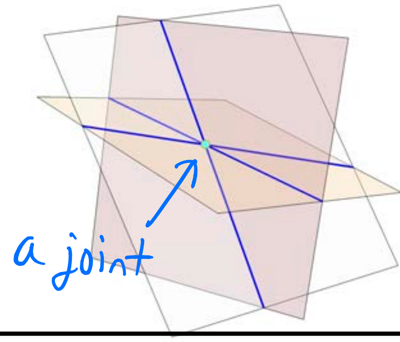
Joints problem What's the max # of joints that N lines in $\mathbb{R}^3$ can make?

A **joint** is a point contained in 3 non-coplanar lines

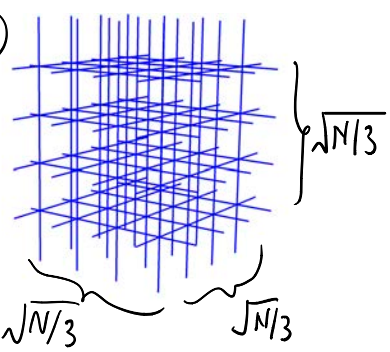


Examples

N lines

$\Theta(N^{3/2})$ joints

①

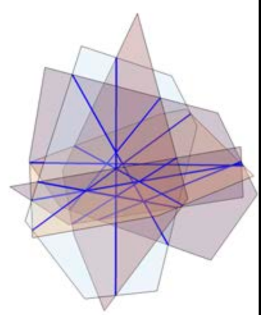


$\left(\frac{N}{3}\right)^{3/2}$ joints

② $k \sim \sqrt{2N}$ generic planes

⇝ pairwise form $\binom{k}{2} \sim N$ lines

& triplewise form $\binom{k}{3} \sim \frac{\sqrt{2}}{3} N^{3/2}$ joints



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Counting and cutting cycles of lines and rods in space*

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Introduced the joints problem & proved $O(N^{7/4})$ upper bd on #joints

... many subsequent improvements on until :

Algebraic methods in discrete analogs of the Kakeya problem

Larry Guth [a], Nets Hawk Katz [b,*]

Theorem 1.1. *Any set of N lines in $\mathbb{R}^3$ form at most $O(N^{\frac{3}{2}})$ joints.*

Recently Hans Yu & I improved the constant factor to optimal : $\leq \frac{\sqrt{2}}{3} N^{3/2}$ joints

Connections

- To Kakeya problem in analysis (Wolff)
- Finite field Kakeya problem (Dvir) ← polynomial method
- Erdős distinct distance problem (Guth–Katz)
- Multilinear Kakeya, "joints of tubes" (Bennett–Carbery–Tao, Guth)

Many extentions & generalizations of the joints problem

[Kaplan–Sharir–Shustin, Quilodrán] Simplification & extension to all dimensions

$\boxed{\text{Thm (joints of lines) } N \text{ lines in } \mathbb{F}^d \text{ have } O_d(N^{\frac{d}{d-1}}) \text{ joints.}}$

↑ arb. field ↑ tight const now known [Yu-Z]

$\boxed{\text{Joints of flats: max \# joints for } N \text{ planes in } \mathbb{F}^6\text{ ?}}$

a point contained in a triple of planes in spanning & indep directions ↑ 2-dim flats

Construction $\Theta(N^{3/2})$ joints : generic 4-flats, $\cap$ pair → planes, $\cap$ triple → joints

Why I like this problem :
- natural extension of joints
- a key step of pf of joints thm fails badly

Incidence geometry for higher dimensional objects

[Solymosi–Tao]: nearly tight ($\boxed{+o(1)}$ in exponent) bound for point-variety incidences in $\mathbb{R}^d$ (in the spirit of Szemerédi–Trotter)

- Extension of the Guth-Katz polynomial partitioning method
  ↳ use bounded degree polynomials

[Walsh]: a different, algebraic, partitioning method
⇝ incidence among higher dim, higher degree varieties in $\mathbb{F}^d$

Previous results:

[Yang] N planes in $\mathbb{R}^6$ have $N^{\frac{3}{2}+o(1)}$ joints
(techinque: bounded deg partitioning, restrict to codim-1 variety)

Limitations ① Error term in exponent ② Only in $\mathbb{R}$

[Yu-Z./Carbery-Iliopoulou] N lines & M planes in $\mathbb{F}^4$ make $O(NM^{1/2})$ joints
(plane–line^2)
line–line–plane, in indep & spanning directions

Our results [Tidor-Yu-Z.]	N planes in $\mathbb{F}^6$ have $O(N^{\frac{3}{2}})$ joints
	N k-flats in $\mathbb{F}^{mk}$ have $O_{m,k}(N^{\frac{m}{m-1}})$ joints

A set of k-dim varieties in $\mathbb{F}^{mk}$ of total degree N has $O_{m,k}(N^{\frac{m}{m-1}})$ joints

$p \in V_1, \ldots, V_m$ regular point
tangent spaces at p spanning & indep directions

A new way to apply polynomial method to higher dim objects

Several sets of lines (& flats, varieties) [Conj by Carbery] and count joints formed by taking one object from each set

Multijoints of lines (Iliopoulou $\mathbb{R}^d$ & $\mathbb{F}^3$; Zhang $\mathbb{F}^d$)

$\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_d$ sets of lines in $\mathbb{F}^d$

joints formed by taking one line from each set is $\lesssim_d \left(|\mathcal{L}_1| \cdots |\mathcal{L}_d|\right)^{\frac{1}{d-1}}$.

[Tidor-Yu-Z.] We extend from lines to varieties of arb dim

Joints of lines with multiplicities (Carbery conj; Iliopoulou $\mathbb{R}^3$; Zhang $\mathbb{F}^d$)

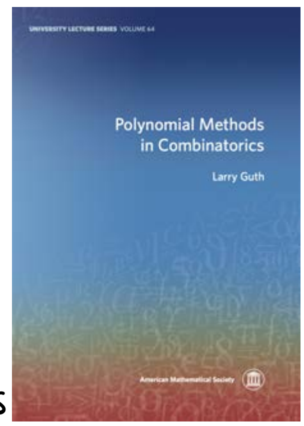
$\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_d$ sets of lines in $\mathbb{F}^d$

$$\sum_{\text{joints } p} \left(\#(l_1,\ldots,l_d) \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_d \text{ making a joint at } p\right)^{\frac{1}{d-1}} \lesssim_d \left(|\mathcal{L}_1| \cdots |\mathcal{L}_d|\right)^{\frac{1}{d-1}}.$$

[Tidor-Yu-Z.] Also ~~lines~~ varieties

Review of the proof of: [Kaplan-Sharir-Shustin, Quilodrán]

N lines in $\mathbb{R}^3$ have $O(N^{3/2})$ joints



① **Parameter counting**:

using $\dim \mathbb{R}[x_1,\ldots,x_d]_{\le n} = \binom{n+d}{d}$ (deg $\le n$)

deduce that $\exists$ non-zero poly g, $\deg \le CJ^{1/3}$, vanishing on joints

Take g with min deg. $J = \#$joints

② **Vanishing lemma**: a single-variable polynomial cannot vanish more times than its degree

③ A joints-specific argument. If all lines have $> CJ^{1/3}$ joints, then vanishing lemma $\Rightarrow$ g vanishes on all lines $\Rightarrow$ ∇g vanishes on all joints

$\Rightarrow$ one of $\partial_x g, \partial_y g, \partial_z g$ is nonzero, lower deg & vanish on all joints

So some line has $\le CJ^{1/3}$ joints. Remove this line & induction ☑

How to generalize vanishing lemma to 2-var polynomials?

Thm (Tidor–Yu–Z.) N planes in $\mathbb{R}^6$ have $O(N^{3/2})$ joints

__Wishful thinking__  only if we had something like ...

- every nonzero $g(x,y)$ of deg $\le n$ has $\lesssim n^2$ zeros

OR

- two polynomials, each deg $\lesssim N^{1/6}$, vanishing at all joints and no common factors when restricted to each plane (related: inverse Bézout)

Method of multiplicities: ask a polynomial to vanish at each joint to some high order

By counting parameters, maybe hope for: 

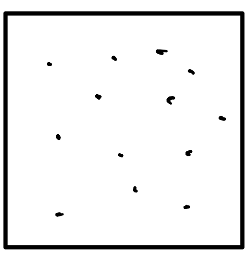
- Every nonzero $g(x,y)$ of deg $\le ns$ vanish to order $\ge s$ at $\lesssim n^2$ pts.

Counterexample: $g(x,y) = y^s$

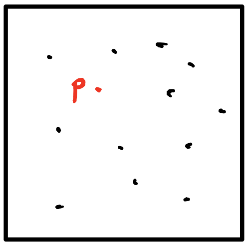
Linear dependencies among vanishing conditions

e.g. $g(p)=0$, $\partial_x g(p)=0$, $(\partial_{xx}-\partial_{xy})g(p)=0$

★ __Key idea 1__ Restricting to a plane for now



We will construct a set of $\dim \mathbb{R}[x,y]_{\le n} = \binom{n+2}{2}$ linearly indep vanishing conditions on $\mathbb{R}[x,y]_{\le n}$

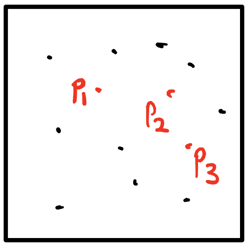


Attached to each point p is a set of vanishing conditions for $g \in \mathbb{R}[x,y]_{\leq n}$:

$g(p)=0$, $\partial_x g(p)=0$, $\partial_y g(p)=0$

$\partial_{xx} g(p)=0$, $\partial_{xy} g(p)=0$, $\partial_{yy} g(p)=0$, $\partial_{xxx} g(p)=0, \ldots$

If we take all these conditions up to order $n+1$ then any satisfying $g \in \mathbb{R}[x,y]_{\leq n}$ must be zero.



The above vanishing conditions attached to several different points are lin. dep. as linear functionals on $\mathbb{R}[x,y]_{\leq n}$

We will select a basis of linear functionals on $\mathbb{R}[x,y]_{\leq n}$ via the following procedure.

First attempt

Cycle through the points on the plane (say 100 pts)

$p_1, p_2, p_3, \ldots, p_1, p_2, p_3, \ldots, p_1, p_2, p_3, \ldots$

p_1 : add vanishing condition $g(p_1)=0$

p_2 : add vanishing condition $g(p_2)=0$, as long as it does not already follow from previously added vanishing cond.

⋮

p_1 : add a <u>nonredundant</u> subset of $\partial_x g(p_1)=0$, $\partial_y g(p_1)=0$

↑ none implied by other added + prev. added
i.e. basis extension

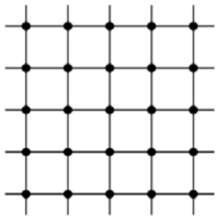
p_2 : add a nonredundant subset of $\partial_x g(p_2)=0$, $\partial_y g(p_2)=0$

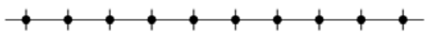
⋮

The process assigns a total of $\binom{n+2}{2}$ vanishing conditions each attached to a point

Can we control the # of vanishing attached to each pt?

<u>Example</u>





← pts on grid get way more vanishing conditions than pts on the line

UNDESIRABLE

(This example also comes up for inverse Bézout; see Tao blog)

☆ <u>Key idea 2</u> Let some points get a head start

e.g. (100 pts) $p_1, p_2, \ldots, p_{50}$, $p_1, \ldots, p_{50}$, $\ldots$, $p_1, \ldots, p_{50}$, $p_1, \ldots, p_{100}$, $p_1, \ldots, p_{100}$, ...

<u>Handicap</u> $\vec{\alpha}$ assigns an integer to each point

e.g.

points	a	b	c	d	e
handicap $\vec{\alpha}$	0	1	3	0	-1

→ order: c c b c a b c d a b c d e a b c d e ...

Modify process of assigning vanishing conditions

c : add a nonredundant set of 0^{th} order derivative vanishing @ c

c ———— 1^{st} ———— c

b ———— 0^{th} ———— b

c ———— 2^{nd} ———— c

a ———— 0^{th} ———— a

b ———— 1^{st} ———— b

c ———— 3^{rd} ———— c

Want a "good" choice of handicaps: treating all joints "fairly"

Hard to compute how # vanishing cond at each pt depends on the handicap. $\vec{\alpha} \to$ distr of vanish cond

① Monotonicity — $\alpha_p \nearrow \Rightarrow$ #van. cond at p cannot $\searrow$

② Lipschitz continuity — small Δ in handicap $\rightsquigarrow$ small Δ in #van cond

③ Bounded domain — suffices to consider handicaps with bounded values (else some pt gets 0 van. cond)

☆ Key idea 3 Decide on handicap choice later, implicitly (existence via compactness/smoothing)

Putting different planes together at joints of planes in $\mathbb{R}^6$

Handicap $\vec{\alpha} \in \mathbb{Z}^J$ assigns an integer to each joint

Separately for each plane F, apply above process to assign vanishing conditions restricted to F to joints on F

$\updownarrow$

assigning derivative operators

A new vanishing lemma Given $0 \neq g \in \mathbb{R}[x_1, \dots, x_6]_{\leq n}$, $\exists$ joint p, contained in planes F_1, F_2, F_3 (indep & spanning directions) & derivative operator D_1 assigned to p on F_1 (& likewise D_2, D_3) s.t. $D_1 D_2 D_3\, g(p) \neq 0$.

By parameter counting,

$$\sum_{\text{joints } p} \left(\#\text{ choices of } (D_1, D_2, D_3) \text{ at } p\right) \geq \dim \mathbb{R}[x_1, \dots, x_6]_{\leq n} = \binom{n+6}{6}$$

By compactness/smoothing, $\exists$ handicap so that these terms are roughly all equal

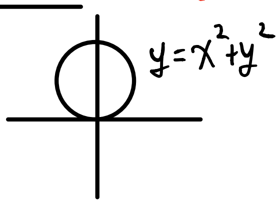
Also recall that # van. cond. on each plane is exactly $\binom{n+2}{2}$

Putting together + AM-GM $\Rightarrow$ theorem $\square$

Joints of varieties

Flats: higher order directional directives along a flat

Varieties: derivatives in local coordinates

e.g. 

on the circle,

completion

$$\begin{aligned} y &= x^2+y^2 \\ &= x^2+(x^2+y^2)^2 \\ &= x^2+(x^2+(x^2+y^2)^2)^2 = \cdots \\ &= x^2+x^4+2x^6+\cdots \end{aligned}$$

Power series in local coord x

2^{nd} order derivative operator at the origin is $\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial y}$ $\left(\text{not } \frac{\partial^2}{\partial x^2}\right)$ so that evaluations give linear functional on the space of regular functions

Extension to arbitrary fields $\mathbb{F}$

When differentiating, we only care about coeff extraction

Hasse derivatives (formal algebraic derivatives)

Question Other applications of this variant of polynomial method for higher dim objects?