Problem Set 5. Due 10/16

Reminder: You must acknowledge your sources and collaborators (even if it is "none", you must write so). Failure to do so on this problem set will result in an automatic 2-point deduction.

You should practice using the compositional formula for the first three problems.

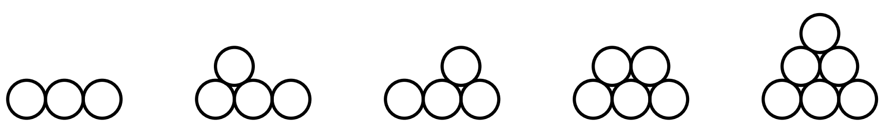
1. Let a_n be the number of ways to choose a composition of n and then color each odd part either red or blue. For instance, when $n = 3$ there are two ways to color the each of compositions 3, $2+1$, and $1+2$ and eight ways to color the composition $1+1+1$. So $a_3 = 14$. Determine the generating function $\sum_{n\geq 0} a_n x^n$ and find a closed-form formula for a_n.

2. Let h_n be the number of ways to tile a $1 \times n$ rectangle with 1×1 tiles that are red or blue and 1×2 tiles that are green, yellow, or white. Determine the generating function $\sum_{n\geq 0} h_n x^n$ and find a closed-form formula for h_n.

3. A permutation of $[n]$ is called *indecomposable* if, when written in one-line form, it cannot be cut into two parts so that everything before the cut is smaller than everything after the cut. For example, 3142 is indecomposable, but 2143 is not as you can cut it after the first two elements.

   Let f_n be the number of indecomposable permutations of length n, and set $f_0 = 0$. Find the generating function $F(x) = \sum_{n\geq 0} f_n x^n$. Express your answer in terms of the generating function $G(x) = \sum_{n\geq 0} n! x^n$ for the number of all permutations.

4. Let a_n denote the number of ways of stacking identical coins in the plane so that the bottom row consists of n consecutive coins. E.g., for $n = 3$, there are five ways (so $a_3 = 5$):

   Prove via bijection that a_n equals the n-th Catalan number.

5. Show that the number of partitions of n for which no part appears more than twice is equal to the number of partitions of n for which no part is divisible by 3. For instance, when $n = 5$ there are five partitions of the first type $(5, 41, 32, 311, 221)$ and five of the second type $(5, 41, 221, 2111, 11111)$. Use generating functions.

6. Show that the number of partitions of n for which no part appears exactly once is equal to the number of partitions of n for which every part is divisible by 2 or 3. For instance, when $n = 6$ there are four partitions of the first type $(111111, 2211, 222, 33)$ and four of the second type $(222, 33, 42, 6)$. Use generating functions.

7. Let $p_{\text{odd}}(n)$ denote the number of partitions of n into an odd number of parts, and let $p_{\text{even}}(n)$ denote the number of partitions of n into an even number of parts.

   Prove that $|p_{\text{even}}(n) - p_{\text{odd}}(n)|$ is equal to the number of partitions of n into distinct odd parts.

   Hint: Consider $\prod_{k\geq 1}(1 - qx^k)^{-1}$. What does the exponent of q encode?