上次已证 (△)：X: top space，$X_0, X_1 \subset X$，$X_{01} = X_0 \cap X_1$，$\mathrm{int}(X_0) \cup \mathrm{int}(X_1) = X$. 则有 cocartesian 图表：

$$\begin{tikzcd} X_{01} \arrow[r, "j_0"] \arrow[d, "j_1"'] & X_0 \arrow[d, "i_0"] \\ X_1 \arrow[r, "i_1"'] & X \end{tikzcd} \qquad \mathrm{Top}.$$

$$\begin{tikzcd} \Pi(X_{01}) \arrow[r, "\Pi(j_0)"] \arrow[d, "\Pi(j_1)"'] & \Pi(X_0) \arrow[d, "\Pi(i_0)"] \\ \Pi(X_1) \arrow[r, "\Pi(i_1)"'] & \Pi(X) \end{tikzcd} \qquad \mathrm{Grpd}.$$

这次：(△)不变，进一步假设 X_0、X_1、X_{01} 道路连通，$p \in X_{01}$，则有 cocartesian 图表：

$$\begin{tikzcd} (X_{01}, p) \arrow[r, "j_0"] \arrow[d, "j_1"'] & (X_0, p) \arrow[d, "i_0"] \\ (X_1, p) \arrow[r, "i_1"'] & (X, p) \end{tikzcd} \qquad \mathrm{Top}^{\circ}$$

$$\begin{tikzcd} \pi_1(X_{01}, p) \arrow[r, "(j_0)_*"] \arrow[d, "(j_1)_*"'] & \pi_1(X_0, p) \arrow[d, "(i_0)_*"] \\ \pi_1(X_1, p) \arrow[r, "(i_1)_*"'] & \pi_1(X, p) \end{tikzcd} \qquad \mathrm{Grp}.$$

$$\Longrightarrow \boxed{\pi_1(X, p) \cong \pi_1(X_0, p) *_{\pi_1(X_{01}, p)} \pi_1(X_1, p)}$$

「将一个群视为一个 groupoid.

$\forall$ 群 G, 定义一个 groupoid (仍记为 G), 如下:

$$Ob(G) = \{pt\} \quad (\text{单点集}).$$

$$Mor(G) = \{g \in G\}.$$

$$g \circ h := h \cdot g \quad (\text{态射的复合}).$$

结合律 $\xrightarrow{\text{保证}}$ 复合的结合律 } $\xrightarrow{\text{保证}}$ G 为一个范畴.

有恒元 $\longrightarrow$ 有 1_{pt}

有逆元 $\longrightarrow$ 每个态射均为同构 $\xrightarrow{\text{保证}}$ G 为一个 groupoid 」

在此导同于图表:

$$\begin{tikzcd}
\pi_1(X_{01}, p) \arrow[r, "(j_0)_*"] \arrow[d, "(j_1)_*"'] & \pi_1(X_0, p) \arrow[d, "(i_0)_*"] \\
\pi_1(X_1, p) \arrow[r, "(i_1)_*"'] & \pi_1(X, p)
\end{tikzcd} \quad (☆')$$

可视为 Grpd 中的交换图表.

$\therefore$ 要证: $(☆')$ 是 cocartesian 的.

想法：

对于：
$$\begin{tikzcd}
\Pi(X_{01}) \arrow[r] \arrow[d] & \Pi(X_0) \arrow[d] \\
\Pi(X_1) \arrow[r] & \Pi(X)
\end{tikzcd}$$
，希望构造一个Grpd中的态射

$r: \Pi(Z) \to \pi_1(Z, p)$，其中 $Z = X_{01}, X_0, X_1, X$.

$\forall x \in Z$，选定一条道路 γ_x，使 γ_x 连接 p 与 x，满足：

① 若 $x \in X_{01}$，$\mathrm{Im}(\gamma_x) \subset X_{01}$.

② 若 $x \in X_0$，$\mathrm{Im}(\gamma_x) \subset X_0$.

③ 若 $x \in X_1$，$\mathrm{Im}(\gamma_x) \subset X_1$.

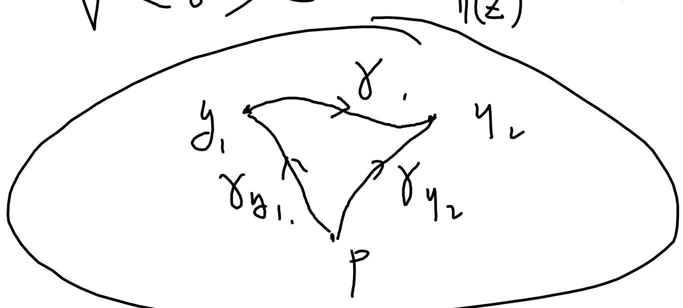
（总可以做到，$\because$ X_{01}, X_0, X_1 都道路连通）

④ $\gamma_p = C_p$（p处常道路）

r 之构造：

on objects: $\forall y \in \mathrm{Ob}(\Pi(Z))$，$r(y) = \{pt\}$.

on morphisms: $\forall \langle\gamma\rangle \in \mathrm{Hom}_{\Pi(Z)}(y_1, y_2)$，$r(\langle\gamma\rangle) := \langle\gamma_{y_1}\cdot\gamma\cdot\gamma_{y_2}^{-1}\rangle$.



Z

r 确为函子：

① $r(\langle C_y\rangle) = \langle\gamma_y\cdot C_y\cdot\gamma_y^{-1}\rangle$

② r 保态射之复合：

$$y_1 \xrightarrow{\gamma_1} y_2 \xrightarrow{\gamma_2} y_3$$

$$r(\langle\gamma_2\rangle\circ\langle\gamma_1\rangle) \overset{?}{=} r(\langle\gamma_2\rangle)\circ r(\langle\gamma_1\rangle)$$

$$r(\langle\gamma_2\rangle\circ\langle\gamma_1\rangle) = r(\langle\gamma_1\cdot\gamma_2\rangle) = \langle\gamma_{y_1}\cdot\gamma_1\cdot\gamma_2\cdot\gamma_{y_3}^{-1}\rangle.$$

$$r(\langle\gamma_2\rangle)\circ r(\langle\gamma_1\rangle) = \langle\gamma_{y_2}\cdot\gamma_2\cdot\gamma_{y_3}^{-1}\rangle\circ\langle\gamma_{y_1}\cdot\gamma_1\cdot\gamma_{y_2}^{-1}\rangle.$$

$$= \langle\gamma_{y_1}\cdot\gamma_1\cdot\gamma_{y_2}^{-1}\rangle\cdot\langle\gamma_{y_2}\cdot\gamma_2\cdot\gamma_{y_3}^{-1}\rangle$$

$$= \langle\gamma_{y_1}\cdot\gamma_1\cdot\gamma_{y_2}^{-1}\cdot\gamma_{y_2}\cdot\gamma_2\cdot\gamma_{y_3}^{-1}\rangle = \langle\gamma_{y_1}\cdot\gamma_1\cdot\gamma_2\cdot\gamma_{y_3}^{-1}\rangle.$$

则有交换图表：

$$\begin{tikzcd}
y_1 \arrow[rr, "\langle\gamma\rangle"] & & y_2 \\
& \bullet \arrow[ul, "\gamma_{y_1}"] \arrow[ur, "\gamma_{y_2}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\Pi(X_{01}) \arrow[r] \arrow[dr] \arrow[dd, "r"] & \Pi(X_0) \arrow[dr] \arrow[dd, "r"] & \\
& \Pi(X_1) \arrow[r] \arrow[dd, "r"] & \Pi(X) \arrow[dd, "r"] \\
\pi_1(X_{01},p) \arrow[r] \arrow[dr] & \pi_1(X_0,p) \arrow[dr] & \\
& \pi_1(X_1,p) \arrow[r] & \pi_1(X,p)
\end{tikzcd}$$

$\boxed{\mathrm{Grpd}}$

反过来，$\forall\, Z = X_{01}, X_0, X_1, X$，又可定义函子：

$$\iota:\ \pi_1(Z, p) \longrightarrow \Pi(Z).$$

显然为函子.

on objects: $\{pt\} \longmapsto p$

on morphisms: $\langle \gamma \rangle \longmapsto \langle \gamma \rangle$

ι 又使下面图表交换：

$$\begin{tikzcd}
 & & \Pi(X_0) \arrow[dr] & \\
\Pi(X_{01}) \arrow[urr] \arrow[dr] & & & \Pi(X) \\
 & \Pi(X_1) \arrow[urr] & \pi_1(X_0, p) \arrow[uu, "\iota"] \arrow[dr] & \\
\pi_1(X_{01}, p) \arrow[uu, "\iota"] \arrow[urr] \arrow[dr] & & & \pi_1(X, p) \arrow[uu, "\iota"'] \\
 & \pi_1(X_1, p) \arrow[uu, "\iota"] \arrow[urr] & &
\end{tikzcd}$$

小结，有交换图表：

（即：将 r 抹去后交换，将 l 抹去后也交换）

$$
\begin{tikzcd}
\Pi(X_{01}) \arrow[rr] \arrow[dr] \arrow[dd, shift left, "r"] & & \Pi(X_0) \arrow[dr] \arrow[dd, shift left, "r"] & \\
& \Pi(X_1) \arrow[rr] \arrow[dd, shift left, "r"] & & \Pi(X) \arrow[dd, shift left, "r"] \\
\pi_1(X_{01},p) \arrow[rr] \arrow[dr] \arrow[uu, shift left, "l"] & & \pi_1(X_0,p) \arrow[dr] \arrow[uu, shift left, "l"] & \\
& \pi_1(X_1,p) \arrow[rr] \arrow[uu, shift left, "l"] & & \pi_1(X,p) \arrow[uu, shift left, "l"]
\end{tikzcd}
$$

且满足：$r\circ l : \pi_1(X_{01},p) \longrightarrow \pi_1(X_{01},p)$

为 $1_{\pi_1(X_{01},p)}$

则顶点图表 cocartesian $\xRightarrow{\text{general nonsense}}$ 底下图表 cocartesian.

「general nonsense：

Lemma：设 $\mathcal{C}$ 为一个范畴，设 $\mathcal{C}$ 中有交换图表：

$$\begin{tikzcd}
W \arrow[r, "p"] \arrow[dr, "g"'] \arrow[dd, shift right, "r"'] & X \arrow[dr, "f"] \arrow[dd, shift right, "r"'] & \\
& Y \arrow[r, "g"] \arrow[dd, shift right, "r"'] & Z \arrow[dd, shift right, "r"'] \\
W_1 \arrow[r, "p_1"] \arrow[dr, "g_1"'] \arrow[uu, shift right, "l"'] & X_1 \arrow[dr, "f_1"] \arrow[uu, shift right, "l"'] & \\
& Y_1 \arrow[r, "g_1"] \arrow[uu, shift right, "l"'] & Z_1 \arrow[uu, shift right, "l"']
\end{tikzcd}$$

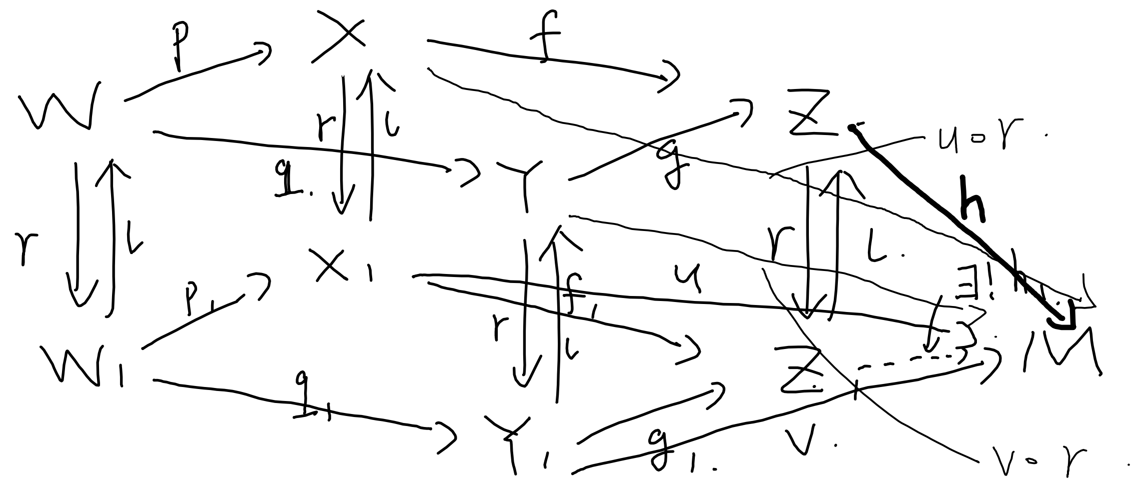
（将 r 隐去后交换，将 l 隐去后交换）.

使 $r\circ l$ 为恒等态射，则顶上图表 cocartesian

$\Longrightarrow$ 底下图表 cocartesian.

Pf. $\forall\ X_1 \xrightarrow{u} M,\ Y_1 \xrightarrow{v} M$ in $\mathcal{C}$.

要证：$\exists!\ h_1 : Z_1 \to M$ 使图表交换.



$u\circ r: X \longrightarrow M$, $v\circ r: Y \longrightarrow M$ 使图表交换.

$\Longrightarrow \exists!\, h: Z \longrightarrow M$ 使图表交换.

(i.e. $h\circ f = u\circ r$, $h\circ g = v\circ r$).

令 $h_1 = h\circ l$ 即可.

$h_1\circ f_1 = h\circ l\circ f_1 = h\circ f\circ l = u\circ \underline{r\circ l} = u$.

类似 $h_1\circ g_1 = v$.

设 $h_2: Z_1 \longrightarrow M$ 也使图表交换, $h_2\circ r \overset{\text{由顶之cocartesian}}{=\!=} h$

$\Longrightarrow h_2\circ r\circ l = h\circ l \Longrightarrow h_2 = h\circ l = h_1$

#

例：$\pi_1(\mathbb{R}^n, p) = \{e\}$.

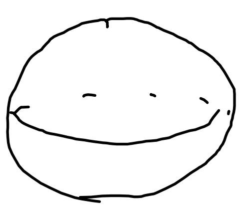
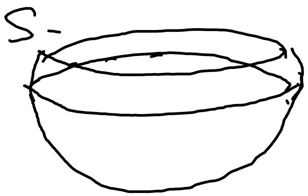
例：$\pi_1(S^n, p) = \{e\}$.　　，$n \geq 2$

定义：设 X 为 top sp. 称 X 是单连通的，若 X 是道路连通的，且 $\pi_1(X, p) = \{e\}$.

命题：设 X 为 top sp. 设 $X_0, X_1 \underset{\text{open}}{\subset} X$. $X_0 \cup X_1 = X$.
X_0、X_1 单连通，且 $X_0 \cap X_1$ 道路连通，则 X 是单连通的.

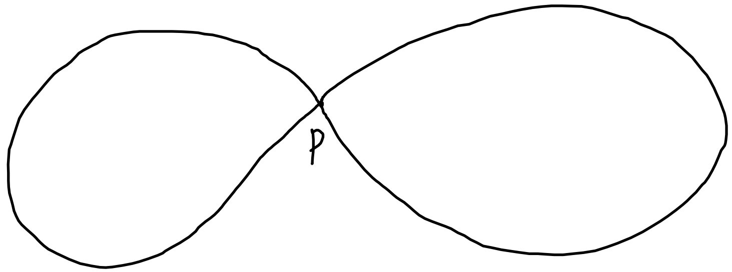
Pf. $\forall p \in X_0 \cap X_1$. Seifert & van Kampen 定理

$$\Rightarrow \pi_1(X, p) = \pi_1(X_0, p) *_{\pi_1(X_0 \cap X_1, p)} \pi_1(X_1, p) = \{e\}$$ #.

$n=2$: S^2.  S_+  S_- 

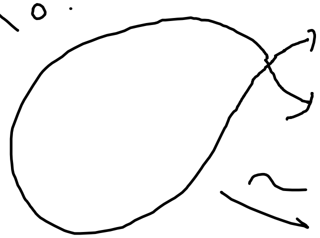
$S^2 = S_+ \cup S_-$.

$S_+ \cap S_-$: 

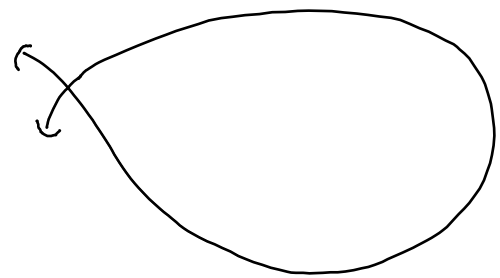


X

X_0.

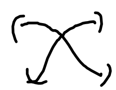


$\simeq S^1$



X_1.

$\simeq S^1$



$X_{01} \simeq \cdot$

$$\pi_1(X,p) = \pi_1(X_0,p) *_{\pi_1(X_{01},p)} \pi_1(X_1,p) \simeq \mathbb{Z} * \mathbb{Z}.$$

① 计算S^1的基本群. $\pi_1(S^1,p) \simeq \mathbb{Z}$.

② 基本群是同伦不变量

§5. S^1的基本群

$S^1=\{z\in\mathbb{C} \mid |z|=1\}\subset\mathbb{C}\cong\mathbb{R}^2$.

$\pi: \mathbb{R}\longrightarrow S^1$, $\quad t\longmapsto e^{2\pi i t}$.

$\forall\, n\in\mathbb{R}$, 选一道路 $\gamma_n:[0,1]\longrightarrow\mathbb{R}$,

$$\gamma_n(0)=0$$
$$\gamma_n(1)=n$$

$\pi\circ\gamma_n$ 则为 S^1 中的以 $z=1$ 为基点的一条圈道路.

定义 $\phi: \mathbb{Z}\longrightarrow\pi_1(S^1, 1)$.

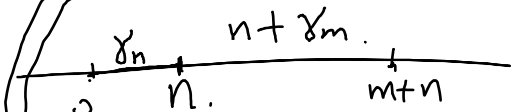
$$n\longmapsto\langle\pi\circ\gamma_n\rangle.$$

Well-defined.

要证:
$$\begin{cases} ① & \phi \text{ 为群同态} \\ ② & \phi \text{ 为满射} \\ ③ & \phi \text{ 为单射}. \end{cases}$$

①: $\phi(m+n)=\phi(m)\cdot\phi(n)$.

$\phi(m+n)=\langle\pi\circ\gamma_{m+n}\rangle$, $\quad \phi(m)\cdot\phi(n)=\langle\pi\circ\gamma_m\rangle\cdot\langle\pi\circ\gamma_n\rangle$.



$\langle\pi\circ\gamma_{m+n}\rangle = \langle\pi\circ(\gamma_n\cdot(n+\gamma_m))\rangle$

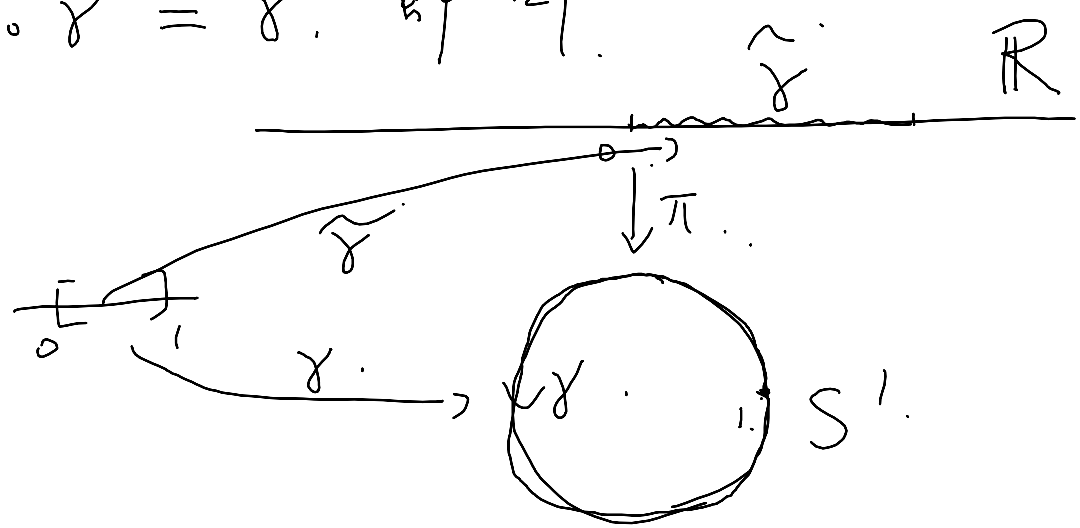
$=\langle(\pi\circ\gamma_n)\cdot\underbrace{\pi\circ(n+\gamma_m)}_{=\pi\circ\gamma_m}\rangle$

② ϕ 为满射.

只要证: $\forall\ \gamma \in L(S^1, 1)$, $\exists\ \tilde{\gamma}: [0,1] \to \mathbb{R}$ 连续,

s.t. $\tilde{\gamma}(0) = 0$, $\pi \circ \tilde{\gamma} = \gamma$. 即可.

$\pi(t) = e^{2\pi i t}$.



「 $\pi \circ \tilde{\gamma}(1) = \gamma(1) = 1$.

$\Rightarrow \tilde{\gamma}(1)$ 为整数. 不妨设 $\tilde{\gamma}(1) = n$.

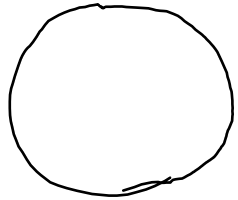
则 $\phi(n) = \langle \pi \circ \gamma_n \rangle = \langle \pi \circ \tilde{\gamma} \rangle = \langle \gamma \rangle$ 」

π 的性质:

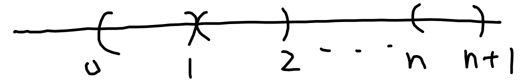
$S^1 = U_0 \cup U_1$, $U_0 = S^1 \setminus \{1\}$, $U_1 = S^1 \setminus \{-1\}$.

$\pi^{-1}(U_0) = \coprod_{n \in \mathbb{Z}} (n, n+1)$. $\pi^{-1}(U_1) = \coprod_{n \in \mathbb{Z}} (n - \frac{1}{2}, n + \frac{1}{2})$.

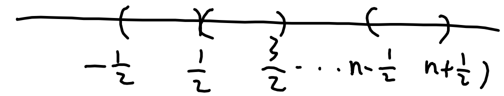
S^1:



$\coprod_n (n, n+1)$

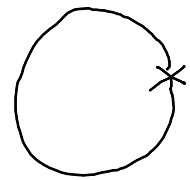


$\coprod_n (n-\frac{1}{2}, n+\frac{1}{2})$

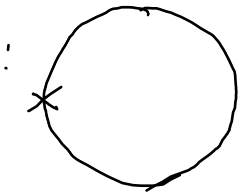


$\downarrow \pi$ $\qquad$ $\downarrow \pi$

$U_0 = S^1 \setminus \{1\}$:

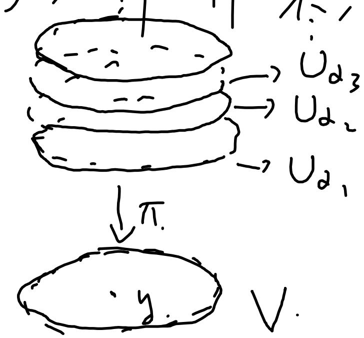


$U_1 = S^1 \setminus \{-1\}$:



$\pi|_{(n,n+1)} : (n, n+1) \xrightarrow{\cong} U_0$ $\qquad$ $\pi|_{(n-\frac{1}{2}, n+\frac{1}{2})} : (n-\frac{1}{2}, n+\frac{1}{2}) \xrightarrow{\cong} U_1$

定义：设 X, Y 为 top spaces，$\pi: X \to Y$ 连续满射，称 π 为一个覆盖映射 (covering map)，if $\forall y \in Y$，$\exists y$ 的开邻域 V，s.t. $\pi^{-1}(V) = \coprod_\alpha U_\alpha$，其中 U_α 均为 X 中开集，且 $\pi|_{U_\alpha} : U_\alpha \to V$ 为一个同胚，$\forall \alpha$.

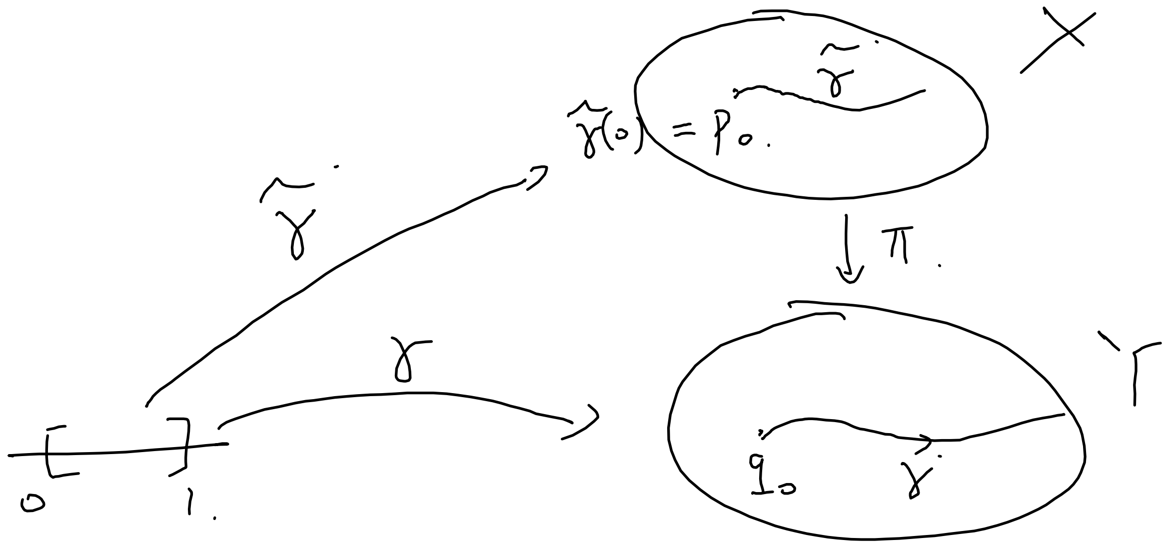


注、要记之事其实是覆盖映射的一般性质.

术语：设 $\pi: X \to Y$ 连续映射，$Z \xrightarrow{f} Y$ 连续，f 的一个提升(lifting)是指一个连续映射 $g: Z \to X$，s.t. $\pi \circ g = f$. i.e. 下面的图表交换：

$$\begin{tikzcd}
 & X \arrow[d, "\pi"] \\
Z \arrow[ur, "g"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

引理：设 $\pi: X \to Y$ 为一个覆盖映射，$q_0 \in Y$，$p_0 \in X$，$\pi(p_0) = q_0$. 则 $\forall$ 道路 $\gamma: [0,1] \to Y$，s.t. $\gamma(0) = q_0$，$\exists!$ 道路 $\tilde{\gamma}: [0,1] \to X$. s.t. $\tilde{\gamma}(0) = p_0$. s.t. $\tilde{\gamma}$ 为 γ 的一个提升.



证. $\forall y \in Y$, $\exists$ y 的开邻域 V_y, s.t. $\pi^{-1}(V_y)=\coprod_\alpha U_{y\alpha}$, 其中

$\forall \alpha$, $\pi|_{U_{y\alpha}}: U_{y\alpha} \longrightarrow V_y$ 为同胚.

$\{V_y \mid y\in Y\}$ 构成 Y 的一个开覆盖.

则 $\{\gamma^{-1}(V_y) \mid y \in Y\}$ 构成了 $[0,1]$ 的一个开覆盖

$[0,1]$ 紧的 metric sp.

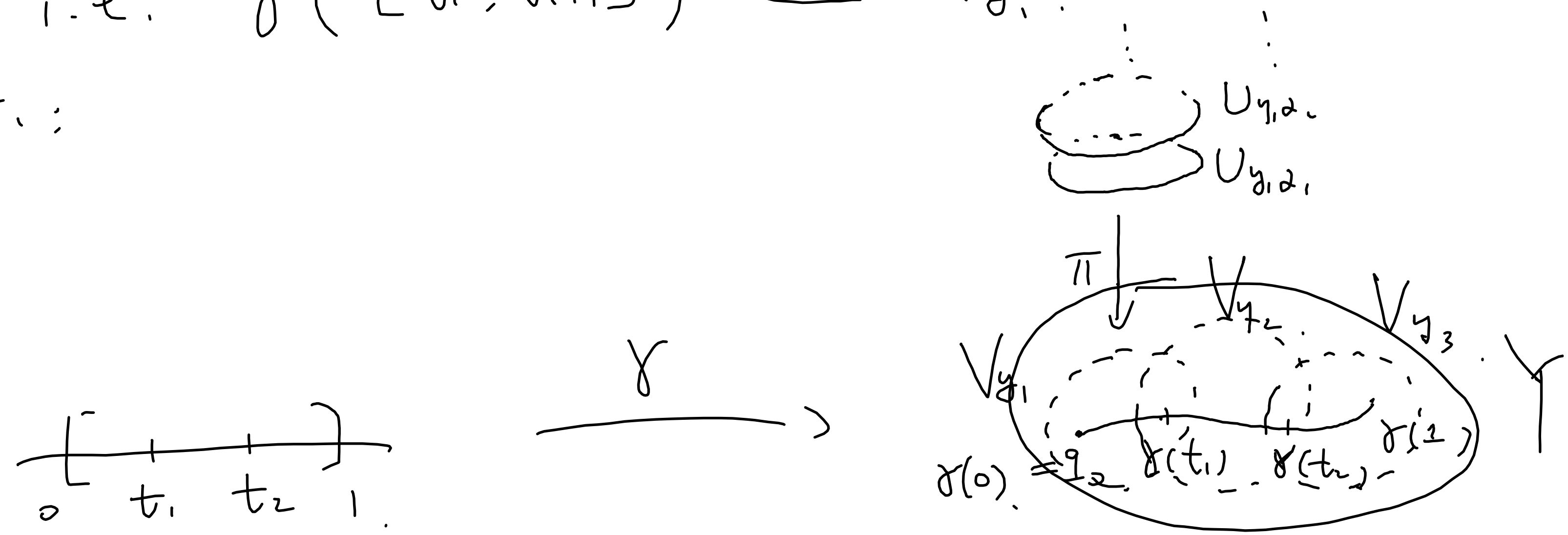
Lebesgue 引理 $\Longrightarrow$ $\exists$ $[0,1]$ 的一个分割 s.t. $\forall i$

$0=t_0<t_1<t_2<\cdots<t_k$.

$[t_i,t_{i+1}] \subset$ 某个 $\gamma^{-1}(V_y)$ 中, 不妨设 $[t_i,t_{i+1}]\subset\gamma^{-1}(V_{y_i})$.

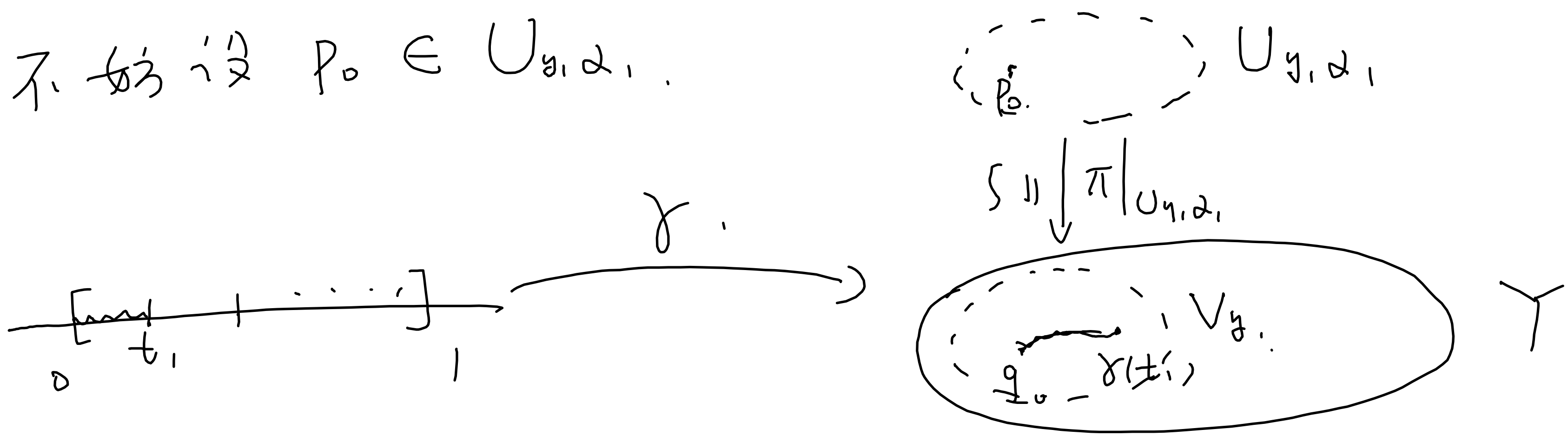
i.e. $\gamma([t_i, t_{i+1}]) \subset V_{y_i}$.

图示：



$\because \pi(p_0) = q_0$, $q_0 \in V_{y_1}$. $p_0 \in \pi^{-1}(V_{y_1})$.

不妨设 $p_0 \in U_{y,\alpha_1}$.



令 $\tilde{\gamma}_1(t) = (\pi|_{U_{y,\alpha_1}})^{-1} \circ \gamma(t)$, $0 \le t \le t_1$.

则 $\tilde{\gamma}_1 : [0, t_1] \longrightarrow X$ 为 $\gamma|_{[0,t_1]}$ 的从 p_0 出发的提升.

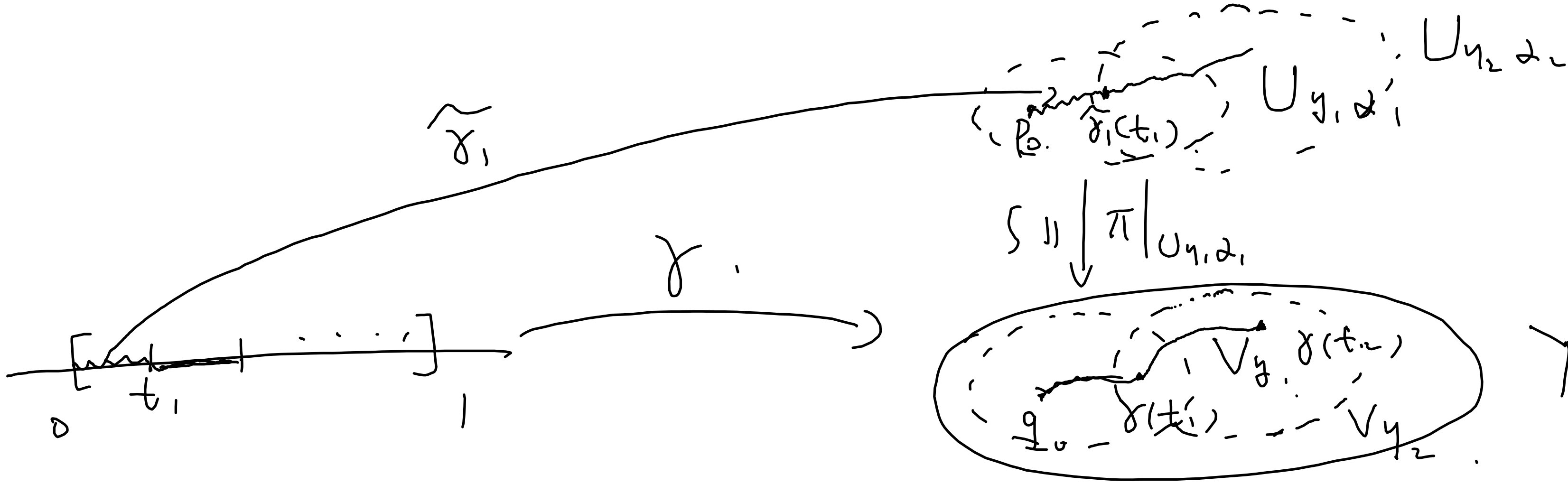
这样的 $\tilde{\gamma}_1$ 是唯一的.

「设 $\tilde{\gamma}_1 : \underline{[0, t_1]} \longrightarrow X$ 也为 $\gamma|_{[0,t_1]}$ 的从 p_0 出发的提升.

$$\left.\begin{array}{l} \tilde{\tilde{\gamma}}_1(0) = p_0 . \\ \tilde{\gamma}_1 \text{ 连续}. \\ p_0 \in U_{y,\alpha_1} \end{array}\right\} \Longrightarrow \tilde{\tilde{\gamma}}_1([0, t_1]) \subset U_{y,\alpha_1}$$

$$\Longrightarrow \quad \begin{tikzcd} {[0, t_1]} \arrow[r, "\tilde{\tilde{\gamma}}_1"] \arrow[dr, "\gamma|_{[0,t_1]}"'] & U_{y,\alpha_1} \arrow[d, "\pi|_{U_{y,\alpha_1}}", "\cong"'] \\ & V_y \end{tikzcd} \Longrightarrow \tilde{\tilde{\gamma}}_1 = (\pi|_{U_{y,\alpha_1}})^{-1} \circ \gamma|_{[0,t_1]} = \tilde{\gamma}_1 .$$

」



#

推论：设 $\pi: X \to Y$ 为一个 covering map，Z 道路连通的，$f: Z \to Y$ 连续，设 $z_0 \in Z$，$y_0 \in Y$，$y_0 = f(z_0)$，$x_0 \in X$，$\pi(x_0) = y_0$。则若 $\tilde{f}_1$，$\tilde{f}_2$ 都为 f 的提升，且 $\tilde{f}_1(z_0) = x_0 = \tilde{f}_2(z_0)$，则 $\tilde{f}_1 = \tilde{f}_2$.

Pf. $\forall z \in Z$, $\because \exists \gamma \in P(z_0, z)$

$\gamma(0) = z_0$, $\gamma(1) = z$.

$$\begin{tikzcd}
 & & X \arrow[d, "\pi_1"] \\
{[0,1]} \arrow[r, "\gamma"] & Z \arrow[ur, bend left, "\tilde{f}_1"] \arrow[ur, "\tilde{f}_2"'] \arrow[r, "f"] & Y
\end{tikzcd}$$

x_0 $\quad \tilde{f}_1 \circ \gamma$ $\quad \tilde{f}_2 \circ \gamma$

$z_0 = \gamma(0)$ $\quad z = \gamma(1)$

y_0 $\quad f \circ \gamma$ $\quad f \circ \gamma(1)$

$0 \quad 1$

$$\Rightarrow \tilde{f}_1 \circ \gamma = \tilde{f}_2 \circ \gamma.$$

$$\Rightarrow \tilde{f}_1(\gamma(1)) = \tilde{f}_2(\gamma(1)) \Rightarrow \tilde{f}_1(z) = \tilde{f}_2(z).$$

$$\Rightarrow \tilde{f}_1 = \tilde{f}_2$$

#.

③ $\phi: \mathbb{Z} \to \pi_1(S^1, 1)$ 为单射. $\mathbb{R} \xrightarrow{\pi} S^1$

$n \longmapsto \langle \pi \circ \gamma_n \rangle$ (π 为 covering map).

即证：若 $\phi(m) = \phi(n)$，则 $m = n$.

即证：若 $\pi \circ \gamma_m \simeq \pi \circ \gamma_n \text{ rel } \{0,1\}$，则 $m = n$

只需证：若 $\gamma_1, \gamma_2 \in L(S^1, 1)$，$\gamma_1 \simeq \gamma_2 \text{ rel } \{0,1\}$，则

它们的提升 $\tilde{\gamma}_1, \tilde{\gamma}_2$ ($\tilde{\gamma}_1(0) = 0$, $\tilde{\gamma}_2(0)$) 的终点相

同

这其实也是 covering map 的一般性质：

Lemma (同伦提升引理). 设 $\pi: X \to Y$ 为 covering map,

设 $\pi(p_0) = q_0$, 设 γ_1, γ_2 为 Y 中 以 q_0 为起点的两条道路, 记 $\tilde{\gamma}_1, \tilde{\gamma}_2$ 为它们的以 p_0 为起点的提升. 则:

若: $\gamma_1 \underset{F}{\simeq} \gamma_2 \quad \text{rel } \{0,1\}$

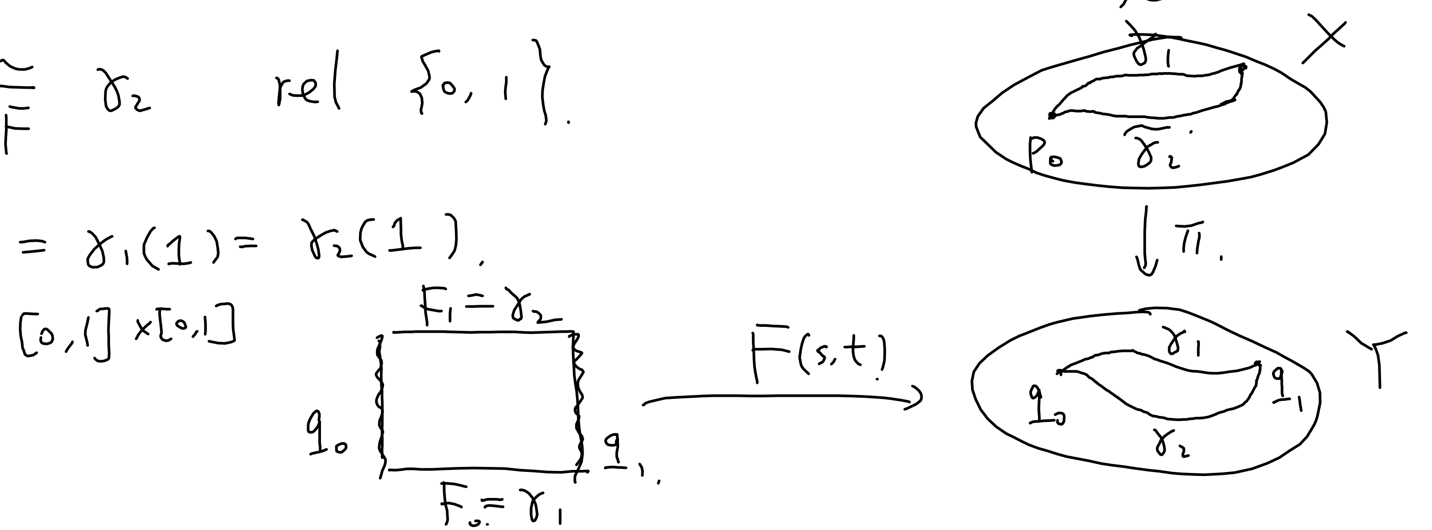
则: $\tilde{\gamma}_1 \underset{\tilde{F}}{\simeq} \tilde{\gamma}_2 \quad \text{rel } \{0,1\}$,

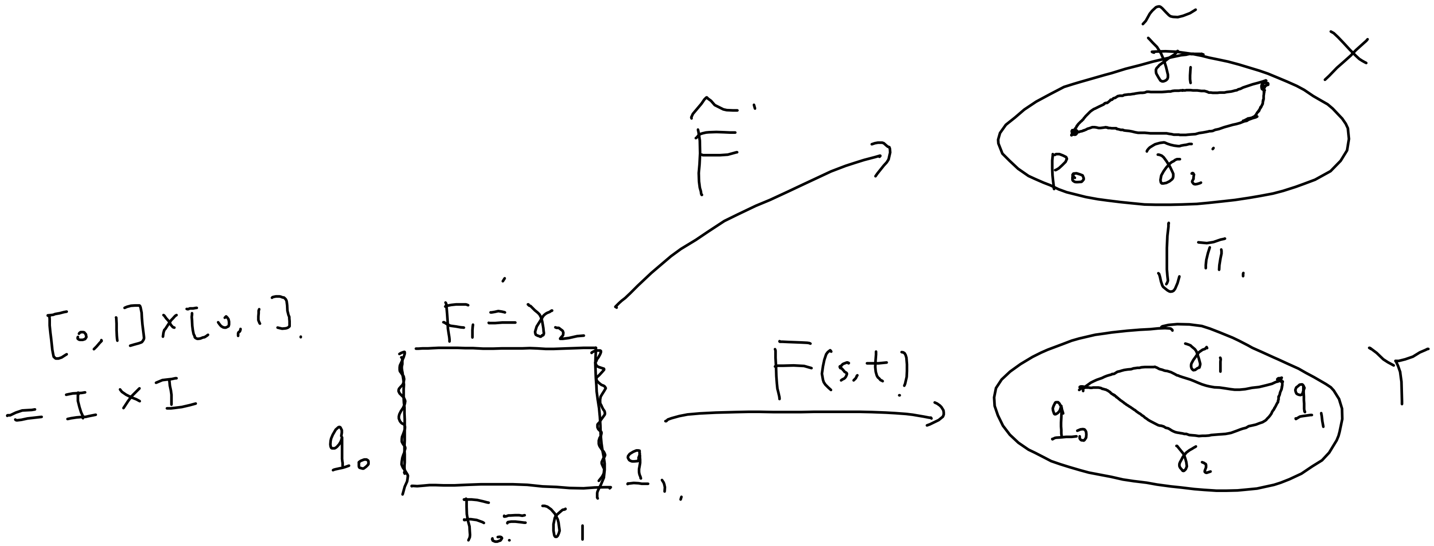
其中 $\pi \circ \tilde{F} = F$.

(此证明出自: [Forster] Lectures on Riemann Surfaces)

Pf. $\gamma_1 \underset{F}{\simeq} \gamma_2 \quad \text{rel } \{0,1\}$.

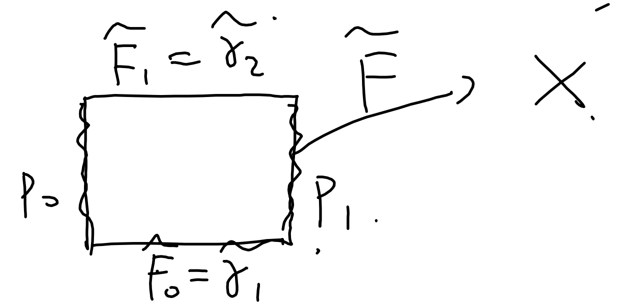
设 $q_1 = \gamma_1(1) = \gamma_2(1)$.





Claim: 只要证明: $\exists\ \tilde{F}: I\times I \longrightarrow X$ 连续. s.t. $\pi\circ\tilde{F} = F$

且 $\tilde{F}(\{0\}\times I) = p_0$. 即可.



⌈ $\pi\circ\tilde{F}_0 = F_0 = \gamma_1$. $\pi\circ\tilde{F}_1 = F_1 = \gamma_2$.

$\Rightarrow$ $\tilde{F}_0$ 为 γ_1 的 以 p_0 为起点之提升.

$\tilde{F}_1$ .. γ_2 . . - - - - .

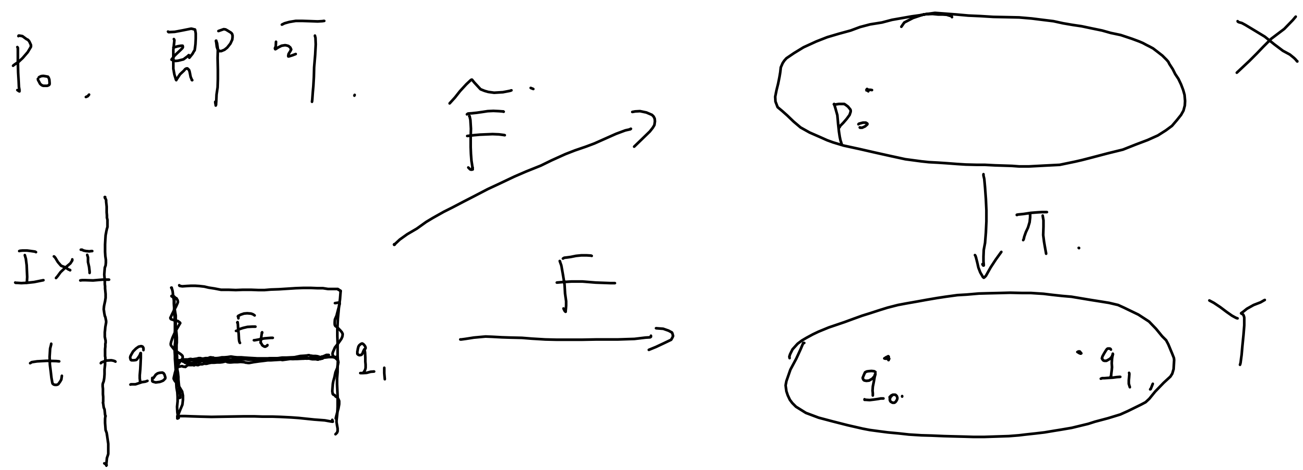
$\pi(\tilde{F}(\{1\}\times I)) = q_1 \Rightarrow \tilde{F}(\{1\}\times I) \subset \pi^{-1}(q_1)$.

$\Rightarrow \exists\ p_1 \in \pi^{-1}(q_1)$. s.t. $\tilde{F}(\{1\}\times I) = p_1$. ⌋

下证: $\exists\ \tilde{F}: I\times I \longrightarrow X$ 连续, s.t. $\pi\circ\tilde{F} = F$

且 $\tilde{F}(\{0\}\times I) = p_0$ 即可.

$\tilde{F}$ 之构造:



定义: $\tilde{F}: I\times I \longrightarrow X$

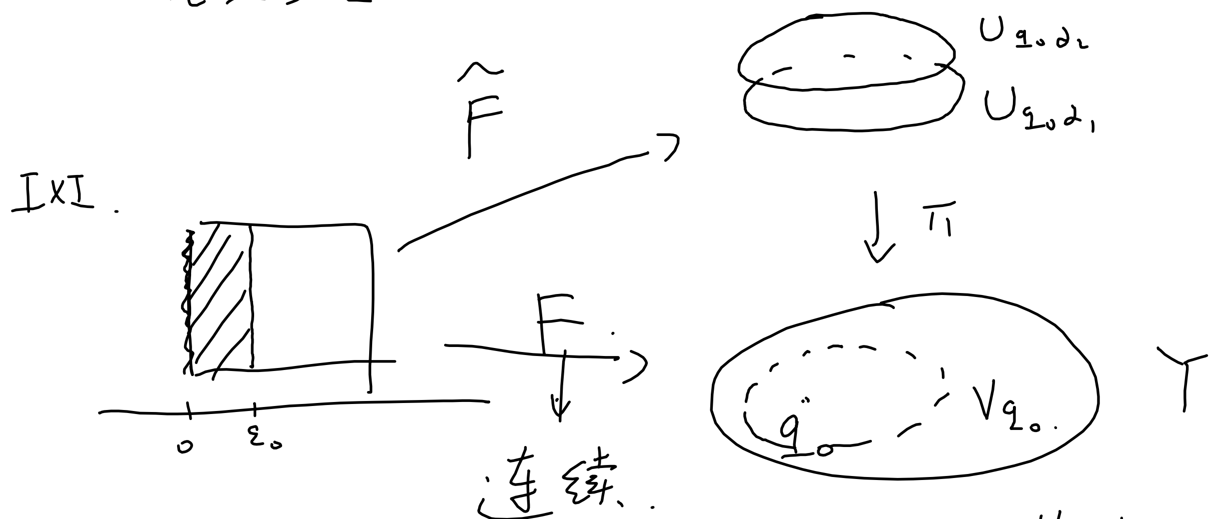
$(s,t) \longmapsto \tilde{F}_t(s)$.

(其中 $\tilde{F}_t: [0,1]\longrightarrow X$ 为 $F_t: [0,1]\longrightarrow Y$, $s\mapsto F(s,t)$ 为以 p_0 为起点的提升)

显然: $\pi\circ\tilde{F} = F$.

只要证: 如此定义的 $\tilde{F}: I\times I \longrightarrow X$ 是连续的.

Step 1. $\exists\, \varepsilon_0 > 0$, s.t. $\tilde{F}|_{[0,\varepsilon_0]\times I}$ 连续的.



$\exists\, q_0$ 的邻域 V_{q_0}, s.t. $\pi^{-1}(V_{q_0}) = \coprod_\alpha U_{q_0\alpha}$, 其中

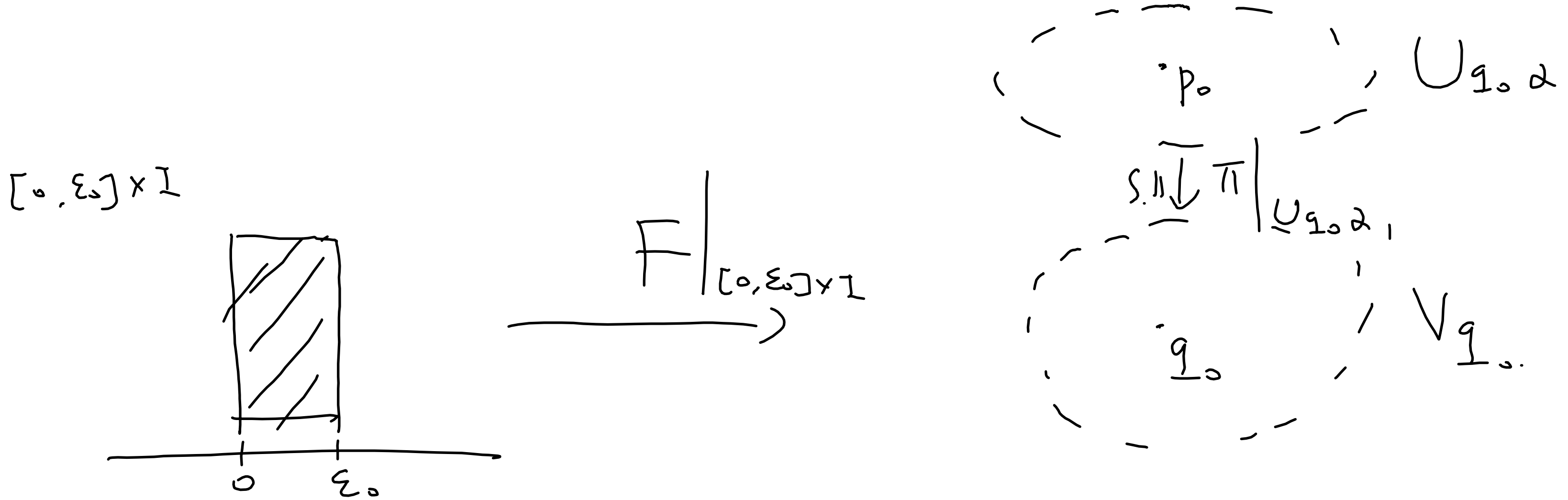
$\pi|_{U_{q_0\alpha}} : U_{q_0\alpha} \xrightarrow{\cong} V_{q_0}$

$F^{-1}(V_{q_0})$ 为包含 $\{0\}\times I$ 的一个开集.

$\Rightarrow \exists\, \varepsilon_0 > 0$, s.t. $[0,\varepsilon_0]\times I \subset F^{-1}(V_{q_0})$.

$\Rightarrow F([0,\varepsilon_0]\times I) \subset V_{q_0}$.

$q_0 \in V_{q_0}$. $\pi(p_0) = q_0$. 不妨设 $p_0 \in U_{q_0\alpha_1}$.



考虑 $\left(\pi|_{U_{q_0\alpha_1}}\right)^{-1}\circ F|_{[0,\varepsilon_0]\times I}$ 为 $F|_{[0,\varepsilon_0]\times I}$ 的一个提升。

Claim: $\left(\pi|_{U_{q_0\alpha_1}}\right)^{-1}\circ F|_{[0,\varepsilon_0]\times I} = \tilde{F}|_{[0,\varepsilon_0]\times I}$.

「$\forall t\in I$, $s \longmapsto \left(\pi|_{U_{q_0\alpha_1}}\right)^{-1}\circ F|_{[0,\varepsilon_0]\times I}(s,t)$ 为 F_t 的一个提升（起点为 p_0）// ← 道路提升引理

$\forall t\in I$, $s \longmapsto \tilde{F}|_{[0,\varepsilon_0]\times I}(s,t)$ 也为 F_t 的一个提升（起点为 p_0）」

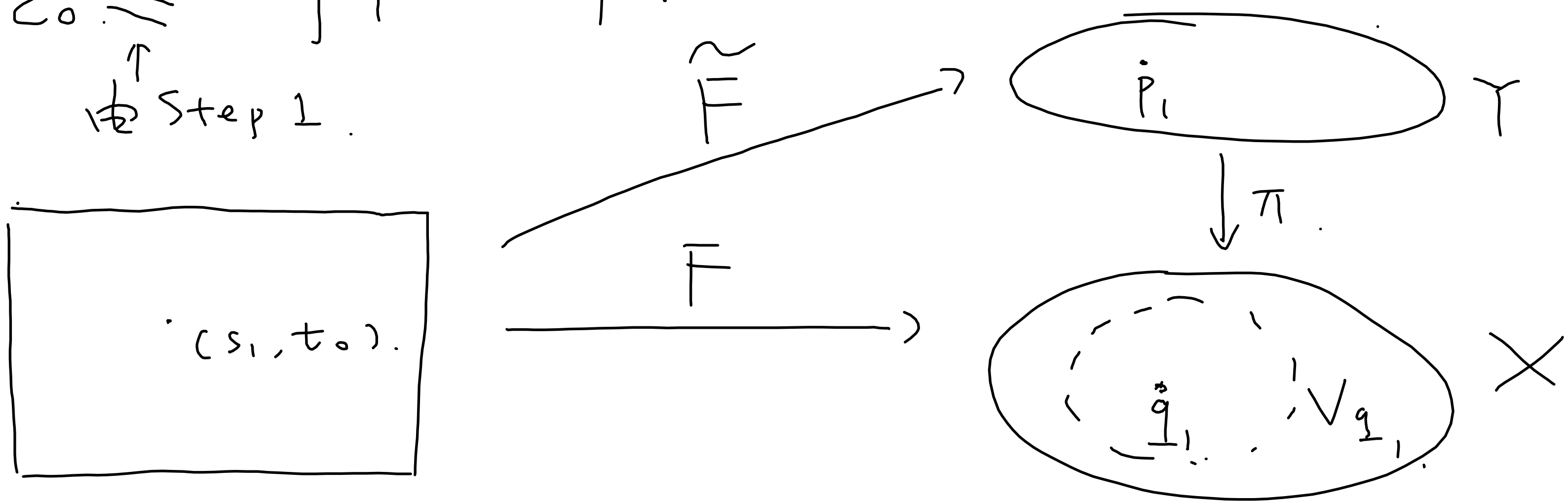
$\Rightarrow \tilde{F}|_{[0,\varepsilon_0]\times I}$ 连续。

Step 2. $\tilde{F}: I \times I \longrightarrow Y$ 连续.

反证: 假设 $\exists\, (s_0, t_0) \in I \times I$, s.t. $\tilde{F}$ 在 (s_0, t_0) 处不连续.

$$0 < \varepsilon_0 \le \inf\{\, s \mid \tilde{F} \text{ 在 } (s, t_0) \text{ 处不连续} \,\} \overset{\text{记为}}{=} s_1 .$$

↑ 由 Step 1.

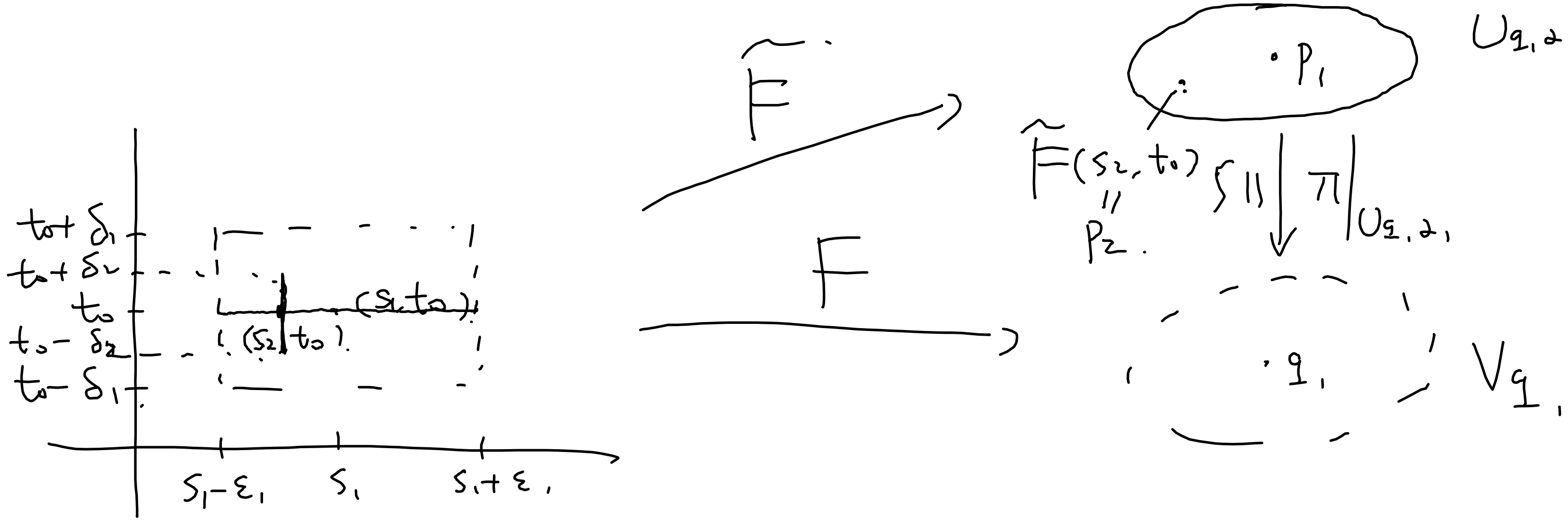


令 $q_1 = F(s_1, t_0)$, $p_1 = \tilde{F}(s_1, t_0)$.

$\exists\, V_{q_1} \underset{\text{open}}{\subset} X$, $q_1 \in V_{q_1}$, s.t. $\pi^{-1}(V_{q_1}) = \coprod_{\alpha} U_{q_1,\alpha}$, 其中

$$\pi\big|_{U_{q_1,\alpha}} : U_{q_1,\alpha} \xrightarrow{\cong} V_{q_1}$$

F 连续, $\Rightarrow \exists\, \varepsilon_1 > 0,\ \delta_1 > 0$, s.t. $F\big((s_1-\varepsilon_1, s_1+\varepsilon_1) \times (t_0-\delta_1, t_0+\delta_1)\big) \subset V_{q_1}$



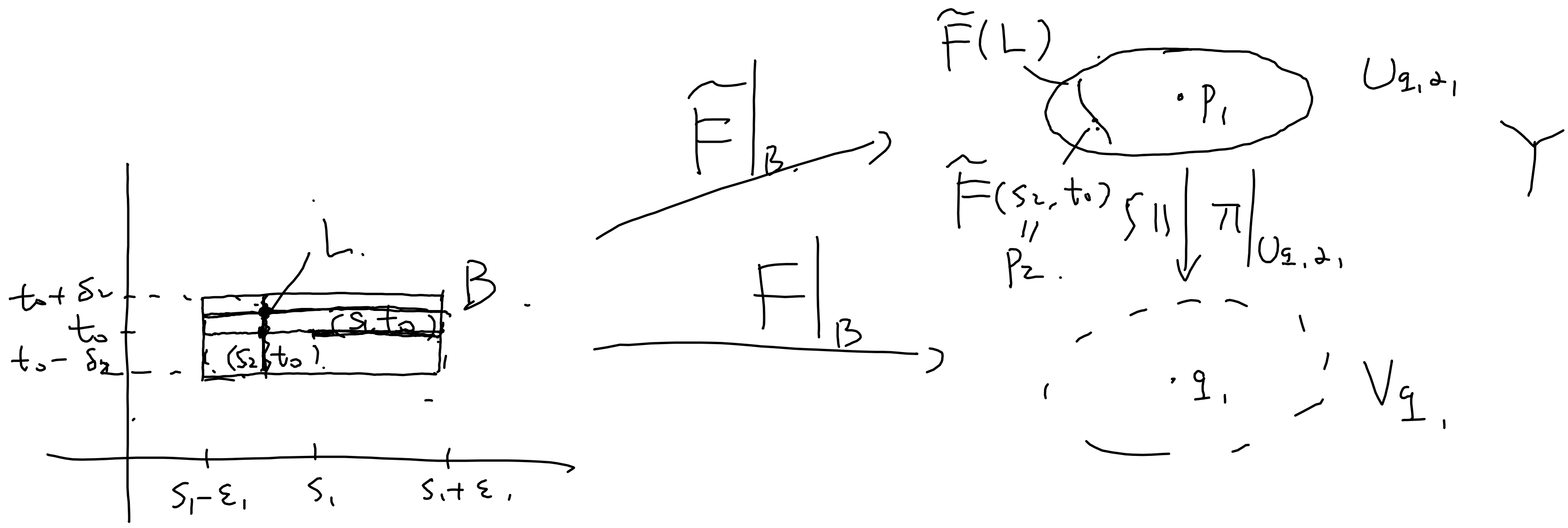
不妨设 $p_1 \in U_{q_1,\alpha_1}$.

$\tilde F_{t_0}$ 为 F_{t_0} 的提升，$\Rightarrow \tilde F_{t_0}$ 连续. $\Rightarrow \exists\, s_1-\varepsilon_1 < s_2 < s_1$, s.t.

$\tilde F(s_2,t_0) \in U_{q_1,\alpha_1}$

$\Rightarrow \tilde F$ 在 (s_2,t_0) 处连续.

$\Rightarrow \exists\, \delta_2>0$, s.t. $\tilde F(\{s_2\}\times(t_0-\delta_2,\ t_0+\delta_2)) \subset U_{q_1,\alpha_1}$.



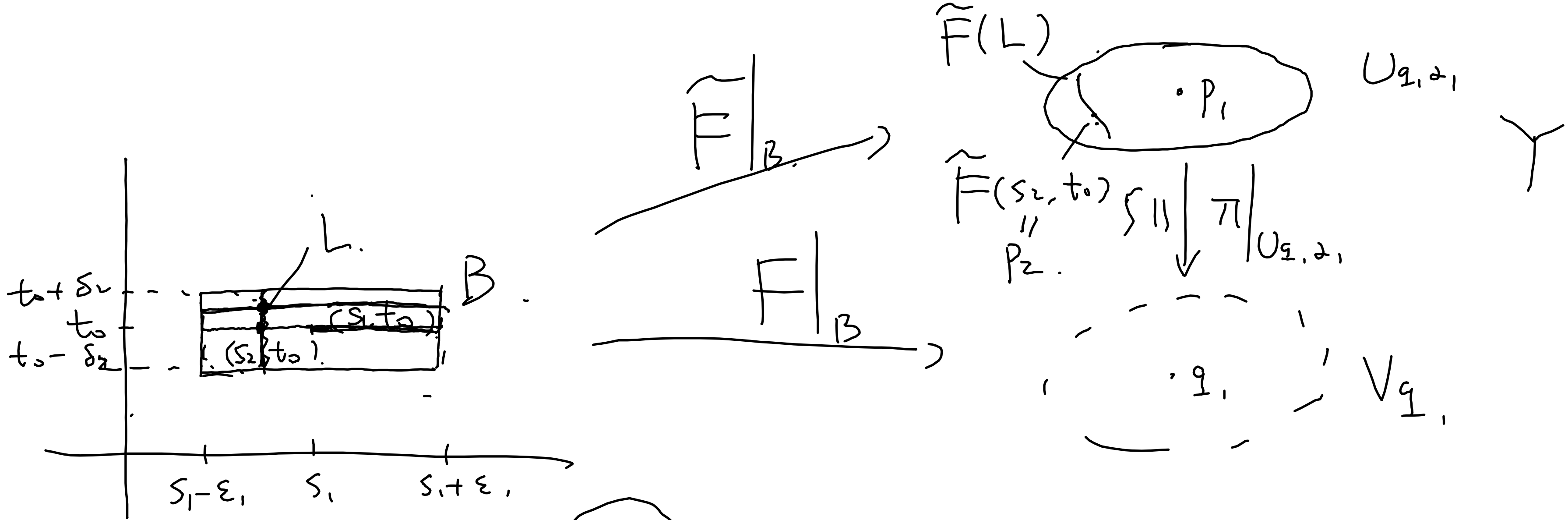
关键点： $\tilde{F}(B) \subset U_{q_1,\alpha_1}$

$[s_1-\varepsilon_1, s_1+\varepsilon_1] \longrightarrow Y \qquad s \longmapsto \tilde{F}(s,t) = \tilde{F}_t(s)$

$\tilde{F}_t([s_1-\varepsilon_1, s_1+\varepsilon_1]) \subset \pi^{-1}(V_{q_1}) = \coprod_{\alpha} U_{q_1,\alpha}$

又 $\tilde{F}_t([s_1-\varepsilon_1, s_1+\varepsilon_1])$ 连通，且 $\tilde{F}_t(s_2) \in U_{q_1,\alpha_1}$

$\Longrightarrow \tilde{F}_t([s_1-\varepsilon_1, s_1+\varepsilon_1]) \subset U_{q_1,\alpha_1}$

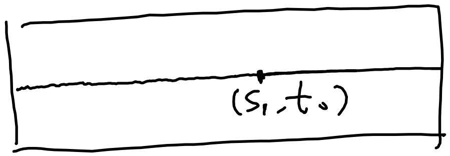


$\tilde{F}(B) \subset U_{q_1,\alpha_1}$, $\quad \underline{\pi|_{U_{q_1,\alpha_1}} \circ \tilde{F}|_B = F|_B}$.

$$(\pi|_{U_{q_1,\alpha_1}})^{-1} \circ F|_B : B \longrightarrow U_{q_1,\alpha_1}.$$

显然：$\quad \underline{\pi|_{U_{q_1,\alpha_1}} \circ \left((\pi|_{U_{q_1,\alpha_1}})^{-1} \circ F|_B \right) = F|_B}$.

$$\Rightarrow \tilde{F}|_B = \underbrace{(\pi|_{U_{q_1,\alpha_1}})^{-1} \circ F}_{\text{连续}}|_B. \Rightarrow \tilde{F}|_B \text{ 连续}.$$



B. $s_1 = \inf\{ s \mid \tilde{F}$ 在 (s, t_0) 处不连续 $\}$.

(细赋之处: $s_1 = 1$)

$\therefore$ 与 s_1 之定义矛盾.

$\therefore$ $\tilde{F}: I \times I \to Y$ 连续. #

应用到 $\pi: \mathbb{R} \to S^1$.

$\Rightarrow \phi: \mathbb{Z} \to \pi_1(S^1, 1)$ 为单射.

总结: $\pi_1(S^1, 1) \cong \mathbb{Z}$.