Ay121 PS1 Solutions

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Based on the solutions from Rachel Theios in 2015.

10/04/2016

1 Getting Comfortable with Temperatures

Note: This question is optional.

(a) Brightness temperature (T_b) is defined as,

$$T_b \equiv \frac{c^2 I_\nu}{2k\nu^2}. \tag{1}$$

We need to find specific intensity (I_ν) based on flux (F_ν).

$$\begin{aligned} F_\nu &= \int_{\text{source}} I_\nu \cos\theta \, \mathrm{d}\Omega \\ &= 2\pi I_\nu \int_0^{\theta/2} \sin\theta' \cos\theta' \, \mathrm{d}\theta' \\ &= \pi I_\nu \sin^2(\theta/2) \\ &\approx \pi(\tfrac{\theta}{2})^2 I_\nu, \end{aligned} \tag{2}$$

which gives $I_\nu = 1.3 \times 10^{-13}$ erg cm^{-2} s^{-1} Hz^{-1} ster^{-1}. Therefore, $\boxed{T_b = 4 \times 10^7 \text{ K}}$.
Based on Wien's displacement law,

$$\begin{aligned} \nu_{\max} &= 5.88 \times 10^{10} \text{ Hz K}^{-1} \\ &= \boxed{2.35 \times 10^{18} \text{ Hz} \gg 100\text{MHz}}. \end{aligned} \tag{3}$$

Therefore, we are in Rayleigh-Jeans limit.

(b) Based on Eq. 1 and 2, $\boxed{T_b \propto I_\nu \propto \theta^{-2}}$. If the source is more compact, T_b will be larger.

(c) Based on Eq. 3, $\boxed{\nu_{\max} = 2.35 \times 10^18\text{Hz}}$.

(d) Assuming the radiation is thermal ($S_\nu = B_\nu(T)$ in Rayleigh-Jeans tail), the radiative transfer equation can be written as,

$$\frac{\mathrm{d}T_b}{\mathrm{d}\tau} = T - T_b. \tag{4}$$

Assuming there is no background illumination ($T_b(\tau = 0) = 0$) and the source has uniform temperature,

$$\boxed{T_b(\tau) = T(1 - e^{-\tau}).} \tag{5}$$

Therefore, the actually temperature of the source $T \geq T_b = 4 \times 10^7$K.

2 Blackbody Luminosity

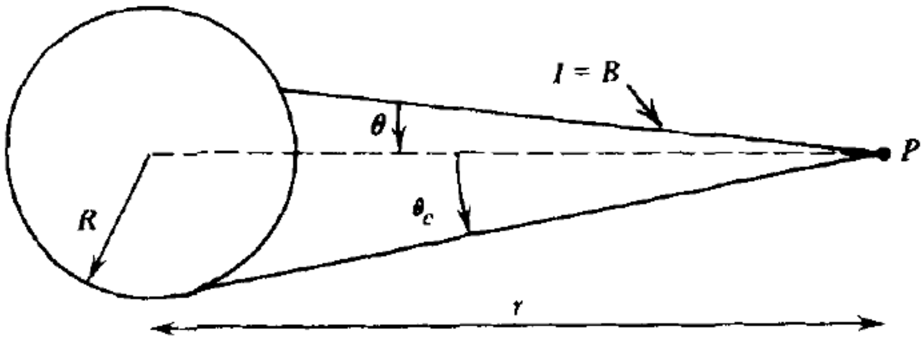

Figure 1: Fig. 1.6 from R&L, Flux from a uniform bright sphere.

Proof. To calculate luminosity (L), we need to get flux.

$$\begin{aligned} F_\nu &= \int I_\nu \cos\theta \, \mathrm{d}\Omega \\ &= 2\pi I_\nu \int_0^{\theta_c} \sin\theta \cos\theta \, \mathrm{d}\theta \\ &= \pi I_\nu \sin^2\theta_c \\ &\approx \boxed{\pi I_v \frac{R^2}{r^2}}. \end{aligned} \tag{6}$$

Since the radiation is blackbody, $I_\nu = B_\nu(T)$. The total flux (F) integrated over all frequencies is,

$$\begin{aligned} F &= \pi \tfrac{R^2}{r^2} \int_0^\infty B_\nu(T) \, \mathrm{d}\nu \\ &= \pi \tfrac{R^2}{r^2} \tfrac{2h}{c^2} \int_0^\infty \tfrac{\nu^3 \, \mathrm{d}\nu}{\exp(h\nu/k_B T) - 1} \\ &= \pi \tfrac{R^2}{r^2} \times \tfrac{2h}{c^2} \times \tfrac{\pi^4}{15} \times (\tfrac{k_B T}{h})^4 \\ &= \boxed{\sigma T^4 \frac{R^2}{r^2}}, \end{aligned} \tag{7}$$

where $\sigma = \frac{2}{15} \frac{\pi^5 k_B^4}{h^3 c^2}$ is the Stefan-Boltzmann constant.

Therefore, we get luminosity

$$\begin{aligned} L &= 4\pi r^2 F \\ &= \boxed{4\pi R^2 \sigma T^4}. \end{aligned} \tag{8}$$

□

3 Limb Darkening

(a) Mean intensity,

$$\begin{aligned} J(\tau_z) &= \tfrac{1}{4\pi} \int I(\tau_z, \mu)\, \mathrm{d}\Omega \\ &= \tfrac{1}{4\pi} \int I_0 \tau_z\, \mathrm{d}\Omega + \int I_1(\tau_z)\mu\, \mathrm{d}\Omega \\ &= \boxed{I_0(\tau_z)}. \end{aligned} \tag{9}$$

Flux,

$$\begin{aligned} F(\tau_z) &= \int I(\tau_z, \mu)\mu\, \mathrm{d}\Omega \\ &= \int I_0(\tau_z)\mu\, \mathrm{d}\Omega + \int I_1(\tau_z)\mu^2\, \mathrm{d}\Omega \\ &= \boxed{\frac{4\pi}{3} I_1(\tau_z)}. \end{aligned} \tag{10}$$

Energy density,

$$\begin{aligned} u(\tau_z) &= \tfrac{1}{c} \int I(\tau_z, \mu)\, \mathrm{d}\Omega \\ &= \tfrac{4\pi}{c} J(\tau_z) \\ &= \boxed{\frac{4\pi}{c} I_0(\tau_z)}. \end{aligned} \tag{11}$$

Radiation pressure,

$$\begin{aligned} p(\tau_z) &= \tfrac{4\pi}{c} \int I(\tau_z, \mu)\mu^2\, \mathrm{d}\Omega \\ &= \tfrac{1}{c} \int I_0(\tau_z)\mu^2\, \mathrm{d}\Omega + \tfrac{1}{c} \int I_1(\tau_z)\mu^3\, \mathrm{d}\Omega \\ &= \boxed{\frac{4\pi}{3c} I_0(\tau_z)} \end{aligned} \tag{12}$$

Thus, $p/u = 1/3$, which was previously saw for isotropic field.

(b) (*Note: If you downloaded the homework before 09/30/16, the Eq. 1 in the problem set should really refer to the equation in the front page, which has no equation number.*)

As instructed, integrate Eq. 1 in the problem set,

$$\begin{aligned} \tfrac{\partial}{\partial \tau_z} \int I(\tau_z, \mu)\mu\, \mathrm{d}\Omega &= \int I(\tau_z, \mu)\, \mathrm{d}\Omega - S(\tau_z)\, \mathrm{d}\Omega \\ \tfrac{\partial}{\partial \tau_z} F(\tau_z) &= 4\pi[J(\tau_z) - S(\tau_z)]. \end{aligned} \tag{13}$$

If flux is independent of optical depth, then $\partial F/\partial\tau_z = 0$. Therefore, $\boxed{J(\tau_z) = S(\tau_z)}$.

(c) Again, as instructed, take the first moment of the radiative transfer equation.

$$\begin{aligned}
\tfrac{\partial}{\partial\tau_z}\textstyle\int I(\tau_z,\mu)\mu^2\,\mathrm{d}\Omega &= \textstyle\int I(\tau_z,\mu)\,\mathrm{d}\Omega - \int S(\tau_z)\mu\,\mathrm{d}\Omega \\
\tfrac{\partial}{\partial\tau_z}cp(\tau_z) &= F(\tau_z) - \textstyle\int J(\tau_z)\mu\,\mathrm{d}\Omega \\
\tfrac{4\pi}{3}\tfrac{\partial}{\partial\tau_z}I_0(\tau_z) &= \tfrac{4\pi}{3}I_1(\tau_z) - 0 \\
\tfrac{\partial}{\partial\tau_z}I_0(\tau_z) &= I_1(\tau_z).
\end{aligned} \tag{14}$$

Based on Eq. 10,

$$I_1(\tau_z) = \frac{3}{4\pi}\sigma T_{\mathrm{eff}}^4. \tag{15}$$

Therefore,

$$I_0(\tau_z) = \frac{3}{4\pi}\sigma T_{\mathrm{eff}}^4(\tau_z + \mathrm{const}). \tag{16}$$

$$\boxed{I(\tau_z,\mu) = I_0(\tau_z) + I_1(\tau_z)\mu = \frac{3}{4\pi}\sigma T_{\mathrm{eff}}^4(\tau_z + \mu + C)} \tag{17}$$

(d) Assuming the ingoing flux at the surface is zero,

$$\int_{\mathrm{inward}} I(\tau_z = 0,\mu)\mu\,\mathrm{d}\Omega = 0, \tag{18}$$

which implies

$$\frac{3}{4\pi}\sigma T_{\mathrm{eff}}^4[\int_{\mathrm{inward}}\mu^2\,\mathrm{d}\Omega + C\int_{\mathrm{inward}}\mu\,\mathrm{d}\Omega] = 0. \tag{19}$$

The inward integration range from $\theta = \pi/2$ to π (see appendix). Therefore, it gives,

$$\frac{2\pi}{3} - \pi C = 0. \tag{20}$$

Therefore, $C = 2/3$.

$$\boxed{I(\tau_z,\mu) = \frac{3}{4\pi}\sigma T_{\mathrm{eff}}^4(\tau_z + \mu + \frac{2}{3})}. \tag{21}$$

$I(\tau_z = 0,\mu)/I(\tau_z = 0,\mu = 1)$ is linearly dependent to $\mu = \cos\theta$ (Fig. 2).

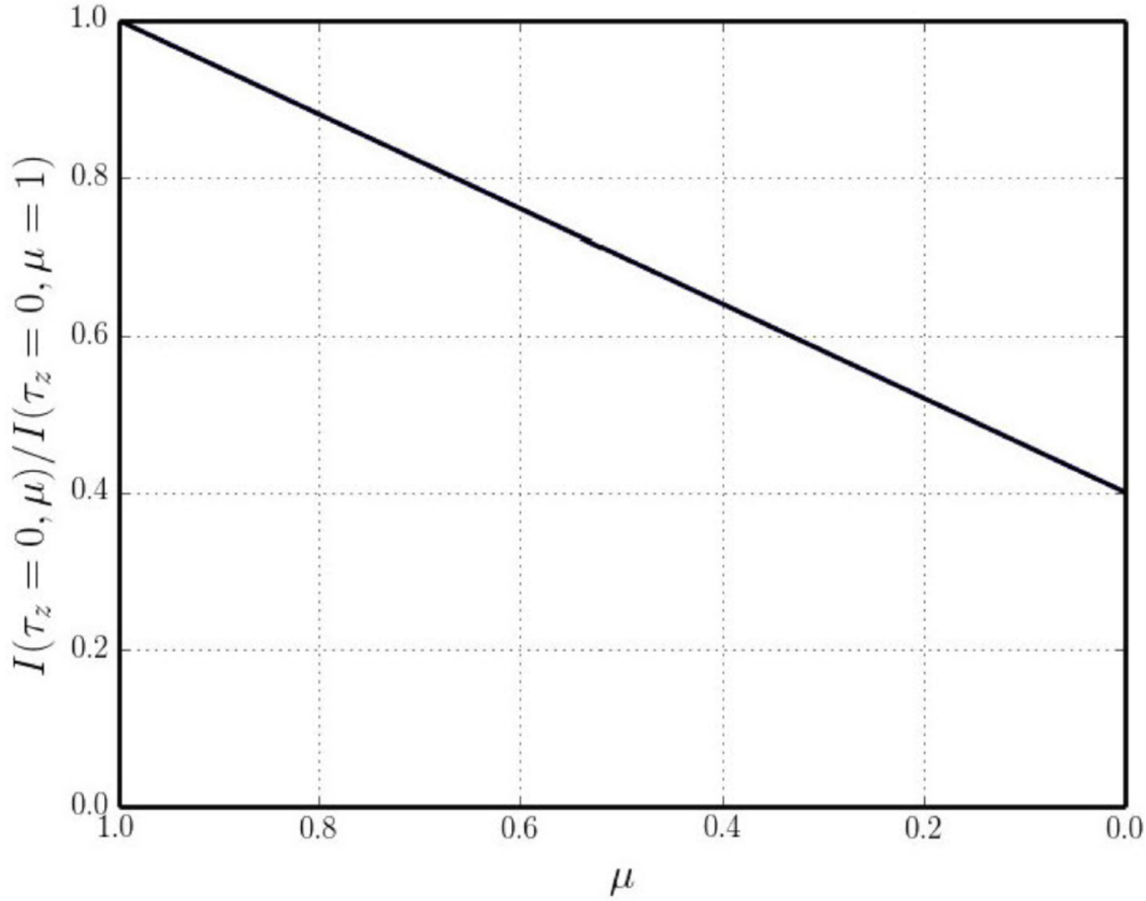

Figure 2: The normalized intensity as a function of $\mu = \cos\theta$.

4 Appendix

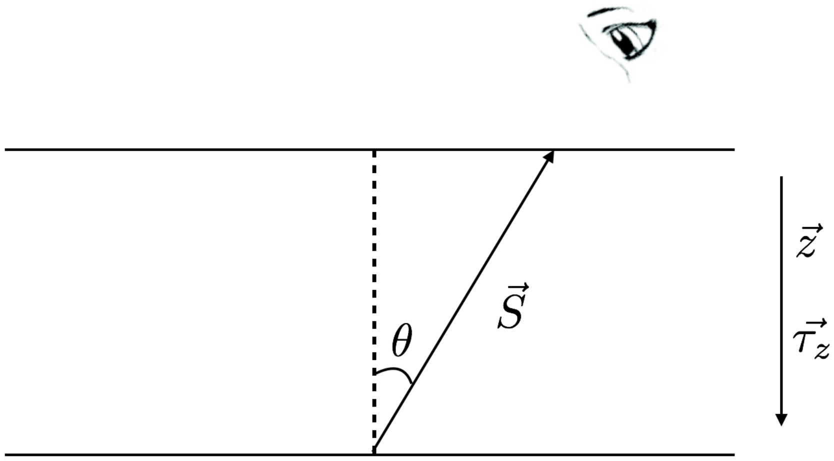

Figure 3: A schematic diagram of the radiative transfer in solar atmosphere.

The way of doing "inward" integration is shown below.

As shown in Fig. 3, $\vec{S}$ indicates the direction of the radiation that is seen by an observer. $\vec{z}$ is the depth as going inward to the atmosphere. $\tau_z \equiv \alpha z$. Therefore,

$$\mathrm{d}z = -\cos\theta\,\mathrm{d}S. \tag{22}$$

$$\mathrm{d}\tau_z = -\alpha\cos\theta\,\mathrm{d}S. \tag{23}$$

Plugging into the radiative transfer equation we learned in class,

$$\begin{aligned} \tfrac{\mathrm{d}I}{\mathrm{d}S} &= -I\alpha + j, \\ -\tfrac{\mathrm{d}I}{\alpha\cos\theta\,\mathrm{d}S}\cos\theta &= I - s, \end{aligned} \tag{24}$$

Therefore,

$$\boxed{\mu\frac{\mathrm{d}I}{\mathrm{d}\tau_z} = I - s}. \tag{25}$$

Thus, integrating all the rays pointing inward refers to the range of θ from $\pi/2$ to π.

Note that even though we flip the direction of the optical depth from pointing outward to the observer to pointing inward to the star which causes the reversal of the signs in the right-hand side, the origin of θ does not follow with the flipping. This version of the radiative transfer equation is frequently used in Ay123 (star) and Ay125 (high energy).