

Malkus Waterwheel

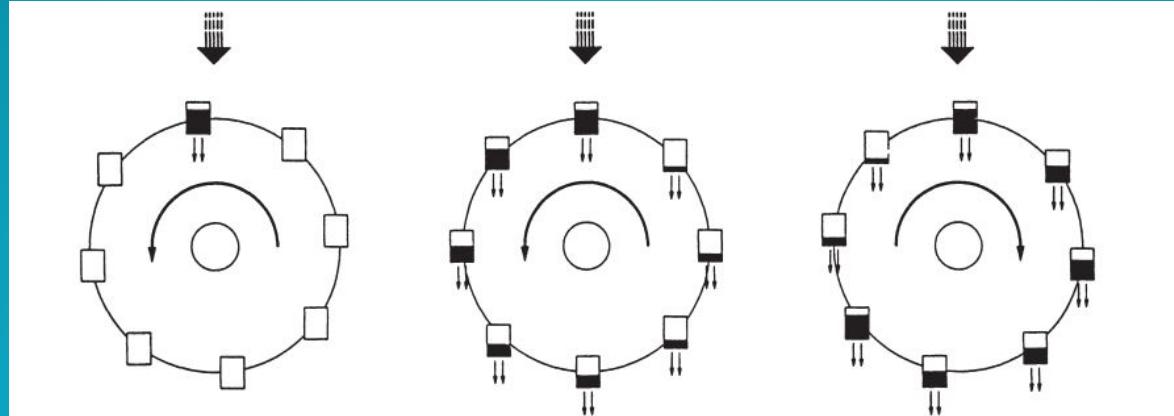
What is it?

PH 567 : Presentation

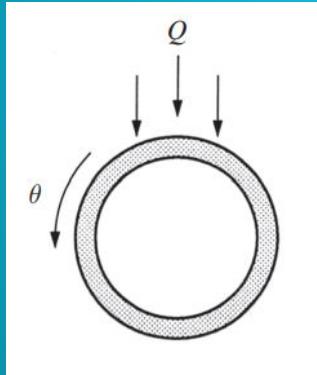


Chaotic/Lorenz Waterwheel

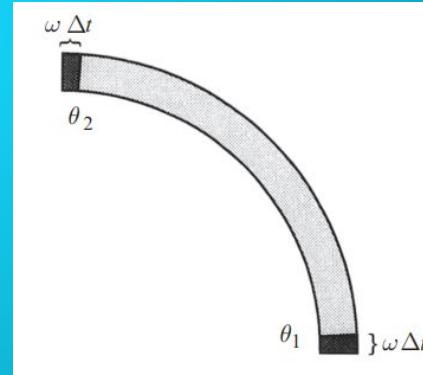
Img by : Harvard Natural Sciences Lecture Demonstration(2020, May 14) Chaotic Waterwheel [Video]. Youtube. URL <https://www.youtube.com/watch?app=desktop&v=Lx8gMBjBlP8>



The motion of waterwheel after onset of tap inflow



Side view of the theoretically modelled wheel



Section of the wheel being analysed

Image credits : Strogatz, S. H. Nonlinear Dynamics and Chaos: With Applications To Physics, Biology, Chemistry, And Engineering

MASS CONSERVATION

$$\Delta M = \Delta t \left[\int_{\theta_1}^{\theta_2} Q d\theta - \int_{\theta_1}^{\theta_2} Km d\theta \right] + m(\theta_1) \omega \Delta t - m(\theta_2) \omega \Delta t.$$

$$\frac{\partial m}{\partial t} = Q - Km \rightarrow \omega \frac{\partial m}{\partial \theta}$$

Rate of mass change in a cup

Mass inflow by the tap

Water leaked due to hole at bottom

Water spilled to neighbouring cups



Cause of Nonlinearity

MOMENT BALANCE

Integro-Differential
Equation



$$\frac{d[I\omega]}{dt} = -\nu\omega + gR \int_0^{2\pi} m(\theta, t) \sin \theta d\theta$$

Net torque on system

Damping due to axle friction

Torque due to gravitational force

HOW TO SOLVE THESE EQUATIONS?



FOURIER SERIES !

Writing the mass and inflow rate as sum of fourier components a, b and q respectively

$$m(\theta, t) = \sum_{n=0}^{\infty} [a_n(t) \sin n\theta + b_n(t) \cos n\theta].$$

$$Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta.$$

Why no $\sin(\theta)$ term ????

The water is being poured symmetrically from top
(Unlike asymmetric flow which drive the rotation in one side only)

Putting the fourier series in the mass conservation equation,

$$\frac{\partial}{\partial t} \left[\sum_{n=0}^{\infty} a_n(t) \sin n\theta + b_n(t) \cos n\theta \right] = -\omega \frac{\partial}{\partial \theta} \left[\sum_{n=0}^{\infty} a_n(t) \sin n\theta + b_n(t) \cos n\theta \right] \\ - K \left[\sum_{n=0}^{\infty} a_n(t) \sin n\theta + b_n(t) \cos n\theta \right] + \sum_{n=0}^{\infty} q_n \cos n\theta$$

Equating the coefficients of sine terms,

$$\dot{a}_n = n\omega b_n - K a_n.$$

Equating the coefficients of cosine terms,

$$\dot{b}_n = -n\omega a_n - K b_n + q_n.$$

Why only sine and cosine terms?

Because sine and cosine are orthogonal in a period

Putting the fourier series in the moment balance equation,

$$\begin{aligned} I\dot{\omega} &= -v\omega + gR \int_0^{2\pi} \left[\sum_{n=0}^{\infty} a_n(t) \sin n\theta + b_n(t) \cos n\theta \right] \sin \theta d\theta \\ &= -v\omega + gR \int_0^{2\pi} a_1 \sin^2 \theta d\theta \\ &= -v\omega + \pi g R a_1 \end{aligned}$$

$\forall n \geq 2$, the three equations are **decoupled**.

Hence, for $n = 1$,

$$\begin{aligned} \dot{a}_1 &= \omega b_1 - K a_1 \\ \dot{b}_1 &= -\omega a_1 - K b_1 + q_1 \\ \dot{\omega} &= (-v\omega + \pi g R a_1)/I. \end{aligned}$$

FIXED POINTS

Q. What is fixed point of a system? $\dot{x} = 0$

fixed points represent **equilibrium** (steady) solutions, means if $x = x^*$ initially, then $x(t) = x^*$ for all time
Can be both **stable** or **unstable**

Q. Why analyse Fixed Points?

We can analyse the stability of the system about the fixed point by linearizing the equation



Fixed points

$$\begin{aligned}\dot{a}_1 = 0 &\implies a_1 = \frac{\omega}{k} b_1 \\ \dot{b}_1 = 0 &\implies \omega a_1 = q_1 - K b_1 \\ \dot{\omega} = 0 &\implies a_1 = \frac{\nu \omega}{\pi g R}\end{aligned}$$

$$\omega^2 = \frac{\pi g R q_1}{\nu} - K^2$$

- $(a_1, b_1, \omega) = (0, q_1/k, 0) \Rightarrow$ Wheel is stationary
- Two solutions, $\pm \omega$ (steady rotation in either direction), iff

$$\frac{\pi g R q_1}{K^2 \nu} > 1$$

Transforming the variables to x, y and z

$$\sigma = \frac{1}{K} \frac{\nu}{I_{total}}$$

$$\rho = \frac{q_1}{K^2} \frac{\pi R g}{\nu}$$

$$x = \frac{\omega}{K}, \quad y = \frac{\rho K}{q_1} a_1, \quad z = \rho - \frac{\rho K}{q_1} b_1$$

The Lorenz equations

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= xy - bz\end{aligned}$$

σ : Prandtl number
 ρ : Rayleigh number
 b : aspect ratio

Transformation credits : Illing et al., 2012

Two types of fixed points in Lorenz system

$$1. (x, y, z) = (0, 0, 0) \quad \forall \varrho$$

$$2. x = y = \pm \sqrt{b(\rho - 1)}, z = \rho - 1, \rho > 1$$



c^+
 c^-

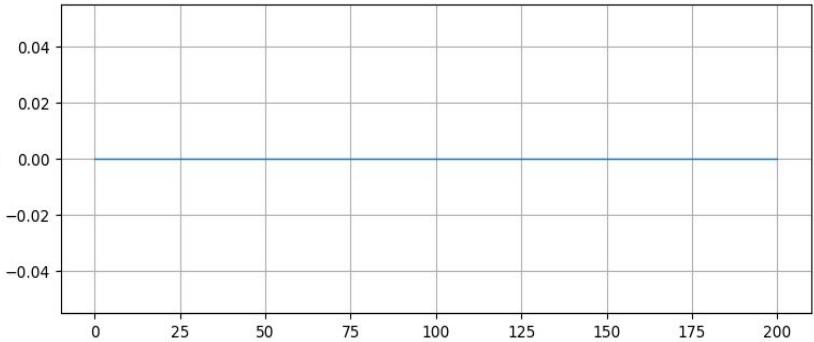
As $\varrho \rightarrow 1^+$, c^+ & $c^- \rightarrow (0,0,0)$

PHYSICAL SANITY CHECK :

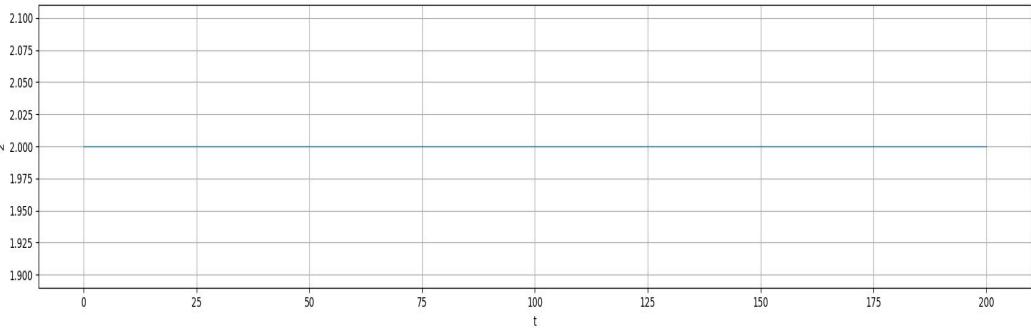
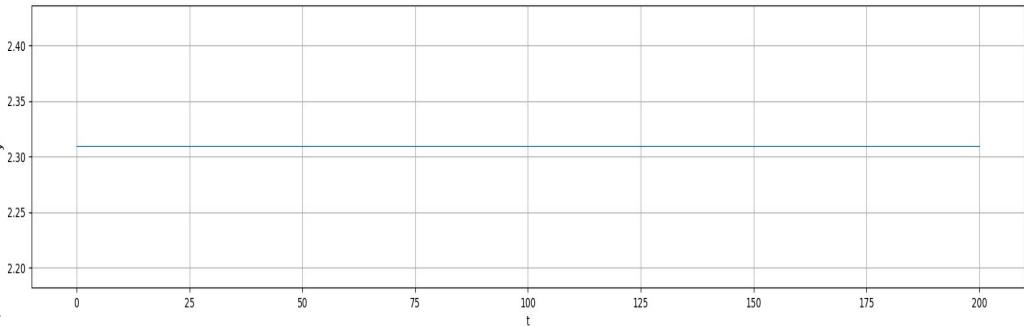
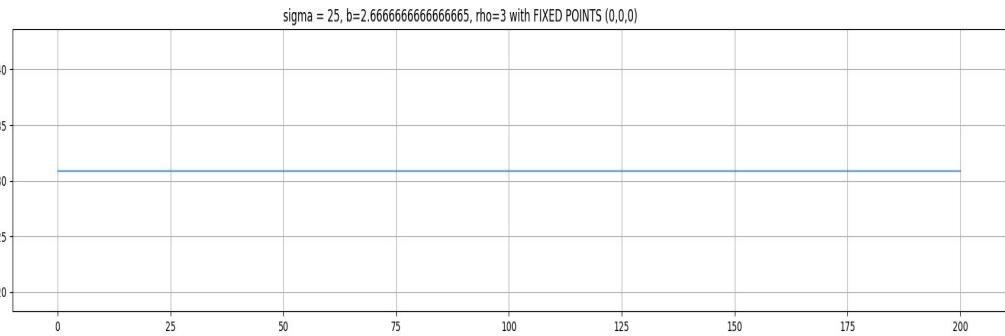
- Small ϱ - angular velocity dies to 0
- Medium ϱ - angular velocity becomes constant
- Large ϱ - angular velocity (x) shows chaos

ϱ is ratio of driving force to resistive force

$\sigma = 25$, $b = 2.6666666666666665$, $\rho = 24.73$ with FIXED POINTS $(0,0,0)$



Fixed point for different
parameters



Let's check the stability about the fixed points -

Linearise 'em equations

1. Linearising the system about origin by omitting the xy and xz nonlinearities in the Lorentz equations,

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y \\ \dot{z} &= -bz\end{aligned}$$



$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ \text{\ding{72}} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalues of the system - all negative => Stable fixed point

- all positive => Unstable fixed point

- both positive and negative => Saddle point

Stable in one direction,
unstable in other

$$\lambda_1 = -b,$$

$$\lambda_{2,3} = \frac{-\sigma - 1 \pm \sqrt{(\sigma + 1)^2 + 4\sigma(\rho - 1)}}{2}$$

For $0 < \rho < 1$, the system has
negative eigenvalues

STABLE F.P.

Credits: Matthew D. Johnston at the University of
Wisconsin – Madison : Math 415 (Fall 2014)

FUN FACT : Origin is globally asymptotically stable when $0 < \rho < 1$

Proved by Lyapunov Function

$$L(x, y, z) = \frac{x^2}{2\sigma} + \frac{y^2}{2} + \frac{z^2}{2}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial x} \frac{dx}{dt} + \frac{\partial L}{\partial y} \frac{dy}{dt} + \frac{\partial L}{\partial z} \frac{dz}{dt}$$

$$= - \left(x - \frac{\rho+1}{2} y \right)^2 - \left(1 - \left(\frac{\rho+1}{2} \right)^2 \right) y^2 - bz^2$$

$L'(t) < 0$ for $0 < \rho < 1 \Rightarrow$ Ellipsoid L collapsing to origin eventually

When $\rho > 1$: 3 Fixed Points exist, but origin is not stable

For -

$$1 < \rho < \rho_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$$

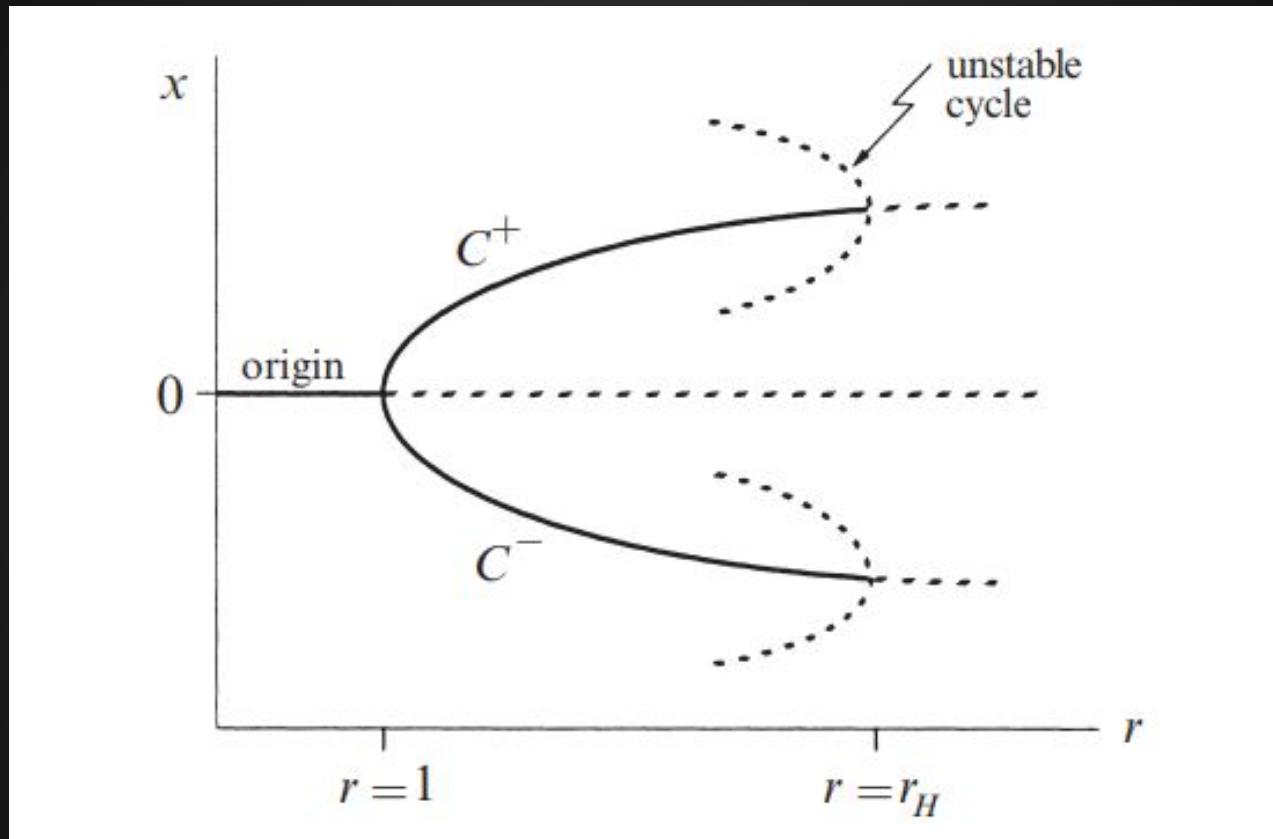
The 2 fixed points are stable - Proof little complicated :(

We observe a “PITCHFORK” bifurcation at $\rho=1$ with the other two fixed points as c^+ & c^- , and a “SUBCRITICAL HOPF” bifurcation at $\rho=\rho_H$

For $\sigma = 10$ and $b=8/3$: standard parameters, $\rho_H = 24.74$

Pitchfork bifurcation vs Hopf bifurcation

- When a stable limit cycle surrounds an unstable equilibrium point:
Supercritical Hopf bifurcation \Rightarrow Periodic behaviour eventually
- Limit cycle is unstable and surrounds a stable equilibrium point : Subcritical Hopf bifurcation
 - Behavior Before Bifurcation - If perturbed, the system might either return to the stable fixed point or diverge away
 - Behavior After Bifurcation - it might exhibit large-amplitude oscillations or be drawn to a distant attractor
- Pitchfork bifurcation - A system's fixed point changes stability, and new fixed points emerge



When $\rho > \rho_H$: Lose the stability of these 2 points also

Where does the solution goes now?

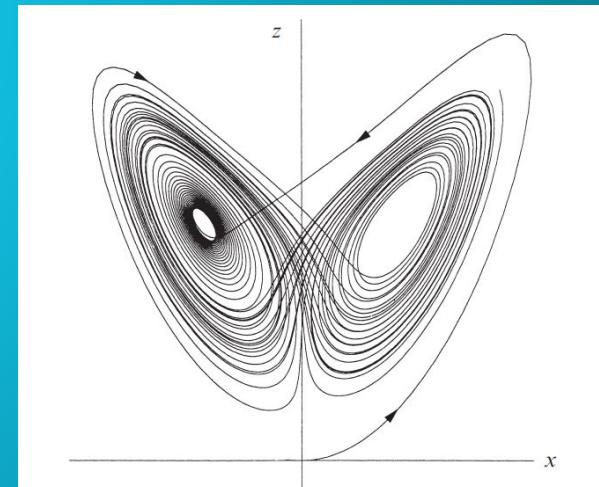
- Stable Fixed Point ? NO
- Remain unbounded ? NO
- Move towards a stable limit cycle? NO

IT GOES TO A **STRANGE DISTANT ATTRACTOR**



STRANGE ATTRACTOR

- After removing initial transients, the solution settles into an irregular/aperiodic oscillation that persists as $t \gg \infty$
- A wonderful structure emerges if the solution is visualized as a trajectory in state space
- After the hopf bifurcation, if ρ is increased further, It enters into a strange distant butterfly attractor.
- The number of rotations made on either side is not fixed and may vary on the conditions



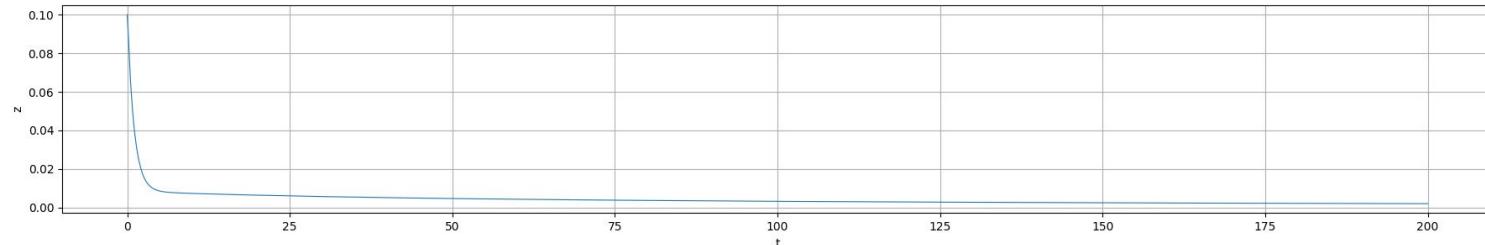
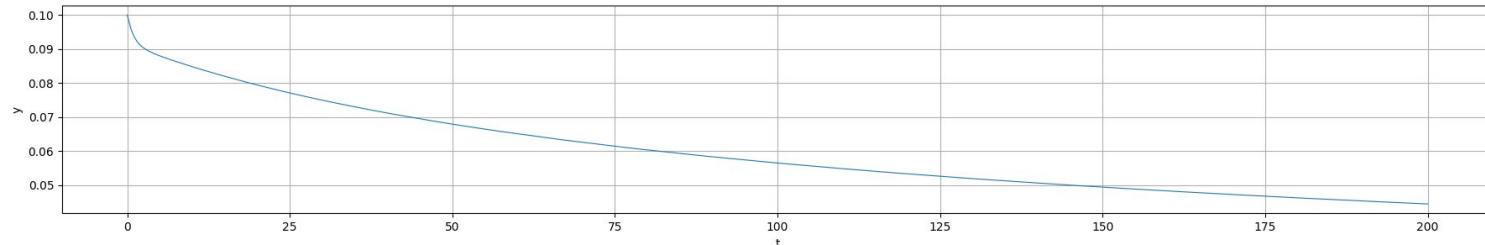
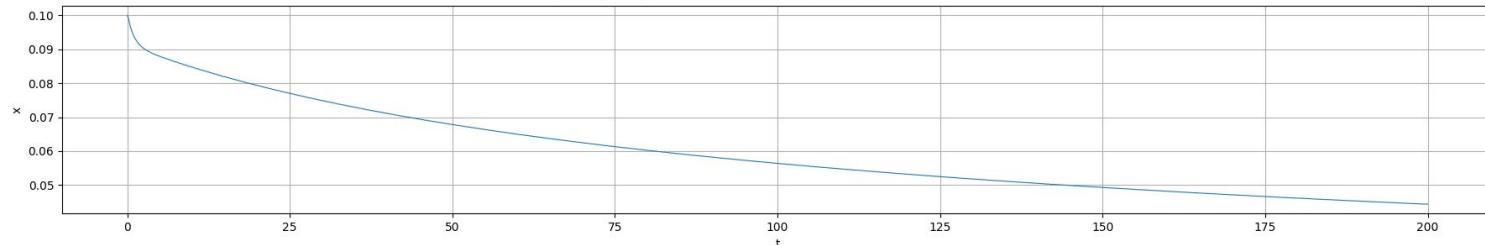


IT'S TIME FOR SIMULATION

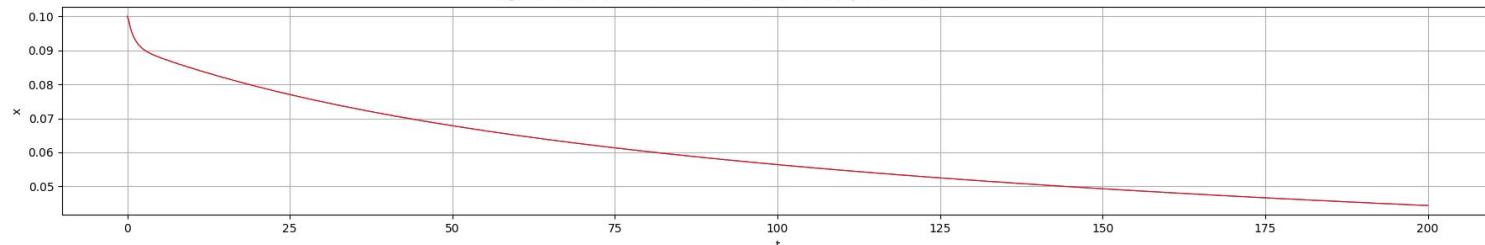
$$\begin{aligned}\sigma &= 10 \\ \varrho &= 28 \\ b &= 8/3\end{aligned}$$

Red - [1.0, 1.0, 1.0]
Blue - [1.1, 1.0, 1.0]
Green - [1.0, 1.1, 1.0]

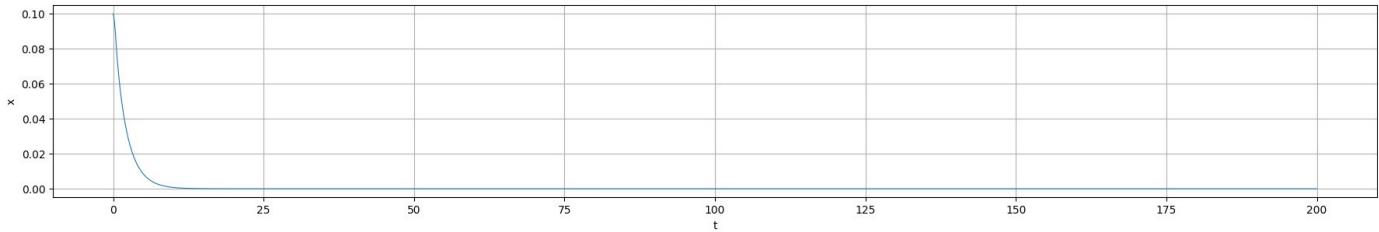
$\sigma = 25, b=2.6666666666666665, \rho=1$



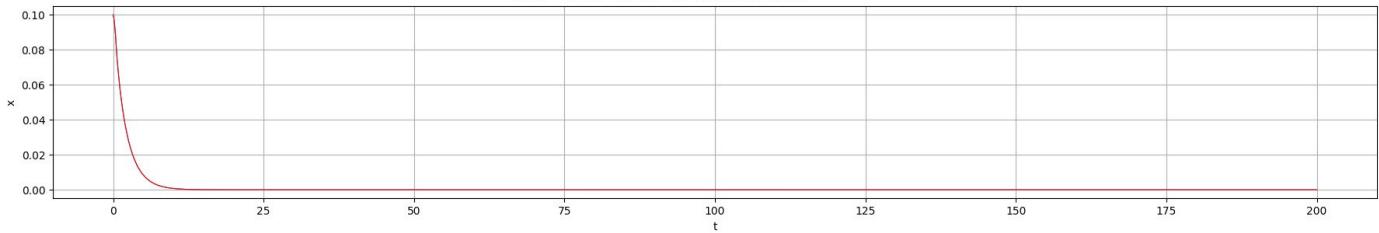
$\sigma = 25, b=2.6666666666666665, \rho=1$ with perturbation



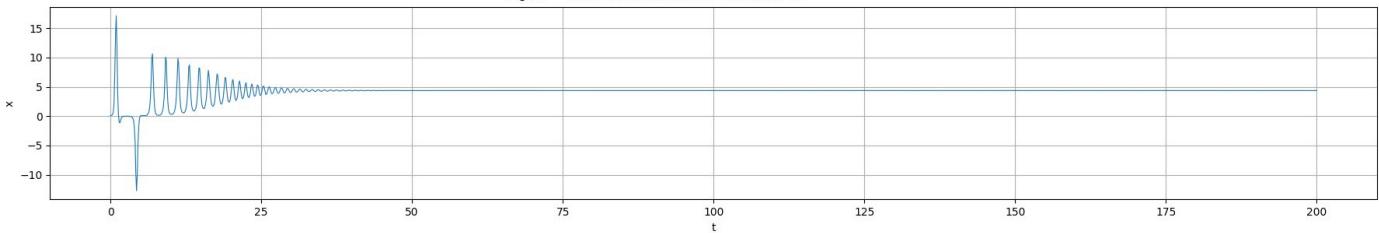
$\sigma = 25, b=8/3, \rho=0.5$



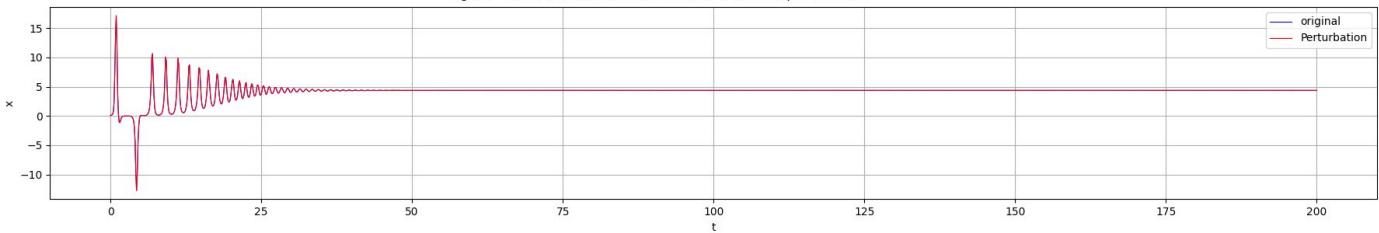
$\sigma = 25, b=8/3, \rho=0.5$ with perturbation



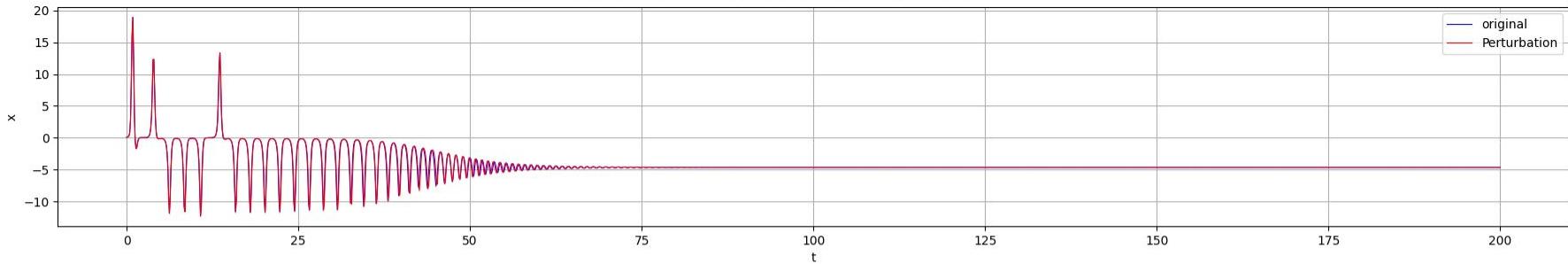
$\sigma = 25, b=2.6666666666666665, \rho=20$



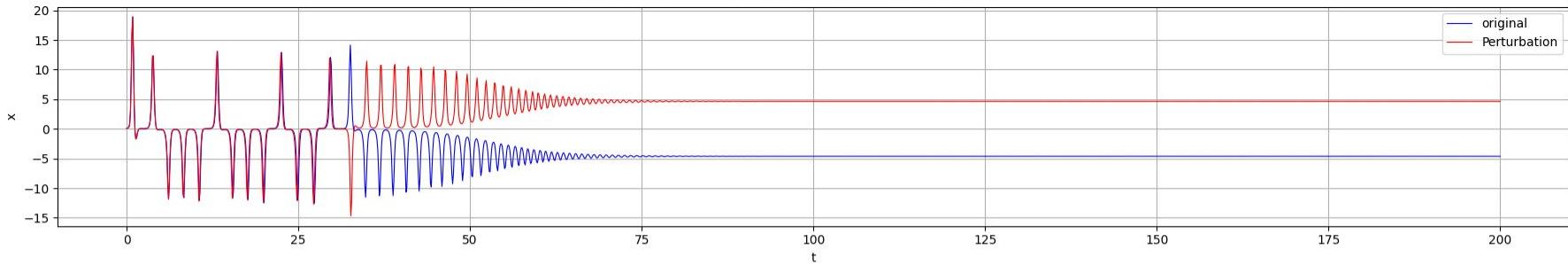
$\sigma = 25, b=2.6666666666666665, \rho=20$ with perturbation



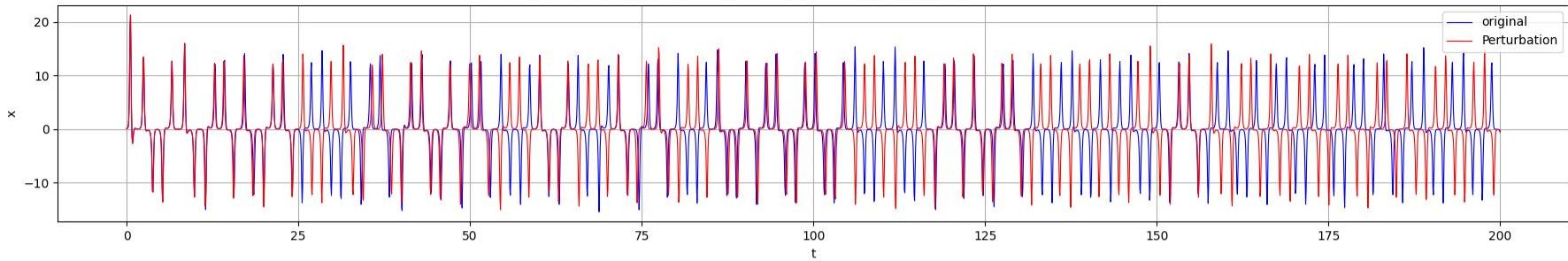
$\sigma = 25$, $b = 2.6666666666666665$, $\rho = 22.43$ with perturbation



$\sigma = 25$, $b = 2.6666666666666665$, $\rho = 22.45$ with perturbation

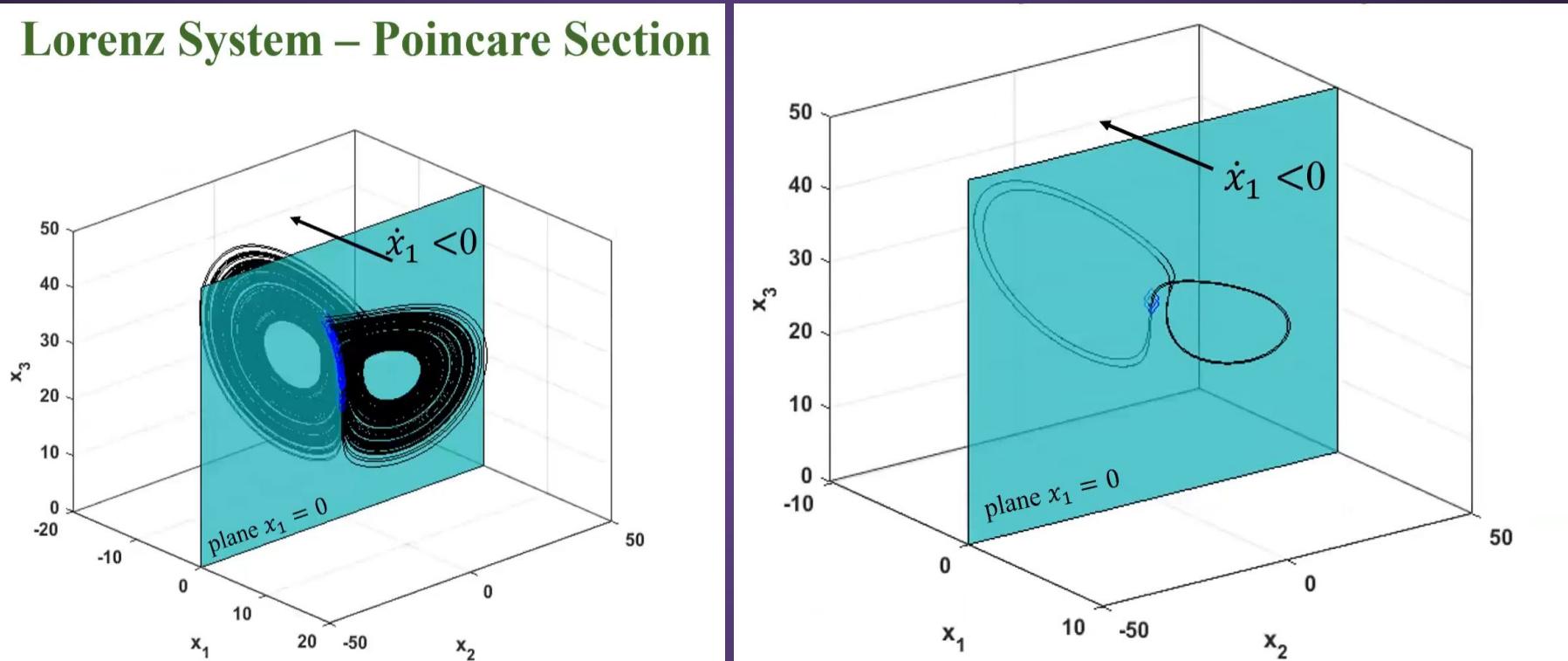
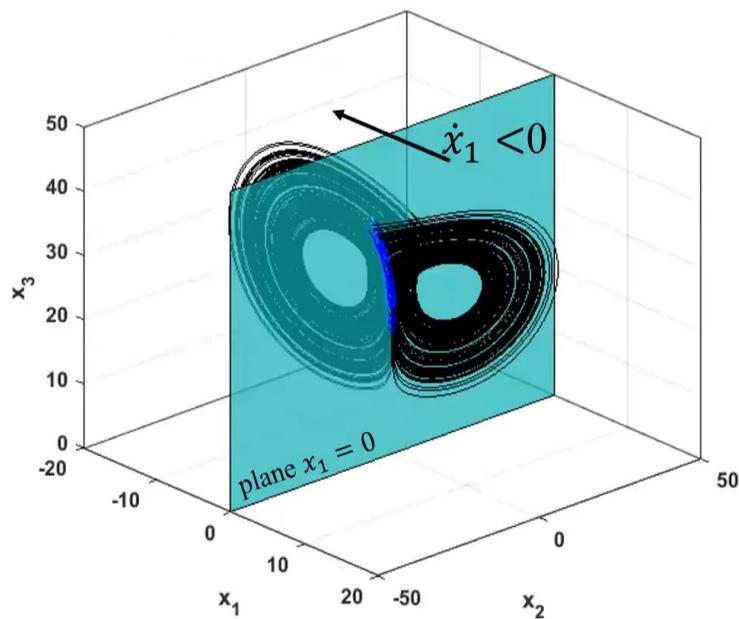


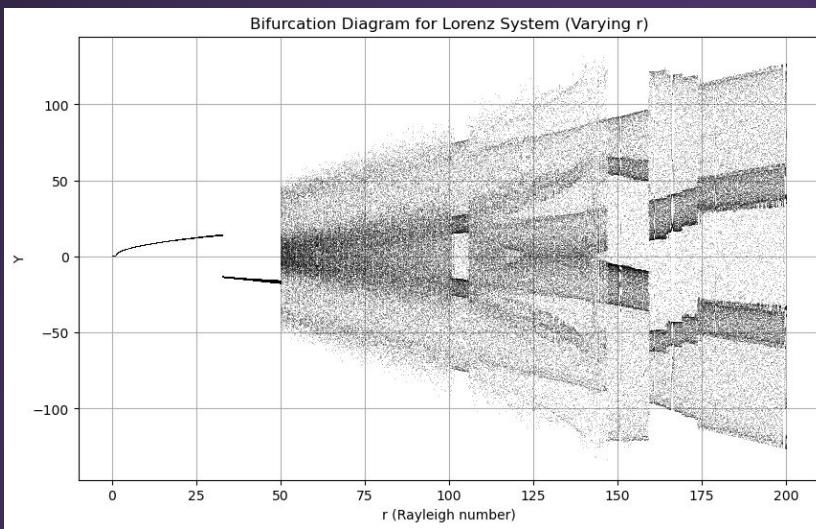
$\sigma = 25$, $b = 2.6666666666666665$, $\rho = 26$ with perturbation



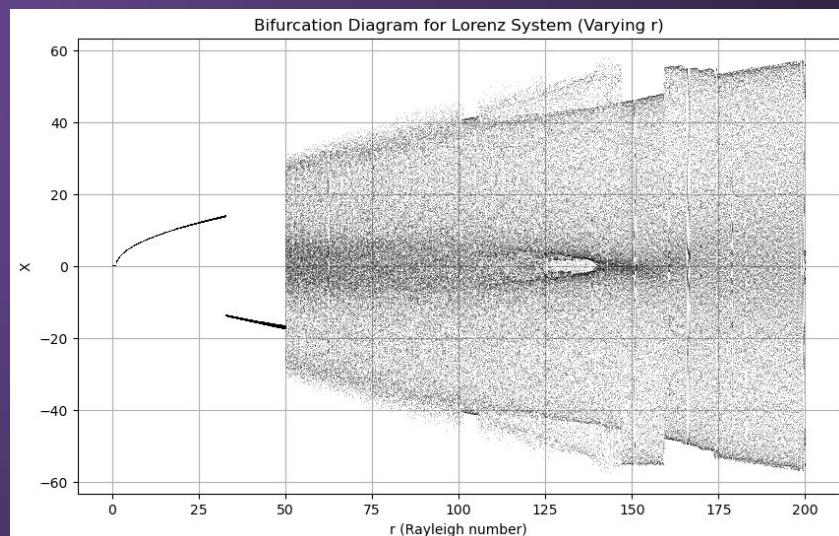
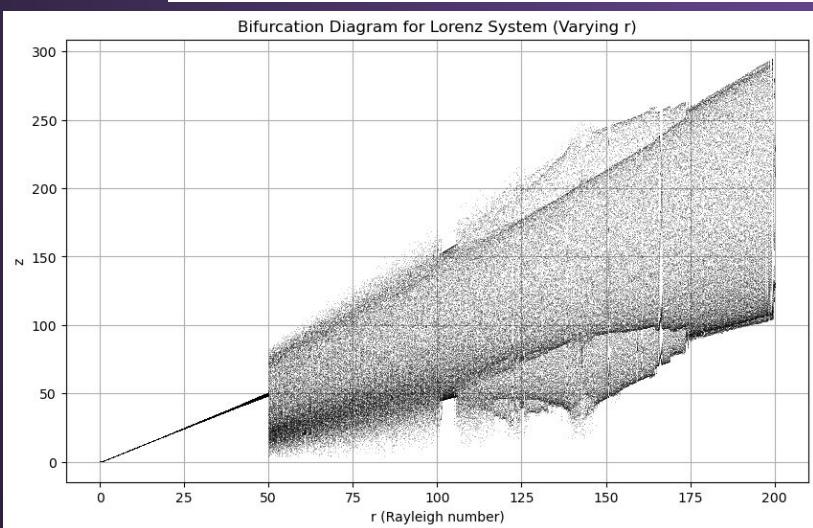
Poincare Cut:

Lorenz System – Poincare Section





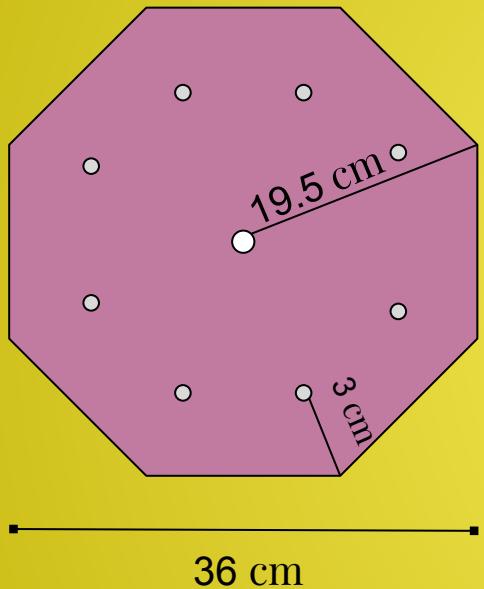
$\sigma=10$, $b = 8/3$, $\rho_H = 63.33$
Point of chaos infliction





IT'S TIME FOR
EXPERIMENTS

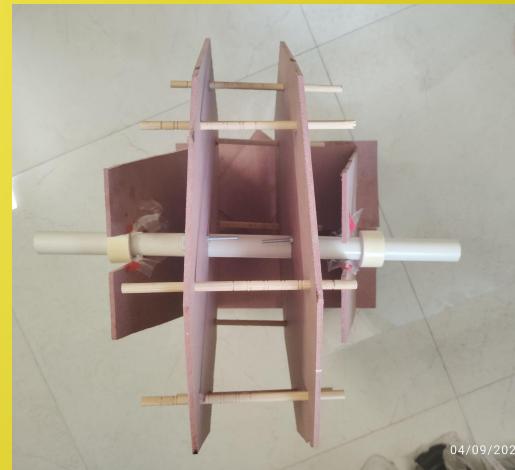
Setup Explanation



Wheel dimensions (not drawn to scale)



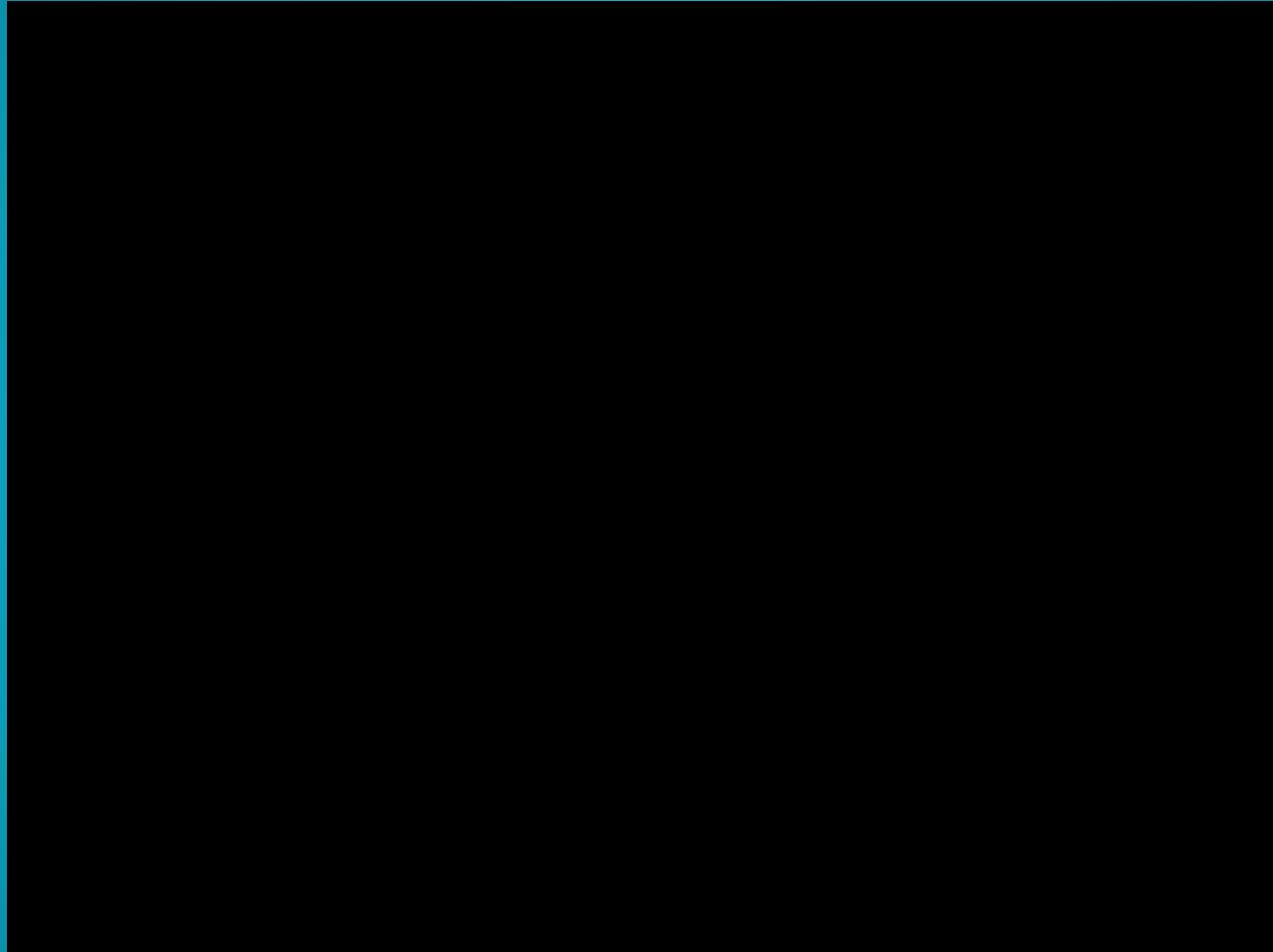
Front view of setup

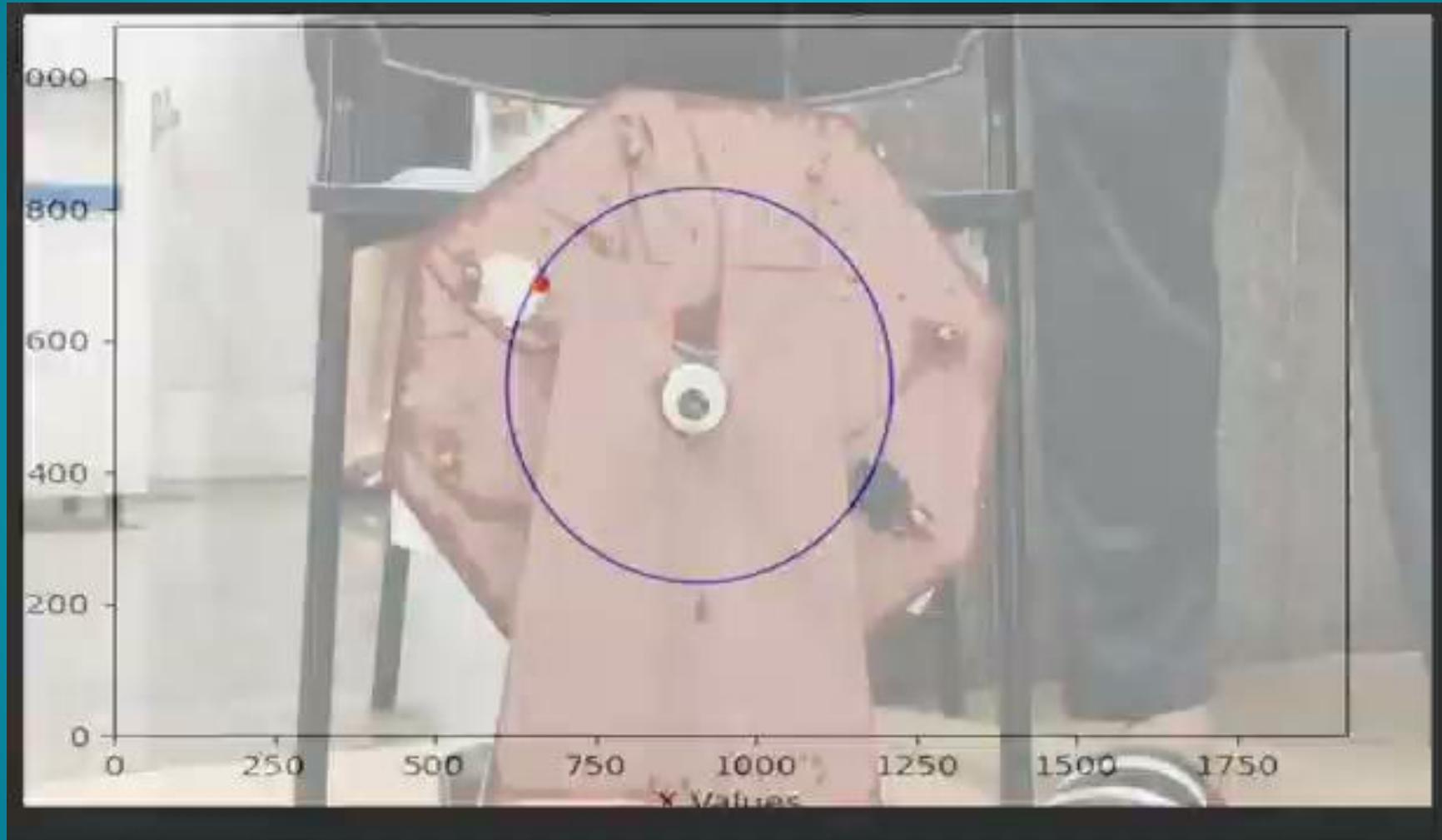


Top view of setup

Properties	Value
Material used for wheel	MDF
Density	600-800 kg/m ³
Thickness of wheel	4mm
Moment of Inertia of wheel + water	0.027255 kg-m ²
Damping coefficient	0.0092669

Experimental Condition	Value
Inflow rate	0.1754 kg/sec
q1 [4]	1.67×10^{-4} kg/sec
Leakage coefficient	0.008793 1/sec





Angular velocity vs frames:

References :-

1. Strogatz, Steven, author. (2015). *Nonlinear dynamics and chaos : with applications to physics, biology, chemistry, and engineering.*
2. Lucas Illing, Rachel F. Fordyce, Alison M. Saunders, Robert Ormond; *Experiments with a Malkus–Lorenz water wheel: Chaos and Synchronization . Am. J. Phys. 1 March 2012; 80 (3): 192–202.*
3. Hirsch, Morris W.; Smale, Stephen; Devaney, Robert (2003). *Differential Equations, Dynamical Systems, & An Introduction to Chaos* (Second ed.). Boston, MA: Academic Press. ISBN 978-0-12-349703-1.
4. Lucas Illing, Rachel F. Fordyce, Alison M. Saunders, Robert Ormond; *Experiments with a Malkus–Lorenz water wheel: Chaos and Synchronization . Am. J. Phys. 1 March 2012; 80 (3): 192–202.*



Behind the scenes :



Presentation by -



Adarsh Prajapati



Yugesh Bhoge



Trideep Basumatary