18.S997 (FALL 2017) PROBLEM SET 1

1. (a) Let s and r be positive integers. Show that there is some integer $n = n(s, r)$ so that if every edge of the complete graph K_n on n vertices is colored with one of r colors, then there is a monochromatic copy of K_s.
   (b) Let $s \geq 3$ be a positive integer. Show that if the edges of the complete graph on $\binom{2s-2}{s-1}$ vertices are colored with 2 colors, then there is a monochromatic copy of K_s.
2. Show that a graph with n vertices and m edges has at least
$$\frac{4m}{3n}\left(m - \frac{n^2}{4}\right)$$
many triangles.
3. Let S be a set of n points in the plane, with the property that no two points are at distance greater than 1. Show that S has at most $\lfloor n^2/3 \rfloor$ pairs of points at distance greater than $1/\sqrt{2}$. Also, show that the bound $\lfloor n^2/3 \rfloor$ is tight (i.e., cannot be improved).
4. Show that for every $r \geq 1$ and $\epsilon > 0$, there is some $c > 0$ so that any graph with at least $\left(1 - \frac{1}{r} + \epsilon\right)\frac{n^2}{2}$ edges contains at least cn^{r+1} copies of K_{r+1}.
5. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that for all sufficiently large n, every K_4-free graph with n vertices and at least $(\frac{1}{3} - \delta)n^2$ edges contains 3 disjoint independent sets each of size at least $(1 - \epsilon)n/3$.
6. Show that, for every $\epsilon > 0$, there exists a graph H with chromatic number $\chi(H) = 3$ such that $\mathrm{ex}(n, H) > \frac{1}{4}n^2 + n^{2-\epsilon}$ for all sufficiently large n.
7. (How *not* to define density) Let $S \subset \mathbb{Z}^2$. Define
$$d_k(S) = \max_{\substack{A,B \subset \mathbb{Z} \\ |A|=|B|=k}} \frac{|S \cap (A \times B)|}{|A||B|}.$$
Show that $\lim_{k\to\infty} d_k(S)$ exists and is always either 0 or 1.

. . . to be continued . . . check back later (last updated: September 19, 2017)