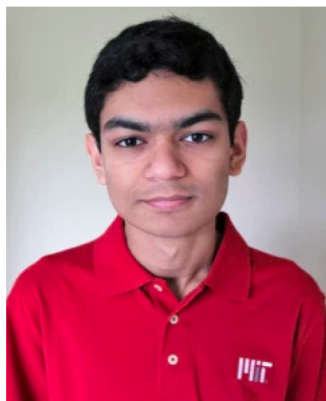


# A reverse Sidorenko inequality

Independent sets, colorings, and graph homomorphisms

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Joint work with



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## Question 1

Fix  $d$ . Which  $d$ -regular graph  $G$  maximizes  $i(G)^{1/v(G)}$ ?

# independent sets

## Question 2

Fix  $d$  and  $q$ . Which  $d$ -regular graph  $G$  maximizes  $c_q(G)^{1/v(G)}$ ?

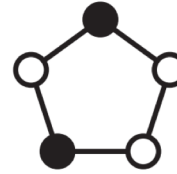
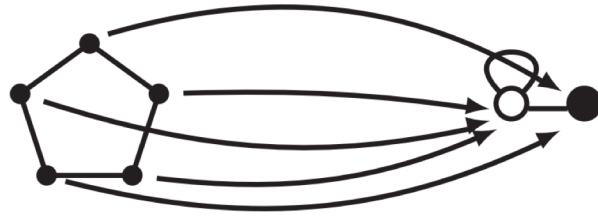
# proper  $q$ -colorings

## Question 3

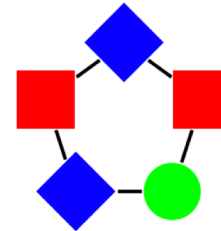
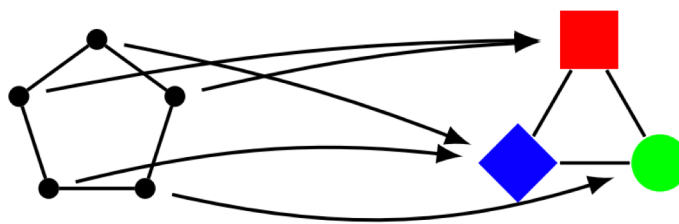
Fix  $d$  and  $H$ . Which  $d$ -regular graph  $G$  maximizes  $\text{hom}(G, H)^{1/v(G)}$ ?

# graph homomorphisms

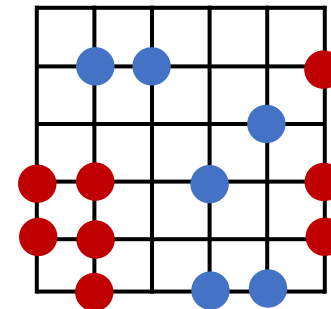
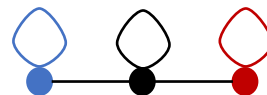
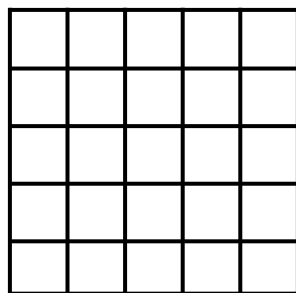
Independent sets:  $i(G) = \text{hom}(G, \text{graph with two nodes and a loop on the first})$



Colorings:  $c_q(G) = \text{hom}(G, K_q)$



Widom–Rowlinson model:  $\text{hom}(G, \text{graph with three nodes and loops on the first and third})$



# Independent sets

**Question 1.** Fix  $d$ . Which  $d$ -regular graph  $G$  maximizes  $i(G)^{1/v(G)}$ ?

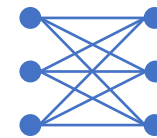
Asked by Granville in 1988 at Banff in an effort to resolve the Cameron–Erdős conjecture on the number of sum-free subsets of  $\{1, \dots, n\}$

Conjectured maximizer:  $K_{d,d}$

Alon (1991) proved an asymptotic version ( $d \rightarrow \infty$ )

Kahn (2001) proved the conjecture for bipartite  $G$  via entropy method

Z. (2010) removed the bipartite hypothesis via “bipartite swapping trick”  $i(G)^2 \leq i(G \times K_2)$



**Theorem (Kahn + Z.).** Let  $G$  be an  $n$ -vertex  $d$ -regular graph. Then

$$i(G) \leq i(K_{d,d})^{n/(2d)} = (2^{d+1} - 1)^{n/(2d)}$$

Davies, Jenssen, Perkins & Roberts (2017) gave a new proof using a novel occupancy method, which found applications in sphere packing and spherical codes [Jenssen, Joos, Perkins 2018]

# Graph homomorphisms

**Question 3.** Fix  $d$  and  $H$ . Which  $d$ -regular graph  $G$  maximizes  $\text{hom}(G, H)^{1/v(G)}$ ?

[Galvin, Tetali 2004] Among bipartite graphs,  $G = K_{d,d}$  is the maximizer (extending [Kahn '01])

**Q.** Can the bipartite hypothesis be dropped?

[Z. 2011] Yes for certain families of  $H$ , such as threshold graphs (generalizing independent sets).

$H = K_q$  ( $q$ -colorings) remained open



The bipartite hypothesis **cannot** always be dropped. E.g.,  $H = \text{cycle of length } 4$ , maximizer is  $K_{d+1}$ , not  $K_{d,d}$ .

[Cohen, Perkins, Tetali 2017] Widom–Rowlinson model ( $H = \text{path of length } 3$ ):  $G = K_{d+1}$  is the maximizer

[Sernau 2017]  $\exists H$ : maximizer is neither  $K_{d,d}$  nor  $K_{d+1}$

**Open:** Among 3-regular graphs, is there a finite list of possible maximizers  $G$  for  $\text{hom}(G, H)^{1/v(G)}$  ?  
(We only know that this list is more than just  $K_{3,3}$  and  $K_4$ )

# Graph homomorphisms

**Question 3.** Fix  $d$  and  $H$ . Which  $d$ -regular graph  $G$  maximizes  $\text{hom}(G, H)^{1/v(G)}$ ?

Wide open in general (see my survey [Extremal regular graphs](#))

**Conjecture** ([Davies, Jenssen, Perkins, Roberts 2017](#)).

For all fixed  $H$ , among [triangle-free](#)  $G$ ,  $G = K_{d,d}$  is always the maximizer

(already known for bipartite  $G$  [[Galvin, Tetali 2004](#)])

# Independent sets in irregular graphs

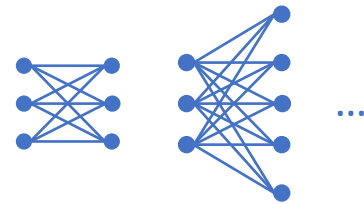
$d_u$  = degree of  $u$  in  $G$

**Degree-degree distribution:** probab. distribution of  $(d_u, d_v)$  for uniformly random edge  $uv$

**Question 1'.** Given the degree-degree distribution, which  $G$  maximizes  $i(G)^{1/v(G)}$ ?

e.g., 20% edges have endpoint degrees (3,4), 30% edges ...

**Conjecture (Kahn '01).** Maximizer is a disjoint union of complete bipartite graphs



We prove this conjecture

**Theorem (Sah, Sawhney, Stoner, Z., '18+).** Let  $G$  be a graph without isolated vertices. Then

$$i(G) \leq \prod_{uv \in E(G)} i(K_{d_u, d_v})^{1/(d_u d_v)}$$

Independent sets are **biclique-maximizing**

**Conjecture (Galvin '06).** An analogous inequality for  $\text{hom}(G, H)$  (False; which  $G$  and  $H$ ?)

# Proper colorings

**Question 2.** Fix  $d$  and  $q$ . Which  $d$ -regular graph  $G$  maximizes  $c_q(G)^{1/v(G)}$ ?

Conjectured answer:  $K_{d,d}$

[Galvin, Tetali '04] True for bipartite  $G$

[Davies, Jenssen, Perkins, Roberts '18] True for  $d = 3$  & [Davies]  $d = 4$  (computer-assisted)

We prove the conjecture

**Theorem** (Sah, Sawhney, Stoner, Z. '18++). Let  $q \in \mathbb{N}$  and  $G$  an  $n$ -vertex  $d$ -regular graph. Then

$$c_q(G) \leq c_q(K_{d,d})^{n/(2d)}$$

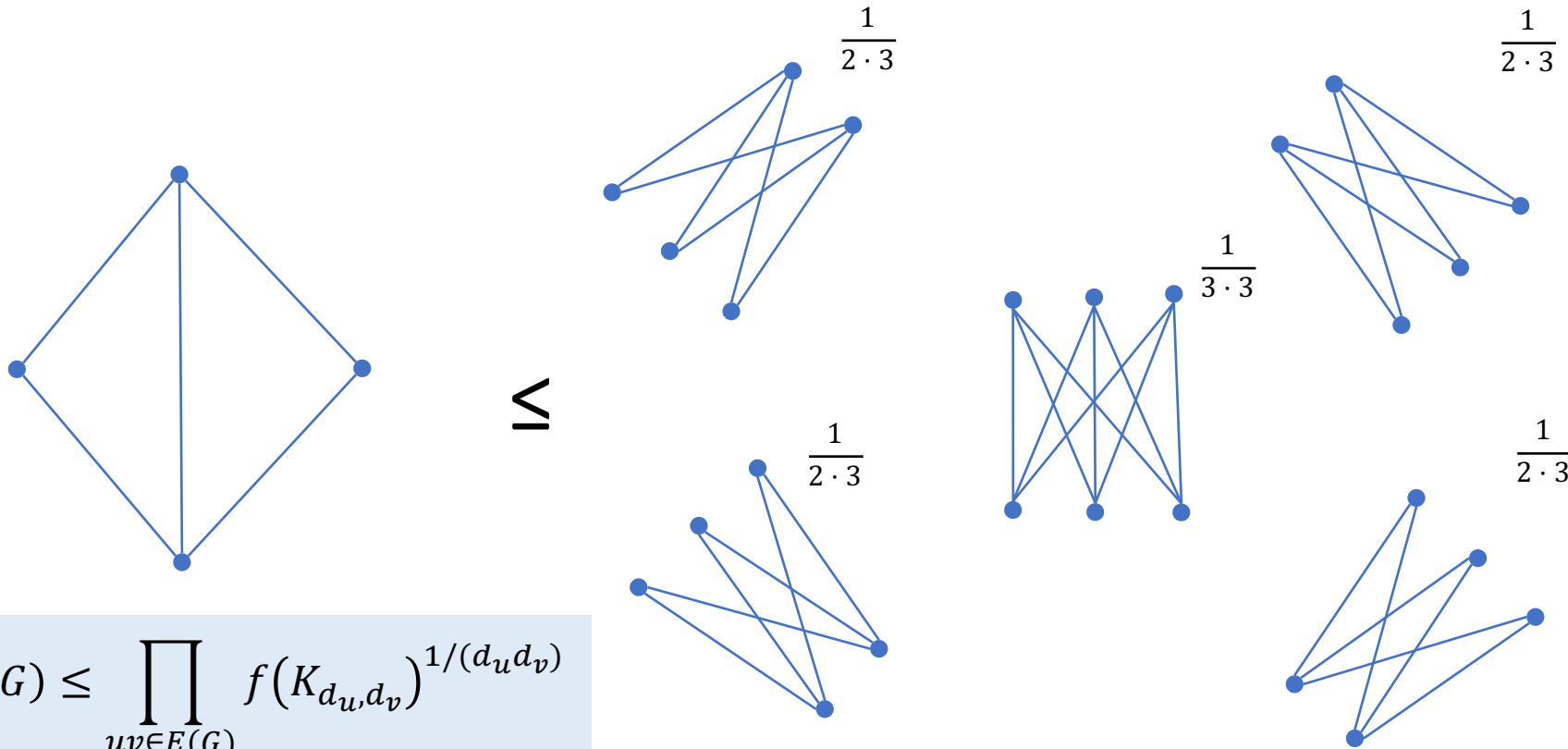
**Theorem** (Sah, Sawhney, Stoner, Z.). Let  $q \in \mathbb{N}$  and  $G$  a graph without isolated vertices. Then

$$c_q(G) \leq \prod_{uv \in E(G)} c_q(K_{d_u, d_v})^{1/(d_u d_v)}$$

Proper colorings are **biclique-maximizing**



# The number of independent sets and proper $q$ -colorings satisfies



$$f(G) \leq \prod_{uv \in E(G)} f(K_{d_u, d_v})^{1/(d_u d_v)}$$

$f$  counts independent sets or proper  $q$ -colorings

# Graph homomorphisms

**Question 3.** Fix  $d$  and  $H$ . Which  $d$ -regular graph  $G$  maximizes  $\text{hom}(G, H)^{1/v(G)}$ ?

**Conjecture** (Davies, Jenssen, Perkins, Roberts '17). Among **triangle-free**  $G$ ,  $G = K_{d,d}$  is always the maximizer (already known for bipartite  $G$  [Galvin, Tetali '04])

We prove this conjecture

**Theorem** (Sah, Sawhney, Stoner, Z.). Let  $G$  be a **triangle-free**  $n$ -vertex  $d$ -regular graph. Then

$$\text{hom}(G, H) \leq \text{hom}(K_{d,d}, H)^{n/(2d)}$$

**Theorem** (SSSZ). Let  $G$  be a **triangle-free** graph without isolated vertices. Then

$$\text{hom}(G, H) \leq \prod_{uv \in E(G)} \text{hom}(K_{d_u, d_v}, H)^{1/(d_u d_v)}$$

Always **biclique-maximizing** among triangle-free graphs

**False** for every  $G$  with a triangle! Counterexample:  $H = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$   as  $\varepsilon \rightarrow 0$

# Reverse Sidorenko inequality

**Sidorenko's conjecture:** for bipartite  $G$ , all  $H$

$$t(G, H) \geq t(K_2, H)^{e(G)}$$

$$t(G, H) = \text{hom}(G, H) / v(H)^{v(G)}$$

[Hatami] [Conlon, Fox, Sudakov] [Li, Szegedy] [Kim, Lee, Lee] [Conlon, Kim, Lee, Lee] [Szegedy] [Conlon, Lee]

Open for  $G = K_{5,5} \setminus C_{10}$  (Möbius strip)

**Our result:** for triangle-free  $d$ -regular  $G$

$$t(G, H) \leq t(K_{d,d}, H)^{e(G)/d^2}$$

$\|\cdot\|_G := t(G, \cdot)^{1/e(G)}$  (Hatami's graph "norm"; [Conlon, Lee]). For graphon  $W: [0,1]^2 \rightarrow [0,1]$ ,

$$\|W\|_{K_2} \leq \|W\|_G \leq \|W\|_{K_{d,d}}$$

bipartite  $G$  (Sidorenko's conjecture)  ?  triangle-free  $d$ -regular  $G$  (our result)

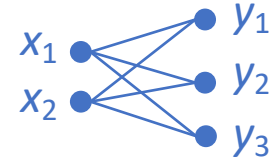
**Theorem** (Sah, Sawhney, Stoner, Z.). Let  $G$  be a triangle-free graph and  $W: [0,1]^2 \rightarrow [0,1]$ . Then

$$t(G, W) \leq \prod_{uv \in E(G)} \|W\|_{K_{d_u, d_v}}$$

# Reverse Sidorenko inequality

Given  $f: \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}$ , e.g.,  $\|f\|_{K_{2,3}} =$

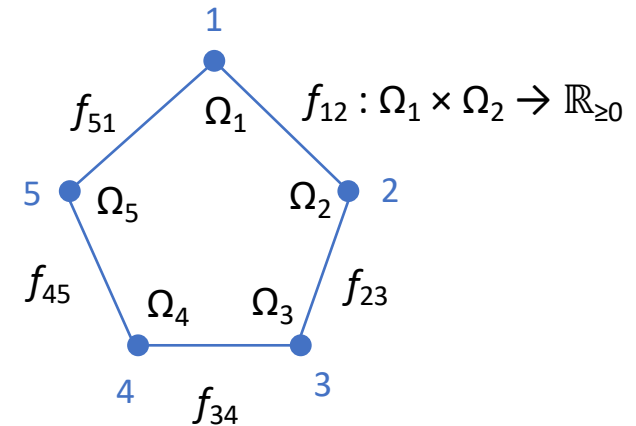
$$\left| \int_{\Omega_1^2 \times \Omega_2^3} f(x_1, y_1) f(x_1, y_2) f(x_1, y_3) f(x_2, y_1) f(x_2, y_2) f(x_2, y_3) dx_1 dx_2 dy_1 dy_2 dy_3 \right|^{1/6}$$



**Theorem** (Sah, Sawhney, Stoner, Z.).

Triangle-free graph  $G = (V, E)$  without isolated vertices

$$\int \prod_{uv \in E} f_{uv}(x_u, x_v) d\mathbf{x}_V \leq \prod_{uv \in E} \|f_{uv}\|_{K_{d_v, d_u}}$$



# Reverse Sidorenko inequality

**Theorem** (Sah, Sawhney, Stoner, Z.).

Triangle-free graph  $G = (V, E)$  without isolated vertices

$$\int \prod_{uv \in E} f_{uv}(x_u, x_v) d\mathbf{x}_V \leq \prod_{uv \in E} \|f_{uv}\|_{K_{d_v, d_u}}$$

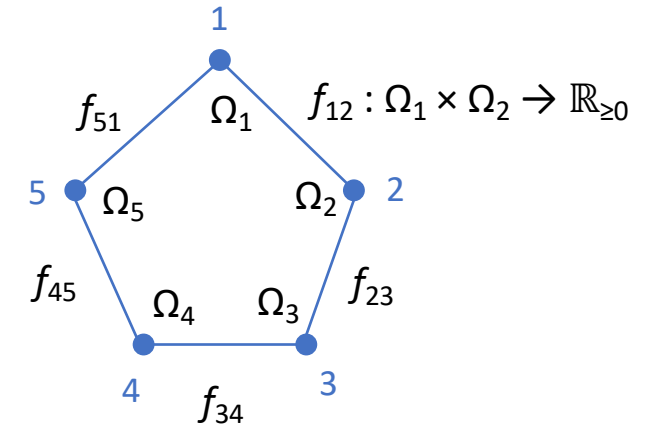
Graphical analogs of Brascamp—Lieb type inequalities:

$$\int f_1(\dots) \dots f_k(\dots) \lesssim \|f_1\|_{L^{p_1}} \dots \|f_k\|_{L^{p_k}}$$

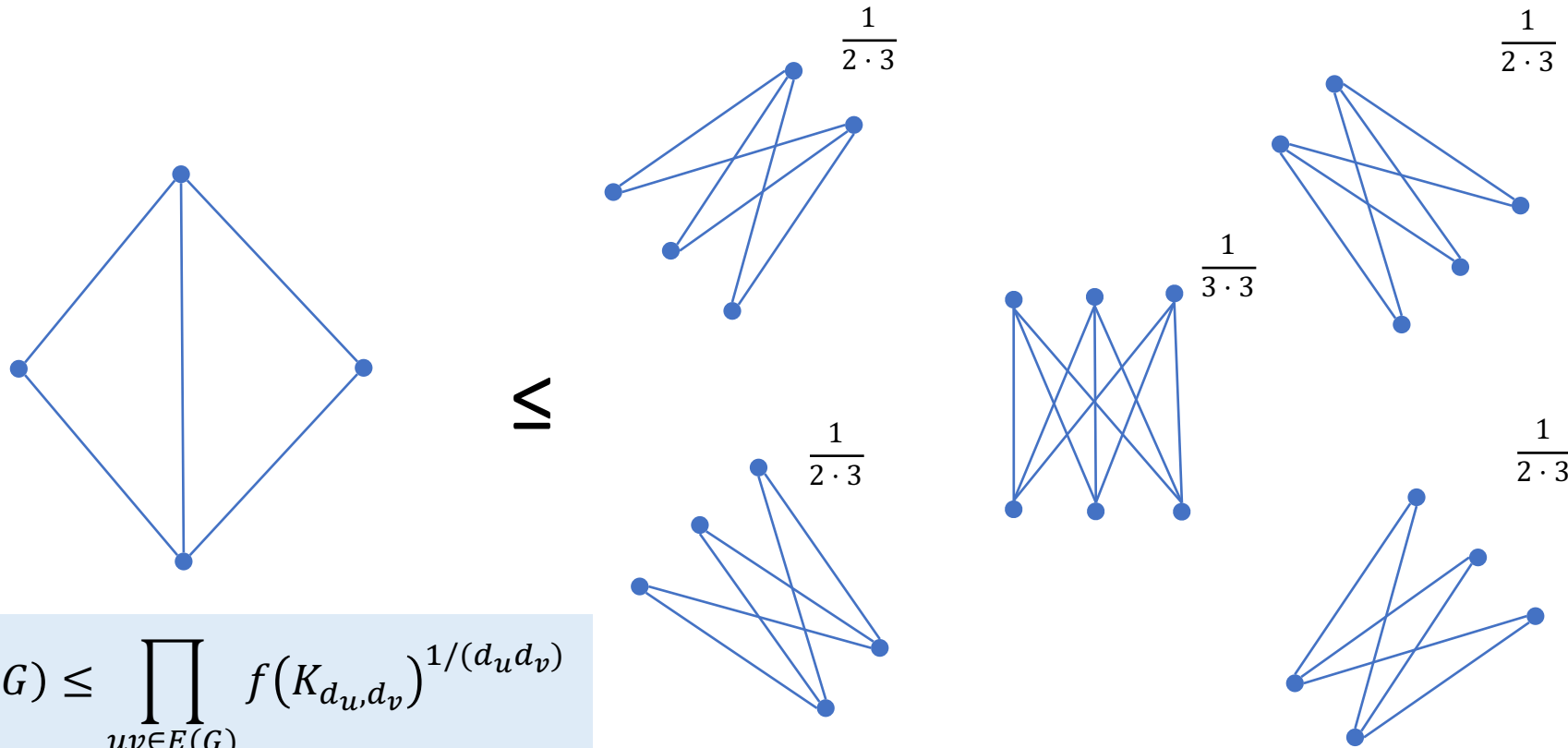
Note that (by Hölder)

$$\|f\|_{K_{a,b}} \leq \|f\|_{L^{ab}}$$

**Future direction:** extensions to simplicial complexes



# The number of independent sets and proper $q$ -colorings

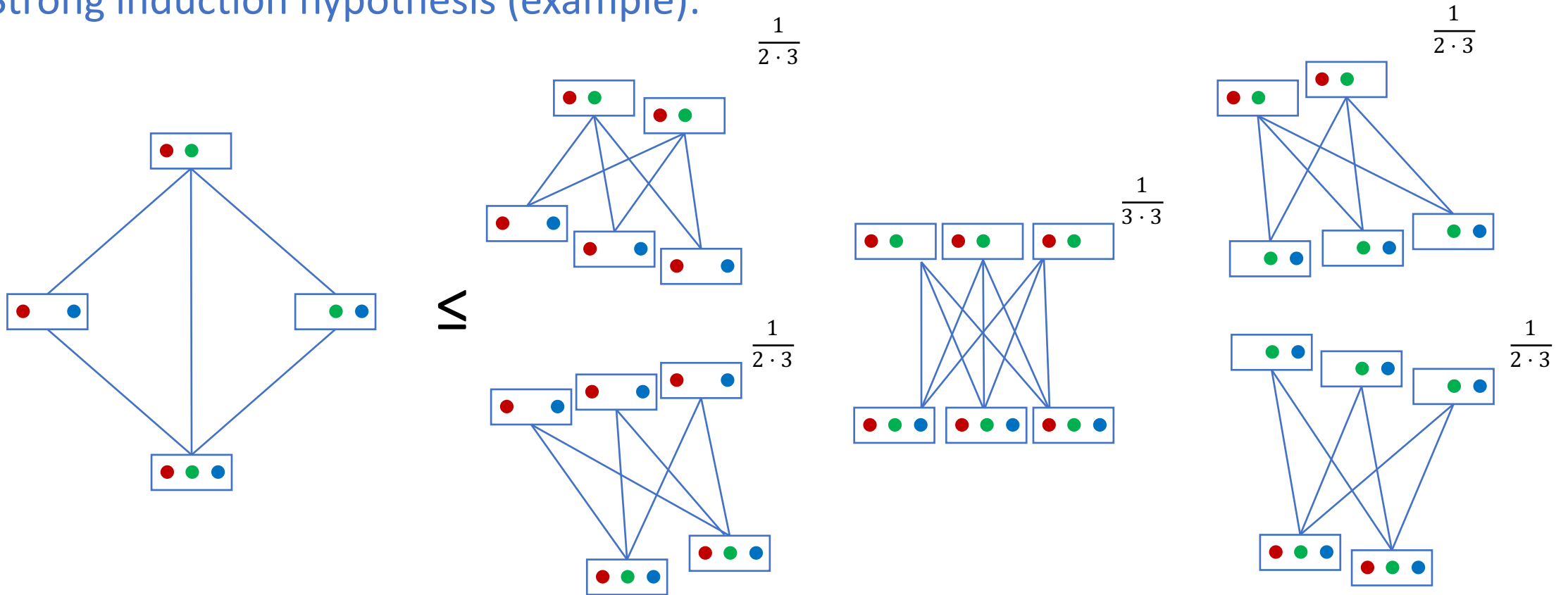


$$f(G) \leq \prod_{uv \in E(G)} f(K_{d_u, d_v})^{1/(d_u d_v)}$$

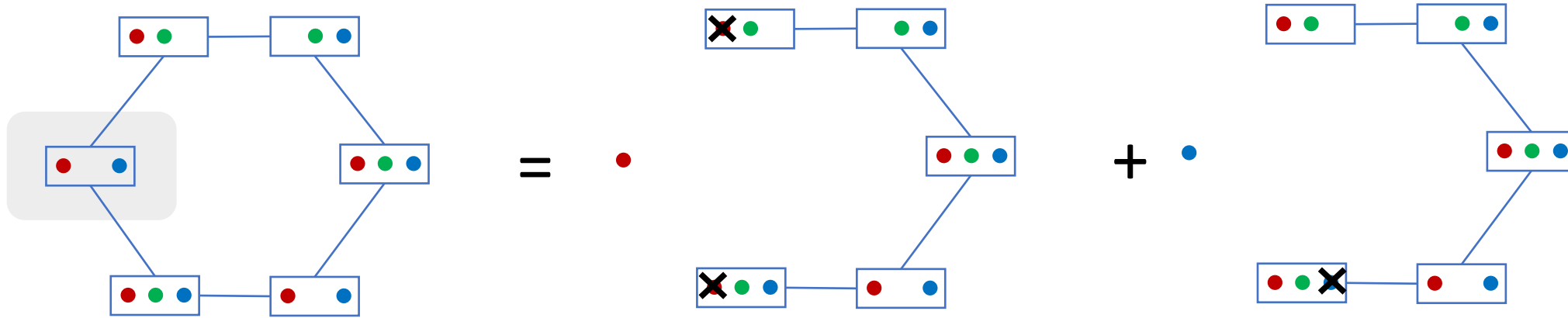
$f$  counts independent sets or proper  $q$ -colorings

# The number of proper list colorings

Strong induction hypothesis (example):

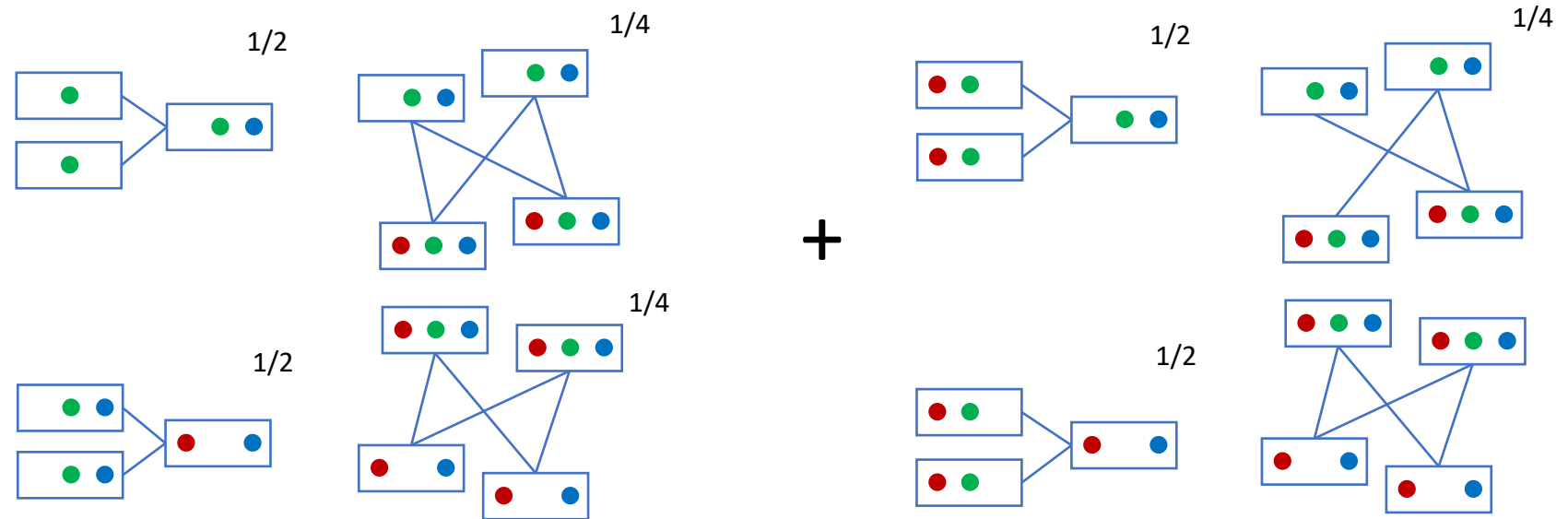


# Proof strategy: Induction



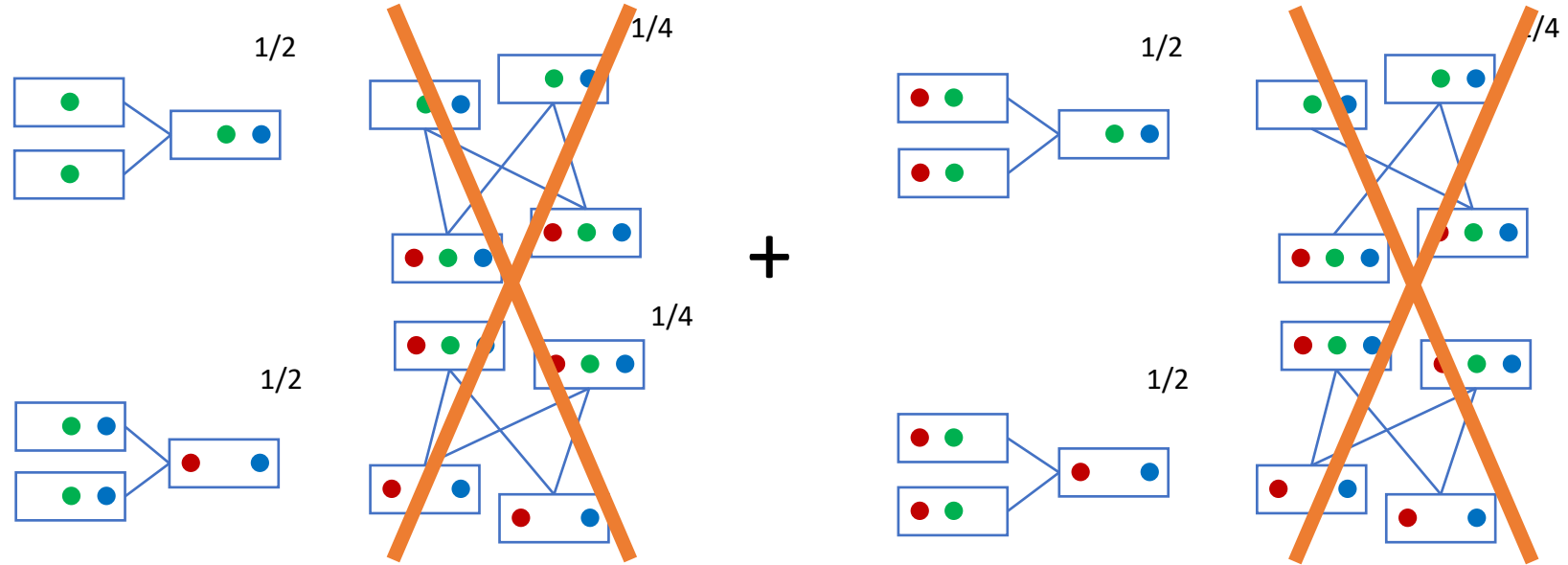
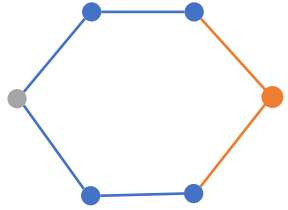
By induction

$\leq$



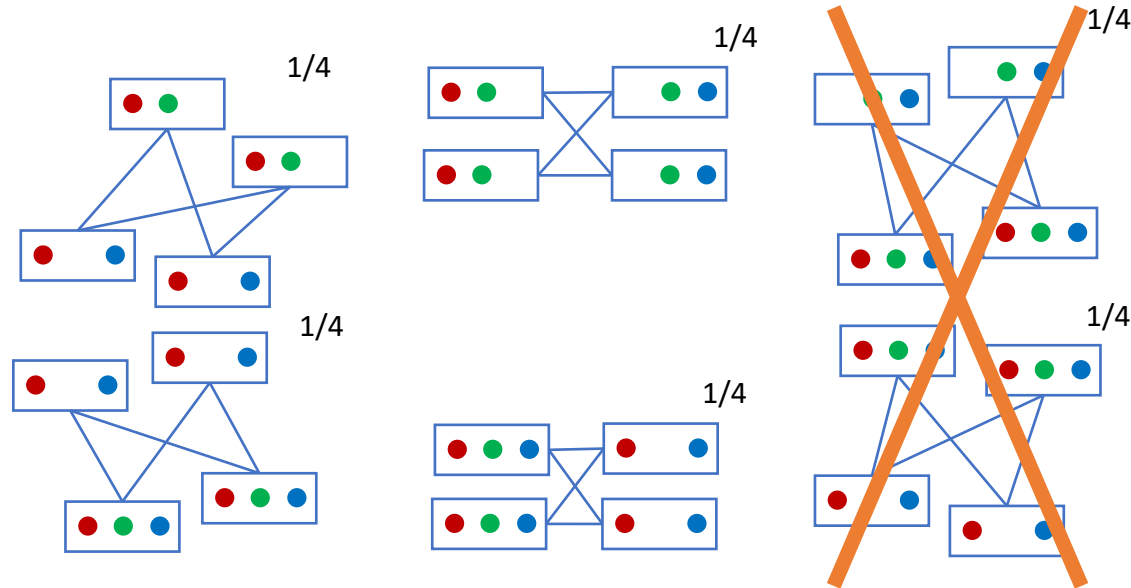


# Proof strategy: Reduction to local inequality

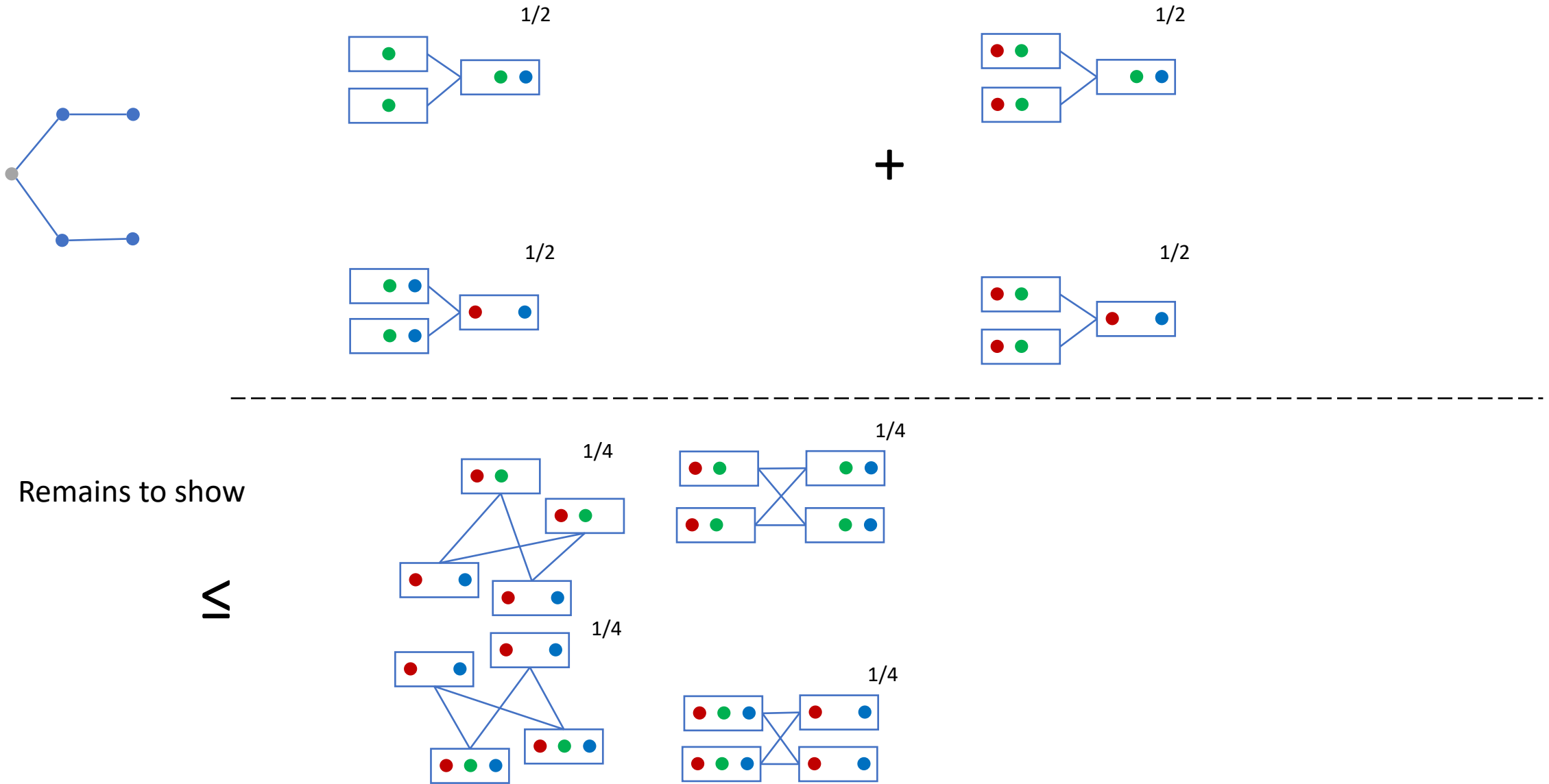


Remains to show

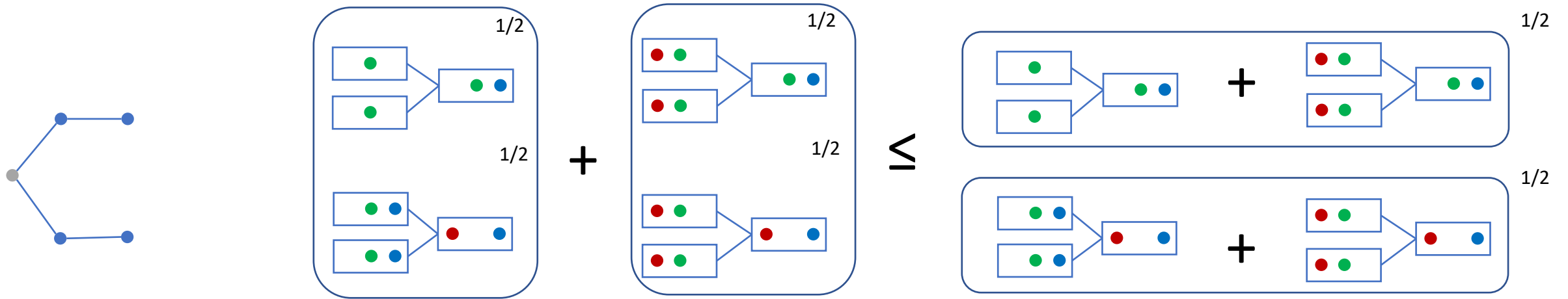
$\leq$



# Proof strategy: Reduction to local inequality



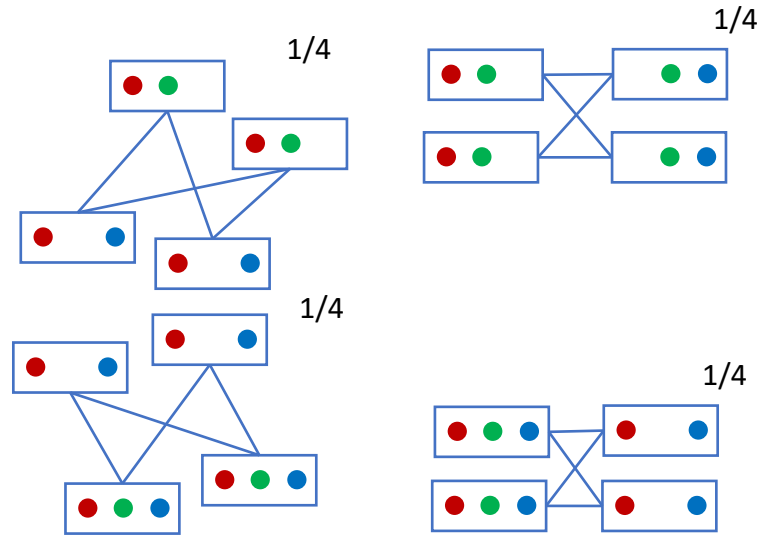
# Proof strategy: Reduction to local inequality



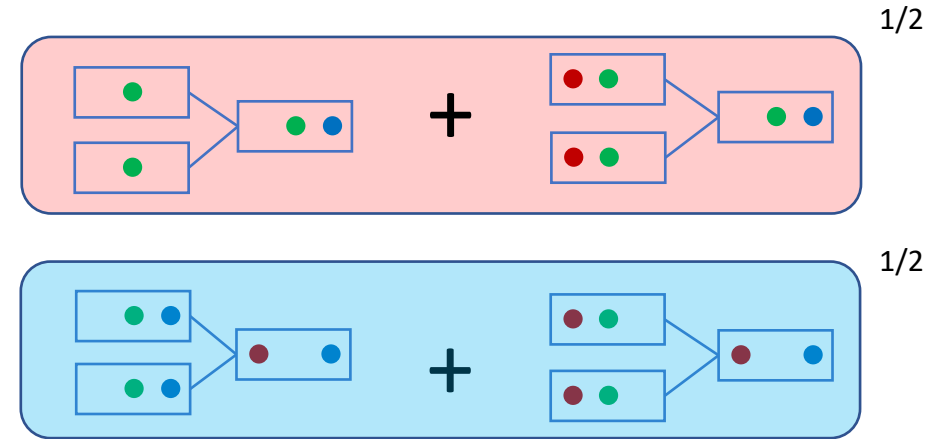
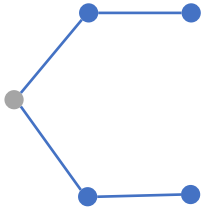
By Cauchy—Schwarz (more generally Hölder):  $\sqrt{AB} + \sqrt{CD} \leq \sqrt{(A + C)(B + D)}$

Remains to show

$\leq$

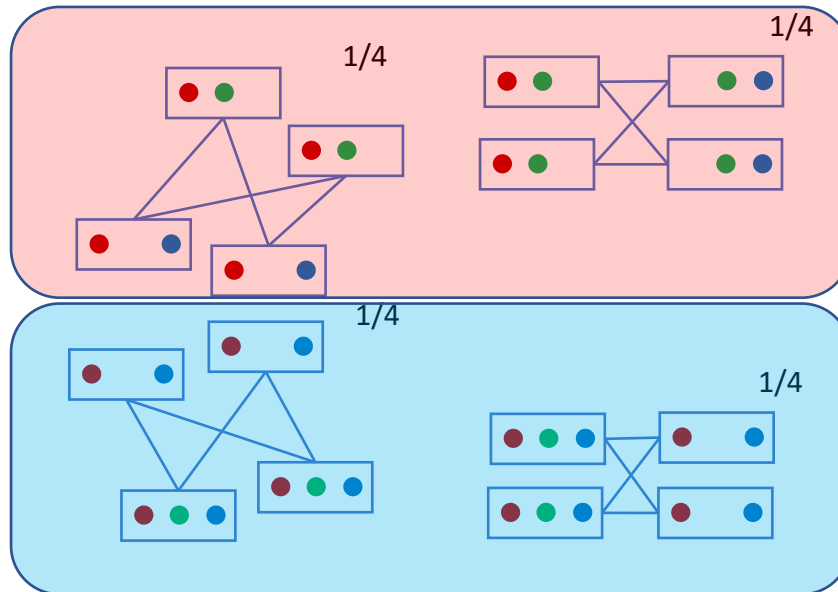


# Proof strategy: Reduction to local inequality



Remains to show

$\leq$



Break inequality into two parts:  
 top & bottom

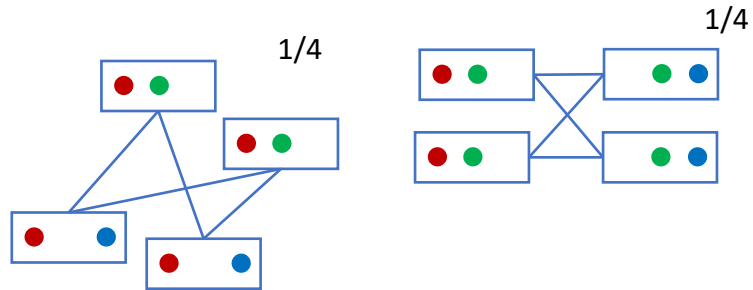
# Proof strategy: Local inequality



$$\left( \begin{array}{c} \boxed{\text{red} \quad \text{blue}} \\ \swarrow \quad \searrow \\ \boxed{\text{red} \quad \text{green}} \quad \boxed{\text{red} \quad \text{green}} \\ \swarrow \quad \searrow \\ \boxed{\text{green} \quad \text{blue}} \end{array} \right)^{1/2} = \left( \begin{array}{c} \boxed{\text{green}} \\ \boxed{\text{green}} \end{array} \rightarrow \boxed{\text{green} \quad \text{blue}} \right) + \left( \begin{array}{c} \boxed{\text{red} \quad \text{green}} \\ \boxed{\text{red} \quad \text{green}} \end{array} \rightarrow \boxed{\text{green} \quad \text{blue}} \right)^{1/2}$$

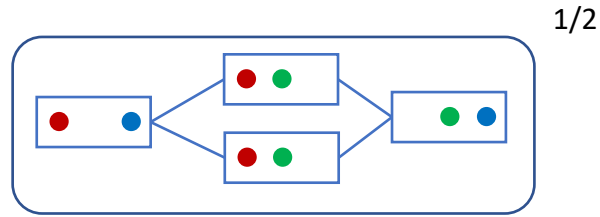
Remains to show

$\leq$



Break inequality into two parts:  
**top** & bottom

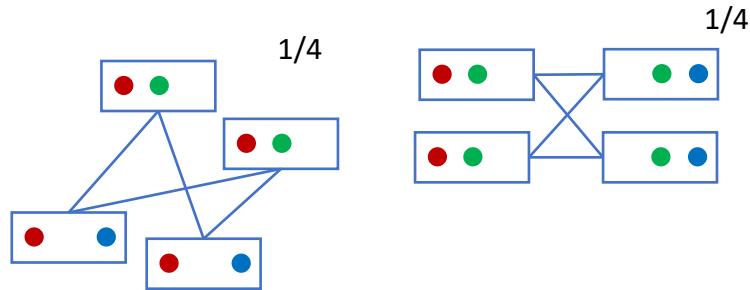
# Proof strategy: Local inequality



This is a **minimal instance**  
of the inequality

Remains to show

$\leq$



Much more difficult if  $G$  has triangles  
(not always true for other models!)

# A useful matrix inequality

Define the **mixed  $\ell^{p,q}$  norm** of matrix  $A = (a_{ij})$  by first taking  $\ell^p$  norm of each row, and then taking  $\ell^q$  norm of the results, i.e.

$$\|A\|_{p,q} := \left( \sum_i \left( \sum_j |a_{ij}|^p \right)^{q/p} \right)^{1/q}$$

**Lemma.** For positive semidefinite (PSD) matrix  $A$  with nonneg entries, and  $q \geq 1$ ,

$$\|A\|_{1,q}^2 \leq \|A\|_{1,1} \|A\|_{q,q}$$

**Question.** Is it true that for all  $1 \leq p \leq q$ ,

$$\|A\|_{p,q}^2 \leq \|A\|_{p,p} \|A\|_{q,q} \quad ?$$

# Graph homomorphisms

**Question 3.** Fix  $d$  and  $H$ . Which  $d$ -regular graph  $G$  maximizes  $\text{hom}(G, H)^{1/v(G)}$ ?

Let  $H$  be a nonneg weighted graph (model)

$\text{hom}(G, H)$  = partition function of some stat. phys. model, e.g., hard-core, Ising, Potts

- $H$  is **biclique-maximizing** if  $Z(G) := \text{hom}(G, H)$  satisfies

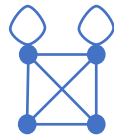
$$Z(G) \leq \prod_{uv \in E(G)} Z(K_{d_u, d_v})^{1/(d_u d_v)} \quad \text{i.e., conditioned on degree-degree distribution}$$

- $H$  is **clique-maximizing** if  $Z(G) := \text{hom}(G, H)$  satisfies

$$Z(G) \leq \prod_{v \in V(G)} Z(K_{d_v+1})^{1/(d_v+1)} \quad \text{i.e., conditioned on degree distribution}$$

**Our results:**  $H = \bullet \text{---} \bullet$  (indep sets) and  $K_q$  (proper colorings) are both **biclique-maximizing**

More generally, every partially looped  $K_q$  (semiproper colorings) is **biclique-maximizing**





# Ferromagnetism and anti-ferromagnetism

Given a nonneg weighted graph/model  $H$ , we say that


- $H$  is **ferromagnetic** if its edge-weight matrix is positive semidefinite, i.e., all eigenvalues are nonnegative:  $0 \leq \dots \leq \lambda_3 \leq \lambda_2 \leq \lambda_1$  (e.g.,  $H = \text{⬤} \text{⬤} \text{⬤}$ )
- $H$  is **antiferromagnetic** if its edge-weight matrix has at most one positive eigenvalue:  $\dots \leq \lambda_3 \leq \lambda_2 \leq 0 \leq \lambda_1$  (e.g., indep sets and colorings)

**Theorem** (Sah, Sawhney, Stoner, Z.). Every **ferromagnetic** model is **clique-maximizing**

**Conjecture 1.** Every **clique-maximizing** model is **ferromagnetic**

**Conjecture 2.** Every **antiferromagnetic** model is **biclique-maximizing**

Our results verify Conj. 2 for independent sets and colorings. Open for Potts model

Widom—Rowlingson model is clique-maximizing among  $d$ -regular  $G$  [Cohen, Perkins, Tetali] but not for irregular  $G$ , and it is not ferromagnetic. 

# Two-spin systems

- An Ising model with nonneg edge-weight matrix  $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$  is  
ferromagnetic if  $ac \geq b^2$  and antiferromagnetic if  $ac \leq b^2$   
E.g., independent set  $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$  is antiferromagnetic

**Corollary** (Sah, Sawhney, Stoner, Z.). A 2-spin model is

- Biclique-maximizing if antiferromagnetic, and
- Clique-maximizing if ferromagnetic

This generalizes the result for independent sets

A similar classification for 3-spin systems is currently out of reach

# Summary of main results

- Independent sets and proper colorings are **biclique-maximizing**
- Every **ferromagnetic** model is **clique-maximizing**
- Every model is **biclique-maximizing** when restricted to **triangle-free graphs**

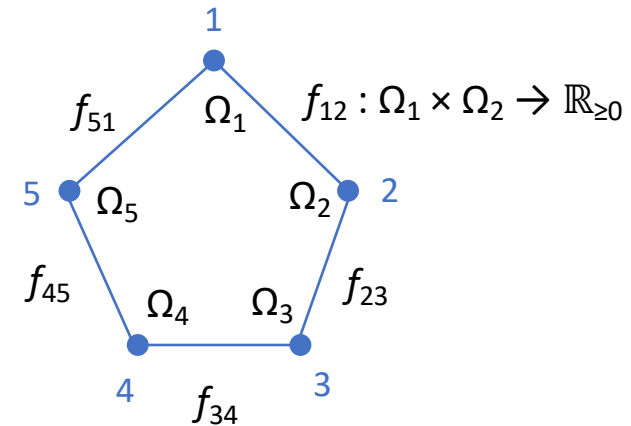
**Reverse Sidorenko inequality** (Sah, Sawhney, Stoner, Z.).

**Triangle-free** graph  $G = (V, E)$  without isolated vertices

$$\int \prod_{uv \in E} f_{uv}(x_u, x_v) d\mathbf{x}_V \leq \prod_{uv \in E} \|f_{uv}\|_{K_{d_v, d_u}}$$

**Corollary.** For **triangle-free**  $G$  without isolated vertices,  $\forall H$

$$\text{hom}(G, H) \leq \prod_{uv \in E(G)} \text{hom}(K_{d_u, d_v}, H)^{1/(d_u d_v)}$$



**Conjecture.** Every antiferromagnetic model is biclique-maximizing (e.g., Potts).