## 18.218 PROBLEM SET

**Instructions:**

- All submissions must be **typed in LaTeX** and submitted as PDF on Stellar (try Overleaf if you are looking for an online LaTeX editor without requiring installations). Please name your file `ps#_Lastname_Firstname.pdf` and remember to include your name in each file.
- At the top of each submission, you must **acknowledge all references and people consulted** (other than lectures and the textbook). Failure acknowledge sources will lead to an automatic 10% penalty. Examples include: names of people you discussed homework with, books, and online resources. If you consulted no additional sources, you should write sources consulted: none.
- Please turn in the problems marked $\boxed{\text{ps1}}$ and $\boxed{\text{ps1}\star}$ for problem set 1, etc., by midnight of each due date (see course homepage for due dates). Do not submit the other problems—they are for you to practice. See course homepage for policies (20% per day late penalty; do not look up solutions; collaboration policy).
- **Bonus problems**, marked by $\star$, are more challenging. A grade of A- may be attained by only solving the non-starred problems. To attain a grade of A or A+, you should solve a substantial number of starred problems.
- Please **do not exceed one page** for each problem (standard 1-inch margins and 11pt font). If you cannot fit your solution within one page, then think about how to better distill your ideas. If necessary, you may skip details of routine calculations (we don't really want to read them either).
- This file will be updated constantly as the term progresses. Please check back regularly. It will be announced when each problem set is complete.
- You are encouraged to include figures whenever they are helpful. Here are some recommended ways to produce figures in decreasing order of learning curve difficulty:
  (1) TikZ or other drawing script
  (2) IPE (which supports LaTeX) or other drawing app
  (3) photo/scan (I recommend the Dropbox app on your phone, which has a nice scanning feature that produces clear monochrome scans)

Problems begin on the next page

1. Verify the following asymptotic calculations used in Ramsey number lower bounds:
   (a) For each $k$, the largest $n$ satisfying $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ has $n = \left(\frac{1}{e\sqrt{2}} + o(1)\right) k2^{k/2}$.
   (b) For each $k$, the maximum value of $n - \binom{n}{k}2^{1-\binom{k}{2}}$ as $n$ ranges over positive integers is $\left(\frac{1}{e} + o(1)\right) k2^{k/2}$.
   (c) For each $k$, the largest $n$ satisfying $e\left(\binom{k}{2}\binom{n}{k-2} + 1\right) 2^{1-\binom{k}{2}} < 1$ satisfies $n = \left(\frac{\sqrt{2}}{e} + o(1)\right) k2^{k/2}$.
2. Prove that, if there is a real $p \in [0,1]$ such that
$$\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$$
then the Ramsey number $R(k,t)$ satisfies $R(k,t) > n$. Using this show that
$$R(4,t) \geq c\left(\frac{t}{\log t}\right)^{3/2}$$
for some constant $c > 0$.
3. (Extension of Sperner's theorem) Let $\mathcal{F}$ be a collection of subset of $[n]$ that does not contain $k+1$ elements forming a chain: $A_1 \subsetneq \cdots \subsetneq A_{k+1}$. Prove that $\mathcal{F}$ is no larger than taking the union of the $k$ levels of the boolean lattice closest to the middle layer.
4. ps1 Let $A_1, \ldots, A_m$ be $r$-element sets and $B_1, \ldots, B_m$ be $s$-element sets. Suppose $A_i \cap B_i = \emptyset$ for each $i$, and for each $i \neq j$, either $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$. Prove that $m \leq (r+s)^{r+s}/(r^r s^s)$.
5. ps1 Prove that for every positive integer $r$, there exists an integer $K$ such that the following holds. Let $S$ be a set of $rk$ points evenly spaced on a circle. If we partition $S = S_1 \cup \cdots \cup S_r$ so that $|S_i| = k$ for each $i$, then, provided $k \geq K$, there exists $r$ congruent triangles where the vertices of the $i$-th triangle lie in $S_i$, for each $1 \leq i \leq r$.
6. ps1 Prove that every set of 10 points in the plane can be covered by a union of disjoint unit disks.
7. ps1⋆ Prove that $[n]^d$ cannot be partitioned into fewer than $2^d$ sets each of the form $A_1 \times \cdots \times A_d$ where $A_i \subsetneq [n]$.
8. Let $k \geq 4$ and $H$ a $k$-uniform hypergraph with at most $4^{k-1}/3^k$ edges. Prove that there is a coloring of the vertices of $H$ by four colors so that in every edge all four colors are represented.
9. ps1 Let $G$ be a graph on $n \geq 10$ vertices. Suppose that adding any new edge to $G$ would create a new clique on 10 vertices. Prove that $G$ has at least $8n - 36$ edges.
   (Hint in white: )
10. Prove that there is an absolute constant $c > 0$ so that for every $n \times n$ matrix with distinct real entries, one can permute its rows so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$. (A subsequence does not need to be selected from consecutive terms. For example, $(1,2,3)$ is an increasing subsequence of $(1,5,2,4,3)$.)
11. Let $G$ be a graph with $n$ vertices and $m$ edges. Prove that $K_n$ can be written as a union of $O(n^2(\log n)/m)$ copies of $G$ (not necessarily edge-disjoint).
12. ps1 Given a set $\mathcal{F}$ of subsets of $[n]$ and $A \subseteq [n]$, write $\mathcal{F}|_A := \{S \cap A : S \in \mathcal{F}\}$ (its *projection* onto $A$). Prove that for every $n$ and $k$, there exists a set $\mathcal{F}$ of subsets of $[n]$ with $|\mathcal{F}| = O(k2^k \log n)$ such that for every $k$-element subset $A$ of $[n]$, $\mathcal{F}|_A$ contains all $2^k$ subsets of $A$.
13. Let $A$ be a subset of the unit sphere in $\mathbb{R}^3$ (centered at the origin) containing no pair of orthogonal points.

ps1 (a) Prove that $A$ occupies at most $1/3$ of the sphere in terms of surface area.

ps1⋆ (b) Prove an upper bound smaller than $1/3$ (give your best bound).

14. Let $\boldsymbol{r} = (r_1, \ldots, r_k)$ be nonzero integers whose sum is nonzero. Prove that there exists a real $c > 0$ (depending on $\boldsymbol{r}$ only) such that the following holds: for every finite set $A$ of nonzero *reals*, there exists a subset $B \subset A$ with $|B| \geq c|A|$ such that there do not exist $b_1, \ldots, b_k \in B$ with $r_1 b_1 + \cdots + r_k b_k = 0$.

ps1 15. Prove that every set $A$ of $n$ nonzero integers contains two disjoint subsets $B_1$ and $B_2$, such that both $B_1$ and $B_2$ are sum-free, and $|B_1| + |B_2| > 2n/3$. Can you do it if $A$ is a set of nonzero reals?

ps1⋆ 16. Let $G = (V, E)$ be a graph with with $|V| = n$ and $|E| = m \geq n^{3/2}$. Show that there exist two distinct edge-subsets $E_1, E_2 \subset E$ (not necessarily disjoint) such that the subgraphs $(V, E_1)$ and $(V, E_2)$ are isomorphic, and $|E_1| = |E_2| \geq cm^{2/3}$, where $c > 0$ is a constant.