Ay121 PS3 Solutions

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Based on the solutions from Rachel Theios in 2015.

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1 Stokes Parameters

(a) Based on the definition of Stokes parameters,

$$\begin{cases} I = \epsilon_1^2 + \epsilon_2^2 = 10.7\ \mathrm{V^2m^{-2}} \\ Q = \epsilon_1^2 - \epsilon_2^2 = -9.75\ \mathrm{V^2m^{-2}} \\ U = 2\epsilon_1\epsilon_2\cos(\phi_1 - \phi_2) = 4.28\ \mathrm{V^2m^{-2}} \\ V = 2\epsilon_1\epsilon_2\sin(\phi_1 - \phi_2) = -1.31\ \mathrm{V^2m^{-2}} \end{cases} \tag{1}$$

(b)

$$\begin{cases} \epsilon_1 = \sqrt{\frac{I+Q}{2}} = 17.8\ \mathrm{V\ m^{-1}} \\ \epsilon_2 = \sqrt{\frac{I-Q}{2}} = 14.1\ \mathrm{V\ m^{-1}} \\ \phi_1 - \phi_2 = \arctan\left(\frac{V}{U}\right) = 1.10\ \mathrm{rad} = 63.0^\circ \end{cases} \tag{2}$$

Note: The solution of ϕ_1 and ϕ_2 is not unique. There is a hidden assumption when writing Eq. 2.38 in R&L.

(c)

$$\begin{aligned} \Pi &= \frac{\sqrt{Q^2+U^2+V^2}}{I} \\ &= 100\% \text{ (for both a and b)} \end{aligned} \tag{3}$$

Note: Π *cannot be larger than 1.*

2 Radiation Spectrum

(a) The Fourier transform of $E(t)$ is,

$$\hat{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{i=1}^{N} E_0(t - t_i)e^{i\omega t}\,\mathrm{d}t. \tag{4}$$

Since Fourier transform is linear, we can pull out the summation.

$$\hat{E}(\omega) = \sum_{i=1}^{N} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(t - t_i) e^{i\omega t} \, \mathrm{d}t. \tag{5}$$

Define $t_i' \equiv t - t_i$,

$$\begin{aligned} \hat{E}(\omega) &= \textstyle\sum_{i=1}^{N} = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(t_i') e^{i\omega t_i'} e^{i\omega t_i} \, \mathrm{d}t_i' \\ &= \textstyle\sum_{i=1}^{N} = e^{i\omega t_i} \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(t_i') e^{i\omega t_i'} \, \mathrm{d}t_i'. \end{aligned} \tag{6}$$

Because,

$$\hat{E}_0(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(t) e^{i\omega t} \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_0(t_i') e^{i\omega t_i'} \, \mathrm{d}t_i' \tag{7}$$

Therefore,

$$\boxed{\hat{E}(\omega) = \hat{E}_0(\omega) \sum_{i=1}^{N} e^{i\omega t}}. \tag{8}$$

(b)

$$\begin{aligned} \left| \textstyle\sum_{i=1}^{N} e^{i\omega t_i} \right|^2 &= \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} e^{i\omega t_i} e^{-i\omega t_j} \\ &= \textstyle\sum_{i=1}^{N} e^{i\omega t_i} e^{-i\omega t_i} + \sum_{i \neq j} e^{i\omega (t_j - t_i)} \\ &= N + \textstyle\sum_{i \neq j} e^{o\omega (t_j - t_i)}. \end{aligned} \tag{9}$$

Since the arrival times are random, the second term averages to zero.

$$\boxed{\left| \sum_{i=0}^{N} e^{i\omega t_i} \right|^2 = N}. \tag{10}$$

(c)

$$\frac{\mathrm{d}W}{\mathrm{d}A \, \mathrm{d}\omega} = c \left| \hat{E}(\omega) \right|^2. \tag{11}$$

Based on results from part (a) and (b),

$$\begin{aligned} \tfrac{\mathrm{d}W}{\mathrm{d}A \, \mathrm{d}\omega} &= c \left| \hat{E}_0(\omega) \textstyle\sum_{i=1}^{N} e^{i\omega t_i} \right|^2 \\ &= c \left| \hat{E}_0(\omega) \right|^2 N \\ &= \boxed{N \left[\frac{\mathrm{d}W}{\mathrm{d}A \, \mathrm{d}\omega} \right]_{\text{single pulse}}}. \end{aligned} \tag{12}$$

(d) In the case when all pulses arrive at the same time, we can set $t_i = 0$.

$$\hat{E}(\omega) = \hat{E}_0(\omega) \sum_{i=1}^{N} e^0 = N \hat{E}_0(\omega). \tag{13}$$

Therefore,

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}A\,\mathrm{d}\omega} &= c\left|N\hat{E}_0(\omega)\right|^2 \\ &= \boxed{N^2\left[\frac{\mathrm{d}W}{\mathrm{d}A\,\mathrm{d}\omega}\right]_{\text{single pulse}}}. \end{aligned} \tag{14}$$

3 Pulsars

(a) In the case of magnetic dipole radiation, the Larmor formula can be expressed as,

$$P = \frac{2\ddot{\mathbf{m}}^2}{3c^3}, \tag{15}$$

where $\ddot{\mathbf{m}}(t) = me^{-i\omega t}$ is the magnetic dipole moment. The component of $\mathbf{m}$ along the rotation axis, $\mathbf{m}\cos\alpha$, is constant with time. Therefore,

$$\ddot{\mathbf{m}} = \omega^2\mathbf{m}\sin\alpha. \tag{16}$$

The magnetic field due to a magnetic dipole m is given by,

$$\mathbf{B} = \frac{m}{r^3}(2\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}). \tag{17}$$

For uniform sphere, evaluated at the magnetic dipole ($\theta = 0$), the magnetic field is

$$B_0 = \frac{2m}{R^3}. \tag{18}$$

Combining the above equations,

$$\boxed{P = \frac{\omega^4 B_0^2 R^6 \sin^2\alpha}{6c^3}}. \tag{19}$$

Note: Since the specific form of the magnetic field is not mentioned in problem, the exact expression of B_0 can have a factor difference.

(b) If the spin down of the pulsar is the source of radiation,

$$P = -\frac{\mathrm{d}E_{\text{rot}}}{\mathrm{d}t}. \tag{20}$$

The rational energy and the momentum of inertia are,

$$E_{\text{rot}} = \frac{1}{2}I\omega^2, \tag{21}$$

$$I = \frac{2}{5}MR^2. \tag{22}$$

The radiation power is then,

$$P = -\frac{2}{5}MR^2\omega\dot{\omega} = -2E_{\text{rot}}\frac{\dot{\omega}}{\omega}. \tag{23}$$

Therefore, the slow-down timescale is,

$$\boxed{\tau = -\frac{\omega}{\dot{\omega}} = \frac{2E_{\text{rot}}}{P} = \frac{12Mc^3}{5R^4B_0^2\omega^2\sin^2\alpha}}. \tag{24}$$

Table 1: Results of Q3(c)

ω	P		τ	
s^{-1}	erg s^{-1}	$L_\odot$	s	years
10^4	6.17×10^{43}	1.60×10^{10}	1.29×10^9	40.9
10^3	6.17×10^{39}	1.60×10^6	1.29×10^{11}	4.09×10^3
10^2	6.17×10^{35}	160	1.29×10^13	4.09×10^5

(c) Plugging values into Eq. 19 and 24, results are shown in Table 1.

Note: This question is closely related to the P–$\dot{P}$ plot of pulsars (Fig. 1). Details will be provided in Ay125 (High Energy). Long story in short, based on Eq. 19 and 24, people can infer the magnetic dipoles and ages of pulsars.

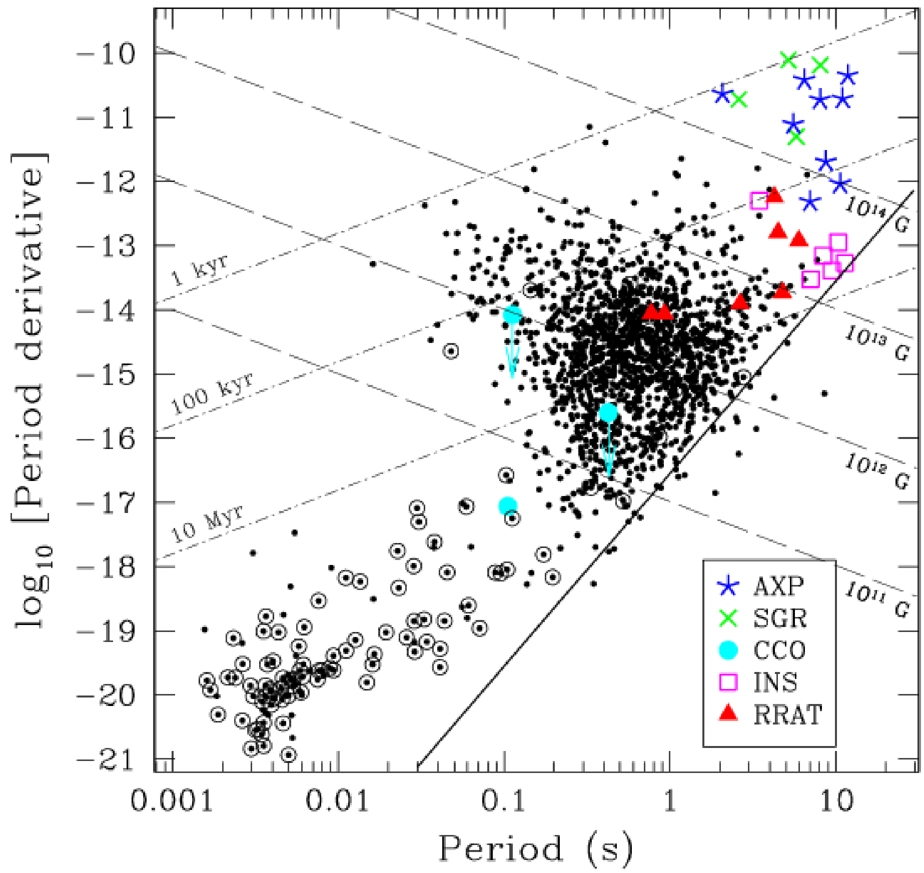

Figure 1: P–$\dot{P}$ diagram. [Credit: [1]]

References

[1] Victoria M Kaspi. Grand unification of neutron stars. *Proceedings of the National Academy of Sciences*, 107(16):7147–7152, 2010.