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Uncertainty estimation and Monte Carlo simulation method

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Abstract

It has been reported that the Monte Carlo Method has many advantages over conventional methods in the estimation of uncertainty, especially that of complex measurement systems' outputs. The method, superficially, is relatively simple to implement, and is slowly gaining industrial acceptance. Unfortunately, very little has been published on how the method works. To those who are uninitiated, this powerful approach remains a 'black art'. This paper demonstrates that the Monte Carlo simulation method is fully compatible with the conventional uncertainty estimation methods for linear systems and systems that have small uncertainties. Monte Carlo simulation has the ability to take account of partial correlated measurement input uncertainties. It also examines the uncertainties of the results of some basic manipulations e.g. addition, multiplication and division, of two input measured variables which may or may not be correlated. For correlated input measurements, the probability distribution of the result could be biased or skewed. These properties cannot be revealed using conventional methods. © 2001 Published by Elsevier Science Ltd.

Keywords: Uncertainty estimation; Monte Carlo; Correlation; Uncertainty propagation

1. Introduction

The application of conventional uncertainty analysis methods, which are basically analytical methods, as is described in ISO/GUM [1], demand the estimation of the separate effect of each input quantity on the final result through a sensitivity analysis. Complex partial derivatives of the output with each of the inputs (sensitivity coefficients) need to be estimated.

When multiple measured input variables in a complex measurement system are correlated (perfectly or partially), sensitivity and, inevitably, uncertainty analysis become extremely difficult and sometimes even unreliable.

The Monte Carlo simulation method has been applied to measurement system uncertainties with many claimed advantages. There is, however, no detailed exposition of the method. To the uninitiated, the method is regarded as a black box. The paper demonstrates the full compatibility of the Monte Carlo method with conventional methods and its ability to take into account even partial correlation effects. It also examines and analyses, both analytically and numerically (by employing the Monte Carlo method), the effects of correlated measurement input uncertainties on measurement output results using simple examples. This helps to understand not only the theoretical background of the method itself but also to understand and visualise the complex mechanism of uncertainty propagation as well.

2. Conventional uncertainty estimation

The functional relationship (i.e. measurement model or equation) between the measurand (quantity being measured) Y and the input quantities X_i in a flow measurement process is given by

$$Y=f(X_1,X_2,X_3\ldots,X_N)$$

The function f includes not only corrections for systematic effects but also accounts for sources of variability, such as those due to different observers, instruments, samples, laboratories and times at which observations are made. Thus, the general functional relationship expresses not only a physical law but also a measurement process. Some of these variables are

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under the direct control of the measurement's 'operator', some are under indirect control, some are observed but not controlled and some are not even observed.

The function f is used to calculate the output estimate, y, of the measurand, Y, using the estimates of $x_1,x_2,\ldots,x_N$ for the values of the N input quantities $X_1,X_2,\ldots,X_N$:

$$y=f(x_1,x_2,x_3,\ldots x_N)$$

The methodology of conventional estimation methods can be illustrated using a simple measurement equation with y as a continuous function of x_1 and x_2. y is approximated using a polynomial approximation or a 2nd order Taylor's series expansion about the means:

$$y=f(\bar{x}_1,\bar{x}_2)+\frac{\partial f}{\partial x_1}(x_1-\bar{x}_1)+\frac{\partial f}{\partial x_2}(x_2-\bar{x}_2)+W \tag{1}$$

where $\bar{x}_1,\bar{x}_2$ are mean observed values and W is the remainder:

$$W=\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_1^2}(x_1-\bar{x}_1)^2 +\frac{\partial^2 f}{\partial x_2^2}(x_2-\bar{x}_2)^2+2\frac{\partial^2 f}{\partial x_1\partial x_2}(x_1-\bar{x}_1)(x_2-\bar{x}_2)\right] \tag{2}$$

As the partial derivatives are computed at the mean values $\bar{x}_1,\bar{x}_2$, they are the same for all i=1,…,N. All the higher terms are normally neglected with W=0. This is acceptable provided that the uncertainties in x_1 and x_2 are small and all values of x_1 and x_2 are close to $\bar{x}_1$ and $\bar{x}_2$ respectively. The square terms in Eq. (2) approach zero more quickly than the first order terms. Additionally, if $f(x_1,x_2)$ is a linear function, then the second order partial derivatives in Eq. (2) are zero and so the remainder W=0. Both linearity and 'small' uncertainty are prerequisites of the conventional method of uncertainty estimation described below.

The standard deviations $\sigma(x_1)$ and $\sigma(x_2)$ are referred to, by the Guide to the Expression of Uncertainty in Measurement (GUM), as the standard uncertainties associated with the input estimates x_1 and x_2. The standard uncertainty in y and can be obtained by Taylor [2]:

$$u(y)=\sigma(y)=\sqrt{\frac{1}{N}\sum_{i=1}^{N}(y_i-y)^2} =\sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2\sigma(x_1)^2+\left(\frac{\partial f}{\partial x_2}\right)^2\sigma(x_2)^2+2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\sigma(x_1x_2)} \tag{3}$$

This equation gives the uncertainty as a standard deviation irrespective of whether or not the measurements of x_1 and x_2 are independent and of the nature of the probability distribution.

Eq. (3) can be written in terms of the correlation coefficient, $\rho_{x_1x_2}$

$$u(y)=\sigma(y)= \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2\sigma(x_1)^2+\left(\frac{\partial f}{\partial x_2}\right)^2\sigma(x_2)^2+2\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\rho_{X_1X_2}\sigma(x_1)\sigma(x_2)} \tag{4}$$

The partial derivatives are called sensitivity coefficients, which give the effects of each input quantity on the final results (or the sensitivity of the output quantity to each input quantity).

The term, expanded uncertainty is used in GUM to express the % confidence interval about the measurement result within which the true value of the measurand is believed to lie and is given by:

$$U(y)=tu(y)$$

where t is the coverage factor on the basis of the confidence required for the interval $y\pm U(y)$. For a level of confidence of approximately 95% the value of t is 2 for normal distributed measurement. In other words, y is between $y\pm 2\sigma(y)$ with 95% confidence.

Coleman and Steele [3], presented a detailed analysis of the subject. The result of a measurement is regarded as only an approximation or estimate of the value of the specific quantity subjected to measurement.

3. Monte Carlo simulation method

With the availability of digital computers, numerical experiments have become an increasingly popular method for analysing physical engineering systems. Simulation is generally defined as the process of replication of the real world based on a set of assumptions and conceived models of reality [4].

Monte Carlo simulation was devised as an experimental probabilistic method to solve difficult deterministic problems since computers can easily simulate a large number of experimental trials that have random outcomes. When applied to uncertainty estimation, random numbers are used to randomly sample parameters' uncertainty space instead of point calculation carried out by conventional methods. Such an analysis is closer with the underlying physics of actual measurement processes that are probabilistic in nature. Nicolis [5] pointed out that in nature the process of measurement, by which the observer communicates with a physical system, is limited by a finite precision. As a result, the 'state' of a system must in reality be understood not as a point in phase space but rather as a small region whose size reflects the finite precision of the measuring apparatus. On the probabilistic view, we look at our system through a ‘window’ (phase space cell). So the application of Monte Carlo simulation in the uncertainty estimation of different states of a system seems to offer a more realistic approach.

Some advantages of Monte Carlo simulation have been described by Basil and Jamieson [6]. The method can handle both small and large uncertainties in the input quantities. Complex partial differentiations to determine the sensitivity coefficients are not necessary. It also takes care of input covariances or dependencies automatically [7].

4. Case studies of simple measurement manipulations

As mentioned in the Introduction and in the last section, Monte Carlo simulation methods are being used in flow measurement. Many advantages of the approach are claimed. There is however, no work published in the open literature examining the compatibility of the Monte Carlo approach with the conventional (including the traditional RSS) methods for linear and small uncertainties. Simple measurement manipulations are presented below to demonstrate this compatibility. The calculations involved in computing flow from measurements are often complex and the manipulation non-linear. It is difficult to examine how uncertainties are propagated. Monte Carlo simulation is used to demonstrate the propagation of uncertainties in simple multiplication and divisions. To help to illustrate the general point, the uncertainties of the input have been deliberately selected to be relatively large.

Great care has been taken in the generation of random numbers in the study to ensure randomness. A special transformation has been used to ensure that with finite samples, the mean and standard deviation are as specified. Details of the transformation can be found in Papadopoulos [8]. The inputs x_1 and x_2 in the examples given below are both normally distributed.

4.1. Sum or difference of two elementary measurements ($y=x_1\pm x_2$)

Three different cases have been examined:

1. the two elementary measurements are independent thus not sharing common uncertainty sources (ρ=0);
2. the two measurements are positively partially correlated uncertainties (ρ=0.7);
3. the two measurements are negatively partial correlated uncertainties ($\rho=-0.7$)

With both the partial derivative equations to unity, in Eq. (3), the uncertainty of $y=x_1+x_2$ is generally:

$$\frac{u(y)}{y}=\frac{\sigma(y)}{y}$$

$$=\left\{\begin{array}{lll}\frac{1}{y}\sqrt{\sigma(x_1)^2+\sigma(x_2)^2+2\rho_{x_1x_2}\sigma(x_1)\sigma(x_2)} & \rightarrow \text{for } -1<\rho_{x_1x_2}<1 & \\ \frac{1}{y}\sqrt{\sigma(x_1)^2+\sigma(x_2)^2} & \rightarrow \text{for } \rho_{x_1x_2}=0 & \text{(RSS)} \\ \frac{1}{y}|\sigma(x_1)\pm\sigma(x_2)| \text{ (for } y=x_1+x_2) & \rightarrow \text{for } \rho_{x_1x_2}=\pm 1 & \text{(ADD)}\end{array}\right\}$$

Taylor [2] has shown analytically that the probability distribution of the result y of the summation of two normal independent inputs is normal and centre at $y=\overline{x_1}+\overline{x_2}$. The value of result y is the best estimate given by the best estimates of $\bar{x}_1$ and $\bar{x}_2$.

4.1.1. Uncorrelated inputs

$$\overline{x_1}=\overline{x_2}=30$$

$$2\sigma(x_1)=0.4$$

$$2\sigma(x_2)=0.9$$

$$\rho_{X1X2}=0$$

$$y=\bar{x}_1+\bar{x}_2=60$$

$$2\sigma(y)=2\sqrt{0.2^2+0.45^2}=0.9849$$

$$\frac{2\sigma(y)}{|y|}=1.6414\%$$

It can be seen from the result that the Monte Carlo uncertainty obtained only differed from the GUM value by −0.03% (Table 1). The result was obtained by using 15 samples of 6000 points each. For each 6000 point sample, the Monte Carlo estimated uncertainty did not differ from the analytical value by more that 0.8%. Even with less sample points (say, 1000 point) the results were still satisfactory. After the simulated result y has been obtained from the simulated inputs x_1 and x_2, the joint probability distributions (bivariate) of the result with each of the inputs $p(y,x_1)$ and $p(y,x_2)$ are obtained.
The joint probability distributions are given by

$$p(y,x_i)$$

$$=\frac{1}{2\pi\sigma(y)\sigma(x_i)\sqrt{(1-\rho_{yx_1}^2)}}\exp\left[-\frac{1}{2(1-\rho_{y,x_i}^2)}\left(\frac{(y-E(y))^2}{\sigma(y)^2}+\frac{(x_i-E(x_i))^2}{\sigma(x_i)^2}-\frac{2\rho_{y,x_i}(y-E(y))(x_i-E(x_i))}{\sigma(y)\sigma(x_i)}\right)\right]$$

where $y=x_1$ and x_2 and i=1 and 2.

Joint probability distributions $p(y,x_1)$ and $p(y,x_2)$ were selected for the presentation of the results as not only it is a very effective way to visualize the distribution of output y but also its relationship with the uncertainties of inputs x_1, x_2, the effects of input correlated uncertainties (if any), and more importantly, the effects of the

Table 1
Result of Monte Carlo simulation

	Average	2×standard deviation	Relative uncertainties %, (95.45% confidence)
Input			
x_1	30.00000000000000	0.40000000000000	1.33333333333333
x_2	30.00000000000000	0.9000000000000	2.99999999999999
Output			
y	60.00000000000023	0.98456218359665	1.64093697266108

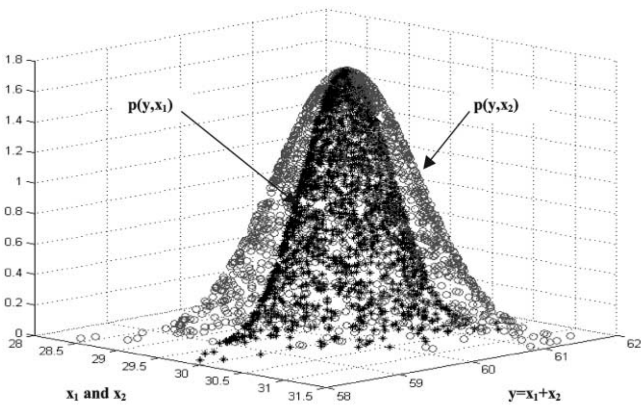

Fig. 1. Joint distributions $p(y,x_1)$ and $p(y,x_2)$ of result $y=x_1+x_2$ with each input x_1, x_2 when $\rho_{x_1x_2}=0$.

mathematical action involved (e.g. addition, multiplication etc.). All these effects and available information can be visualised in one 3-D graph, as in Fig. 1.

The joint distributions $p(y,x_i)$, as seen from the output's view is shown in Fig. 2, and clearly demonstrates that y is normally distributed.

4.1.2. Correlated inputs

$$\overline{x_1}=6,\ \overline{x_2}=5$$

$$2\sigma(x_1)=0.3$$

$$2\sigma(x_2)=0.1$$

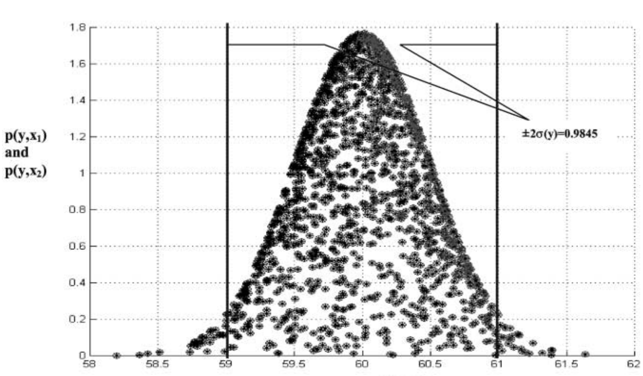

Fig. 2. Joint distributions $p(y,x_I)$ of $y=x_1+x_2$ for uncorrelated inputs.

$\rho_{x_1x_2}=+0.7$	$\rho_{x_1x_2}=-0.7$
$y=x_x+x_2=6+5=11$	$y=x_x+x_2=6+5=11$
$2\sigma(y)=2\sqrt{0.15^2+0.05^2+2\times0.7\times0.15\times0.05}$	$2\sigma(y)=2\sqrt{0.15^2+0.05^2-2\times0.7\times0.15\times0.05}$
$=0.3768$	$=0.2408$
$\frac{\sigma(y)}{\lvert y\rvert}=3.426\%$	$\frac{\sigma(y)}{\lvert y\rvert}=2.189\%$

Both the analytical method and the Monte Carlo simulation show that the negative correlated inputs reduce the uncertainty of the result (Table 2). The distribution of the result is again normal as shown in Fig. 3.

An interesting situation could arise when the uncertainties of the two inputs and the nature of their correlation combined result in zero uncertainty of the result. The probability of y is a delta function, see Fig. 4.

$$\overline{x_1}=6$$

$$\overline{x_2}=5$$

$$2\sigma(x_1)=0.2$$

$$2\sigma(x_2)=0.2$$

$\rho_{x_1x_2}=1$	$\rho_{x_1x_2}=-1$
$2\sigma(y)=2\sqrt{0.1^2+0.1^2+2\times1\times0.1\times0.1}$	$2\sigma(y)=2\sqrt{0.1^2+0.1^2-2\times1\times0.1\times0.1}$
$=0.4$	$=0$
$\frac{2\sigma(y)}{\lvert y\rvert}=3.636\%$	$\frac{2\sigma(y)}{\lvert y\rvert}=0\%$

So far all the examples are concerned with the addition of two inputs. For the case of subtraction, $y=x_1-x_2$, the opposite results can be obtained in each case. Uncertainty, is 'amplified' for negative correlation.

For addition and subtraction, the result y is a linear function in terms of the inputs x_1,x_2. The first order partial derivatives are all equal to ±1, the square of which is equal to unity. The second order partial derivatives are both equal to zero, i.e.

$$\frac{\partial^2 y}{\partial x_1^2}=\frac{\partial^2 y}{\partial x_2^2}=0.$$

The uncertainty propagation law as expressed by Eqs. (3) and (4) is no longer an approximation. It is valid for all cases and not limited by conditions such as small

Table 2
Result of Monte Carlo simulation $y=x_1+x_2$ $\rho_{x_1x_2}=\pm 0.7$

	Average	2×standard deviation	Relative uncertainties (%) 95.45% confidence
Input			
x_1	6.00000000000000	0.30000000000000	5.00000000000000
x_2	5.00000000000001	0.10002239715478	2.00044794309560
Output			
$Y(\rho_{x_1x_2}=+0.7)$	11.0000000000001	0.37686029194342	3.42600265403112
$Y(\rho_{x_1,x_2}=-0.7)$	11.0000000000001	0.24086244991386	2.18965863558052

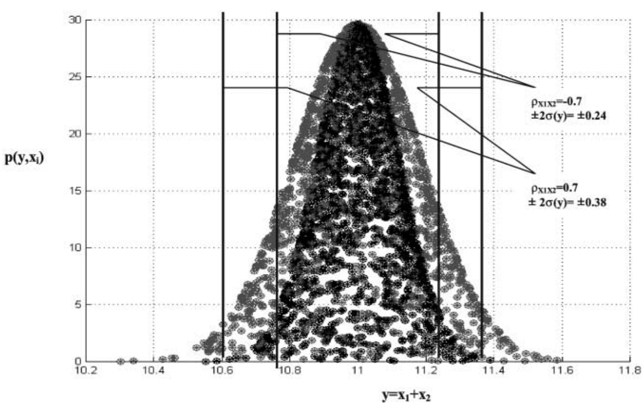

Fig. 3. Joint distributions $p(y,x_I)$ of $y=x_1+x_2$ for $\rho_{x_1x_2}=\pm 0.7$.

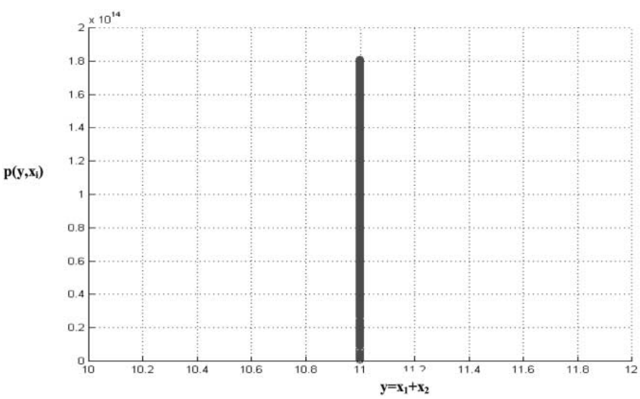

Fig. 4. Distribution of $y=x_1+x_2$ with $\rho_{x_1x_2}=-1$.

uncertainties. This means that a measurement system where the measured output is obtained by the addition or subtraction of elementary measurement processes is always additive. This is quite important since in such a measurement process the principle of superposition always holds. It does not matter if elementary measurements are processed before averaging or averaged before processing. This particular characteristic could be very useful in practice if it is considered in the decentralisation of the elementary measurements' electronic processing.

As the sensitivity coefficients

$$\frac{\partial y}{\partial x_1}=\frac{\partial y}{\partial x_2}=\pm 1$$

the uncertainty of the y depends solely on the uncertainties of the inputs and on their combined uncertainty (covariance). The absolute uncertainty is independent of the magnitudes of the inputs. This is somehow misleading, as uncertainty $u(y)$ as an absolute value does not actually gives any feeling of its significance if it does not compare with the magnitude of the output result y. In this respect, the relative uncertainty

$$\frac{u(y)}{|y|}$$

is the most useful quality index of a measurement system.

4.2. Multiplication of two elementary measurements ($y=x_1x_2$)

By assuming small uncertainties and ignoring the second order effects, the sensitivity coefficients obtained at the mean values $\bar{x}_1$ and $\bar{x}_2$ are:

$$\frac{\partial y}{\partial x_1}=x_2=\frac{y}{x_1} \text{ and } \frac{\partial y}{\partial x_2}=x_1=\frac{y}{x_2}.$$

and

$$2\sigma(y)=2\sqrt{(x_2\sigma(x_1))^2+(x_1\sigma(x_2))^2+2\rho_{x_1x_2}\sigma(x_1)\sigma(x_2)}$$

The relative uncertainty of y (for 95.45% confidence) is:

$$\frac{2\sigma(y)}{|y|}=2\sqrt{\left(\frac{\sigma(x_1)}{x_1}\right)^2+\left(\frac{\sigma(x_2)}{x_2}\right)^2+2\frac{\rho_{X_1X_2}\sigma(x_1)\sigma(x_2)}{x_1x_2}}$$

4.2.1. Uncorrelated inputs

$$\bar{x}_1=6$$

$$\bar{x}_2=5$$

$$2\sigma(x_1)=0.3$$

Table 3
Results of Monte Carlo simulation $y=x_1x_2$ for $\rho_{x_1x_2}=0$

	Average	2×standard deviation	Relative uncertainties %, (95.45% confidence)
Input			
x_1	6.00000000000000	0.30000000000000	5.00000000000000
x_2	5.00000000000000	0.10000000000000	2.00000000000000
Output			
y	30.00000335838123	1.61483667662676	5.38278831950736

$$2\sigma(x_2)=0.1$$

$$2\sigma(y)=2\sqrt{(5\times0.15)^2+(6\times0.05)^2}=1.616$$

$$\frac{2\sigma(y)}{|y|}=2\sqrt{\left(\frac{0.15}{6}\right)^2+\left(\frac{0.05}{5}\right)^2}=5.385\%$$

The computed results from one of many single samples of 10,000 points are shown in Table 3.

The Monte Carlo output uncertainty average result of 10 samples was, in more than 95% of the cases, within 0.1% of the analytical calculation result. It must be noted that there is a small insignificant bias in the estimated y. The bias corresponds to some residual covariance left due to the single sample used and it is averaged down to zero with a rate $n^{1/2}$ where n is the number of samples. The joint distributions $p(y,x_i)$ of the result is shown in Fig. 5 below.

4.2.2. Correlated inputs

$$\overline{x_1}=6$$

$$\overline{x_2}=5$$

$$2\sigma(x_1)=0.3$$

$$2\sigma(x_2)=0.1$$

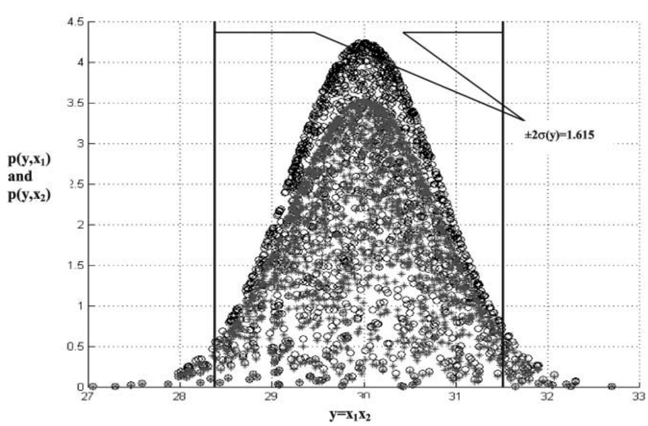

Fig. 5. Joint distributions $p(y,x_I)$ of $y=x_1x_2$ for uncorrelated inputs $\rho_{x_1,x_2}=0$.

$$\rho_{x_1x_2}=1$$

$$2\sigma(y)=2.1$$

$$\frac{2\sigma(y)}{|y|}=2\sqrt{\left(\frac{0.15}{6}\right)^2\left(\frac{0.05}{5}\right)^2+2\times\frac{1\times0.15\times0.05}{6\times5}}=7\%$$

Monte Carlo simulation shows that the output y=30.0075 which is 0.0075 higher than the product of the average of x_1 and x_2 (Table 4). The difference is not random but represents a systematic ‘error’. This is due to the neglect of the residual term W in Eq. (1). The mean residual term is associated with the inputs covariance (or correlation). The output y can be obtained from Eq. (1)

$$\bar{y}=\overline{f(x_1,x_2)}=\frac{1}{N}\sum_{i=1}^{N}\left[\overline{x_1x_2}+\frac{\partial f}{\partial x_1}(x_{1i}-\overline{x_1})+\frac{\partial f}{\partial x_2}(x_{2i}-\overline{x_2})+W\right]=\overline{x_1x_2}+\bar{W}=\overline{x_1x_2}+\rho_{x_1x_2}\sigma(x_1)\sigma(x_2)=5\times6+1\times0.05\times0.15=30.0075 \quad (5)$$

Perfect positive correlation between the input increased the relative uncertainty of the result from 5.4% for the uncorrelated case to 7%. This change is predicted by Eq. (4) when the correlation coefficient is included.

The uncertainty can be expressed as:

$$2\sigma(y)=2\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial f}{\partial x_1}(x_{1i}-\bar{x}_1)+\frac{\partial f}{\partial x_2}(x_{2i}-\bar{x}_2)+W-\bar{W}\right]^2} \quad (6)$$

The last two mean residual terms in Eq. (6) can be zeroed out, leaving the random uncertainty component the same (as in uncertainty propagation law) as Eq. (3). The standard deviation or uncertainty alone does not reveal that the probability of y is skewed. Fig. 6 shows the probabilities $p(y,x_i)$ for perfectly positive and negative correlated inputs from the Monte Carlo simulation. It is clear that $\rho_{x_1x_2}<0$ result in positive skewness, i.e. long tail on the right. For $\rho_{x_1x_2}>0$ the result shows negative skewness, i.e. long tail on the left. Measurement of skew and bending of the distribution are given by

Table 4
Result of Monte Carlo simulation $y=x_1x_2$ for $\rho_{x_1x_2}=1$

	Average	2×standard deviation	Relative uncertainties %, (95.45% confidence)
Input			
x_1	6.00000000000000	0.30000000000000	5.00000000000000
x_2	5.00000000000000	0.10000000000000	2.00000000000000
Output			
Y	30.007495000000	2.10027879309718	7.00092931032421

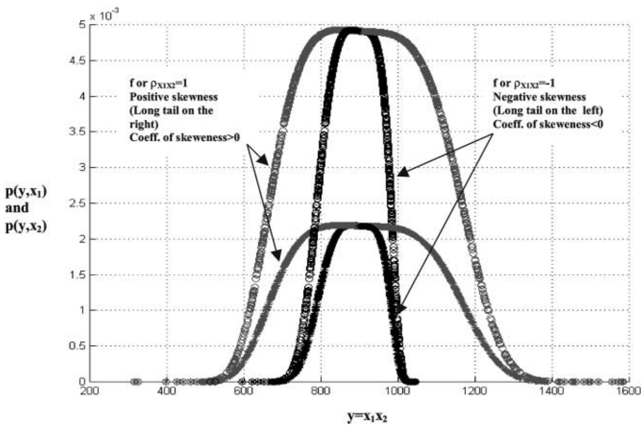

Fig. 6. Joint distributions $p(y,x_I)$ of $y=x_1x_2$ for positive and negative correlated inputs.

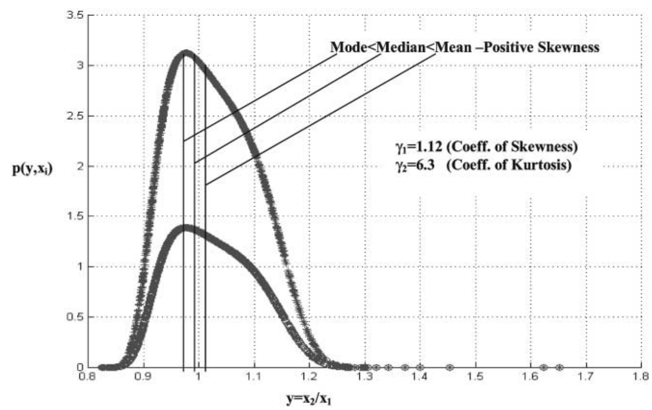

Fig. 7. Joint distributions $p(y,x_i)$ of $y=x_2/x_1$ for perfect positive correlated input $\rho_{x_1x_2}=1$.

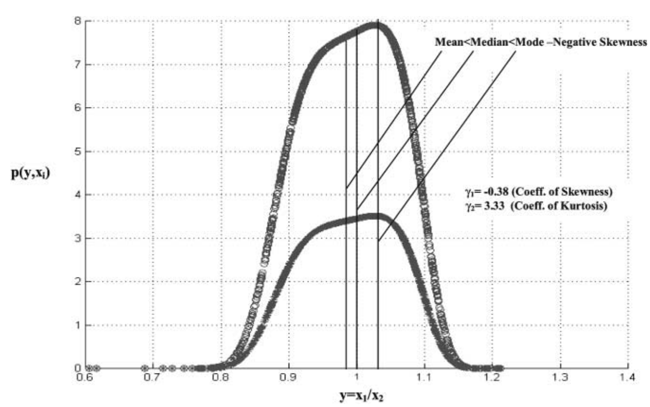

Fig. 8. Joint distributions $p(y,x_I)$ of $y=x_1/x_2$ for perfect positive correlated inputs $\rho_{x_1x_2}=1$.

Coefficient of skewness $\gamma_{1y}=\frac{1}{N\sigma(y)^3}\sum_{i=1}^{N}(y_i-\bar{y})^3$

Coefficient of kurtosis $\gamma_{2y}=\frac{1}{N\sigma(y)^4}\sum_{i=1}^{N}(y_i-\bar{y})^4$

4.3. Division of two elementary measurements ($y=x_1/x_2$ and $y=x_2/x_1$)

Simulation has been carried out for $y=x_1/x_2$ and $y=x_2/x_1$ for cases of perfectly positive and negative correlated inputs. The corresponding joint distributions $p(y,x_I)$ are given in Figs. 7 and 8. The skewness is much exaggerated.

The distributions are skewed and flattened. For $y=x_2/x_1$ with perfect positive correlated input uncertainties, the distribution is positively skewed with a long tail on the right. The coefficient of skewness, $\gamma_{1y}=1.12>0$. The opposite is true for $y=x_1/x_2$, the coefficient of skewness is negative, $\gamma_{1y}=-0.38>0$.

The coefficient of skewness does not review anything about the flattening (compared to Gaussian) and the bending (non-linear) of the distribution. The coefficient of kurtosis, $y_{2y}=3.33>3$ and $\gamma_{2y}=6.3\gg 3$ for Figs. 7 and 8 respectively.

5. Conclusions

1. The compatibility of Monte Carlo and the conventional method has been demonstrated.
2. For non-linear functions, errors are introduced as a consequence of the neglect of the higher order terms. The Monte Carlo method readily takes into account all non-linearities.
3. Partial and perfect correlated uncertainties in measurement inputs affect differently the measurement results. For linear functions (addition and

subtraction), the output is unbiased. When the inputs are Gaussian, the result is also Gaussian but the width of the distribution is changed. For non-linear functions (multiplication and division, with correlated inputs), the result has a systematic shift. The resulting distribution became asymmetric and flattened (compared to Gaussian). Non-linearity is revealed as a bending of the joint distributions $p(y,x_i)$.
4. These effects can be readily demonstrated and visualised with the Monte Carlo simulation method.

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