Ay121 PS7 Solutions

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Based on the solutions from Rachel Theios in 2015.

11/29/2016

1 Selection Rules

The dipole matrix elements are given by

$$\mathbf{d}_{fi} = e \int \phi_f^* \sum_j \mathbf{r}_j \phi_i \, \mathrm{d}^3 x. \tag{1}$$

So the dipole matrix element in polar direction $z = r\cos\theta$ between state m and m' is,

$$\mathbf{d} = e \int \phi_{m'}^* r \cos\theta \phi_m \, \mathrm{d}^3 x. \tag{2}$$

The wave function,

$$\phi = r^{-1} R(r) Y(\theta, \varphi) \propto r^{-1} R(r) P^m(\cos\theta) e^{im\varphi}. \tag{3}$$

So Eq. 2 is proportional to,

$$\begin{aligned} & \int P_l^m(\cos\theta) \cos\theta P_{l'}^{m'}(\cos\theta) e^{i(m-m')\varphi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\ = \; & \int_{-1}^{1} P_l^m(\mu) \mu P_{l'}^{m'}(\mu) \, \mathrm{d}\mu \int_0^{2\pi} e^i (m - m')\varphi \, \mathrm{d}\varphi, \end{aligned} \tag{4}$$

where $\mu = \cos\theta$. I the equation, we ignored the r integral, since it is always positive.

The φ integral vanishes unless $\boxed{m = m'}$. Thus, we need $\int_{-1}^{1} P_l^m(\mu) \mu P_{l'}^m(\mu) \, \mathrm{d}\mu$ to be non-zero. Plugging in the recurrence relation,

$$(2l+1) P_l^m \cos\theta = (l - m + 1) P_{l+1}^m + (l + m) P_{l-1}^m. \tag{5}$$

So the integral vanishes unless $\boxed{l' = l \pm 1}$.

Similarly, the matrix element of $x \pm iy \equiv r \sin\theta e^{\pm i\varphi}$ is proportional to,

$$\begin{aligned} & \int P_{l'}^{m'}(\cos\theta) \sin\theta P_l^m(\cos\theta) e^{i(m-m'\pm 1)\varphi} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \\ = \; & \int_{-1}^{1} P_{l'}^{m'}(\mu) \sqrt{1-\mu^2} \, \mathrm{d}\mu \int_0^{2\pi} e^{i(m-m'\pm 1)\varphi} \, \mathrm{d}\varphi. \end{aligned} \tag{6}$$

The φ integral vanishes unless $\boxed{m' = m \pm 1}$. Plugging in another recurrence relation,

$$(2l+1)\sqrt{1-\mu^2} P_l^{m-1} = P_{l-1}^m - P_{l+1}^m. \tag{7}$$

So if $m' = m + 1$,

$$(2l+1)\sqrt{q-\mu^2}P_l^m = P_{l-1}^{m'} - P_{l+1}^{m'}. \tag{8}$$

If $m' = m - 1$,

$$(2l+1)\sqrt{q-\mu^2}P_l^{m'} = P_{l-1}^{m} - P_{l+1}^{m}. \tag{9}$$

Both of the cases give $\boxed{l' = l \pm 1}$.

2 Temperature Curve of Growth

(a) Since the source is optically thin, $I_\nu \propto j_\nu$, where

$$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu). \tag{10}$$

So the emission line has the same shape as the line profile function $\phi(\nu)$. The line profile is described by Voigt function. In low-T limit, the line profile is dominated by natural broadening,

$$\phi(\nu) = \frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}. \tag{11}$$

The observed half-width of the line is where $\phi(\nu) = \frac{1}{2}\phi(\nu_0)$:

$$\phi(\nu_0) = \frac{\gamma/4\pi^2}{(\gamma/4\pi)^2} = \frac{4}{\gamma}. \tag{12}$$

Thus,

$$\nu = \nu_0 \pm \frac{\gamma}{4\pi}. \tag{13}$$

$\gamma = \sum_{n'} A_{nn'}$, is the spontaneous decay rate, independent of temperature. Therefore, $\boxed{\text{FWHM} = \frac{\gamma}{2\pi}}$, is independent of temperature.

(b) In the high-T limit, the line profile is dominated by Doppler broadening:

$$\phi(\nu) = \frac{1}{\nu_D\sqrt{\pi}} = e^(\nu-\nu_0)^2/(\Delta\nu_D)^2, \tag{14}$$

where $\Delta\nu_D = \frac{\nu 0}{c}\sqrt{\frac{2kT}{m}}$. Therefore,

$$\boxed{\text{FHWM} = \frac{\nu_0}{c}\sqrt{\frac{8kT\ln 2}{m}} \propto \sqrt{T}}. \tag{15}$$

(c) The critical temperature is where these two line widths are equal, or

$$\Delta\nu_D(T_c) = \frac{\gamma}{4\pi}, \tag{16}$$

$$kT_c = \left(\frac{\gamma}{4\pi}\frac{c}{\nu_0}\right)^2 \frac{m_H}{2} = \frac{1}{8}\left(\frac{\gamma}{\omega}\right)^2 m_H c^2. \tag{17}$$

We can find γ by setting $\gamma = A_{21}$ and using the Einstein relations,

$$\begin{aligned} A_{ul} &= \tfrac{2h\nu^3}{c^2}\tfrac{g_l}{g_u}B_{lu} \\ &= \tfrac{8\pi^2 e^2 \nu_{ul}^2}{mc^3}\tfrac{g_l f_{lu}}{A_{ul}}. \end{aligned} \tag{18}$$

For Lyα,

$$\gamma = \frac{2e^2\omega_0^2}{m_e c^3}\left(\frac{g_1 f_{12}}{g_2}\right). \tag{19}$$

The degeneracies are

$$\begin{cases} g_1 = 2, \\ g_2 = 2(2l+1) = 6 \qquad \text{(for 2p electron).} \end{cases} \tag{20}$$

and $g_1 f_{12} = \frac{2^{14}}{3^9}$. Also plugging in the wavelength of Lyα (1216 Å), we get,

$$kT_c = \frac{2^{19}}{3^{18}}\frac{e^{12}}{\hbar^6 c^6} m_H c^2. \tag{21}$$

Therefore,

$$\boxed{T_c = 2.2 \times 10^{-3}\ \mathrm{K}}. \tag{22}$$

For most of the astronomical applications, $T \gg T_c$, Doppler broadening dominates the line center. However, the Lorentz component falls off rapidly at the line wings. Thus, the natural broadening dominates the regions far from the line center.

3 Carbon Monoxide

(a) The angular momentum is quantized as

$$L^2 = \hbar^2 J(J+1). \tag{23}$$

From classical mechanics, the rotational kinetic energy of a barbell is

$$E = L^2/2I, \tag{24}$$

where $I = \sum m_i r_i^2 \sim 28 m_p a_0^2$ (assuming the width of the barbell is twice as the Bohr radius).

Rearranging, we find,

$$E \approx \frac{\hbar^2 J(J+1)}{2 \cdot 28 m_p a_0^2} \sim \frac{hc}{\lambda_{\mathrm{CO}}} \tag{25}$$

For $J = 1 \to J = 0$,

$$\boxed{\lambda_{\mathrm{CO}} \sim \frac{2\pi(28m_p)a_0^2 c}{\hbar} \sim 2.3\ \mathrm{mm}}. \tag{26}$$

(b) Larmor formula,

$$P = \frac{2\ddot{d}}{3c^3}. \tag{27}$$

Taking the sinusoidal form for d,

$$d = d_0 \sin \omega t, \tag{28}$$

where $d_0 = 0.1$ Debye. Plugging in Larmor Formula,

$$P = \frac{2\omega^4 d_0^2 \sin^2 \omega t}{3c^3}. \tag{29}$$

Taking $\langle \sin^2 \omega t \rangle = 1/2$, and one rotation is equivalent to two oscillations,

$$\boxed{A \sim \frac{2\omega^4 d_0^2}{3c^3 \hbar \omega} \sim 8.7 \times 10^{-8}\ \mathrm{s}^{-1}}. \tag{30}$$

(c) The cross section at the line center is,

$$\sigma = \frac{h\nu_0}{4\pi} B_{12} \phi(\nu_0) = \frac{1}{\Delta\nu_D \sqrt{\pi}} \frac{h\nu}{4\pi} B_{12}, \tag{31}$$

where $\Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m}}$. Using Einstein relations,

$$\sigma = \frac{1}{\sqrt{\pi}} \frac{\lambda_0^3}{8\pi} \sqrt{\frac{m}{2k_B T}} A_{21}. \tag{32}$$

Plugging in $\lambda_0 = 2.6$ mm, $m_a = 30 m_p$, $T = 20$ K, and $A_{21} = 8.7 \times 10^{-8}\ \mathrm{s}^{-1}$,

$$\boxed{\sigma \sim 3.2 \times 10^{-15}\ \mathrm{cm}^2}. \tag{33}$$

(d) Number density of CO in the cloud is

$$n_{\mathrm{CO}} = \frac{n_{\mathrm{CO}}}{n_{\mathrm{H_2}}} n_{\mathrm{H_2}} \sim (10^{-4})(10^4\ \mathrm{cm}^{-3}) \sim 1\ \mathrm{cm}^{-3}. \tag{34}$$

When optical depth $\tau \sim 1$,

$$\boxed{s \sim (n_{\mathrm{CO}}\sigma)^{-1} \sim 3.1 \times 10^{14}\ \mathrm{cm} \sim 10^{-4}\ \mathrm{pc}}. \tag{35}$$

So for a typical molecular cloud (~ 100 pc), we expect the column density is optically thick however the line-of-sight passes through the cloud.

(e) Since the molecular cloud is so optically thick, we can only observe the very surface of the cloud, and we can't tell how much material lies beneath the surface. In future (Ay126, ISM), we will know that the way we resolve this is by making an assumption that the cloud is virialized, so the motion of the gas on surface reflects the total mass inside. In general, that is not a good assumption.

Alternative ways of observing molecular clouds include,

- using ^{13}CO or C^{18}O, which are much less abundant than ^{12}CO;
- measuring higher order transitions;
- using other less abundant species, e.g. C^+.