STAT6018 Research Frontiers in Data Science

Topic II: Introduction to empirical process theory

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Course Logistics

Course website: https://yugu-stat.github.io/teaching/stat6018

Lectures: Attendance is required

Final presentation: At Week 4, present an arbitrary theorem/lemma and its proof from the references within 20 mins (including Q & A).

References:

- van der Vaart, A. W. & Wellner, J. A. (1996). Weak Convergence and Empirical Processes. New York: Springer.
- Sen, B. (2018). A gentle introduction to empirical process theory and applications.
- Kosorok, M. R. (2008). Introduction to empirical processes and semiparametric inference. New York: Springer.

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What is an empirical process?

- A *stochastic process* is a collection of random variables $\{X(t), t \in T\}$ on the same probability space, indexed by an arbitrary index set T.
- In general, an *empirical process* is a stochastic process based on a random sample, usually of n i.i.d. random variables $X_1, \ldots, X_n$.

Example: empirical distribution function

Let $X_1, \ldots, X_n$ be i.i.d. real-valued random variables with cumulative distribution function (c.d.f.) F. Then the *empirical distribution function* (e.d.f.) is defined as

$$\mathbb{F}_n(t) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(X_i \le t), \quad t \in \mathbb{R}.$$

$\mathbb{F}_n(t)$ is one of the simplest examples of an empirical process.

Example: Kaplan-Meier estimator

Let $(X_1, \delta_1), \ldots, (X_n, \delta_n)$ be a sample of right-censored failure time observations. Then the *Kaplan-Meier estimator* of the survival function is given by

$$\widehat{S}(t) = \prod_{k:\, T_k^0 \leq t} \left\{ 1 - \frac{\sum_{i=1}^n \delta_i \mathbf{1}(X_i = T_k^0)}{\sum_{i=1}^n \mathbf{1}(X_i \geq T_k^0)} \right\},$$

where $T_1^0 < T_2^0 < \cdots < T_K^0$ are unique observed failure times.

$\widehat{S}(t)$ is another simple example of an empirical process.

General features of an empirical process

- The i.i.d. sample $X_1, \ldots, X_n$ is drawn from a probability measure P on an arbitrary sample space $\mathcal{X}$.
- Define the *empirical measure* to be $\mathbb{P}_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$, where δ_x denotes the Dirac measure at x.
- For a measurable function $f : \mathcal{X} \mapsto \mathbb{R}$, define

$$\mathbb{P}_n f := \int f d\mathbb{P}_n = \frac{1}{n} \sum_{i=1}^n f(X_i).$$

- For any class $\mathcal{F}$ of such real-valued functions on $\mathcal{X}$, $\{\mathbb{P}_n f : f \in \mathcal{F}\}$ is the empirical process indexed by $\mathcal{F}$.

Start with the classical e.d.f. $\mathbb{F}_n$

- Setting $\mathcal{X} = \mathbb{R}$, $\mathbb{F}_n$ can be re-expressed as the empirical process $\{\mathbb{P}_n f : f \in \mathcal{F}\}$, where $\mathcal{F} = \{\mathbb{1}(x \leq t), t \in \mathbb{R}\}$.
- By the law of large numbers, $\mathbb{F}_n(t) \overset{a.s.}{\to} F(t)$ for each $t \in \mathbb{R}$.
- By the central limit theorem, for each $t \in \mathbb{R}$,

$$\mathbb{G}_n(t) := \sqrt{n}\,(\mathbb{F}_n(t) - F(t)) \overset{d}{\to} N\Big(0, F(t)(1 - F(t))\Big).$$

- From the functional perspective, uniform results over $t \in \mathbb{R}$ would be more appealing.
  - Need theory of empirical processes

Strengthened results on $\mathbb{F}_n$ and $\mathbb{G}_n$

- Glivenko (1933) and Cantelli (1933) demonstrated that the previous result could be strengthened to

$$\|\mathbb{F}_n - F\|_\infty = \sup_{t \in \mathbb{R}} |\mathbb{F}_n(t) - F(t)| \xrightarrow{a.s.} 0.$$

- Donsker (1952) showed that

$$\mathbb{G}_n \xrightarrow{d} \mathbb{B}(F) \quad \text{in } \ell^\infty(\mathbb{R}),$$

where $\mathbb{B}$ is the standard Brownian bridge process on $[0, 1]$; for any index set T, $\ell^\infty(T)$ denotes the space of all bounded functions $f : T \mapsto \mathbb{R}$.

Extend to general empirical processes

- Properties of the approximation of Pf by $\mathbb{P}_n f$, uniformly in $\mathcal{F}$
  - the random quantity $\|\mathbb{P}_n - P\|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} |\mathbb{P}_n f - Pf|$
  - the empirical process $\mathbb{G}_n := \sqrt{n}(\mathbb{P}_n - P)$
- Two special classes
  - **Glivenko-Cantelli:** $\mathcal{F}$ is P-Glivenko-Cantelli if

$$\|\mathbb{P}_n - P\|_{\mathcal{F}} \overset{a.s.}{\to} 0.$$

  - **Donsker:** $\mathcal{F}$ is P-Donsker if

$$\mathbb{G}_n \overset{d}{\to} \mathbb{G} \quad \text{in } \ell^\infty(\mathcal{F}),$$

where $\mathbb{G}$ is a mean zero Gaussian process indexed by $\mathcal{F}$, and $\ell^\infty(\mathcal{F}) = \left\{x : \mathcal{F} \mapsto \mathbb{R} \middle| \, \|x\|_{\mathcal{F}} < \infty\right\}$.

Remarks

- Glivenko-Cantelli (GC): uniform almost surely convergence
- Donsker: uniform central limit theorem
- Donsker $\Rightarrow$ GC
- GC or Donsker properties depend crucially on the complexity of $\mathcal{F}$.

Complexity of $\mathcal{F}$

For a given norm $\|\cdot\|$, such as the $L_r(Q)$-norms, define the covering and bracketing numbers as follows:

Covering number

- denoted by $N(\epsilon, \mathcal{F}, \|\cdot\|)$
- minimum number of balls $B(f;\epsilon) := \{g : \|g - f\| \leq \epsilon\}$ needed to cover $\mathcal{F}$
- *entropy without bracketing*: $\log N(\epsilon, \mathcal{F}, \|\cdot\|)$

Bracketing number

- denoted by $N_{[]}(\epsilon, \mathcal{F}, \|\cdot\|)$
- minimum number of brackets $[\ell, u]$ with $\|\ell - u\| < \epsilon$ needed to cover $\mathcal{F}$
- *entropy with bracketing*: $\log N_{[]}(\epsilon, \mathcal{F}, \|\cdot\|)$

GC theorems

Theorem 1 (GC with bracketing)

A function class $\mathcal{F}$ is a P-Glivenko-Cantelli if

$$N_{[]}(\epsilon, \mathcal{F}, L_1(P)) < \infty, \qquad \text{for every } \epsilon > 0.$$

Theorem 2 (GC without bracketing)

A function class $\mathcal{F}$ is a P-Glivenko-Cantelli if

$$\sup_Q N(\epsilon \|F\|_{Q,1}, \mathcal{F}, L_1(Q)) < \infty, \qquad \text{for every } \epsilon > 0,$$

where F is an envelope function[a] of $\mathcal{F}$, and the supremum is over all probability measures Q on $\mathcal{X}$.

[a] *An envelope function of a class $\mathcal{F}$ is any function $x \mapsto F(x)$ such that $|f(x)| \leq F(x)$, for every x and $f \in \mathcal{F}$.*

Donsker theorems

Theorem 3 (Donsker with bracketing entropy integral)

A function class $\mathcal{F}$ is a P-Donsker if

$$\int_0^\infty \sqrt{\log N_{[]}\left(\epsilon, \mathcal{F}, L_2(P)\right)} d\epsilon < \infty.$$

Theorem 4 (Donsker with uniform entropy integral)

A function class $\mathcal{F}$ is a P-Donsker if

$$\int_0^\infty \sup_Q \sqrt{\log N\left(\epsilon \|F\|_{Q,2}, \mathcal{F}, L_2(Q)\right)} d\epsilon < \infty,$$

where F is an envelope function of $\mathcal{F}$, and the supremum is over all probability measures Q on $\mathcal{X}$.

M-estimators

- Definition:
  - Metric space: (Θ, d)
  - $m_\theta : \mathcal{X} \to \mathbb{R}$, for each $\theta \in \Theta$
  - "Empirical gain": $M_n(\theta) = \mathbb{P}_n m_\theta$
  - M-estimator: $\hat{\theta}_n = \arg\max_{\theta \in \Theta} M_n(\theta)$
- Examples:
  - Maximum (penalized) likelihood estimator
  - Least squares estimator
  - Nonparametric maximum likelihood estimator

Application: consistency of M-estimators

- Two assumptions:
  1. $\mathcal{F} := \{m_\theta(\cdot) : \theta \in \Theta\}$ is P-GC
  2. θ_0 is a well-separated maximizer of $M(\theta) = Pm_\theta$, i.e., for every $\delta > 0$, $M(\theta_0) > \sup_{\theta \in \Theta : d(\theta,\theta_0) \geq \delta} M(\theta)$.
- For fixed $\delta > 0$, let $\psi(\delta) = M(\theta_0) - \sup_{\theta \in \Theta : d(\theta,\theta_0) \geq \delta} M(\theta) > 0$

$$
\begin{aligned}
\left\{ d(\hat{\theta}_n, \theta_0) \geq \delta \right\} &\Rightarrow M(\hat{\theta}_n) \leq \sup_{\theta \in \Theta : d(\theta,\theta_0) \geq \delta} M(\theta) \\
&\Leftrightarrow M(\hat{\theta}_n) - M(\theta_0) \leq -\psi(\delta) \\
&\Rightarrow M(\hat{\theta}_n) - M(\theta_0) + \left(M_n(\theta_0) - M_n(\hat{\theta}_n) \right) \leq -\psi(\delta) \\
&\Rightarrow 2 \sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| \geq \psi(\delta)
\end{aligned}
$$

$$
\Rightarrow \mathbb{P}\left(d(\hat{\theta}_n, \theta_0) \geq \delta \right) \leq \mathbb{P}\left(\sup_{\theta \in \Theta} |M_n(\theta) - M(\theta)| \geq \psi(\delta)/2 \right) \to 0.
$$

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Covering and packing numbers

Let (Θ, d) be an arbitrary semi-metric space.

Definition 5 (Covering number)

The ϵ-covering number $N(\epsilon, \Theta, d)$ is the minimal number of balls $B(x;\epsilon) := \{y \in \Theta : d(x,y) \leq \epsilon\}$ of radius ϵ needed to cover the set Θ. The corresponding entropy number is $\log N(\epsilon, \Theta, d)$.

Definition 6 (Packing number)

Call a collection of points ϵ-separated if the distance between each pair of points is larger than ϵ. The packing number $D(\epsilon, \Theta, d)$ is the maximum number of ϵ-separated points in Θ.

Covering and packing numbers (cont.)

Lemma 7 (Covering vs packing numbers)

$$D(2\epsilon, \Theta, d) \leq N(\epsilon, \Theta, d) \leq D(\epsilon, \Theta, d), \quad \forall \epsilon > 0.$$

Thus, packing and covering numbers have the same scaling in the radius ϵ.

- The first inequality can be easily proved by contradiction.
- The second inequality follows by the fact that Θ can be covered by the balls $B(\theta_i; \epsilon)$ $(i = 1, \ldots, D)$, where $\theta_1, \ldots, \theta_D$ are the ϵ-separated points associated with the packing number D.

Example: bounded sets on Euclidean space

Example 8 (Bounded sets on Euclidean space)

For any bounded subset $\Theta \subset \mathbb{R}^p$, there exist constants $c < C$ such that

$$c\left(\frac{1}{\epsilon}\right)^p \leq N(\epsilon, \Theta, \|\cdot\|) \leq C\left(\frac{1}{\epsilon}\right)^p, \quad \forall \epsilon \in (0,1).$$

Proof.

The union of $D(\epsilon, \Theta, \|\cdot\|)$ number of ϵ-separated balls of radius $\epsilon/2$ is contained in the set $\Theta' := \{\theta \in \mathbb{R}^p : \|\theta - \Theta\| < \epsilon/2\}$. Thus, $D(\epsilon, \Theta, \|\cdot\|) v_p \left(\frac{\epsilon}{2}\right)^p \leq Vol(\Theta')$, where v_p is the volume of the unit ball. On the other hand, $D(2\epsilon, \Theta, \|\cdot\|)$ number of 2ϵ-separated balls cover the set Θ. Thus, $D(2\epsilon, \Theta, \|\cdot\|) v_p (2\epsilon)^p \geq Vol(\Theta)$. The desired inequalities then follow by the above results and Lemma 7. $\square$

Example: bounded Lipschitz functions

Example 9 (Bounded Lipschitz functions)

Let $\mathcal{F} := \{f : [0,1] \mapsto [0,1] \mid f \text{ is 1-Lipschitz}\}$. Then there exists some constant A such that

$$\log N(\epsilon, \mathcal{F}, \|\cdot\|_\infty) \leq \frac{A}{\epsilon}, \quad \forall \epsilon > 0.$$

Proof.

If $\epsilon \geq 1$, take $f_0 \equiv 0$ and observe that $\forall f \in \mathcal{F}$, $\|f - f_0\|_\infty \leq 1 \leq \epsilon$. Then $N(\epsilon, \mathcal{F}, \|\cdot\|_\infty) = 1$.
Let $0 < \epsilon < 1$. Define a ϵ-grid of the interval $[0,1]$ (for both axes), i.e. $0 = a_0 < a_1 < \cdots, a_N = 1$ where $N = \lfloor 1/\epsilon \rfloor + 1$ and $a_k = k\epsilon$ for $k = 1, \cdots, N-1$.
Let $B_1 := [a_0, a_1]$ and $B_k := (a_{k-1}, a_k]$ for $k = 2, \cdots, N$.

Example: bounded Lipschitz functions (cont.)

Proof (cont.)

For each $f \in \mathcal{F}$, define the step function $\tilde{f} : [0,1] \mapsto \mathbb{R}$ as

$$\tilde{f}(x) = \sum_{k=1}^{N} \epsilon \left\lfloor \frac{f(a_k)}{\epsilon} \right\rfloor \mathbb{1}_{B_k}(x).$$

Clearly, $\tilde{f}$ is constant on each interval B_k and can only take values of the form $i\epsilon$ for $i = 0, \cdots, N-1$.
For any $x \in [0,1]$, suppose that $x \in B_k$. By the Lipschitz property of f and the construction of $\tilde{f}$,

$$\left|f(x) - \tilde{f}(x)\right| \leq |f(x) - f(a_k)| + \left|f(a_k) - \tilde{f}(a_k)\right| \leq 2\epsilon.$$

Therefore, $\left\|f - \tilde{f}\right\|_\infty \leq 2\epsilon$.

Example: bounded Lipschitz functions (cont.)

Proof (cont.)

Now we count the number of distinct $\tilde{f}$'s obtained as f varies over $\mathcal{F}$. There are at most N choices for $\tilde{f}(a_1)$. Further, note that for any $\tilde{f}$ and $k = 2, \cdots, N$,

$$\begin{aligned} &\left|\tilde{f}(a_k) - \tilde{f}(a_{k-1})\right| \\ \le &\left|\tilde{f}(a_k) - f(a_k)\right| + \left|f(a_k) - f(a_{k-1})\right| + \left|f(a_{k-1}) - \tilde{f}(a_{k-1})\right| \le 3\epsilon. \end{aligned}$$

Thus, for fixed $\tilde{f}(a_{k-1})$, there are at most 7 choices left for $\tilde{f}(a_k)$. Therefore,

$$N(\epsilon, \mathcal{F}, \|\cdot\|_\infty) \le \left(\lfloor 1/\epsilon \rfloor + 1\right) 7^{\lfloor 1/\epsilon \rfloor},$$

which completes the proof. $\square$

Bracketing numbers

Let $(\mathcal{F}, \|\cdot\|)$ be a subset of a normed space of real functions $f : \mathcal{X} \mapsto \mathbb{R}$ on some set $\mathcal{X}$.

Definition 10 (Bracketing number)

Given two functions $l(\cdot)$ and $u(\cdot)$, the bracket $[l, u]$ is the set of all functions $f \in \mathcal{F}$ with $l(x) \leq f(x) \leq u(x), \forall x \in \mathcal{X}$. An ϵ-bracket is a bracket $[l, u]$ with $\|l - u\| < \epsilon$. The bracketing number $N_{[]}(\epsilon, \mathcal{F}, \|\cdot\|)$ is the minimum number of the ϵ-brackets needed to cover $\mathcal{F}$. The entropy with bracketing is $\log N_{[]}(\epsilon, \mathcal{F}, \|\cdot\|)$.

Bracketing numbers (cont.)

Theorem 11 (Bracketing vs covering numbers)

Suppose that $\|\cdot\|$ has the Riesz property[a]. Then

$$N(\epsilon, \mathcal{F}, \|\cdot\|) \leq N_{[]}(2\epsilon, \mathcal{F}, \|\cdot\|), \quad \forall \epsilon > 0.$$

[a] $|f| \leq |g|$ *implies that* $\|f\| \leq \|g\|$.

- The proof uses the fact that every f within the 2ϵ-bracket $[l, u]$ falls within the ball $B(\frac{l+u}{2}; \epsilon)$.
- In general, there is no converse inequality, so that bracketing numbers are bigger than covering numbers.
- A bracket gives pointwise control over a function.
- A ball under the $L_r(Q)$-norm gives integrated control over a function.

Example: distribution functions

Example 12 (Distribution functions)

Recall that the function class relevant to the e.d.f. $\mathbb{F}_n$ *is* $\mathcal{F} = \{\mathbb{1}_{(-\infty,t]} \mid t \in \mathbb{R}\}$. *The bracketing numbers of* $\mathcal{F}$ *are of polynomial orders:*

$$N_{[]}(\epsilon, \mathcal{F}, L_1(P)) \leq \frac{2}{\epsilon},$$
$$N_{[]}(\epsilon, \mathcal{F}, L_2(P)) \leq \frac{2}{\epsilon^2}.$$

Proof.

Consider the brackets of the form $[\mathbb{1}_{(-\infty,t_{i-1}]}, \mathbb{1}_{(-\infty,t_i]}]$ for a grid of points $-\infty = t_0 < t_1 < \cdots < t_N = \infty$ such that $F(t_i) - F(t_{i-1}) < \epsilon$ for $i = 1, \ldots, N$, where $N = \lfloor 1/\epsilon \rfloor + 1 < 2/\epsilon$.
Clearly, these brackets can cover $\mathcal{F}$. Moreover, these brackets have $L_1(P)$-size ϵ and $L_2(P)$-size bounded by $\sqrt{\epsilon}$ (since $Pf^2 \leq Pf$ for every $0 \leq f \leq 1$). $\square$

Example: classes Lipschitz in a parameter

Example 13 (Classes Lipschitz in a parameter)

Consider a function class $\mathcal{F} = \{m_\theta : \theta \in \Theta\}$ *which has a Lipschitz dependence on* θ*, i.e., there exists some function* $F : \mathcal{X} \mapsto \mathbb{R}$ *such that*

$$|m_{\theta_1}(x) - m_{\theta_2}(x)| \leq F(x)d(\theta_1, \theta_2), \quad \forall \theta_1, \theta_2 \in \Theta, \forall x \in \mathcal{X}.$$

Then, for any norm $\|\cdot\|$*,*

$$N_{[]}(2\epsilon \|F\|, \mathcal{F}, \|\cdot\|) \leq N(\epsilon, \Theta, d).$$

Proof.

Let $\theta_1, \cdots, \theta_p$ be an ϵ-cover of Θ (under the metric d).
Then for every $\theta \in B(\theta_i; \epsilon)$, $|m_\theta(x) - m_{\theta_i}(x)| \leq \epsilon F(x)$.
Thus, the brackets $[m_{\theta_i} - \epsilon F, m_{\theta_i} + \epsilon F]$ $(i = 1, \cdots, p)$, each of size $2\epsilon \|F\|$, can cover $\mathcal{F}$. $\square$

Monotone functions

Theorem 14 (Monotone functions)

The class $\mathcal{F}$ of monotone functions $f : \mathbb{R} \mapsto [0,1]$ satisfies

$$\log N_{[]}(\epsilon, \mathcal{F}, L_r(Q)) \leq K(\frac{1}{\epsilon}), \quad \forall \epsilon > 0,$$

for every probability measure Q, every $r \geq 1$, and some constant K that depends on r only.

- The result implies that $\mathcal{F}$ is Donsker (by Theorem 3).
- See Theorem 2.7.5 of VW for the proof.

Smooth functions

- $\mathcal{X}$: bounded, convex subset of $\mathbb{R}^p$ with nonempty interior
- $\underline{\alpha}$: largest integer smaller than α, for any $\alpha > 0$
- D^k: differential operator of order k
- For a function $f : \mathcal{X} \mapsto \mathbb{R}$, define

$$\|f\|_\alpha = \max_{k \le \underline{\alpha}} \sup_{D^k, x} |D^k f(x)| + \sup_{D^{\underline{\alpha}}, x, y} \frac{|D^{\underline{\alpha}} f(x) - D^{\underline{\alpha}} f(y)|}{\|x - y\|^{\alpha - \underline{\alpha}}}$$

- $C_M^\alpha(\mathcal{X})$: set of all continuous functions $f : \mathcal{X} \mapsto \mathbb{R}$ with $\|f\|_\alpha \le M$ (f has uniformly bounded partial derivatives and the highest partial derivatives are Lipschitz)

Smooth functions (cont.)

Theorem 15 (Smooth functions)

There exists a constant K depending only on α, diam$\mathcal{X}$, and p such that

$$\log N(\epsilon, C_1^\alpha(\mathcal{X}), \|\cdot\|_\infty) \le K \left(\frac{1}{\epsilon}\right)^{p/\alpha},$$

$$\log N_{[]}(\epsilon, C_1^\alpha(\mathcal{X}), L_r(Q)) \le K \left(\frac{1}{\epsilon}\right)^{p/\alpha},$$

for every $\epsilon > 0$, $r \ge 1$, and probability measure Q.

See Theorem 2.7.1 and Corollary 2.7.2 of VW for the proofs.

Convex functions

Theorem 16 (Convex functions)

For a compact, convex subset $C \subset \mathbb{R}^p$, the class $\mathcal{F}$ of all convex functions $f : C \mapsto [0,1]$ that are L-Lipschitz satisfies

$$\log N(\epsilon, \mathcal{F}, \|\cdot\|_\infty) \leq K(1+L)^{p/2} \left(\frac{1}{\epsilon}\right)^{p/2},$$

for some constant K depending on p and C only.

See Corollary 2.7.10 of VW for the proof.

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Tail probability of random variables

- **Markov's inequality**
  Let $Z \geq 0$ be a random variable. Then for any $t > 0$,
  $$P(Z \geq t) \leq \frac{EZ}{t}.$$
- **Cheyshev's inequality**
  If Z has a finite variance $Var(Z)$, then
  $$P(|Z - EZ| \geq t) \leq \frac{Var(Z)}{t^2}.$$

But these inequalities can only yield a tail bound of order t^{-2}, which may be too relaxed. The tail bound can be improved to an exponential decrease in t^2 by Hoeffding's inequality.

Hoeffding's inequality

Lemma 17 (Hoeffding's inequality)

Let $X_1, \ldots, X_n$ be independent bounded random variables such that $X_i \in [a_i, b_i]$ with probability 1. Let $S_n = \sum_{i=1}^n X_i$. Then,

$$P(S_n - ES_n \geq t) \leq e^{-2t^2/\sum_{i=1}^n (b_i - a_i)^2},$$
$$P(S_n - ES_n \leq -t) \leq e^{-2t^2/\sum_{i=1}^n (b_i - a_i)^2}.$$

The proof uses Markov's inequality and the following lemma:

Lemma 18

Let X be a random variable with $EX = 0$ and $X \in [a, b]$ with probability 1. Then for any $\lambda > 0$,

$$E(e^{\lambda X}) \leq e^{\lambda^2 (b-a)^2/8}.$$

Sub-Gaussian random variables

Definition 19 (Sub-Gaussian random variables)

A random variable X is called sub-Gaussian if there exist constants $C, v > 0$ such that $P(|X| > t) \leq Ce^{-vt^2}$ for every $t > 0$.

Some equivalent characterizations of sub-Gaussian random variables:

- There exists $a > 0$ such that $E[e^{aX^2}] < \infty$.
- Laplace transform condition: $\exists B, b > 0$ such that $\forall \lambda \in \mathbb{R}, Ee^{\lambda(X - E[X])} \leq Be^{\lambda^2 b}$.
- Moment condition: $\exists K > 0$ such that $\forall p \geq 1$, $(E|X|^p)^{1/p} \leq K\sqrt{p}$.
- Union bound condition: $\exists c > 0$ such that $\forall n \geq c$,

  $$E[\max\{|X_1 - E[X]|, \ldots, |X_n - E[X]|\}] \leq c\sqrt{\log n}$$

  where $X_1, \ldots, X_n$ are i.i.d. copies of X.

Sub-Gaussian processes

Definition 20 (Sub-Gaussian processes)

Let (T, d) be a semi-metric space and $\{X_t, t \in T\}$ be a stochastic process indexed by T. Then X_t is called sub-Gaussian w.r.t. the semi-metric d if

$$P(|X_s - X_t| > u) \leq 2\exp\left(-\frac{u^2}{2d(s,t)^2}\right), \quad \forall s, t \in T, u > 0.$$

Any Gaussian process is sub-Gaussian w.r.t. the standard deviation semi-metric $d(s,t) = \sqrt{\mathrm{Var}(X_s - X_t)}$.

Rademacher process and Hoeffding's inequality

Consider the *Rademacher process*

$$X_a = \sum_{i=1}^{n} a_i \varepsilon_i, \quad a = (a_1, \ldots, a_n) \in \mathbb{R}^n, \tag{1}$$

where ε_i's are independent Radermacher variables which take values $+1$ and -1 with probability $1/2$.

By the following special case of Hoeffding's inequality, Rademacher process is also sub-Gaussian (w.r.t. the Euclidean distance).

Lemma 21 (Hoeffding's inequality)

The Rademacher process $\{X_a : a \in \mathbb{R}^n\}$ *defined in* (1) *satisfies*

$$P(|X_a| > t) \le 2e^{-t^2/(2\|a\|^2)}.$$

Bernstein's inequality

The following result gives tail bounds for random variables with larger than normal tails.

Lemma 22 (Bernstein's inequality)

For independent random variables $Y_1, \ldots, Y_n$ with zero means and bounded ranges $[-M, M]$, there exists a constant $v \geq Var(\sum_{i=1}^n Y_i)$ such that

$$P(|\sum_{i=1}^{n} Y_i| > t) \leq 2e^{-\frac{t^2}{2(v+Mt/3)}}.$$

- See page 855 of Shorack and Wellner (1986)[1] for the proof.
- Compared to the normal tail bound $e^{-t^2/(2v)}$, the extra term $2Mt/3$ can be seen as a penalty for the non-normality.
- When $n \to \infty$, $Mt/3$ is typically negligible w.r.t. v.

[1] *Shorack, G. R., & Wellner, J. A. (1986). Empirical Processes with Applications to Statistics. Wiley, New York.*

Maximal inequalities

Lemma 23 (Maximal inequality for sub-Gaussian variables)

Suppose that $Y_1, \ldots, Y_N$ (not necessarily independent) are sub-Gaussian in the sense that $Ee^{\lambda Y_i} \leq e^{\lambda^2 \sigma^2 / 2}$ for all $\lambda > 0$ and $i = 1, \ldots, N$. Then,

$$E \max_{i=1,\ldots,N} Y_i \leq \sigma \sqrt{2 \log N}.$$

Proof.

By Jensen's inequality, we have

$$e^{\lambda E \max_{i=1,\ldots,N} Y_i} \leq E e^{\lambda \max_{i=1,\ldots,N} Y_i} \leq \sum_{i=1}^{N} E e^{\lambda Y_i} \leq N e^{\lambda^2 \sigma^2 / 2}.$$

Tanking logarithms yields

$$E \max_{i=1,\ldots,N} Y_i \leq \frac{\log N}{\lambda} + \frac{\lambda \sigma^2}{2} \leq \sigma \sqrt{2 \log N}.$$

□

Maximal inequalities (cont.)

Corollary 24

Let ψ be a strictly increasing, convex, non-negative function. Suppose that $\xi_1, \ldots, \xi_N$ are random variables such that $E[\psi(|\xi_i|/c_i)] \leq L$ for $i = 1, \ldots, N$ and some constant L. Then,

$$E \max_{i=1,\ldots,N} |\xi_i| \leq \psi^{-1}(LN) \max_{1 \leq i \leq N} c_i.$$

Proof.

By the properties of ψ,

$$\psi\left(\frac{E \max |\xi_i|}{\max c_i}\right) \leq \psi\left(E \max \frac{|\xi_i|}{c_i}\right) \leq \sum_{i=1}^{N} E\psi\left(\frac{|\xi_i|}{c_i}\right) \leq LN.$$

Apply ψ^{-1} to both sides. $\square$

Symmetrization

Symmetrized empirical process:

$$f \mapsto \mathbb{P}_n^o f = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i),$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. Rademacher random variables.

- $\varepsilon_1, \ldots, \varepsilon_n$ are independent of $(X_1, \ldots, X_n)$
- $E(\mathbb{P}_n^o f) = 0$
- For fixed $(X_1, \ldots, X_n)$, $\mathbb{P}_n^o$ is a Rademacher process (hence sub-Gaussian).

Symmetrization result

Theorem 25 (Symmetrization)

For any class $\mathcal{F}$ of measurable functions,

$$E\left\|\mathbb{P}_n - P\right\|_{\mathcal{F}} \leq 2E\left\|\mathbb{P}_n^o\right\|_{\mathcal{F}}.$$

Proof.

Let Y_i be independent copies of X_i. For fixed $(X_1, \ldots, X_n)$,

$$\left\|\mathbb{P}_n - P\right\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} \frac{1}{n}\left|\sum_{i=1}^{n}[f(X_i) - Ef(Y_i)]\right| \leq E_Y \sup_{f \in \mathcal{F}} \frac{1}{n}\left|\sum_{i=1}^{n}[f(X_i) - f(Y_i)]\right|.$$

Taking expectation with respect to $(X_1, \ldots, X_n)$, we obtain

$$E\left\|\mathbb{P}_n - P\right\|_{\mathcal{F}} \leq E\left\|\frac{1}{n}\sum_{i=1}^{n}\left[f\left(X_i\right) - f\left(Y_i\right)\right]\right\|_{\mathcal{F}}.$$

Symmetrization result (cont.)

Proof (cont.)

We can see that adding a minus sign in front of $[f(X_i) - f(Y_i)]$ just exchanges X's and Y's, so the expectation remains unchanged. Thus, $E\frac{1}{n}\|\sum_{i=1}^n e_i[f(X_i) - f(Y_i)]\|_{\mathcal{F}}$ is the same for any $(e_1, \ldots, e_n) \in \{-1, +1\}^n$. Hence,

$$
\begin{aligned}
E\|\mathbb{P}_n - P\|_{\mathcal{F}} &\leq E_\varepsilon E_{X,Y} \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i [f(X_i) - f(Y_i)] \right\|_{\mathcal{F}} \\
&\leq E_\varepsilon E_X \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i f(X_i) \right\|_{\mathcal{F}} + E_\varepsilon E_Y \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i f(Y_i) \right\|_{\mathcal{F}} \\
&= 2E \left\|\mathbb{P}_n^o\right\|_{\mathcal{F}}.
\end{aligned}
$$

$\square$