18.S997 (FALL 2017) PROBLEM SET 4

1. In class we showed that "Fourier controls 3-AP counts." In this problem, you will work out an example showing that "Fourier does not control 4-AP counts". Let $A = \{x \in \mathbb{F}_5^n : x \cdot x = 0\}$.[1] Write $N = 5^n$.
   (a) Show that $|A| = (1/5 + o(1))N$ and $|\widehat{1_A}(r)| = o(1)$ for all $r \neq 0$.
   (b) Show that $|\{(x, y) \in \mathbb{F}_5^n : x, x + y, x + 2y \in A\}| = (5^{-3} + o(1))N^2$.
   (c) Show that $|\{(x, y) \in \mathbb{F}_5^n : x, x + y, x + 2y, x + 3y \in A\}| = (5^{-3} + o(1))N^2$ (in particular, it is *not* $(5^{-4} + o(1))N^2$, which would be the case for a random subset A of density $1/5$).
   (d) Explain (no need to give all the details) why $A = \{x \in \mathbb{Z}/N\mathbb{Z} : x^2 \in [\alpha N]\} \subset \mathbb{Z}/N\mathbb{Z}$ is Fourier-uniform but has "too many" 4-APs (here α can be thought of as a small fixed number).

. . . to be continued . . . check back later (last updated: November 9, 2017).

[1] Why $\mathbb{F}_5$?