

18.S997 (FALL 2017) PROBLEM SET 3

1. Fix $0 < p < 1$. Let G be a graph on n vertices with average degree at least pn . Prove:
 - (a) The number of labeled 6-cycles in G is at least $(p^6 - o(1))n^6$.
 - (b) The number of labeled copies of $K_{3,3}$ in G is at least $(p^9 - o(1))n^6$.
 - (c) The number of labeled copies of $Q_3 = \begin{array}{c} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array}$ in G is at least $(p^{12} - o(1))n^8$.
 - (d) (Bonus) The number of labeled paths on 4 vertices in G is at least $(p^3 - o(1))n^4$.
2. Deduce from the quasirandom Cayley graphs theorem the following corollary for vertex transitive graphs: If an n -vertex d -regular vertex-transitive graph G satisfies

$$|e(X, Y) - \frac{d}{n}|X||Y|| \leq \epsilon dn \quad \text{for all } X, Y \subseteq V(G),$$

then all the eigenvalues of the adjacency matrix of G , other than the largest one, are at most $8\epsilon dn$ in absolute value.

3. Define $W: [0, 1]^2 \rightarrow \mathbb{R}$ by $W(x, y) = 2 \cos(2\pi(x - y))$. Let G be a graph. Show that $t(G, W)$ is the number of ways to orient all edges of G so that every vertex has the same number of incoming edges as outgoing edges.
4. Let W be a $\{0, 1\}$ -valued graphon. Suppose graphons W_n satisfy $\|W_n - W\|_{\square} \rightarrow 0$ as $n \rightarrow \infty$. Show that $\|W_n - W\|_1 \rightarrow 0$ as $n \rightarrow \infty$.
5. Let $\epsilon > 0$. Show that for every graphon $W: [0, 1]^2 \rightarrow [0, 1]$, there exist measurable sets $S_1, \dots, S_k, T_1, \dots, T_k \subseteq [0, 1]$ and reals $a_1, \dots, a_k \in \mathbb{R}$, with $k < 1/\epsilon^2$, such that

$$\left\| W - \sum_{i=1}^k a_i \mathbf{1}_{S_i \times T_i} \right\|_{\square} \leq \epsilon.$$

... to be continued ... check back later (last updated: October 26, 2017)