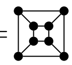
18.S997 (FALL 2017) PROBLEM SET 3

1. Fix $0 < p < 1$. Let G be a graph on n vertices with average degree at least pn. Prove:
   (a) The number of labeled 6-cycles in G is at least $(p^6 - o(1))n^6$.
   (b) The number of labeled copies of $K_{3,3}$ in G is at least $(p^9 - o(1))n^6$.
   (c) The number of labeled copies of $Q_3 =$  in G is at least $(p^{12} - o(1))n^8$.
   (d) (Bonus) The number of labeled paths on 4 vertices in G is at least $(p^3 - o(1))n^4$.
2. Deduce from the quasirandom Cayley graphs theorem the following corollary for vertex transitive graphs: If an n-vertex d-regular vertex-transitive graph G satisfies
$$\left|e(X,Y) - \tfrac{d}{n}|X||Y|\right| \le \epsilon dn \qquad \text{for all } X, Y \subseteq V(G),$$
then all the eigenvalues of the adjacency matrix of G, other than the largest one, are at most $8\epsilon dn$ in absolute value.
3. Define $W\colon [0,1]^2 \to \mathbb{R}$ by $W(x,y) = 2\cos(2\pi(x-y))$. Let G be a graph. Show that $t(G,W)$ is the number of ways to orient all edges of G so that every vertex has the same number of incoming edges as outgoing edges.
4. Let W be a $\{0,1\}$-valued graphon. Suppose graphons W_n satisfy $\|W_n - W\|_\square \to 0$ as $n \to \infty$. Show that $\|W_n - W\|_1 \to 0$ as $n \to \infty$.
5. (a) Let $\epsilon > 0$. Show that for every graphon $W\colon [0,1]^2 \to [0,1]$, there exist measurable sets $S_1, \dots, S_k, T_1, \dots, T_k \subseteq [0,1]$ and reals $a_1, \dots, a_k \in \mathbb{R}$, with $k < 1/\epsilon^2$, such that
$$\left\| W - \sum_{i=1}^k a_i \mathbf{1}_{S_i \times T_i} \right\|_\square \le \epsilon.$$
   (b) Let $\mathcal{P}$ be a partition of $[0,1]$ into measurable sets. Let U be a graphon that is constant on $S \times T$ for each $S, T \in \mathcal{P}$. For that for every graphon W, one has
$$\|W - W_{\mathcal{P}}\|_\square \le 2\|W - U\|_\square.$$
   (c) Use (a) and (b) to give a different proof of the weak regularity lemma (with slightly worse bounds than the one given in class): show that for every $\epsilon > 0$ and every graphon W, there exists partition $\mathcal{P}$ of $[0,1]$ into $2^{O(1/\epsilon^2)}$ measurable sets such that $\|W - W_{\mathcal{P}}\|_\square \le \epsilon$.

. . . to be continued . . . check back later (last updated: October 26, 2017). Some hints on next page

Hints

4. Every measurable set can be arbitrarily well approximated (in measure) as a union of boxes.