Lecture notes (MIT 18.226, Fall 2020)

Probabilistic Methods in Combinatorics

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These notes were created primarily for my own lecture preparation. The writing style is far below that of formal writing and publications (in terms of complete sentences, abbreviations, citations, etc.). The notes are not meant to be a replacement of the lectures. Please let me know if you spot any errors.

Asymptotic notation

Each line means the same for positive functions f and g (as some parameter, usually n, tends to infinity)

- $f \lesssim g$, $f = O(g)$, $g = \Omega(f)$, $f \le Cg$ (for some constant $C > 0$)
- $f/g \to 0$, $f \ll g$, $f = o(g)$ (and sometimes $g = \omega(f)$)
- $f = \Theta(g)$, $f \asymp g$, $g \lesssim f \lesssim g$
- $f \sim g$, $f = (1 + o(1))g$
- *whp* (= *with high probability*) means with probability $1 - o(1)$

Warning: analytic number theorists use $\ll$ to mean $O(\cdot)$ (Vinogradov notation)

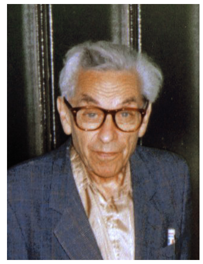

Figure 1: Paul Erdős (1913–1996) is considered the father of the probabilistic method. He published around 1,500 papers during his lifetime, and had more than 500 collaborators. To learn more about Erdős, see his biography *The man who loved only numbers* by Hoffman and the documentary *N is a number*.

1 Introduction

Probabilistic method: to prove that an object exists, show that a random construction works with positive probability

Tackle combinatorics problems by introducing randomness

Theorem 1.0.1. Every graph $G = (V, E)$ contains a bipartite subgraph with at least $|E|/2$ edges.

Proof. Randomly color every vertex of G with black or white, iid uniform

Let E' = edges with one end black and one end white

Then (V, E') is a bipartite subgraph of G

Every edge belongs to E' with probability $\frac{1}{2}$, so by linearity of expectation, $\mathbb{E}[|E'|] = \frac{1}{2}|E|$.

Thus there is some coloring with $|E'| \geq \frac{1}{2}|E|$, giving the desired bipartite subgraph. □

1.1 Lower bound to Ramsey number

Ramsey number $R(k, \ell)$ = smallest n such that in every red-blue edge coloring of K_n, there exists a red K_k or a blue K_ℓ.

e.g., $R(3, 3) = 6$

Ramsey (1929) proved that $R(k, \ell)$ exists and is finite

The probabilistic method started with:

P. Erdős, Some remarks on the theory of graphs, BAMS, 1947

Remark (Hungarian names). Typing "Erdős" in LaTeX: `Erd\H{o}s` and *not* `Erd\"os`

Hungarian pronunciations: `s` = `/sh/` and `sz` = `/s/`, e.g., Erdős, Szekeres, Lovász

Figure 2: Frank Ramsey (1903–1930) wrote seminal papers in philosophy, economics, and mathematical logic, before his untimely death at the age of 26 from liver problems. See a recent profile of him in the New Yorker.

Theorem 1.1.1 (Erdős 1947)**.** If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. In other words, there exist a red-blue edge-coloring of K_n without a monochromatic K_k.

Proof. Color edges uniformly at random

For every fixed subset R of k vertices, let A_R denote the event that R induces a monochromatic K_k. Then $\mathbb{P}(A_R) = 2^{1-\binom{k}{2}}$.

$$\mathbb{P}(\text{there exists a monochromatic } K_k) = \mathbb{P}\left(\bigcup_{R\in\binom{[n]}{k}} A_R\right) \le \sum_{R\in\binom{[n]}{k}} \mathbb{P}(A_R) = \binom{n}{k}2^{1-\binom{k}{2}} < 1.$$

Thus, with positive probability, the random coloring gives no monochromatic K_k. □

Remark. By optimizing n (using Stirling's formula) above, we obtain

$$R(k,k) > \left(\frac{1}{e\sqrt{2}} + o(1)\right) k2^{k/2}$$

Can be alternatively phrased as counting: of all $2^{\binom{n}{2}}$ possible colorings, not all are bad (this was how the argument was phrased in the original Erdős 1947 paper.

In this course, we almost always only consider finite probability spaces. While in principle the finite probability arguments can be rephrased as counting, but some of the later more involved arguments are impractical without a probabilistic perspective.

Constructive lower bounds? Algorithmic? Open! "Finding hay in a haystack"

Remark (Ramsey number upper bounds)*.* Erdős–Szekeres (1935):

$$R(k+1, \ell+1) \le \binom{k+\ell}{k}.$$

Recent improvements by Conlon (2009), and most recently Sah (2020+):

$$R(k+1,k+1) \le e^{-c(\log k)^2}\binom{2k}{k}.$$

All these bounds have the form $R(k,k) \le (4+o(1))^k$. It is a major open problem whether $R(k,k) \le (4-c)^k$ is true for some constant $c > 0$ and all sufficiently large k.

1.2 Alterations

Two steps: (1) randomly color (2) get rid of bad parts

Theorem 1.2.1. For any k, n, we have $R(k,k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$.

Proof. Construct in two steps:
(1) Randomly 2-color the edges of K_n
(2) Delete a vertex from every monochromatic K_k
Final graph has no monochromatic K_k
After step (1), every fixed K_k is monochromatic with probability $2^{1-\binom{k}{2}}$, let X be the number of monochromatic K_k's. $\mathbb{E}X = \binom{n}{k}2^{1-\binom{k}{2}}$.
We delete at most $|X|$ vertices in step (2). Thus final graph has size $\ge n - |X|$, which has expectation $n - \binom{n}{k}2^{1-\binom{k}{2}}$.
Thus with positive probability, the remaining graph has size at least $n - \binom{n}{k}2^{1-\binom{k}{2}}$ (and no monochromatic K_k by construction) $\square$

Remark. By optimizing the choice of n in the theorem, we obtain

$$R(k,k) > \left(\frac{1}{e} + o(1)\right) k 2^{k/2},$$

which improves the previous bound by a constant factor of $\sqrt{2}$.

1.3 Lovász local lemma

We give one more improvement to the lower bound, using the Lovász local lemma, which we will prove later in the course
Consider "bad events" $E_1, \dots, E_n$. We want to avoid all.

If all $\mathbb{P}(E_i)$ small, say $\sum_i \mathbb{P}(E_i) < 1$, then can avoid all bad events.

Or, if they are all independent, then the probability that none of E_i occurs is $\prod_{i=1}^n (1 - \mathbb{P}(E_i)) > 0$ (provided that all $\mathbb{P}(E_i) < 1$).

What if there are some weak dependencies?

Theorem 1.3.1 (Lovász local lemma)**.** Let $E_1, \ldots, E_n$ be events, with $\mathbb{P}[E_i] \le p$ for all i. Suppose that each E_i is independent of all other E_j except for at most d of them. If

$$ep(d+1) < 1,$$

then with some positive probability, none of the events E_i occur.

Remark. The meaning of "independent of ..." is actually somewhat subtle (and easily mistaken). We will come back to this issue later on when we discuss the local lemma in more detail.

Theorem 1.3.2 (Spencer 1975)**.** If $e\left(\binom{k}{2}\binom{n}{k-2}+1\right)2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.

Proof. Random 2-color edges of K_n

For each k-vertex subset R, let E_R be the event that R induces a monochromatic K_k. $\mathbb{P}[E_R] = 2^{1-\binom{k}{2}}$.

E_R is independent of all E_S other than those such that $|R \cap S| \ge 2$

For each R, there are at most $\binom{k}{2}\binom{n}{k-2}$ choices S with $|S| = k$ and $|R \cap S| \ge 2$.

Apply Lovász local lemma to the events $\left\{E_R : R \in \binom{V}{k}\right\}$ and $p = 2^{1-\binom{k}{2}}$ and $d = \binom{k}{2}\binom{n}{k-2}$, we get that with positive probability none of the events E_R occur, which gives a coloring with no monochromatic K_k's. □

Remark. By optimizing the choice of n, we obtain

$$R(k,k) > \left(\frac{\sqrt{2}}{e} + o(1)\right) k 2^{k/2}$$

once again improving the previous bound by a constant factor of $\sqrt{2}$. This bound was given by Spencer in 1975. It is the best known lower bound to $R(k,k)$ to date.