Problem Set 4. Due 10/2

1. How many compositions does the positive integer n have in which neither the first nor the last entry is 1?

2. How many positive integers are there that are not larger than 1000 and are neither perfect squares nor perfect cubes?

3. Find a closed formula (no summation signs) for

$$\sum_{i=0}^{n} \binom{n}{i} D(i).$$

   Here $D(n)$ is the number of derangements (permutations without fixed points) of $[n]$

4. Prove that, for nonnegative integers m, n, k with $k \leq n$,

$$\binom{n}{k} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n+m-k}{k-j}.$$

5. Using generating functions, find an explicit formula for a_n if $a_0 = 1$ and $a_{n+1} = 3a_n + 2^n$ for all $n \geq 0$.

6. Using generating functions, find an explicit formula for the Fibonacci numbers f_n defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \geq 1$.

7. Let a_n be the number of compositions of n into parts that are larger than 1. Find a closed form for the generating function $A(x) = \sum_{n \geq 0} a_n x^n$.