Practice Midterm 1

Closed book. No notes/calculators/phones.
Time: 80 minutes.
6 problems worth 10 points each.
You must provide justification in your solutions (not just answers). Simplify all answers and express in closed form whenever possible.

1. Determine the number of solutions to $x + y + z \leq n$ with integers $x, y, z \geq 1$.

   Solution. Let $a = x - 1$, $b = y - 1$, $c = z - 1$, and introduce an additional "slack" variable d. There is a bijection with solutions to $a + b + c + d = n - 3$ with nonnegative integers a, b, c, d. This is the number of weak compositions of $n - 3$ into four parts, which we solved in class using a "stars and bars" argument (counting linear arrangements of $n - 3$ stars and 3 bars). Thus number of solutions is $\boxed{\binom{n}{3}}$.

2. Prove that for all positive integers n,

   $$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

   Solution. Consider the number of ways to choose a committee of n persons from n men and n women. There are $\binom{2n}{n}$ ways.

   On the other hand, for each $k = 0, \ldots, n$, the number of ways of forming the committee using k men and $n - k$ women is exactly $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$, and summing over k gives the total number of ways.

3. Let $D(n)$ denote the number of derangements (permutations without fixed points) of $[n]$. Give a combinatorial proof of the identity

   $$D(n+1) = n(D(n) + D(n-1)), \quad \text{for all } n \geq 1.$$

   Do not use the formula for the numbers $D(n)$ derived in class.

   Solution. Let us count the number of derangements of $[n + 1]$ by considering the cycle that the number $n + 1$ lies in.

   Case 1: $n + 1$ lies in a cycle of length 2. There are n choices for the other number in the same cycle as $n+1$. After removing these two elements, the number of ways to permuting the remaining elements without fixed points is $D(n-1)$. Thus there are $nD(n-1)$ derangements where $n + 1$ lies in a 2-cycle.

   Case 2: $n + 1$ lies in a cycle of length greater than 2. Consider the cycle decomposition. If we remove the number $n + 1$ from its cycle, it does not result in a fixed point (since the cycle containing the number $n + 1$ has length at least 3), so we are left with a derangement of $[n]$. On the other hand, for each derangement of $[n]$, viewed as a cycle decomposition, there are n places where we can insert the number $n + 1$ into one of the existing cycles (by picking that number the comes before the spot where we would like to insert $n+1$ in the cycle decomposition). Thus there are $nD(n)$ derangements where the number $n + 1$ lies in a cycle of length greater than 2.

4. Let $n \geq 4$. How many permutations of $[n]$ are there such that some cycle contains both 1 and 2 and a different cycle contains both 3 and 4?

   Solution. Let us relabel elements of $[n]$ so that $1, 2, 3, 4$ become $n-3, n-2, n-1, n$ respectively. Consider the canonical cycle form and dropping the parentheses. We have n and $n-1$ in one cycle and $n-2$ and $n-3$ in a different cycle if and only if the numbers $n-2, n-3, n, n-1$ must appear in this order. Indeed, since the cycles are listed with its largest element first, and in increasing order of its initial element, the rightmost cycle starts with n and must contain $n-1$, and the second-rightmost cycle must start with $n-2$ and must contain $n-1$.

   We proved in class that the operation of writing a permutation in its canonical cycle form and dropping the parentheses is a bijection on the set of permutations of $[n]$. The numbers $n-3, n-2, n-1, n$ are equally likely to appear in each of the $4! = 24$ orders among all $n!$ permutations. Therefore, the number of permutation have $n-2, n-3, n, n-1$ appearing in this specific order is $\boxed{n!/24}$, which is also the answer to the original question due to the bijection.

5. Let $a_0 = 0$ and $a_{n+1} = 3a_n + n$ for all $n \geq 0$.

   (a) Express the generating function $A(x) = \sum_{n \geq 0} a_n x^n$ in closed form.

   (b) Find a closed form formula for a_n.

   Solution. (a) Multiplying the recurrence by x^{n+1} and summing over all $n \geq 0$, we have

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = \sum_{n \geq 0} 3a_n x^{n+1} + \sum_{n \geq 0} n x^{n+1}.$$

   We have

$$\sum_{n \geq 0} a_{n+1} x^{n+1} = A(x) - a_0 = A(x)$$

$$\sum_{n \geq 0} 3a_n x^{n+1} = 3xA(x)$$

   and

$$\sum_{n \geq 0} n x^{n+1} = x^2 \frac{d}{dx} \sum_{n \geq 0} x^n = x^2 \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{x^2}{(1-x)^2}.$$

   Thus

$$A(x) = 3xA(x) + \frac{x^2}{(1-x)^2}.$$

   Solving for $A(x)$, we find

$$\boxed{A(x) = \frac{x^2}{(1-3x)(1-x)^2}}.$$

   (b) We have the following partial fraction decomposition:

$$\begin{aligned} A(x) &= \frac{1/4}{1-3x} + \frac{1/4}{1-x} - \frac{1/2}{(1-x)^2} \\ &= \sum_{n \geq 0} \left(\frac{3^n}{4} + \frac{1}{4} - \frac{n+1}{2} \right) x^n. \\ &= \sum_{n \geq 0} \left(\frac{3^n - 2n - 1}{4} \right) x^n. \end{aligned}$$

Thus

$$\boxed{a_n = \frac{3^n - 2n - 1}{4}, \quad \text{for all } n \geq 0.}$$

6. Let n be a positive integer.

- Let a_n be the number of partitions of n whose parts differ by at least two. For instance, when $n = 10$ the partitions are (10), $(9,1)$, $(8,2)$, $(7,3)$, $(6,4)$, $(6,3,1)$.
- Let b_n be the number of partitions of n whose smallest part is at least as large as the number of parts. For instance, when $n = 10$ the partitions are (10), $(8,2)$, $(7,3)$, $(6,4)$, $(5,5)$, $(4,3,3)$.

Give a bijective proof that $a_n = b_n$.
Hint. Consider $1 + 3 + 5 + \cdots + (2k-1)$.

Solution. Consider a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n whose smallest part λ_k satisfies $\lambda_k \geq k$. Let $\mu = (\lambda_1 - k, \lambda_2 - k, \dots, \lambda_k - k)$. Note that $1 + 3 + 5 + \cdots + (2k-1) = k^2$. Let

$$\nu = (\mu_1 + 2k - 1, \mu_2 + 2k - 3, \mu_3 + 2k - 5, \dots, \mu_{k-1} + 3, \mu_k + 1).$$

Then ν is a partition of n whose parts differ by at least two.

The process can be reversed. Starting with partition $\nu = (\nu_1, \dots, \nu_k)$ satisfying $\nu_i - \mu_{i+1} \geq 2$ for each $1 \leq i \leq k-1$. We have $\nu_k \geq 1$, $\nu_{k-1} \geq 3$, ..., $\nu_1 \geq (2k-1)$. Let

$$\mu = (\nu_1 - (2k-1), \nu_2 - (2k-3), \dots, \nu_k - 1).$$

(This is the same μ as in the forward map!) And let $\lambda = (\mu_1 + k, \mu_2 + k, \dots, \mu_k + k)$. Then λ is a partition of n with $\lambda_k \geq k$, and this map is the inverse of the earlier map.

This gives the desired bijection.