

### 18.S997 (FALL 2017) PROBLEM SET 3

1. Fix  $0 < p < 1$ . Let  $G$  be a graph on  $n$  vertices with average degree at least  $pn$ . Prove:
  - (a) The number of labeled 6-cycles in  $G$  is at least  $(p^6 - o(1))n^6$ .
  - (b) The number of labeled copies of  $K_{3,3}$  in  $G$  is at least  $(p^9 - o(1))n^6$ .
  - (c) The number of labeled copies of  $Q_3 = \begin{array}{c} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array}$  in  $G$  is at least  $(p^{12} - o(1))n^8$ .
  - (d) (Bonus) The number of labeled paths on 4 vertices in  $G$  is at least  $(p^3 - o(1))n^4$ .
2. Deduce from the quasirandom Cayley graphs theorem the following corollary for vertex transitive graphs: If an  $n$ -vertex  $d$ -regular vertex-transitive graph  $G$  satisfies

$$|e(X, Y) - \frac{d}{n}|X||Y|| \leq \epsilon dn \quad \text{for all } X, Y \subseteq V(G),$$

then all the eigenvalues of the adjacency matrix of  $G$ , other than the largest one, are at most  $8\epsilon d$  in absolute value.

3. Define  $W: [0, 1]^2 \rightarrow \mathbb{R}$  by  $W(x, y) = 2 \cos(2\pi(x - y))$ . Let  $G$  be a graph. Show that  $t(G, W)$  is the number of ways to orient all edges of  $G$  so that every vertex has the same number of incoming edges as outgoing edges.
4. Let  $W$  be a  $\{0, 1\}$ -valued graphon. Suppose graphons  $W_n$  satisfy  $\|W_n - W\|_{\square} \rightarrow 0$  as  $n \rightarrow \infty$ . Show that  $\|W_n - W\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ .
5. (a) Let  $\epsilon > 0$ . Show that for every graphon  $W: [0, 1]^2 \rightarrow [0, 1]$ , there exist measurable sets  $S_1, \dots, S_k, T_1, \dots, T_k \subseteq [0, 1]$  and reals  $a_1, \dots, a_k \in \mathbb{R}$ , with  $k < 1/\epsilon^2$ , such that

$$\left\| W - \sum_{i=1}^k a_i \mathbf{1}_{S_i \times T_i} \right\|_{\square} \leq \epsilon.$$

- (b) Let  $\mathcal{P}$  be a partition of  $[0, 1]$  into measurable sets. Let  $U$  be a graphon that is constant on  $S \times T$  for each  $S, T \in \mathcal{P}$ . For that for every graphon  $W$ , one has

$$\|W - W_{\mathcal{P}}\|_{\square} \leq 2\|W - U\|_{\square}.$$

- (c) Use (a) and (b) to give a different proof of the weak regularity lemma (with slightly worse bounds than the one given in class): show that for every  $\epsilon > 0$  and every graphon  $W$ , there exists partition  $\mathcal{P}$  of  $[0, 1]$  into  $2^{O(1/\epsilon^2)}$  measurable sets such that  $\|W - W_{\mathcal{P}}\|_{\square} \leq \epsilon$ .
6. In this problem, you will give an alternate proof of the strong regularity lemma with explicit bounds.

Let  $\epsilon = (\epsilon_1, \epsilon_2, \dots)$  be a sequence of positive reals. By repeatedly applying the weak regularity lemma, show that there is some  $M = M(\epsilon)$  such that for every graphon  $W$ , there is a pair of partitions  $\mathcal{P}$  and  $\mathcal{Q}$  of  $[0, 1]$  into measurable sets, such that  $\mathcal{Q}$  refines  $\mathcal{P}$ ,  $|\mathcal{Q}| \leq M$  (here  $|\mathcal{Q}|$  denotes the number of parts of  $\mathcal{Q}$ ),

$$\|W - W_{\mathcal{Q}}\|_{\square} \leq \epsilon_{|\mathcal{P}|} \quad \text{and} \quad \|W_{\mathcal{Q}}\|_2^2 \leq \|W_{\mathcal{P}}\|_2^2 + \epsilon_1^2.$$

Furthermore, deduce the strong regularity lemma in the form given in class: one can write

$$W = W_{\text{str}} + W_{\text{psr}} + W_{\text{sml}}$$

where  $W_{\text{str}}$  is a  $k$ -step-graphon with  $k \leq M$ ,  $\|W_{\text{psr}}\|_{\square} \leq \epsilon_k$ , and  $\|W_{\text{sml}}\|_1 \leq \epsilon_1$ . State your bounds<sup>1</sup> on  $M$  explicitly in terms of  $\epsilon$ .

7. (Generalized maximum cut) For symmetric measurable functions  $W, U: [0, 1]^2 \rightarrow \mathbb{R}$ , define

$$\mathcal{C}(W, U) := \sup_{\varphi} \langle W, U^{\varphi} \rangle = \sup_{\varphi} \int W(x, y) U(\varphi(x), \varphi(y)) dx dy,$$

where  $\varphi$  ranges over all measure-preserving bijections on  $[0, 1]$ . Extend the definition of  $\mathcal{C}(\cdot, \cdot)$  to graphs:  $\mathcal{C}(G, \cdot) := \mathcal{C}(W_G, \cdot)$  etc.

- (a) Show that if  $W_1$  and  $W_2$  are graphons such that  $\mathcal{C}(W_1, U) = \mathcal{C}(W_2, U)$  for all graphons  $U$ , then  $\delta_{\square}(W_1, W_2) = 0$ .
- (b) Let  $G_1, G_2, \dots$  be a sequence of graphs such that  $\mathcal{C}(G_n, U)$  converges as  $n \rightarrow \infty$  for every graphon  $U$ . Show that  $G_1, G_2, \dots$  is convergent.
- (c) Can the hypothesis in (b) be replaced by “ $\mathcal{C}(G_n, H)$  converges as  $n \rightarrow \infty$  for every graph  $H$ ”?

*... to be continued ... check back later (last updated: October 31, 2017). Some hints on next page*

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<sup>1</sup>With  $\epsilon_k = \epsilon/k^2$  (corresponding to Szemerédi’s regularity lemma), your bound on  $M$  should be  $\text{tower}(\epsilon^{-O(1)})$ ; if not then you are doing something wrong.

## HINTS

4. Every measurable set can be arbitrarily well approximated (in measure) as a union of boxes.
7. Remember that  $\|\cdot\|_{\square} \leq \|\cdot\|_1 \leq \|\cdot\|_2$ .