18.218 PROBLEM SET

Instructions:

- All submissions must be **typed in LaTeX** and submitted as PDF on Stellar (try Overleaf if you are looking for an online LaTeX editor without requiring installations). Please name your file `ps#_Lastname_Firstname.pdf` and remember to include your name in each file.
- At the top of each submission, you must **acknowledge all references and people consulted** (other than lectures and the textbook). Failure acknowledge sources will lead to an automatic 10% penalty. Examples include: names of people you discussed homework with, books, and online resources. If you consulted no additional sources, you should write sources consulted: none.
- Please turn in the problems marked ⟨ps1⟩ and ⟨ps1⋆⟩ for problem set 1, etc., by midnight of each due date (see course homepage for due dates). Do not submit the other problems—they are for you to practice. See course homepage for policies (20% per day late penalty; do not look up solutions; collaboration policy).
- **Bonus problems**, marked by ⋆, are more challenging. A grade of A- may be attained by only solving the non-starred problems. To attain a grade of A or A+, you should solve a substantial number of starred problems.
- Please **do not exceed one page** for each problem (standard 1-inch margins and 11pt font). If you cannot fit your solution within one page, then think about how to better distill your ideas. If necessary, you may skip details of routine calculations (we don't really want to read them either).
- This file will be updated constantly as the term progresses. Please check back regularly. It will be announced when each problem set is complete.
- You are encouraged to include figures whenever they are helpful. Here are some recommended ways to produce figures in decreasing order of learning curve difficulty:
  (1) TikZ or other drawing script
  (2) IPE (which supports LaTeX) or other drawing app
  (3) photo/scan (I recommend the Dropbox app on your phone, which has a nice scanning feature that produces clear monochrome scans)

Problems begin on the next page

Last updated: March 4, 2019

1. Verify the following asymptotic calculations used in Ramsey number lower bounds:
   (a) For each k, the largest n satisfying $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ has $n = \left(\frac{1}{e\sqrt{2}} + o(1)\right)k2^{k/2}$.

   (b) For each k, the maximum value of $n - \binom{n}{k}2^{1-\binom{k}{2}}$ as n ranges over positive integers is $\left(\frac{1}{e} + o(1)\right)k2^{k/2}$.

   (c) For each k, the largest n satisfying $e\left(\binom{k}{2}\binom{n}{k-2} + 1\right)2^{1-\binom{k}{2}} < 1$ satisfies $n = \left(\frac{\sqrt{2}}{e} + o(1)\right)k2^{k/2}$.

2. Prove that, if there is a real $p \in [0, 1]$ such that
$$\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{t}(1 - p)^{\binom{t}{2}} < 1$$
then the Ramsey number $R(k, t)$ satisfies $R(k, t) > n$. Using this show that
$$R(4, t) \geq c\left(\frac{t}{\log t}\right)^{3/2}$$
for some constant $c > 0$.

3. (Extension of Sperner's theorem) Let $\mathcal{F}$ be a collection of subset of $[n]$ that does not contain $k + 1$ elements forming a chain: $A_1 \subsetneq \cdots \subsetneq A_{k+1}$. Prove that $\mathcal{F}$ is no larger than taking the union of the k levels of the boolean lattice closest to the middle layer.

$\boxed{\text{ps1}}$ 4. Let $A_1, \ldots, A_m$ be r-element sets and $B_1, \ldots, B_m$ be s-element sets. Suppose $A_i \cap B_i = \emptyset$ for each i, and for each $i \neq j$, either $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$. Prove that $m \leq (r+s)^{r+s}/(r^r s^s)$.

$\boxed{\text{ps1}}$ 5. Prove that for every positive integer r, there exists an integer K such that the following holds. Let S be a set of rk points evenly spaced on a circle. If we partition $S = S_1 \cup \cdots \cup S_r$ so that $|S_i| = k$ for each i, then, provided $k \geq K$, there exist r congruent triangles where the vertices of the i-th triangle lie in S_i, for each $1 \leq i \leq r$.

$\boxed{\text{ps1}}$ 6. Prove that every set of 10 points in the plane can be covered by a union of disjoint unit disks.

$\boxed{\text{ps1}\star}$ 7. Prove that $[n]^d$ cannot be partitioned into fewer than 2^d sets each of the form $A_1 \times \cdots \times A_d$ where $A_i \subsetneq [n]$.

8. Let $k \geq 4$ and H a k-uniform hypergraph with at most $4^{k-1}/3^k$ edges. Prove that there is a coloring of the vertices of H by four colors so that in every edge all four colors are represented.

$\boxed{\text{ps1}}$ 9. Let G be a graph on $n \geq 10$ vertices. Suppose that adding any new edge to G would create a new clique on 10 vertices. Prove that G has at least $8n - 36$ edges.
   (Hint in white:)

10. Prove that there is an absolute constant $c > 0$ so that for every $n \times n$ matrix with distinct real entries, one can permute its rows so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$. (A subsequence does not need to be selected from consecutive terms. For example, $(1, 2, 3)$ is an increasing subsequence of $(1, 5, 2, 4, 3)$.)

11. Let G be a graph with n vertices and m edges. Prove that K_n can be written as a union of $O(n^2(\log n)/m)$ copies of G (not necessarily edge-disjoint).

$\boxed{\text{ps1}}$ 12. Given a set $\mathcal{F}$ of subsets of $[n]$ and $A \subseteq [n]$, write $\mathcal{F}|_A := \{S \cap A : S \in \mathcal{F}\}$ (its *projection* onto A). Prove that for every n and k, there exists a set $\mathcal{F}$ of subsets of $[n]$ with $|\mathcal{F}| = O(k2^k \log n)$ such that for every k-element subset A of $[n]$, $\mathcal{F}|_A$ contains all 2^k subsets of A.

13. Let A be a subset of the unit sphere in $\mathbb{R}^3$ (centered at the origin) containing no pair of orthogonal points.

`ps1`   (a) Prove that A occupies at most $1/3$ of the sphere in terms of surface area.

`ps1⋆`   (b) Prove an upper bound smaller than $1/3$ (give your best bound).

14. Let $\boldsymbol{r} = (r_1, \ldots, r_k)$ be a vector of nonzero integers whose sum is nonzero. Prove that there exists a real $c > 0$ (depending on $\boldsymbol{r}$ only) such that the following holds: for every finite set A of nonzero *reals*, there exists a subset $B \subset A$ with $|B| \geq c|A|$ such that there do not exist $b_1, \ldots, b_k \in B$ with $r_1 b_1 + \cdots + r_k b_k = 0$.

`ps1` 15. Prove that every set A of n nonzero integers contains two disjoint subsets B_1 and B_2, such that both B_1 and B_2 are sum-free, and $|B_1| + |B_2| > 2n/3$. Can you do it if A is a set of nonzero reals?

`ps1⋆` 16. Prove that every graph with n vertices and $m \geq n^{3/2}$ edges contains a pair of vertex-disjoint and isomorphic subgraphs (not necessarily induced) each with at least $cm^{2/3}$ edges, where $c > 0$ is a constant.

17. Let $M(n)$ denote the maximum number of edges in a 3-uniform hypergraph on n vertices without a clique on 4 vertices.

   (a) Prove that $M(n+1)/\binom{n+1}{3} \leq M(n)/\binom{n}{3}$ for all n, and conclude that $M(n)/\binom{n}{3}$ approaches some limit α as $n \to \infty$.

   (This limit is called the *Turán density* of the hypergraph $K_4^{(3)}$, and its exact value is currently unknown and is a major open problem.)

   (b) Prove that for every $\delta > 0$ there exists $\epsilon > 0$ and n_0 so that every 3-uniform hypergraph with $n \geq n_0$ vertices and at least $(\alpha + \delta)\binom{n}{3}$ edges must contain at least $\epsilon\binom{n}{4}$ copies of the clique on 4 vertices.

18. Using the alteration method, prove that the Ramsey number $R(4, k)$ satisfies $R(4, k) \geq c(k/\log k)^2$ for some constant $c > 0$.

19. Prove that every 3-uniform hypergraph with n vertices and $m \geq n$ edges contains an independent set (i.e., a set of vertices containing no edges) of size at least $cn^{3/2}/\sqrt{m}$, where $c > 0$ is a constant.

20. (Zarankiewicz problem) Prove that for every positive integer $k \geq 2$, there exists a constant $c > 0$ such that for every n there exists an $n \times n$ matrix with $\{0, 1\}$ entries, with at least $cn^{2-2/(k+1)}$ 1's, such that there is no $k \times k$ submatrix consisting of all 1's.

`ps2` 21. Fix k. Prove that there exists a constant $c_k > 1$ so that for every sufficiently large n, there exists a collection $\mathcal{F}$ of at least c_k^n subsets of $[n]$ such that for every k distinct $F_1, \ldots, F_k \in \mathcal{F}$, all 2^k intersections $\bigcap_{i=1}^k G_i$ are nonempty, where each G_i is either F_i or $[n] \setminus F_i$.

22. *Acute sets in $\mathbb{R}^n$*

   (a) Prove that there exists a family of $\Omega((2/\sqrt{3})^n)$ subsets of $[n]$ containing no three distinct members A, B, C satisfying $A \cap B \subseteq C \subseteq A \cup B$.

   (b) Prove that there exists a set of $\Omega((2/\sqrt{3})^n)$ points in $\mathbb{R}^n$ so that all angles determined by three points from the set are acute.

   Remark: The current best lower and upper bounds for the maximum size of an "acute set" in $\mathbb{R}^n$ (i.e., spanning only acute angles) are $2^{n-1} + 1$ and $2^n - 1$ respectively.

`ps2`   (c) Prove that there exists a constant $c > 1$ such that for every n there are at least c^n points in $\mathbb{R}^n$ so that the angle spanned by every three distinct points is at most $61°$.

ps2⋆ 23. *Covering complements of sparse graphs by cliques*

    (a) Prove that every graph with n vertices and minimum degree $n - d$ can be written as a union of $O(d^2 \log n)$ cliques.

    (b) Prove that every bipartite graph with n vertices on each side of the vertex bipartition and minimum degree $n - d$ can be written as a union of $O(d \log n)$ complete bipartite graphs (assume $d \geq 1$).

ps2⋆ 24. Let $G = (V, E)$ be a graph with n vertices and minimum degree $\delta \geq 2$. Prove that there exists $A \subseteq V$ with $|A| \leq Cn(\log \delta)/\delta$, where $C > 0$ is a constant, so that every vertex in $V \setminus A$ contains at least one neighbor in A and at least one neighbor not in A.

25. Let X be a nonnegative real-valued random variable. Suppose $\mathbb{P}(X = 0) < 1$. Prove that

$$\mathbb{P}(X = 0) \leq \frac{\operatorname{Var} X}{\mathbb{E}[X^2]}.$$

ps2 26. Let X be a random variable with mean μ and variance σ^2. Prove that for all $\lambda > 0$,

$$\mathbb{P}(X \geq \mu + \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

27. What is the threshold function for $G(n, p)$ to contain a cycle?

ps2 28. Show that, for each fixed k, there is a sequence p_n such that

$$\mathbb{P}(G(n, p_n) \text{ has a connected component with exactly } k \text{ vertices}) \to 1 \quad \text{as } n \to \infty.$$

29. Let $p = (\log n + f(n))/n$. Show that, as $n \to \infty$,

$$\mathbb{P}(G(n, p) \text{ has no isolated vertices}) \to \begin{cases} 0 & \text{if } f(n) \to -\infty, \\ 1 & \text{if } f(n) \to \infty. \end{cases}$$

ps2 30. Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n) \in \mathbb{Z}^2$ with $|x_i|, |y_i| \leq 2^{n/2}/(100\sqrt{n})$ for all $i \in [n]$. Show that there are two disjoint sets $I, J \subseteq [n]$ such that $\sum_{i \in I} v_i = \sum_{j \in J} v_j$.

ps2⋆ 31. Prove that there is an absolute constant $c > 0$ so that the following holds. For every prime p and every $A \subseteq \mathbb{Z}/p\mathbb{Z}$ with $|A| = k$, there exists an integer x so that $\{xa : a \in A\}$ intersects every interval of length at least $cp/\sqrt{k}$ in $\mathbb{Z}/p\mathbb{Z}$.

ps2⋆ 32. Let $S_1, \ldots, S_k$ be subsets of $[n]$. Prove that if $k \leq 1.99n/\log_2 n$ and n is sufficiently large, then there are two distinct subsets $X, Y \subseteq [n]$ such that $|X \cap S_i| = |Y \cap S_i|$ for all $i \in [k]$.

    In addition, show that there is some constant C such that the claim is false for $k \geq Cn/\log_2 n$. What is the best constant C?

ps2⋆ 33. Let X be a collection of pairwise orthogonal unit vectors in $\mathbb{R}^n$ and suppose that the projection of each of these vectors on the first k coordinates has norm at least ϵ. Show that $|X| \leq k/\epsilon^2$, and show that this is tight when $\epsilon^2 = k/2^r < 1$ for some integer r.

ps2⋆ 34. Prove that there is a constant $c > 0$ so that every hyperplane containing the origin in $\mathbb{R}^n$ intersects at least c-fraction of the 2^n closed unit balls centered at $\{-1, 1\}^n$.

    (Give your best c. Can you get $c \geq 3/8$? It is conjectured that $c = 1/2$ works.)

ps2 35. Prove that, with probability approaching 1 as $n \to \infty$, $G(n, n^{-1/2})$ has at least $cn^{3/2}$ edge-disjoint triangles, where $c > 0$ is some constant.

(Hint in white:)

ps2 36. *Simple nibble.* Prove that for some constant C, with probability approaching 1 as $n \to \infty$,

(a) $G(n, Cn^{-2/3})$ has at least $n/100$ vertex-disjoint triangles.

(b) $G(n, Cn^{-2/3})$ has at least $0.33n$ vertex-disjoint triangles

(Hint in white:)

(You are asked to solve the above problem using the second moment method. Later in the course we will learn a different method to solve this problem.)

37. Let $X \sim \text{Binomial}(n, p)$. Prove that for $q \in [0, p]$, $\mathbb{P}(X \le nq) \le e^{-nH(q\|p)}$ and for $q \in [p, 1]$, $\mathbb{P}(X \ge nq) \le e^{-nH(q\|p)}$, where

$$H(q\|p) := q \log \frac{q}{p} + (1 - q) \log \frac{1 - q}{1 - p}.$$

(This quantity is known as the *relative entropy* or *Kullback-Leibler divergence*, in this case, between two Bernoulli distributions. They are the best possible exponents as $n \to \infty$.)

38. Prove that there is a constant $C > 0$ so that, with probability $1 - o(1)$ as $n \to \infty$, the maximum number of edges in a bipartite subgraph of $G(n, 1/2)$ is at most $n^2/8 + Cn^{3/2}$.

39. (a) Prove that there is some constant $c > 1$ so that there exists $S \subset \{0, 1\}^n$ with $|S| \ge c^n$ so that every pair of points in S differ in at least $n/4$ coordinates.

(b) Prove that there is some constant $c > 1$ so that the the unit sphere in $\mathbb{R}^n$ contains at least c^n points, where each pair of points is at distance at least 1 apart.

40. *Planted clique.* Let $t = \lfloor 100\sqrt{n \log n} \rfloor$. Give a polynomial-time algorithm that, given an n-vertex unlabeled graph G created as the union of $G(n, 1/2)$ and a clique on t-vertex subset, finds a clique of size t in G, in polynomial time that succeeds with probability approaching 1 as $n \to \infty$.