QUADRATIC EQUATION

1. INTRODUCTION :

The algebraic expression of the form $ax^2 + bx + c$, $a \neq 0$ is called a quadratic expression, because the highest order term in it is of second degree. Quadratic equation means, $ax^2 + bx + c = 0$. In general whenever one says zeroes of the expression $ax^2+bx + c$, it implies roots of the equation $ax^2 + bx + c = 0$, unless specified otherwise.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

2. SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS :

(a) The general form of quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

The roots can be found in following manner :

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \Rightarrow \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This expression can be directly used to find the two roots of a quadratic equation.

(b) The expression $b^2 - 4ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If $\alpha \,\&\, \beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$, then :

(i) $\alpha + \beta = -b/a$ (ii) $\alpha\beta = c/a$ (iii) $|\alpha - \beta| = \sqrt{D}/|a|$

(d) A quadratic equation whose roots are $\alpha \,\&\, \beta$ is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. x^2 – (sum of roots) x + product of roots = 0.

Illustration 1 : If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is -

(A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$ (C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Solution : Since α, β are the roots of equation $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\beta^2 - 3\beta + 5 = 0$

$\therefore$ $\alpha^2 - 3\alpha = -5$

$\beta^2 - 3\beta = -5$

Putting in $(\alpha^2 - 3\alpha + 7)$ & $(\beta^2 - 3\beta + 7)$ ....(i)

$-5 + 7, -5 + 7$

$\therefore$ 2 and 2 are the roots.

$\therefore$ The required equation is

$x^2 - 4x + 4 = 0$. **Ans. (B)**

Illustration 2 : If α and β are the roots of $ax^2 + bx + c = 0$, find the value of $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$.

Solution : We know that $\alpha + \beta = -\frac{b}{a}$ & $\alpha\beta = \frac{c}{a}$

$$(a\alpha + b)^{-2} + (a\beta + b)^{-2} = \frac{1}{(a\alpha + b)^2} + \frac{1}{(a\beta + b)^2}$$

$$= \frac{a^2\beta^2 + b^2 + 2ab\beta + a^2\alpha^2 + b^2 + 2ab\alpha}{(a^2\alpha\beta + ba\beta + ba\alpha + b^2)^2} = \frac{a^2(\alpha^2 + \beta^2) + 2ab(\alpha + \beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha + \beta) + b^2)^2}$$

($\alpha^2 + \beta^2$ can always be written as $(\alpha + \beta)^2 - 2\alpha\beta$)

$$= \frac{a^2\left[(\alpha+\beta)^2 - 2\alpha\beta\right] + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2} = \frac{a^2\left[\frac{b^2-2ac}{a^2}\right] + 2ab\left(-\frac{b}{a}\right) + 2b^2}{\left(a^2\frac{c}{a} + ab\left(-\frac{b}{a}\right) + b^2\right)^2} = \frac{b^2-2ac}{a^2c^2}$$

Alternatively :

Take $b = -(\alpha+\beta)a$

$$(a\alpha + b)^{-2} + (a\beta + b)^{-2} = \frac{1}{a^2}\left[\frac{1}{(\alpha-\alpha-\beta)^2} + \frac{1}{(\beta-\alpha-\beta)^2}\right]$$

$$= \frac{1}{a^2}\left[\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}\right] = \frac{1}{a^2}\left[\frac{b^2-2ac}{a^2.\frac{c^2}{a^2}}\right] = \frac{b^2-2ac}{a^2c^2}$$

Do yourself - 1 :

(i) Find the roots of following equations :

(a) $x^2 + 3x + 2 = 0$ (b) $x^2 - 8x + 16 = 0$ (c) $x^2 - 2x - 1 = 0$

(ii) Find the roots of the equation $a(x^2 + 1) - (a^2 + 1)x = 0$, where $a \neq 0$.

(iii) Solve : $\frac{6-x}{x^2-4} = 2 + \frac{x}{x+2}$

(iv) If the roots of $4x^2 + 5k = (5k + 1)x$ differ by unity, then find the values of k.

3. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then ;

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. $(p, q \in R \ \& \ i = \sqrt{-1})$.

(b) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

Illustration 3 : If the coefficient of the quadratic equation are rational & the coefficient of x^2 is 1, then find the equation one of whose roots is $\tan\frac{\pi}{8}$.

Solution : We know that $\tan\frac{\pi}{8} = \sqrt{2} - 1$

Irrational roots always occur in conjugational pairs.

Hence if one root is $(-1 + \sqrt{2})$, the other root will be $(-1 - \sqrt{2})$. Equation is

$(x - (-1+\sqrt{2}))\ (x - (-1-\sqrt{2})) = 0 \quad \Rightarrow \quad x^2 + 2x - 1 = 0$

Illustration 4 : Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Solution : Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow \quad D = (a - 10)^2 - 4 \quad \Rightarrow \quad 4 = (a - 10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow \quad (a - 10) = \pm 2 \quad \Rightarrow \quad a = 12, 8$ **Ans.**

Do yourself - 2 :

(i) If $2+\sqrt{3}$ is a root of the equation $x^2 + bx + c = 0$, where $b, c \in Q$, find b, c.

(ii) For the following equations, find the nature of the roots (real & distinct, real & coincident or imaginary).

(a) $x^2 - 6x + 10 = 0$

(b) $x^2 - (7+\sqrt{3})x + 6(1+\sqrt{3}) = 0$

(c) $4x^2 + 28x + 49 = 0$

(iii) If ℓ, m are real and $\ell \neq m$, then show that the roots of $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are real and unequal.

4. ROOTS UNDER PARTICULAR CASES :

Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(a) If $b = 0 \Rightarrow$ roots are equal in magnitude but opposite in sign

(b) If $c = 0 \Rightarrow$ one root is zero other is $-b/a$

(c) If $a = c \Rightarrow$ roots are reciprocal to each other

(d) If $\left.\begin{matrix} a > 0 \;\; c < 0 \\ a < 0 \;\; c > 0 \end{matrix}\right\} \Rightarrow$ roots are of opposite signs

(e) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are negative.

(f) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\} \Rightarrow$ both roots are positive.

(g) If sign of a = sign of b $\neq$ sign of c $\Rightarrow$ Greater root in magnitude is negative.

(h) If sign of b = sign of c $\neq$ sign of a $\Rightarrow$ Greater root in magnitude is positive.

(i) If $a + b + c = 0 \Rightarrow$ one root is 1 and second root is c/a or $(-b-a)/a$.

Illustration 5 : If equation $\frac{x^2 - bx}{ax - c} = \frac{k-1}{k+1}$ has roots equal in magnitude & opposite in sign, then the value of k is -

(A) $\frac{a+b}{a-b}$ (B) $\frac{a-b}{a+b}$ (C) $\frac{a}{b}+1$ (D) $\frac{a}{b}-1$

Solution : Let the roots are α & $-\alpha$.

given equation is

$(x^2 - bx)(k + 1) = (k - 1)(ax - c)$ {Considering, $x \neq c/a$ & $k \neq -1$}

$\Rightarrow \quad x^2(k + 1) - bx(k + 1) = ax(k - 1) - c(k - 1)$

$\Rightarrow \quad x^2(k + 1) - bx(k + 1) - ax(k - 1) + c(k - 1) = 0$

Now sum of roots = 0 $\quad (\because \; \alpha - \alpha = 0)$

$\therefore \quad b(k + 1) + a(k - 1) = 0 \quad \Rightarrow \quad k = \frac{a-b}{a+b}$ **Ans. (B)**

***Illustration 6 :** If roots of the equation $(a - b)x^2 + (c - a)x + (b - c) = 0$ are equal, then a, b, c are in

(A) A.P. (B) H.P. (C) G.P. (D) none of these

Solution : $(a - b)x^2 + (c - a)x + (b - c) = 0$

As roots are equal so

$B^2 - 4AC = 0 \Rightarrow (c - a)^2 - 4(a - b)(b - c) = 0 \Rightarrow (c - a)^2 - 4ab + 4b^2 + 4ac - 4bc = 0$

$\Rightarrow (c - a)^2 + 4ac - 4b(c + a) + 4b^2 = 0 \Rightarrow (c + a)^2 - 2 \,.\, (2b)(c + a) + (2b)^2 = 0$

$\Rightarrow [c + a - 2b]^2 = 0 \Rightarrow c + a - 2b = 0 \Rightarrow c + a = 2b$

Hence a, b, c are in A. P.

Alternative method :

$\because$ Sum of the coefficients = 0

Hence one root is 1 and other root is $\frac{b-c}{a-b}$.

Given that both roots are equal, so

$1 = \frac{b-c}{a-b} \Rightarrow a - b = b - c \Rightarrow 2b = a + c$

Hence a, b, c are in A.P. **Ans. (A)**

Do yourself - 3 :

(i) Consider $f(x) = x^2 + bx + c$.

(a) Find c if $x = 0$ is a root of $f(x) = 0$.

(b) Find c if $\alpha, \frac{1}{\alpha}$ are roots of $f(x) = 0$.

(c) Comment on sign of b & c, if $\alpha < 0 < \beta$ & $|\beta| > |\alpha|$, where α, β are roots of $f(x) = 0$.

5. IDENTITY :

An equation which is true for every value of the variable within the domain is called an identity , for example : $5(a - 3) = 5a - 15$, $(a + b)^2 = a^2 + b^2 + 2ab$ for all $a, b \in R$.

Note : A quadratic equation cannot have three or more roots & if it has , it becomes an identity.

If $ax^2 + bx + c = 0$ is an identity $\Leftrightarrow a = b = c = 0$

Illustration 7 : If the equation $(\lambda^2 - 5\lambda + 6)x^2 + (\lambda^2 - 3\lambda + 2)x + (\lambda^2 - 4) = 0$ has more than two roots, then find the value of λ ?

Solution : As the equation has more than two roots so it becomes an identity. Hence

$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2, 3$

and $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$

and $\lambda^2 - 4 = 0 \Rightarrow \lambda = 2, -2$

So $\lambda = 2$ **Ans. $\lambda = 2$**

6. COMMON ROOTS OF TWO QUADRATIC EQUATIONS :

(a) Only one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then

$a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$

Therefore, $\alpha = \frac{ca' - c'a}{ab' - a'b} = \frac{bc' - b'c}{a'c - ac'}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

(b) If both roots are same then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Illustration 8 : Find p and q such that $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots in common.

Solution : $a_1 = p,\ b_1 = 5,\ c_1 = 2$

$a_2 = 3,\ b_2 = 10,\ c_2 = q$

We know that :

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{p}{3} = \frac{5}{10} = \frac{2}{q} \Rightarrow p = \frac{3}{2}\ ;\ q = 4$$

****Illustration 9 :*** The equations $5x^2 + 12x + 13 = 0$ and $ax^2 + bx + c = 0$ $(a,b,c \in R)$ have a common root, where a, b, c are the sides of the Δ ABC. Then find $\angle C$.

(A) 45 (B) 60 (C) 90 (D) 30

Solution : As we can see discriminant of the equation $5x^2 + 12x + 13 = 0$ is negative so roots of the equation are imaginary. We know that imaginary roots always occurs in pair. So this equation can not have single common roots with any other equation having real coefficients. So both roots are common of the given equations.

Hence $\frac{a}{5} = \frac{b}{12} = \frac{c}{13} = \lambda(\text{let})$

then $a = 5\lambda,\ b = 12\lambda\ ,\ c = 13\lambda$

Now $\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{25\lambda^2 + 144\lambda^2 - 169\lambda^2}{2(5\lambda)(12\lambda)} = 0$

$\therefore \quad \angle C = 90$ **Ans. (C)**

Do yourself - 4 :

(i) If $x^2 + bx + c = 0$ & $2x^2 + 9x + 10 = 0$ have both roots, find b & c.

(ii) If $x^2 - 7x + 10 = 0$ & $x^2 - 5x + c = 0$ have a common root, find c.

(iii) Show that $x^2 + (a^2 - 2)x - 2a^2 = 0$ and $x^2 - 3x + 2 = 0$ have exactly one common root for all $a \in R$.

7. REMAINDER THEOREM :

If we divide a polynomial f(x) by $(x - \alpha)$ the remainder obtained is $f(\alpha)$. If $f(\alpha)$ is 0 then $(x - \alpha)$ is a factor of f(x).

Consider $f(x) = x^3 - 9x^2 + 23x - 15$

$f(1) = 0 \quad \Rightarrow \quad (x - 1)$ is a factor of f(x).

$f(x) = (x - 2)(x^2 - 7x + 9) + 3$. Hence $f(2) = 3$ is remainder when f(x) is divided by $(x - 2)$.

8. SOLUTION OF RATIONAL INEQUALITIES :

Let $y = \frac{f(x)}{g(x)}$ be an expression in x where f(x) & g(x) are polynomials in x. Now, if it is given that $y > 0$ (or < 0 or ≥ 0 or ≤ 0), this calls for all the values of x for which y satisfies the constraint. This solution set can be found by following steps :

Step I : Factorize f(x) & g(x) and generate the form :

$$y = \frac{(x - a_1)^{n_1}(x - a_2)^{n_2}........(x - a_k)^{n_k}}{(x - b_1)^{m_1}(x - b_2)^{m_2}.......(x - b_p)^{m_p}}$$

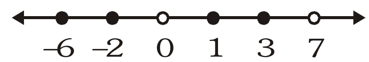
where $n_1 n_2.........n_k$, $m_1, m_2.......m_p$ are natural numbers and $a_1, a_2.......a_k$, $b_1, b_2......b_p$ are real numbers. Clearly, here $a_1, a_2......a_k$ are roots of $f(x)=0$ & $b_1, b_2,.....b_p$ are roots of $g(x) = 0$.

Step II : Here y vanishes (becomes zero) for $a_1, a_2, \ldots\ldots\ldots a_k$. These points are marked on the number line with a black dot. They are solution of y=0.

If g(x)=0, $y = \dfrac{f(x)}{g(x)}$ attains an undefined form, hence $b_1, b_2 \ldots\ldots b_k$ are excluded from the solution.

These points are marked with white dots.

e.g. $f(x) = \dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$



Step-III : Check the value of y for any real number greater than the right most marked number on the number line. If it is positive, then y is positive for all the real numbers greater than the right most marked number and vice versa.

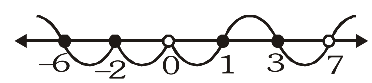
Step-IV : If the exponent of a factor is odd, then the point is called simple point and if the exponent of a factor is even, then the point is called double point

$$\frac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$$

Here 1,3,–6 and 7 are simple points and –2 & 0 are double points.

From right to left, beginning above the number line (if y is positive in step 3 otherwise from below the line), a wavy curve should be drawn which passes through all the marked points so that when passing through a simple point, the curve intersects the number line and when passing through a double point, the curve remains on the same side of number line.

$f(x) = \dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$



As exponents of (x + 2) and x are even, the curve does not cross the number line. This method is called wavy curve method.

Step-V : The intervals where the curve is above number line, y will be positive and the intervals where the curve is below the number line, y will be negative. The appropriate intervals are chosen in accordance with the sign of inequality & their union represents the solution of inequality.

Note :

(i) Points where denominator is zero will never be included in the answer.

(ii) If you are asked to find the intervals where f(x) is non-negative or non-positive then make the intervals closed corresponding to the roots of the numerator and let it remain open corresponding to the roots of denominator.

(iii) Normally we cannot cross-multiply in inequalities. But we cross multiply if we are sure that quantity in denominator is always positive.

(iv) Normally we cannot square in inequalities. But we can square if we are sure that both sides are non negative.

(v) We can multiply both sides with a negative number by changing the sign of inequality.

(vi) We can add or subtract equal quantity to both sides of inequalities without changing the sign of inequality.

Illustration 10 : Find x such that $3x^2 - 7x + 6 < 0$

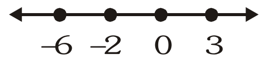
Solution : $D = 49 - 72 < 0$

As $D < 0$, $3x^2 - 7x + 6$ will always be positive. Hence $x \in \phi$.

Illustration 11 : $(x^2 - x - 6)(x^2 + 6x) \geq 0$

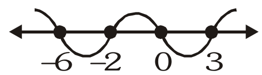
Solution : $(x-3)(x+2)(x)(x+6) \geq 0$

Consider $E = x(x-3)(x+2)(x+6)$, $E = 0 \Rightarrow x = 0, 3, -2, -6$ (all are simple points)



For $x \geq 3$ $E = \underbrace{x}_{+ve}\ \underbrace{(x-3)}_{+ve}\ \underbrace{(x+2)}_{+ve}\ \underbrace{(x+6)}_{+ve}$

= positive



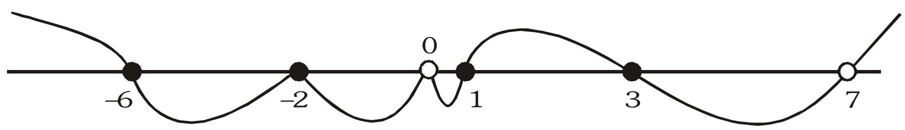
Hence for $x(x-3)(x+2)(x+6) \geq 0$

$x \in (-\infty,-6] \cup [-2,0] \cup [3,\infty)$

Illustration 12 : Let $f(x) = \dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$. Solve the following inequality

(i) $f(x) > 0$ (ii) $f(x) \geq 0$ (iii) $f(x) < 0$ (iv) $f(x) \leq 0$

Solution : We mark on the number line zeros of the function : 1, –2, 3 and –6 (with black circles) and the points of discontinuity 0 and 7 (with white circles), isolate the double points : –2 and 0 and draw the wavy curve :



From graph, we get

(i) $x \in (-\infty, -6) \cup (1, 3) \cup (7, \infty)$

(ii) $x \in (-\infty, -6] \cup \{-2\} \cup [1, 3] \cup (7, \infty)$

(iii) $x \in (-6, -2) \cup (-2, 0) \cup (0, 1) \cup (3, 7)$

(iv) $x \in [-6, 0) \cup (0, 1] \cup [3, 7)$

Do yourself - 5 :

(i) Find range of x such that

(a) $(x - 2)(x + 3) \geq 0$

(b) $\dfrac{x}{x+1} > 2$

(c) $\dfrac{3x-1}{4x+1} \leq 0$

(d) $\dfrac{(2x-1)(x+3)(2-x)(1-x)^2}{x^4(x+6)(x-9)(2x^2+4x+9)} < 0$

(e) $\dfrac{7x-17}{x^2-3x+4} \geq 1$

(f) $x^2 + 2 \leq 3x \leq 2x^2 - 5$

9. QUADRATIC EXPRESSION AND IT'S GRAPHS :

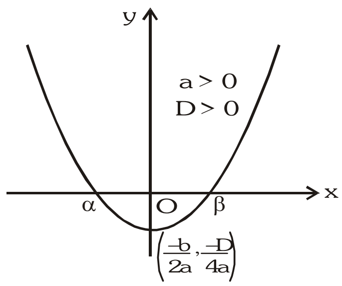
Consider the quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ & $a, b, c \in R$ then ;

(a) The graph between x, y is always a parabola. If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.

(b) The graph of $y = ax^2 + bx + c$ can be divided in 6 broad categories which are as follows :

(Let the real roots of quadratic equation $ax^2 + bx + c = 0$ be α & β where $\alpha \leq \beta$).

Fig. 1

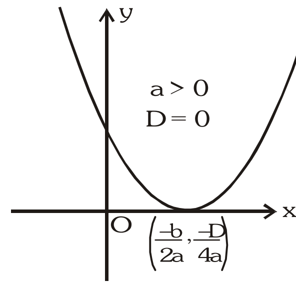


Roots are real & distinct

$ax^2 + bx + c > 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

$ax^2 + bx + c < 0 \ \forall \ x \in (\alpha, \beta)$

Fig. 2

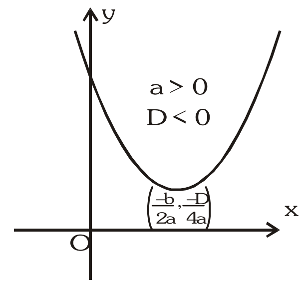


Roots are coincident

$ax^2 + bx + c > 0 \ \forall \ x \in R - \{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

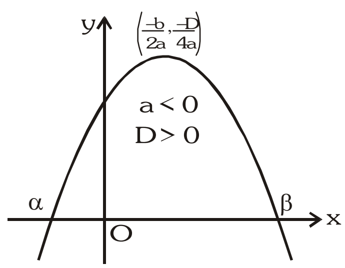
Fig.3



Roots are complex conjugate

$ax^2 + bx + c > 0 \ \forall \ x \in R$

Fig. 4

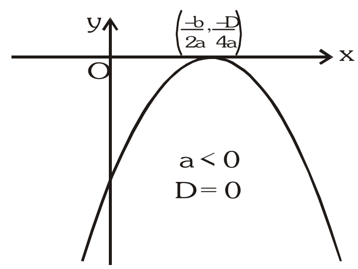


Roots are real & distinct

$ax^2 + bx + c > 0 \ \forall \ x \in (\alpha, \beta)$

$ax^2 + bx + c < 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

Fig. 5

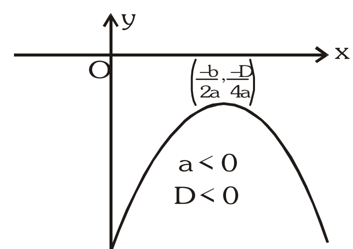


Roots are coincident

$ax^2+bx + c < 0 \ \forall \ x \in R-\{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

Fig.6



Roots are complex conjugate

$ax^2 + bx + c < 0 \ \forall \ x \in R$

Important Note :

(i) The quadratic expression $ax^2 + bx + c > 0$ for each $x \in R \Rightarrow a > 0, D < 0$ & vice-versa (Fig. 3)

(ii) The quadratic expression $ax^2 + bx + c < 0$ for each $x \in R \Rightarrow a < 0, D < 0$ & vice-versa (Fig. 6)

10. MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSIONS : $y = ax^2 + bx + c$:

We know that $y = ax^2 + bx + c$ takes following form : $y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a^2}\right]$,

which is a parabola. $\therefore$ vertex $= \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

When $a > 0$, y will take a minimum value at vertex; $x = \frac{-b}{2a}$; $y_{min} = \frac{-D}{4a}$

When $a < 0$, y will take a maximum value at vertex; $x = \frac{-b}{2a}$; $y_{max} = \frac{-D}{4a}$.

If quadratic expression ax^2+bx+c is a perfect square, then $a > 0$ and $D = 0$

***Illustration13** : If f(x) is a quadratic expression such that $f(x) > 0 \ \forall \ x \in R$, and if $g(x) = f(x) + f'(x) + f''(x)$, then prove that $g(x) > 0 \ \forall \ x \in R$.

Solution : Let $f(x) = ax^2 + bx + c$

Given that $f(x) > 0$ so $a > 0$, $b^2 - 4ac < 0$

Now $g(x) = ax^2 + bx + c + 2ax + b + 2a \ = ax^2 + (b + 2a)x + (b + c + 2a)$

For this quadratic expression $a > 0$ and discriminant

$D = (b + 2a)^2 - 4a(b + c + 2a) = b^2 + 4a^2 + 4ab - 4ab - 4ac - 8a^2 \ = b^2 - 4ac - 4a^2 < 0$

So $g(x) > 0 \ \forall \ x \in R$.

Illustration 14 : The value of the expression $x^2+2bx+c$ will be positive for all real x if -

(A) $b^2-4c>0$ (B) $b^2-4c<0$ (C) $c^2<b$ (D) $b^2<c$

Solution : As a > 0, so this expression will be positive if D < 0

so $4b^2 - 4c < 0$

$b^2 < c$ **Ans. (D)**

Illustration 15 : The minimum value of the expression $4x^2 + 2x + 1$ is -

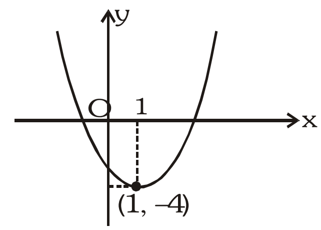
(A) 1/4 (B) 1/2 (C) 3/4 (D) 1

Solution : Since a = 4 > 0 therefore its minimum value is = $\dfrac{4(4)(1)-(2)^2}{4(4)}=\dfrac{16-4}{16}=\dfrac{12}{16}=\dfrac{3}{4}$ **Ans. (C)**

**Illustration16* : If $y = x^2 - 2x - 3$, then find the range of y when :

(i) $x \in R$ (ii) $x \in [0,3]$ (iii) $x \in [-2,0]$

Solution : We know that minimum value of y will occur at



$$x=-\frac{b}{2a}=-\frac{(-2)}{2\times1}=1$$

$$y_{min}= -\frac{D}{4a}=\frac{-(4+3\times4)}{4} = -4$$

(i) $x \in R$;

$y \in [-4,\infty)$ **Ans.**

(ii) $x \in [0, 3]$

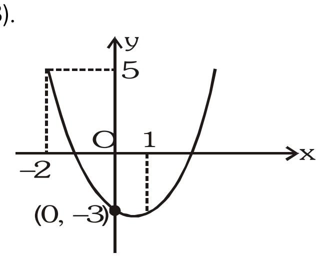
$f(0) = -3,\ f(1) = -4, f(3) = 0$

$\because$ $f(3) > f(0)$

$\therefore$ y will take all the values from minimum to f(3).

$y \in [-4, 0]$ **Ans.**

(iii) $x \in [-2, 0]$



This interval does not contain the minimum value of y for $x \in R$.

y will take values from f(0) to f(–2)

$f(0) = -3$

$f(-2) = 5$

$y \in [-3, 5]$ **Ans.**

Illustration 17 : If $ax^2 + bx + 10 = 0$ does not have real & distinct roots, find the minimum value of 5a–b.

Solution : Either $f(x) \ge 0\ \forall\ x \in R$ or $f(x) \le 0\ \forall\ x \in R$

$\because$ $f(0) = 10 > 0 \Rightarrow f(x) \ge 0\ \forall\ x \in R$

$\Rightarrow$ $f(-5) = 25a - 5b + 10 \ge 0$

$\Rightarrow$ $5a - b \ge -2$ **Ans.**

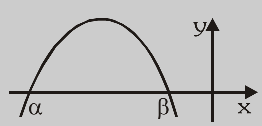
Do yourself - 6

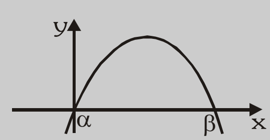
(i) Find the minimum value of :

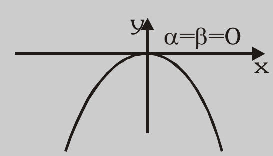
(a) $y = x^2 + 2x + 2$ (b) $y = 4x^2 - 16x + 15$

(ii) For following graphs of $y = ax^2 + bx + c$ with $a,b,c \in R$, comment on the sign of :

(i) a (ii) b (iii) c (iv) D (v) $\alpha + \beta$ (vi) $\alpha\beta$

(1)


(2)


(3)


(iii) Given the roots of equation $ax^2 + bx + c = 0$ are real & distinct, where $a,b,c \in R^+$, then the vertex of the graph will lie in which quadrant.

***(iv)** Find the range of 'a' for which :

(a) $ax^2 + 3x + 4 > 0\quad \forall\ x \in R$ (b) $ax^2 + 4x - 2 < 0\quad \forall\ x \in R$

11. INEQUALITIES INVOLVING MODULUS FUNCTION :

Properties of modulus function :

(i) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$, where a is positive.

(ii) $|x| \leq a \Rightarrow x \in [-a, a]$, where a is positive

(iii) $|x| > |y| \Rightarrow x^2 > y^2$

(iv) $\left||a|-|b|\right| \leq |a \pm b| \leq |a|+|b|$

(v) $|x + y| = |x| + |y| \Rightarrow xy \geq 0$

(vi) $|x - y| = |x| + |y| \Rightarrow xy \leq 0$

Illustration 18 : If x satisfies $|x - 1| + |x - 2| + |x - 3| \geq 6$, then

(A) $0 \leq x \leq 4$ (B) $x \leq -2$ or $x \geq 4$

(C) $x \leq 0$ or $x \geq 4$ (D) none of these

Solution :

Case I : $x \leq 1$, then

$1 - x + 2 - x + 3 - x \geq 6 \Rightarrow x \leq 0$

Hence $x \leq 0$ ...(i)

Case II : $1 < x \leq 2$, then

$x - 1 + 2 - x + 3 - x \geq 6 \Rightarrow x \leq -2$

But $1 < x \leq 2 \Rightarrow$ No solution. ...(ii)

Case III : $2 < x \leq 3$, then

$x - 1 + x - 2 + 3 - x \geq 6 \Rightarrow x \geq 6$

But $2 < x \leq 3 \Rightarrow$ No solution. ...(iii)

Case IV : $x > 3$, then

$x - 1 + x - 2 + x - 3 \geq 6 \Rightarrow x \geq 4$

Hence $x \geq 4$ ...(iv)

From (i), (ii), (iii) and (iv) the given inequality holds for $x \leq 0$ or $x \geq 4$.

Illustration 19 : Solve for x : (a) $||x - 1| + 2| \leq 4$. (b) $\dfrac{x-4}{x+2} \leq \left|\dfrac{x-2}{x-1}\right|$

Solution :

(a) $||x - 1| + 2| \leq 4 \Rightarrow -4 \leq |x - 1| + 2 \leq 4$

$\Rightarrow -6 \leq |x - 1| \leq 2$

$\Rightarrow |x - 1| \leq 2 \Rightarrow -2 \leq x - 1 \leq 2$

$\Rightarrow -1 \leq x \leq 3 \Rightarrow x \in [-1, 3]$

(b) **Case 1 :** Given inequation will be statisfied for all x such that

$\dfrac{x-4}{x+2} \leq 0 \Rightarrow x \in (-2, 4] - \{1\}$ ....(i)

(Note : {1} is not in domain of RHS)

Case 2 : $\dfrac{x-4}{x+2} > 0 \Rightarrow x \in (-\infty, -2) \cup (4, \infty)$ ....(ii)

Given inequation becomes

$\dfrac{x-2}{x-1} \geq \dfrac{x-4}{x+2}$ or $\dfrac{x-2}{x-1} \leq -\dfrac{x-4}{x+2}$

on solving we get $x \in \left(-2, 4/5\right) \cup (1, \infty)$; on solving we get $x \in (-2, 0] \cup (1, 5/2]$

taking intersection with (ii) we get $x \in (4, \infty)$ ....(iii) ; taking intersection with (ii) we get $x \in \phi$

Hence, solution of the original inequation : $x \in (-2,\infty) - \{1\}$ (taking union of (i) & (iii))

Illustration 20 : The equation $|x|+\left|\frac{x}{x-1}\right| = \frac{x^2}{|x-1|}$ is always true for x belongs to

(A) $\{0\}$ (B) $(1, \infty)$ (C) $(-1, 1)$ (D) $(-\infty, \infty)$

Solution : $\frac{x^2}{|x-1|} = \left|x+\frac{x}{x-1}\right|$

$\because\ |x|+\left|\frac{x}{x-1}\right| = \left|x+\frac{x}{x-1}\right|$ is true only if $\left(x.\frac{x}{x-1}\right) \geq 0 \Rightarrow x \in \{0\} \cup (1, \infty)$. **Ans (A, B)**

12. IRRATIONAL INEQUALITIES :

Illustration 21 : Solve for x, if $\sqrt{x^2-3x+2} > x-2$

Solution :

$$\left[\begin{array}{l} \begin{cases} x^2-3x+2 \geq 0 \\ x-2 \geq 0 \\ (x^2-3x+2) > (x-2)^2 \end{cases} \\ \quad\quad \text{or} \\ \begin{cases} x^2-3x+2 \geq 0 \\ x-2 < 0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} (x-1)(x-2) \geq 0 \\ (x-2) \geq 0 \\ x-2 > 0 \end{cases} \Rightarrow x > 2 \\ \quad\quad \text{or} \\ \begin{cases} (x-1)(x-2) \geq 0 \\ x-2 < 0 \end{cases} \Rightarrow x \leq 1 \end{array}\right.$$

Hence, solution set of the original inequation is $x \in R - (1,2]$

Do yourself - 7 :

(i) Solve for x if $\frac{|x^2-4|}{x^2+x-2} > 1$

(ii) Solve for x if $\sqrt{x^2-x} > (x-1)$

13. LOGARITHMIC INEQUALITIES :

Points to remember :

(i) $\log_a x < \log_a y \Leftrightarrow \left[\begin{array}{lll} x < y & \text{if} & a > 1 \\ x > y & \text{if} & 0 < a < 1 \end{array}\right.$

(ii) If $a > 1$, then (a) $\log_a x < p \Rightarrow 0 < x < a^p$ (b) $\log_a x > p \Rightarrow x > a^p$

(iii) If $0 < a < 1$, then (a) $\log_a x < p \Rightarrow x > a^p$ (b) $\log_a x > p \Rightarrow 0 < x < a^p$

Illustration 22 : Solve for x : (a) $\log_{0.5}(x^2 - 5x + 6) \geq -1$ (b) $\log_{1/3}(\log_4(x^2 - 5)) > 0$

Solution : (a) $\log_{0.5}(x^2 - 5x + 6) \geq -1 \Rightarrow 0 < x^2 - 5x + 6 \leq (0.5)^{-1}$

$\Rightarrow 0 < x^2 - 5x + 6 \leq 2$

$$\begin{cases} x^2-5x+6 > 0 \\ x^2-5x+6 \leq 2 \end{cases} \Rightarrow x \in [1,2) \cup (3,4]$$

Hence, solution set of original inequation : $x \in [1,2) \cup (3,4]$

(b) $\log_{1/3}(\log_4(x^2-5)) > 0 \quad \Rightarrow \quad 0 < \log_4(x^2-5) < 1$

$$\begin{cases} 0 < \log_4(x^2-5) \Rightarrow x^2-5>1 \\ \log_4(x^2-5) < 1 \Rightarrow 0 < x^2-5 < 4 \end{cases} \quad \Rightarrow \quad 1 < (x^2-5) < 4$$

$$\Rightarrow \quad 6 < x^2 < 9 \quad \Rightarrow \quad x \in \left(-3,-\sqrt{6}\right) \cup \left(\sqrt{6},3\right)$$

Hence, solution set of original inequation : $x \in \left(-3,-\sqrt{6}\right) \cup \left(\sqrt{6},3\right)$

Illustration 23 : Solve for x : $\log_2 x \le \dfrac{2}{\log_2 x - 1}$.

Solution : Let $\log_2 x = t$

$$t \le \frac{2}{t-1} \Rightarrow t - \frac{2}{t-1} \le 0$$

$$\Rightarrow \quad \frac{t^2-t-2}{t-1} \le 0 \Rightarrow \frac{(t-2)(t+1)}{(t-1)} \le 0$$

$\Rightarrow \quad t \in (-\infty, -1] \cup (1,2]$

or $\quad \log_2 x \in (-\infty,-1] \cup (1,2]$

or $\quad x \in \left(0, \frac{1}{2}\right] \cup (2,4]$

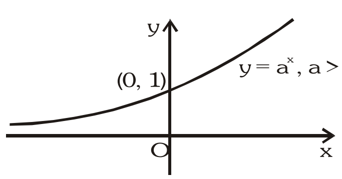
Illustration 24 : Solve the inequation : $\log_{2x+3} x^2 < \log_{2x+3}(2x+3)$

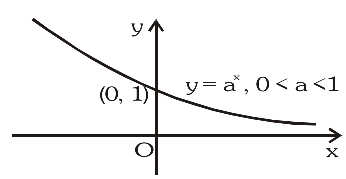
Solution : This inequation is equivalent to the collection of the systems

$$\left[\begin{array}{l} \begin{cases} 2x+3>1 \\ 0<x^2<2x+3 \end{cases} \\ \text{or} \\ \begin{cases} 0<2x+3<1 \\ x^2>2x+3>0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x>-1 \\ (x-3)(x+1)<0 \ \& \ x \ne 0 \end{cases} \\ \text{or} \\ \begin{cases} -\frac{3}{2}<x<-1 \\ (x-3)(x+1)>0 \end{cases} \end{array}\right. \Rightarrow \left[\begin{array}{l} \begin{cases} x>-1 \\ -1<x<3 \end{cases} \quad \Rightarrow \ -1<x<3 \ \& \ x \ne 0 \\ \text{or} \qquad\qquad\qquad\qquad \text{or} \\ \begin{cases} -\frac{3}{2}<x<-1 \\ x<-1 \text{ or } x>3 \end{cases} \quad \Rightarrow \ -\frac{3}{2}<x<-1 \end{array}\right.$$

Hence, solution of the original inequation is $x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 0) \cup (0, 3)$

14. EXPONENTIAL INEQUATIONS :





If $a^{f(x)} > b \Rightarrow \left[\begin{array}{ll} f(x) > \log_a b & \text{when} \quad a > 1 \\ f(x) < \log_a b & \text{when} \quad 0 < a < 1 \end{array}\right.$

Illustration 25 : Solve for x : $2^{x+2} > \left(\frac{1}{4}\right)^{\frac{1}{x}}$

Solution : We have $2^{x+2} > 2^{-2/x}$. Since the base 2 > 1, we have $x + 2 > -\frac{2}{x}$ (the sign of the inequality is retained).

Now $x+2+\frac{2}{x}>0 \quad \Rightarrow \quad \frac{x^2+2x+2}{x}>0$

$\Rightarrow \quad \frac{(x+1)^2+1}{x}>0 \quad \Rightarrow \quad x \in (0, \infty)$

Illustration 26 : Solve for x : $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$

Solution : We have $\left(\frac{5}{4}\right)^{1-x} < \left(\frac{16}{25}\right)^{2(1+\sqrt{x})}$ or $\left(\frac{4}{5}\right)^{x-1} < \left(\frac{4}{5}\right)^{4(1+\sqrt{x})}$

Since the base $0<\frac{4}{5}<1$, the inequality is equivalent to the inequality $x - 1 > 4(1+\sqrt{x})$

$\Rightarrow \quad \frac{x-5}{4} > \sqrt{x}$

Now, R.H.S. is positive

$\Rightarrow \quad \frac{x-5}{4} > 0 \quad \Rightarrow \quad x > 5$ ....(i)

we have $\frac{x-5}{4} > \sqrt{x}$

both sides are positive, so squaring both sides

$\Rightarrow \quad \frac{(x-5)^2}{16} > x$ or $\frac{(x-5)^2}{16} - x > 0$

or $x^2 - 26x + 25 > 0$ or $(x - 25)(x - 1) > 0$

$\Rightarrow \quad x \in (-\infty, 1) \cup (25, \infty)$ ....(ii)

intersection (i) & (ii) gives $x \in (25, \infty)$

Do yourself-8 :

(i) Solve for x : (a) $\log_{0.3}(x^2 + 8) > \log_{0.3}(9x)$ (b) $\log_7\left(\frac{2x-6}{2x-1}\right) > 0$

(ii) Solve for x : $\left(\frac{2}{3}\right)^{\frac{|x|-1}{|x|+1}} > 1$

15. MAXIMUM & MINIMUM VALUES OF RATIONAL ALGEBRAIC EXPRESSIONS :

$y = \frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}, \ \frac{1}{ax^2+bx+c}, \ \frac{a_1x+b_1}{a_2x^2+b_2x+c_2}, \ \frac{a_1x^2+b_1x+c_1}{a_2x+b_2}$:

Sometime we have to find range of expression of form $\frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$. The following procedure is used :

Step 1 : Equate the given expression to y i.e. $y = \frac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$

Step 2 : By cross multiplying and simplifying, obtain a quadratic equation in x.
$(a_1 - a_2y)x^2 + (b_1 - b_2y)x + (c_1 - c_2y) = 0$

Step 3 : Put Discriminant ≥ 0 and solve the inequality for possible set of values of y.

Illustration 27 : For $x \in R$, find the set of values attainable by $\dfrac{x^2-3x+4}{x^2+3x+4}$.

Solution : Let $y = \dfrac{x^2-3x+4}{x^2+3x+4}$

$x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0$

Case- I : $y \neq 1$

For $y \neq 1$ above equation is a quadratic equation.

So for $x \in R$, $D \geq 0$

$\Rightarrow \quad 9(y + 1)^2 - 16(y - 1)^2 \geq 0 \quad \Rightarrow \quad 7y^2 - 50y + 7 \leq 0$

$\Rightarrow \quad (7y - 1)(y - 7) \leq 0 \quad \Rightarrow \quad y \in \left[\dfrac{1}{7}, 7\right] - \{1\}$

Case II : when $y = 1$

$\Rightarrow \quad 1 = \dfrac{x^2-3x+4}{x^2+3x+4}$

$\Rightarrow \quad x^2 + 3x + 4 = x^2 - 3x + 4$

$\Rightarrow \quad x = 0$

Hence $y = 1$ for real value of x.

so range of y is $\left[\dfrac{1}{7}, 7\right]$

Illustration 28 : Find the values of a for which the expression $\dfrac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.

Solution : Let $y = \dfrac{ax^2+3x-4}{3x-4x^2+a}$

$x^2(a + 4y) + 3(1 - y)x - (4 + ay) = 0$

If $x \in R$, $D \geq 0$

$\Rightarrow \quad 9(1 - y)^2 + 4(a + 4y)(4 + ay) \geq 0 \quad \Rightarrow \quad (9 + 16a)y^2 + (4a^2 + 46)y + (9 + 16a) \geq 0$

for all $y \in R$, $(9 + 16a) > 0$ & $D \leq 0$

$\Rightarrow \quad (4a^2 + 46)^2 - 4(9 + 16a)(9 + 16a) \leq 0 \Rightarrow \quad 4(a^2 - 8a + 7)(a^2 + 8a + 16) \leq 0$

$\Rightarrow \quad a^2 - 8a + 7 \leq 0 \quad \Rightarrow \quad 1 \leq a \leq 7$

$9 + 16a > 0$ & $1 \leq a \leq 7$

Taking intersection, $a \in [1, 7]$

Now, checking the boundary values of a

For $a = 1$

$$y = \frac{x^2+3x-4}{3x-4x^2+1} = -\frac{(x-1)(x+4)}{(x-1)(4x+1)}$$

$\because \quad x \neq 1 \Rightarrow y \neq -1$

$\Rightarrow \quad a = 1$ is not possible.

if $\quad a = 7$

$$y = \frac{7x^2+3x-4}{3x-4x^2+7} = \frac{(7x-4)(x+1)}{(7-4x)(x+1)} \qquad \because x \neq -1 \Rightarrow y \neq -1$$

So y will assume all real values for some real values of x.

So $\quad a \in (1,7)$

Do yourself - 9 :

(i) Prove that the expression $\dfrac{8x-4}{x^2+2x-1}$ cannot have values between 2 and 4, in its domain.

(ii) Find the range of $\dfrac{x^2+2x+1}{x^2+2x+7}$, where x is real

16. LOCATION OF ROOTS :

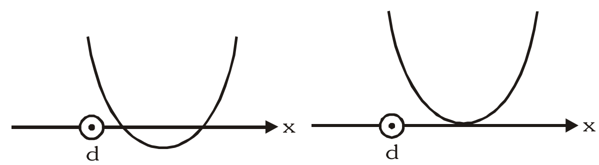
This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :

Consider the quadratic equation **$ax^2 + bx + c = 0$ with $a > 0$ and let $f(x) = ax^2 + bx + c$**

Type-1 : Both roots of the quadratic equation are greater than a specific number (say d).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) f (d) > 0 ; (iii) $-\dfrac{b}{2a} > d$



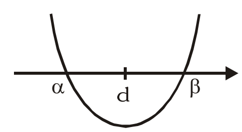
Note : When both roots of the quadratic equation are less than a specific number d than the necessary and sufficient condition will be :

(i) $D \geq 0$; (ii) f (d) > 0 ; (iii) $-\dfrac{b}{2a} < d$

Type-2 :

Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than 'd' and other root less than 'd' or 'd' lies between the roots of the given equation.

The necessary and sufficient condition for this are : f(d) < 0



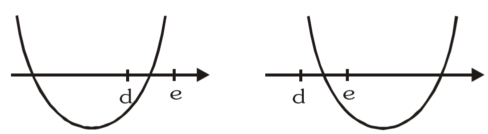
Note : Consideration of discriminant is not needed.

Type-3 :

Exactly one root lies in the interval (d, e).

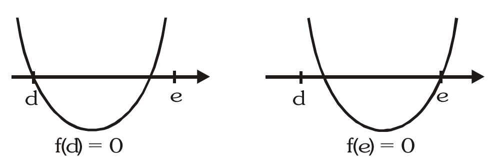
The necessary and sufficient condition for this are :

f (d) . f (e) < 0



Note : The extremes of the intervals found by given conditions give 'd' or 'e' as the root of the equation.

Hence in this case also check for end points.



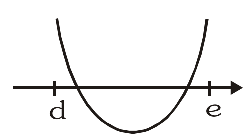
Type-4 :

When both roots are confined between the number d and e (d < e).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) f (d) > 0 ; (iii) f (e) > 0

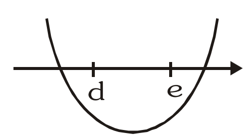
(iv) $d < -\dfrac{b}{2a} < e$



Type-5 :

One root is greater than e and the other roots is less than d (d < e).

The necessary and sufficient condition for this are : f(d) < 0 and f(e) < 0



Note : If a < 0 in the quadratic equation $ax^2 + bx + c = 0$ then we divide the whole equation by 'a'. Now assume $x^2 + \dfrac{b}{a}x + \dfrac{c}{a}$ as f(x). This makes the coefficient of x^2 positive and hence above cases are applicable.

Illustration 29 : Find the values of the parameter 'a' for which the roots of the quadratic equation $x^2 + 2(a - 1)x + a + 5 = 0$ are

(i) real and distinct (ii) equal

(iii) opposite in sign (iv) equal in magnitude but opposite in sign

(v) positive (vi) negative

(vii) greater than 3 (viii) smaller than 3

(ix) such that both the roots lie in the interval (1, 3)

Solution : Let $f(x) = x^2 + 2(a - 1)x + a + 5 = Ax^2 + Bx + C$ (say)

$\Rightarrow \quad A = 1,\ B = 2(a - 1),\ C = a + 5.$

Also $D = B^2 - 4AC = 4(a - 1)^2 - 4(a + 5) = 4(a + 1)(a - 4)$

(i) $D > 0$

$\Rightarrow \quad (a + 1)(a - 4) > 0 \quad \Rightarrow \quad a \in (-\infty\ ,\ -1)\cup(4,\ \infty).$

(ii) $D = 0$

$\Rightarrow \quad (a + 1)(a - 4) = 0 \quad \Rightarrow \quad a = -1,\ 4.$

(iii) This means that 0 lies between the roots of the given equation.

$\Rightarrow \quad f(0) < 0$ and $D > 0$ i.e. $a \in (-\infty,\ -1) \cup (4,\ \infty)$

$\Rightarrow \quad a + 5 < 0 \Rightarrow a < -5 \ \Rightarrow\ a \in (-\infty\ ,\ -5).$

(iv) This means that the sum of the roots is zero

$\Rightarrow \quad -2(a - 1) = 0$ and $D > 0$ i.e. $a \in -(-\infty,\ -1) \cup (4,\ \infty) \Rightarrow \quad a = 1$

which does not belong to $(-\infty,\ -1)\cup(4,\ \infty)$

$\Rightarrow \quad a \in \phi.$

(v) This implies that both the roots are greater than zero

$\Rightarrow \quad -\dfrac{B}{A} > 0,\ \dfrac{C}{A} > 0, D \geq 0. \qquad \Rightarrow \quad -(a - 1) > 0,\ a + 5 > 0,\ a \in (-\infty,\ -1]\cup[4,\ \infty)$

$\Rightarrow \quad a < 1,\ -5 < a,\ a \in (-\infty,\ -1]\cup[4,\ \infty) \qquad \Rightarrow \quad a \in (-5,\ -1].$

(vi) This implies that both the roots are less than zero

$\Rightarrow \quad -\dfrac{B}{A} < 0,\ \dfrac{C}{A} > 0, D \geq 0. \qquad \Rightarrow \quad -(a - 1) < 0,\ a + 5 > 0,\ a \in (-\infty,\ -1]\cup[4,\ \infty)$

$\Rightarrow \quad a > 1,\ a > -5,\ a \in (-\infty,\ -1]\cup[4,\ \infty) \qquad \Rightarrow \quad a \in [4,\ \infty).$

(vii) In this case

$-\dfrac{B}{2a} > 3$, A.f(3) > 0 and $D \geq 0$.

$\Rightarrow \quad -(a - 1) > 3,\ 7a + 8 > 0$ and $a \in (-\infty,\ -1]\cup[4,\ \infty)$

$\Rightarrow \quad a < -2,\ a > -8/7$ and $a \in (-\infty,\ -1]\cup[4,\ \infty)$

Since no value of 'a' can satisfy these conditions simultaneously, there can be no value of a for which both the roots will be greater than 3.

(viii) In this case

$-\dfrac{B}{2a} < 3$, A.f(3) > 0 and $D \geq 0$.

$\Rightarrow \quad a > -2,\ a > -8/7$ and $a \in (-\infty,\ -1]\cup[4,\ \infty) \qquad \Rightarrow \quad a \in (-8/7,\ -1]\cup[4,\ \infty)$

(ix) In this case

$1 < -\dfrac{B}{2A} < 3,\ A.f(1) > 0,\ A.f(3) > 0,\ D \geq 0.$

$\Rightarrow \quad 1 < -1(a - 1) < 3,\ 3a + 4 > 0,\ 7a + 8 > 0,\ a \in (-\infty,\ -1]\cup[4,\ \infty)$

$\Rightarrow \quad -2 < a < 0,\ a > -4/3,\ a > -8/7,\ a \in (-\infty,\ -1]\cup[4,\ \infty) \qquad \Rightarrow\ a \in \left(-\dfrac{8}{7}, -1\right]$

Illustration 30 : Find value of k for which one root of equation $x^2 - (k+1)x + k^2 + k - 8 = 0$ exceeds 2 & other is less than 2.

Solution : $4-2(k+1) + k^2 + k-8 < 0 \quad \Rightarrow k^2 - k - 6 < 0$

$(k-3)(k+2) < 0 \quad \Rightarrow \quad -2 < k < 3$

Taking intersection, $k \in (-2, 3)$.

Illustration 31 : Find all possible values of a for which exactly one root of $x^2 -(a+1)x + 2a = 0$ lies in interval (0,3).

Solution : $f(0) . f(3) < 0$

$\Rightarrow \quad 2a\,(9-3(a+1)+2a) < 0 \quad \Rightarrow \quad 2a\,(-a+6) < 0$

$\Rightarrow \quad a(a-6) > 0 \quad \Rightarrow \quad a< 0$ or $a > 6$

Checking the extremes.

If $a = 0$, $\quad x^2 - x = 0$

$x = 0, 1$

$1 \in (0, 3)$

If $a = 6$, $\quad x^2 - 7x + 12 = 0$

$x = 3, 4 \quad$ But $4 \notin (0, 3)$

Hence solution set is

$a \in (-\infty,0] \cup (6,\infty)$

Illustration 32 : If α is a root of the equation $ax^2 + bx + c = 0$ and β is a root of the equation $-ax^2 + bx + c = 0$, then prove that there will be a root of the equation $\frac{a}{2}x^2 + bx + c = 0$ lying between α and β.

Solution : Let $f(x) = \frac{a}{2}x^2 + bx + c$

$f(\alpha) = \frac{a}{2}\alpha^2 + b\alpha + c = a\alpha^2 + b\alpha + c - \frac{a}{2}\alpha^2$

$= -\frac{a}{2}\alpha^2 \quad$ (As α is a root of $ax^2 + bx + c = 0$)

And $f(\beta) = \frac{a}{2}\beta^2 + b\beta + c = -a\beta^2 + b\beta + c + \frac{3a}{2}\beta^2$

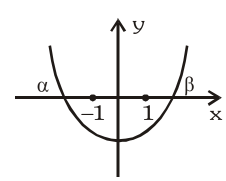
$= \frac{3a}{2}\beta^2 \quad$ (As β is a root of $-ax^2 + bx + c = 0$)

Now $f(\alpha) . f(\beta) = \frac{-3}{4}a^2\alpha^2\beta^2 < 0$

$\Rightarrow \quad f(x) = 0$ has one real root between α and β.

Illustration 33 : Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β where $\alpha < -1$ and $\beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$.

Solution : Let $f(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$



from graph $f(-1) < 0$ and $f(1) < 0$

$\Rightarrow \quad 1 + \frac{c}{a} - \frac{b}{a} < 0$ and $1 + \frac{c}{a} + \frac{b}{a} < 0$

Multiplying these two, we get $\left(1 + \frac{c}{a}\right)^2 - \frac{b^2}{a^2} > 0$

$\Rightarrow \quad \left|1 + \frac{c}{a}\right| > \left|\frac{b}{a}\right| \qquad \{\alpha\beta < -1 \Rightarrow \frac{c}{a} < -1\}$

$\Rightarrow \quad 1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$

Do yourself - 10 :

(i) If α, β are roots of $7x^2 + 9x - 2 = 0$, find their position with respect to following $(\alpha < \beta)$:

(a) -3 (b) 0 (c) 1

(ii) If $a > 1$, roots of the equation $(1 - a)x^2 + 3ax - 1 = 0$ are -

(A) one positive one negative (B) both negative

(C) both positive (D) both non-real

(iii) Find the set of value of a for which the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are less than 3.

(iv) If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.

(v) If α, β are roots of $4x^2 - 16x + \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then find the range of λ.

17. GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES :

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Illustration 34 : If $x^2 + 2xy + 2x + my - 3$ have two linear factor then m is equal to -

(A) 6, 2 (B) –6, 2 (C) 6, –2 (D) –6, –2

Solution : Here $a = 1$, $h = 1$, $b = 0$, $g = 1$, $f = m/2$, $c = -3$

$$\text{So} \quad \Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & m/2 \\ 1 & m/2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow -\frac{m^2}{4} - (-3 - m/2) + m/2 = 0 \qquad \Rightarrow -\frac{m^2}{4} + m + 3 = 0$$

$$\Rightarrow m^2 - 4m - 12 = 0 \qquad \Rightarrow m = -2, 6$$

Ans. (C)

Do yourself - 11 :

(i) Find the value of k for which the expression $x^2 + 2xy + ky^2 + 2x + k = 0$ can be resolved into two linear factors.

18. THEORY OF EQUATIONS :

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ are roots of the equation, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots.\ a_{n-1}x + a_n = 0$, where $a_0, a_1, \ldots\ldots\ldots a_n$ are constants and $a_0 \neq 0$.

$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots\ldots\ldots (x - \alpha_n)$

$\therefore \quad a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2) \ldots.. (x - \alpha_n)$

Comparing the coefficients of like powers of x, we get

$$\sum \alpha_i = -\frac{a_1}{a_0} = S_1 \quad \text{(say)}$$

$$\text{or} \quad S_1 = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$$

$$S_2 = \sum_{i \neq j} \alpha_i \alpha_j = (-1)^2 \frac{a_2}{a_0}$$

$S_3 = \sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k = (-1)^3 \frac{a_3}{a_0}$

$\vdots$

$S_n = \alpha_1 \alpha_2 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coefficient of } x^n}$

where S_k denotes the sum of the product of root taken k at a time.

Quadratic equation : If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Cubic equation : If α, β, γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$\alpha + \beta + \gamma = -\frac{b}{a}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ and $\alpha\beta\gamma = -\frac{d}{a}$

Note :

(i) If α is a root of the equation f (x) = 0, then the polynomial f(x) is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of f(x) and conversely.

(ii) Every equation of nth degree ($n \geq 1$) has exactly n root & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation f (x) = 0 are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs.**

(iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x)=0 must have atleast one real root between 'a' and 'b'.

(vi) Every equation f(x) = 0 of degree odd has atleast one real root of a sign opposite to that of its last term.

Descartes rule of signs :

The maximum number of positive real roots of polynomial equation f (x) = 0 is the number of changes of signs in f (x).

Consider $x^3 + 6x^2 + 11x - 6 = 0$

The signs are : + + + –

As there is only one change of sign, the equation has atmost one positive real root.

The maximum number of negative real roots of a polynomial equation f (x) = 0 is the number of changes of signs in f (–x)

Consider $f(x) = x^4 + x^3 + x^2 - x - 1 = 0$

$f(-x) = x^4 - x^3 + x^2 + x - 1 = 0$

3 sign changes, hence atmost 3 negative real roots.

Illustration 35 : If two roots are equal, find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$.

Solution : Let roots be α, α and β

$\therefore \quad \alpha + \alpha + \beta = -\frac{20}{4} \quad \Rightarrow \quad 2\alpha + \beta = -5 \quad \ldots\ldots\ldots\ldots \text{(i)}$

$\therefore \quad \alpha \cdot \alpha + \alpha\beta + \alpha\beta = -\frac{23}{4} \quad \Rightarrow \quad \alpha^2 + 2\alpha\beta = -\frac{23}{4} \;\&\; \alpha^2\beta = -\frac{6}{4}$

from equation (i)

$\alpha^2 + 2\alpha(-5 - 2\alpha) = -\frac{23}{4} \quad \Rightarrow \quad \alpha^2 - 10\alpha - 4\alpha^2 = -\frac{23}{4} \quad \Rightarrow 12\alpha^2 + 40\alpha - 23 = 0$

$\therefore \quad \alpha = 1/2, -\frac{23}{6}$

when $\alpha = \dfrac{1}{2}$

$\alpha^2\beta = \dfrac{1}{4}\ (-5-1) = -\dfrac{3}{2}$

when $\alpha = -\dfrac{23}{6} \Rightarrow \alpha^2\beta = \dfrac{23\times 23}{36}\left(-5-2\times\left(-\dfrac{23}{6}\right)\right) \neq -\dfrac{3}{2} \quad \Rightarrow \quad \alpha = \dfrac{1}{2} \qquad \beta = -6$

Hence roots of equation $= \dfrac{1}{2}, \dfrac{1}{2}, -6$ **Ans.**

Illustration 36 : If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find :

(i) $\sum \alpha^3$ (ii) $\alpha^2(\beta+\gamma)+\beta^2(\gamma+\alpha)+\gamma^2(\alpha+\beta)$

Solution : We know that $\alpha+\beta+\gamma = p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = r$

(i) $\alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma + (\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma)^2 - 3(\alpha\beta+\beta\gamma+\gamma\alpha)\}$

$= 3r + p\{p^2 - 3q\} = 3r + p^3 - 3pq$

(ii) $\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta) = \alpha^2(p-\alpha)+\beta^2(p-\beta)+\gamma^2(p-\gamma)$

$= p(\alpha^2+\beta^2+\gamma^2) - 3r - p^3 + 3pq = p(p^2 - 2q) - 3r - p^3 + 3pq = pq - 3r$

Illustration 37 : If $b^2 < 2ac$ and $a, b, c, d \in R$, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

Solution : Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

Then $\alpha+\beta+\gamma = -\dfrac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{c}{a}$

$\alpha\beta\gamma = \dfrac{-d}{a}$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = \dfrac{b^2}{a^2} - \dfrac{2c}{a} = \dfrac{b^2-2ac}{a^2}$

$\Rightarrow$ $\alpha^2+\beta^2+\gamma^2 < 0$, which is not possible if all α, β, γ are real. So atleast one root is non-real, but complex roots occurs in pair. Hence given cubic equation has two non-real and one real roots.

Illustration 38 : If q, r, s are positive, show that the equation $f(x) \equiv x^4 + qx^2 + rx - s = 0$ has one positive, one negative and two imaginary roots.

Solution : Product $= -s < 0$

let roots be $\alpha,\beta,\gamma,\delta$

$\Rightarrow$ $\alpha\beta\gamma\delta < 0$

this is possible when -

(i) one root is negative & three are positive

(ii) three roots are negative & one is positive

(iii) one root negative, one positive & two roots imaginary.

$f(x) \equiv x^4 + qx^2 + rx - s$

As there is only one change of sign, the equation has atmost one positive root.

$f(-x) \equiv x^4 + qx^2 - rx - s$

Again there is only one change of sign, the equation has atmost only one negative root.

so (i), (ii) can't be possible.

Hence there is only one negative root, one positive root & two imaginary roots.

Do yourself - 12 :

(i) Let α, β be two of the roots of the equation $x^3 - px^2 + qx - r = 0$. If $\alpha + \beta = 0$, then show that $pq = r$

(ii) If two roots of $x^3 + 3x^2 - 9x + c = 0$ are equal, then find the value of c.

(iii) If α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$, then find the value of

(a) $\sum \alpha^2$ (b) $\sum \frac{1}{\alpha}$ (c) $\sum \alpha^2(\beta+\gamma)$

19. TRANSFORMATION OF THE EQUATION :

Let $ax^2 + bx + c = 0$ be a quadratic equation with two roots α and β. If we have to find an equation whose roots are $f(\alpha)$ and $f(\beta)$, i.e. some expression in α & β, then this equation can be found by finding α in terms of y. Now as α satisfies given equation, put this α in terms of y directly in the equation.

$y = f(\alpha)$

By transformation, $\alpha = g(y)$

$a(g(y))^2 + b(g(y)) + c = 0$

This is the required equation in y.

Illustration 39 : If the roots of $ax^2 + bx + c = 0$ are α and β, then find the equation whose roots are :

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$ (b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (c) α^2, β^2

Solution : (a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$

put, $y = \frac{-2}{\alpha} \Rightarrow \alpha = \frac{-2}{y}$

$a\left(-\frac{2}{y}\right)^2 + b\left(\frac{-2}{y}\right) + c = 0 \quad \Rightarrow \quad cy^2 - 2by + 4a = 0$

Required equation is $cx^2 - 2bx + 4a = 0$

(b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$

put, $y = \frac{\alpha}{\alpha+1} \Rightarrow \alpha = \frac{y}{1-y}$

$\Rightarrow \quad a\left(\frac{y}{1-y}\right)^2 + b\left(\frac{y}{1-y}\right) + c = 0 \quad \Rightarrow \quad (a + c - b)y^2 + (-2c + b)y + c = 0$

Required equation is $(a + c - b)\, x^2 + (b - 2c)\, x + c = 0$

(c) α^2, β^2

put $y = \alpha^2 \Rightarrow \alpha = \sqrt{y}$

$ay + b\sqrt{y} + c = 0$

$b^2y = a^2y^2 + c^2 + 2acy$

$\Rightarrow \quad a^2y^2 + (2ac - b^2)\, y + c^2 = 0$

Required equation is $a^2x^2 + (2ac - b^2)\, x + c^2 = 0$

Illustration 40 : If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ then find equation whose roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$.

Solution : Put $y = \frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = -\frac{a\gamma}{d} \quad \left(\because \alpha\beta\gamma = -\frac{d}{a}\right)$

Put $x = -\dfrac{dy}{a}$

$$\Rightarrow \quad a\left(-\frac{dy}{a}\right)^3 + b\left(-\frac{dy}{a}\right)^2 + c\left(-\frac{dy}{a}\right) + d = 0$$

Required equation is $d^2x^3 - bdx^2 + acx - a^2 = 0$

Do yourself - 13 :

(i) If α, β are the roots of $ax^2 + bx + c = 0$, then find the equation whose roots are

(a) $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}$ (b) $\dfrac{1}{a\alpha+b}, \dfrac{1}{a\beta+b}$ (c) $\alpha + \dfrac{1}{\beta}, \quad \beta + \dfrac{1}{\alpha}$

(ii) If α, β are roots of $x^2 - px + q = 0$, then find the quadratic equation whose root are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$ and $\alpha^2\beta^3 + \alpha^3\beta^2$.

Miscellaneous Illustrations :

Illustrations 41 : A polynomial in x of degree greater than three, leaves remainders 2, 1 and –1 when divided, respectively, by $(x - 1)$, $(x + 2)$ and $(x + 1)$. What will be the remainder when it is divided by $(x - 1)(x + 2)(x + 1)$.

Solution : Let required polynomial be $f(x) = p(x)(x - 1)(x + 2)(x + 1) + a_0x^2 + a_1x + a_2$

By remainder theorem, $f(1) = 2$, $f(-2) = 1$, $f(-1) = -1$.

$$\Rightarrow \quad a_0 + a_1 + a_2 = 2$$
$$4a_0 - 2a_1 + a_2 = 1$$
$$a_0 - a_1 + a_2 = -1$$

Solving we get, $a_0 = \dfrac{7}{6}$, $a_1 = \dfrac{3}{2}$, $a_2 = \dfrac{2}{3}$

Remainder when $f(x)$ is divided by $(x - 1)(x + 2)(x + 1)$

will be $\dfrac{7}{6}x^2 + \dfrac{3}{2}x + \dfrac{2}{3}$.

Illustrations 42 : If α, β are the roots of $x^2 + px + q = 0$, and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equations have a common root.

Solution : α, β are the roots of $x^2 + px + q = 0$

$\therefore \quad \alpha + \beta = -p, \ \alpha\beta = q$ ....(1)

and γ, δ are the roots of $x^2 + rx + s = 0$

$\therefore \quad \gamma + \delta = -r, \ \gamma\delta = s$ ....(2)

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$

$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s(\alpha^2 + \beta^2) + sr(\alpha + \beta) + s^2$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s((\alpha + \beta)^2 - 2\alpha\beta)) + sr(\alpha + \beta) + s^2$

$= q^2 - pqr + r^2q + s(p^2 - 2q) + sr(-p) + s^2$

$= (q - s)^2 - rpq + r^2q + sp^2 - prs$

$= (q - s)^2 - rq(p - r) + sp(p - r)$

$= (q - s)^2 + (p - r)(sp - rq)$

For a common root (Let $\alpha = \gamma$ or $\beta = \delta$) ....(3)

then $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$ ....(4)

from (3) and (4), we get

$(q - s)^2 + (p - r)(sp - rq) = 0$

$\Rightarrow \quad (q - s)^2 = (p - r)(rq - sp)$, which is the required condition.

Illustrations 43 : If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then find the interval in which y lies.

Solution : $(y^2 - 5y + 3)(x^2 + x + 1) < 2x,\ \forall\ x \in R$

$\Rightarrow\ y^2 - 5y + 3 < \dfrac{2x}{x^2+x+1}$

Let $\dfrac{2x}{x^2+x+1} = P$

$\Rightarrow\ px^2 + (p - 2)x + p = 0$

(1) Since x is real, $(p - 2)^2 - 4p^2 \geq 0$

$\Rightarrow\ -2 \leq p \leq \dfrac{2}{3}$

(2) The minimum value of $2x/(x^2 + x + 1)$ is -2. So,

$y^2 - 5y + 3 < -2 \quad \Rightarrow \quad y^2 - 5y + 5 < 0$

$\Rightarrow\ y \in \left(\dfrac{5-\sqrt{5}}{2}, \dfrac{5+\sqrt{5}}{2}\right)$

ANSWERS FOR DO YOURSELF

1 : (i) (a) $-1, -2$; (b) 4 ; (c) $1 \pm \sqrt{2}$; (ii) $a, \dfrac{1}{a}$; (iii) $\dfrac{7}{3}$ (iv) $3, -\dfrac{1}{5}$

2 : (i) $b = -4, c = 1$; (ii) (a) imaginary; (b) real & distinct ; (c) real & coincident

3 : (i) (a) $c = 0$; (b) $c = 1$; (c) $b \rightarrow$ negative, $c \rightarrow$ negative

4 : (i) $b = \dfrac{9}{2}, c = 5$; (ii) $c = 0, 6$

5 : (i) (a) $x \in (-\infty, -3] \cup [2, \infty)$; (b) $x \in (-2, -1)$; (c) $\left(-\dfrac{1}{4}, \dfrac{1}{3}\right]$;

(d) $x \in (-6, -3) \cup \left(\dfrac{1}{2}, 2\right) - \{1\} \cup (9, \infty)$; (e) [3,7]; (f) ϕ

6 : (i) (a) $1, x = -1$; (b) $-1, x = 2$

(ii) (1) (i) $a < 0$ (ii) $b < 0$ (iii) $c < 0$ (iv) $D > 0$ (v) $\alpha + \beta < 0$ (vi) $\alpha\beta > 0$

(2) (i) $a < 0$ (ii) $b > 0$ (iii) $c = 0$ (iv) $D > 0$ (v) $\alpha + \beta > 0$ (vi) $\alpha\beta = 0$

(3) (i) $a < 0$ (ii) $b = 0$ (iii) $c = 0$ (iv) $D = 0$ (v) $\alpha + \beta = 0$ (vi) $\alpha\beta = 0$

(iii) Third quadrant

(iv) (a) $a > 9/16$ (b) $a < -2$

7 : (i) $x \in (-\infty, -2) \cup (1, 3/2)$ (ii) $x \in R - (0,1]$

8 : (i) (a) $x \in (1,8)$ (b) $x \in (-\infty, 1/2)$ (ii) $x \in (-1,1)$

9 : (ii) least value = 0, greatest value = 1.

10 : (i) $-3 < \alpha < 0 < \beta < 1$; (ii) C ; (iii) $a < 2$; (iv) $a < 2$; (v) $12 < \lambda < 16$

11 : (i) 0, 2

12 : (ii) $-27, 5$; (iii) (a) $\dfrac{1}{a^2}(b^2 - 2ac)$, (b) $-\dfrac{c}{d}$, (c) $\dfrac{1}{a^2}(3ad - bc)$

13 : (i) (a) $c^2y^2 + y(2ac - b^2) + a^2 = 0$; (b) $acx^2 - bx + 1 = 0$; (c) $acx^2 + (a + c)bx + (a + c)^2 = 0$

(ii) $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. The roots of the quadratic equation $(a + b - 2c)x^2 - (2a - b - c)x + (a - 2b + c) = 0$ are -

(A) $a + b + c$ & $a - b + c$ (B) $1/2$ & $a - 2b + c$

(C) $a - 2b + c$ & $1/(a + b - 2c)$ (D) none of these

2. If the A.M. of the roots of a quadratic equation is $\frac{8}{5}$ and A.M. of their reciprocals is $\frac{8}{7}$, then the quadratic equation is -

(A) $5x^2 - 8x + 7 = 0$ (B) $5x^2 - 16x + 7 = 0$ (C) $7x^2 - 16x + 5 = 0$ (D) $7x^2 + 16x + 5 = 0$

3. If $\sin\alpha$ & $\cos\alpha$ are the roots of the equation $ax^2 + bx + c = 0$ then -

(A) $a^2 - b^2 + 2ac = 0$ (B) $a^2 + b^2 + 2ac = 0$

(C) $a^2 - b^2 - 2ac = 0$ (D) $a^2 + b^2 - 2ac = 0$

4. If one root of the quadratic equation $px^2 + qx + r = 0$ $(p \neq 0)$ is a surd $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{a-b}}$, where p, q, r; a, b are all rationals then the other root is -

(A) $\frac{\sqrt{b}}{\sqrt{a}-\sqrt{a-b}}$ (B) $a + \frac{\sqrt{a(a-b)}}{b}$

(C) $\frac{a+\sqrt{a(a-b)}}{b}$ (D) $\frac{\sqrt{a}-\sqrt{a-b}}{\sqrt{b}}$

5. A quadratic equation with rational coefficients one of whose roots is $\tan\left(\frac{\pi}{12}\right)$ is -

(A) $x^2 - 2x + 1 = 0$ (B) $x^2 - 2x + 4 = 0$ (C) $x^2 - 4x + 1 = 0$ (D) $x^2 - 4x - 1 = 0$

6. $ax^2 + bx + c = 0$ has real and distinct roots α and $\beta(\beta > \alpha)$. Further $a > 0$, $b < 0$ and $c < 0$, then -

(A) $0 < \beta < |\alpha|$ (B) $0 < |\alpha| < \beta$ (C) $\alpha + \beta < 0$ (D) $|\alpha| + |\beta| = \left|\frac{b}{a}\right|$

7. If the roots of $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$ are equal then a, b, c are in

(A) A.P. (B) G.P. (C) H.P. (D) none of these

8. If $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ has equal root, then a, b, c are in

(A) A.P. (B) G.P. (C) H.P. (D) none of these

9. Let $p, q \in \{1, 2, 3, 4\}$. Then number of equation of the form $px^2 + qx + 1 = 0$, having real roots, is

(A) 15 (B) 9 (C) 7 (D) 8

10. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are imaginary then for all values of a, b, c and $x \in R$, the expression $a^2x^2 + abx + ac$ is -

(A) positive (B) non-negative

(C) negative (D) may be positive, zero or negative

11. If x, y are rational number such that $x + y + (x - 2y)\sqrt{2} = 2x - y + (x - y - 1)\sqrt{6}$, then

(A) x and y connot be determined (B) $x = 2, y = 1$

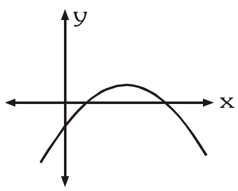
(C) $x = 5, y = 1$ (D) none of these

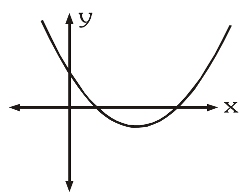
12. Graph of the function $f(x) = Ax^2 - BX + C$, where

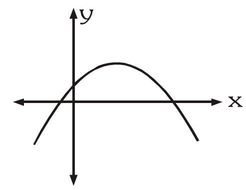
$A = (\sec\theta - \cos\theta)(\mathrm{cosec}\theta - \sin\theta)(\tan\theta + \cot\theta)$,

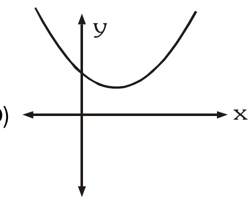
$B = (\sin\theta + \mathrm{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 - (\tan^2\theta + \cot^2\theta)$ &

$C = 12$, is represented by

(A) 

(B) 

(C) 

(D) 

13. The equation whose roots are the squares of the roots of the equation $ax^2 + bx + c = 0$ is -

(A) $a^2x^2 + b^2x + c^2 = 0$ (B) $a^2x^2 - (b^2 - 4ac)x + c^2 = 0$

(C) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$ (D) $a^2x^2 + (b^2 - ac)x + c^2 = 0$

14. If $\alpha \neq \beta$, $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then the equation whose roots are α/β & β/α, is

(A) $x^2 + 5x - 3 = 0$ (B) $3x^2 + 12x + 3 = 0$ (C) $3x^2 - 19x + 3 = 0$ (D) none of these

15. If α, β are the roots of the equation $x^2 - 3x + 1 = 0$, then the equation with roots $\dfrac{1}{\alpha-2}, \dfrac{1}{\beta-2}$ will be

(A) $x^2 - x - 1 = 0$ (B) $x^2 + x - 1 = 0$ (C) $x^2 + x + 2 = 0$ (D) none of these

16. If $x^2 - 11x + a$ and $x^2 - 14x + 2a$ have a common factor then 'a' is equal to

(A) 24 (B) 1 (C) 2 (D) 12

17. The smallest integer x for which the inequality $\dfrac{x-5}{x^2+5x-14} > 0$ is satisfied is given by -

(A) – 7 (B) – 5 (C) – 4 (D) – 6

18. The number of positive integral solutions of the inequation $\dfrac{x^2(3x-4)^3\ (x-2)^4}{(x-5)^5\ (2x-7)^6} \leq 0$ is -

(A) 2 (B) 0 (C) 3 (D) 4

19. The value of 'a' for which the sum of the squares of the roots of $2x^2 - 2(a - 2)x - a - 1 = 0$ is least is -

(A) 1 (B) 3/2 (C) 2 (D) – 1

20. If the roots of the quadratic equation $x^2 + 6x + b = 0$ are real and distinct and they differ by atmost 4 then the least value of b is -

(A) 5 (B) 6 (C) 7 (D) 8

21. The expression $\dfrac{x^2+2x+1}{x^2+2x+7}$ lies in the interval ; $(x \in R)$ -

(A) $[0, -1]$ (B) $(-\infty, 0] \cup [1, \infty)$ (C) $[0, 1)$ (D) none of these

22. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real & less than 3 then -

(A) $a < 2$ (B) $2 \leq a \leq 3$ (C) $3 < a \leq 4$ (D) $a > 4$

23. The number of integral values of m, for which the roots of $x^2 - 2mx + m^2 - 1 = 0$ will lie between – 2 and 4 is -

(A) 2 (B) 0 (C) 3 (D) 1

24. If the roots of the equation, $x^3 + Px^2 + Qx - 19 = 0$ are each one more than the roots of the equation, $x^3 - Ax^2 + Bx - C = 0$, where A, B, C, P & Q are constants then the value of A + B + C =

(A) 18 (B) 19 (C) 20 (D) none

25. If $\alpha, \beta, \gamma, \delta$ are roots of $x^4 - 100x^3 + 2x^2 + 4x + 10 = 0$, then $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}+\dfrac{1}{\delta}$ is equal to -

(A) $\dfrac{2}{5}$ (B) $\dfrac{1}{10}$ (C) 4 (D) $-\dfrac{2}{5}$

26. Number of real solutions of the equation $x^4 + 8x^2 + 16 = 4x^2 - 12x + 9$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 4

27. The complete solution set of the inequation $\sqrt{x+18} < 2-x$ is -

(A) $[-18, -2]$ (B) $(-\infty, -2) \cup (7, \infty)$ (C) $(-18, 2) \cup (7, \infty)$ (D) $[-18, -2)$

28. If $\log_{1/3}\dfrac{3x-1}{x+2}$ is less than unity then x must lie in the interval -

(A) $(-\infty, -2) \cup (5/8, \infty)$ (B) $(-2, 5/8)$

(C) $(-\infty, -2) \cup (1/3, 5/8)$ (D) $(-2, 1/3)$

29. Exhaustive set of value of x satisfying $\log_{|x|}(x^2 + x + 1) \geq 0$ is -

(A) $(-1, 0)$ (B) $(-\infty, 1) \cup (1, \infty)$

(C) $(-\infty, \infty) - \{-1, 0, 1\}$ (D) $(-\infty, -1) \cup (-1, 0) \cup (1, \infty)$

30. Solution set of the inequality, $2 - \log_2(x^2 + 3x) \geq 0$ is -

(A) $[-4, 1]$ (B) $[-4, -3) \cup (0, 1]$ (C) $(-\infty, -3) \cup (1, \infty)$ (D) $(-\infty, -4) \cup [1, \infty)$

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

31. If α is a root of the equation $2x(2x + 1) = 1$, then the other root is -

(A) $3\alpha^3 - 4\alpha$ (B) $-2\alpha(\alpha + 1)$ (C) $4\alpha^3 - 3\alpha$ (D) none of these

32. If $b^2 \geq 4ac$ for the equation $ax^4 + bx^2 + c = 0$, then all roots of the equation will be real if -

(A) $b > 0, a < 0, c > 0$ (B) $b < 0, a > 0, c > 0$

(C) $b > 0, a > 0, c > 0$ (D) $b > 0, a < 0, c < 0$

33. Let α, β be the roots of $x^2 - ax + b = 0$, where a & $b \in R$. If $\alpha + 3\beta = 0$, then -

(A) $3a^2 + 4b = 0$ (B) $3b^2 + 4a = 0$ (C) $b < 0$ (D) $a < 0$

34. For $x \in [1, 5]$, $y = x^2 - 5x + 3$ has -

(A) least value = -1.5 (B) greatest value = 3

(C) least value = -3.25 (D) greatest value = $\dfrac{5+\sqrt{13}}{2}$

35. Integral real values of x satisfying $\log_{1/2}(x^2 - 6x + 12) \geq -2$ is -

(A) 2 (B) 3 (C) 4 (D) 5

36. If $\dfrac{1}{2} \leq \log_{0.1} x \leq 2$, then -

(A) the maximum value of x is $\dfrac{1}{\sqrt{10}}$ (B) x lies between $\dfrac{1}{100}$ and $\dfrac{1}{\sqrt{10}}$

(C) x does not lie between $\dfrac{1}{100}$ and $\dfrac{1}{\sqrt{10}}$ (D) the minimum value of x is $\dfrac{1}{100}$

CHECK YOUR GRASP — ANSWER KEY — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	B	A	C	C	B	B	C	C	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	B	C	C	A	A	D	C	B	A
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	C	A	C	A	D	A	D	A	D	B
Que.	31	32	33	34	35	36				
Ans.	B,C	B,D	A,C	B,C	A,B,C	A,B,D				

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. The equation whose roots are $\sec^2\alpha$ & $\text{cosec}^2\alpha$ can be -

(A) $2x^2 - x - 1 = 0$ (B) $x^2 - 3x + 3 = 0$ (C) $x^2 - 9x + 9 = 0$ (D) $x^2 + 3x + 3 = 0$

2. If $\cos\alpha$ is a root of the equation $25x^2 + 5x - 12 = 0$, $-1 < x < 0$, then the value of $\sin 2\alpha$ is -

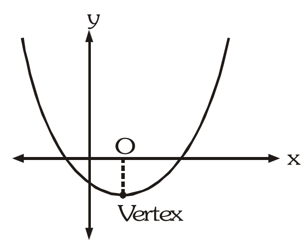
(A) 12/25 (B) $-12/25$ (C) $-24/25$ (D) 24 / 25

3. If the roots of the equation $\dfrac{1}{x+p} + \dfrac{1}{x+q} = \dfrac{1}{r}$ are equal in magnitude and opposite in sign, then -

(A) $p + q = r$ (B) $p + q = 2r$

(C) product of roots $= -\dfrac{1}{2}(p^2 + q^2)$ (D) sum of roots = 1

4. Graph of $y = ax^2 + bx + c = 0$ is given adjacently. What conclusions can be drawn from this graph -



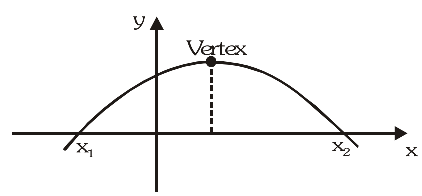
(A) $a > 0$ (B) $b < 0$

(C) $c < 0$ (D) $b^2 - 4ac > 0$

5. If a, b, c are real distinct numbers satisfying the condition $a + b + c = 0$ then the roots of the quadratic equation $3ax^2 + 5bx + 7c = 0$ are -

(A) positive (B) negative (C) real and distinct (D) imaginary

6. The adjoining figure shows the graph of $y = ax^2 + bx + c$. Then -



(A) $a > 0$ (B) $b > 0$

(C) $c > 0$ (D) $b^2 < 4ac$

7. If $x^2 + Px + 1$ is a factor of the expression $ax^3 + bx + c$ then -

(A) $a^2 + c^2 = -ab$ (B) $a^2 - c^2 = -ab$ (C) $a^2 - c^2 = ab$ (D) none of these

8. The set of values of 'a' for which the inequality $(x - 3a)(x - a - 3) < 0$ is satisfied for all x in the interval $1 \le x \le 3$

(A) (1/3, 3) (B) (0, 1/3) (C) (–2, 0) (D) (–2, 3)

9. Let p(x) be the cubic polynomial $7x^3 - 4x^2 + K$. Suppose the three roots of p(x) form an arithmetic progression. Then the value of K, is -

(A) $\dfrac{4}{21}$ (B) $\dfrac{16}{147}$ (C) $\dfrac{16}{441}$ (D) $\dfrac{128}{1323}$

10. If the quadratic equation $ax^2 + bx + 6 = 0$ does not have two distinct real roots, then the least value of $2a + b$ is -

(A) 2 (B) -3 (C) -6 (D) 1

11. If p & q are distinct reals, then $2\{(x - p)(x - q) + (p - x)(p - q) + (q - x)(q - p)\} = (p - q)^2 + (x - p)^2 + (x - q)^2$ is satisfied by -

(A) no value of x (B) exactly one value of x (C) exactly two values of x (D) infinite values of x

12. The value of 'a' for which the expression $y = x^2 + 2a\sqrt{a^2 - 3}\,x + 4$ is perfect square, is -

(A) 4 (B) $\pm\sqrt{3}$

(C) ± 2 (D) $a \in (-\infty, -\sqrt{3}\,] \cup [\,\sqrt{3}, \infty)$

13. Set of values of 'K' for which roots of the quadratic $x^2 - (2K - 1)x + K(K - 1) = 0$ are -

(A) both less than 2 is $K \in (2, \infty)$
(B) of opposite sign is $K \in (-\infty, 0) \cup (1, \infty)$
(C) of same sign is $K \in (-\infty, 0) \cup (1, \infty)$
(D) both greater than 2 is $K \in (2, \infty)$

14. The correct statement is / are -

(A) If x_1 & x_2 are roots of the equation $2x^2 - 6x - b = 0$ $(b > 0)$, then $\frac{x_1}{x_2} + \frac{x_2}{x_1} < -2$

(B) Equation $ax^2 + bx + c = 0$ has real roots if $a < 0$, $c > 0$ and $b \in R$

(C) If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$, where $ac \neq 0$ and $a, b, c \in R$, then $P(x).Q(x)$ has at least two real roots.

(D) If both the roots of the equation $(3a + 1)x^2 - (2a + 3b)x + 3 = 0$ are infinite then $a = 0$ & $b \in R$

15. If $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_6$, then the equation $(x - \alpha_1)(x - \alpha_3)(x - \alpha_5) + 3(x - \alpha_2)(x - \alpha_4)(x - \alpha_6) = 0$ has -

(A) three real roots
(B) no real root in $(-\infty, \alpha_1)$
(C) one real root in (α_1, α_2)
(D) no real root in (α_5, α_6)

16. Equation $2x^2 - 2(2a + 1)x + a(a + 1) = 0$ has one root less than 'a' and other root greater than 'a', if

(A) $0 < a < 1$ (B) $-1 < a < 0$ (C) $a > 0$ (D) $a < -1$

17. The value(s) of 'b' for which the equation, $2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2)$ has coincident roots, is/are -

(A) $b = -12$ (B) $b = 4$ (C) $b = 4$ or $b = -12$ (D) $b = -4$ or $b = 12$

18. For every $x \in R$, the polynomial $x^8 - x^5 + x^2 - x + 1$ is -

(A) positive
(B) never positive
(C) positive as well as negative
(D) negative

19. If α, β are the roots of the quadratic equation $(p^2 + p + 1)x^2 + (p - 1)x + p^2 = 0$ such that unity lies between the roots then the set of values of p is -

(A) ϕ (B) $p \in (-\infty, -1) \cup (0, \infty)$ (C) $p \in (-1, 0)$ (D) $(-1, 1)$

20. Three roots of the equation, $x^4 - px^3 + qx^2 - rx + s = 0$ are tanA, tanB & tanC where A, B, C are the angles of a triangle. The fourth root of the biquadratic is -

(A) $\frac{p-r}{1-q+s}$ (B) $\frac{p-r}{1+q-s}$ (C) $\frac{p+r}{1-q+s}$ (D) $\frac{p+r}{1+q-s}$

21. If $\log_{\left(\frac{x^2-12x+30}{10}\right)}\left(\log_2 \frac{2x}{5}\right) > 0$ then x belongs to interval -

(A) $\left(\frac{5}{2}, 6+\sqrt{6}\right)$ (B) $\left(\frac{5}{2}, 6-\sqrt{6}\right)$ (C) $(6, 6+\sqrt{6})$ (D) $(10, \infty)$

BRAIN TEASERS — **ANSWER KEY** — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C,D	B,C	A,B,C,D	C	B,C	C	B	D	B
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	C	C	A,B,C	A,B,C	A,C,D	B	A	C	A
Que.	21									
Ans.	B,D									

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

TRUE / FALSE

1. If a, b, c $\in$ Q, then roots of $ax^2 + 2(a + b)x - (3a + 2b) = 0$ are rational.

2. The necessary and sufficient condition for which a fixed number 'd' lies between the roots of quadratic equation $f(x) = ax^2 + bx + c = 0$; (a, b, c $\in$ R), is $f(d) < 0$.

3. If $0 < p < \pi$ then the quadratic equation, $(\cos p - 1)x^2 + x\cos p + \sin p = 0$ has real roots.

4. The necessary and sufficient condition for the quadratic function $f(x) = ax^2 + bx + c$, to take both positive and negative values is, $b^2 > 4ac$, where a, b, c $\in$ R & $a \neq 0$.

FILL IN THE BLANKS

1. If $a + b + c = 0$ & $a^2 + b^2 + c^2 = 1$ then the value of $a^4 + b^4 + c^4$ is ....

2. If $x^2 - 4x + 5 - \sin y = 0$, $y \in (0, 2\pi)$ then x = ... & y = ...

3. If α, β be the roots of the equation $ax^2 + bx + c = 0$ then the value of $\frac{a\alpha^2}{b\alpha + c} + \frac{a\beta^2}{b\beta + c}$ is equal to ....

MATCH THE COLUMN

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE OR MORE** statement(s) in **Column-II**.

1. Consider the equation $x^2 + 2(a - 1)x + a + 5 = 0$, where 'a' is a parameter. Match of the real values of 'a' so that the given equation has

	Column-I		Column-II
(A)	imaginary roots	(p)	$\left(-\infty, -\frac{8}{7}\right)$
(B)	one root smaller than 3 and other root greater than 3	(q)	$(-1, 4)$
(C)	exactly one root in the interval (1, 3) & 1 and 3 are not the root of the equation	(r)	$\left(-\frac{4}{3}, -\frac{8}{7}\right)$
(D)	one root smaller than 1 and other root greater than 3	(s)	$\left(-\infty, -\frac{4}{3}\right)$

ASSERTION & REASON

These questions contains, Statement-I (assertion) and Statement-II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I

(C) Statement-I is true, Statement-II is false

(D) Statement-I is false, Statement-II is true

1. **Statement-I** : If equation $ax^2 + bx + c = 0$; (a, b, c $\in$ R) and $2x^2 + 3x + 4 = 0$ have a common root, then a : b : c = 2 : 3 : 4.

Because

Statement-II : If p + iq is one root of a quadratic equation with real coefficients then p − iq will be the other root ; p, q $\in$ R, $i = \sqrt{-1}$

(A) A (B) B (C) C (D) D

2. **Statement-I** : If f(x) is a quadratic expression such that f(1) + f(2) = 0. If –1 is a root of f(x) = 0 then the other root is $\frac{8}{5}$.

Because

Statement-II : If $f(x) = ax^2 + bx + c$ then sum of roots $= -\frac{b}{a}$ and product of roots $= \frac{c}{a}$

(A) A (B) B (C) C (D) D

3. **Statement-I** : If a + b + c > 0 and a < 0 < b < c, then the roots of the equation a(x – b) (x – c) + b(x – c)(x – a) + c(x – a) (x – b) = 0 are of both negative.

Because

Statement-II : If both roots are negative, then sum of roots < 0 and product of roots > 0

(A) A (B) B (C) C (D) D

4. **Statement-I** : Let $(a_1, a_2, a_3, a_4, a_5)$ denote a re-arrangement of (1, –4, 6, 7, –10). Then the equation $a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0$ has at least two real roots.

Because

Statement-II : If $ax^2 + bx + c = 0$ and a + b + c = 0, (i.e. in a polynomial the sum of coefficients is zero) then x = 1 is root of $ax^2 + bx + c = 0$.

(A) A (B) B (C) C (D) D

5. **Statement-I** : If roots of the equation $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c = 1$.

Because

Statement-II : If a, b, c are odd integer then the roots of the equation $4\,abc\,x^2 + (b^2 - 4ac)x - b = 0$ are real and distinct.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

If α, β, γ be the roots of the equation $ax^3 + bx^2 + cx + d = 0$. To obtain the equation whose roots are f(α), f(β), f(γ), where f is a function, we put y = f(α) and simplify it to obtain α = g(y) (some function of y). Now, α is a root of the equation $ax^3 + bx^2 + cx + d = 0$, then we obtain the desired equation which is $a\{g(y)\}^3 + b\{g(y)\}^2 + c\{g(y)\} + d = 0$

For example, if α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$. To find equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ we put $y = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{y}$

As α is a root of $ax^3 + bx^2 + cx + d = 0$

we get $\frac{a}{y^3} + \frac{b}{y^2} + \frac{c}{y} + d = 0 \Rightarrow dy^3 + cy^2 + by + a = 0$

This is desired equation.

On the basis of above information, answer the following questions :

1. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then the roots of the equation $a(2x + 1)^2 + b(2x + 1)(x - 1) + c(x - 1)^2 = 0$ are-

(A) $\frac{2\alpha+1}{\alpha-1}, \frac{2\beta+1}{\beta-1}$ (B) $\frac{2\alpha-1}{\alpha+1}, \frac{2\beta-1}{\beta+1}$ (C) $\frac{\alpha+1}{\alpha-2}, \frac{\beta+1}{\beta-2}$ (D) $\frac{2\alpha+3}{\alpha-1}, \frac{2\beta+3}{\beta-1}$

2. If α, β are the roots of the equation $2x^2 + 4x - 5 = 0$, the equation whose roots are the reciprocals of 2α – 3 and 2β – 3 is -

(A) $x^2 + 10x - 11 = 0$ (B) $11x^2 + 10x + 1 = 0$

(C) $x^2 + 10x + 11 = 0$ (D) $11x^2 - 10x + 1 = 0$

3. If α, β are the roots of the equation $px^2 - qx + r = 0$, then the equation whose roots are $\alpha^2 + \frac{r}{p}$ and $\beta^2 + \frac{r}{p}$ is-

(A) $p^3x^2 + pq^2x + r = 0$ (B) $px^2 - qx + r = 0$
(C) $p^3x^2 - pq^2x + q^2r = 0$ (D) $px^2 + qx - r = 0$

4. If α, β, γ are the roots of the equation $x^3 - x - 1 = 0$, then the value of $\Pi\left(\frac{1+\alpha}{1-\alpha}\right)$ is equal to -

(A) -7 (B) -5 (C) -3 (D) -1

Comprehension # 2

Let $(a + \sqrt{b})^{Q(x)} + (a - \sqrt{b})^{Q(x) - 2\lambda} = A$, where $\lambda \in N$, $A \in R$ and $a^2 - b = 1$

$\therefore$ $(a + \sqrt{b})(a - \sqrt{b}) = 1 \Rightarrow (a + \sqrt{b}) = (a - \sqrt{b})^{-1}$ and $(a - \sqrt{b}) = (a + \sqrt{b})^{-1}$

ie, $(a \pm \sqrt{b}) = \left(a+\sqrt{b}\right)^{\pm 1}$ or $\left(a-\sqrt{b}\right)^{\mp 1}$

By substituting $(a+\sqrt{b})^{Q(x)}$ as t in the equation we get a quadratic in t.

Also $a + ar + ar^2 \ldots\ldots\ldots\ \infty = \frac{a}{1-r}$ where $-1 < r < 1$

On the basis of above information, answer the following questions :

1. Solution of $(2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \frac{4}{2-\sqrt{3}}$ are-

(A) $1 \pm \sqrt{3}$, 1 (B) $1 \pm \sqrt{2}$, 1 (C) $1 \pm \sqrt{3}$, 2 (D) $1 \pm \sqrt{2}$, 2

2. The number of real solutions of the equation $(15 + 4\sqrt{14})^t + (15 - 4\sqrt{14})^t = 30$ are - where $t = x^2 - 2|x|$

(A)0 (B) 2 (C) 4 (D) 6

3. If $\left(\sqrt{(49+20\sqrt{6})}\right)^{\sqrt{a\sqrt{a\sqrt{a\ldots\ldots\infty}}}} + (5-2\sqrt{6})^{x^2+x-3-\sqrt{x\sqrt{x\sqrt{x\ldots\ldots\infty}}}} = 10$ where $a = x^2 - 3$, then x is -

(A) $-\sqrt{2}$ (B) $\sqrt{2}$ (C) -2 (D) 2

MISCELLANEOUS TYPE QUESTION — **ANSWER KEY** — EXERCISE -3

- *True / False*

  1. T 2. F 3. T 4. T

- *Fill in the Blanks*

  1. 1/2 2. $x = 2$ & $y = \pi/2$ 3. -2

- *Match the Column*

  1. (A) $\to$ (q), (B) $\to$ (p, r, s), (C) $\to$ (r), (D) $\to$ (s)

- *Assertion & Reason*

  1. A 2. A 3. D 4. A 5. B

- *Comprehension Based Questions*

  Comprehension # 1 : 1. C 2. B 3. C 4. D

  Comprehension # 2 : 1. B 2. C 3. D

EXERCISE - 04 [A]

CONCEPTUAL SUBJECTIVE EXERCISE

1. If α, β are the roots of the equation $x^2 - 2x + 3 = 0$ obtain the equation whose roots are $\alpha^3 - 3\alpha^2 + 5\alpha - 2$, $\beta^3 - \beta^2 + \beta + 5$.

2. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, prove that $b^3 + a^2c + ac^2 = 3abc$.

3. Show that if p, q , r & s are real numbers & $pr = 2(q + s)$, then at least one of the equations $x^2 + px + q = 0$, $x^2 + rx + s = 0$ has real roots.

4. Let a, b, c, d be distinct real numbers and a and b are the roots of quadratic equation $x^2 - 2cx - 5d = 0$. If c and d are the roots of the quadratic equation $x^2 - 2ax - 5b = 0$ then find the numerical values of $a + b + c + d$.

5. Find the product of the real roots of the equation, $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$

6. Find the range of values of a, such that $f(x) = \dfrac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32}$ is always negative.

7. Find the values of 'a' for which $-3 < \dfrac{x^2 + ax - 2}{x^2 + x + 1} < 2$ is valid for all real x .

8. If the quadratic equations $x^2 + bx + ca = 0$ & $x^2 + cx + ab = 0$ have a common root, prove that the equation containing their other roots is $x^2 + ax + bc = 0$.

9. The equation $x^2 - ax + b = 0$ & $x^3 - px^2 + qx = 0$, where $b \neq 0$, $q \neq 0$, have one common root & the second equation has two equal roots. Prove that $2(q + b) = ap$.

Find the solutions of following inequations : (10 to 14)

10. $\dfrac{(x-2)(x-4)(x-7)}{(x+2)(x+4)(x+7)} > 1$

11. $(x^2 - x - 1)(x^2 - x - 7) < -5$.

12. $(x^2 - 2x)(2x - 2) - 9\dfrac{2x-2}{x^2 - 2x} \leq 0$

13. $\dfrac{1}{x-2} + \dfrac{1}{x-1} > \dfrac{1}{x}$

14. $\dfrac{20}{(x-3)(x-4)} + \dfrac{10}{x-4} + 1 > 0$

Find the solutions of following miscellaneous inequations : (15 to 20)

15. $|x^2 - 2x - 3| < |x^2 - x + 5|$

16. $x - 3 < \sqrt{x^2 + 4x - 5}$

17. $\log_{\frac{5}{8}}\left(2x^2 - x - \dfrac{3}{8}\right) \geq 1$

18. $\left(\dfrac{3}{4}\right)^{6x+10-x^2} < \dfrac{27}{64}$

19. $\log_{1/2}(x + 1) > \log_2(2 - x)$.

20. $\log_x 2 \cdot \log_{2x} 2 \cdot \log_2 4x > 1$.

21. Find all values of a for which the inequality $(a + 4)x^2 - 2ax + 2a - 6 < 0$ is satisfied for all $x \in R$.

22. Find all values of a for which both roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ are greater than 3.

23. Find all the values of the parameter 'a' for which both roots of the quadratic equation $x^2 - ax + 2 = 0$ belong to the interval (0, 3).

24. Find the values of K so that the quadratic equation $x^2 + 2(K - 1)x + K + 5 = 0$ has atleast one positive root.

25. If $a < b < c < d$ then prove that the roots of the equation ; $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are real & distinct.

26. Two roots of a biquadratic $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ have their product equal to (–32). Find the value of k.

CONCEPTUAL SUBJECTIVE EXERCISE — **ANSWER KEY** — EXERCISE-4(A)

1. $x^2 - 3x + 2 = 0$ **4.** 30 **5.** 20 **6.** $a \in \left(-\infty, -\frac{1}{2}\right)$

7. $-2 < a < 1$ **10.** $(-\infty, -7) \cup (-4, -2)$ **11.** $(-2, -1) \cup (2, 3)$ **12.** $(-\infty, -1] \cup (0, 1] \cup (2, 3]$

13. $\left(-\sqrt{2}, 0\right) \cup \left(1, \sqrt{2}\right) \cup (2, +\infty)$ **14.** $(-\infty, -2) \cup (-1, 3) \cup (4, +\infty)$ **15.** $(-8, \infty)$

16. $(-\infty, -5] \cup [1, \infty)$ **17.** $\left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right]$ **18.** 7

19. $-1 < x < \frac{1-\sqrt{5}}{2}$ or $\frac{1+\sqrt{5}}{2} < x < 2$ **20.** $2^{-\sqrt{2}} < x < 2^{-1}$; $1 < x < 2^{\sqrt{2}}$

21. For all $a \in (-\infty, -6)$ **22.** For all $a \in (11/9, +\infty)$ **23.** $2\sqrt{2} \le a < \frac{11}{3}$

24. $K \le -1$ **26.** $k = 86$

EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the n^{th} power of the other, then show that $(ac^n)^{1/(n+1)} + (a^n c)^{1/(n+1)} + b = 0$.

2. Let $P(x) = x^2 + bx + c$, where b and c are integer. If P(x) is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, find the value of P(1).

3. Find the true set of values of p for which the equation : $p.2^{\cos^2 x} + p.2^{-\cos^2 x} - 2 = 0$ has real roots.

4. If the coefficients of the quadratic equation $ax^2 + bx + c = 0$ are odd integers then prove that the roots of the equation cannot be rational number.

5. If the three equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common positive root, find a and b and the roots of the equations.

6. If the quadratic equation $ax^2 + bx + c = 0$ has real roots, of opposite sign in the interval (–2,2) then prove that

$$1 + \frac{c}{4a} - \left|\frac{b}{2a}\right| > 0 .$$

7. Show that the function $z = 2x^2 + 2xy + y^2 - 2x + 2y + 2$ is not smaller than – 3.

8. For $a \le 0$, determine all real roots of the equation $x^2 - 2a\,|x - a| - 3a^2 = 0$.

9. The equation $x^n + px^2 + qx + r = 0$, where $n \ge 5$ & $r \ne 0$ has roots $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots\ldots \alpha_n$.

   Denoting $\sum\limits_{i=1}^{n} \alpha_i^k$ by S_k.

   (a) Calculate S_2 & deduce that the roots cannot all be real.

   (b) Prove that $S_n + pS_2 + qS_1 + nr = 0$ & hence find the value of S_n.

10. Find the values of 'b' for which the equation $2\log_{\frac{1}{25}}(bx + 28) = -\log_5(12 - 4x - x^2)$ has only one solution.

11. Solve the inequality : $\log_3 \dfrac{|x^2 - 4x| + 3}{x^2 + |x - 5|} \ge 0$

BRAIN STORMING SUBJECTIVE EXERCISE	ANSWER KEY	EXERCISE-4(B)

2. P(1) = 4 **3.** [4/5, 1] **5.** a = – 7, b = – 8 ; (3, 4) ; (3 , 5) and (3 , 12)

8. $x = (1 - \sqrt{2})\,a$ or $(\sqrt{6} - 1)\,a$ **9.** (a) $S_2 = 0$, (b) $S_n = -nr$ **10.** $(-\infty, -14) \cup \{4\} \cup \left[\frac{14}{3}, \infty\right)$

11. $x \le \frac{-2}{3}$, $\frac{1}{2} \le x \le 2$

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. If the roots of the equation $x^2 - 5x + 16 = 0$ are α, β and the roots of the equation $x^2 + px + q = 0$ are $(\alpha^2 + \beta^2)$ and $\frac{\alpha\beta}{2}$, then- [AIEEE-2002]

(1) p = 1 and q = 56 (2) p = 1 and q = –56
(3) p = –1 and q = 56 (4) p = –1 and q = –56

2. If α and β be the roots of the equation $(x - a)(x - b) = c$ and $c \neq 0$, then roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are - [AIEEE-2002]

(1) a and c (2) b and c (3) a and b (4) a + b and b + c

3. If $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$ then the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (where $\alpha \neq \beta$) is- [AIEEE-2002]

(1) 19/3 (2) 25/3 (3) –19/3 (4) none of these

4. The value of a for which one roots of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other is [AIEEE-2003]

(1) – 2/3 (2) 1/3 (3) – 1/3 (4) 2/3

5, If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the square of their reciprocals, then $\frac{a}{c}, \frac{b}{a}$ and $\frac{c}{b}$ are in [AIEEE-2003]

(1) geometric progression (2) harmonic progression
(3) arithmetic-geometric progression (4) arithmetic progression

6. The number of real solutions of the equation $x^2 - 3|x| + 2 = 0$, is- [AIEEE-2003]

(1) 4 (2) 1 (3) 3 (4) 2

7. The real number x when added to its inverse gives the minimum value of the sum at x equal to- [AIEEE-2003]

(1) 1 (2) –1 (3) –2 (4) 2

8. Let two numbers have arithmetic mean 9 and geometric mean 4. Then these numbers are the roots of the quadratic equation- [AIEEE-2004]

(1) $x^2 + 18x - 16 = 0$ (2) $x^2 - 18x + 16 = 0$
(3) $x^2 + 18x + 16 = 0$ (4) $x^2 - 18x - 16 = 0$

9. If $(1 - p)$ is a root of quadratic equation $x^2 + px + (1 - p) = 0$ then its roots are- [AIEEE-2004]

(1) 0, – 1 (2) – 1, 1 (3) 0, 1 (4) – 1, 2

10. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then the value of 'q' is- [AIEEE-2004]

(1) 3 (2) 12 (3) 49/4 (4) 4

11. If value of a for which the sum of the squares of the roots of the equation $x^2 - (a - 2)x - a - 1 = 0$ assume the least value is- [AIEEE-2005]

(1) 2 (2) 3 (3) 0 (4) 1

12. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then $b^2 - 4c$ equals- [AIEEE-2005]

(1) 1 (2) 2 (3) 3 (4) –2

13. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval- [AIEEE-2005]

(1) [4, 5] (2) $(-\infty, 4)$ (3) $(6, \infty)$ (4) (5, 6)

14. If the equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x = 0$, $a_1 \neq 0$, $n \geq 2$, has a positive root $x = \alpha$, then the equation $na_n x^{n-1} + (n - 1) a_{n-1} x^{n-2} + \ldots + a_1 = 0$ has a positive root, which is- [AIEEE-2005]

(1) equal to α (2) greater than or equal to α
(3) smaller than α (4) greater than α

15. All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4, lie in the interval- **[AIEEE-2006]**

(1) $-1 < m < 3$ (2) $1 < m < 4$ (3) $-2 < m < 0$ (4) $m > 3$

16. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30 and tan 15 , respectively then the value of $2 + q - p$ is- **[AIEEE-2006]**

(1) 0 (2) 1 (3) 2 (4) 3

17. If x is real, then maximum value of $\dfrac{3x^2+9x+17}{3x^2+9x+7}$ is- **[AIEEE-2006]**

(1) 1 (2) $\dfrac{17}{7}$ (3) $\dfrac{1}{4}$ (4) 41

18. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is **[AIEEE-2007]**

(1) $(-3, \infty)$ (2) $(3, \infty)$

(3) $(-\infty, -3)$ (4) $(-3, -2) \cup (2, 3)$

19. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4:3. Then the common root is **[AIEEE-2008]**

(1) 1 (2) 4 (3) 3 (4) 2

20. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is :- **[AIEEE-2009]**

(1) Greater than $-4ab$ (2) Less than $-4ab$

(3) Greater than $4ab$ (4) Less than $4ab$

21. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009} =$ **[AIEEE-2010]**

(1) -2 (2) -1 (3) 1 (4) 2

22. Let for $a \neq a_1 \neq 0$, $f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and $p(x)=f(x) - g(x)$. If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then the value of $p(2)$ is: **[AIEEE-2011]**

(1) 18 (2) 3 (3) 9 (4) 6

23. Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are: **[AIEEE-2011]**

(1) $-4, -3$ (2) 6, 1 (3) 4, 3 (4) $-6, -1$

24. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has : **[AIEEE-2012]**

(1) exactly four real roots. (2) infinite number of real roots.

(3) no real roots. (4) exactly one real root.

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	4	3	1	4	2	1	1	2	1	3
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	4	1	2	3	1	4	4	4	4	1
Que.	21	22	23	24						
Ans.	3	1	2	3						

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β.

[JEE 2001, Mains, 5 out of 100]

2. The set of all real numbers x for which $x^2 - |x + 2| + x > 0$, is

(A) $(-\infty, -2) \cup (2, \infty)$ (B) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(C) $(-\infty, -1) \cup (1, \infty)$ (D) $(\sqrt{2}, \infty)$ [JEE 2002 (screening), 3]

3. If $x^2 + (a - b)x + (1 - a - b) = 0$ where $a, b \in R$ then find the values of 'a' for which equation has unequal real roots for all values of 'b'. [JEE 2003, Mains-4 out of 60]

4. (a) If one root of the equation $x^2 + px + q = 0$ is the square of the other, then

(A) $p^3 + q^2 - q(3p + 1) = 0$ (B) $p^3 + q^2 + q(1 + 3p) = 0$

(C) $p^3 + q^2 + q(3p - 1) = 0$ (D) $p^3 + q^2 + q(1 - 3p) = 0$

(b) If $x^2 + 2ax + 10 - 3a > 0$ for all $x \in R$, then

(A) $-5 < a < 2$ (B) $a < -5$ (C) $a > 5$ (D) $2 < a < 5$

[JEE 2004 (Screening)]

5. Find the range of values of t for which $2 \sin t = \dfrac{1 - 2x + 5x^2}{3x^2 - 2x - 1}$, $t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

[JEE 2005(Mains), 2]

6. (a) Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then

(A) $\lambda < \dfrac{4}{3}$ (B) $\lambda > \dfrac{5}{3}$ (C) $\lambda \in \left(\dfrac{1}{3}, \dfrac{5}{3}\right)$ (D) $\lambda \in \left(\dfrac{4}{3}, \dfrac{5}{3}\right)$

[JEE 2006, 3]

(b) If roots of the equation $x^2 - 10cx - 11d = 0$ are a, b and those of $x^2 - 10ax - 11b = 0$ are c, d, then find the value of $a + b + c + d$. (a, b, c and d are distinct numbers) [JEE 2006, 6]

7. (a) Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2$, 2β be the roots of the equation $x^2 - qx + r = 0$. Then the value of 'r' is

(A) $\dfrac{2}{9}(p-q)(2q - p)$ (B) $\dfrac{2}{9}(q - p)(2p - q)$

(C) $\dfrac{2}{9}(q - 2p)(2q - p)$ (D) $\dfrac{2}{9}(2p-q)(2q - p)$

MATCH THE COLUMN :

(b) Let $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6}$

Match the expressions / statements in **Column I** with expressions / statements in **Column II**.

Column I		Column II	
(A)	If $-1 < x < 1$, then $f(x)$ satisfies	(P)	$0 < f(x) < 1$
(B)	If $1 < x < 2$, the $f(x)$ satisfies	(Q)	$f(x) < 0$
(C)	If $3 < x < 5$, then $f(x)$ satisfies	(R)	$f(x) > 0$
(D)	If $x > 5$, then $f(x)$ satisfies	(S)	$f(x) < 1$

[JEE 2007, 3+6]

ASSERTION & REASON :

8. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and α, $1/\beta$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$

STATEMENT-1 : $(p^2 - q)(b^2 - ac) \geq 0$

and

STATEMENT-2 : $b \neq pa$ or $c \neq qa$

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is **NOT** a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True [JEE 2008, 3 (–1)]

9. The smallest value of k, for which both the roots of the equation, $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is [JEE 2009, 4 (–1)]

10. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is [JEE 2010, 3]

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$ (B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$ (D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

11. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is [JEE 2011]

(A) 1 (B) 2 (C) 3 (D) 4

12. A value of b for which the equations

$$x^2 + bx - 1 = 0$$
$$x^2 + x + b = 0,$$

have one root in common is - [JEE 2011]

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [B]

1. $\gamma = \alpha^2\beta$ and $\delta = \alpha\beta^2$ or $\gamma = \alpha\beta^2$ and $\delta = \alpha^2\beta$ 2. B 3. $a > 1$

4. (a) D ; (b) A 5. $\left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$ 6. (a) A; (b) 1210

7. (a) D; (b) (A) P, R, S; (B) Q, S; (C) Q, S; (D) P, R, S 8. B 9. 2

10. B 11. C 12. B