Practice Midterm 1

Closed book. No notes/calculators/phones.
Time: 80 minutes.
6 problems worth 10 points each.
You must provide justification in your solutions (not just answers). Simplify all answers and express in closed form whenever possible.

1. Determine the number of solutions to $x + y + z \le n$ with integers $x, y, z \ge 1$.

2. Prove that for all positive integers n,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

3. Let $D(n)$ denote the number of derangements (permutations without fixed points) of $[n]$. Give a combinatorial proof of the identity

$$D(n+1) = n(D(n) + D(n-1)), \quad \text{for all } n \ge 1.$$

   Do not use the formula for the numbers $D(n)$ derived in class.

4. Let $n \ge 4$. How many permutations of $[n]$ are there such that some cycle contains both 1 and 2 and a different cycle contains both 3 and 4?

5. Let $a_0 = 0$ and $a_{n+1} = 3a_n + n$ for all $n \ge 0$.
   (a) Express the generating function $A(x) = \sum_{n \ge 0} a_n x^n$ in closed form.
   (b) Find a closed form formula for a_n.

6. Let n be a positive integer.
   - Let a_n be the number of partitions of n whose parts differ by at least two. For instance, when $n = 10$ the partitions are (10), $(9,1)$, $(8,2)$, $(7,3)$, $(6,4)$, $(6,3,1)$.
   - Let b_n be the number of partitions of n whose smallest part is at least as large as the number of parts. For instance, when $n = 10$ the partitions are (10), $(8,2)$, $(7,3)$, $(6,4)$, $(5,5)$, $(4,3,3)$.

   Give a bijective proof that $a_n = b_n$.
   HINT. Consider $1 + 3 + 5 + \cdots + (2k-1)$.