Some B2 Putnam Problems

(1995) An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin\left(\frac{x}{a}\right)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?

(1996) Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

(1997) Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

(1998) Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

(1999) Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

(2000) Prove that the expression

$$\frac{gcd(m, n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

(2001) Find all pairs of real numbers (x, y) satisfying the system of equations

$$\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2)$$
$$\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4).$$

(2002) Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game:

> Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex.

Show that the player who signs first will always win by playing as well as possible.

(2003) Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n-1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$ by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n-2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n < 2/n$.

(2004) (56, 41, 39, 0, 0, 0, 0, 0, 22, 0, 28, 10) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

(2005) (148, 11, 4, 0, 0, 0, 0, 0, 9, 1, 13, 10) Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

(2006) (123, 28, 16, 0, 0, 0, 0, 0, 3, 0, 13, 15) Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

(2007) (80, 11, 9, 0, 0, 0, 0, 0, 27,21,34,24) Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left| \int_0^\alpha f(x)\,dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

(2008) (83, 17, 6, 0, 0, 0, 0, 0, 45, 4, 15, 19) Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate

$$\lim_{n \to \infty} \frac{n! F_n(1)}{\ln n}.$$

(2009) (62, 3, 0, 0, 0, 0, 0, 0, 63, 13, 49, 10) A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

(2010) (133, 5, 31, 0, 0, 0, 0, 0, 10, 2, 4, 3) Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

(2011) (141, 7, 14, 0, 0, 0, 0, 0, 6, 15, 12, 2) Let S be the set of all ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?