Chapter 1 Lecture Notes

MATH 1104

Winter 2023

1 System of Linear Equations

Some definitions to start with

Definition 1. A linear equation in variables $x_1, x_2, \cdots x_n$ is an equation that can be written in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $a_1, a_2, \cdots a_n \in \mathbb{R}$

Definition 2. A system of linear equations (or linear system) is a collection of one or more linear equations involving the same variables , $x_1, x_2, \cdots, x_n$

$$a_{1_1}x_1 + a_{1_2}x_2 + \cdots + a_{1_n}x_n = b_1$$

$$a_{m_1}x_1 + a_{m_2}x_2 + \cdots + a_{m_n}x_n = b_m$$

Definition 3. A solution of the system is a list $(S_1, S_2, \cdots S_n)$ of numbers that makes each equation true when $S_1, S_2, \cdots S_n$are substituted for $x_1, x_2, \cdots x_n$

Example 1.

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

A solution here would be $(3, 2)$, where $x_1 = 3$ and $x_2 = 2$. Note that this satisfies BOTH equations.

To draw the graph set all the variables but one to 0. Repeat for the remaining variables. Only works for linear equations

A system of linear equations has either

1. No solutions (Inconsistent) (Parallel)

2. Unique solution (Consistent)

3. Infinite solutions (Consistent) (Same equation)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

A= $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ This is a coefficient matrix

x = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b = $\begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$

Combine the A and b

Fundamental matrix questions:

1. Does a solution of a linear system exist?

2. How many solutions does it have if it is consistent?

Definition 4. The set of all solutions is called the solution set of the linear system.

Example 2.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned}$$

The equation on the bottom is easier to solve than the other. But what if you want to solve the top? Convert the complex system into an equivalent simpler system.

Definition 5. Two linear systems are equivalent if they have the same solution set.

There are 3 elementary operations.

Example 3.

$$\begin{aligned} x + y &= 3 \\ 2x - y &= 4 \end{aligned}$$

1. Replace one equation by the sum of itself and a multiple of another equation

$$2x - y + (x + y) = 4 + 3$$

$$3x = 7$$

2. Multiplying by a non-zero constant

3. Interchanging 2 equations

Perform Gaussian Elimination. Review it perhaps.

2 Elementary Row Operations

1. Replacement ($R_1 + 3R_2$ means $R_1 \leftarrow R_1 + 3R_2$)

2. Interchange ($R_1 \leftrightarrow R_2$)

3. Scaling (multiply the row by a non zero constant)

Definition 6. A rectangular matrix is in Echelon Form or Row Echelon Form, if it has the following properties:

1. All nonzero rows are above any row of all zeros

2. Each leading entry of a row is in a column to the right of leading entry of the row above it

3. All entries in a column below a leading entry are zeros

Example 4. $\begin{bmatrix} 0 & 0 & 0 \\ 8 & 1 & 3 \\ 9 & 4 & 0 \end{bmatrix}$ is not in REF

Definition 7. Reduced Row Echelon Form is REF, but with the leading entry equal to 1.