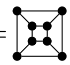
18.S997 (FALL 2017) PROBLEM SET 3

1. Fix $0 < p < 1$. Let G be a graph on n vertices with average degree at least pn. Prove:
   (a) The number of labeled 6-cycles in G is at least $(p^6 - o(1))n^6$.
   (b) The number of labeled copies of $K_{3,3}$ in G is at least $(p^9 - o(1))n^6$.
   (c) The number of labeled copies of $Q_3 =$  in G is at least $(p^{12} - o(1))n^8$.
   (d) (Bonus) The number of labeled paths on 4 vertices in G is at least $(p^3 - o(1))n^4$.
2. Deduce from the quasirandom Cayley graphs theorem the following corollary for vertex transitive graphs: If an n-vertex d-regular vertex-transitive graph G satisfies

$$\left|e(X, Y) - \tfrac{d}{n}|X||Y|\right| \le \epsilon dn \quad \text{for all } X, Y \subseteq V(G),$$

   then all the eigenvalues of the adjacency matrix of G, other than the largest one, are at most $8\epsilon dn$ in absolute value.
3. Define $W\colon [0,1]^2 \to \mathbb{R}$ by $W(x, y) = 2\cos(2\pi(x - y))$. Let G be a graph. Show that $t(G, W)$ is the number of ways to orient all edges of G so that every vertex has the same number of incoming edges as outgoing edges.

. . . to be continued . . . check back later (last updated: October 19, 2017)