

## 18.226 PROBLEM SET (FALL 2020)

### Helpful tips:

- Please read the [course homepage](#) carefully regarding homework policies (due dates, lateness, acknowledging sources on each problem, etc.)
- Only turn in problems marked `ps1` and `ps1★` for problem set 1, etc. You are recommended to try the other problems for practice, but do not submit them.
- In a multipart problem, if a later part is marked for submission, it may be helpful to think about the earlier unassigned parts first.
- **Bonus problems**, marked by ★, are more challenging. A grade of A- may be attained by only solving the non-starred problems. To attain a grade of A or A+, you should solve a substantial number of starred problems. No hints will be given for bonus problems, e.g., during office hours.
- **Start each solution on a new page**, and try to **fit your solution within one page** for each unstarred problem/part (without abusing font/margins). The spirit of this policy is to encourage you to think first before you write. Distill your ideas, structure your arguments, and eliminate unnecessary steps. If necessary, some details of routine calculations may be skipped provided that you give precise statements and convincing explanations.
- This file will be updated as the term progresses. Please check back regularly. There will be an announcement whenever each problem set is complete.
- You are encouraged to include figures whenever they are helpful. Here are some recommended ways to produce figures in decreasing order of learning curve difficulty:
  - (1) [TikZ](#)
  - (2) [IPE](#) (which supports LaTeX), Powerpoint, or other drawing app
  - (3) drawing on a tablet (e.g., Notability on iPad)
  - (4) photo/scan (I recommend the Dropbox app on your phone, which has a nice scanning feature that produces clear monochrome scans)

## A. INTRODUCTION AND LINEARITY OF EXPECTATIONS

A1. Verify the following asymptotic calculations used in Ramsey number lower bounds:

(a) For each  $k$ , the largest  $n$  satisfying  $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$  has  $n = \left(\frac{1}{e\sqrt{2}} + o(1)\right) k 2^{k/2}$ .

(b) For each  $k$ , the maximum value of  $n - \binom{n}{k} 2^{1-\binom{k}{2}}$  as  $n$  ranges over positive integers is  $\left(\frac{1}{e} + o(1)\right) k 2^{k/2}$ .

(c) For each  $k$ , the largest  $n$  satisfying  $e \left(\binom{k}{2} \binom{n}{k-2} + 1\right) 2^{1-\binom{k}{2}} < 1$  satisfies  $n = \left(\frac{\sqrt{2}}{e} + o(1)\right) k 2^{k/2}$ .

A2. Prove that, if there is a real  $p \in [0, 1]$  such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$$

then the Ramsey number  $R(k, t)$  satisfies  $R(k, t) > n$ . Using this show that

$$R(4, t) \geq c \left(\frac{t}{\log t}\right)^{3/2}$$

for some constant  $c > 0$ .

ps1

A3. Let  $G$  be a graph with  $n$  vertices and  $m$  edges. Prove that  $K_n$  can be written as a union of  $O(n^2(\log n)/m)$  copies of  $G$  (not necessarily edge-disjoint).

A4. *Generalization of Sperner's theorem.* Let  $\mathcal{F}$  be a collection of subset of  $[n]$  that does not contain  $k+1$  elements forming a chain:  $A_1 \subsetneq \cdots \subsetneq A_{k+1}$ . Prove that  $\mathcal{F}$  is no larger than taking the union of the  $k$  levels of the boolean lattice closest to the middle layer.

ps1

A5. Let  $A_1, \dots, A_m$  be  $r$ -element sets and  $B_1, \dots, B_m$  be  $s$ -element sets. Suppose  $A_i \cap B_i = \emptyset$  for each  $i$ , and for each  $i \neq j$ , either  $A_i \cap B_j \neq \emptyset$  or  $A_j \cap B_i \neq \emptyset$ . Prove that  $m \leq (r+s)^{r+s}/(r^r s^s)$ .

ps1

A6. Let  $G$  be a graph on  $n \geq 10$  vertices. Suppose that adding any new edge to  $G$  would create a new clique on 10 vertices. Prove that  $G$  has at least  $8n - 36$  edges.

Hint in white:

ps1★

A7. Prove that for every positive integer  $r$ , there exists an integer  $K$  such that the following holds. Let  $S$  be a set of  $rk$  points evenly spaced on a circle. If we partition  $S = S_1 \cup \cdots \cup S_r$  so that  $|S_i| = k$  for each  $i$ , then, provided  $k \geq K$ , there exist  $r$  congruent triangles where the vertices of the  $i$ -th triangle lie in  $S_i$ , for each  $1 \leq i \leq r$ .

ps1

A8. Prove that  $[n]^d$  cannot be partitioned into fewer than  $2^d$  sets each of the form  $A_1 \times \cdots \times A_d$  where  $A_i \subsetneq [n]$ .

A9. Let  $k \geq 4$  and  $H$  a  $k$ -uniform hypergraph with at most  $4^{k-1}/3^k$  edges. Prove that there is a coloring of the vertices of  $H$  by four colors so that in every edge all four colors are represented.

ps1

A10. Prove that there is an absolute constant  $C > 0$  so that for every  $n \times n$  matrix with distinct real entries, one can permute its rows so that no column in the permuted matrix contains an increasing subsequence of length at least  $C\sqrt{n}$ . (A subsequence does not need to be selected from consecutive terms. For example,  $(1, 2, 3)$  is an increasing subsequence of  $(1, 5, 2, 4, 3)$ .)

ps1

A11. Given a set  $\mathcal{F}$  of subsets of  $[n]$  and  $A \subseteq [n]$ , write  $\mathcal{F}|_A := \{S \cap A : S \in \mathcal{F}\}$  (its *projection* onto  $A$ ). Prove that for every  $n$  and  $k$ , there exists a set  $\mathcal{F}$  of subsets of  $[n]$  with  $|\mathcal{F}| = O(k 2^k \log n)$  such that for every  $k$ -element subset  $A$  of  $[n]$ ,  $\mathcal{F}|_A$  contains all  $2^k$  subsets of  $A$ .

- ps1★** A12. Show that in every non-2-colorable  $n$ -uniform hypergraph, one can find at least  $\frac{n}{2} \binom{2n-1}{n}$  unordered pairs of edges that intersect in exactly one vertex.
- A13. Let  $A$  be a subset of the unit sphere in  $\mathbb{R}^3$  (centered at the origin) containing no pair of orthogonal points.
- ps1** (a) Prove that  $A$  occupies at most  $1/3$  of the sphere in terms of surface area.
- ps1★** (b) Prove an upper bound smaller than  $1/3$  (give your best bound).
- ps1★** A14. Prove that every set of 10 points in the plane can be covered by a union of disjoint unit disks.
- A15. Let  $\mathbf{r} = (r_1, \dots, r_k)$  be a vector of nonzero integers whose sum is nonzero. Prove that there exists a real  $c > 0$  (depending on  $\mathbf{r}$  only) such that the following holds: for every finite set  $A$  of nonzero reals, there exists a subset  $B \subseteq A$  with  $|B| \geq c|A|$  such that there do not exist  $b_1, \dots, b_k \in B$  with  $r_1 b_1 + \dots + r_k b_k = 0$ .
- ps1** A16. Prove that every set  $A$  of  $n$  nonzero integers contains two disjoint subsets  $B_1$  and  $B_2$ , such that both  $B_1$  and  $B_2$  are sum-free, and  $|B_1| + |B_2| > 2n/3$ . Can you do it if  $A$  is a set of nonzero reals?
- A17. Let  $M(n)$  denote the maximum number of edges in a 3-uniform hypergraph on  $n$  vertices without a clique on 4 vertices.
- (a) Prove that  $M(n+1)/\binom{n+1}{3} \leq M(n)/\binom{n}{3}$  for all  $n$ , and conclude that  $M(n)/\binom{n}{3}$  approaches some limit  $\alpha$  as  $n \rightarrow \infty$ .  
(This limit is called the *Turán density* of the hypergraph  $K_4^{(3)}$ , and its exact value is currently unknown and is a major open problem.)
- (b) Prove that for every  $\delta > 0$ , there exists  $\epsilon > 0$  and  $n_0$  so that every 3-uniform hypergraph with  $n \geq n_0$  vertices and at least  $(\alpha + \delta)\binom{n}{3}$  edges must contain at least  $\epsilon\binom{n}{4}$  copies of the clique on 4 vertices.
- A18. Prove that every graph with  $n$  vertices and  $m \geq n^{3/2}$  edges contains a pair of vertex-disjoint and isomorphic subgraphs (not necessarily induced) each with at least  $cm^{2/3}$  edges, where  $c > 0$  is a constant.

## B. ALTERATION METHOD

- B1. Using the alteration method, prove the Ramsey number bound

$$R(4, k) \geq c(k/\log k)^2$$

for some constant  $c > 0$ .

- B2. Prove that every 3-uniform hypergraph with  $n$  vertices and  $m \geq n$  edges contains an independent set (i.e., a set of vertices containing no edges) of size at least  $cn^{3/2}/\sqrt{m}$ , where  $c > 0$  is a constant.
- ps2** B3. Prove that every  $k$ -uniform hypergraph with  $n$  vertices and  $m$  edges has a transversal (i.e., a set of vertices intersecting every edge) of size at most  $n(\log k)/k + m/k$ .
- ps2** B4. *Zarankiewicz problem*. Prove that for every positive integer  $k \geq 2$ , there exists a constant  $c > 0$  such that for every  $n$ , there exists an  $n \times n$  matrix with  $\{0, 1\}$  entries, with at least  $cn^{2-2/(k+1)}$  1's, such that there is no  $k \times k$  submatrix consisting of all 1's.

- ps2** B5. Fix  $k$ . Prove that there exists a constant  $c_k > 1$  so that for every sufficiently large  $n > n_0(k)$ , there exists a collection  $\mathcal{F}$  of at least  $c_k^n$  subsets of  $[n]$  such that for every  $k$  distinct  $F_1, \dots, F_k \in \mathcal{F}$ , all  $2^k$  intersections  $\bigcap_{i=1}^k G_i$  are nonempty, where each  $G_i$  is either  $F_i$  or  $[n] \setminus F_i$ .
- B6. *Acute sets in  $\mathbb{R}^n$ .* Prove that, for some constant  $c > 0$ , for every  $n$ , there exists a family of at least  $c(2/\sqrt{3})^n$  subsets of  $[n]$  containing no three distinct members  $A, B, C$  satisfying  $A \cap B \subseteq C \subseteq A \cup B$ .  
Deduce that there exists a set of at least  $c(2/\sqrt{3})^n$  points in  $\mathbb{R}^n$  so that all angles determined by three points from the set are acute.  
*Remark.* The current best lower and upper bounds for the maximum size of an “acute set” in  $\mathbb{R}^n$  (i.e., spanning only acute angles) are  $2^{n-1} + 1$  and  $2^n - 1$  respectively.
- ps2★** B7. *Covering complements of sparse graphs by cliques*  
(a) Prove that every graph with  $n$  vertices and minimum degree  $n - d$  can be written as a union of  $O(d^2 \log n)$  cliques.  
(b) Prove that every bipartite graph with  $n$  vertices on each side of the vertex bipartition and minimum degree  $n - d$  can be written as a union of  $O(d \log n)$  complete bipartite graphs (assume  $d \geq 1$ ).
- ps2★** B8. Let  $G = (V, E)$  be a graph with  $n$  vertices and minimum degree  $\delta \geq 2$ . Prove that there exists  $A \subseteq V$  with  $|A| = O(n(\log \delta)/\delta)$  so that every vertex in  $V \setminus A$  contains at least one neighbor in  $A$  and at least one neighbor not in  $A$ .
- ps2★** B9. Prove that every graph  $G$  without isolated vertices has an induced subgraph  $H$  on at least  $\alpha(G)/2$  vertices such that all vertices of  $H$  have odd degree. Here  $\alpha(G)$  is the size of the largest independent set in  $G$ .

## C. SECOND MOMENT METHOD

- C1. Let  $X$  be a nonnegative real-valued random variable. Suppose  $\mathbb{P}(X = 0) < 1$ . Prove that

$$\mathbb{P}(X = 0) \leq \frac{\text{Var } X}{\mathbb{E}[X^2]}.$$

- ps2** C2. Let  $X$  be a random variable with mean  $\mu$  and variance  $\sigma^2$ . Prove that for all  $\lambda > 0$ ,

$$\mathbb{P}(X \geq \mu + \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

- ps2** C3. Let  $x, y \in \mathbb{R}^n$  be two unit vectors each orthogonal to the all-1 vector. For a permutation  $\sigma$  of  $[n]$ , write  $x^\sigma \in \mathbb{R}^n$  for the vector whose  $i$ -th coordinate is  $x_{\sigma(i)}$ . Write  $\langle \cdot, \cdot \rangle$  for dot product.  
(a) Compute  $\text{Var} \langle x^\sigma, y \rangle$  where  $\sigma$  is a uniformly chosen permutation of  $[n]$ .  
(b) Prove that  $\max_\sigma \langle x^\sigma, y \rangle - \min_\sigma \langle x^\sigma, y \rangle \geq 2/\sqrt{n-1}$ , and that equality can be achieved for even  $n$ .
- ps2** C4. Show that, for each fixed positive integer  $k$ , there is a sequence  $p_n$  such that

$$\mathbb{P}(G(n, p_n) \text{ has a connected component with exactly } k \text{ vertices}) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Hint in white:

C5. Let  $p_n = (\log n + c_n)/n$ . Show that, as  $n \rightarrow \infty$ ,

$$\mathbb{P}(G(n, p_n) \text{ has no isolated vertices}) \rightarrow \begin{cases} 0 & \text{if } c_n \rightarrow -\infty, \\ 1 & \text{if } c_n \rightarrow \infty. \end{cases}$$

ps3 C6. Prove that, with probability approaching 1 as  $n \rightarrow \infty$ ,  $G(n, n^{-1/2})$  has at least  $cn^{3/2}$  edge-disjoint triangles, where  $c > 0$  is some constant.

ps3 C7. *Simple nibble*. Prove that, with probability approaching 1 as  $n \rightarrow \infty$ ,  
 (a)  $G(n, n^{-2/3})$  has at least  $n/100$  vertex-disjoint triangles.  
 (b)  $G(n, Cn^{-2/3})$  has at least  $0.33n$  vertex-disjoint triangles, for some constant  $C$ .

Hint in white:

ps3★ C8. Is the threshold for the bipartiteness of a random graph coarse or sharp?

(You are not allowed to use any theorems that we did not prove in class/notes.)

ps3★ C9. Prove that there is an absolute constant  $C > 0$  so that the following holds. For every prime  $p$  and every  $A \subseteq \mathbb{Z}/p\mathbb{Z}$  with  $|A| = k$ , there exists an integer  $x$  so that  $\{xa : a \in A\}$  intersects every interval of length at least  $Cp/\sqrt{k}$  in  $\mathbb{Z}/p\mathbb{Z}$ .

ps3★ C10. Prove that there is a constant  $c > 0$  so that every hyperplane containing the origin in  $\mathbb{R}^n$  intersects at least  $c$ -fraction of the  $2^n$  closed unit balls centered at  $\{-1, 1\}^n$ .

ps3 C11. Let  $v_1 = (x_1, y_1), \dots, v_n = (x_n, y_n) \in \mathbb{Z}^2$  with  $|x_i|, |y_i| \leq 2^{n/2}/(100\sqrt{n})$  for all  $i \in [n]$ . Show that there are two disjoint sets  $I, J \subseteq [n]$ , not both empty, such that  $\sum_{i \in I} v_i = \sum_{j \in J} v_j$ .

#### D. CHERNOFF BOUND

D1. Let  $X \sim \text{Binomial}(n, p)$ . Prove that for  $0 < q \leq p < 1$ ,

$$\mathbb{P}(X \leq nq) \leq e^{-nH(q||p)} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X \leq nq) = -H(q||p)$$

and for  $0 < p \leq q < 1$ ,

$$\mathbb{P}(X \geq nq) \leq e^{-nH(q||p)} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(X \geq nq) = -H(q||p),$$

where

$$H(q||p) := q \log \frac{q}{p} + (1 - q) \log \frac{1 - q}{1 - p}.$$

is known as the *relative entropy* or *Kullback-Leibler divergence*, in this case, between two Bernoulli distributions.

ps3 D2. Prove that with probability  $1 - o(1)$  as  $n \rightarrow \infty$ , every bipartite subgraph of  $G(n, 1/2)$  has at most  $n^2/8 + 10n^{3/2}$  edges.

D3. Show that for every  $k$  there is some  $m = O(k \log k)$  such that for sufficiently large  $n > n_0(k)$ , there exists a collection  $\mathcal{F}$  of  $m$  subsets of  $[nk]$  such that for every partition  $[nk] = A_1 \cup \dots \cup A_k$  with  $|A_i| = n$  for each  $i \in [k]$ , there exists some  $F \in \mathcal{F}$  with  $|A_i \cap F| \in [0.4n, 0.6n]$  for each  $i \in [k]$ .

D4. (a) Prove that there is some constant  $c > 1$  so that there exists  $S \subset \{0, 1\}^n$  with  $|S| \geq c^n$  so that every pair of points in  $S$  differ in at least  $n/4$  coordinates.

(b) Prove that there is some constant  $c > 1$  so that the the unit sphere in  $\mathbb{R}^n$  contains at least  $c^n$  points, where each pair of points is at distance at least 1 apart.

ps3★

D5. Prove that there exists a constant  $c > 1$  such that for every  $n$ , there are at least  $c^n$  points in  $\mathbb{R}^n$  so that the angle spanned by every three distinct points is at most  $61^\circ$ .

ps3

D6. *Planted clique*. Give a deterministic polynomial-time algorithm solving the following problem so that it succeeds over the random input with probability approaching 1 as  $n \rightarrow \infty$ :

Input: an  $n$ -vertex unlabeled graph  $G$  created as the union of  $G(n, 1/2)$  and a clique on vertex subset of size  $t = \lfloor 100\sqrt{n \log n} \rfloor$

Output: a clique in  $G$  of size  $t$

D7. *Weighing coins*

ps3

(a) Prove that if  $k \leq 1.99n/\log_2 n$  and  $n$  is sufficiently large, then for every  $S_1, \dots, S_k \subseteq [n]$ , there are two distinct subsets  $X, Y \subseteq [n]$  such that  $|X \cap S_i| = |Y \cap S_i|$  for all  $i \in [k]$ .

ps3★

(b) Show that there is some constant  $C$  such that (a) is false if 1.99 is replaced by  $C$ .

(Bonus competition: additional points to the student who correctly proves the smallest  $C$  in the class; if  $m$  students achieve the same lowest  $C$ , then each is awarded  $\min\{10, \lfloor 24/m \rfloor\}$  points additional to the 10 points for a correct solution. You are not allowed to cheat by looking up the solution.)

### E. LOVÁSZ LOCAL LEMMA

ps4

E1. Show that it is possible to color the edges of  $K_n$  with at most  $3\sqrt{n}$  colors so that there are no monochromatic triangles.

E2. Prove that it is possible to color the vertices of every  $k$ -uniform  $k$ -regular hypergraph using at most  $k/\log k$  colors so that every color appears at most  $O(\log k)$  times on each edge.

ps4

E3. Prove that there is some constant  $c > 0$  so that given a graph and a set of  $k$  acceptable colors for each vertex such that every color is acceptable for at most  $ck$  neighbors of each vertex, there is always a proper coloring where every vertex is assigned one of its acceptable colors.

ps4★

E4. Prove that there is a constant  $C > 0$  so that for every sufficiently small  $\epsilon > 0$ , one can choose exactly one point inside each grid square  $[n, n+1) \times [m, m+1) \subset \mathbb{R}^2$ ,  $m, n \in \mathbb{Z}$ , so that every rectangle of dimensions  $\epsilon$  by  $(C/\epsilon) \log(1/\epsilon)$  in the plane (not necessarily axis-aligned) contains at least one chosen point.

E5. Prove that there exists  $k_0$  and a red/blue coloring of  $\mathbb{Z}$  without any monochromatic  $k$ -term arithmetic progressions with  $k \geq k_0$  and common difference less than  $1.99^k$ .

ps4

E6. Prove that, for every  $\epsilon > 0$ , there exists  $\ell_0$  and some  $(a_1, a_2, \dots) \in \{0, 1\}^{\mathbb{N}}$  such that for every  $\ell > \ell_0$  and every  $i > 1$ , the vectors  $(a_i, a_{i+1}, \dots, a_{i+\ell-1})$  and  $(a_{i+\ell}, a_{i+\ell+1}, \dots, a_{i+2\ell-1})$  differ in at least  $(\frac{1}{2} - \epsilon)\ell$  coordinates.

E7. A *periodic path* in a graph  $G$  with respect to a vertex coloring  $f: V(G) \rightarrow [k]$  is a path  $v_1 v_2 \dots v_{2\ell}$  for some positive integer  $\ell$  with  $f(v_i) = f(v_{i+\ell})$  for each  $i \in [\ell]$  (reminder: no repeated vertices allowed in a path).

Prove that for every  $\Delta$ , there exists  $k$  so that every graph with maximum degree at most  $\Delta$  has a vertex-coloring using  $k$  colors with no periodic paths.

ps4

E8. Prove that every graph with maximum degree  $\Delta$  can be edge-colored using  $O(\Delta)$  colors so that every cycle contains at least three colors and no two adjacent edges have the same color.

ps4★ E9. Prove that for every  $\Delta$ , there exists  $g$  so that every bipartite graph with maximum degree  $\Delta$  and girth at least  $g$  can be properly edge-colored using  $\Delta + 1$  colors so that every cycle contains at least three colors.

ps4★ E10. Prove that for every positive integer  $r$ , there exists  $C_r$  so that every graph with maximum degree  $\Delta$  has a *proper* vertex coloring using at most  $C_r \Delta^{1+1/r}$  colors so that every vertex has at most  $r$  neighbors of each color.

E11. Let  $H = (V, E)$  be a hypergraph satisfying, for some  $\lambda > 1/2$ ,

$$\sum_{f \in E: v \in f} \lambda^{|f|} \leq \frac{1}{2} - \frac{1}{4\lambda} \quad \text{for every } v \in V$$

(here  $|f|$  is then number of vertices in the edge  $f$ ). Prove that  $H$  is 2-colorable.

E12. *Vertex-disjoint cycles in digraphs.* (Recall that a directed graph is *k-regular* if all vertices have in-degree and out-degree both equal to  $k$ . Also, cycles cannot repeat vertices.)

ps4 (a) Prove that every  $k$ -regular directed graph has at least  $ck/\log k$  vertex-disjoint directed cycles, where  $c > 0$  is some constant.

ps4★ (b) Prove that every  $k$ -regular directed graph has at least  $ck$  vertex-disjoint directed cycles, where  $c > 0$  is some constant.

Hint in white: