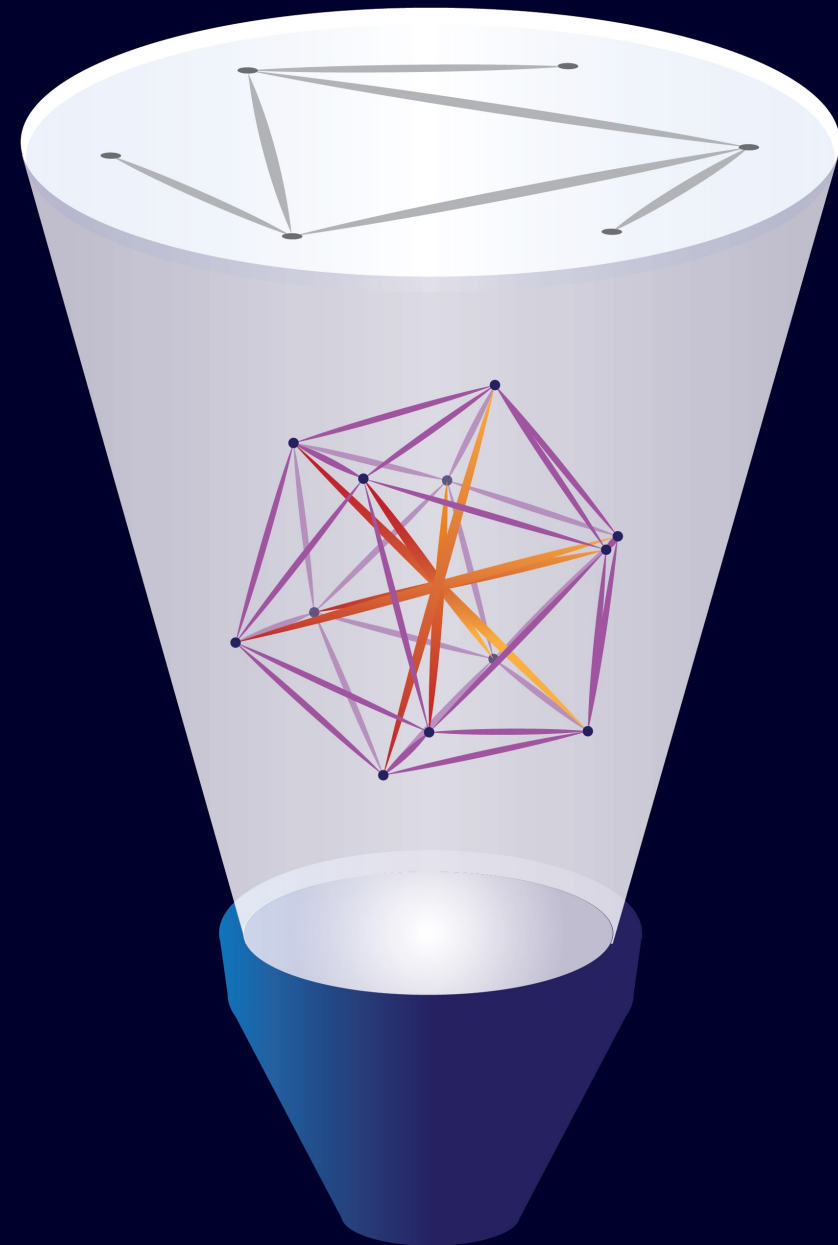


Equiangular Lines and Eigenvalue Multiplicities

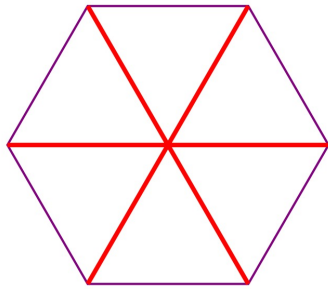
Yufei Zhao
MIT

August 2022

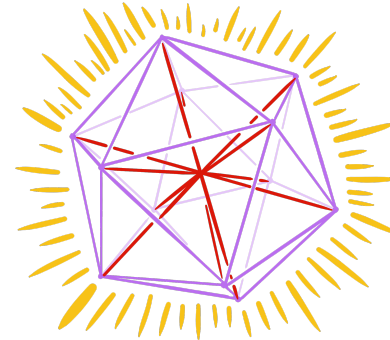


Equiangular lines

$N(d)$ = max # of lines in \mathbb{R}^d with pairwise equal angles



$$N(2) = 3$$



$$N(3) = 6$$

Exact answer known for finitely many d

General bounds:

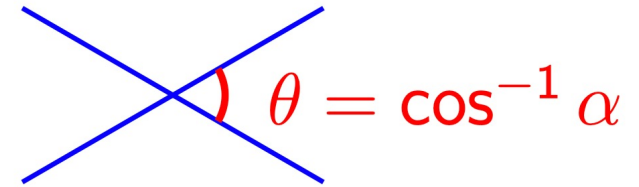
$$[\text{de Caen '00}] \quad cd^2 \leq N(d) \leq \binom{d+1}{2} \quad [\text{Gerzon '73}]$$

In lower bound constructions, pairwise angles $\rightarrow 90^\circ$ as $d \rightarrow \infty$

Equiangular lines with a fixed angle

$N_\alpha(d)$ = max # of lines in \mathbb{R}^d with pairwise angle $\cos^{-1} \alpha$
(focus: $\alpha > 0$ fixed, $d \rightarrow \infty$)

- $N_\alpha(d)$ grows linearly in d
 - in contrast to $N(d) = \Theta(d^2)$
- **Problem:** determine



$$\lim_{d \rightarrow \infty} \frac{N_\alpha(d)}{d}$$

Equiangular lines with a fixed angle: history

[Lemmens, Seidel '73] $N_{1/3}(d) = 2(d - 1) \quad \forall d \geq 15$

[Neumaier '89] $N_{1/5}(d) = \left\lfloor \frac{3}{2}(d - 1) \right\rfloor$ for sufficiently large d

“the next interesting case will require substantially stronger techniques”




[Bukh '16] $N_\alpha(d) \leq C_\alpha d$

[Balla, Dräxler, Keevash, Sudakov '18] $\limsup_{d \rightarrow \infty} N_\alpha(d)/d$ maximized at $\alpha = \frac{1}{3}$

[Jiang, Polyanskii '20] Determined $\lim_{d \rightarrow \infty} N_\alpha(d)/d \quad \forall \alpha > 0.196$

[Jiang, Tidor, Yao, Zhang, Z. '21] Solved! Determined $\lim_{d \rightarrow \infty} N_\alpha(d)/d$ for all fixed α



Zilin Jiang

Jonathan Tidor

me

Yuan Yao

Shengtong Zhang

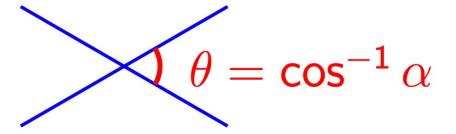
Annals of Mathematics **194** (2021), 729–743
<https://doi.org/10.4007/annals.2021.194.3.3>

Equiangular lines with a fixed angle

By ZILIN JIANG, JONATHAN TIDOR, YUAN YAO, SHENG TONG ZHANG,
and YUFEI ZHAO

Main result [Jiang, Tidor, Yao, Zhang, Z. '21]

$N_\alpha(d)$ = max # of lines in \mathbb{R}^d with pairwise angle $\cos^{-1} \alpha$



For every integer $k \geq 2$

$$N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1} (d-1) \right\rfloor \quad \forall d \geq d_0(k)$$

Proof needs $d \geq 2^{2^{k^{1+o(1)}}}$
Conjecture: $\forall d \geq k^C$

Other angles: \forall fixed $\alpha \in (0,1)$, setting $\lambda = (1 - \alpha)/(2\alpha)$ and

spectral radius order $k = k(\lambda)$

= min # vertices in a graph with top eigval exactly λ (adj. matrix)




- If $k < \infty$, $N_\alpha(d) = \left\lfloor \frac{k}{k-1} (d-1) \right\rfloor \quad \forall d \geq d_0(\alpha)$
- If $k = \infty$, $N_\alpha(d) = d + o(d) \quad \text{as } d \rightarrow \infty$

Spectral radius order

spectral radius order $k = k(\lambda)$

= min # vertices in a graph with top eigval exactly λ (adj. matrix)

Examples

α	λ	k	G
$1/3$	1	2	
$1/5$	2	3	
$1/7$	3	4	
$\frac{1}{1+2\sqrt{2}}$	$\sqrt{2}$	3	

Key new result on eigenvalue multiplicity

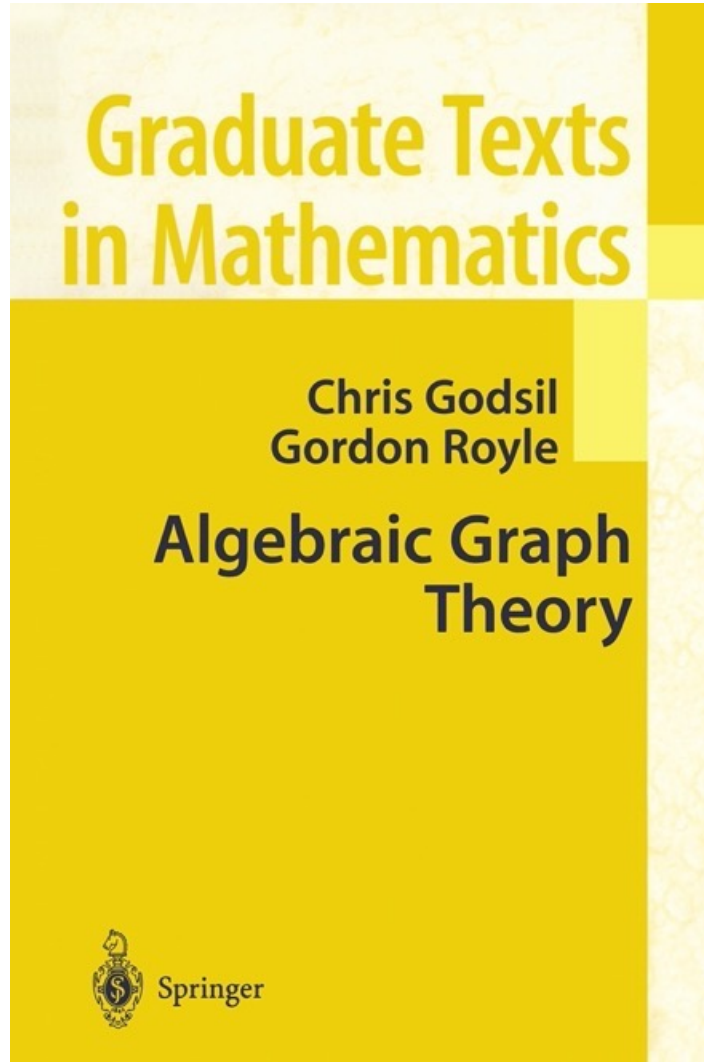
A connected bounded degree graph has sublinear second eigenvalue multiplicity (always referring to the adjacency matrix)

Theorem. (Jiang, Tidor, Yao, Zhang, Z. '21)

A connected n -vertex graph with maximum degree Δ has second largest eigenvalue with multiplicity

$$\leq C \log \Delta \frac{n}{\log \log n}$$

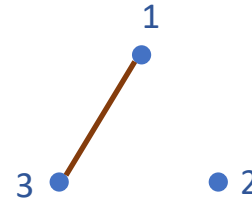
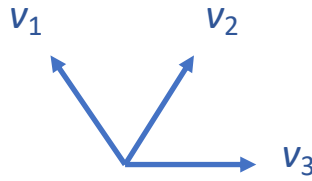
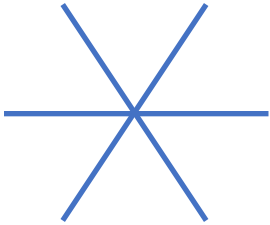
Connection to spectral graph theory



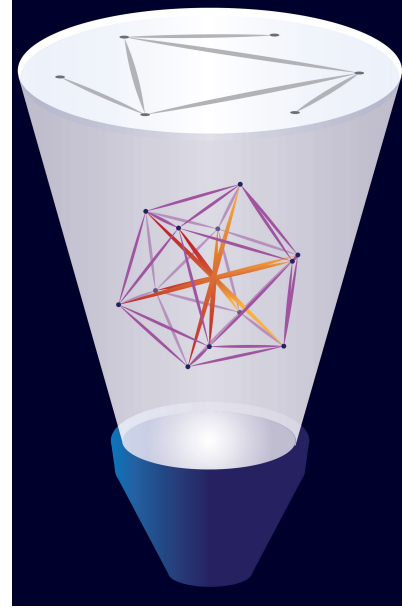
The problem that we are about to discuss is one of the founding problems of algebraic graph theory, despite the fact that at first sight it has little connection to graphs. A *simplex* in a metric space with distance function d is a subset S such that the distance $d(x, y)$ between any two distinct points of S is the same. In \mathbb{R}^d , for example, a simplex contains at most $d + 1$ elements. However, if we consider the problem in real projective space then finding the maximum number of points in a simplex is not so easy. The points of this space are the lines through the origin of \mathbb{R}^d , and the distance between two lines is determined by the angle between them. Therefore, a simplex is a set of lines in \mathbb{R}^d such that the angle between any two distinct lines is the same. We call this a set of *equiangular lines*. In this chapter we show how the problem of determining the maximum number of equiangular lines in \mathbb{R}^d can be expressed in graph-theoretic terms.

Connection to spectral graph theory

Equiangular lines in $\mathbb{R}^d \rightarrow$ unit vectors in $\mathbb{R}^d \rightarrow$ graph G



Edge = obtuse
Non-edge = acute



Given a list of vectors $v_1, \dots, v_n \in \mathbb{R}^d$, Gram matrix is PSD and rank $\leq d$:

$$\text{Gram matrix} = \begin{pmatrix} v_1 \cdot v_1 & \cdots & v_1 \cdot v_n \\ \vdots & \ddots & \vdots \\ v_n \cdot v_1 & \cdots & v_n \cdot v_n \end{pmatrix} = (1 - \alpha)I - 2\alpha A_G + \alpha J$$

$J = \text{all 1s matrix}$

Equivalent problem: given α, d , find graph G with max # vertices N s.t.

$$(1 - \alpha)I - 2\alpha A_G + \alpha J \text{ is PSD and rank } \leq d.$$

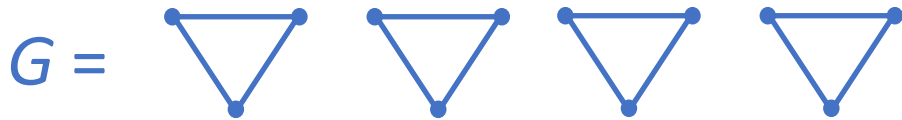
Connection to spectral graph theory

Problem: Given α, d , find graph G with max # vertices N s.t.

$\text{Gram} = (1 - \alpha)I - 2\alpha A_G + \alpha J$ is PSD and rank $\leq d$.

Example: recall $N_{1/5}(d) = \left\lfloor \frac{3}{2}(d - 1) \right\rfloor$ for all large d .

To verify $N_{1/5}(9) \geq 12$, check



$$(1 - \alpha)I - 2\alpha A_G + \alpha J =$$

is PSD and rank 9

($\alpha = 1/5$)

1	$-\alpha$	$-\alpha$	α	α	α	α	α	α	α	α	α
$-\alpha$	1	$-\alpha$	α	α	α	α	α	α	α	α	α
$-\alpha$	$-\alpha$	1	α	α	α	α	α	α	α	α	α
α	α	α	1	$-\alpha$	$-\alpha$	α	α	α	α	α	α
α	α	α	$-\alpha$	1	$-\alpha$	α	α	α	α	α	α
α	α	α	$-\alpha$	$-\alpha$	1	α	α	α	α	α	α
α	α	α	α	α	α	1	$-\alpha$	$-\alpha$	α	α	α
α	α	α	α	α	α	$-\alpha$	1	$-\alpha$	α	α	α
α	α	α	α	α	α	$-\alpha$	$-\alpha$	1	α	α	α
α	α	α	α	α	α	α	α	α	1	$-\alpha$	$-\alpha$
α	α	α	α	α	α	α	α	α	$-\alpha$	1	$-\alpha$
α	α	α	α	α	α	α	α	α	$-\alpha$	$-\alpha$	1

Recap

- Equiangular lines in $\mathbb{R}^d \rightarrow$ unit vectors in $\mathbb{R}^d \rightarrow$ graph G

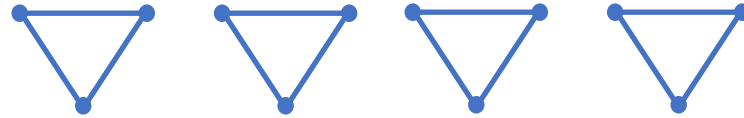
$N =$ # lines

vertices

- $N \leq d + \text{mult}(\lambda, A_G) + 1$

- Optimal configuration (for large d) turns out to be

$G =$ disjoint copies of a fixed graph with top eigval exactly $\lambda = \frac{1-\alpha}{2\alpha}$



- What happens if λ is the 2nd eigval of G ?
 - Can assume G is connected from now on
- Want to show that $\text{mult}(\lambda_2, A_G)$ is small

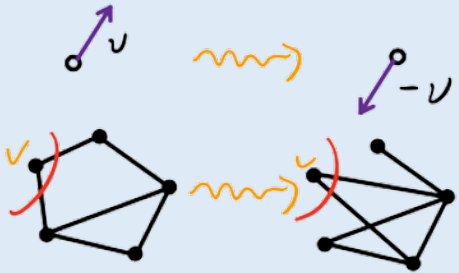
Second eigenvalue multiplicity

Q: must all connected graphs have small 2nd eigval multiplicity?

No. k -clique has eigvals $k - 1$ (once) and -1 ($k - 1$ times)

Not all graphs can arise from equiangular lines

Switching operation:



Theorem (Balla, Dräxler, Keevash, Sudakov '18)

$\forall \alpha \exists \Delta = \Delta(\alpha) : \text{can switch so that max degree} \leq \Delta$

[Balla '21+] $\Delta = O(\alpha^{-4})$ & tight

Theorem. (Jiang, Tidor, Yao, Zhang, Z. '21) A connected n -vertex graph with $\text{max deg} \leq \Delta$ has 2nd eigval multiplicity $O_{\Delta} \left(\frac{n}{\log \log n} \right)$

Sublinear second eigenvalue multiplicity

Theorem. (Jiang, Tidor, Yao, Zhang, Z. '21) A connected n -vertex graph with $\max \deg \leq \Delta$ has 2nd eigval multiplicity $O_{\Delta} \left(\frac{n}{\log \log n} \right)$

More generally, j^{th} eigval multiplicity $O_{\Delta, j} \left(\frac{n}{\log \log n} \right)$ for fixed j

Near miss examples

- Strongly regular graphs (e.g., complete graph, Paley graph)

- Not bounded degree

-  has eigval 0 with linear multiplicity

- 0 is not the 2nd largest eigval

-  has top eigval with linear multiplicity

- not connected

Second eigenvalue multiplicity

Open: Maximum 2^{nd} eigval mult of conn. bounded degree graph on n vertices?

- [Jiang, Tidor, Yao, Zhang, Z. '21]

$$\text{mult}(\lambda_2, G) \leq \frac{C_{\Delta} n}{\log \log n}$$

- [Haiman, Schildkraut, Zhang, Z. '21+]

\exists infinite family of bounded degree graphs with

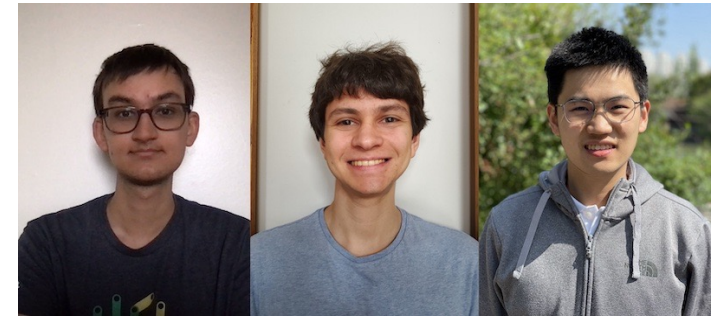
Construction.

Step 1. Cayley graph on $\text{Aff}(\mathbb{F}_q)$ generated by:

- (I) a multiplicative generator, and
- (II) an additive shift.

Step 2. Each (II) edge \rightsquigarrow path of length $\log q$

$$\text{mult}(\lambda_2, G) \geq \sqrt{\frac{n}{\log_2 n}}$$



Milan
Haiman

Carl
Schildkraut

Shengtong
Zhang

- Eigenvalues tend not to collide “by accident”
- Relies on group representations to get multiple eigval. Barrier at \sqrt{n}
- **Open:** $\text{mult}(\lambda_2, G) < n^{1-c}$? (\Rightarrow equiangular lines theorem for dimension $d > k^c$)

Second eigenvalue multiplicity

Q: Maximum 2nd eigval multiplicity of connected bounded degree n -vertex graph?

Main theorem (Jiang, Tidor, Yao, Zhang, Z. '21) $\text{mult}(\lambda_2, G) = O(n/\log \log n)$

- For **expander graphs** ($N(A) \geq (1 + c)|A| \quad \forall |A| \leq n/2$),
 $\text{mult}(\lambda_2, G) = O(n/\log n)$
- [Lee–Makarychev '08, building on Gromov, Colding–Minicozzi, Kleiner]
For **non-expanding Cayley graphs**, $\text{mult}(\lambda_2, G) = O(1)$
- [McKenzie, Rasmussen, Srivastava '21] For **regular graphs**
 $\text{mult}(\lambda_2, G) = O(n/(\log n)^c)$
 - A typical length $2k$ closed walk covers $\geq k^c$ vertices
- [Haiman, Schildkraut, Zhang, Z. '21+] Lower bounds (constructions)
Irregular: $\geq \sqrt{n/\log_2 n}$ Cayley: $\geq n^{2/5} - 1$

Proof sketch (moment method & vertex removal)

Theorem. $\text{mult}(\lambda_2, G) \leq C_\Delta n / \log \log n$ for connected G

$$r = c \log \log n \quad s = c \log n \quad \lambda = \lambda_2(G)$$

- $H = G$ with a small r -net removed
- s -balls in H typically have spectral radius $< \lambda - \varepsilon$
 - By counting length $2s$ closed walks
- Bound 2^{nd} eigval multiplicity in H via moments:

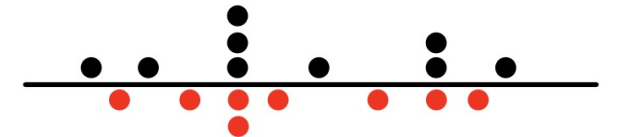
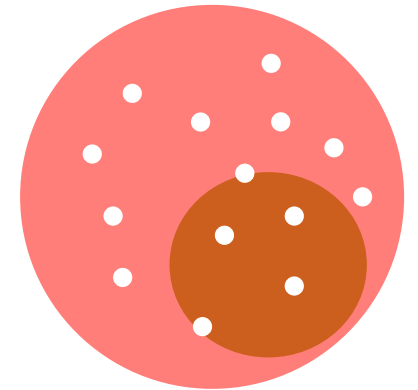
$$\text{mult}(\lambda, H) \lambda^{2s} \leq \sum_i \lambda_i(H)^{2s} = \text{tr} A_H^{2s} = \# \text{closed length } 2s \text{ walks in } H$$

$$\leq \sum_{v \in V(H)} \lambda_1(s\text{-ball around } v \text{ in } H)^{2s} \leq n(\lambda - \varepsilon)^{2s}$$

- $\Rightarrow \text{mult}(\lambda, H) = o(n)$
- $\Rightarrow \text{mult}(\lambda, G) \leq \text{mult}(\lambda, H) + |\text{net}| = o(n)$ by Cauchy eigval interlacing

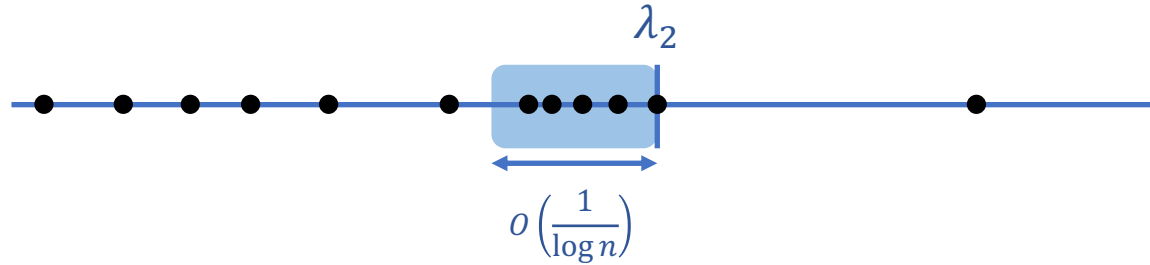


vertex removal
lowers spectral
radius



Approximate 2nd eigenvalue multiplicity

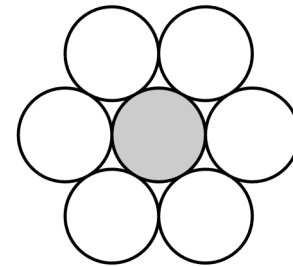
- Proof also bounds the “approximate 2nd eigval multiplicity”, showing at most $O\left(\frac{n}{\log \log n}\right)$ eigenvalues (incl. mult.) within $O\left(\frac{1}{\log n}\right)$ of λ_2



- [Haiman, Schildkraut, Zhang, Z. '21+]
A construction showing the above bounds are tight
 - Demonstrates a limitation of the trace method

Spherical codes

- $L \subseteq [-1,1)$. An **L -code** in \mathbb{R}^d is a set of unit vectors whose pairwise inner products lie in L
- **$N_L(d)$** = size of largest L -code in \mathbb{R}^d
- Points on a sphere with pairwise angle $\geq \theta$: **$L = [-1, \cos \theta]$**
 - Kissing number \mathbb{R}^d : **$L = [-1, \frac{1}{2}]$**
 - Sphere packing upper bounds in high dimensions
 - Linear programming bound (Delsarte '73)
- **Equiangular lines**: **$L = \{-\alpha, \alpha\}$**



Beyond linear programming bounds ...

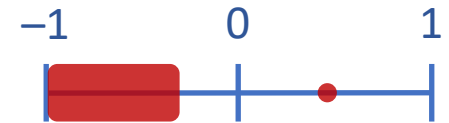
Spherical codes

$N_L(d)$ = size of largest L -code in \mathbb{R}^d , i.e., unit vec, pairwise inner products lie in L

- [Bukh '05]

For fixed $L = [-1, -\beta] \cup \{\alpha\}$ with $\beta > 0$

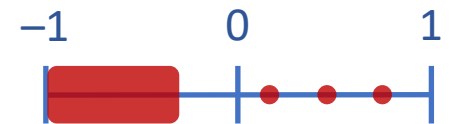
$$N_L(d) = O_L(d)$$



- [Balla, Dräxler, Keevash, Sudakov '18]

For fixed $L = [-1, -\beta] \cup \{\alpha_1, \dots, \alpha_k\}$ with $\beta > 0$

$$N_L(d) = O_L(d^k)$$



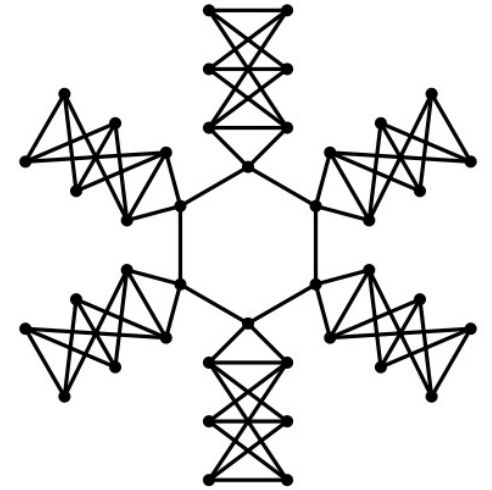
Spherical two-distance sets

[Jiang, Tidor, Yao, Zhang, Z. '20+] [Jiang, Polyanskii '21+]

- **Problem.** For fixed $\alpha, \beta > 0$, determine

$$\lim_{d \rightarrow \infty} \frac{N_{\{-\beta, \alpha\}}(d)}{d}$$

- A **conjectural limit** in terms of **eigenvalue of signed graphs**
- Solved in special cases: $\alpha < 2\beta$ or $(1 - \alpha)/(\alpha + \beta) < 2.019 \dots$
- Open in general
- **Obstacle:** **Sublinear eigenvalue multiplicity FALSE** for signed graphs
 - E.g., \exists bounded degree graph whose most negative eigval multiplicity is linear



Solution Framework

I. Forbidden local configurations / subgraphs

- Using Gram matrix is PSD

II. Global graph structure

- Graph theory, Ramsey theory
- Equiangular lines: bounded degree graph
- Spherical two-dist sets: complete multipartite XOR bounded degree

III. Extremal result

- **Spectral graph theory**, eigenvalue multiplicity
- [JTYZZ] Sublinear eigenvalue multiplicity in connected bounded degree graphs
- [Jiang Polyanski '21+] {signed graphs with largest eigenvalue $\leq \lambda$ } is characterized by forbidding a finite set of induced subgraphs iff $\lambda < 2.019 \dots$

Complex equiangular lines

Unrestricted angles

- **Zauner's conjecture:** $N^{\mathbb{C}}(d) = d^2$ for all d (known: $N^{\mathbb{C}}(d) \leq d^2$)
i.e., $\exists d^2$ unit vec. in \mathbb{C}^d with equal abs. of pairwise inner product
 - “SIC-POVM” from quantum mechanics
 - Verified in small dim. (exactly for $d \leq 53$, numerically for $d \leq 193$)

Restricted angles

- Determine $\lim_{d \rightarrow \infty} N_{\alpha}^{\mathbb{C}}(d)/d$

Equiangular subspaces in \mathbb{R}^d

- Configs of k -dim. subspaces in \mathbb{R}^d with given pairwise angles

Equiangular lines and eigenvalue multiplicities

Equiangular lines with a fixed angle.

$N_\alpha(d) = \max \# \text{ of lines in } \mathbb{R}^d \text{ with pairwise angle } \cos^{-1} \alpha$

$\forall \text{ integer } k \geq 2, N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(k)$

Other angles: \forall fixed $\alpha \in (0,1)$, setting $\lambda = (1-\alpha)/(2\alpha)$

spectral radius order $k = k(\lambda)$

= min # vertex in a graph with top eigval exactly λ

- If $k < \infty$, $N_\alpha(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(\alpha)$
- If $k = \infty$, $N_\alpha(d) = d + o(d) \quad \text{as } d \rightarrow \infty$

Sublinear eigenvalue multiplicity of bounded degree graphs.

A connected n -vertex graph with maximum degree Δ has second largest eigenvalue with multiplicity $\leq C \log \Delta \frac{n}{\log \log n}$

