graph on n vertices with at least $(\frac{1}{8} + \epsilon)n^2$ edges contains an independent set of size at least δn.
7. For this problem you may assume either the tetrahedron[1] removal lemma for 3-uniform hypergraphs or its following corollary:

   A 3-uniform hypergraph with n vertices, where every hyperedge is contained in a unique tetrahedron, has $o(n^3)$ hyperedges.

   Deduce that if $A \subset [N]^2$ contains no axes-aligned squares (i.e., four points of the form $(x, y), (x + d, y), (x, y + d), (x + d, y + d)$, where $d \ne 0$), then $|A| = o(N^2)$.
8. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if an n-vertex graph G satisfies

$$|e(X, Y) - p|X||Y|| \le \delta p^2 n \sqrt{|X||Y|} \qquad \text{for all } X, Y \subset V(G)$$

   for some p, then the number of triangles in G is at least $(1 - \epsilon)p^3 \binom{n}{3}$.
9. Let G be an n-vertex d-regular graph. Suppose n is divisible by k. Color the vertices of G with k colors (not necessarily a proper coloring) such that each color appears exactly n/k times. Suppose that all eigenvalues, except the top one, of the adjacency matrix of G are at most d/k in absolute value. Show that there is a vertex of G whose neighborhood contains all k colors.

Problem set complete. Some hints on next page (but try the problems yourself first)

[1] A tetrahedron is the set of all 3-element subsets of a 4-element vertex set—think the faces of a geometric tetrahedron

Hints

5. You may find the following estimate helpful: $\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{ne}{k}\right)^k$ for all $1 \le k \le n$.
6. Given an ϵ'-regular pair of vertex sets with edge-density slightly above $1/2$, find either a K_4 or a large independent set.
7. Compare to the proof in class for Szemerédi's theorem for 4-term APs. Re-parameterize the square using 4 variables so that each point of the square uses exactly 3 of the 4 variables.