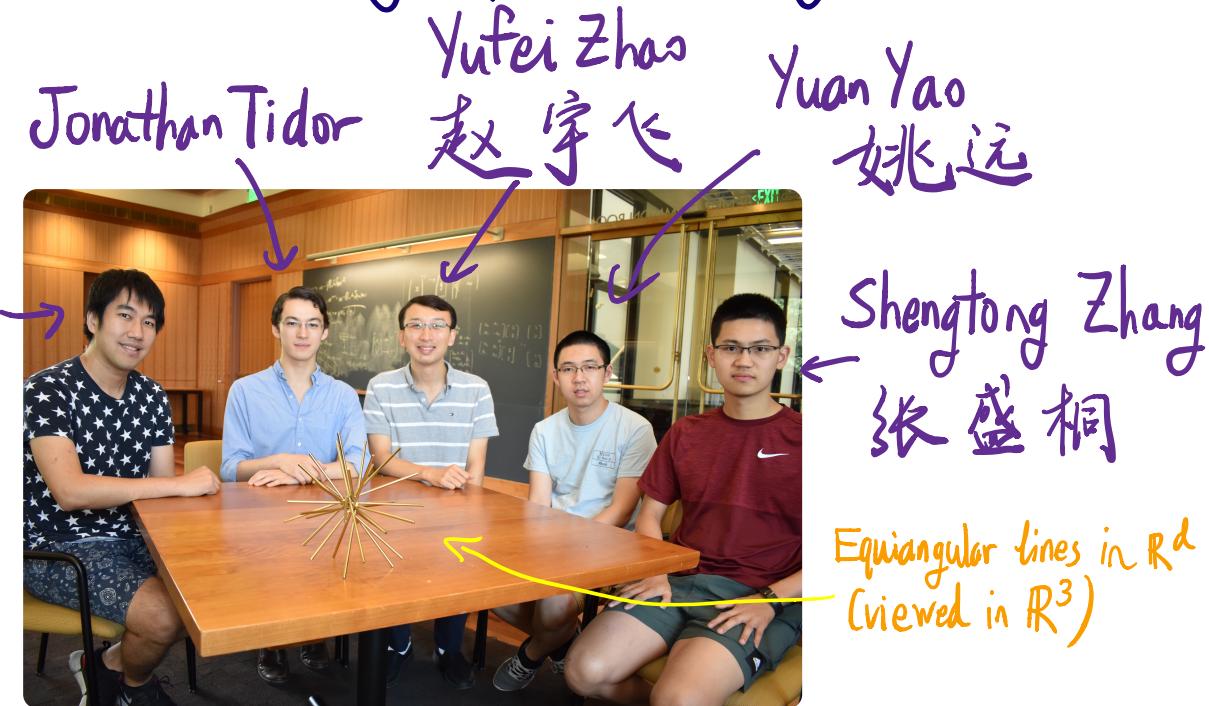


Equiangular lines, spherical two-distance sets & spectral graph theory

Zilin Jiang
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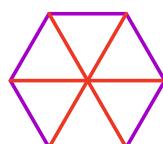
Based on
arXiv:1907.12466
& 2006.06633



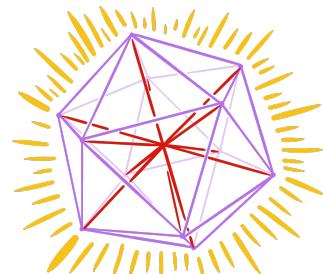
Equiangular lines

$N(d) = \max \# \text{lines in } \mathbb{R}^d \text{ pairwise same angle}$

e.g. $N(2) = 3$



$N(3) = 6$



de Caen '00 $c d^2 \leq N(d) \leq \binom{d+1}{2}$

↑ angles $\rightarrow 90^\circ$
as $d \rightarrow \infty$

Gerzon '73

Equiangular lines with a fixed angle

$N_\alpha(d) = \max \# \text{equiangular lines with angle } \cos^{-1} \alpha$

Some angles are more special

Lemmens-Seidel
'73

$$N_{1/3}(d) = 2(d-1) \quad \forall d \geq 15$$

Neumann '73

$$N_\alpha(d) \leq 2d \quad \text{unless } \alpha = \frac{1}{\text{odd integer}}$$

Neumaier '89

$$N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor \quad \forall d \geq d_0$$



Next interesting case $\alpha = 1/7$?

Finally, we remark that the recent result of Shearer [13], that every number $t \geq t^* = (2 + \sqrt{5})^{1/2} \approx 2.058$ is a limit point from above of the set of largest eigenvalues of graphs, makes it likely that the hypothesis of Theorem 2.6 can be satisfied if and only if $t < t^*$. (As communicated to me by Professor J. J. Seidel, Eindhoven, this has indeed been verified by A. J. Hoffman and J. Shearer.) Thus the next interesting case, $t = 3$, will require substantially stronger techniques.

⌚⌚⌚ Some decades later ...

Bukh '16

$$N_\alpha(d) \leq C_\alpha d$$

Balla-Dräxler
-Keerash-Sudakov '18

$$N_\alpha(d) \leq 1.93d \quad \forall d \geq d_0(\alpha) \quad \text{if } \alpha \neq 1/3$$

Problem: determine, for each α

$$\lim_{d \rightarrow \infty} \frac{N_\alpha(d)}{d}$$

Our work completely solves this problem

Lemmens-Seidel '73 $N_{1/3}(d) = 2(d-1)$ $\forall d$ suff. large

Neuhauser '89 $N_{1/5}(d) = \left\lfloor \frac{3}{2}(d-1) \right\rfloor$ \sim

Our result: $N_{1/7}(d) = \left\lfloor \frac{4}{3}(d-1) \right\rfloor$ \sim

Thm (JTYZZ) \forall integer $k \geq 2$

$$N_{\frac{1}{2k-1}}(d) = \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor \quad \forall d \geq d_0(k)$$

And for other angles \forall fixed $\alpha \in (0, 1)$

Set $\lambda = \frac{1-\alpha}{2\alpha}$ $k = k(\lambda)$
 (reparameterization) "spectral radius order"

Then

$$N_\alpha(d) = \begin{cases} \left\lfloor \frac{k}{k-1}(d-1) \right\rfloor & \forall d \geq d_0(\alpha) \text{ if } k < \infty \\ d + o(d) & \text{if } k = \infty \end{cases}$$

$k(\lambda) = \text{spectral radius order}$
 $= \min k \text{ s.t. } \exists k\text{-vertex graph } G \text{ with } \lambda_1(G) = \lambda$
 (set $k(\lambda) = \infty$ if \nexists such G)
 \uparrow
 spectral radius of G
 = top eigenvalue
 of adjacency mat of G

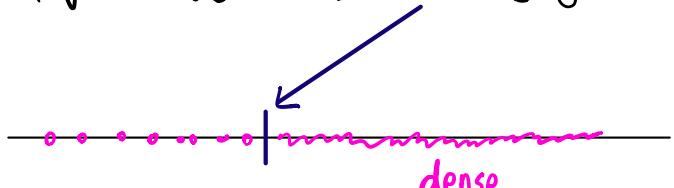
Examples

α	λ	k	G
$1/3$	1	2	---
$1/5$	2	3	Δ
$1/7$	3	4	\square
$1/(1+2\sqrt{2})$	$\sqrt{2}$	3	---

$\lim_{d \rightarrow \infty} \frac{N_\alpha(d)}{d} = \frac{k(\lambda)}{k(\lambda) - 1}$ was conjectured by Jiang - Polyanskiy

who proved it for $\lambda < \sqrt{2 + \sqrt{5}} \approx 2.058$

$$\{\lambda_1(G) : G \in \mathbb{R}$$



Hoffman '72 + Shearer '89

Spherical two distance sets

A set of unit vectors in \mathbb{R}^d whose inner products take only two values α, β

(equiangular lines : $\alpha = -\beta$)

Delsarte, Goethals, Seidel '77 max size $\leq \frac{1}{2}d(d+3)$

Taking midpoints of a regular simplex $\sim \frac{1}{2}d(d+1)$

Glazyrin-Yu '18 : tight if $d \geq 7$ &
 $d+3$ not odd perf sq

From now on let's consider fixed angles

More generally : spherical A-code , $A \subset [-1, 1]$

$N_A(d) = \max$ # unit vectors in \mathbb{R}^d
whose pairwise inner products lie in A

Equiangular lines $\leadsto A = \{-\alpha, \alpha\}$

We consider $A = \{\alpha, \beta\}$ for fixed $-1 \leq \beta < 0 < \alpha < 1$

$$\text{Neumaier '81} \quad N_{\alpha, \beta}(d) \leq 2d+1 \quad \text{unless } \frac{1-\alpha}{\alpha-\beta} \in \mathbb{Z}$$

$$\text{Bukh '16} \quad N_{[-1, \beta] \cup \{\alpha\}}(d) = O_\beta(d)$$

$$\begin{array}{ll} \text{Balla-Dräxler} & N_{[-1, \beta] \cup \{\alpha_1, \alpha_2, \dots, \alpha_k\}}(d) \leq 2^k (k-1)! \left(1 + \frac{\alpha_1}{\beta} + \dots + \frac{\alpha_k}{\beta}\right) n^k \\ - \text{Keevash-Sudakov '18} & \alpha_1 < \alpha_2 < \dots < \alpha_k \end{array}$$

& $\exists \alpha_1, \dots, \alpha_k, \beta$ s.t. bound tight up to constant factor

Problem Determine, for fixed $-1 \leq \beta < 0 < \alpha < 1$,

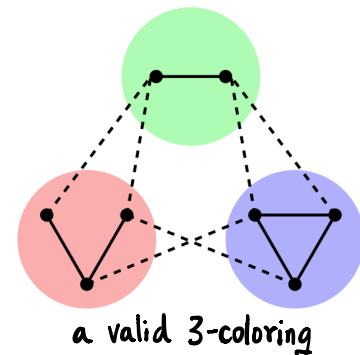
$$\lim_{d \rightarrow \infty} \frac{N_{\alpha, \beta}(d)}{d}$$

We conjecturally relate this problem to eigenvalues of signed graphs

$$G^\pm \quad + \quad \begin{matrix} & & & \\ & & & \\ & & & \\ & & & \end{matrix} \\ A_{G^\pm} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix}$$

Defn A valid t -coloring:

- + edges join identical colors —
- - edges join distinct colors -----



$$k_p(\lambda) = \inf \left\{ \frac{|G^\pm|}{\text{mult}(\lambda, G^\pm)} : G^\pm \text{ has valid } p\text{-coloring} \right. \\ \left. \& \lambda_1(G^\pm) = \lambda \right\}$$

If G^+ has valid 2-coloring
then G^+ is isospectral with its underlying graph

Thus $k_1(\lambda) = k_2(\lambda) = k(\lambda) = \{|G| : \lambda_i(G) = \lambda\}$

Determining $k_p(\lambda)$, $p \geq 3$ seems hard

Main conjecture on spherical two-distance sets:

Fix $-1 \leq \beta < 0 < \alpha < 1$. Set

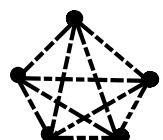
$$\lambda = \frac{1-\alpha}{\alpha-\beta} \quad \& \quad p = \lfloor \frac{-\alpha}{\beta} \rfloor + 1$$

Then

$$\lim_{d \rightarrow \infty} \frac{N_{\alpha, \beta}(d)}{d} = \frac{k_p(\lambda)}{k_p(\lambda) - 1} \quad (\lambda = 1 \text{ if } k_p(\lambda) = \infty)$$

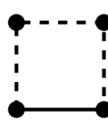
Thm (JTYZZ) Conj true if $p \leq 2$ OR $\lambda \in \{1, \sqrt{2}, \sqrt{3}\}$.

$$k_p(1) = \begin{cases} 2 & \text{if } p=1,2 \\ \frac{p}{p-1} & \text{if } p \geq 2 \end{cases}$$



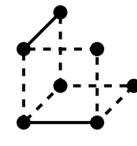
$$-4, 1, 1, 1, 1 \\ k_5(1) = \frac{5}{4}$$

$$k_p(\sqrt{2}) = \begin{cases} 3 & \text{if } p=1,2 \\ 2 & \text{if } p \geq 2 \end{cases}$$



$$k_p(\sqrt{2}) = 2 \quad \forall p \geq 2$$

$$k_p(\sqrt{3}) = \begin{cases} 4 & \text{if } p=1,2 \\ \frac{7}{3} & \text{if } p=3 \\ 2 & \text{if } p \geq 4 \end{cases}$$



$$k_3(\sqrt{3}) = \frac{7}{3}$$



$$k_p(\sqrt{3}) = 2 \quad \forall p \geq 4$$

cf Huang pf of sensitivity conj

E.g. $(\alpha, \beta) = \left(\frac{2}{5}, \frac{-1}{5}\right)$, $\lambda = 1$, $p = 3$,

$$\rightarrow k_p(\lambda) = 3, \quad N_{\alpha, \beta}(d) = 3d + O(1)$$

(contrasting $N_{\alpha, -\alpha}(d) \leq (2+o(1))d$ for eq-ang lines)

Graduate Texts in Mathematics

Chris Godsil
Gordon Royle

Algebraic Graph Theory

 Springer

The problem that we are about to discuss is one of the **founding problems** of algebraic graph theory, despite the fact that at first sight it has little connection to graphs. A *simplex* in a metric space with distance function d is a subset S such that the distance $d(x, y)$ between any two distinct points of S is the same. In \mathbb{R}^d , for example, a simplex contains at most $d + 1$ elements. However, if we consider the problem in real projective space then finding the maximum number of points in a simplex is not so easy. The points of this space are the lines through the origin of \mathbb{R}^d , and the distance between two lines is determined by the angle between them. Therefore, a simplex is a set of lines in \mathbb{R}^d such that the angle between any two distinct lines is the same. We call this a set of **equiangular lines**. In this chapter we show how the problem of determining the maximum number of equiangular lines in \mathbb{R}^d can be expressed in graph-theoretic terms.

Unit vectors

$$v_1, v_2, \dots, v_N \in \mathbb{R}^d \quad \langle v_i, v_j \rangle = \pm \alpha \quad \forall i \neq j$$

Associated graph: vtx $[N]$, $i \sim j$ if $\langle v_i, v_j \rangle = -\alpha$ (obtuse)

Observation:

$\exists N$ equiangular lines in \mathbb{R}^d with common angle $\cos^{-1}\alpha$



$\exists N$ -vtx graph G s.t. $\lambda I - A_G + \frac{1}{2}J$ is positive semidef
 $(\lambda = \frac{1-\alpha}{2\alpha})$ ($J = A \otimes J$) and rank $\leq d$
 $= \frac{1}{2\alpha}$ Gram matrix

Recall GOAL: maximize N given λ , d

Construction

Starting with some H with $\lambda_1(H) = \lambda$ & $|H| = k(\lambda)$
take $G = H \cup H \cup H \dots \cup H \cup H$ (N vtx total)

Upper bound on N

$$\begin{aligned} N &= \text{rank}(\lambda I - A_G + \frac{1}{2}J) + \text{null}(\lambda I - A_G + \frac{1}{2}J) \\ &\leq d + \text{null}(\lambda I - A_G + \frac{1}{2}J) \\ &\leq d + \underbrace{\text{null}(\lambda I - A_G)}_{\text{mult}(\lambda, G)} + 1 \end{aligned}$$

eigenvalue multiplicity

Difficult/interesting case to rule out :

a large connected G with high $\text{mult}(\lambda, G)$

Note that since $\lambda I - A_G + \frac{1}{2}J$ is psd

$\lambda = 1^{\text{st}}$ or 2^{nd} largest eigenvalue of A_G

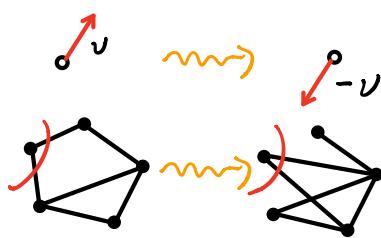
If $\lambda = \lambda_1(G)$, Perron-Frobenius $\Rightarrow \text{mult}(\lambda, G) = 1$

So focus on the case $\lambda = \lambda_2$

Q: Must all connected graphs have small 2^{nd} eigenvalue multiplicity?

Not all graphs can arise from equiangular lines

Switching operation



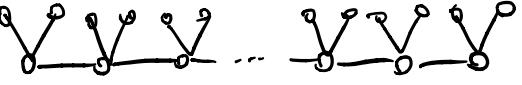
Thm (Balla-Dräxler-Keevash-Sudakov)

$\forall \alpha \exists \Delta$: can switch G to $\max \deg \leq \Delta$

We prove a new result in spectral graph theory :

Ihm [JTYZZ] A connected n -vertex graph with $\max \deg \leq \Delta$ has second largest eigenvalue with multiplicity $O_{\Delta}\left(\frac{n}{\log \log n}\right)$

Near miss examples

- Strongly regular graphs
e.g. complete graphs, Paley graphs Not bounded degree
-  mult(0, G) linear, 0: a middle eigenval
-  not connected

Open problem Max. possible 2nd eigenvalue multiplicity
of a connected bounded degree graph?

Interesting to consider restrictions to (bdd deg)

- regular graphs
- Cayley graphs

Example: a Cayley graph on $\mathrm{PSL}(2, p)$
gives 2nd eigenvalue multiplicity $\gtrsim n^{1/3}$

For expander graphs, $\mathrm{mult}(\lambda_2, G) = O\left(\frac{n}{\log n}\right)$

For non-expanding Cayley graphs, $\mathrm{mult}(\lambda_2, G) = O(1)$

Lee-Makarychev, building on Gromov, Colding-Minicozzi, Kleiner

Recently: McKenzie-Rasmussen-Srivastava
for a connected d-reg graph, $\mathrm{mult}(\lambda_2, G) \leq O_d\left(\frac{n}{\log^{\frac{1}{d}-\epsilon} n}\right)$

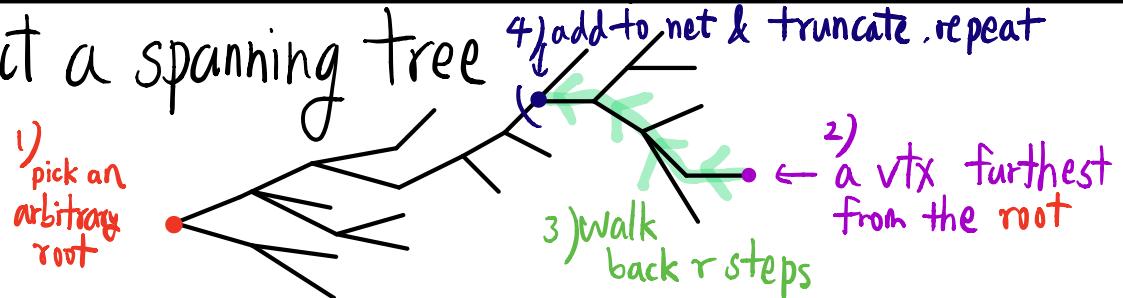
Let's prove:

Thm If G is connected, n vtx, $\max \deg \leq \Delta$
then its 2nd largest eigenvalue has multiplicity
 $O\left(\frac{n}{\log \log n}\right)$

Lem 1 (Finding a small net)

Every connected n -vtx graph has an r -net of size $\lceil \frac{n}{r+1} \rceil \quad \forall n, r$

Pf Select a spanning tree



Lem 2 (Net removal significantly reduces spectral radius)

$$\text{If } H = G_1 - (\text{an } r\text{-net of } G_1) \\ \text{then } \lambda_1(H)^{2r} \leq \lambda_1(G_1)^{2r} - 1$$

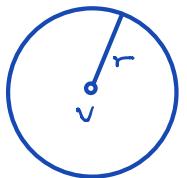
Pf $A_H^{2r} \leq A_G^{2r} - I$ entrywise ($A_H = A_G$ with the deleted edges zero'd)

to check diagonal entries, count closed walks
Suffice to exhibit a closed walk $v \circ$ in G not in H

Lem 3 (Local versus global spectra)

$$\sum_{i=1}^{|H|} \lambda_i(H)^{2r} \leq \sum_{v \in V(H)} \lambda_1(B_H(v, r))^{2r}$$

r-neighborhood



Pf //

closed walks of length $2r$ in H

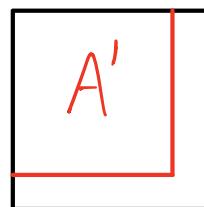
such walks starting at v (necessarily stays in $B_H(v, r)$)

$$= 1_v^T A_{B_H(v, r)}^{2r} 1_v$$

$$\leq \lambda_1(B_H(v, r))^{2r}$$

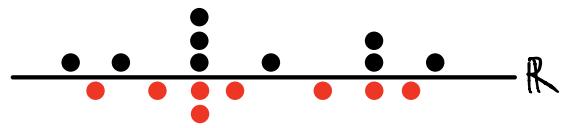
Tool: Cauchy eigenvalue interlacing theorem

Real sym matrix A



Then eigenvalues of A & A' interlace

*remove last row
& column $\rightarrow A'$*



\Rightarrow Deleting a vertex cannot reduce $\text{mult}(\lambda, G)$ by more than 1

Proof sketch that $\text{mult}(\lambda_2, G) = o(n)$

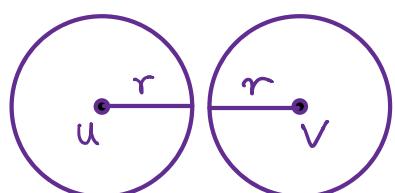
$$\lambda = \lambda_2(G) > 0 \\ (\text{easy if } < 0)$$

$$\text{Let } r = r_1 + r_2, \quad r_1 = c \log \log n \\ r_2 = c \log n$$

$$U = \{v \in V(G) : \lambda_1(B_G(v, r)) > \lambda\}$$

(vtx with large local spectral radius)

If U contains u, v with $d(u, v) \geq 2r+2$



then G restricted to these two balls has ≥ 2 eigenvals $> \lambda$. Contradiction.

$$\text{Thus } U \subset \text{a } (2r+1)\text{-ball} \Rightarrow |U| \leq \Delta^{2r+2} = o(n)$$

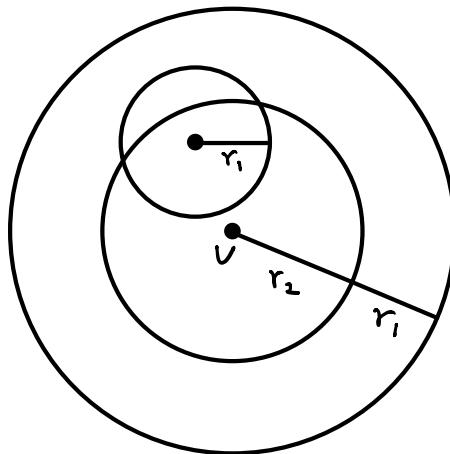
Net removal: let V_0 be an r_1 -net of G with $|V_0| \leq \lceil \frac{n}{r_{1+1}} \rceil$ by Lem 1

Set $H = G - U \cup V_0$

$\forall v \in V(H)$

$$B_G(v, r) \setminus B_H(v, r_2)$$

is an r_1 -net of $B_G(v, r)$



By Lem 2, $\forall v \in V(H)$,

$$\begin{aligned} \lambda_1(B_H(v, r_2))^{2r_1} &\leq \lambda_1(B_G(v, r))^{2r_1} - 1 \\ &\leq \lambda^{2r_1} - 1 \quad (\text{since } v \notin U) \end{aligned}$$

By Lem 3, $\sum_{i=1}^{|H|} \lambda_i(H)^{2r_2} \leq \sum_{v \in V(H)} \lambda_1(B_H(v, r_2))^{2r_2}$

$$\begin{aligned} \text{mult}(\lambda, H) \lambda^{2r_2} &\leq \prod_{i=1}^{|H|} (\lambda^{2r_1} - 1)^{\frac{r_2}{r_1}} n \\ \Rightarrow \text{mult}(\lambda, H) &= o(n) \end{aligned}$$

By interlacing, $\text{mult}(\lambda, G) \leq \text{mult}(\lambda, H) + |U| + |V| = o(n)$

Summary:

- bound moment by counting closed $2r_2$ -walks
- net removal significantly reduces local closed $2r_1$ -walks
- relate these via local spectral radii

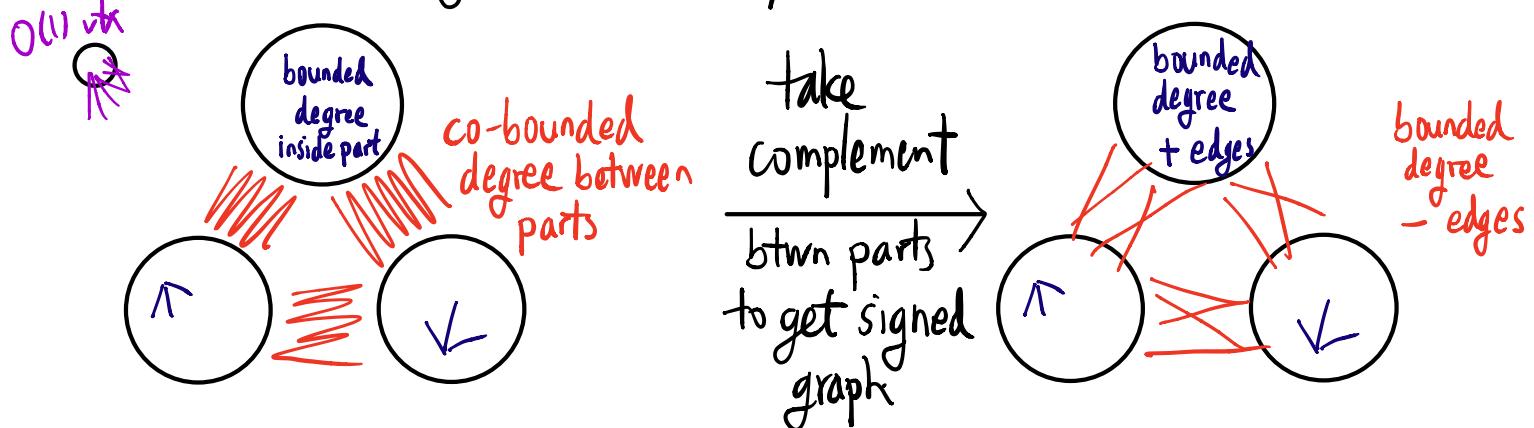
Spherical 2-dist set: unit vectors $v_1, \dots, v_N \in \mathbb{R}^d$
(fixed $-1 < \beta < 0 < \alpha < 1$) $\langle v_i, v_j \rangle \in \{\alpha, \beta\} \quad \forall i \neq j$

Associated graph G : $i \sim j$ if $\langle v_i, v_j \rangle = \beta$ (obtuse)

"Switching" no longer valid 

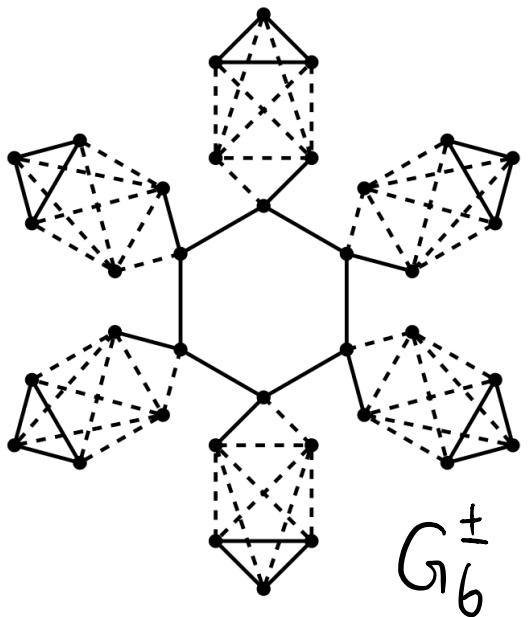
Structure theorem

After deleting $O(1)$ vtx, G can be modified into a complete p -partite graph, $p = \lfloor \frac{-d}{\beta} \rfloor + 1$ where $O(1)$ edges are added/removed at each vtx



We would be able to proceed the same as eq-ang lines if all such signed graphs have $\text{mult}(\lambda_{p+1}, G) = o(n)$

But this is not true



$\exists G_n^{\pm}$ signed graph
 6n vertices
 max deg 5
 has a valid 3-coloring
 BUT: largest eigenvalue appears
 with multiplicity n

But there is still hope

Our structure theorem actually gives additional forbidden subgraphs not mentioned above

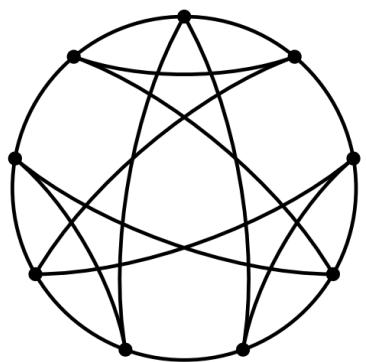
To solve the spherical 2-dist set problem
 for $\lambda = \frac{1-\alpha}{\alpha-\beta}$ & $p = \lfloor \frac{1-\alpha}{\beta} \rfloor + 1$, it suffices to

① Determine

$$k_p(\lambda) = \inf \left\{ \frac{|G^\pm|}{\text{mult}(\lambda, G^\pm)} : G^\pm \text{ has valid } p\text{-coloring} \& \lambda_1(G^\pm) = \lambda \right\}$$

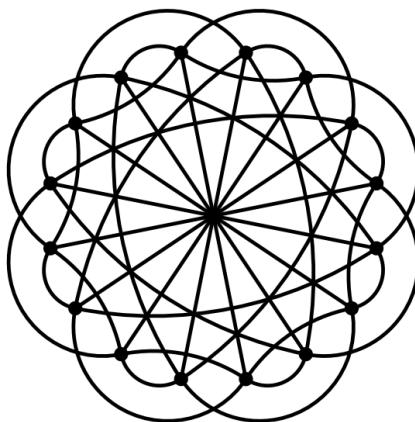
② Give a sufficiently good linear upper bound
 on $\text{mult}(\lambda_{p+1}, G^\pm)$ for certain classes of
 signed graphs G^\pm given by forbidden subgraphs

Next interesting case $\lambda=2$ (we solved $\lambda \in \{1, \sqrt{2}, \sqrt{3}\}$)



Paley graph of order 9

$$\Rightarrow k_3(2) \leq \frac{9}{4}$$



Clebsch graph

$$\Rightarrow k_4(2) \leq \frac{8}{5}$$