Propositional Logic

COMP 1805

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Understanding how statements work. Foundations are known as propositions (a declarative sentence that is either true or false). Math equations involving variables are not propositions(within reason).

Atomic propositions are the simplest things present in a proposition. They are the individuals that vary in true or false values.

1 Operations

1.1 Negation

Negation: if p is a proposition.

Negation of p, denoted by $\neg p$ is statement: It is not the case that p

P	$\neg p$
T	F
F	T

1.2 AND

If p and q are propositions. Proposition p and q are denoted by $p \wedge q$, os a statement which is true if both p and q are true, otherwise false. "$\wedge$" is called conjunction

P	Q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

1.3 OR

if p and q are propositions. Disjunction of p and q, denoted by $p \vee q$ is the proposition p or q.

P	Q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

XOR is the same as OR, only that both cannot be True (if they are both the same bool then False). Denoted by $p \oplus q$

1.4 Implication

If p and q are two propositions. Proposition $p \to q$ is statement of "if p then q" and its truth value is False if p is True and q is False, otherwise True.

P	Q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse of $p \to q$: $q \to p$
Contraposition of $p \to q$: $q \to \neg p$
Reverse of $p \to q$: $\neg p \to \neg q$

Definition 1. When two compound propositions have the same truth value for each assignment of values to the atomic proposition we say they are equivalent.

1.5 Bi-implication (if and only if)

if p and q are two propositions. Proposition $p \longleftrightarrow q$ is statement of "p if and only if$q$" and its truth value is true if p and q have same truth value, otherwise false.

P	Q	$p \longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

1.6 Priorties (like bedmas)

1. $\neg$
2. $\wedge$
3. $\vee$
4. $\oplus$
5. $\leftarrow$
6. $\longleftrightarrow$

2 Truth Tables

Prove that $\neg q \to \neg p \equiv \neg p \vee q$

Proof. $\neg q \to \neg p \equiv \neg p \vee q$

p	q	$p \to q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

□

Build the truth table of proposition $(p \vee \neg q) \to (p \wedge q)$

Example 1.

p	q	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	T	F	F

Sometimes truth value doesn't depend on the other truth values: the compound proposition is always true or always false, regardless of the truth assignments of the propositions. For example, $p \vee \neg p$ is always true, regardless of whether p is true or false. This is known as **tautology**.

On the other hand, $p \wedge \neg p$ is always false, regardless of whether p is true or false. This is known as a **contradiction**.

If a statement is neither of these, then it is known as a **contingency**.

3 Logical Equivalences

Rule 1. Fundamental laws in logical operations. Let T represent True and F represent False

1. Identity Law: $p \wedge T \equiv p$ and $p \vee F \equiv p$
2. Domination Law: $p \vee T \equiv T$ and $p \wedge F \equiv F$
3. Idempotent Law: $p \vee p \equiv p$ and $p \wedge p \equiv p$
4. Commutative Law: $p \vee q \equiv q \vee p$ and $p \wedge q \equiv q \wedge p$

5. Associative Law: $(p \vee q) \vee r \equiv p \vee (q \vee r)$ and $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
6. Distributive Law: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ and $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
7. Absorption Law: $p \vee (p \wedge q) \equiv p$ and $p \wedge (p \vee q) \equiv p$
8. De Morgan's Law: $\neg(p \wedge q) \equiv \neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Implication Equivalence: $p \rightarrow q \equiv \neg p \vee q$
10. Biconditional Equivalence: $p \longleftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Example 2. Prove that $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Proof.

$$\begin{aligned} \neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{(Implication)} \\ &\equiv \neg(\neg p) \wedge \neg q && \text{(De Morgan)} \\ &\equiv p \wedge \neg q && \text{(Double Negation)} \end{aligned}$$

Definition 2. Satisfiability is when a compound proposition p, can find truth values for the atoms of p such that evaluation of p on those atom values equal to T. As long as we get one truth in the final truth table, then it is satisfiable. Any tautology and contingency is satisfiable. Any contradiction is not satisfiable.

Example 3. $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is satisfiable by $p = T, r = T, q = T$

Definition 3. if we represent $T = 1$ and $F = 0$ a bit string is a sequence of zeros and ones.

Example 4. $p = 0010011$ and $q = 1110110$. Evaluate $p \wedge q$, $p \vee q$ and $p \oplus q$

$p \wedge q$

$$\begin{array}{r} 0010011 \\ \wedge 1110110 \\ 0010010 \end{array}$$

Treat it as an addition table, where you evaluate each digit separately.

4 Predicate Logic

5 Proof Scheme (Rules of Inference)

How to write proofs with logical statements. Given the rules of inference:

1. Addition: given p, conclude $p \vee q$

2. Conjunction: given p and q, conclude $p \wedge q$
3. Simplification: given $p \wedge q$, conclude p and q
4. Modus Ponens: given p and $p \to q$, conclude q
5. Modus Tollens: given $\neg q$ and $p \to q$, conclude $\neg p$
6. Hypothetical Syllogism: given $p \to q$ and $q \to r$, conclude $p \to r$
7. Disjunctive Syllogism: given $p \vee q$ and $\neg p$, conclude q
8. Resolution: given $p \vee q$ and $\neg p \vee r$, conclude $q \vee r$

Example 5. $s \to t$, $\neg p \wedge q$, $r \to p$, $\neg r \to s$, then prove that t is true.

Proof. Using the Rules of Inference.

1. $s \to t$
2. $\neg p \wedge q$
3. $r \to p$
4. $\neg r \to s$
5. $\neg p$ (simplification on Row 2)
6. $\neg p \to \neg r$ (contraposition on Row 3)
7. $\neg r$ (Modus ponens 6)
8. $\neg r \to t$ (Hypothetical Syllogism on Row 1 and 4)
9. t (using Modus ponens on 7,8)

$\square$