§3. 群的自由积与 fibered coproduct.

· 群的自由积.

设 $\{G_\alpha\}_{\alpha\in I}$ 为一族群. $\{G_\alpha\}_{\alpha\in I}$ 中的一个长度为 m 的文字(word), $m\geq 1$, 是指一个 $\underbrace{\coprod_{\alpha\in I} G_\alpha \times \cdots \times \coprod_{\alpha\in I} G_\alpha}_{m\text{个}}$ 中的元素, i.e. 形如 $(g_1,\cdots,g_m)$, 其中 $g_i\in$ 某个 G_{α_i} 的元素. 额外地. 记长度为 0 的文字为 $(\ )$. (空文字 empty word).

定义. 文字的初等约化 (elementary reduction) 是指下面两种操作之一.

① $(g_1,\cdots,g_i,g_{i+1},\cdots,g_m)\longmapsto(g_1,\cdots,g_ig_{i+1},\cdots,g_m)$

其中 $g_i, g_{i+1}\in$ 同一个 G_α.

② $(g_1,\cdots,g_{i-1},1_\alpha,g_{i+1},\cdots,g_m)\longmapsto(g_1,\cdots,g_{i-1},g_{i+1},\cdots,g_m)$

(G_α 中的单位元记为 1_α).

(规定: $(1_\alpha)\longmapsto(\ )$).

定义. 若一个文字不能再由初等约化变短, 则称该文字是既约的 (reduced)

例. $G_1=\{x^n \mid n\in\mathbb{Z}\}$, $G_2=\{y^n \mid n\in\mathbb{Z}\}$

$(x, x, y, y^{-2}) \sim (x^2, y^2, y^{-3})$

$\{G_1, G_2\}$ 中的文字.

$(x^2, y, y^{-2}, x^{-3}, y)$

$\downarrow$

(x^2, y^{-1}, x^{-3}, y) 既约.

$(x^3, x, y^{-1}, y^2, x, x^0, y)$,

$\downarrow$

$(x^4, y^{-1}, y^2, x, x^0, y)$

$\downarrow$

(x^4, y, x, x^0, y).

$\downarrow$

(x^4, y, x, y). 既约.

事实: 任何一个文字由有限步初等约化可变为一个既约文字. 且如此变换得到的既约文字是唯一的. (不证)

记 $\mathrm{Word}(\{G_\alpha\}_{\alpha\in I}) = \{G_\alpha\}_{\alpha\in I}$ 中的文字全体.

定义 $\mathrm{Word}(\{G_\alpha\}_{\alpha\in I})$ 上的一个关系"$\sim$".

$\forall$ 两文字 w_1, w_2. 定义"$w_1 \sim w_2$" iff w_1 与 w_2 经有限步初等约化后所变为的既约文字是同一个文字.

由事实, 显然"$\sim$"为一个等价关系.

记 $*_\alpha G_\alpha = \mathrm{Word}(\{G_\alpha\}_{\alpha\in I})/\sim$.

Rmk. 由事实, $*_\alpha G_\alpha$ 中的每一个等价类有且仅有一个既约文字.

$$*_\alpha G_\alpha \xleftrightarrow{1:1} \{\{G_\alpha\}_{\alpha\in I} \text{ 中的既约文字}\}$$

例. $G_1 = \{x^n \mid n\in\mathbb{Z}\}$, $G_2 = \{y^n \mid n \in \mathbb{Z}\}$.

$$G_1 * G_2 \xleftrightarrow{1:1} \{x^\square y^\square \cdots x^\square y^\square,\ x^\square y^\square \cdots x^\square,\ y^\square x^\square \cdots y^\square,\ y^\square x^\square \cdots x^\square \mid \text{其中 } \square \text{ 为非零整数}\}$$

定义: $\cdot : \{\text{文字}\}\times\{\text{文字}\} \longrightarrow \{\text{文字}\}$ 　　规定 $(\ )\cdot w = w$, $w\cdot(\ ) = w$

$$((g_1,\cdots,g_n),(h_1,\cdots,h_m)) \longmapsto (g_1,\cdots,g_n,h_1,\cdots,h_m)$$

定义：$\cdot$: $*_\alpha G_\alpha \times *_\alpha G_\alpha \longrightarrow *_\alpha G_\alpha$.　(良好定义)

$(\overline{w_1}, \overline{w_2}) \longmapsto \overline{w_1 \cdot w_2}$.

$$\begin{tikzcd}[column sep=small]
w_1' \arrow[dr] & \sim & w_1, \arrow[dl] & & w_2' \arrow[dr] & \sim & w_2. \arrow[dl] \\
& w_{10} & & & & w_{20}. &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
w_1' w_2' \arrow[dr] & & w_1 \cdot w_2 \arrow[dl] \\
& w_{10} \cdot w_{20} \arrow[d] & \\
& w_3 \ (\text{既约}) &
\end{tikzcd}
\quad \Rightarrow \quad \overline{w_1' \cdot w_2'} = \overline{w_1 \cdot w_2}.$$

命题：$(*_\alpha G_\alpha, \cdot)$ 构成一个群.

Pf. (1) 结合律.

(2) 有恒元，$1 = \overline{(\ )}$，$\forall\, \overline{w} \in *_\alpha G_\alpha$，$1 \cdot \overline{w} = \overline{(\ ) \cdot w} = \overline{w}$

(3) 有逆元，$\forall\, \overline{w} \in *_\alpha G_\alpha$，$w = (g_1, \cdots, g_n)$.

令 $w^{-1} = (g_n^{-1}, \cdots, g_1^{-1})$，$\overline{w^{-1}} \cdot \overline{w} = \overline{w^{-1} w} = \overline{(\ )} = 1$.

#

称 $(*_\alpha G_\alpha, \cdot)$ 为 $\{G_\alpha\}_{\alpha\in I}$ 的自由积 (free product)

universal property of $*_\alpha G_\alpha$.

$\forall \alpha\in I$, 有典型映射, $\iota_\alpha : G_\alpha \longrightarrow *_\alpha G_\alpha$

$g \longmapsto \overline{(g)}$

ι_α 显然为单群同态.

数据 $(*_\alpha G_\alpha, \iota_\alpha)$ 满足性质:

$\forall$ 群 H, 以及群同态 $\varphi_\alpha : G_\alpha \to H$, $\forall \alpha\in I$,

则存在唯一的群同态 $\varphi : *_\alpha G_\alpha \to H$, s.t. $\varphi\circ\iota_\alpha = \varphi_\alpha$.

i.e. 有下面的交换图表:

$$
\begin{tikzcd}
G_{\alpha_1} \arrow[r, "\iota_{\alpha_1}"] \arrow[drr, "\varphi_{\alpha_1}"'] & *_\alpha G_\alpha \arrow[dr, dashed, "\exists!\varphi"] & \\
G_{\alpha_2} \arrow[ur, "\iota_{\alpha_2}"] \arrow[rr, "\varphi_{\alpha_2}"'] & & H \\
\vdots & &
\end{tikzcd}
$$

Pf. $\forall w \in *_\alpha G_\alpha$,

设 $w = \overline{(g_1, \cdots, g_n)}$, 其中 $g_i \in G_{\alpha_i}$,

定义 $\varphi(w) = \varphi_{\alpha_1}(g_1)\cdots\varphi_{\alpha_n}(g_n)$

① 良好定义 ($\because \varphi_\alpha$ 为群同态)

② φ 为群同态.

③ $\varphi\circ\iota_\alpha = \varphi_\alpha$.

④ 如此的 φ 是唯一的

#

④: 假设 $\psi : *_\alpha G_\alpha \to H$ 为同态, $\psi\circ\iota_\alpha = \varphi_\alpha$.

$\Rightarrow \forall g\in G_\alpha$, $\psi(\overline{(g)}) = \varphi_\alpha(g)$

$\forall w = \overline{(g_1, \cdots, g_n)} \in *_\alpha G_\alpha$, $g_i \in G_{\alpha_i}$.

$\psi(w) = \psi(\overline{(g_1)}\cdots\overline{(g_n)})$

$= \psi(\overline{(g_1)})\cdots\psi(\overline{(g_n)}) = \varphi_{\alpha_1}(g_1)\cdots\varphi_{\alpha_n}(g_n)$.

Rmk. 此 universal property 唯一地刻画了自由积群

i.e. 设有群 $\tilde{G}$, 以及 $\tilde{\iota}_\alpha : G_\alpha \to \tilde{G}$, 假设 $(\tilde{G}, \tilde{\iota}_\alpha)$

同样满足: $\forall$ 群 H, 以及群同态 $\varphi_\alpha : G_\alpha \to H$,

则 $\exists!$ 群同态 $\varphi : \tilde{G} \to H$ 使图表交换.

$$
\begin{tikzcd}
G_{\alpha_1} \arrow[r, "\tilde{\iota}_{\alpha_1}"] \arrow[rr, bend left=10, "\varphi_{\alpha_1}"] & \tilde{G} \arrow[r, dashed, "\exists!\varphi"] & H \\
G_{\alpha_2} \arrow[ur, "\tilde{\iota}_{\alpha_2}"] \arrow[urr, "\varphi_{\alpha_2}"'] & & \\
\vdots & &
\end{tikzcd}
$$

则 $\tilde{G} \cong *_\alpha G_\alpha$.

pf. 设 $(\tilde{G}, \tilde{\iota}_\alpha)$ 满足如上的 universal property

$$
\begin{tikzcd}
G_{\alpha_1} \arrow[r, "\tilde{\iota}_{\alpha_1}"] \arrow[rr, bend left, "\iota_{\alpha_1}"] & \tilde{G} \arrow[r, dashed, bend left=10, "\exists! f"] & *_\alpha G_\alpha \arrow[l, dashed, bend left=10, "\exists! g"] \\
G_{\alpha_2} \arrow[ur, "\tilde{\iota}_{\alpha_2}"] \arrow[urr, bend right, "\iota_{\alpha_2}"'] & & \\
\vdots & &
\end{tikzcd}
$$

Claim: $g \circ f = id$, $f \circ g = id$.

$$
\begin{tikzcd}
G_{\alpha_1} \arrow[r, "\tilde{\iota}_{\alpha_1}"] \arrow[dr, "\tilde{\iota}_{\alpha_1}"'] & \tilde{G} \arrow[d, "g \circ f"] \\
G_{\alpha_2} \arrow[ur, "\tilde{\iota}_{\alpha_2}"] \arrow[r, "\tilde{\iota}_{\alpha_2}"'] & \tilde{G} \\
\vdots &
\end{tikzcd}
$$

universal property $\Rightarrow$ $g \circ f = id$

#

· 用生成元和关系来表示一个群

G为一个群，$S \subset G$ 为一个子集，定义 S 在 G 中生成的群为 G 中包含 S 的最小子群，将之记为 $\langle S \rangle$.

$$\langle S \rangle = \{ g_1^{k_1} \cdots g_n^{k_n} \mid n \in \mathbb{Z}_{\geq 1},\ k_i \in \mathbb{Z},\ g_i \in S,\ \forall i \}.$$

若 $\langle S \rangle = G$，则称 S 为 G 的一组生成元.

例：$(\mathbb{Z}, +)$. $\langle 1 \rangle = \mathbb{Z}$，$\langle -1 \rangle = \mathbb{Z}$，$\langle 2 \rangle = 2\mathbb{Z}$

定义：设 S 为一个集合. S 上的自由群 (free group on S)

$F(S)$ 定义为：

$$F(S) := \ast_{\alpha \in S} \langle \alpha \rangle.$$

其中 $\langle \alpha \rangle = \{ \alpha^n \mid n \in \mathbb{Z} \}$. 若 S 为有限集，比如 $S = \{\alpha_1, \cdots, \alpha_n\}$，

$F(S)$ 又记为 $F(\alpha_1, \cdots, \alpha_n)$.

例：$\langle x, y \rangle = \langle x \rangle \ast \langle y \rangle \xleftrightarrow{1:1} \{ x^{n_1} y^{m_1} \cdots x^{n_k} y^{m_k} \mid k \in \mathbb{Z}_{\geq 1},\ n_i, m_j$ 中 $n_1, m_k \in \mathbb{Z}$，其余为非零整数 $\}$

有典型嵌入: $\iota: S \longrightarrow F(S)$.

$$x \longmapsto \overline{(x)}$$

数据: $(F(S), \iota)$ 满足下列的 universal mapping property:

$\forall$ 群 H, 以及映射 $f: S \longrightarrow H$, 存在唯一群同态 $\varphi_f: F(S) \longrightarrow H$, s.t. $\varphi_f \circ \iota = f$, i.e. 下列图表交换:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\circlearrowleft"] & & H \\ F(S) \arrow[urr, dashed, "\exists!\varphi_f"'] & & \end{tikzcd}$$

Rmk. 此 universal property 唯一地(差一个同构的意义下)决定 $F(S)$.

设 $\tilde{G}$ 为一个群, $\tilde{\iota}: S \longrightarrow \tilde{G}$, 设 $(\tilde{G}, \tilde{\iota})$ 满足: 数据

$\forall f: S \longrightarrow H$ 群, $\exists! \varphi_f: \tilde{G} \longrightarrow S$ s.t. $\varphi_f \circ \tilde{\iota} = f$.

则 $\tilde{G} \cong F(S)$. (留作练习)

设 $F(S)$ 为 S 上的自由群，$R \subset F(S)$ 为子集，定义群：

$$\langle S \mid R \rangle := F(S) / \overline{R}$$

其中 $\overline{R}$ 为 $F(S)$ 中包含 R 的最小的正规子群。

$$\left(\overline{R} = \bigcap_{S \subset N \triangleleft F(S)} N = \bigcup_{g \in F(S)} g \cdot \langle R \rangle g^{-1} . \right).$$

$\forall x \in S$，

$$S \xrightarrow{\iota} F(S) \xrightarrow{\pi} \langle S \mid R \rangle$$

$$x \longmapsto \pi \circ \iota(x) \left(\overset{\text{记为}}{=} \overline{x} \overset{\text{记为}}{=} x \right).$$

注意到 $\langle \{ \overline{x} \mid x \in S \} \rangle = \langle S \mid R \rangle$.

故而称 S 为 $\langle S \mid R \rangle$ 的生成元.

称 R 为 $\langle S \mid R \rangle$ 的关系，

称 $\langle S \mid R \rangle$ 为在关系 R 下，由 S 生成的群.

问：是否每个群都可写为 $\langle S \mid R \rangle$ 的形式.

答：是的。

Pf. 设 G 为群，取 S 为 G 的一组生成元。

$$i: S \longrightarrow G.$$

由 universal property of $F(S)$，$\exists !\ \underbrace{\varphi: F(S) \longrightarrow G}_{\text{群同态}}$，

s.t. $\underline{\varphi \circ \iota = i}$ $\quad (\iota: S \to F(S))$.

$\Rightarrow \varphi$ 为满同态

$$\Rightarrow F(S)/\ker\varphi \cong G.$$

$$F(S)/\ker\varphi = \langle S \mid \ker\varphi \rangle$$

若 $G \cong \langle S \mid R \rangle$，则把 $\langle S \mid R \rangle$ 称为 G 的一个 presentation.

若 S, R 为有限集，称 G is f finitely presented.

例：$\mathbb{Z}/n\mathbb{Z} = \langle \alpha \mid \alpha^n \rangle$

$$\langle \alpha \rangle \longrightarrow \mathbb{Z}/n\mathbb{Z} \qquad \alpha^k \longmapsto \overline{k}$$

例. $\mathbb{Z}\times\mathbb{Z} = \langle \alpha, \beta \mid \alpha\cdot\beta = \beta\cdot\alpha \rangle$

$$\left(= \langle \alpha, \beta \mid \alpha\cdot\beta\cdot\alpha^{-1}\cdot\beta^{-1} \rangle \right.$$

$$\left. = \langle \alpha, \beta \rangle \Big/ \overline{\{\alpha\cdot\beta\cdot\alpha^{-1}\cdot\beta^{-1}\}} \right)$$

原先在 $\langle \alpha, \beta \rangle$ 中, $(\alpha, \beta) \neq (\beta, \alpha)$.

$$\begin{array}{ccc} (\alpha,\beta) & \neq & (\beta,\alpha) \\ \| & & \| \\ (\alpha)\cdot(\beta) & & (\beta)\cdot(\alpha) \end{array}$$

记 $\alpha = (\alpha)$, $\beta = (\beta)$.

$\alpha\cdot\beta \neq \beta\cdot\alpha$.

在 $\langle \alpha, \beta \mid \alpha\cdot\beta = \beta\cdot\alpha \rangle$ 中, $\alpha\cdot\beta = \beta\cdot\alpha$.

$\forall\, x \in \langle \alpha, \beta \mid \alpha\cdot\beta = \beta\cdot\alpha \rangle$, $x = \alpha^n\cdot\beta^m$.

练习: 证明: $\mathbb{Z}\times\mathbb{Z} \cong \langle \alpha, \beta \mid \alpha\cdot\beta = \beta\cdot\alpha \rangle$.

提示: $\varphi: \langle \alpha, \beta \rangle \longrightarrow \mathbb{Z}\times\mathbb{Z}$ by $\alpha \mapsto (1,0)$, $\beta \mapsto (0,1)$.

$\ker\varphi \supset \{\alpha\cdot\beta\cdot\alpha^{-1}\cdot\beta^{-1}\} \Rightarrow \varphi$ 诱导 $\overline{\varphi}: \langle \alpha, \beta \mid \alpha\cdot\beta = \beta\cdot\alpha \rangle \longrightarrow \mathbb{Z}\times\mathbb{Z}$

再定义 $\psi: \mathbb{Z}\times\mathbb{Z} \longrightarrow \langle \alpha, \beta \mid \alpha\cdot\beta = \beta\cdot\alpha \rangle$. 证: $\varphi\circ\psi = id$, $\psi\circ\varphi = id$

· fibered coproduct.

定义. 设 $\mathcal{C}$ 为一个范畴, 设 $f: Z \to X$, $g: Z \to Y$, f 与 g 的 fibered coproduct 是指数据 (W, p_1, p_2), 其中 $W \in Ob(\mathcal{C})$, $p_1: X \to W$, $p_2: Y \to W$, 满足:

① 图表.

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "p_1"] \\ Y \arrow[r, "p_2"] & W \end{tikzcd}$$

② $\forall$ 使下面图表交换的 $u: X \to U$, $v: Y \to U$,

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "p_1"] \arrow[ddr, bend left, "u"] & \\ Y \arrow[r, "p_2"] \arrow[drr, bend right, "v"'] & W \arrow[dr, dashed, "\exists! h"] & \\ & & U \end{tikzcd}$$

$\exists!$ 态射 $h: W \to U$, s.t. $h \circ p_1 = u$, $h \circ p_2 = v$.

此时, 称 $\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & W \end{tikzcd}$ 为一个 cocartesian 的图表.

练习：证明 $\mathcal{C}$ 中 fibered coproduct of $\left(\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \\ Z & \end{tikzcd}\right)$ 若存在，则在差一个同构的意义下唯一。

Rmk. 将上面定义中箭头全部反向，则得 fibered product.

例. Top 拓扑空间范畴，$X \in Ob(Top)$.

设 X_0，X_1 为 X 的子空间，且 $int(X_0) \cup int(X_1) = X$.

记 $X_{01} = X_0 \cap X_1$，记 $i_0 : X_0 \hookrightarrow X$，$i_1 : X_1 \hookrightarrow X$，

$j_0 : X_{01} \hookrightarrow X_0$，$j_1 : X_{01} \hookrightarrow X_1$ 均为包含映射. 则

$$\begin{tikzcd} X_{01} \arrow[r, "j_0"] \arrow[d, "j_1"'] & X_0 \arrow[d, "i_0"] \\ X_1 \arrow[r, "i_1"] & X \end{tikzcd}$$

为一个 cocartesian 的图表

Pf. 设 $f: X_0 \to Y$，$g: X_1 \to Y$ in Top. 满足 $f \circ j_0 = g \circ j_1$

$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \arrow[ddr, bend left, "f"] & \\ X_1 \arrow[r] \arrow[drr, bend right, "g"'] & X & \\ & & Y \end{tikzcd}$$

定义 $h: X \to Y$

$x \mapsto h(x)$

$$h(x) = \begin{cases} f(x), & \text{if } x \in X_0 \\ g(x), & \text{if } x \in X_1 \end{cases}$$

$X = int(X_0) \cup int(X_1)$

$h|_{int(X_0)} = f|_{int(X_0)}$ $\Rightarrow$ h 连续. #

例. Top° 带点拓扑空间范畴.

X, X_0, X_1, X_{01} 同上, $p \in X_{01}$.

$$\begin{tikzcd} (X_{01}, p) \arrow[r, "j_0"] \arrow[d, "j_1"'] & (X_0, p) \arrow[d, "i_0"] \\ (X_1, p) \arrow[r, "i_1"] & (X, p) \end{tikzcd}$$

为 cocartesian 的图表.

例. Grp 群范畴. $P, G, H \in Ob(Grp)$!

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "j"'] & G \\ H . & \end{tikzcd}$$

Claim: i 与 j 的 fibered coproduct 存在的.

$G * H$: 自由积, $\iota_1: G \to G * H$, $\iota_2: H \to G * H$.

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[d, "\iota_1"] \\ H \arrow[r, "\iota_2"'] & G * H \end{tikzcd}$$

$N = \overline{\{\iota_1 \circ i(x) \cdot (\iota_2 \circ j(x))^{-1} \mid x \in P\}}$.

定义 $G *_P H = G * H / N$.

$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[dd, "j"'] & G \arrow[d, "\iota_1"'] \arrow[ddr, "\tau"] & \\
& G*H \arrow[dr, "\pi"] & \\
H \arrow[ur, "\iota_2"] \arrow[rr, bend right, "v"'] & & G*_P H
\end{tikzcd}$$

Claim:

$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[d, "\tau"] \\
H \arrow[r, "v"] & G*_P H
\end{tikzcd}$$

是 Grp 中的 cocartesian 的图表

「Check.

$$\begin{tikzcd}
P \arrow[r, "i"] \arrow[dd, "j"'] & G \arrow[d, "\iota_1"'] \arrow[dd, "\tau"] \arrow[dddrr, "f"] & & \\
& G*H \arrow[d, "\pi"] \arrow[ddrr, bend left, "\exists!\tilde{h}"] & & \\
H \arrow[ur, "\iota_2"] \arrow[r, "v"] \arrow[drrr, "g"'] & G*_P H \arrow[drr, dashed, "h"] & & \\
& & & K
\end{tikzcd}$$

$\forall f: G \to K$, $g: H \to K$, s.t. 左图表交换，要证 $\exists!\, h: G*_P H \to K$, s.t. 加入 h 后，图表仍交换

由 $G*H$ 的 universal property, $\exists!\ \tilde{h}: G*H \to K$

s.t. $\tilde{h}\circ\iota_1 = f$, $\tilde{h}\circ\iota_2 = g$. $\ker\pi = \{\iota_1\circ i(x)\cdot(\iota_2\circ j(x))^{-1} \mid x \in P\}$.

$\forall x \in P$, $\tilde{h}(\iota_1\circ i(x)\cdot(\iota_2\circ j(x))^{-1}) = \underbrace{\tilde{h}\circ\iota_1}_{=f}\circ i(x)\cdot\underbrace{\tilde{h}\circ\iota_2}_{=g}\circ j(x^{-1}) = 1$.

$\Rightarrow \operatorname{Ker}\tilde{h} \supset \operatorname{Ker}\pi$.

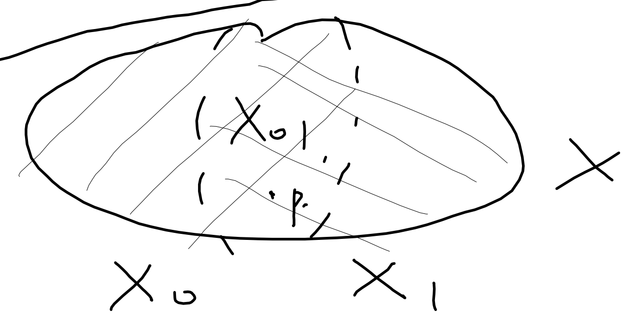
$$\begin{tikzcd} G * H \arrow[r, "\tilde{h}"] \arrow[d, "\pi"'] & K \\ G *_p H & \end{tikzcd}$$

$\Rightarrow \exists!\ h : G *_p H \to K$, s.t. 图表交换. #

§4. Seifert & van Kampen 定理 (0).

X : top space, $X_0, X_1 \subset X$, $X_{01} = X_0 \cap X_1$

$\operatorname{int}(X_0) \cup \operatorname{int}(X_1) = X$



Seifert & van Kampen 定理:

$\pi_1(X, p)$ 可由 $\pi_1(X_0, p)$, $\pi_1(X_1, p)$, $\pi_1(X_{01}, p)$ 来计算.

设 X, X_0, X_1, X_{01} 如 (○).

$$\begin{tikzcd} X_{01} \arrow[r, "j_0"] \arrow[d, "j_1"'] & X_0 \arrow[d, "i_0"] \\ X_1 \arrow[r, "i_1"] & X \end{tikzcd} \qquad (\#)$$

取 $p \in X_{01}$.

$$\begin{tikzcd} (X_{01}, p) \arrow[r, "j_0"] \arrow[d, "j_1"'] & (X_0, p) \arrow[d, "i_0"] \\ (X_1, p) \arrow[r, "i_1"] & (X, p) \end{tikzcd} \qquad (\#')$$

上一次: $(\#)$ 与 $(\#')$ 为 cocartesian.

定理1 (Seifeit & van-Kampen) 设 X, X_0, X_1, X_{01} 如 (○), 进一步假设: $p \in X_{01}$, X_0, X_1, X_{01} 均道路连通, 则函子 $\pi_1 : \mathcal{T}op^{\circ} \to Grp$. 把 $(\#')$ 变为 Grp 中的 cocartesian 图表:

$$\begin{tikzcd} \pi_1(X_{01}, p) \arrow[r, "(j_0)_*"] \arrow[d, "(j_1)_*"'] & \pi_1(X_0, p) \arrow[d, "(i_0)_*"] \\ \pi_1(X_1, p) \arrow[r, "(i_1)_*"] & \pi_1(X, p) \end{tikzcd}.$$

推论：$\pi_1(X,p) \cong \pi_1(X_0,p) *_{\pi_1(X_{01},p)} \pi_1(X_1,p)$.

「上节课：

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "j"'] & G \arrow[d, "\tau"] \\ H \arrow[r, "v"'] & G *_P H \end{tikzcd}$$

cocartesian.」

定理α (R. Brown) 设 X_0, X_1, X_{01}, X 如 (○). (注意不需要假定 X_0, X_1, X_{01} 道路连通). 则函子 $\Pi: \mathrm{Top} \to \mathrm{Grpd}$, 把 cocartesian 图表 (#) 变为 Grpd 中的 cocartesian 图表.

i.e.

$$\begin{tikzcd} \Pi(X_{01}) \arrow[r, "\Pi(j_0)"] \arrow[d, "\Pi(j_1)"'] & \Pi(X_0) \arrow[d, "\Pi(i_0)"] \\ \Pi(X_1) \arrow[r, "\Pi(i_1)"'] & \Pi(X) \end{tikzcd}$$

为 cocartesian.

定理α之证明：设 $\mathcal{G} \in \mathrm{Ob}(\mathrm{Grpd})$ 为一个 groupoid. 且有函子 $F: \Pi(X_0) \to \mathcal{G}$, $G: \Pi(X_1) \to \mathcal{G}$, 使 $F \circ \Pi(j_0) = G \circ \Pi(j_1)$.

$$
\begin{tikzcd}
\Pi(X_{01}) \arrow[r, "\Pi(j_0)"] \arrow[d, "\Pi(j_1)"'] & \Pi(X_0) \arrow[d, "\Pi(i_0)"] \arrow[rd, "F"] & \\
\Pi(X_1) \arrow[r, "\Pi(i_1)"] \arrow[rr, bend right, "G"'] & \Pi(X) & \mathcal{G}
\end{tikzcd}
$$

要证：$\exists!$ 函子 $K: \Pi(X) \to \mathcal{G}$, s.t. 加入 K 后上图表仍交换.

定义 K:

① on objects: $\forall\, x \in Ob(\Pi(X)) = X$, $K(x) = \begin{cases} F(x), & \text{if } x \in X_0 \\ G(x), & \text{if } x \in X_1 \end{cases}$.

(良好定义 由图表交换)

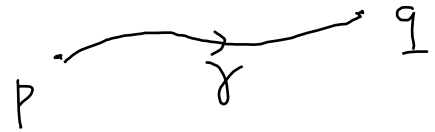
② on morphism: $\forall\, p, q \in X$, $\langle \gamma \rangle : p \to q$ in $Hom_{\Pi(X)}(p, q)$

要定义 $K(\langle \gamma \rangle) \in Hom_{\mathcal{G}}(K(p), K(q))$.

「策略：(1) $\forall\, \gamma \in P(p, q)$, 定义 $\hat{K}(\gamma) \in Hom_{\mathcal{G}}(K(p), K(q))$.

(2) $\hat{K}(\gamma) = \hat{K}(\gamma')$, 若 $\gamma \simeq \gamma'$ rel $\{0, 1\}$

(3) 定义 $K(\langle \gamma \rangle) = \hat{K}(\gamma)$ 」



$\gamma: [0,1] \longrightarrow X$, $\gamma(0) = p$, $\gamma(1) = q$.

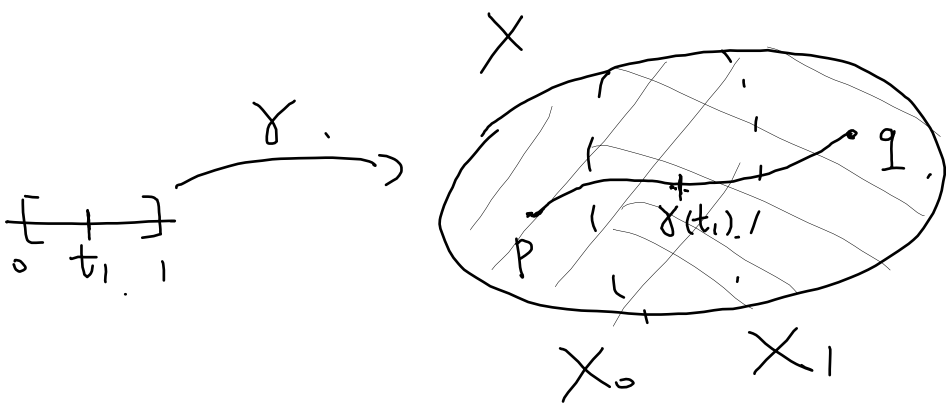
$\mathrm{int}(X_0) \cup \mathrm{int}(X_1) = X$.

$\gamma^{-1}(\mathrm{int}(X_0)) \cup \gamma^{-1}(\mathrm{int}(X_1)) = [0,1]$.

Lebesgue 引理. $\Longrightarrow$ 可以找到 $[0,1]$ 的一个分割：

$$0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1.$$

s.t. $\forall\, i = 1, \dots, m$, $\gamma([t_{i-1}, t_i])$ 要么 $\subset \mathrm{int}(X_0)$，要么 $\subset \mathrm{int}(X_1)$.



令 $\gamma_i : [0,1] \longrightarrow X$, $i = 1, \dots, m$.

$$t \longmapsto \gamma((1-t)t_{i-1} + t\, t_i)$$

$\gamma_i \in P_{X_0}$ or P_{X_1}.

$$\widetilde{K}(\gamma) := \widetilde{K}(\gamma_m) \circ \widetilde{K}(\gamma_{m-1}) \circ \cdots \circ \widetilde{K}(\gamma_1) \in \mathrm{Hom}_G(K(p), K(q))$$

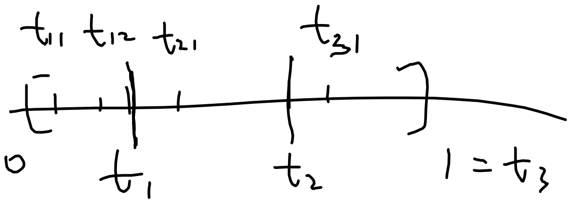
其中 $\widetilde{K}(\gamma_i) = \begin{cases} F(\langle \gamma_i \rangle), & \text{if } \gamma_i \in P_{X_0} \\ G(\langle \gamma_i \rangle), & \text{if } \gamma_i \in P_{X_1}. \end{cases}$

「需验证：$\hat{K}(\gamma)$ 的定义不依赖于分割的选取.

只要证：对于分割 $0=t_0<t_1<t_2<\cdots<t_{m-1}<t_m=1$

取任一加细.

$$0=t_0=t_{10}<t_{11}<\cdots<t_{1k_1}=t_1=t_{20}<t_{21}<\cdots<\cdots<\cdots=t_{m-1}=t_{m0}<t_{m1}<\cdots<t_{mk_m}=t_m=1.$$



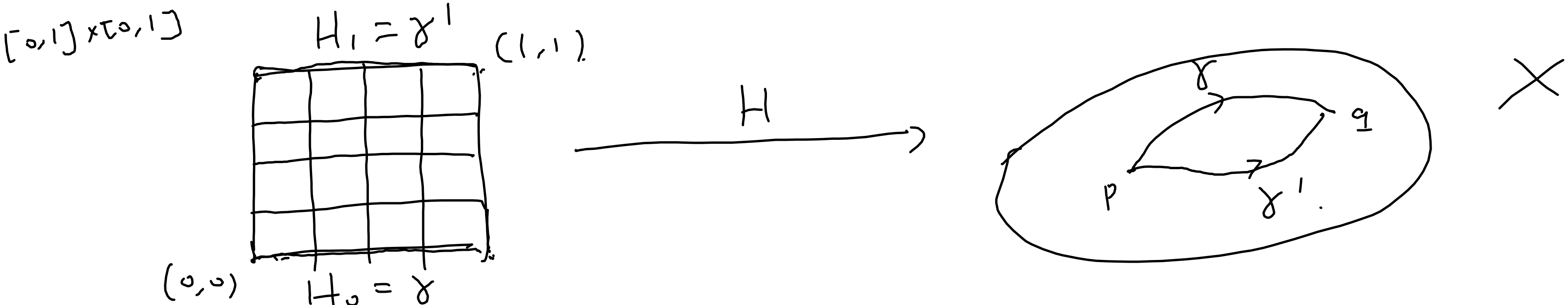
记 $\gamma_{ij}:[0,1]\longrightarrow X$

$$t\longmapsto \gamma((1-t)t_{ij-1}+t\,t_{ij}).$$

$$\underline{\hat{K}(\gamma_3)}\circ\hat{K}(\gamma_2)\circ\hat{K}(\gamma_1)=\underline{\hat{K}(\gamma_{32})\circ\hat{K}(\gamma_{31})}\circ\hat{K}(\gamma_{22})\circ\hat{K}(\gamma_{21})\circ\hat{K}(\gamma_{13})\circ\hat{K}(\gamma_{12})\circ\hat{K}(\gamma_{11})$$

由 F,G 为函子立得. 」

下面：要证：若 $\gamma \underset{H}{\simeq} \gamma'$ rel $\{0,1\}$，$\widehat{F}(\gamma)=\widehat{F}(\gamma')$



由 Lebesgue 引理，∃ $[0,1]\times[0,1]$ 的一个分割 $\mathcal{P}$，使分割所得的小正方形在 H 下的像或者落在 X_0 中 / X_1。

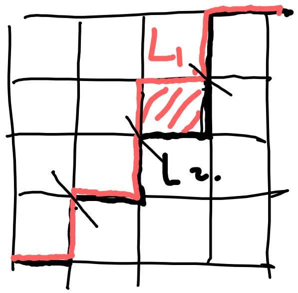
考虑：从 $(0,0)$ 出发到 $(1,1)$ 的折线，(要求不向左，不向下) (Z)

$\forall$ 折线 L 满足 (Z)，考虑 $\gamma_L : [0,1] \longrightarrow L$ (匀速率定义)

$t \longmapsto \gamma_L(t)$.

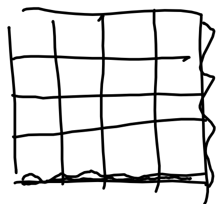
$\Gamma_L : H|_L \circ \gamma_L : [0,1] \longrightarrow X$，$\Gamma_L \in P(p,q)$

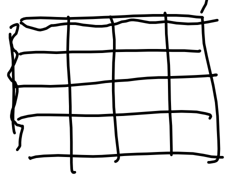
考虑满足(Z)的折线 L_1, L_2. L_1 与 L_2 仅相差一个小正方形.



$\Gamma_{L_1}, \Gamma_{L_2} \in P(p,q)$.

$\tilde{K}(\Gamma_{L_1}) = \tilde{K}(\Gamma_{L_2})$.

记 Γ_0 为  最下方的满足(Z)的折线

Γ_1 为  最上方的 - - - —— -

$$\Rightarrow \quad \tilde{K}(\Gamma_0) = \tilde{K}(\Gamma_1).$$

$\|$ ← 由 $\tilde{K}$ 的定义 → $\|$

$$\tilde{K}(\gamma) \overset{\text{显然}}{=} \tilde{K}(\gamma').$$

小结: $\gamma \simeq \gamma' \ \mathrm{rel}\ \{0,1\}$ $\quad \tilde{K}(\gamma) = \tilde{K}(\gamma')$

∴ 若定义 $K(\langle\gamma\rangle) = \widetilde{F}(\gamma)$，则 $K: \mathrm{Mor}(\Pi(X)) \to \mathrm{Mor}(\mathcal{G})$

良好定义.

小结2：目前定义：$K: \mathrm{Ob}(\Pi(X)) \longrightarrow \mathrm{Ob}(\mathcal{G})$.

$$K: \mathrm{Hom}_{\Pi(X)}(p,q) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(K(p),K(q))$$

Claim 1：K 定义了函子：$K: \Pi(X) \longrightarrow \mathcal{G}$.

<u>这是显然的</u>.

Claim 2：图表交换：

$$\begin{tikzcd}
\Pi(X_{01}) \arrow[r, "\Pi(j_0)"] \arrow[d, "\Pi(j_1)"'] & \Pi(X_0) \arrow[d, "\Pi(i_0)"] \arrow[ddr, "F"] & \\
\Pi(X_1) \arrow[r, "\Pi(i_1)"] \arrow[drr, "G"'] & \Pi(X) \arrow[dr, "K"] & \\
 & & \mathcal{G}
\end{tikzcd}$$

<u>显然</u>

Claim 3. 满足Claim 1与Claim 2的 K 是唯一的

#.