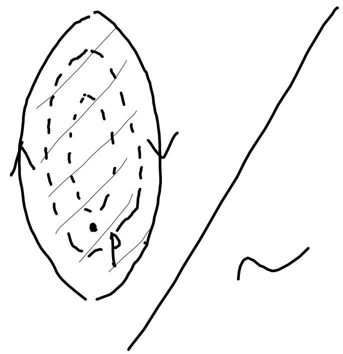
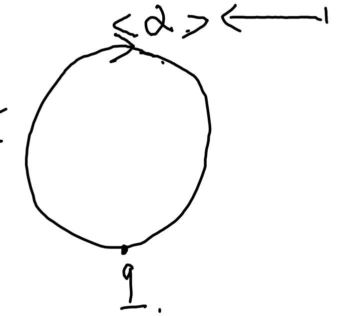
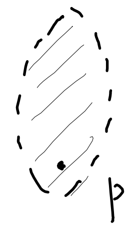
例4. $\mathbb{RP}^2 \left(= M(1)\right)$.

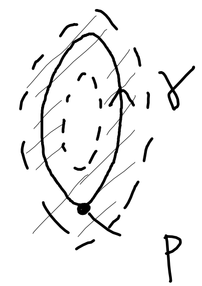


X_0 : 

$\simeq$ 

X_1 : 

单连通.

$X_{01} = X_0 \cap X_1$: 

$\pi_1(X_0, p) \cong \pi_1(S^1, q) \cong \mathbb{Z}$.

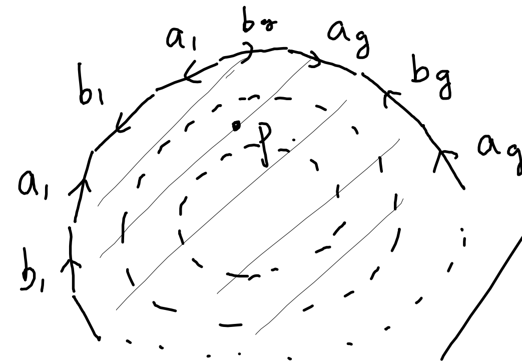
$\langle \alpha \rangle \mapsfrom \langle \alpha \rangle \mapsfrom 1$

$j_0 : X_{01} \hookrightarrow X_0$

$$\pi_1(\mathbb{RP}^2, p) \cong \pi_1(X_0, p) \Big/ \overline{\{(j_0)_*(\langle \gamma \rangle)\}}$$

$$\cong \mathbb{Z} / 2\mathbb{Z}$$

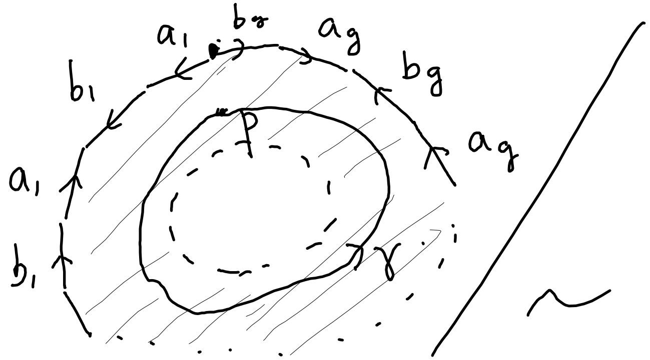
例5. $H(g)$.



$\sim$

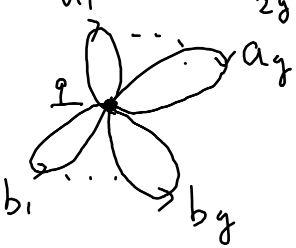
$$\pi_1(H(g), p) \cong \left\langle a_1, \cdots, a_g, b_1, \cdots, b_g \,\middle|\, a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \right\rangle$$

X_0 :

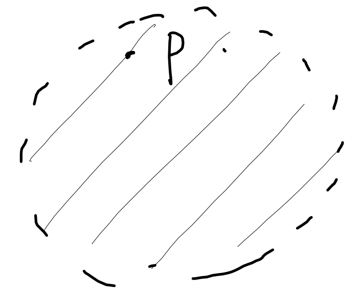


$\sim \;\simeq$

$\bigvee_{2g} S^1$



X_1 :



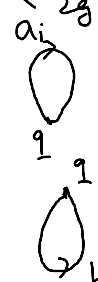
单连通

X_{01} :

$$\pi_1(H(g), p) \cong \pi_1(X_0, p) \Big/ \{(j_0)_*(\langle \gamma \rangle)\}$$

$$\cong \langle a_1, \cdots, a_g, b_1, \cdots, b_g \rangle \Big/ \{a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdots a_g \cdot b_g \cdot a_g^{-1} \cdot b_g^{-1}\}$$

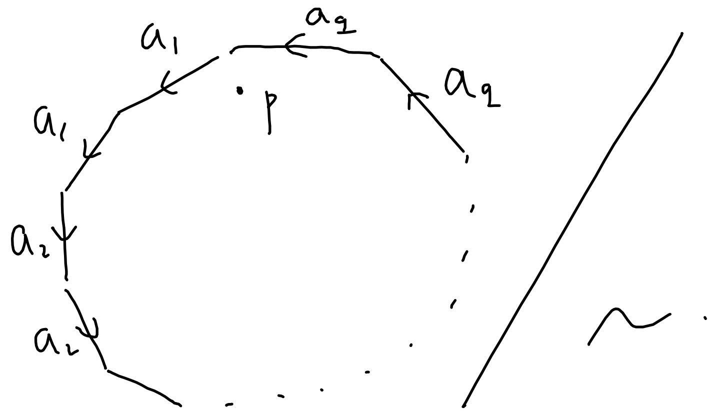
$$\pi_1\left(\bigvee_{2g} S^1, q\right) \cong \langle a_1, \cdots, a_g, b_1, \cdots, b_g \rangle$$



$\longleftarrow\!\shortmid\; a_i$

$\longleftarrow\!\shortmid\; b_j$

例6. $M(q)$.



$$\pi_1(M(q), p) \cong \langle a_1, a_2, \dots, a_q \mid a_1^2 a_2^2 \cdots a_q^2 \rangle$$

Abel化: $\forall$ 群 G, $G^{(1)} := \langle \{ a\cdot b\cdot a^{-1}\cdot b^{-1} \mid a, b \in G \} \rangle$. (换位子群).

$$G^{(1)} \lhd G. \qquad G/G^{(1)} =: G^{ab}.$$

例: 设 $G = \langle a_1, \dots, a_n \rangle$, $G^{ab} \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n\text{个}}$.

构造 $\varphi: G \longrightarrow \mathbb{Z} \times \cdots \times \mathbb{Z}$ by: $\varphi(a_i) = (0, \dots, \overset{\text{第}i\text{位}}{1}, \dots, 0)$

$i = 1, \dots, n$.

$$\varphi(a\cdot b\cdot a^{-1}\cdot b^{-1}) = \varphi(a) + \varphi(b) - \varphi(a) - \varphi(b) = 0.$$

$$\{ a\cdot b\cdot a^{-1}\cdot b^{-1} \mid a, b \in G \} \subset \ker\varphi \lhd G$$

$$\Rightarrow G^{(1)} \subset \ker\varphi.$$

$$\rightsquigarrow \bar{\varphi}: G^{ab} \longrightarrow \mathbb{Z} \times \cdots \times \mathbb{Z}.$$

反过来：构造：$\psi: \mathbb{Z}\times\cdots\times\mathbb{Z} \longrightarrow G^{ab}$.

$$(k_1,\dots,k_n) \longmapsto \overline{a_1^{k_1}\cdots a_n^{k_n}}$$

ψ 显然为群同态.

$$G^{ab} \underset{\psi}{\overset{\overline{\varphi}}{\rightleftarrows}} \mathbb{Z}\times\cdots\times\mathbb{Z}$$

Claim: $\psi\circ\overline{\varphi} = id$, $\overline{\varphi}\circ\psi = id$.

$\psi\circ\overline{\varphi}$: $G^{ab} \xrightarrow{\overline{\varphi}} \mathbb{Z}\times\cdots\times\mathbb{Z} \xrightarrow{\psi} G^{ab}$.

$$\overline{a_1^{k_1}\cdots a_n^{k_n}} \longmapsto (k_1,\dots,k_n) \longmapsto \overline{a_1^{k_1}\cdots a_n^{k_n}}$$

$\overline{\varphi}\circ\psi$: $\mathbb{Z}\times\cdots\times\mathbb{Z} \xrightarrow{\psi} G^{ab} \xrightarrow{\varphi} \mathbb{Z}\times\cdots\times\mathbb{Z}$

$$(k_1,\dots,k_n) \longmapsto \overline{a_1^{k_1}\cdots a_n^{k_n}} \longmapsto (k_1,\dots,k_n)$$

综上：$\langle a_1,\dots,a_n\rangle^{ab} \cong \underbrace{\mathbb{Z}\times\cdots\times\mathbb{Z}}_{n\text{个}}$ #.

$$\pi_1(M(g), p) \cong \langle a_1, \dots, a_g \mid a_1^2 \cdots a_g^2 \rangle$$

$$\pi_1(H(g), p) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle .$$

$$\left(\langle a_1, \dots, a_g \rangle \Big/ \overline{\{a_1^2 \cdots a_g^2\}} \right) \Big/ \text{“换位子群”} = (G/N) \Big/ \left((H\cdot N)/N\right) = G / H\cdot N .$$

$$= \boxed{(G/H) \Big/ \left((H\cdot N)/H\right)} = G^{ab} \Big/ \left((H\cdot N)/H\right)$$

记 $G = \langle a_1, \dots, a_g \rangle$. $N = \overline{\{a_1^2 \cdots a_g^2\}} \lhd G$

$H = G^{(1)} = \overline{\{a\cdot b\cdot a^{-1}\cdot b^{-1} \mid a, b \in G\}} \lhd G$.

$\pi : G \longrightarrow G/N$ $\qquad \pi(H) = H\cdot N / N$.

“换位子群” $= \pi(H) \lhd G/N$.

「

$\pi(H) \supset \{\pi(a)\cdot\pi(b)\cdot\pi(a)^{-1}\cdot\pi(b)^{-1} \mid a, b \in G\}$

$\qquad\qquad = \{a\cdot b\cdot a^{-1}\cdot b^{-1} \mid a, b \in G/N\}$

$\pi(H) \supset$ “换位子群” $\qquad \pi^{-1}(\text{“换位子群”}) \supset \{a\cdot b\cdot a^{-1}\cdot b^{-1} \mid a, b \in G\}$

反之，$\pi(H) \subset$ “换位子群”.

“换位子群” $= \pi(\pi^{-1}(\text{“换位子群”})) \supset \pi(H)$.

」

$$G^{ab} = a_1, \cdots, a_q \text{ 生成的自由Abel群}$$

$$\pi_1(M(q), p)^{ab} \cong a_1, \cdots, a_q \text{ 生成的自由Abel群} \Big/ \langle a_1^2 \cdots a_q^2 \rangle.$$

$$\overset{t = a_1 \cdots a_q}{=\!=} t, a_2, \cdots, a_q \text{ 生成的自由Abel群} \Big/ \langle t^2 \rangle.$$

$$\cong \mathbb{Z}_2 \times \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{q-1\text{个}}.$$

$$\pi_1(H(g), p)^{ab} = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{2g\text{个}}$$

$\pi_1(M(q), p)^{ab}$, $q \geq 1$, $\pi_1(H(g), p)^{ab}$, $g \geq 0$. 互不同构.

$\Rightarrow$ $\pi_1(M(q), p)$, $q \geq 1$, $\pi_1(H(g), p)$, $g \geq 0$, 互不同构.

$\Rightarrow$ $M(q)$, $q \geq 1$, $H(g)$, $g \geq 0$ 互不同胚.

例7 group of finite presentation G,

综合 设 $G = \langle a_1, \cdots, a_g \mid b_1, \cdots, b_r \rangle$

要找 X, s.t. $\pi_1(X, p) \cong G$.

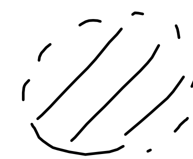
「提示: Case 0, $r=0$, $G = \langle a_1, \cdots, a_g \rangle$. $X = \bigvee_g S^1$

Case 1, $r=1$, $G = \langle a_1, \cdots, a_g \mid b_1 \rangle$.

$$X = (\bigvee_g S^1) \cup_f \overline{B^2}.$$

其中 $f: \underset{(S^1)}{\overset{\shortparallel}{\partial \overline{B^2}}} \longrightarrow \bigvee_g S^1$.

$\left.\overline{B^2}.\right\} X_0$.

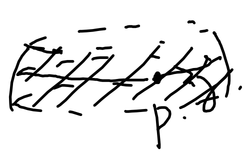
X_0:  单连通

X

X_1  $\bigvee_g S^1$.

$$\pi_1(X, p) \cong \pi_1(X_1, p) / \{(j_0)_*(\langle \gamma \rangle)\}$$

「$X_1 \simeq \bigvee_g S^1$. $\pi_1(X_1, p) \cong \langle a_1, \cdots, a_g \rangle$」

X_{01}:  $\simeq S^1$. $\pi_1(X, p) \cong \langle a_1, \cdots, a_g \rangle / \{c_1\}$

$$= \langle a_1, \cdots, a_g \mid c \rangle$$

」

8. 覆盖映射 (covering map)

回忆域扩张的 Galois 理论:

设 $F \subset E$ 为域扩张，定义 $\mathrm{Gal}(E/F) = \{\sigma: E \to E \mid \sigma \text{同构}, \sigma|_F = 1_F\}$

(E/F)

称为 E/F 的 Galois 群

反过来，记 $\mathrm{Aut}(E) = \{\sigma: E \to E \mid \sigma \text{为同构}\}$.

$\forall H < \mathrm{Aut}(E)$, $\mathrm{In}(H) := \{x \in E \mid \sigma(x) = x, \forall \sigma \in H\}$

$\mathrm{In}(H) \subset E$ 为一个域扩张.

小结:

固定域 E.

Galois 算子.

不动域算子

$$\{F \mid F < E, F \text{为子域}\} \underset{\mathrm{In}}{\overset{\mathrm{Gal}}{\rightleftarrows}} \{H \mid H < \mathrm{Aut}(E)\}.$$

$$F \longmapsto \mathrm{Gal}(E/F)$$

$$(\text{不动域}) \quad \mathrm{In}(H) \longleftarrow\!\shortmid \; H$$

自然的问题：$\mathrm{In} \circ \mathrm{Gal} = \mathrm{Id}$ $\quad \mathrm{Gal} \circ \mathrm{In} = \mathrm{Id}$.

定理（Artin）设 E 为一个域，$H < \mathrm{Aut}(E)$，$F = \mathrm{In}(H)$.

设 E/F 为一个有限扩张，则：$\underline{\mathrm{Gal}(E/\mathrm{In}(H)) = H}$.

$$\mathrm{Gal}(E/\mathrm{In}(H)) = \mathrm{Gal} \circ \mathrm{In}(H)$$

再看 $\mathrm{In} \circ \mathrm{Gal} = \mathrm{Id}$：

两边作用在 $F\ (\subset E)$：

$$\underline{\mathrm{In}\,\mathrm{Gal}(E/F) = F}.$$ 不一定对.

定义：设 $F \subset E$ 为域扩张，E/F 称为 Galois 扩张，若

$$\mathrm{In}\,\mathrm{Gal}(E/F) = F.$$

$$F \rightsquigarrow \mathrm{Gal}(E/F) \rightsquigarrow \mathrm{In}\,\mathrm{Gal}(E/F).$$

域$_F$ $\rightsquigarrow$ 群 $\rightsquigarrow$ 域'.

§1. Galois covering.

定义：$p: Y \to X$ 称为一个 covering，if p 为连续满射，

$\forall x \in X$，$\exists$ x 的开邻域 U，s.t. $p^{-1}(U) = \coprod_\alpha V_\alpha$，$V_\alpha \underset{\text{open}}{\subset} Y$，

且 $\forall \alpha$，$p|_{U_\alpha}: U_\alpha \xrightarrow{\cong} U$． (local homeomorphism)．

Rmk．设 $p: Y \to X$ 为一个 covering，则 p 为一个局部同胚．

i.e. $\forall y \in Y$，$\exists$ y 的一个开邻域 U_y，s.t. $p(U_y) \underset{\text{open}}{\subseteq} X$，且

$$p|_{U_y}: U_y \xrightarrow{\cong} p(U_y).$$

「 $\forall y \in Y$，设 $x = p(y)$，$\exists$ x 的开邻域 U，s.t. $p^{-1}(U) = \coprod_\alpha V_\alpha$，

$V_\alpha \underset{\text{open}}{\subseteq} Y$，且 $p|_{V_\alpha}: V_\alpha \xrightarrow{\cong} U$．不妨设 $y \in V_{\alpha_0}$，$p|_{V_{\alpha_0}}: V_{\alpha_0} \xrightarrow{\cong} p(V_{\alpha_0})$ 」

反之，局部同胚不一定是 covering．

反例：$\mathbb{R}_{>0} \xrightarrow{\pi} S^1 \quad x \longmapsto e^{2\pi i x}$．

推论：设 $p: Y \to X$ 为一个 covering，则 p 为开映射.

Pf. $\forall\, U \underset{\text{open}}{\subset} Y$，要证：$p(U)$ 为 X 中开集.

$\forall\, y \in U$，$\exists\, U_y \subset U$，s.t. $p|_{U_y}: U_y \xrightarrow{\cong} p(U_y) \underset{\text{open}}{\subset} X$.

$$U = \bigcup_{y \in U} U_y. \qquad p(U) = \bigcup_{y \in U} p(U_y). \qquad \#$$

定义：设 $p: Y \to X$，$p': Y' \to X$，为两个 covering，从 p 到 p' 的一个等价 (equivalence) 是指一个同胚 $\varphi: Y \to Y'$，s.t. 图表交换：

$$\begin{tikzcd}
Y \arrow[rr, "\varphi"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & Y' \arrow[dl, "p'"] \\
& X &
\end{tikzcd}$$

定义：给定 $p: Y \to X$ 为一个 covering，定义：

$$\mathrm{Deck}(Y/X) := \{\sigma: Y \to Y \mid \sigma \text{ 为 } p \text{ 到 } p \text{ 的等价}\}.$$

$$= \{\sigma: Y \to Y \mid \sigma \text{ 为同胚，且 } p \circ \sigma = p\} < \mathrm{Aut}(Y)$$

$\{\sigma: Y \to Y \mid \sigma \text{ 为同胚}\} = \mathrm{Aut}(Y)$

称为 $p: Y \to X$ 的覆盖变换群.

Rmk. 有显然的群作用:

$$\mathrm{Deck}(Y/X) \curvearrowright Y$$

y在映射h下的像

$$\forall h \in \mathrm{Deck}(Y/X),\quad y \in Y,\quad \underline{h(y)} = \underline{h(y)}.$$

命题1: $\mathrm{Deck}(Y/X) \curvearrowright Y$ 是 even 的 (Y假设是连通的)

Pf. $p: Y \to X$ 为一个覆盖.

$\forall y \in Y$, 要找 y 的开邻域 U, s.t. $U \cap g(U) = \emptyset$, $\forall g \in \mathrm{Deck}(Y/X)$, $g \neq e$.

下证之:

$\forall y \in Y$, $p(y) = x \in X$. $\exists$ x 的开邻域 V, s.t.

$$p^{-1}(V) = \coprod_{\alpha} U_\alpha,\quad U_\alpha \underset{\text{open}}{\subset} Y,\ 且\ p|_{U_\alpha}: U_\alpha \xrightarrow{\cong} V.$$

不妨设 $y \in U_\alpha$.

Claim U_α 即满足条件.

In fact:

假设 $\exists\, g \in \mathrm{Deck}(Y/X)$, $g \neq e$, 使 $U_{\alpha_0} \cap g(U_{\alpha_0}) \neq \emptyset$.

取 $x_1 \in U_{\alpha_0} \cap g(U_{\alpha_0})$. $\exists\, x_2 \in U_{\alpha_0}$, s.t. $x_1 = g(x_2)$.

$$\left.\begin{array}{l} \Rightarrow \underline{p(x_1) = p(x_2)} \\ \qquad\ \ \| \qquad\qquad \| \\ p(g(x_2)) = p(x_2) \\ x_1, x_2 \in U_{\alpha_0}. \\ p|_{U_{\alpha_0}}: U_{\alpha_0} \xrightarrow{\sim} V \end{array}\right\} \Rightarrow x_1 = x_2 \Rightarrow x_1 = g(x_1)$$

$\Rightarrow$ g 为有一个不动点的 covering transformation.

$\xrightarrow{\text{Lemma 1}}$ $g = e$ $\Rightarrow$ 矛盾.

#

Lemma 1. 设 Y 连通的, $p: Y \to X$ 为一个 covering. $\forall\, g \in \mathrm{Deck}(Y/X)$.

g 有不动点 $\Rightarrow g = e$.

Pf. 不妨设 $g(y_1)=y_1$.

$$\begin{tikzcd} & y_1 \in Y \arrow[d, "p"] \\ y_1 \in Y \arrow[ur, "g"] \arrow[ur, bend right, "id"'] \arrow[r, "p"'] & X \end{tikzcd}$$

$\left\{\begin{array}{l} g \text{ 与 } id \text{ 都为 } Y \xrightarrow{p} X \text{ 的提升} \\ \text{且 } g(y_1)=id(y_1). \end{array}\right.$ $\xRightarrow[\text{(提升的唯一性)}]{\text{Lemma2.}}$ $g=id$.

#

Lemma2. 设 $p: Y \to X$ 为一个 covering. Z 连通. $Z \xrightarrow{f} X$ 连续. $g_1, g_2: Z \to Y$ 为 f 的两个提升. 且 $\exists z_0 \in Z$, $g_1(z_0)=g_2(z_0)$, 则 $g_1=g_2$.

Pf.

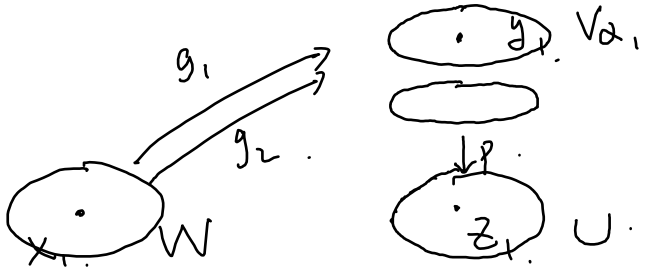
$$\begin{tikzcd} & z_0 \in Y \arrow[d, "p"] \\ z_0 \in Z \arrow[ur, "g_1"] \arrow[ur, bend right, "g_2"'] \arrow[r, "f"'] & X \end{tikzcd}$$

$S=\{z \in Z \mid g_1(z)=g_2(z)\}$

$S \neq \phi$, $z_0 \in S$.

只要证: S 既开又闭.

· S 是开的. $\forall x_1 \in S$, $g_1(x_1)=g_2(x_1)=y_1$.



$z_1 = f(x_1) \in X$.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ x_1 \in Z \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] \arrow[r, "f"] & X \\ S \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

∵ p 为一个 covering，∴ ∃ z_1 的开邻域 U，

s.t. $p^{-1}(U) = \coprod_{\alpha} V_\alpha$，$V_\alpha \underset{open}{\subset} Y$，$p|_{V_\alpha}: V_\alpha \xrightarrow{\cong} U$.

不妨设 $y_1 \in V_{\alpha_1}$.

由 g_1, g_2 连续，∃ x_1 的开邻域 W，$g_1(W) \subset V_{\alpha_1}$，$g_2(W) \subset V_{\alpha_1}$

$$\begin{tikzcd} & V_{\alpha_1} \arrow[d, "\cong", "p|_{V_{\alpha_1}}"'] \\ W \arrow[ur, shift left, "g_1|_W"] \arrow[ur, shift right, "g_2|_W"'] \arrow[r, "f"'] & U \end{tikzcd}$$

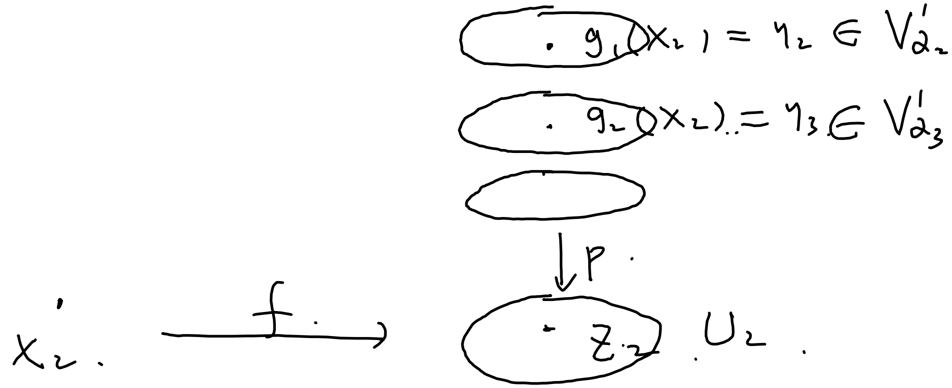
$\Rightarrow g_1|_W = g_2|_W$.

$\Rightarrow W \subset S$

• $Z \setminus S$ 开集.

∀ $x_2 \in Z \setminus S$，$\underset{\underset{y_2}{\parallel}}{g_1(x_2)} \neq \underset{\underset{y_3}{\parallel}}{g_2(x_2)}$. 记 $z_2 = f(x_2)$.

∃ z_2 的开邻域 U_2，s.t. $p^{-1}(U_2) = \coprod_{\alpha} V'_\alpha$. $V'_\alpha \underset{open}{\subset} Y$，$p|_{V'_\alpha}: V'_\alpha \xrightarrow{\cong} U_2$



$$\begin{tikzcd} & Y \arrow[d, "p"] \\ z_2 \in Z \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] \arrow[r, "f"'] & X \end{tikzcd}$$

∵ $y_2, y_3 \in \coprod_{\alpha} V'_{\alpha}$，不妨设 $y_2 \in V'_{\alpha_2}$，$y_3 \in V'_{\alpha_3}$.

又由 g_1, g_2 之连续性 $\Rightarrow \exists\ x_2$ 在 Z 中开邻域 W'，s.t.

$g_1(W') \subset V'_{\alpha_2}$，$g_2(W') \subset V'_{\alpha_3}$

$\Rightarrow W' \subset Z \setminus S \Rightarrow Z \setminus S$ 为开 $\Rightarrow S$ 闭.

综上：$S = Z \Rightarrow g_1 = g_2$.

#

推论：设 $p: Y \to X$ 为一个 covering，Y 连通，$\forall\ H < Deck(Y/X)$，

自然投射 $\pi: Y \to Y/H$ 为一个 covering.

固定连通的拓扑空间 Y.

$$\left\{ Y \xrightarrow{p} X \;\middle|\; p \text{ 为一个 covering} \right\} \Big/ \sim \quad \underset{/}{\overset{\text{Deck}}{\rightleftarrows}} \quad \left\{ H \;\middle|\; H < \mathrm{Aut}(Y),\ H \curvearrowright Y \text{ even} \right\}$$

「: "$Y \xrightarrow{p} X$" $\sim$ "$Y \xrightarrow{p'} X'$" $\iff$ $\exists$ 同胚 $\varphi: X \to X'$, s.t. 图表交换:

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ X \arrow[rr, "\varphi"'] & \circlearrowleft & X' \end{tikzcd}$$

」

$$Y \xrightarrow{p} X \longmapsto \mathrm{Deck}(Y/X)$$

$$Y \xrightarrow{\pi} Y/H \longleftarrow\!\shortmid\ H$$

自然的问题: $\underline{\mathrm{Deck} \circ / = \mathrm{id}}$, $/ \circ \mathrm{Deck} = \mathrm{id}$.

先看 $\mathrm{Deck} \circ / = \mathrm{id}$. i.e. $\forall H < \mathrm{Aut}(Y)$, $H \curvearrowright Y$ even, 令 $\pi: Y \to \overset{X}{\overset{\parallel}{Y/H}}$ 为自然投射, $\underline{\mathrm{Deck}(Y/X) = H}$

下证: $\mathrm{Deck}(Y/X) = H$.

$Y \xrightarrow{\pi} Y/H (= X)$.

$H < \mathrm{Aut}(Y)$

$H \curvearrowright Y$ even.

$$\mathrm{Deck}(Y/X) = \{ \sigma \in \mathrm{Aut}(Y) \mid \pi \circ \sigma = \pi \} < \mathrm{Aut}(Y)$$

$$H < \mathrm{Aut}(Y).$$

"$H \subset \mathrm{Deck}(Y/X)$"

$\forall h \in H$, 要证 $\pi \circ h = \pi$.

"$\mathrm{Deck}(Y/X) \subset H$"

$\forall \varphi \in \mathrm{Deck}(Y/X)$.

$$\begin{tikzcd} y_1 \arrow[rr, mapsto] & & \varphi(y_1) \\ Y \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi"] \\ & Y/H & \end{tikzcd}$$

$\pi(\varphi(y_1)) = \pi(y_1)$.

$\Longleftrightarrow \exists h \in H$, $\varphi(y_1) = h(y_1)$.

$\Rightarrow \underbrace{h^{-1} \circ \varphi}_{\in \mathrm{Deck}(Y/X)} (y_1) = y_1 \xRightarrow{\text{Lemma 1}} h^{-1} \circ \varphi = e \Rightarrow \varphi = h$.

#

再看：$/ \circ \mathrm{Deck} = \mathrm{id}$.

"$/ \circ \mathrm{Deck}(Y \xrightarrow{p} X) = \mathrm{id}(Y \xrightarrow{p} X)$".

$$/ \circ \mathrm{Deck}(Y \xrightarrow{p} X) = /(\mathrm{Deck}(Y/X)) = \left(Y \to Y/\mathrm{Deck}(Y/X)\right)$$

$$\mathrm{id}(Y \xrightarrow{p} X) = \left(Y \xrightarrow{p} X\right).$$

"E 固定. 若 $F \subset E$ 满足 $\mathrm{In}\,\mathrm{Gal}(E/F) = F$, 则称 E/F 为一个 Galois 扩张." $F = \mathrm{Id}(F) = \mathrm{In} \circ \mathrm{Gal}(F)$.

$\Longleftrightarrow$ "$\exists$ 同胚 $\varphi: Y/\mathrm{Deck}(Y/X) \xrightarrow{\cong} X$, s.t. 图表交换："

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "p"] \arrow[d, phantom, "\circlearrowleft"] & \\
Y/\mathrm{Deck}(Y/X) \arrow[rr, "\varphi", "\cong"'] & & X
\end{tikzcd}$$

$\Longleftrightarrow$ $\bar{p}: Y/\mathrm{Deck}(Y/X) \to X$ 为一个同胚.

注意到：$\exists!\ \bar{p}: Y/\mathrm{Deck}(Y/X) \to X$ 连续 使图表交换：

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
Y/\mathrm{Deck}(Y/X) \arrow[rr, "\exists!\, \bar{p}"] & & X
\end{tikzcd}$$

$\bar{p}(\bar{y}) := p(y)$.

定义：设Y连通，$Y \xrightarrow{p} X$ 为一个 covering，p 称为一个 Galois 覆盖，若 $\underline{/\circ \mathrm{Deck}(Y \xrightarrow{p} X) = \text{“} Y \xrightarrow{p} X \text{”}}$.

$\Updownarrow$.

$\bar{p}: Y/\mathrm{Deck}(Y/X) \longrightarrow X$ 为一个同胚，

其中 $\bar{p}$ 为使下面图表交换的唯一的连续映射：

$$
\begin{tikzcd}
Y \arrow[d, "\pi"'] \arrow[dr, "p"] \arrow[dr, phantom, "\circlearrowleft", near start] & \\
Y/\mathrm{Deck}(Y/X) \arrow[r, "\exists!\,\bar{p}"] & X
\end{tikzcd}
$$

设 $Y \xrightarrow{p} X$ 为一个 covering.

$\bar{p}: Y/\mathrm{Deck}(Y/X) \longrightarrow X$

$$
\begin{tikzcd}
Y \arrow[d, "\pi"'] \arrow[dr, "p"] & \\
Y/\mathrm{Deck}(Y/X) \arrow[r, "\bar{p}"'] & X
\end{tikzcd}
$$

$\Rightarrow$ $\bar{p}$ 为连续满射，开映射.

$\ulcorner$ $\forall U \underset{\text{open}}{\subseteq} Y/\mathrm{Deck}(Y/X)$. $\bar{p}(U) = \bar{p}(\pi(\pi^{-1}(U))) = p(\underbrace{\pi^{-1}(U)}_{\underset{\text{open}}{\subseteq} Y})$ $\underset{\text{open}}{\subseteq} X$. $\lrcorner$

Rmk. 设 Y 连通，$p: Y \to X$ 为一个 covering.

则 p 为 Galois covering $\Longleftrightarrow$ $\bar{p}: Y/_{\mathrm{Deck}(Y/X)} \to X$ 为单射.

$\mathrm{Deck}(Y/X) \circlearrowright Y$.

$\mathrm{Deck}(Y/X) \circlearrowright p^{-1}(x)$，$\forall x$.

$\forall h \in \mathrm{Deck}(Y/X)$

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} y \arrow[rr, mapsto] \arrow[dr, mapsto] & & h(y) \arrow[dl, mapsto] \\ & x & \end{tikzcd}$$

命题：Y 连通，$p: Y \to X$ 为一个 covering，则

p 为 Galois covering $\Longleftrightarrow$ $\mathrm{Deck}(Y/X) \circlearrowright p^{-1}(x)$，$\forall x$ 均为可迁作用.

Pf. 由 rmk. p 为 Galois covering $\Longleftrightarrow$ $\bar{p}: Y/_{\mathrm{Deck}(Y/X)} \to X$ 单. #

例：设 Y 连通，$G \circlearrowright Y$ even，$Y \xrightarrow{\pi} Y/G$ 为 Galois covering.

回 #2：Galois 扩张基本定理.

定理：设 E/F 为有限 Galois 扩张，则

(i). $\forall$ 中间域 $F \subset L \subset E$，E/L 为 Galois 扩张.

(ii).
$$\{\text{中间域}\} \underset{\mathrm{In}}{\overset{\mathrm{Gal}}{\rightleftarrows}} \{H \mid H < \mathrm{Gal}(E/F)\}$$
$$L \longmapsto \mathrm{Gal}(E/L) \ (< \mathrm{Gal}(E/F)).$$
$$\mathrm{In}(H) \longleftarrow\!\shortmid \ H$$

为 1-1 对应，i.e. $\mathrm{Gal} \circ \mathrm{In} = \mathrm{Id}$，$\mathrm{In} \circ \mathrm{Gal} = \mathrm{Id}$.

(iii). 在 (ii) 的 1-1 对应下，对于中间域 $F \subset L \subset E$，

L/F 为 Galois 扩张 $\Longleftrightarrow \mathrm{Gal}(E/L) \lhd \mathrm{Gal}(E/F)$

下面：固定一个 Galois covering $p: Y \longrightarrow X$
Y：连通． X：局部连通．

(X：局部连通，if $\forall x \in X$，$\forall x$ 的开邻域 U，$\exists x$ 的连通的开邻域 V，s.t. $x \in V \subset U$．(简言之，X 中任意一点都有连通的邻域基))

$$\left\{ \text{中间覆盖 } \begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & X \end{tikzcd} \;\middle|\; f,\ q \text{ 为covering} \right\} \Big/ \sim \quad \underset{}{\overset{\mathrm{Deck}}{\rightleftarrows}} \quad \left\{ H \;\middle|\; H < \mathrm{Deck}(Y/X) \right\}$$

「 " $\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & X \end{tikzcd} \sim \begin{tikzcd} Y \arrow[r, "f'"] \arrow[dr, "p"'] & Z' \arrow[d, "q'"] \\ & X \end{tikzcd}$ " $\Longleftrightarrow$ $\exists$ 同胚 $\varphi: Z \to Z'$，使图表交换：

$$\begin{tikzcd} & & Z' \arrow[ddl, "q'"] \\ Y \arrow[r, "f"] \arrow[urr, "f'"] \arrow[dr, "p"'] & Z \arrow[ur, "\varphi"'] \arrow[d, "q"] & \\ & X & \end{tikzcd}$$

」

$$\left\{ \text{中间覆盖 } \begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & X \end{tikzcd} \;\middle|\; f,\ q \text{ 为 covering} \right\} \Big/ \sim \quad \underset{\longleftarrow}{\overset{\mathrm{Deck}}{\longrightarrow}} \quad \{ H \mid H < \mathrm{Deck}(Y/X) \}$$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & X \end{tikzcd} \longmapsto \mathrm{Deck}(Y/Z) \ (< \mathrm{Deck}(Y/X))$$

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[dr, "p"'] & Y/H \arrow[d, "q"] \\ & X \end{tikzcd} \longleftarrow\!\shortmid H$$

验证: $q: Y/H \to X$, $\bar{y} \mapsto p(y)$, 为 covering.

$\forall x \in X$, $\exists x$ 的连通开邻域 U, s.t. $p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$, $V_\alpha \underset{\text{open}}{\subset} Y$,

且 $p|_{V_\alpha}: V_\alpha \xrightarrow{\cong} U$, $\forall \alpha$.

$q^{-1}(U) = \underline{\pi\left(\coprod_{\alpha \in I} V_\alpha\right)} = \coprod_{\alpha \in I} V_\alpha \Big/ H$

$\forall h \in H$, $h(V_\alpha) \subset V_{\alpha'}$, 对某个 $\alpha' \in I$ 成立. $\Rightarrow$ $h|_{V_\alpha}: V_\alpha \xrightarrow{\cong} V_{\alpha'}$

(设 $h(V_\alpha) \cap V_{\alpha'} \neq \phi$,
$\Rightarrow h(V_\alpha) \subset V_{\alpha'}$)

$$\begin{tikzcd} V_\alpha \arrow[r, "h", "\cong"'] \arrow[dr, "p|_{V_\alpha}"', "\cong"] & h(V_\alpha) \arrow[d, "p|_{h(V_\alpha)}", "\cong"'] & \subset & V_{\alpha'} \arrow[dll, "p|_{V_{\alpha'}}", "\cong"'] \\ & U & & \end{tikzcd}$$

$\Rightarrow h(V_\alpha) = V_{\alpha'}$

且 $h|_{V_\alpha} : V_\alpha \xrightarrow{\cong} V_{\alpha'}$. $\perp$

定理：(Galois covering 基本定理). 设 $p: Y \to X$ 为 Galois 覆盖

局部连通

(i). $\forall$ 覆盖 $g: Z \to X$, 其中 Z 连通, s.t. 对某 $f: Y \to Z$ 组成

如下交换图表：

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "g"] \arrow[dl, phantom, "\circlearrowleft"] \\ & X \end{tikzcd}$$

, 则 f 必为 Galois 覆盖.

(ii). $\mathrm{Deck} \circ / = \mathrm{id}$, $/ \circ \mathrm{Deck} = \mathrm{id}$.

(iii). 在 Deck, / 下, 中间覆盖 $Z \to X$ 为 Galois covering.

$\Longleftrightarrow \mathrm{Deck}(Y/Z) \lhd \mathrm{Deck}(Y/X)$.

Pf. (ii). $\mathrm{Deck} \circ / = \mathrm{id}$ 天生成立. $\mathrm{Deck}(Y/(Y/G)) = G$

$/ \circ \mathrm{Deck} = \mathrm{id}$. 已分析过. $/ \circ \mathrm{Deck}(Y \xrightarrow{f} Z) = $ "$Y \xrightarrow{f} Z$"

$\Longleftrightarrow Y \xrightarrow{f} Z$ 为 Galois covering.

Pf. (i)
$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$
Y 连通, Z 连通, X 局部连通

p: Galois covering

q: covering.

要证: f 为 Galois covering.

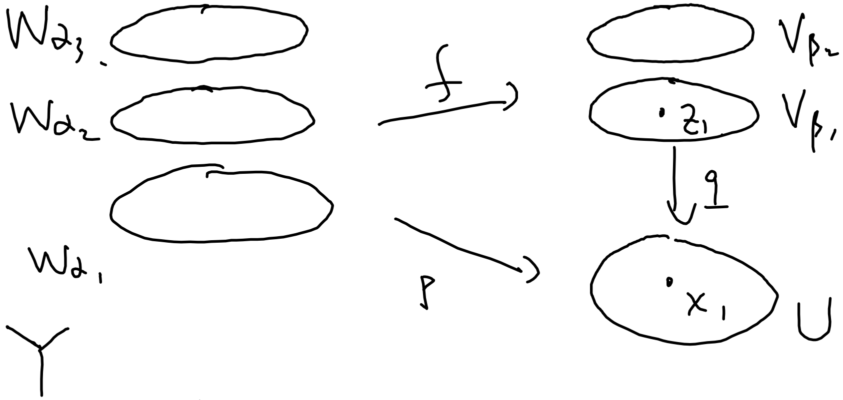
Step 1. 先证 f 为 covering.

Lemma. 设有交换图表:
$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$
其中 X: 局部连通, Z 连通, p, q 为 covering, 则 f 为 covering.

Pf of Lemma. 只要证 f 为满射.

$\forall z_1 \in Z$, $\exists y_1 \in Y$, s.t. $f(y_1) = z_1$. 记 $x_1 = q(z_1)$.
$$\begin{tikzcd} y_1 \in Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \ni z_1 \arrow[d, "q"] \\ & X \ni x_1 \end{tikzcd}$$
$\therefore$ $\exists$ x_1 的连通的开邻域 U, s.t. $p^{-1}(U) = \coprod_{\alpha\in I} W_\alpha$, $W_\alpha \underset{\text{open}}{\subset} Y$.

$q^{-1}(U) = \coprod_{\beta \in J} V_\beta$. $V_\beta \underset{\text{open}}{\subset} Z$, $p|_{W_\alpha}: W_\alpha \xrightarrow{\cong} U$, $q|_{V_\beta}: V_\beta \xrightarrow{\cong} U$.



不妨设 $z_1 \in V_{\beta_1}$.

看 $\underline{f^{-1}(V_{\beta_1})}$.

$f\big|_{\coprod_\alpha W_\alpha}: \coprod_{\alpha\in I} W_\alpha \longrightarrow \coprod_{\beta\in J} V_\beta$ 具有特性:

特性: 若 $f(W_\alpha)\cap V_\beta \neq \phi$, 则 $f(W_\alpha)\subset V_\beta$, 且 $f\big|_{W_\alpha}: W_\alpha \xrightarrow{\cong} V_\beta$.

「 由 W_α 连通 $\Longrightarrow f(W_\alpha)\subset V_\beta$.

$$\begin{tikzcd} & V_\beta \arrow[d, "g|_{V_\beta}", "\cong"'] \\ W_\alpha \arrow[ur, "f|_{W_\alpha}"] \arrow[r, "\cong", "p|_{W_\alpha}"'] & U \end{tikzcd}$$

」

由此特性: $\exists\, I' \subset I$, s.t. $f^{-1}(V_{\beta_1}) = \coprod_{\alpha\in I'} W_\alpha$,

$\Longrightarrow f$ 为一个 covering map. 」

下证: $f: Y \longrightarrow Z$ 为满射.

只要证 $f(Y)$ 既开又闭.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "g"] \\ & X \end{tikzcd}$$

• $f(Y)$ 开:

$\forall z_1 \in f(Y)$, 设 $z_1 = f(y_1)$, $x_1 = g(z_1)$

已证: $\exists$ z_1 的邻域 V_{β_1}, s.t. $f^{-1}(V_{\beta_1}) = \coprod_{\alpha \in I'} U_\alpha$, $U_\alpha \underset{\text{open}}{\subset} Y$.

且 $f|_{U_\alpha}: U_\alpha \xrightarrow{\cong} V_{\beta_1}$.

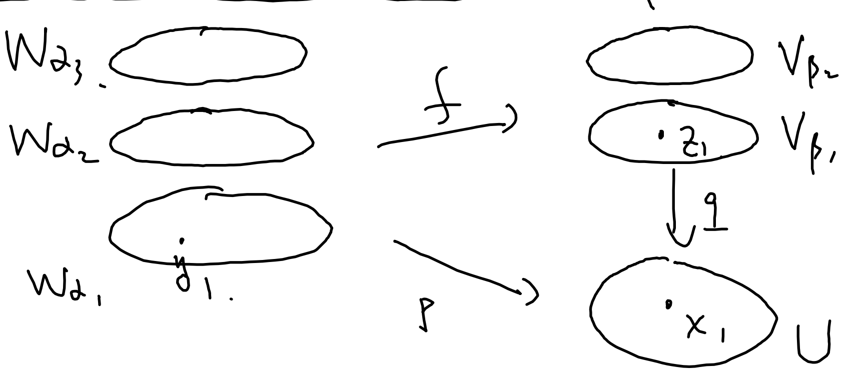
$\Rightarrow V_{\beta_1} \subset f(Y)$

• $Z \setminus f(Y)$ 开:

$\forall z_1 \in Z \setminus f(Y)$, 令 $x_1 = g(z_1)$. $\exists$ x_1 的连通开邻域 U,

s.t. $p^{-1}(U) = \coprod_{\alpha \in I} W_\alpha$, $W_\alpha \underset{\text{open}}{\subset} Y$, $p|_{W_\alpha}: W_\alpha \xrightarrow{\cong} U$

$g^{-1}(U) = \coprod_{\beta \in J} V_\beta$. $V_\beta \underset{\text{open}}{\subset} Z$, $g|_{V_\beta}: V_\beta \xrightarrow{\cong} U$.



Claim: $V_{\beta_1} \subset Z \setminus f(Y)$.

「若 $\exists z \in V_{\beta_1}$, s.t. $z = f(y_1)$.

$\Rightarrow f(W_{\alpha_1}) \cap V_{\beta_1} \neq \phi$.

$\Rightarrow f(W_{\alpha_1}) = V_{\beta_1}$. 」

又 Z 是连通的，$\Rightarrow f(Y) = Z$. #

Step 2. f 为 Galois covering.

$$\begin{tikzcd}
\text{连通} & Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "g"] & \text{连通} \\
& & X & & \\
& \text{局部连通} & & &
\end{tikzcd}$$

p: Galois covering

g: covering.

要证：$\forall\, y_1, y_2 \in Y$, $f(y_1) = f(y_2)$, 则 $\exists\, g \in \mathrm{Deck}(Y/Z)$,

s.t. $g(y_1) = y_2$. i.e. $\exists$ 同胚 $g: Y \to Y$. s.t.

$g(y_1) = y_2$, 且图表交换：

$$\begin{tikzcd}
Y \arrow[rr, "g"] \arrow[dr, "f"'] & \circlearrowleft & Y \arrow[dl, "f"] \\
& Z &
\end{tikzcd}$$

由 p 为 Galois covering, $\exists\, h: Y \xrightarrow{\cong} Y$. s.t.

图表交换：

$$\begin{tikzcd}
Y \arrow[rr, "h"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

, 且 $h(y_1) = y_2$.

$$
\begin{tikzcd}
y_2 \in Y \arrow[r, "f"] \arrow[dr, "p"] & Z \arrow[d, "q"] \\
y_1 \in Y \arrow[u, "h"] \arrow[r, "p"] & X
\end{tikzcd}
$$

Claim: $h \in \mathrm{Deck}(Y/Z)$, i.e. $\underline{f\circ h = f}$, i.e. 图表交换:

$$
\begin{tikzcd}
\overset{y_1}{Y} \arrow[rr, "h"] \arrow[dr, "f"'] & & \overset{y_2}{Y} \arrow[dl, "f"] \\
& Z &
\end{tikzcd}
$$

┌.

$$
\begin{tikzcd}
& Z \arrow[d, "q"] \\
\text{连通} \rightarrow Y \arrow[ur, shift left, "f\circ h"] \arrow[ur, shift right, "f"'] \arrow[r, "p"] & X
\end{tikzcd}
$$

$\Rightarrow$ $f\circ h$, f 均为 $p: Y \to X$ 在 covering q 下的提升.

$$
\begin{cases}
f\circ h(y_1) = f(y_2) \\
f(y_1)
\end{cases}
\quad f\circ h(y_1) = f(y_1)
$$

由提升唯一性引理 $\Rightarrow$ $f\circ h = f$.

┘

(iii).
$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "g"] \\ (\triangle) & X \end{tikzcd}$$

要证：g 为 Galois covering

$\Longleftrightarrow \mathrm{Deck}(Y/Z) \triangleleft \mathrm{Deck}(Y/X)$

不妨设 $(\triangle)$ 长成：

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[dr, "p"'] & Y/H \arrow[d, "\bar{p}"] & \bar{y} \arrow[d, mapsto] \\ & X & p(y) \end{tikzcd}$$

其中 $H < \mathrm{Deck}(Y/X)$.

要证：$\bar{p}: Y/H \to X$ 为 Galois covering $\Longleftrightarrow H \triangleleft \underline{\mathrm{Deck}(Y/X)}$ ($= G$)

"$\Longleftarrow$". $H \triangleleft G$.

$G \circlearrowright Y$, 且 G 可迁地作用在 p 的纤维上,

($G \circlearrowright p^{-1}(x)$, $\forall x \in X$ 是可迁的)

定义：$G \circlearrowright Y/H$, $\forall g \in G$, $\bar{y} \in Y/H$, 定义

$g(\bar{y}) := \overline{g(y)}$. (设 $\bar{y} = \bar{y}'$, i.e. $\exists h \in H$, $y = h(y')$.

$\underline{\overline{g(y)} = \overline{g(y')}} \Longleftarrow g(y) = g \cdot h(y')$
$= \underline{g \cdot h \cdot g^{-1}}(g(y'))$.)

由 $G \curvearrowright Y/H$ 之定义，有交换图表：

$\forall g \in G$.

$G = \mathrm{Deck}(Y/X)$

$$\begin{tikzcd}
Y/H \arrow[rr, "g"] \arrow[drr, "\bar{p}"'] & & Y/H \arrow[d, phantom] \arrow[dl, "\bar{p}"] \\
Y \arrow[u, "\pi"] \arrow[rr, "g"] \arrow[dr, "p"'] & & Y \arrow[u, "\pi"'] \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

$\forall\ \bar{y}_1, \bar{y}_2 \in Y/H$. $\bar{p}(\bar{y}_1) = \bar{p}(\bar{y}_2)$. $p(y_1) = p(y_2)$

$\Rightarrow \exists\, h \in G$, s.t. $h(y_1) = h(y_2)$.

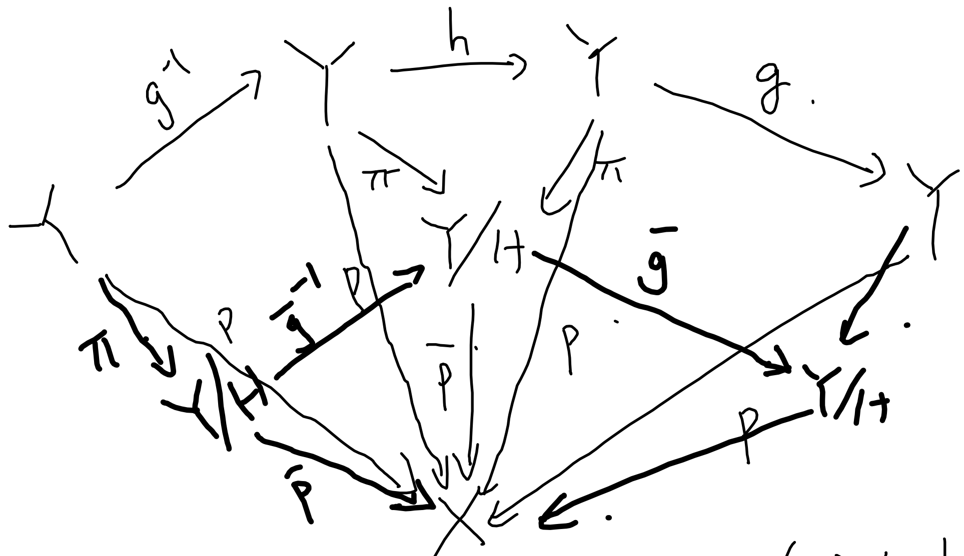
$$h(\bar{y}_1) = \overline{h(y_1)} = \overline{h(y_2)} = h(\bar{y}_1).$$

"$\Longrightarrow$" $H < G = \mathrm{Deck}(Y/X)$.

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[dr, "p"'] & Y/H \arrow[d, "\bar{p}"] \\ & X \end{tikzcd}$$

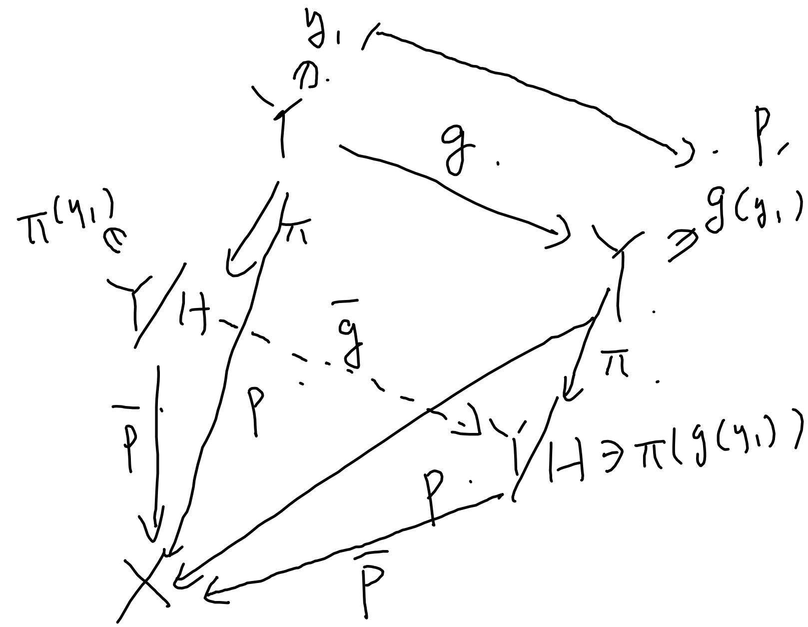
设 $\bar{p}$ 为 Galois covering

要证: $\underline{\forall h \in H,\ g \in G,\ g\cdot h\cdot g^{-1} \in H}$.



要证: $\pi\circ(g\cdot h\cdot g^{-1}) = \pi$. $\left(\because H = \mathrm{Deck}(Y/(Y/H))\right)$.

只需证：



$p, \bar{p}, \pi$: Galois 覆盖

$$p(y_1) = p(g(y_1)).$$

$$\Rightarrow \pi(y_1)、\pi(g(y_1)) \in \bar{p}^{-1}(p(y_1))$$

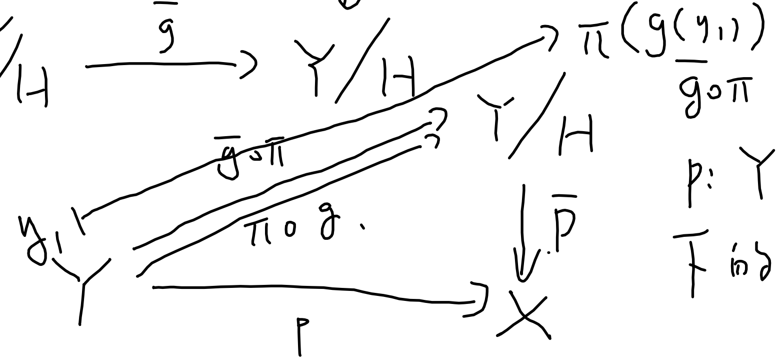
由 $\bar{p}$ 为 Galois covering，$\exists\, \bar{g} \in \mathrm{Deck}((Y/H)/X)$，s.t. $\bar{g}(\pi(y_1)) = \pi(g(y_1))$.

下面只剩需证：

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"] \\ Y/H \arrow[r, "\bar{g}"] & Y/H \end{tikzcd}$$

交换.

i.e. $\bar{g} \circ \pi = \pi \circ g$.



$\bar{g}\circ\pi$ 与 $\pi\circ g$ 同为 $p: Y \to X$ 在 $\bar{p}$ 下的提升.

由于 $\bar{g}\circ\pi(y_1) = \pi\circ g(y_1)$. 且 Y 连通.

由提升唯一性引理, $\Rightarrow$ $\bar{g}\circ\pi = \pi\circ g$. #