18.226 PROBLEM SET (FALL 2020)

Helpful tips:

- Please read the course homepage carefully regarding homework policies (due dates, lateness, acknowledging sources on each problem, etc.)
- Only turn in problems marked $\boxed{\textsf{ps1}}$ and $\boxed{\textsf{ps1}\star}$ for problem set 1, etc. You are recommended to try the other problems for practice, but do not submit them.
- In a multipart problem, if a later part is marked for submission, it may be helpful to think about the earlier unassigned parts first.
- **Bonus problems**, marked by $\star$, are more challenging. A grade of A- may be attained by only solving the non-starred problems. To attain a grade of A or A+, you should solve a substantial number of starred problems. No hints will be given for bonus problems, e.g., during office hours.
- **Start each solution on a new page**, and try to **fit your solution within one page** for each unstarred problem/part (without abusing font/margins). The spirit of this policy is to encourage you to think first before you write. Distill your ideas, structure your arguments, and eliminate unnecessary steps. If necessary, some details of routine calculations may be skipped provided that you give precise statements and convincing explanations.
- This file will be updated as the term progresses. Please check back regularly. There will be an announcement whenever each problem set is complete.
- You are encouraged to include figures whenever they are helpful. Here are some recommended ways to produce figures in decreasing order of learning curve difficulty:
  (1) TikZ
  (2) IPE (which supports LaTeX), Powerpoint, or other drawing app
  (3) drawing on a tablet (e.g., Notability on iPad)
  (4) photo/scan (I recommend the Dropbox app on your phone, which has a nice scanning feature that produces clear monochrome scans)

Problems begin on the next page Last updated: September 21, 2020

A. Introduction and linearity of expectations

A1. Verify the following asymptotic calculations used in Ramsey number lower bounds:

(a) For each k, the largest n satisfying $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ has $n = \left(\frac{1}{e\sqrt{2}} + o(1)\right) k2^{k/2}$.

(b) For each k, the maximum value of $n - \binom{n}{k}2^{1-\binom{k}{2}}$ as n ranges over positive integers is $\left(\frac{1}{e} + o(1)\right) k2^{k/2}$.

(c) For each k, the largest n satisfying $e\left(\binom{k}{2}\binom{n}{k-2} + 1\right) 2^{1-\binom{k}{2}} < 1$ satisfies $n = \left(\frac{\sqrt{2}}{e} + o(1)\right) k2^{k/2}$.

A2. Prove that, if there is a real $p \in [0,1]$ such that

$$\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$$

then the Ramsey number $R(k,t)$ satisfies $R(k,t) > n$. Using this show that

$$R(4,t) \geq c\left(\frac{t}{\log t}\right)^{3/2}$$

for some constant $c > 0$.

ps1 A3. Let G be a graph with n vertices and m edges. Prove that K_n can be written as a union of $O(n^2(\log n)/m)$ copies of G (not necessarily edge-disjoint).

A4. *Generalization of Sperner's theorem.* Let $\mathcal{F}$ be a collection of subset of $[n]$ that does not contain $k+1$ elements forming a chain: $A_1 \subsetneq \cdots \subsetneq A_{k+1}$. Prove that $\mathcal{F}$ is no larger than taking the union of the k levels of the boolean lattice closest to the middle layer.

ps1 A5. Let $A_1, \ldots, A_m$ be r-element sets and $B_1, \ldots, B_m$ be s-element sets. Suppose $A_i \cap B_i = \emptyset$ for each i, and for each $i \neq j$, either $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$. Prove that $m \leq (r+s)^{r+s}/(r^r s^s)$.

ps1 A6. Let G be a graph on $n \geq 10$ vertices. Suppose that adding any new edge to G would create a new clique on 10 vertices. Prove that G has at least $8n - 36$ edges.

Hint in white:

ps1⋆ A7. Prove that for every positive integer r, there exists an integer K such that the following holds. Let S be a set of rk points evenly spaced on a circle. If we partition $S = S_1 \cup \cdots \cup S_r$ so that $|S_i| = k$ for each i, then, provided $k \geq K$, there exist r congruent triangles where the vertices of the i-th triangle lie in S_i, for each $1 \leq i \leq r$.

ps1⋆ A8. Prove that $[n]^d$ cannot be partitioned into fewer than 2^d sets each of the form $A_1 \times \cdots \times A_d$ where $A_i \subsetneq [n]$.

A9. Let $k \geq 4$ and H a k-uniform hypergraph with at most $4^{k-1}/3^k$ edges. Prove that there is a coloring of the vertices of H by four colors so that in every edge all four colors are represented.

ps1 A10. Prove that there is an absolute constant $C > 0$ so that for every $n \times n$ matrix with distinct real entries, one can permute its rows so that no column in the permuted matrix contains an increasing subsequence of length at least $C\sqrt{n}$. (A subsequence does not need to be selected from consecutive terms. For example, $(1,2,3)$ is an increasing subsequence of $(1,5,2,4,3)$.)

ps1 A11. Given a set $\mathcal{F}$ of subsets of $[n]$ and $A \subseteq [n]$, write $\mathcal{F}|_A := \{S \cap A : S \in \mathcal{F}\}$ (its *projection* onto A). Prove that for every n and k, there exists a set $\mathcal{F}$ of subsets of $[n]$ with $|\mathcal{F}| = O(k2^k \log n)$ such that for every k-element subset A of $[n]$, $\mathcal{F}|_A$ contains all 2^k subsets of A.

ps1⋆ A12. Show that in every non-2-colorable n-uniform hypergraph, one can find at least $\frac{n}{2}\binom{2n-1}{n}$ unordered pairs of edges that intersect in exactly one vertex.

A13. Let A be a subset of the unit sphere in $\mathbb{R}^3$ (centered at the origin) containing no pair of orthogonal points.

ps1 (a) Prove that A occupies at most $1/3$ of the sphere in terms of surface area.

ps1⋆ (b) Prove an upper bound smaller than $1/3$ (give your best bound).

ps1⋆ A14. Prove that every set of 10 points in the plane can be covered by a union of disjoint unit disks.

A15. Let $\boldsymbol{r} = (r_1, \dots, r_k)$ be a vector of nonzero integers whose sum is nonzero. Prove that there exists a real $c > 0$ (depending on $\boldsymbol{r}$ only) such that the following holds: for every finite set A of nonzero *reals*, there exists a subset $B \subseteq A$ with $|B| \geq c|A|$ such that there do not exist $b_1, \dots, b_k \in B$ with $r_1 b_1 + \cdots + r_k b_k = 0$.

ps1 A16. Prove that every set A of n nonzero integers contains two disjoint subsets B_1 and B_2, such that both B_1 and B_2 are sum-free, and $|B_1| + |B_2| > 2n/3$. Can you do it if A is a set of nonzero reals?

A17. Let $M(n)$ denote the maximum number of edges in a 3-uniform hypergraph on n vertices without a clique on 4 vertices.

(a) Prove that $M(n+1)/\binom{n+1}{3} \leq M(n)/\binom{n}{3}$ for all n, and conclude that $M(n)/\binom{n}{3}$ approaches some limit α as $n \to \infty$.
(This limit is called the *Turán density* of the hypergraph $K_4^{(3)}$, and its exact value is currently unknown and is a major open problem.)

(b) Prove that for every $\delta > 0$, there exists $\epsilon > 0$ and n_0 so that every 3-uniform hypergraph with $n \geq n_0$ vertices and at least $(\alpha + \delta)\binom{n}{3}$ edges must contain at least $\epsilon\binom{n}{4}$ copies of the clique on 4 vertices.

A18. Prove that every graph with n vertices and $m \geq n^{3/2}$ edges contains a pair of vertex-disjoint and isomorphic subgraphs (not necessarily induced) each with at least $cm^{2/3}$ edges, where $c > 0$ is a constant.

B. Alteration method

B1. Using the alteration method, prove the Ramsey number bound

$$R(4, k) \geq c(k/\log k)^2$$

for some constant $c > 0$.

B2. Prove that every 3-uniform hypergraph with n vertices and $m \geq n$ edges contains an independent set (i.e., a set of vertices containing no edges) of size at least $cn^{3/2}/\sqrt{m}$, where $c > 0$ is a constant.

ps2 B3. Prove that every k-uniform hypergraph with n vertices and m edges has a transversal (i.e., a set of vertices intersecting every edge) of size at most $(n/k)\log(ekm/n)$.

ps2 B4. *Zarankiewicz problem.* Prove that for every positive integer $k \geq 2$, there exists a constant $c > 0$ such that for every n, there exists an $n \times n$ matrix with $\{0, 1\}$ entries, with at least $cn^{2-2/(k+1)}$ 1's, such that there is no $k \times k$ submatrix consisting of all 1's.

ps2

B5. Fix k. Prove that there exists a constant $c_k > 1$ so that for every sufficiently large $n > n_0(k)$, there exists a collection $\mathcal{F}$ of at least c_k^n subsets of $[n]$ such that for every k distinct $F_1, \dots, F_k \in \mathcal{F}$, all 2^k intersections $\bigcap_{i=1}^k G_i$ are nonempty, where each G_i is either F_i or $[n] \setminus F_i$.

B6. *Acute sets in* $\mathbb{R}^n$. Prove that, for some constant $c > 0$, for every n, there exists a family of at least $c(2/\sqrt{3})^n$ subsets of $[n]$ containing no three distinct members A, B, C satisfying $A \cap B \subseteq C \subseteq A \cup B$.

Deduce that there exists a set of at least $c(2/\sqrt{3})^n$ points in $\mathbb{R}^n$ so that all angles determined by three points from the set are acute.

Remark. The current best lower and upper bounds for the maximum size of an "acute set" in $\mathbb{R}^n$ (i.e., spanning only acute angles) are $2^{n-1} + 1$ and $2^n - 1$ respectively.