# 18.S997 (FALL 2017) PROBLEM SET 2

1. Let the *half-graph* $\mathsf{H}_n$ be the bipartite graph on $2n$ vertices $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ with edges $\{a_i b_j : i \le j\}$.
   (a) For every $\epsilon > 0$, explicitly construct an $\epsilon$-regular partition of $\mathsf{H}_n$ into $O(1/\epsilon)$ parts.
   (b) Show that there is some $c > 0$ such that for every sufficiently small $\epsilon > 0$, every integer $k$ and sufficiently large multiple $n$ of $k$, every partition of the vertices of $H_n$ into $k$ equal-sized parts contains at least $ck$ pairs of parts which are not $\epsilon$-regular.
2. Show that there is some absolute constant $C > 0$ such that for every $\epsilon > 0$, every graph on $n$ vertices contains an $\epsilon$-regular pair of vertex subsets each with size at least $\delta n$, where $\delta = 2^{-\epsilon^{-C}}$.
3. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that every graph on $n$ vertices contains an $\epsilon$-regular subset of vertices of size at least $\delta n$. (Here a vertex subset $X$ is called an *$\epsilon$-regular set* if the pair $(X, X)$ is $\epsilon$-regular, i.e., for all $A, B \subset X$ with $|A|, |B| \ge \epsilon |X|$, one has $|d(A, B) - d(X, X)| \le \epsilon$.)
4. Show that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subset [n]$ has fewer than $\delta n^2$ many triples $(x, y, z) \in A^3$ with $x + y = z$, then there is some $B \subset A$ with $|A \setminus B| \le \epsilon n$ such that $B$ is sum-free, i.e., there do not exist $x, y, z \in B$ with $x + y = z$.
5. Show that the number of triangle-free graphs on $n$ labeled vertices is $2^{(1/4+o(1))n^2}$.
6. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that every $K_4$-free graph on $n$ vertices with at least $(\frac{1}{8} + \epsilon)n^2$ edges contains an independent set of size at least $\delta n$.

*... to be continued ... check back later (last updated: October 3, 2017)*
*Some hints on next page*

## Hints

6. Given an $\epsilon'$-regular pair of vertex sets with edge-density slightly above $1/2$, find either a $K_4$ or a large independent set.