# A semi-numerical algorithm for the homology lattice and periods of complex elliptic surfaces over $\mathbb{P}^1$

## Eric Pichon-Pharabod

#### Abstract

We provide an algorithm for computing an effective basis of homology of elliptic surfaces over  $\mathbb{P}^1_{\mathbb{C}}$  on which integration of periods can be carried out. This allows the heuristic recovery of several algebraic invariants of the surface, notably the Néron-Severi lattice, the transcendental lattice, the Mordell-Weil group and the Mordell-Weil lattice. This algorithm comes with a SageMath implementation.

### Contents

1	Introduction	1
2	Elliptic surfaces	3
3	Homology and periods of elliptic surfaces	4
4	Application: Mordell-Weil group and lattice	12

# 1. Introduction

An elliptic surface is a surface S together with a map  $f:S\to C$  to a curve, such that the generic fibre is an elliptic curve. Such surfaces benefit from a rich structure, bridging the elegance of elliptic curves with properties of higher-dimensional varieties. As such, they exhibit a weave of algebraic and topological phenomena.

The set of rational points of an elliptic curve benefits from a group structure, called the Mordell-Weil group. This group's structure and properties have far-reaching implications in various fields, such as algebraic number theory with the modular theorem [43]; cryptography with Elliptic Curve Cryptography [30, 22]; or the study of Diophantine equation [27]. The Mordell-Weil theorem, a central result in this context, establishes that this group is finitely generated [34, 42] in the case of elliptic curves over a number field. Determining the Mordell-Weil group, and in particular determining its rank, however, remains a notoriously difficult problem [18].

Viewing elliptic surfaces as elliptic curves over the function field of the base space reveals a map between rational points of the curve and sections of the fibration. In this context the Mordell-Weil group gains additional structure, induced from the lattice structure of the second homology group of the surface. Furthermore in this setting, Shioda provided an isomorphism between the Mordell-Weil group of S and a group derived from the Néron-Severi lattice of the surface [39]. This isomorphism not only simplifies the computation of the group's rank, yielding the Shioda-Tate formula, but also imparts a lattice structure to the Mordell-Weil group.

In light of these observations, we provide in this paper a semi-numerical algorithm for efficiently computing the homology of elliptic surfaces, as well as high precision numerical approximations of its holomorphic periods. In turn, we obtain a heuristic method for computing their Mordell-Weil group and lattice. All in all, this yields practical computational methods to fields where elliptic surfaces naturally arise, such as the study of Feynman integrals [1, 2, 9] and mirror symmetry [5, 20].

#### Contributions

We provide an algorithm for computing the full homology lattice, with its intersection product, of an elliptic surface S given its defining equation as a polynomial  $P_t \in \mathbb{Q}[t, X, Y, Z]$ . When S is not isotrivial, the algorithm also provides:

- the holomorphic period map  $H^{2,0}(S) \times H_2(S) \to \mathbb{C}$  given numerically with certified precision bounds in quasilinear time with respect to precision;
- an embedding of the Néron-Severi lattice in  $H_2(S)$ , obtained heuristically as the kernel of the holomorphic period map;
- the Mordell-Weil group, with the lattice structure of its torsion free part, also depending on the heuristic of the Néron-Severi lattice.

By "heuristically", we mean that it is possible that the algorithm misses elements of the kernel or finds fake ones, in the case of numerical coincidence – we provide more detail at the end of Section 4.1. The algorithm presented here is implemented in Sagemath, as part of the package *lefschetz-family*<sup>1</sup>.

#### Previous works

Periods of algebraic curves are well studied as they coincide with Riemann surfaces [8, 41, 3, 33, 36]. Computation of the periods of some varieties realised as double cover ramified along hyperplane arrangements have been carried out in two distinct cases: in [7] in the case of Calabi-Yau manifolds given as double covers of  $\mathbb{P}^3$  ramified along 8 planes; and in [12] in the case of K3 surfaces given as a double cover of  $\mathbb{P}^2$  ramified along 6 lines.

A method for arbitrary dimensions was first given by [38] in the case of hypersurfaces. A computationally cheaper and more general approach was developed in [24]. The work presented in this paper is an improvement on the methods of [24], adapted to the case of elliptic surfaces.

# Outline

We consider elliptic surfaces with section. In the case where the elliptic surface only has simple singular fibres – i.e. singular fibres of type  $I_1$  – semi-numerical methods for computing the homology and periods were developed in [24]. One may always reduce to this case by deforming the elliptic surface in a way that separates its singular fibres into  $I_1$  fibres, thus obtaining a Lefschetz fibration. By continuity, the homology of this deformation is the same as that of the initial surface. As the monodromy representation of the deformation only depends on the type of the singular fibres, this can be done formally, without having to consider an explicit realisation of such a morsification.

This provides a way to compute an effective basis of the full homology lattice of an elliptic surface. Certain cycles can be expressed as *extensions*, i.e. lifts of paths in  $\mathbb{P}^1$  avoiding the critical values to  $H_1(S_t)$ . The lattice generated by these cycles and the fibre components is a full rank sub-lattice of "primary" cycles  $\operatorname{Prim}(S) \subset H_2(S)$ . Numerical approximations of the holomorphic periods of primary cycles can be computed through numerical integration. We may thus recover a numerical approximation of the full holomorphic period mapping  $H^{2,0}(S) \times H_2(S) \to \mathbb{C}$ . In turn, we may use the LLL algorithm to heuristically recover the Néron-Severi group in our basis of homology. This allows the computation of several algebraic invariants of the fibration, notably the Mordell-Weil group, and the lattice associated to its torsion-free part.

https://gitlab.inria.fr/epichonp/lefschetz-family

### 2. Elliptic surfaces

In this section, we recall the definition of an elliptic surface, as well as related notions that are useful to our discussion, notably the action of monodromy and Kodaira's classification of singular fibres. For further reading on elliptic surfaces, we recommend [37, 31].

**Definition 1.** Let V be a Riemann surface. An elliptic surface over V is an algebraic surface S along with a proper surjective map  $f: S \to V$  such that

- for all but finitely many  $t \in V$ , the fibre  $F_t = f^{-1}(t)$  is a smooth genus 1 algebraic curve (i.e. an elliptic curve);
- no fibre contains a smooth rational curve of self-intersection -1.

The second condition is there to ensure that the surface is minimal, as such rational curves can always be blown down. We denote by  $\Sigma$  the finite set of values  $t \in V$  over which the fibre  $F_t$  is not a smooth elliptic curve

In the following, we consider an elliptic surface  $f: S \to \mathbb{P}^1$  over  $V = \mathbb{P}^1$ .

**Definition 2.** A section of S is a map  $\pi: \mathbb{P}^1 \to S$  such that  $f \circ \pi = \mathrm{id}_V$ .

Sections of S are in bijection with  $\mathbb{C}(t)$ -points on the generic fibre E, which is an elliptic curve over  $\mathbb{C}(t)$ . Indeed, given a section  $\pi$ , the intersection of its image with the generic fibre im  $\pi \cap E$  yields a point in E. Conversely, given a point  $P \in E(\mathbb{C}(t))$ , its specialisation to any smooth fibre yields a point of the fiber. The closure  $\Gamma$  of the union of all these points yields a birational isomorphism  $f|_{\Gamma} : \Gamma \to \mathbb{P}^1$ . The inverse of this map gives the section associated to P, which we denote  $\bar{P}$ .

Throughout this paper, we will only consider elliptic surfaces with section. We will notably fix a section O of S, which we call the *zero section*. It will serve as the zero of the group of rational points  $E(\mathbb{C}(t))$  of the generic elliptic curve. Furthermore, to avoid the trivial case of a product  $E \times \mathbb{P}^1$ , we require in the rest of this text that the elliptic surface has at least one singular fibre.

# 2.1. Monodromy and extensions

We briefly recall the notions of monodromy and extensions, following [26] and Section 2.1.2 of [24]. The restriction of f to  $f^{-1}(\mathbb{P}^1 \setminus \Sigma)$  is a locally trivial fibration: if  $U \subset \mathbb{P}^1 \setminus \Sigma$  is open and simply connected, there is a trivialisation  $f^{-1}(U) \simeq F_b \times U$  of the fibration, for all  $b \in U$ . In particular a non-self-intersecting path  $\ell : [0,1] \to \mathbb{P}^1 \setminus \Sigma$  induces a diffeomorphism  $F_{\ell(0)} \simeq F_{\ell(1)}$  which is unique up to some automorphism of  $F_{\ell(1)}$  that is isotopic to the identity. Thus  $\ell$  induces an isomorphism  $\ell_* : H_1(F_{\ell(0)}) \to H_1(F_{\ell(1)})$ . For  $\ell$ ,  $\ell'$  two non-intersecting such paths compatible for concatenation that do not intersect, one may show that

$$(\ell'\ell)_* = \ell'_* \circ \ell_* \,, \tag{1}$$

(where  $\ell'\ell$  is the path that goes through  $\ell$  first, then through  $\ell'$ ). Using this formula, we can extend the notion of monodromy to self intersecting paths, and to loops. Furthermore, one may show that the map  $\ell_*$  depends only on the homotopy class of  $\ell$ .

Let  $b \in \mathbb{P}^1 \setminus \Sigma$ . The above construction yields a map

$$\begin{cases} \pi_1(\mathbb{P}^1 \setminus \Sigma, b) \to \operatorname{Aut}(H_1(F_b)) \\ [\ell] \mapsto \ell_* \end{cases} , \tag{2}$$

where  $[\ell]$  denotes the homotopy class of  $\ell$ . The map  $\ell_*$  is called the action of monodromy along  $\ell$  on  $H_1(F_b)$ . As  $F_b$  is an elliptic curve, its first homology group  $H_1(F_b)$  is a lattice of rank 2, equipped with a skew-symmetric intersection product. As readily seen from the trivialisation of the fibration, monodromy preserves the intersection product. Therefore, a simple computation shows that the matrix of the action of monodromy in a symplectic basis of  $H_1(F_b)$  belongs to  $\mathrm{SL}_2(\mathbb{Z})$ . It is possible to compute this matrix with semi-numerical computations involving the Picard-Fuchs equation of  $F_t$ , see [24, §3.5.2].

A related notion to monodromy is that of extensions. Given a non-intersecting path  $\ell$ , a simply connected neighbourhood V of im  $\ell$  and a 1-chain  $\Delta$  of  $F_{\ell(0)}$ , the identification of  $\Delta \times \operatorname{im} \ell$  in  $f^{-1}(V) \subset S$  produces a 2-chain with boundary in  $F_{\ell(0)} \cup F_{\ell(1)}$ . Once again, the relative homology class of this 2-chain is unique in the relative homology group  $H_2(S, F_{\ell(0)} \cup F_{\ell(1)})$ , and only depends on the homotopy class of  $\ell$  and the homology class of  $\Delta$ . Therefore we define the extension map  $\tau_{\ell}: H_1(F_{\ell(0)}) \to H_2(S, F_{\ell(0)} \cup F_{\ell(1)})$ . Similarly to monodromy, extensions satisfy a composition rule which allows to extend their definition to self-intersecting paths:

$$\tau_{\ell'\ell}(\gamma) = \tau_{\ell}(\gamma) + \tau_{\ell'}(\ell_*(\gamma)), \tag{3}$$

in  $H_2(S, F_{\ell(0)} \cup F_{\ell(1)} \cup F_{\ell'(1)})$ . In particular, when  $\ell$  is a loop pointed at b, we obtain a map

$$\tau: \begin{cases} \pi_1(\mathbb{P}^1 \setminus \Sigma, b) \times H_1(F_b) \to H_2(S, F_b) \\ [\ell], \Delta \mapsto \tau_\ell(\Delta) \end{cases}$$
 (4)

Extensions and monodromy are closely related by the formula

$$\delta(\tau_{\ell}(\gamma)) = \ell_* \gamma - \gamma \,, \tag{5}$$

where  $\delta: H_2(S, F_b) \to H_1(F_b)$  is the boundary map. This is represented in Fig. 1b

#### 2.2. The Kodaira classification

Kodaira provides a classification of the singular fibres of an elliptic fibration [23]: Let  $\sigma \in \Sigma$  be a critical value and  $\ell$  be the counter-clockwise simple loop around  $\sigma$  pointed at b. As stated in the previous section, the monodromy action  $\ell_*$  is represented by a matrix  $M \in \mathrm{SL}_2(\mathbb{Z})$  in a symplectic basis of  $H_1(F_b)$ . The  $\mathrm{SL}_2(\mathbb{Z})$ -conjugation class of M determines the type of the singular fibre.

These conjugacy classes are classified in two infinite families  $I_{\nu}$  and  $I_{\nu}^*$ ,  $\nu \in \mathbb{N}$  and six classes II, III, IV,  $II^*$ ,  $III^*$  and  $IV^*$ . Representatives  $M_T$  of these conjugacy classes are given in Table 1 (which is a reproduction of [4, Table 1]), along with a factorisation as a product of  $I_1$ -type monodromy matrices which will prove useful in Section 3.2.1, and the Euler characteristic of the fibre. These singular fibres have been extensively studied – for further reading on this topic, we recommend [13, Chap. 7].

# 3. Homology and periods of elliptic surfaces

In this section, we discuss means to recover the full homology lattice from the knowledge of the monodromy matrices of an elliptic fibration. Let us first fix some notations. Let  $f: S \to \mathbb{P}^1$  be an elliptic fibration.

- $F_v$  denotes the fibre above  $v \in \mathbb{P}^1$ . We will also use this notation for the homology class of  $F_v$  in  $H_2(S)$  when there is no ambiguity;
- ullet O denotes the zero section of S;
- $c_1, \ldots, c_r \in \mathbb{P}^1$  denote the critical values of f, and  $\Sigma = \{c_1, \ldots, c_r\}$ ;
- $m_v$  denotes the number of irreducible components of the fibre  $F_v$ ;
- $\Theta_0^v$  denotes the zero component of  $F_v$ , i.e. the irreducible component intersecting the zero section.
- $\Theta_1^v, \ldots, \Theta_{m_v-1}^v$  denote the irreducible singular components of  $F_v$  that are orthogonal to O;
- $\mathcal{T} \subset H_2(S, F_b)$  denotes the image of extension maps. For a basis  $\ell_1, \ldots, \ell_{r-1}$  of  $\pi_1(\mathbb{P}^1 \setminus \Sigma, b)$ , we have  $\mathcal{T} = \bigoplus_{i=1}^{r-1} \operatorname{im} \tau_{\ell_i}$  as a direct consequence of (3);

Type	$M_T$	Minimal normal factorisation	Euler characteristic of the fibre
$I_{\nu}, \nu \geq 1$	$\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$	$U^{ u}$	ν
II	$\left(\begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array}\right)$	VU	2
III	$\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$	VUV	3
IV	$\left(\begin{array}{cc} 0 & 1 \\ -1 & -1 \end{array}\right)$	$(VU)^2$	4
$I_{\nu}^*, \nu \geq 0$	$\left(\begin{array}{cc} -1 & -\nu \\ 0 & -1 \end{array}\right)$	$U^{\nu}(VU)^3$	$\nu + 6$
$II^*$	$\left(\begin{array}{cc} 0 & -1 \\ 1 & 1 \end{array}\right)$	$(VU)^5$	10
$III^*$	$\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right)$	$VUV(VU)^3$	9
$IV^*$	$\left(\begin{array}{cc} -1 & -1 \\ 1 & 0 \end{array}\right)$	$(VU)^4$	8

Table 1: The singular fibre types of the Kodaira classification, representatives of their  $\mathrm{SL}_2(\mathbb{Z})$  conjugacy class of the monodromy matrix, and the minimal normal factorisation of this representative in terms of the  $I_1$ -type matrices  $U=\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$  and  $V=\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$ .

We begin by a simple lemma connecting the homology groups  $H_2(S)$  and  $H_2(S, F_b)$ .

**Lemma 1.** Let  $\delta: H_2(S, F_b) \to H_1(F_b)$  be the boundary map. We have

$$H_2(S)/\langle F_b \rangle \simeq \ker \delta$$
. (6)

*Proof.* This is a direct consequence of the long exact sequence of relative homology of the pair  $(S, F_b)$ :

$$\langle F_b \rangle = H_2(F_b) \to H_2(S) \to H_2(S, F_b) \to^{\delta} H_1(F_b)$$
 (7)

Several cycles in  $H_2(S)$  are distinguished in that their associated holomorphic periods (that is, periods of holomorphic forms) are directly computable, see Section 4.1 below. In the next paragraph, we define a lattice  $Prim(S) \subset H_2(S)$  of such cycles. When the fibration is Lefschetz, Prim(S) coincides with the full homology lattice  $H_2(S)$ . In general, however, Prim(S) may be a proper sublattice of  $H_2(S)$ . Nevertheless, we will show in Section 4.1 that Prim(S) always has full rank. In particular, all the periods of S can be recovered from the periods of S.

More precisely, the periods associated to singular components and the section are 0. Furthermore, [24] provides a way to compute the periods of extensions. We call such cycles primary and define Prim(S) as follows:

**Definition 3.** The primary lattice Prim(S) is the sublattice of  $H_2(S)$  generated by extensions, fibre components and the zero section:

$$Prim(S) = \phi(\mathcal{T} \cap \ker \delta) \oplus \langle O, F_b \rangle \oplus \bigoplus_{v \in \Sigma} \bigoplus_{i=1}^{m_v - 1} \Theta_i^v,$$
(8)

where  $\phi : \ker \delta \to H_2(S)$  is a lift of the quotient of Lemma 1. Note that the choice of this lift does not matter.

**Remark 2.** Although not apparent in the notation, Prim(S) depends on the fibration of the surface, and not solely on S.

# 3.1. The Lefschetz case

When all the singular fibres are of type  $I_1$ , f is said to be a *Lefschetz fibration* of S. In this setting, [24] provides an algorithm for computing an effective basis of  $H_2(S)$  from the list of the monodromy matrices of a certain basis of the homotopy group  $\pi_1(\mathbb{C}\setminus\Sigma)$  (where  $\mathbb{C}\simeq\mathbb{P}^1\setminus\{\infty\}$  with  $\infty$  a regular value). In particular, we have the following theorem.

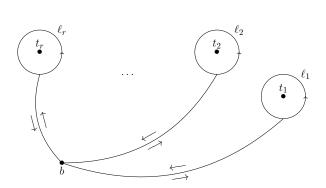
**Theorem 3.** When  $f: S \to \mathbb{P}^1$  is a Lefschetz fibration,  $Prim(S) = H_2(S)$ .

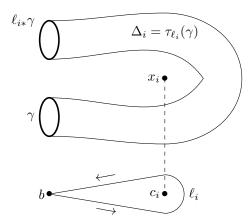
In order to prove this theorem, we first recall the elements of the construction in [24] that are relevant to our discussion. Let  $V \subset \mathbb{P}^1$  be diffeomorphic to a disk, let  $S^* = f^{-1}(V)$ , let  $\Sigma_V = \Sigma \cap V$ ,  $s = |\Sigma_V|$  and pick  $b \in V \setminus \Sigma_V$ .

**Definition 4.** A basis  $\ell_1, \ldots, \ell_s$  of  $\pi_1(V \setminus \Sigma_V, b)$  is distinguished if for every  $i, \ell_i$  is isotopic to a simple (i.e. with winding number 1) counterclockwise loop around a critical value of  $\Sigma_V$ , and the composition  $\ell_r \cdots \ell_1$  is isotopic to a simple counterclockwise loop around all the points of  $\Sigma_V$ . This is represented in Fig. 1a.

If  $\ell_1, \ldots, \ell_s$  is a distinguished basis of  $\pi_1(V \setminus \Sigma, b)$ , then for each i, im  $\tau_{\ell_i} \subset H_2(S^*, F_b)$  has rank 1 – its generator (up to sign) is called the *thimble* above  $c_i$ , denoted  $\Delta_i$ . The border of  $\Delta_i$  is called the *vanishing cycle* at  $c_i$  and is a generator of the image of  $\ell_{i*}$  – id, as can be readily seen from (5). An illustration of a thimble is given in Fig. 1b.

Thimbles serve as building blocks for  $H_2(S^*)$ , as the following lemma demonstrates.





(a) A distinguished basis. The composition  $\ell_1, \ldots, \ell_r$  is represented by the loop encircling all the critical values once counterclockwise.

(b) The thimble  $\Delta_i \in H_2(S, F_b)$  above a critical values  $c_i$  is the nontrivial extension of a 1-cycle  $\gamma \in H_1(F_b)$  along a loop  $\ell_i$  around a unique critical value  $c_i$ . For a given such loop  $\ell_i$ , there is a unique thimble (up to sign). Its boundary is  $\delta \Delta_i = \ell_{i*} \gamma - \gamma$ .

Figure 1: Distinguished bases and thimbles.

Lemma 4 ([26, Main lemma]).

$$H_2(S^*, F_b) = \bigoplus_{i=1}^s \mathbb{Z}\Delta_i \quad and \quad H_1(S^*, F_b) = 0.$$
 (9)

In particular when  $V = \mathbb{P}^1 \setminus \{\infty\}$  for a regular point  $\infty \notin \Sigma$ , the following lemmas shows how to recover  $H_2(S)$  from  $H_2(S^*, F_b)$ .

**Lemma 5.** ker  $(\iota: H_2(S^*, F_b) \to H_2(S, F_b)) = \operatorname{im} \tau_{\infty}$ , where  $\tau_{\infty} = \tau_{\ell_r \cdots \ell_1}$ . In other words, an element of  $H_2(S^*, F_b)$  is trivial in  $H_2(S, F_b)$  if and only if it is a sum of extensions along isotopy classes of  $\pi_1(V \setminus \Sigma, b)$  that are trivial in  $\pi_1(\mathbb{P}^1 \setminus \Sigma, b)$ .

*Proof.* This is the second line of diagram (16) in [24]. 
$$\Box$$

Lemma 6. There is a split short exact sequence

$$0 \to \mathcal{T} \to H_2(S, F_b) \to H_0(F_b) \to 0, \tag{10}$$

where the first map is the inclusion and the second map is the intersection with the generic fibre  $F_b$ . In other words,  $H_2(S, F_b) = \mathcal{T} \oplus \langle O \rangle$ .

*Proof.* We have the long exact sequence of the triplet  $(S, S^*, F_b)$ :

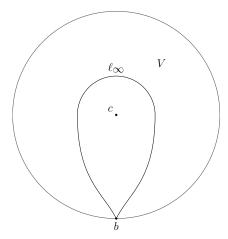
$$H_2(S^*, F_b) \to H_2(S, F_b) \to H_2(S, S^*) \to H_1(S^*, F_b)$$
 (11)

It follows from Lemma 4 that the image of the first map is  $\mathcal{T}$ . As  $(S, S^*) \simeq F_b \times (D, S^1)$ , the Künneth formula yields

$$H_2(S, S^*) \simeq H_0(F_b)$$
, (12)

where the identification is the intersection product with the generic fibre  $F_b$ . Finally, from Lemma 4,  $H_1(S^*, F_b) = 0$ . The sequence splits because  $H_0(F_b)$  is free.

Proof of Proposition 3. The proposition follows from Lemma 6 and Lemma 1.  $\Box$ 



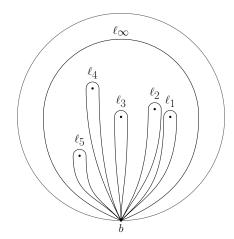


Figure 2: The morsification of a neighbourhood of a single critical value. Left: A neighbourhood V of a single critical value of the elliptic fibration f, along with a chosen basepoint b. The homotopy group  $\pi_1(V \setminus \{c\}, b)$  is generated by the counterclockwise loop  $\ell_{\infty}$ . Right: The neighbourhood after morsification. The critical fibre has split into five Lefschetz fibres. The homotopy group  $\pi_1(V \setminus \Sigma)$  is generated by the 5 counterclockwise loops  $\ell_1, \ldots, \ell_5$ . Thus  $H_2(f^{-1}(V), F_b)$  has rank 5 – it follows from Table 1 that the original singular fibre was of type  $I_5$ . Furthermore we see that  $\tau_{\ell_{\infty}} = \tau_{\ell_5...\ell_1}$ .

### 3.2. Morsification

We have seen in the previous section that the data of the monodromy matrices of the elliptic surface is sufficient to recover the homology in the Lefschetz case. In this section we extend this result to general elliptic surfaces.

In a nutshell, it is possible to deform the elliptic surface in a way that splits the non-Lefschetz singular fibres into several Lefschetz fibres. Therefore it is possible to compute the homology of the deformed elliptic surface from its monodromy matrices, which is by construction diffeomorphic to the original one. Interestingly, the monodromy matrices of the Lefschetz fibres of the deformation are determined by the monodromy matrices of the initial surface. In particular, this implies that we do not have to do any computations with (or even find) an explicit realisation of such a deformation. Instead it only serves as a formal computational tool.

More precisely, let  $V \subset \mathbb{P}^1$  be open and  $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$  denote the unit disk.

**Definition 5.** Let  $S \to V$  be an elliptic surface. Consider a commutative diagram

$$\tilde{S} \xrightarrow{\tilde{f}} V \times D \xrightarrow{p} D , \qquad (13)$$

where p is the projection onto the second coordinate. For  $u \in D$ , denote  $S_u = \eta^{-1}(u)$  and  $V_u = p^{-1}(u)$ . Such a diagram is a morsification of  $S \to V$  if

- $\eta: \tilde{S} \to D$  is a locally trivial smooth fibration;
- $\tilde{f}|_{S_0}: S_0 \to V_0$  coincides with  $S \to V$ ;
- for  $u \in D \setminus \{0\}$ ,  $\tilde{f}|_{S_u} : S_u \to V_u$  is a Lefschetz elliptic surface;
- $\eta$  has no critical values.

Remark 7. Such a deformation is sometimes also called a splitting deformation of the elliptic surface.

Morsifications of elliptic surfaces are useful as they allow use of the results for Lefschetz fibrations to obtain information about general elliptic surfaces. Indeed as  $\eta$  is a trivial fibration, it induces an isometry  $H_2(S_0) \simeq H_2(S_u)$  for all  $u \in D \setminus \{0\}$ , and we may apply the results of Section 3.1 to the describe the latter. The existence of a morsification of any elliptic surface is guaranteed by the following theorem of [32].

**Theorem 8** ([32, Thm. 8]). Let  $S \to V$  be an elliptic surface. There exists a morsification of S. Moreover, the number of singular fibres of  $S_u$  for  $u \in D \setminus \{0\}$  does not depend on u.

**Remark 9.** In our setting, the existence of a section prevents the possibility of multiple fibres mentioned in [32].

We will apply this result to neighbourhoods of each critical value to obtain local morsifications. For  $c \in \Sigma$ , define  $D_c$  a disk around c in  $\mathbb{P}^1$  such that  $b \notin D_c$  Let  $\ell_c$  be a path connecting b to a point  $b_c \in \partial D_c$ . Assume that for  $c \neq c'$ ,  $D_c \cap D_{c'} = \emptyset$  and the interior of  $\ell_c$  and  $\ell_{c'}$  do not intersect. Let  $\infty \notin \bigcup_{c \in \Sigma} D_c \cup \{b\}$ , identify  $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ , and let  $S^* = f^{-1}(\mathbb{C})$ . Define  $T_c = f^{-1}(D_c)$  and  $F_{b_c} = f^{-1}(b_c)$ . Then we have the following lemma.

Lemma 10. The inclusion yields an isomorphism

$$\bigoplus_{c \in \Sigma} H_*(T_c, F_{b_c}) \to H_*(S^*, F_b) , \qquad (14)$$

where the identification  $F_{bc} \simeq F_b$  is given by  $\ell_c$ .

*Proof.* The main line of the argument is that the retraction of  $\mathbb{C}$  to  $\bigcup_{c \in \Sigma} D_c \cup \ell_c$  lifts through the fibration. For more details, see [26, §5.3].

Notice that  $T_c \to D_c$  is an elliptic surface with a single singular fibre.

## 3.2.1. Local morsification

Let  $V \subset \mathbb{P}^1$  be a disk in  $\mathbb{P}^1$  and consider an elliptic surface  $S \to V$  with a single singular fibre. Let c denote the critical value and b a regular value on the boundary of V. Fix a symplectic basis of  $H_1(F_b)$  and let  $M_{\infty}$  be the monodromy matrix around c in this basis. From Theorem 8, there exists a morsification of  $S \to V$ 

$$\tilde{S} \xrightarrow{\tilde{f}} V \times D \xrightarrow{p} D . \tag{15}$$

Let  $t \neq c$  and  $S' = \eta^{-1}(t)$ . Then  $S' \to V_t$  is a Lefschetz fibration. Denote by r its number of singular fibres and by  $\Sigma$  its set of critical values.

**Lemma 11.** Following the terminology of [4], the number r of singular fibres of  $S' \to D$  is the number of factors in the minimal normal factorisation of  $M_T$  (see Table 1). Let  $G_r \ldots G_1$  be this factorisation. Let  $A \in \operatorname{SL}_2(\mathbb{Z})$  be a matrix such that  $M_{\infty} = AM_TA^{-1}$ . Then there is a distinguished basis of  $\pi_1(D \setminus \Sigma_t, b)$  such that the corresponding monodromy matrices  $M_1, \ldots, M_r$  are given by  $M_i = AG_iA^{-1}$ .

*Proof.* The first part is simply the observation that

$$r = \chi(S) = \chi(S') = \sum_{v \in \Sigma} \chi(F_v), \qquad (16)$$

and  $\chi(F_v) = 1$  for every v as  $F_v$  is of type  $I_1$ . For the second part, let  $\ell_1, \ldots, \ell_r$  be a distinguished basis of  $\pi_1(D \setminus \Sigma, b)$ . Let  $1 \le i \le r - 1$  and notice that the bases

$$(\ell_1, \dots, \ell_{i-1}, \ell_i \ell_{i+1} \ell_i^{-1}, \ell_i, \ell_{i+2}, \dots, \ell_r)$$
 (17)

and

$$(\ell_1, \dots, \ell_{i-1}, \ell_{i+1}, \ell_{i+1}^{-1}, \ell_i \ell_{i+1}, \ell_{i+2}, \dots, \ell_r)$$
(18)

are also distinguished bases. Then the result is a direct application of [4, Thm.19], as Hurwitz moves on the product  $M_r \dots M_i$  can be achieved by the above changes of basis, see (1).

**Remark 12.** As shown by [4, Thm. 21], the choice of the factorisation of  $M_T$  as a product of  $I_1$  monodromy matrices does not matter. We could equivalently take any such minimal factorisation, such as the ones in [35, Table 5].

Pick such a distinguished basis  $\ell_1, \ldots, \ell_r$  of  $\pi_1(D \setminus \Sigma, b)$  and let  $\Delta_1, \ldots, \Delta_r$  be the corresponding thimbles. Then the trivialisation of  $\tilde{S}$  through  $\eta$  yields an isomorphism  $H_2(S, F_b) \simeq H_2(S', F_b') = \bigoplus_{i=1}^r \mathbb{Z}\Delta_i$ . We conclude with two lemmas linking extensions and singular components of S to this basis of thimbles.

**Lemma 13.** Let  $\ell_{\infty}$  be the simple loop around c pointed at b. Then

$$\tau_{\ell_{\infty}} = \sum_{i=1}^{r} \tau_{\ell_i} \circ \ell_{i-1_*} \circ \cdots \circ \ell_{1_*}. \tag{19}$$

*Proof.* For  $t \in D$ , let  $\Sigma_t$  be the set of critical values of  $\tilde{S}_t \to D$ . Define  $\tilde{\Sigma} = \bigcup_{t \in D} \Sigma_t \times \{t\}$ .  $\tilde{\Sigma}$  is an analytic set of  $V \times D$  and the projection onto D is a finite morphism of degree r, totally ramified at c. Clearly,  $\ell_{\infty}$  and  $\ell_r \dots \ell_1$  have the same homotopy class in  $\pi_1((V \times D) \setminus \tilde{\Sigma})$ . The lemma is then a direct application of (3).

**Lemma 14.** The inclusion of the singular components of  $F_0$  in  $H_2(S, F_b)$  coincides with the kernel of the boundary map:

$$\bigoplus_{i=1}^{m_0-1} \langle \Theta_i^c \rangle = \ker \left( \delta : H_2(S', F_b') \to H_1(F_b') \right) . \tag{20}$$

In particular,  $r = m_0$  if  $F_0$  has type  $I_{\nu}$  and  $m_0 + 1$  otherwise.

*Proof.* The direct inclusion is clear. The rank of this kernel is r-1 in the case of fibres of type  $I_{\nu}$  and r-2 in the other cases. Therefore, if the mentioned equality holds, the last statement follows.

Let us detail the proof of the equality in the case of a fibre of type  $I_3$ . Its morsification splits it into 3 fibres of type  $I_1$ , for each of which the monodromy matrix is (up to a global conjugation) U. There are thus 3 thimbles, the restriction of the boundary map to these thimbles has rank 1, and the kernel thus has rank 3-1=2. The intersection matrix of (the lift in  $H_2(S)$  of) this kernel is

$$\left(\begin{array}{cc} -2 & -1 \\ -1 & -2 \end{array}\right),\tag{21}$$

and it is thus a sublattice of discriminant 3. As this coincides with the discriminant of the sublattice generated by the singular components [23], and as one is contained in the other, these sublattices are equal. A similar direct computation gives the same result for each possible fibre type.

An illustration of the effects of a morsification is provided in Fig. 2.

## 3.3. Global homology and periods

We are now ready to give an algorithm for computing  $H_2(S)$  (with its lattice structure) from the action of monodromy on  $F_b$  of the fibration  $S \to \mathbb{P}^1$ .

Note that per Lemma 14 the sublattice generated by the singular components of a given singular fibre is explicitly identified in this description of homology; and per Lemma 13, the extensions of  $\mathcal{T} \cap \ker \delta$  are also explicitly identified.

To conclude this section, we provide a way to compute the periods of certain 2-forms on this basis of homology. Let  $\omega \in H^2(S)$  and assume that  $\omega = \omega_t \wedge dt$  for some 1-form  $\omega_t \in H^1(S)$ , where t denotes the dependance on a coordinate of  $\mathbb{P}^1$ . Then the integral of  $\omega$  on an extension can be obtained as a path integral of a period of the elliptic fibre via the observation that

$$\int_{\tau_{\ell}(\eta)} \omega = \int_{\ell} \left( \int_{\eta_t} \omega_t \right) dt \,, \tag{22}$$

## **Algorithm 1** Homology of elliptic surface

**Input:** the monodromy matrices  $M_1, \ldots, M_r \in \mathrm{SL}_2(\mathbb{Z})$ 

**Output:** a description of  $H_2(S)$  with its lattice structure

$$N \leftarrow []$$

for  $1 \le i \le r$  decreasing do

Find  $A \in SL_2(\mathbb{Z})$  and T such that  $M_i = AM_TA^{-1}$ 

for W in the minimal normal factorisation of  $M_T$  do

Append  $AWA^{-1}$  to N

for  $M_i$  in N do

Compute  $d_i \in \mathbb{Z}^{2 \times 1}$  and  $m_i \in \mathbb{Z}^{1 \times 2}$  such that  $M_i = I_2 + d_i m_i$ .

$$T_i \leftarrow (-1)^{n-1} \begin{pmatrix} \mathbf{0} \\ \mathbf{m}_i \\ \mathbf{0} \end{pmatrix}$$
, where  $m_i$  is the *i*-th line.

$$B \leftarrow \left(\begin{array}{c|c} d_1 & \cdots & d_r \end{array}\right)$$

$$T_{\infty} \leftarrow T_1 + T_2 M_1 + T_3 M_2 M_1 + \cdots + T_r M_{r-1} \cdots M_1$$

$$k \leftarrow \ker R$$

 $i \leftarrow \operatorname{im} T_{\infty}$ 

 $H \leftarrow k/i$ 

H is identified to the subspace of  $H_2(S^*)/H_2(F_b)$  generated by extensions. The (representatives of) vectors of H are the coordinates in the basis of thimbles of  $H_2(S^*, F_b)$ . For more details, see [24, §§3,5].

 $\triangleright$  See Table 1 for  $M_T$ 

where  $\eta_t \in H_1(F_t)$  is the unique deformation of  $\eta \in H_1(F_t)$  along  $\ell$ . In particular, it is possible to recover the periods of  $\omega$  on  $\mathcal{T}$  with high precision from the Picard-Fuchs equation of  $\omega_t$  using numerical integration methods with quasilinear algorithmic complexity with respect to precision alone [19, 28]. In practice, we rely on the implementation provided in Sagemath by [29] in the ore\_algebra package [21]. For further details on the computation of periods on thimbles, see [24, §3.7].

As the fibre components are localised in a single fibre, the periods of  $\omega$  on a fibre component are all zero. The following lemma shows that this information, i.e. the periods on extension and fibre components, is sufficient to recover the full period mapping.

**Lemma 15.** The primary lattice Prim(S) has full rank.

*Proof.* For each  $c \in \Sigma$ ,  $H_2(T_c, F_{b_c})_{\mathbb{Q}} = (\ker \delta_c)_{\mathbb{Q}} \oplus (\operatorname{im} \tau_c)_{\mathbb{Q}}$ . Furthermore,  $\mathcal{T} = \bigoplus_{c \in \Sigma} \operatorname{im} \tau_c$ . For a  $\mathbb{Z}$ -module A, denote by  $A_{\mathbb{Q}}$  the tensor product  $A \otimes \mathbb{Q}$ . From Lemma 14 and Lemma 10, we have that

$$(\mathcal{T} \cap \ker \delta)_{\mathbb{Q}} \oplus \bigoplus_{\substack{c \in \Sigma \\ 1 \le i \le m_c - 1}} \langle \Theta_i^c \rangle_{\mathbb{Q}}$$

$$= (\ker \delta)_{\mathbb{Q}} \cap \bigoplus_{c \in \Sigma} (\operatorname{im} \tau_c)_{\mathbb{Q}} \oplus (\ker \delta_c)_{\mathbb{Q}},$$

$$= (\ker \delta)_{\mathbb{Q}}.$$

$$(23)$$

Lemma 1 allows to conclude.

Let  $\mathcal{B} = (\eta_1, \dots, \eta_s, F_b, O)$  be the basis of  $H_2(S)$  obtained from Algorithm 1, where  $\eta_i$  lies in  $\phi(H_2(S, F_b))$ for a lift  $\phi : \ker \delta \to H_2(S)$ . Let

$$\mathcal{B}' = (\Gamma_1, \dots, \Gamma_t, \Theta_1, \dots, \Theta_{s-t}, F_b, O) \tag{24}$$

be a basis of Prim(S), such that for each  $i, \Gamma_i \in \mathcal{T} \cap \ker \delta$  is an extension and  $\Theta_i \in \ker \delta_c$  for some  $c \in \Sigma$  is a fibre component. Over  $\mathbb{Q}$ ,  $\mathcal{B}'$  is also a basis of  $H_2(S)_{\mathbb{Q}}$ , and the matrix of change of basis  $M_{\mathcal{B}'\to\mathcal{B}}\in\mathrm{GL}(\mathbb{Q})$ has integer coefficients and can be computed with Lemma 13 and Lemma 14.

Using the methods of [24, §3.7], we may numerically compute the periods  $\int_{\Gamma_i} \omega$ . Furthermore, the periods  $\int_{\Theta_i} \omega$ ,  $\int_{F_b} \omega$  and  $\int_O \omega$  are all zero<sup>2</sup> [40, §1]. Let  $\pi_{\mathcal{B}} = \left(\int_{\gamma} \omega\right)_{\gamma \in \mathcal{B}}$  be the vector of periods of  $\omega$  on a basis  $\mathcal{B}$ . Then

$$\pi_{\mathcal{B}'} = \left( \int_{\Gamma_1} \omega, \dots, \int_{\Gamma_r} \omega, 0, \dots, 0 \right) \tag{25}$$

and

$$\pi_{\mathcal{B}} = M_{\mathcal{B}' \to \mathcal{B}}^{-1} \pi_{\mathcal{B}'} \,. \tag{26}$$

# 4. Application: Mordell-Weil group and lattice

In this section, we expalin how to compute explicit embeddings of the Néron-Severi lattice in the description of  $H_2(S)$  given in the previous section. We then use this to recover the Mordell-Weil group, and the lattice structure of its torsion-free part, the Mordell-Weil lattice.

We start by recalling generalities about these lattices.

**Definition 6.** The Néron-Severi lattice NS(S) is the sublattice of  $H_2(S)$  generated by classes of divisors. Its rank is called the Picard rank or Picard number or Néron-Severi rank.

**Definition 7.** The trivial lattice Triv(S) is the sublattice of NS(S) generated by the zero section and the fibre components. Its orthogonal complement is the essential lattice  $L(S) = Triv(S)^{\perp}$ .

**Definition 8.** The Mordell-Weil group  $E(\mathbb{C}(t))$  of the elliptic curve  $E/\mathbb{C}(t)$  is the group of its  $\mathbb{C}(t)$ -rational points.

As mentioned in the beginning of Section 2, sections of  $S \to \mathbb{P}^1$  are in bijection with  $E(\mathbb{C}(t))$ . The following lemma shows that the group structure of  $E(\mathbb{C}(t))$  coïncides with the lattice structure of  $H_2(S)$  modulo the trivial lattice.

**Theorem 16** ([37, Thm 6.5]). The map  $P \mapsto \bar{P} \mod \operatorname{Triv}(S)$  is an isomorphism from  $E(\mathbb{C}(t))$  to  $\operatorname{NS}(S)/\operatorname{Triv}(S)$ .

In particular, this equips the torsion-free part of the Mordell-Weil group with a lattice structure, inherited from the lattice structure on  $\mathrm{NS}(S) \subset H_2(S)$ . More precisely, the orthogonal projection of  $NS(S)_{\mathbb{Q}} = NS(S) \otimes \mathbb{Q}$  onto  $L(S)_{\mathbb{Q}} = L(S) \otimes \mathbb{Q}$  defines a map  $\phi : E(\mathbb{C}(t)) \to L(S)_{\mathbb{Q}}$ . The kernel of this map is the torsion subgroup, and thus  $\phi$  equips  $E(\mathbb{C}(t))/E(\mathbb{C}(t))_{tor}$  with a rational lattice structure.

**Definition 9.** The Mordell-Weil lattice MWL(S) of S is the resulting lattice  $E(\mathbb{C}(t))/E(\mathbb{C}(t))_{tor}$ .

For further reading on this topic, we recommend [37].

### 4.1. Computing the Néron-Severi lattice

The first step toward computing the Mordell-Weil lattice is to compute the Néron-Severi lattice NS(S). By Lefschetz's (1,1) theorem [16, §1.2], it is entirely characterised as the kernel of the holomorphic period mapping.

**Theorem 17** (Lefschetz (1,1) theorem). Let  $\omega_1, \ldots, \omega_s$  be a basis of the space of holomorphic 2-forms of S,  $H^{2,0}(S)$ , and consider the period map  $\pi: H_2(S) \to \mathbb{C}^s, \gamma \mapsto (\int_{\gamma} \omega_1, \ldots, \int_{\gamma} \omega_s)$ . Then

$$NS(S) = \ker \pi. \tag{27}$$

<sup>&</sup>lt;sup>2</sup>The vanishing of the periods on the zero section is a consequence of the fact that parabolic cohomology is the orthogonal complement of the trivial lattice, see [6].

We compute this kernel heuristically using the LLL method. In order to do this we need two things: a basis of  $H^{2,0}(S)$  and numerical approximations of the associated periods.

Let  $\omega$  be a holomorphic 2-form on S. As an element of  $H^0(S,\Omega_S^2)$ , it can be written as

$$\omega = f(t)\omega_t \wedge dt, \qquad (28)$$

where  $\omega_t \in H^1(E)$  is a rational section of the holomorphic 1-form bundle of the generic fibre, and  $f \in \mathbb{Q}(t)$  is a rational function. This representation is well adapted to the integration algorithm of [24, §3.7]. Of course the converse is not true: not every rational function will yield a holomorphic 2-form on S. In fact the rational functions for which this is true are very tightly controlled by the *Picard-Fuchs equation*  $\Lambda$  of  $\omega_t$  – that is, the minimal differential equation satisfied by  $\omega_t$  with respect to the connection inherited from the derivation on  $\mathbb{C}(t)$  through the fibration over  $\mathbb{P}^1$ .

Before focusing on the rational functions, let us briefly recall how  $\omega_t$  and its Picard-Fuchs equation can be computed. Let  $P_t$  be the defining equation of the elliptic fibration. In particular, the defining equation for  $F_a$  is  $P_a$  whenever a is regular.

**Proposition 1.** There is a natural isomorphism Res :  $H^2(\mathbb{P}^2 \setminus F_a) \to H^1(F_a)$ . It is called the residue mapping.

Since  $\mathbb{P}^2 \setminus F_a$  is affine, its De Rham cohomology can be computed using algebraic forms directly [17]. More precisely, it is the quotient of the space of homogeneous rational fractions of the form  $\frac{A}{P^k}$  with degree -3 by derivatives of the same form. Furthermore, the holomorphic form of  $H^1(F_a)$  coincides with the residue of the image of  $\frac{1}{P_a}$  in this description [15, (8.6)]. All in all,  $\omega_t$  can be taken to be Res  $\frac{1}{P_t}$ . Its Picard-Fuchs equation can then be computed using Griffiths-Dwork reduction [5, §5.3]. For a more detailed discussion, see [24, §3.2].

We now turn back to the computation of valid rational coefficients. The following result of [40] gives a way to compute rational functions  $f_1, \ldots, f_r \in \mathbb{Q}(t)$  such that:

- $f_i(t)\omega_t \wedge dt$  defines a holomorphic 2-form on S;
- and  $f_1(t)\omega_t \wedge dt, \ldots, f_r(t)\omega_t \wedge dt$  is a basis of  $H^{2,0}(S)$ .

**Theorem 18** ([40, §3]). There is a divisor  $\mathfrak{A}_0$  on  $\mathbb{P}^1$  depending only on  $\Lambda$  such that if  $Z \in L(\mathfrak{A}_0)$ , then  $\frac{Z}{W}\omega_t \wedge \mathrm{d}t$  is a holomorphic 2-form on S, where  $W \in \mathbb{Q}(t)$  is the Wronskian of  $\Lambda$ . Furthermore the map  $L(\mathfrak{A}_0) \to H^{2,0}(S)$  is an isomorphism.

An algorithm for computing  $\mathfrak{A}_0$  is given in [40, §3], and we recall it here for the sake of completeness. Assume that the Picard-Fuchs equation  $\Lambda$  has order two<sup>3</sup>. In particular at any point  $p \in \mathbb{P}^1$ , the space of local solutions in a slit neighbourhood of p is generated by two solutions, that are locally of the form

$$(t-p)^{q}(h_{1}(t-p)\log(t-p)+h_{2}(t-p)), (29)$$

with  $q \in \mathbb{Q}$ ,  $h_1$  and  $h_2$  two holomorphic functions in a neighbourhood of 0. Let  $r \leq s$  be the respective leading exponents of these two solutions. Then the order of  $\mathfrak{A}_0$  at p is

$$\operatorname{ord}_{p} \mathfrak{A}_{0} = \begin{cases} -\lfloor s \rfloor - 3 & \text{if } p = \infty \\ -\lfloor s \rfloor + 1 & \text{otherwise} \end{cases}$$
(30)

In particular, if p is a finite regular point of  $\Lambda$ ,  $\operatorname{ord}_p \mathfrak{A}_0 = 0$ , meaning we can consider solely the singular points of  $\Lambda$ . For further reading on the topic, we recommend [40] and [10, §4.3].

The local exponents of  $\Lambda$  can be obtained symbolically (see [14] and [28, §4] for instance), and we thus have a method to compute a basis of the holomorphic 2-forms of S, with a presentation that is well suited

 $<sup>^{3}</sup>$ this may happen when S is an isotrivial elliptic surface. In that case, the author knows does not know of a way to identify the holomorphic form.

for the integration methods of the periods mentioned in Section 3.3. We can thus compute high precision numerical approximations of the holomorphic periods of S. The Néron-Severi group can be heuristically computed by recovering integer linear relations between these periods. Indeed, let  $\alpha_i$  be integers. Then

$$\int_{\sum_{i} \alpha_{i} \gamma_{i}} \omega = \sum_{i} \alpha_{i} \left( \int_{\gamma_{i}} \omega \right) = 0 \text{ for all } \omega \in H^{2,0}(S)$$
(31)

if and only if the cycle  $\sum_i \alpha_i \gamma_i \in \text{NS}(S)$ , where the  $\gamma_i$ 's form a basis of  $H_2(S)$ . Thus integer linear relations between the holomorphic period vectors  $(\int_{\gamma_i} \omega_1, \dots, \int_{\gamma_i} \omega_s)$  are in bijection with NS(S).

In order to recover these linear relations, we use the LLL algorithm. This computation is not certified,

In order to recover these linear relations, we use the LLL algorithm. This computation is not certified, and may fail in two ways: the algorithm may miss integer relations with large coefficients, or may recover "fake" linear relations that hold up to very high precision. More precisely, the algorithm provides a sublattice  $\Lambda \subset H_2(X)$  and positive numbers B, N and  $\varepsilon$  that depend on precision such that

- 1.  $\Lambda = NS(X)$ ; or
- 2. NS(X) is not generated by elements of the form  $\sum_i \alpha_i \gamma_i$  with  $\sum_i \alpha_i^2 \leq B$ ; or
- 3. There  $\sum_i \alpha_i \gamma_i \notin NS(X)$  such that

$$\sum_{j} \left| \sum_{i} \alpha_{i} \int_{\gamma_{i}} \omega_{j} \right|^{2} \leq \varepsilon^{2} \quad \text{and} \quad \sum_{i} \alpha_{i}^{2} \leq N^{2}.$$
 (32)

In practice, for 300 recovered decimal digits of precision for the periods of an elliptic K3 surface (which can be obtained in a few seconds), we find  $B \simeq 10^{132}$ , N = 3, and  $\varepsilon \simeq 10^{-271}$ . For further discussion on these issues, see [25].

## 4.2. Example: elliptic curve with high Mordell-Weil rank over Q

In this section we detail the workings of our algorithm on an explicit example. A Sagemath worksheet reproducing the results mentioned here is available at  $example\_paper\_elliptic.ipynb^4$ . The elliptic surface S we consider is an elliptic K3 with Picard rank 19 used in [11, §9] (with u = 5 in their notations) to find the elliptic curve with highest known Mordell-Weil rank over  $\mathbb{Q}$  for which the Mordell-Weil torsion subgroup is  $\mathbb{Z}/2\mathbb{Z}$ . Its defining equation is

$$X^{3} + 4A(t)X^{2}Z + 512B(t)XZ^{2} - Y^{2}Z$$
(33)

where

$$A(t) = 93273t^4 + 58840t^3 + 102618t^2 + 35680t + 14485$$
(34)

and

$$B(t) = -8590032t^8 - 78412620t^7 + 17011856t^6 + 241822775t^5 - 19459741t^4 - 127136490t^3 + 16161642t^2 + 15406335t - 2083725$$
(35)

The homology lattice of S

This elliptic fibration has 16 singular fibres above points  $c_1, \ldots, c_{16}$ . We pick a basepoint b as well as a distinguished basis  $\ell_1, \ldots, \ell_{16}$  of  $\pi_1(\mathbb{C}^1 \setminus \{c_1, \ldots, c_{16}\}, b)$ . The corresponding monodromy matrices in a chosen symplectic basis  $\gamma_1, \gamma_2$  of the homology of the fibre are given by

$$M_{1} = \begin{pmatrix} 7 & 9 \\ -4 & -5 \end{pmatrix}, \quad M_{i} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } i = 2, 3, 15$$

$$M_{i} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \text{ for } i = 4, 9, 11, 12, \text{ and}$$

$$M_{i} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \text{ for } i = 5, 6, 7, 8, 10, 13, 14, 16.$$

$$(36)$$

 $<sup>^4 {\</sup>tt https://nbviewer.org/urls/gitlab.inria.fr/epichonp/eplt-support/-/raw/main/example\_paper\_elliptic.ipynb} \\$ 

Computing the  $SL_2(\mathbb{Z})$  conjugation class of these matrices, one finds that eight fibres (those for which the monodromy matrix is  $M_5$ ) are  $I_2$  fibres, and the remaining eight are Lefschetz, i.e. of type  $I_1$ . Indeed, we have

$$M_5 = AM_{I_2}A^{-1} \text{ with } A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}).$$
 (37)

Per the minimal normal factorisation of Table 1,  $M_{I_2} = U^2$ . Therefore in a morsification S' of S, there is a distinguished basis  $\ell'_1, \ldots, \ell'_{24}$  of  $\pi_1(\mathbb{C} \setminus \Sigma', b)$  consisting of  $8 + 8 \times 2 = 24$  elements, where  $\Sigma'$  is the set of critical values of the morsification, and such that the associated monodromy matrices are given by

$$M'_{1} = \begin{pmatrix} 7 & 9 \\ -4 & -5 \end{pmatrix}, \quad M'_{i} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } i = 2, 3, 22,$$

$$M'_{i} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \text{ for } i = 4, 13, 16, 17, \text{ and}$$

$$M'_{i} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = AUA^{-1} \text{ for all other } i.$$

$$(38)$$

We may then use the methods of [24, §§3,5] to compute an effective basis  $\Gamma_1, \ldots, \Gamma_{22}$  of the homology of S' in terms of the thimbles  $\Delta'_1, \ldots, \Delta'_{24}$ , the fibre class and the zero section. For instance, we find that a non trivial homology class is given by  $\Delta'_2 - \Delta'_{22}$ : as  $M'_2 = M'_{22}$ , this relative homology class has empty boundary and thus lifts to a class in  $H_2(S')$ . It is non-trivial as  $\ell'_{22}\ell'_2^{-1}$  is non-trivial in  $\pi_1(\mathbb{P}^1 \setminus \Sigma')$ .

With this description, the singular components coming from an  $I_2$  fibre of S above a critical value c can be obtained as the kernel of the thimbles of critical points flowing together at c. More explicitly, the component of the fibre above  $c_5$  is the homology class corresponding to the lift of  $\Delta'_5 - \Delta'_6$ .

Furthermore, extensions of S can also be described in this basis. For example

$$\tau_{\ell_6^{-1}\ell_5}(\gamma_2) = \tau_{\ell_8^{-1}\ell_7^{-1}\ell_6\ell_5}(\gamma_2) = \Delta_5 + \Delta_6 - \Delta_7 - \Delta_8.$$
(39)

All in all we obtain the coordinates of a basis of Prim(S) in the basis of  $H_2(S)$  obtained from the morsification S'. From the Picard-Fuchs equation of the surface, we recover using Theorem 18 that the space of holomorphic forms of S is generated by

$$\omega = \operatorname{Res} \frac{1}{P_t} \wedge \mathrm{d}t \,. \tag{40}$$

From then on, we can compute the periods on the primary lattice, and recover the full period mapping using the coordinates computed above. For example, we find that the holomorphic period of the first element of the basis of homology we computed is

$$\int_{\Gamma_1} \omega = -0.0007064447191 \dots - i0.0002821239749 \dots, \tag{41}$$

with certified precision bounds of around 150 digits.

Using the LLL algorithm, we find that the Néron-Severi lattice has rank 19 as expected. Finally the Mordell-Weil group is obtained as the quotient of the Néron-Severi lattice by the trivial lattice. We find

$$MW(S) \simeq \mathbb{Z}^9 \times \mathbb{Z}/2\mathbb{Z}$$
. (42)

It should be noted that the result of [11, §9] is stronger than what we have computed here. First our approach for computing the Néron-Severi group is heuristic as it relies on the LLL algorithm – in particular it is not a certified computation, and thus does not provide a proof. Secondly we have merely shown a result for the Mordell-Weil group over  $\mathbb{C}(t)^5$ , whereas [11] shows that this computation holds over  $\mathbb{Q}$  for one of the fibres.

<sup>&</sup>lt;sup>5</sup>or rather over  $\bar{\mathbb{Q}}(t)$ .

#### Acknowledgements

I would like to thank Charles Doran, Pierre Lairez, Erik Panzer, Duco van Straten, and Pierre Vanhove for valuable and insightful discussions.

# **Funding**

This work has been supported by the Agence nationale de la recherche (ANR), grant agreement ANR-19-CE40-0018 (De Rerum Natura), grant agreement ANR-20-CE40-0026-01 (Symmetries and moduli spaces in algebraic geometry and physics); and the European Research Council (ERC) under the European Union's Horizon Europe research and innovation programme, grant agreement 101040794 (10000 DIGITS)

#### References

- [1] Bloch, S., Kerr, M., Vanhove, P., 2018. Local mirror symmetry and the sunset Feynman integral. Adv. Theor. Math. Phys. 21, 1373–1453. doi:10.4310/ATMP.2017.v21.n6.a1.
- [2] Bönisch, K., Fischbach, F., Klemm, A., Nega, C., Safari, R., 2021. Analytic structure of all loop banana integrals. J. High Energy Phys. 2021, 41. doi:10.1007/JHEP05(2021)066. id/No 66.
- [3] Bruin, N., Sijsling, J., Zotine, A., 2019-01-28. Numerical computation of endomorphism rings of Jacobians, 155–171doi:10/ggck8d.
- [4] Cadavid, C.A., Vélez, J.D., 2009. Normal factorization in  $SL(2,\mathbb{Z})$  and the confluence of singular fibers in elliptic fibrations. Beitr. Algebra Geom. 50, 405–423.
- [5] Cox, D.A., Katz, S., 1999. Mirror symmetry and algebraic geometry. volume 68 of Math. Surv. Monogr. Providence, RI: American Mathematical Society.
- [6] Cox, D.A., Zucker, S., 1979. Intersection numbers of sections of elliptic surfaces. Invent. Math. 53, 1–44. doi:10.1007/ BF01403189.
- [7] Cynk, S., van Straten, D., 2019. Periods of rigid double octic Calabi-Yau threefolds. Ann. Pol. Math. 123, 243–258. doi:10.4064/ap180608-23-10.
- [8] Deconinck, B., van Hoeij, M., 2001. Computing Riemann matrices of algebraic curves. Physica D 152-153, 28-46. doi:10.1016/S0167-2789(01)00156-7.
- [9] Doran, C.F., Harder, A., Pichon-Pharabod, E., Vanhove, P., 2023. Motivic geometry of two-loop feynman integrals.
- [10] Doran, C.F., Kostiuk, J., 2023. Geometric variations of local systems and elliptic surfaces. Israel Journal of Mathematics . 1–79.
- [11] Elkies, N.D., Klagsbrun, Z., 2020. New rank records for elliptic curves having rational torsion, in: ANTS XIV. Proceedings of the fourteenth algorithmic number theory symposium, Auckland, New Zealand, virtual event, June 29 July 4, 2020. Berkeley, CA: Mathematical Sciences Publishers (MSP), pp. 233–250. doi:10.2140/obs.2020.4.233.
- [12] Elsenhans, A.S., Jahnel, J., 2022. Real and complex multiplication on k 3 surfaces via period integration. Experimental Mathematics , 1–32.
- [13] Esole, M., 2017. Introduction to Elliptic Fibrations. Springer International Publishing, Cham. pp. 247–276. doi:10.1007/978-3-319-65427-0\_7.
- [14] Frobenius, G., 1873. On the integration of linear differential equations by means of series. J. Reine Angew. Math. 76, 214–235. doi:10.1515/crll.1873.76.214.
- [15] Griffiths, P.A., 1969. On the periods of certain rational integrals. I, II. Ann. Math. (2) 90, 460–495, 496–541. doi:10. 2307/1970746.
- [16] Griffiths, P.A., Harris, J., 1978. Principles of algebraic geometry.
- [17] Grothendieck, A., 1966. On the De Rham cohomology of algebraic varieties. Publ. Math., Inst. Hautes Étud. Sci. 29, 95–103. doi:10.1007/BF02684807.
- [18] Hindry, M., 2007. Why is it difficult to compute the mordell-weil group. Diophantine geometry 4, 197-219.
- [19] van der Hoeven, J., 1999. Fast evaluation of holonomic functions. Theor. Comput. Sci. 210, 199-215. doi:10.1016/ S0304-3975(98)00102-9.
- [20] Hori, K., Katz, S., Klemm, A., Pandharipande, R., Thomas, R., Vafa, C., Vakil, R., Zaslow, E., 2003. Mirror symmetry. volume 1 of *Clay Math. Monogr.* Providence, RI: American Mathematical Society (AMS).
- [21] Kauers, M., Jaroschek, M., Johansson, F., 2015. Ore polynomials in Sage, in: Computer algebra and polynomials. Applications of algebra and number theory. Berlin: Springer, pp. 105–125. doi:10.1007/978-3-319-15081-9\_6.
- [22] Koblitz, N., 1987. Elliptic curve cryptosystems. Math. Comput. 48, 203–209. doi:10.2307/2007884.
- [23] Kodaira, K., 1963. On compact analytic surfaces. II. Ann. Math. (2) 77, 563-626. doi:10.2307/1970131.
- [24] Lairez, P., Pichon-Pharabod, E., Vanhove, P., 2023. Effective homology and periods of complex projective hypersurfaces. arXiv:2306.05263.
- [25] Lairez, P., Sertöz, E.C., 2019. A numerical transcendental method in algebraic geometry: computation of Picard groups and related invariants. SIAM J. Appl. Algebra Geom. 3, 559–584. doi:10.1137/18M122861X.

- [26] Lamotke, K., 1981. The topology of complex projective varieties after S. Lefschetz. Topology 20, 15–51. doi:10.1016/ 0040-9383(81)90013-6.
- [27] Lang, S., 1978. Elliptic curves: Diophantine analysis. volume 231 of Grundlehren Math. Wiss. Springer, Cham.
- [28] Mezzarobba, M., 2010. NumGfun: a package for numerical and analytic computation with D-finite functions, in: Proceedings of the 35th international symposium on symbolic and algebraic computation, ISSAC 2010, Munich, Germany, July 25–28, 2010. New York, NY: Association for Computing Machinery (ACM), pp. 139–145. doi:10.1145/1837934.1837965.
- [29] Mezzarobba, M., 2016. Rigorous multiple-precision evaluation of d-finite functions in sagemath. CoRR abs/1607.01967. URL: http://arxiv.org/abs/1607.01967, arXiv:1607.01967.
- [30] Miller, V.S., 1986. Use of elliptic curves in cryptography. Advances in cryptology CRYPTO '85, Proc. Conf., Santa Barbara/Calif. 1985, Lect. Notes Comput. Sci. 218, 417-426 (1986).
- [31] Miranda, R., 1989. The basic theory of elliptic surfaces. Notes of lectures. Pisa: ETS Editrice.
- [32] Moishezon, B., 1977. Complex surfaces and connected sums of complex projective planes. volume 603 of Lect. Notes Math. Springer, Cham.
- [33] Molin, P., Neurohr, C., 2019. Computing period matrices and the Abel-Jacobi map of superelliptic curves 88, 847–888. doi:10/ggck8t.
- [34] Mordell, L.J., 1922. On the rational solutions of the indeterminate equations of the third and fourth degrees. Proc. Camb. Philos. Soc. 21, 179–192.
- [35] Naruki, I., 1987. On confluence of singular fibers in elliptic fibrations. Publ. Res. Inst. Math. Sci. 23, 409–431. doi:10. 2977/prims/1195176546.
- [36] Neurohr, C., 2018. Efficient integration on Riemann surfaces & applications. Ph.D. thesis.
- [37] Schütt, M., Shioda, T., 2010. Elliptic surfaces, in: Algebraic geometry in East Asia Seoul 2008. Proceedings of the 3rd international conference "Algebraic geometry in East Asia, III", Seoul, Korea, November 11–15, 2008. Tokyo: Mathematical Society of Japan, pp. 51–160.
- [38] Sertöz, E.C., 2019. Computing periods of hypersurfaces. Math. Comput. 88, 2987–3022. doi:10.1090/mcom/3430.
- [39] Shioda, T., et al., 1990. On the mordell-weil lattices. Rikkyo Daigaku sugaku zasshi 39, 211–240.
- [40] Stiller, P.F., 1987. The Picard numbers of elliptic surfaces with many symmetries. Pac. J. Math. 128, 157–189. doi:10. 2140/pjm.1987.128.157.
- [41] Swierczewski, C., 2017. Abelfunctions: A library for computing with Abelian functions, Riemann surfaces, and algebraic curves. URL: https://github.com/abelfunctions/abelfunctions.
- [42] Weil, A., 1929. L'arithmétique sur les courbes algébriques. Acta mathematica 52, 281-315.
- [43] Wiles, A., 1995. Modular elliptic curves and Fermat's Last Theorem. Ann. Math. (2) 141, 443–551. doi:10.2307/2118559.