Problems from Rice: *Mathematical statistics and data analysis*

11.6. Respond to the following:

I have two batches of numbers and I have a corresponding $\bar{x}$ and $\bar{y}$. Why should I test whether they are equal when I can just see whether they are or not?

Explanation. A test of significance allows us to make inferences about the populations X and Y from which samples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ were randomly drawn. If the means $\bar{x}$ and $\bar{y}$ were means calculated from random samples, then they are random variables that may be equal or not by chance alone. If we do not wish to make an inference about the populations, then there would be no need for the test of significance.

11.8a. An experiment to determine the efficacy of a drug for reducing high blood pressure is performed using 4 subjects in the following way: 2 of the subjects are chosen at random for the control group and 2 for the treatment group. During the course of treatment with the drug, the blood pressure of each of the subjects in the treatment group is measured for 10 consecutive days as is the blood pressure of each of the subjects in the control group.
In order to test whether the treatment has an effect, do you think it is appropriate to use the two-sample t-test with $n = m = 20$?

Explanation. No, it is not appropriate to use the two-sample t-test because the 10 observations within individuals are not independent.

11.2. The difference of the means of two normal distributions with equal variance is to be estimated by sampling an equal number of observations from each distribution. If it were possible, would it be better to halve the standard deviations of the populations or double the sample sizes?

Explanation. In estimation, minimum standard error is preferred. The standard error of the difference in sample means, assuming independence is

Halving the population standard deviations gives us a new Z-statistic of

$$\mathrm{SE}\left(\bar{X} - \bar{Y}\right) = \sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{n}} = \sigma\sqrt{2/n}$$

Suppose we have $\sigma^* = \sigma/2$. Then the new standard error under the halved population standard deviation is

$$\mathrm{SE}_{\sigma^*}\left(\bar{X} - \bar{Y}\right) = \sigma/2\sqrt{2/n} = \mathrm{SE}\left(\bar{X} - \bar{Y}\right)/2$$

Suppose we have $n^* = 2n$. Then the new standard error under the doubled sample size is

$$\mathrm{SE}_{n^*}\left(\bar{X} - \bar{Y}\right) = \sigma\sqrt{2/(2n)} = \mathrm{SE}\left(\bar{X} - \bar{Y}\right)/\sqrt{2}$$

Hence, halving the population standard deviation is better.

11.15. Suppose that n measurements are to be taken under a treatment condition and another n measurements are to be taken independently under a control condition. It is thought that the standard deviation of a single observation is about 10 under both conditions. How large should n be so that a 95% confidence interval for $\mu_X - \mu_Y$ has a width of 2? Use the normal distribution rather than the t distribution, since n will turn out to be rather large.

Solution. The radius of a 95% CI assuming a normally distributed difference in means test statistic is

$$\begin{aligned}
&Z_{1-0.05/2} \cdot \mathrm{SE}(\bar{X} - \bar{Y}) \\
&= \texttt{qnorm(0.975)} \cdot \sigma\sqrt{2/n} \\
&= 19.6\sqrt{2/n}
\end{aligned}$$

But, we want n such that

$$\begin{aligned}
2 &= 2\left(Z_{1-0.05/2} \cdot \mathrm{SE}(\bar{X} - \bar{Y})\right) \\
2 &= 2\left(19.6\sqrt{2/n}\right) \\
\Rightarrow n &= 19.6^2 \cdot 2 \\
&= 768.32
\end{aligned}$$

11.16. Referring to Problem 11.15, how large should n be so that the test of $H_0 : \mu_X = \mu_Y$ against the one-sided alternative $H_A : \mu_X > \mu_Y$ has a power of 0.5 if $\mu_X - \mu_Y = 2$ and $\alpha = 0.10$?

Solution. Statistical power is the probability that a test of significance correctly rejects the null hypothesis when a specific alternative hypothesis is true. Usually, we use notation $1 - \beta$ to represent is quantity where β is the probability of failing to reject the null hypothesis when it is false.

In the present setup, we have

$$(\bar{X} - \bar{Y}) \overset{a}{\sim} \text{normal}\left(\mu_X - \mu_Y, 2\sigma^2/n\right)$$

In order to reject H_A, we need a Z-statistic z such that

$$\begin{aligned}
\mathbb{P}\left(Z > z\right) &\leq \alpha \\
1 - \mathbb{P}\left(Z \leq z\right) &\leq \alpha \\
\mathbb{P}\left(Z \leq z\right) &\geq 1 - \alpha
\end{aligned}$$

where $Z \sim \text{Normal}\,(0,1)$. So, we need

$$\begin{aligned}
(1-\beta) &= \mathbb{P}\left(z \geq Z_{1-\alpha}\right)\\
&= \mathbb{P}\left(\frac{(\bar{X}-\bar{Y})-0}{\sigma\sqrt{2/n}} \geq Z_{1-\alpha}\right)\\
&= \mathbb{P}\left(\frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sigma\sqrt{2/n}} \geq Z_{1-\alpha} - \frac{\mu_X-\mu_Y}{\sigma\sqrt{2/n}}\right)\\
&= \mathbb{P}\left(Z \geq Z_{1-\alpha} - \frac{\mu_X-\mu_Y}{\sigma\sqrt{2/n}}\right)\\
&= 1-\mathbb{P}\left(Z \leq Z_{1-\alpha} - \frac{\mu_X-\mu_Y}{\sigma\sqrt{2/n}}\right)\\
\Rightarrow \beta &= \mathbb{P}\left(Z \leq Z_{1-\alpha} - \frac{\mu_X-\mu_Y}{\sigma\sqrt{2/n}}\right)
\end{aligned}$$

Hence, we need

$$\begin{aligned}
Z_\beta &= Z_{1-\alpha} - \frac{\mu_X-\mu_Y}{\sigma\sqrt{2/n}}\\
\Rightarrow n &= 2\left(\frac{\sigma(Z_{1-\alpha}-Z_\beta)}{\mu_X-\mu_Y}\right)^2\\
&= \frac{\left(10(\texttt{qnorm(1 - 0.1)} - \texttt{qnorm(0.5)})\right)^2}{2}\\
&= 82.12
\end{aligned}$$

Additional problems

1. Let X have a binomial distribution with parameters $n = 10$ and $p \in \{p : p = \frac{1}{2}, \frac{1}{4}\}$. The (simple) hypothesis $H_0 : p = \frac{1}{2}$ is rejected and the alternative (simple) hypothesis $H_1 : p = \frac{1}{4}$ is accepted if the observed value of X, a random sample of size 1, is less than or equal to 3. Find the significance level and power of the test.

   Solution. Let $Y \sim \text{binomial}\,(10, 1/2)$. Then we have significance level

$$\begin{aligned}
\alpha &= \mathbb{P}\left(Y \leq 3\right)\\
&= \sum_{k=0}^{3}\binom{10}{k}\left(\frac{1}{2}\right)^k\left(1-\frac{1}{2}\right)^{10-k}\\
&= \texttt{pbinom(3, 10, 1/2)}\\
&= 0.171875
\end{aligned}$$

and power

$$\begin{aligned}1-\beta &= \mathbb{P}\left(X \le 3\right)\\ &= \texttt{pbinom(3, 10, 1/4)}\\ &= 0.7758751\end{aligned}$$

2. Let X have the binomial distribution with parameters n and p. We reject $H_0 : p = \frac{1}{2}$, and accept $H_1 : p > \frac{1}{2}$ if $X \ge c$. Find n and c so that $\alpha = 0.10$ and the power of the test against the alternative $p = \frac{2}{3}$ is $1-\beta = 0.95$.

   Solution. Let $Y \sim \text{binomial}\left(n, 1/2\right)$ and assume $Y \overset{a}{\sim} \text{normal}\left(n/2, n/4\right)$. Applying the definition of the critical value, we have

$$\begin{aligned}\alpha &= \mathbb{P}\left(Y \ge c\right)\\ \alpha &= 1-\mathbb{P}\left(Y \le c-1\right)\\ \mathbb{P}\left(Y \le c-1\right) &= 1-\alpha\\ \mathbb{P}\left(Z \le \frac{c-1+1/2-n/2}{\sqrt{n/4}}\right) &\approx 1-\alpha\\ \Rightarrow \frac{c-1/2-n/2}{\sqrt{n/4}} &\approx Z_{1-\alpha}\end{aligned}$$

   By the definition of power, we have

$$\begin{aligned}1-\beta &= \mathbb{P}\left(X \ge c\right)\\ \beta &= \mathbb{P}\left(X \le c-1\right)\\ &\approx \mathbb{P}\left(Z \le \frac{c-1+0.5-2n/3}{\sqrt{2n/9}}\right)\end{aligned}$$

   Hence, we have system of equations

$$\begin{cases} c-n/2-Z_{1-\alpha}\sqrt{n/4} = 1/2\\ c-2n/3-Z_{\beta}\sqrt{2n/9} = 1/2\end{cases}$$

   Solving the first equation for c, we have

$$c = 1/2+n/2+Z_{1-\alpha}\sqrt{n/4}$$

   Substituting into the second equation gives

$$\begin{aligned}1/2+n/2+Z_{1-\alpha}\sqrt{n/4}-2n/3-Z_{\beta}\sqrt{2n/9} &= 1/2\\ n &= \left(\frac{Z_{1-\alpha}/2-Z_{\beta}\sqrt{2}/3}{(2/3-1/2)}\right)^2\\ &= 72.20\end{aligned}$$

Hence,

$$\begin{aligned} c &= 1/2 + n/2 + Z_{1-\alpha}\sqrt{n/4} \\ &= 42.04 \end{aligned}$$

Verifying the exact probabilities, we have

$$\begin{cases} \mathbb{P}\left(Y \leq 42 - 1\right) = \texttt{pbinom(41, 72, 1/2)} = 0.9027 \\ \mathbb{P}\left(X \leq 42 - 1\right) = \texttt{pbinom(41, 72, 2/3)} = 0.0541 \end{cases}$$