Ay121 PS5 Solutions

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Based on the solutions from Rachel Theios in 2015.

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1 The Inverse Compton Catastrophe

(a) For an ultra-relativistic population of electrons with $N(E)\,\mathrm{d}E \propto E^{-p}\,\mathrm{d}E$ (ignoring the exponential cutoff at high frequencies due to cooling), we have

$$I_\nu \propto \begin{cases} \nu^{5/2} & \text{if } \nu < \nu^* \text{ (optically thick)} \\ \nu^{-(p-1)/2} & \text{if } \nu > \nu^* \text{ (optically thin)} \end{cases} \tag{1}$$

Thus, since $T_\mathrm{b} \propto I_\nu \nu^{-2}$, we have

$$T_\mathrm{b} \propto \begin{cases} \nu^{1/2} & \text{if } \nu < \nu^* \\ \nu^{-(p+3)/2} & \text{if } \nu > \nu^* \end{cases} \tag{2}$$

(b) The energy density of the magnetic field is

$$\boxed{U_\mathrm{B} = \frac{B^2}{8\pi}}. \tag{3}$$

The energy density of radiation at frequency ν, assuming the source is emitting isotropically, is

$$u_\nu = \frac{4\pi}{c} J_\nu = \frac{4\pi}{c} I_\nu = \frac{8\pi\nu^2 k T_\mathrm{b}}{c^3}. \tag{4}$$

This has units of energy density per frequency. Strictly speaking, to obtain the total energy density in radiation we need to integrate over a frequency range, taking into account the dependence of T_b on ν, but for an order-of-magnitude estimate we can just multiply by frequency:

$$\boxed{U_\mathrm{rad} \approx \frac{4\pi}{c}\nu I_\nu = \frac{8\pi\nu^3 k T_\mathrm{b}}{c^3}}. \tag{5}$$

(c) The critical frequency for synchrotron radiation is

$$\nu_\mathrm{crit} = \frac{3}{2}\gamma^2 \nu_\mathrm{cyc} = \frac{3}{4\pi}\gamma^2 \frac{eB}{m_e c} \tag{6}$$

where a factor of $1/2\pi$ has come from converting ω to ν. Then setting $\nu = \nu_\mathrm{crit}$, we can write B as

$$B = \frac{4\pi}{3}\frac{m_e c}{e}\frac{\nu_\mathrm{crit}}{\gamma^2}. \tag{7}$$

Now setting $\nu^* = \nu_{\rm crit}$ and substituting this into Equation 3, we have

$$\boxed{U_{\rm B} = \frac{2\pi}{9}\frac{m_e^2 c^2}{e^2}\frac{\nu^{*2}}{\gamma^4}}. \tag{8}$$

(d) Now setting $\gamma m_e c^2 \approx 3kT$ and $T = T_{\rm b} = T^*$, we find

$$\boxed{U_{\rm B} = \frac{2\pi}{3^6}\frac{m_e^6 c^{10}}{e^2 k^4}\frac{\nu^{*2}}{T^{*4}}}. \tag{9}$$

(e) We therefore find

$$\frac{P_{\rm IC}}{P_{\rm syn}} = \frac{U_{\rm rad}}{U_{\rm B}} = 4\cdot 3^6 \frac{e^2 k^5}{m_e^6 c^{13}}\nu^* T^{*5}$$

$$\boxed{\frac{P_{\rm IC}}{P_{\rm syn}} = \left(\frac{\nu^*}{100\ \text{GHz}}\right)\left(\frac{T^*}{10^{11.5}\ \text{K}}\right)^5} \tag{10}$$

There must be a limit on the brightness temperature so the energy in radiation field will not go to infinitely large. If $T_{\rm b}$ exceeds $T_{\rm limit}$, the system loses more energy through inverse Compton radiation than it receives through synchrotron, and the gas must cool down.

2 Comptonization

(a) The energy transfer per scattering to a photon of energy ϵ is,

$$\Delta\epsilon = \epsilon\left(\frac{4kT}{mc^2} - \frac{\epsilon}{mc^2}\right). \tag{11}$$

Thus, for photons with $\epsilon \ll 4kT$, the energy gain per scattering can be put into differential form,

$$\frac{{\rm d}\epsilon}{{\rm d}N} \sim \epsilon\frac{4kT}{mc^2}, \tag{12}$$

where ${\rm d}N$ is the differential number of scatterings. After N scatterings, the energy of a photon of initial energy ϵ_i is thus

$$\frac{\epsilon_N}{\epsilon_i} \sim e^{(4kT/mc^2)N} \quad \text{for } \epsilon_N \ll 4kT. \tag{13}$$

In the case when $\tau_{\rm es} \gg 1$, $N = \tau_{\rm es}^2$ (random walk). Thus,

$$\boxed{\frac{\epsilon_f}{\epsilon_i} \sim e^y}, \tag{14}$$

where $\boxed{y \equiv \frac{4kT}{mc^2}\tau_{\rm es}^2}$.

(b) The process saturates when photons stop gaining energy from electrons, i.e. $\epsilon \sim 4kT$. Thus, to obtain $\tau_{\rm crit}$,

$$\frac{4kT}{\epsilon_i} \sim e^{y_{\rm crit}}, \tag{15}$$

Plugging in the definition of y,

$$\boxed{\tau_{\rm crit} = \sqrt{\frac{mc^2}{4kT} \ln\left(\frac{4kT}{\epsilon_i}\right)}} \tag{16}$$

(c) From part (a), the parameter is $\boxed{y = \frac{4kT}{mc^2}\tau_{\rm es}^2}$

3 Validity of Thomson Scattering in the Rest Frame

(a) The characteristic synchrotron frequency,

$$\omega_c = \frac{3\gamma^2 eB\sin\alpha}{2mc}. \tag{17}$$

Taking $|\sin\alpha| = 3^{-1/2}$,

$$h\nu_c \approx 0.10 \text{ eV} \left(\frac{\gamma}{10^4}\right)^2 \left(\frac{B}{0.1 \text{ G}}\right). \tag{18}$$

The ratio of photon's energy to the electron's rest mass, in electron's rest frame, is given by,

$$\boxed{\frac{\gamma h\nu_c}{mc^2} \approx 2.0 \times 10^{-3} \left(\frac{\gamma}{10^4}\right)^3 \left(\frac{B}{0.1 \text{ G}}\right)}. \tag{19}$$

(b) The blackbody spectrum of CMB peaks at ~ 2.8 K, which correspond to $\sim 2.4 \times 10^{-4}$ eV. In electron's rest frame, the ratio of CMB photon energy to electron rest mass is, thus,

$$\boxed{\frac{\gamma h\nu}{mc^2} \approx 1.4 \times 10^{-5} \left(\frac{\gamma}{10^4}\right)}. \tag{20}$$

Note that, when $\gamma \sim 10^4$, the ratio of the *second scattering* is γ^2 higher. Therefore, for both case (a) and (b), the rest frame Thomson scattering approximation is no longer valid.