Ay121 PS4 Solutions

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Based on the solutions from Rachel Theios in 2015.

11/01/2016

1 Bremsstrahlung in Galaxy Clusters

(a) When optically thin, Bremsstrahlung spectrum has $F_\nu \propto e^{-h\nu/kT}$ (Eq. 6.15 in lecture note). In log-log space, the spectrum appears flat up until a cut-off at $h\nu \sim kT$. In this figure, the cut-off occurs at $h\nu \sim 3$ keV. Therefore,

$$\boxed{kT \sim 3 \text{ keV}}, \tag{1}$$

$$\boxed{T \sim 3.5 \times 10^7 \text{K}}. \tag{2}$$

The total X-ray flux can be estimated by integrating $F_\nu = F_0 e^{-h\nu/kT}$, where $F_0 \sim 0.4$ keV cm^{-2} s^{-1} keV^{-1} is the flux at $h\nu \sim 3$ keV. Then, the integrated X-ray flux is,

$$\begin{aligned} F &= \int_{3\text{ keV}}^{50\text{ keV}} F_0 e^{-h\nu/kT}\,\mathrm{d}(h\nu). \\ &= \boxed{0.44 \text{ keV s}^{-1} \text{ cm}^{-2}} \qquad (3) \\ &= \boxed{7 \times 10^{-10} \text{ erg s}^{-1} \text{ cm}^{-2}}. \end{aligned}$$

Note: Since the question is lack of detailed description, it is also reasonable to use the model decomposed Bremsstrahlung spectrum (dashed green curve in figure). In this case, the characteristic $kT \sim 10$ keV, $T \sim 1.2 \times 10^8$ K, and F is still 0.44 keV s^{-1} cm^{-2}.

(b) Integrating emissivity, ϵ_ν^{ff} over frequency,

$$\epsilon^{\text{ff}} = \left(\frac{2\pi kT}{3m}\right)^{1/2} \frac{32\pi e^6}{3hmc^3} Z^2 n_e n_i \bar{g}^{\text{ff}}. \tag{4}$$

With $\bar{g}^{\text{ff}} = 1$,

$$\epsilon^{\text{ff}} = 1.4 \times 10^{-27} T^{1/2} Z^2 n_e n_i. \tag{5}$$

Therefore,

$$\boxed{j^{\text{ff}} = \frac{\epsilon^{\text{ff}}}{4\pi} = 1.1 \times 10^{-28} T^{1/2} Z^2 n_e n_i}. \tag{6}$$

(c) Since the cluster is optically thin, we can ignore the absorption, so the Bremsstrahlung luminosity is given by,

$$L = \frac{4\pi}{3} R^3 \epsilon^{\mathrm{ff}}, \tag{7}$$

where R is the radius of the cluster. For fully ionized hydrogen,

$$n_e = n_i = \frac{M_g}{\frac{4\pi}{3} R^3 m_H}. \tag{8}$$

Plugging in Eq. 7,

$$\boxed{L = \frac{4\pi}{3} R^3 \times 1.4 \times 10^{-27} T^{1/2} \left(\frac{M_g}{\frac{4\pi}{3} R^3 m_H} \right)^2}. \tag{9}$$

(d) Let $L = 4\pi d^2 F$,

$$M_g = m_H \left(\frac{16\pi^2 d^2 R^3 F}{3 \times 1.4 \times 10^{-27} T^{1/2}} \right)^{1/2}. \tag{10}$$

Plugging in $d = 70$ Mpc, $R = 0.5$ Mpc, $F = 7 \times 10^{-10}$ erg cm^{-2} s^{-1} and $T = 3.5 \times 10^7$ K, we find,

$$\boxed{M_g = 2 \times 10^{13} M_\odot}. \tag{11}$$

(e) The optical depth of free-free absorption is

$$\tau_\nu = \int_{-R}^{R} \alpha_\nu^{\mathrm{ff}} \, \mathrm{d}s = 2\alpha_\nu^{\mathrm{ff}} R, \tag{12}$$

where α_ν^{ff} is the free-free absorption coefficient,

$$\alpha_\nu^{\mathrm{ff}} = 3.7 \times 10^8 T^{-1/2} Z^2 n_e^2 \nu^{-3} (1 - e^{h\nu/kT}) g_\nu^{\mathrm{ff}}. \tag{13}$$

To get rid of the frequency dependence, we use the Rosseland mean absorption coefficient,

$$\alpha_R^{\mathrm{ff}} = 1.7 \times 10^{-25} T^{-7/2} Z^2 n_e^2 \bar{g}_R^{\mathrm{ff}}. \tag{14}$$

Taking $\bar{g}_R^{\mathrm{ff}} \approx 1$, $Z = 1$ and n_e in terms of gas mass,

$$\alpha_R^{\mathrm{ff}} = 1.7 \times 10^{-25} T^{-7/2} \left(\frac{M_g}{\frac{4\pi}{3} R^3 m_H} \right)^2. \tag{15}$$

Plugging in Eq. 12,

$$\boxed{\tau = 1.7 \times 10^{-25} T^{-7/2} \left(\frac{M_g}{\frac{4\pi}{3} R^3 m_H} \right)^2 2R \approx 10^{-34}}. \tag{16}$$

So the optically thin assumption is justified.

(f) For virialized objects, $2T + V = 0$, where T is the kinetic energy of the galaxy cluster, V is the gravitational potential. Taking $T = \frac{1}{2} M\sigma^2$,

$$2\left(\frac{1}{2} M\sigma^2\right) - \frac{GM^2}{R} = 0. \tag{17}$$

Therefore,

$$\boxed{M = \frac{\sigma^2 R}{G} = 2 \times 10^{14} M_\odot}. \tag{18}$$

Thus, the gas mass only takes $\sim 10\%$ of the total mass of the galaxy cluster. An even smaller fraction of this mass is in stars. Most of the mass is in dark matter. You will learn more details in Ay124 (Galaxies), Ay126 (ISM) and Ay127 (Cosmology).

2 Synchrotron Cooling

(a)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -P_{\mathrm{rad}}, \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma mc^2) = \dot{\gamma} mc^2 = \frac{4}{3}\sigma_T c \beta^2 \gamma^2 U_B. \tag{20}$$

Therefore,

$$\dot{\gamma} = -A\beta^2\gamma^2 \approx -A\gamma^2, \tag{21}$$

by assuming $\beta \approx 1$ and setting $A = \frac{9e^4 B^2}{4m^3c^5}$. Since $B_\perp = B \sin\alpha$, we can replace $\frac{2}{3}B^2$ (which is averaged over all angles) with $B_\perp^2$,

$$A = \frac{2e^4 B_\perp^2}{3m^3c^5}. \tag{22}$$

When integrated, $-\gamma^{-1} = At + \mathrm{const}$. Given that $\gamma(t = t_0) = \gamma_0$,

$$\boxed{\gamma = \gamma_0(1 + A\gamma_0 t)^{-1}}. \tag{23}$$

(b)

$$E(t_{1/2}) = \frac{1}{2}E(t = 0), \tag{24}$$

$$\gamma(t_{1/2})mc^2 = \frac{1}{2}\gamma_0 mc^2, \tag{25}$$

$$\gamma_0(1 + A\gamma_0 t_{1/2})^{-1} = \frac{1}{2}\gamma_0 \tag{26}$$

$$\boxed{t_{1/2} = (A\gamma_0)^{-1} = \frac{5.1 \times 10^8 \text{ sec}}{\gamma_0 B_\perp^2}}. \tag{27}$$

3 What is Powering the Crab Nebula?

(a) The bolometric luminosity is given by,

$$L = 4\pi d^2 F. \tag{28}$$

The flux F is the specific flux integrated over the frequency range of interest. The upper limit is $\nu = (20 \text{ keV})/h = 4.8 \times 10^{18} Hz$.

The specific flux is

$$F_\nu = (1000\text{ Jy})\left(\frac{\nu}{\text{GHz}}\right)^{-0.3}. \tag{29}$$

So the luminosity is given by

$$L = 4\pi(2\text{ kpc})^2 \times 1000\text{ Jy} \times \left(\frac{1\text{ GHz}}{0.7}\right)\left[\left(\frac{\nu}{\text{GHz}}\right)^{0.7}\right]_{10\text{ MHz}}^{4.8\times10^{18}\text{ Hz}}, \tag{30}$$

$$\boxed{L = 4.1\times10^{40}\text{ erg s}^{-1} \approx 10^7 L^{\odot}}. \tag{31}$$

(b) The synchrotron cooling time can be expressed as,

$$t_{\text{cool}} = \frac{E}{P} = \frac{\gamma m_e c^2}{\frac{4}{3}\sigma_T c\gamma^2\frac{B^2}{8\pi}}. \tag{32}$$

Assuming all electrons radiate at the critical frequency,

$$\nu_{\text{crit}} \sim \frac{1}{2\pi}\gamma^2\frac{eB}{m_e c}. \tag{33}$$

Therefore,

$$\gamma = \sqrt{\frac{2\pi\nu_{\text{crit}} m_e c}{eB}}. \tag{34}$$

By plugging $B = 300\mu\text{G}$ (Googling) and $\nu_{\text{crit}} = 20\text{ keV}/h = 4.8\times10^{18}$ Hz in,

$$t_{\text{cool}} = 1.3\times10^8\text{ s} \sim 4\text{ years}. \tag{35}$$

We know the Crab nebula went off $\sim 10^3$ years ago, so there is something powering the Crab nebula. The actually source is the dipole radiation from the Crab pulsar.