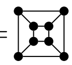
18.S997 (FALL 2017) PROBLEM SET 3

1. Fix $0 < p < 1$. Let G be a graph on n vertices with average degree at least pn. Prove:
   (a) The number of labeled 6-cycles in G is at least $(p^6 - o(1))n^6$.
   (b) The number of labeled copies of $K_{3,3}$ in G is at least $(p^9 - o(1))n^6$.
   (c) The number of labeled copies of $Q_3 =$  in G is at least $(p^{12} - o(1))n^8$.
   (d) (Bonus) The number of labeled paths on 4 vertices in G is at least $(p^3 - o(1))n^4$.
2. Deduce from the quasirandom Cayley graphs theorem the following corollary for vertex transitive graphs: If an n-vertex d-regular vertex-transitive graph G satisfies
$$\left|e(X,Y) - \tfrac{d}{n}|X||Y|\right| \le \epsilon dn \quad \text{for all } X, Y \subseteq V(G),$$
then all the eigenvalues of the adjacency matrix of G, other than the largest one, are at most $8\epsilon d$ in absolute value.
3. Define $W\colon [0,1]^2 \to \mathbb{R}$ by $W(x,y) = 2\cos(2\pi(x-y))$. Let G be a graph. Show that $t(G,W)$ is the number of ways to orient all edges of G so that every vertex has the same number of incoming edges as outgoing edges.
4. Let W be a $\{0,1\}$-valued graphon. Suppose graphons W_n satisfy $\|W_n - W\|_\square \to 0$ as $n \to \infty$. Show that $\|W_n - W\|_1 \to 0$ as $n \to \infty$.
5. (a) Let $\epsilon > 0$. Show that for every graphon $W\colon [0,1]^2 \to [0,1]$, there exist measurable sets $S_1, \dots, S_k, T_1, \dots, T_k \subseteq [0,1]$ and reals $a_1, \dots, a_k \in \mathbb{R}$, with $k < 1/\epsilon^2$, such that
$$\left\| W - \sum_{i=1}^{k} a_i \mathbf{1}_{S_i \times T_i} \right\|_\square \le \epsilon.$$
   (b) Let $\mathcal{P}$ be a partition of $[0,1]$ into measurable sets. Let U be a graphon that is constant on $S \times T$ for each $S, T \in \mathcal{P}$. For that for every graphon W, one has
$$\|W - W_\mathcal{P}\|_\square \le 2\|W - U\|_\square.$$
   (c) Use (a) and (b) to give a different proof of the weak regularity lemma (with slightly worse bounds than the one given in class): show that for every $\epsilon > 0$ and every graphon W, there exists partition $\mathcal{P}$ of $[0,1]$ into $2^{O(1/\epsilon^2)}$ measurable sets such that $\|W - W_\mathcal{P}\|_\square \le \epsilon$.
6. In this problem, you will give an alternate proof of the strong regularity lemma with explicit bounds.

   Let $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \dots)$ be a sequence of positive reals. By repeatedly applying the weak regularity lemma, show that there is some $M = M(\boldsymbol{\epsilon})$ such that for every graphon W, there is a pair of partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[0,1]$ into measurable sets, such that $\mathcal{Q}$ refines $\mathcal{P}$, $|\mathcal{Q}| \le M$ (here $|\mathcal{Q}|$ denotes the number of parts of $\mathcal{Q}$),
$$\|W - W_\mathcal{Q}\|_\square \le \epsilon_{|\mathcal{P}|} \qquad \text{and} \qquad \|W_\mathcal{Q}\|_2^2 \le \|W_\mathcal{P}\|_2^2 + \epsilon_1^2.$$

Furthermore, deduce the strong regularity lemma in the form given in class: one can write

$$W = W_{\mathrm{str}} + W_{\mathrm{psr}} + W_{\mathrm{sml}}$$

where W_{str} is a k-step-graphon with $k \le M$, $\|W_{\mathrm{psr}}\|_\square \le \epsilon_k$, and $\|W_{\mathrm{sml}}\|_1 \le \epsilon_1$. State your bounds[1] on M explicitly in terms of $\boldsymbol{\epsilon}$.

7. (Generalized maximum cut) For symmetric measurable functions $W, U \colon [0,1]^2 \to \mathbb{R}$, define

$$\mathcal{C}(W,U) := \sup_{\varphi} \langle W, U^{\varphi} \rangle = \sup_{\varphi} \int W(x,y) U(\varphi(x), \varphi(y))\, dxdy,$$

where φ ranges over all measure-preserving bijections on $[0,1]$. Extend the definition of $\mathcal{C}(\cdot,\cdot)$ to graphs: $\mathcal{C}(G,\cdot) := \mathcal{C}(W_G, \cdot)$ etc.

   (a) Show that if W_1 and W_2 are graphons such that $\mathcal{C}(W_1,U) = \mathcal{C}(W_2,U)$ for all graphons U, then $\delta_\square(W_1,W_2) = 0$.
   (b) Let $G_1, G_2, \dots$ be a sequence of graphs such that $\mathcal{C}(G_n, U)$ converges as $n \to \infty$ for every graphon U. Show that $G_1, G_2, \dots$ is convergent.
   (c) Can the hypothesis in (b) be replaced by "$\mathcal{C}(G_n, H)$ converges as $n \to \infty$ for every graph H"?

8. Using the moments lemma ($t(F,U) = t(F,W)$ for all F implies $\delta_\square(U,W) = 0$) and compactness of the space of graphons, deduce:

   Inverse counting lemma. For every $\epsilon > 0$, there exist $k \in \mathbb{N}$ and $\eta > 0$ such that whenever two graphons U and W satisfy

$$|t(F,U) - t(F,W)| \le \eta \quad \text{for all graphs } F \text{ on } k \text{ vertices},$$

   we must have $\delta_\square(U,W) \le \epsilon$.

9. (a) Given a function $f \colon \mathbb{Z} \to \mathbb{C}$ with finite support, define $\widehat{f} \colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by

$$\widehat{f}(t) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n t}.$$

   Let $c_1, \dots, c_k \in \mathbb{Z}$. Let $A \subset \mathbb{Z}$ be a finite set. Show that

$$|\{(a_1, \dots, a_k) \in A^k : c_1 a_1 + \cdots + c_k a_k = 0\}| = \int_0^1 \widehat{1_A}(c_1 t) \widehat{1_A}(c_k t) \dots \widehat{1_A}(c_k t)\, dt.$$

   (b) Show that if a finite set A of integers contains $\beta |A|^2$ solutions $(a,b,c) \in A^3$ to $a + 2b = 3c$, then it contains at least $\beta^2 |A|^3$ solutions $(a,b,c,d) \in A^4$ to $a + b = c + d$.

Problem set complete. Some hints on next page

Problems 1 and 7 are worth 0.5 points per part.

[1] With $\epsilon_k = \epsilon/k^2$ (corresponding to Szemerédi's regularity lemma), your bound on M should be an exponential tower of 2's of height $\epsilon^{-O(1)}$; if not then you are doing something wrong.

Hints

4. Every measurable set can be arbitrarily well approximated (in measure) as a union of boxes.
7. Remember that $\| \ \|_\square \le \| \ \|_1 \le \| \ \|_2$.