§13. 胞腔同调.

内容预览:

设定、记号同上一节，设 X 为 CW 复形(满足定义 α)，$X = \coprod_{\substack{n\geq 0\\ \alpha\in I_n}} e_\alpha^n$,

$K_n = \bigoplus_{\alpha\in I_n} \mathbb{Z} e_\alpha^n$.

① 定义 $d_n: K_n \longrightarrow K_{n-1}$, $\forall n$, 使 $(K_\cdot, d)$ 成为一个链复形,

定义 $H_n^{CW}(X) = H_n(K_\cdot)$

② $H_n^{CW}(X) \cong H_n(X)$.

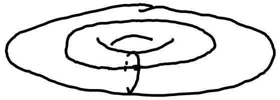
② 计算举例.

观察: $\bigoplus_{\alpha\in I_n} \mathbb{Z} e_\alpha^n \cong H_n(X^n, X^{n-1})$

Lemma 1: $H_k(X^n, X^{n-1}) \cong \begin{cases} \bigoplus_{\alpha\in I_n}\mathbb{Z}e_\alpha^n, & k=n \\ 0, & \text{else} \end{cases}$

Proof. 回忆: (X^n, X^{n-1}) is a good pair $\Rightarrow H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1})$

$X^n/X^{n-1} = \bigvee_{\alpha\in I_n} S_\alpha^n$, 其中 S_α^n 为 n 维球面.

(例:  X^2/X^1, $X^1/X^0 \cong$  S_2^1 S_1^1, $X^2 = X^1 \sqcup_\varphi B^2$, $X^2/X^1 \cong B^2/\partial B^2 \cong$ S^2 )

$\tilde{H}_k(\underbrace{\bigvee_\alpha S^n_\alpha}_{\cong \coprod_{\alpha\in I_n} S^n_\alpha / \coprod_{\alpha\in I_n}\{x_\alpha\}}) = ?$

其中 $x_\alpha \in S^n_\alpha$

注意到 $(\coprod_{\alpha\in I_n} S^n_\alpha, \coprod_{\alpha\in I_n}\{x_\alpha\})$ 为一个 good pair.（自行验证）

$$H_k(\coprod_{\alpha\in I_n} S^n_\alpha, \coprod_{\alpha\in I_n}\{x_\alpha\}) \cong \tilde{H}_k(\bigvee_\alpha S^n_\alpha)$$

$$\cong \bigoplus_{\alpha\in I_n} H_k(S^n_\alpha, \{x_\alpha\}).$$

$$\cong \bigoplus_{\alpha\in I_n} \tilde{H}_k(S^n_\alpha) \cong \begin{cases} \bigoplus_{\alpha\in I_n} \mathbb{Z} e^n_\alpha, & k = n \\ 0, & k \neq n \end{cases}$$

#

练习：设 $X_\alpha, \alpha\in I$，为一族拓扑空间，$Y_\alpha \subset X_\alpha$，$\forall \alpha$，证明：$H_k(\coprod_\alpha X_\alpha, \coprod_\alpha Y_\alpha) \cong \bigoplus_\alpha H_k(X_\alpha, Y_\alpha)$.

由 lemma 1，只需构造 $d_n: H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$

回忆：

$$\begin{array}{ccc} H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1}) \\ & \searrow & \downarrow j_{n-1} \\ & & H_{n-1}(X^{n-1}, X^{n-2}). \end{array}$$

定义 $d_n = j_{n-1}\circ \partial_n$.

$H_n(X^{n-1}) \to H_n(X^n) \xrightarrow{j_n} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \to H_{n-1}(X^n) \to H_{n-1}(X^n, X^{n-1}) \to$

为验证 $\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$

为一个链复形，还需验证：$d_n \circ d_{n+1} = 0$.

$$
\begin{tikzcd}
& H_n(X^n) \arrow[dr, "j_n"] & & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
& & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] &
\end{tikzcd}
$$

$$d_n \circ d_{n+1} = j_{n-1} \circ \underbrace{(\partial_n \circ j_n)}_{\substack{\parallel \\ 0}} \circ \partial_{n+1} = 0.$$

定义：$H_n^{CW}(X) := \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$.（称为 X 的第 n 个胞腔同调群）

下面：$H_n^{CW}(X) \cong H_n(X)$, $\forall n$.

Lemma 2. a). $H_k(X^n) = 0$, $\forall k > n$

b). 由包含映射 $X^n \to X$ 所诱导的群同态 $H_k(X^n) \to H_k(X)$

为 $\begin{cases} \text{一个同构}, & \text{if } k < n \\ \text{满同构}, & \text{if } k = n. \end{cases}$

Proof.

$$H_{k+1}(X^{n-1}) \longrightarrow H_{k+1}(X^{n}) \longrightarrow H_{k+1}(X^{n}, X^{n-1})$$

$\neq 0$, only if $k = n-1$

$$\longrightarrow H_{k}(X^{n-1}) \longrightarrow H_{k}(X^{n}) \longrightarrow H_{k}(X^{n}, X^{n-1})$$

$\neq 0$, only if $k = n$.

$$\longrightarrow H_{k-1}(X^{n-1}) \longrightarrow H_{k-1}(X^{n}) \longrightarrow H_{k-1}(X^{n}, X^{n-1}) \longrightarrow$$

$$\Rightarrow \begin{cases} H_k(X^{n-1}) \xrightarrow{\cong} H_k(X^n) & \text{if } k \neq n, n-1. \\ H_{n-1}(X^{n-1}) \longrightarrow H_{n-1}(X^n) & \text{为满同态.} \\ H_n(X^{n-1}) \longrightarrow H_n(X^n) & \text{为单同态.} \end{cases}$$

考虑.

$$\bigoplus_{\alpha\in I_0} H_k(pt) \cong H_k(X^0) \xrightarrow{\cong} H_k(X^1) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X^{k-2}) \xrightarrow{\cong} H_k(X^{k-1}) \xrightarrow{\text{单}} H_k(X^{k}) \xrightarrow{\text{满}} H_k(X^{k+1}) \xrightarrow{\cong} H_k(X^{k+2}) \xrightarrow{\cong} \cdots$$

X^0：离散空间.

$\bigoplus_{\alpha\in I_0} H_k(pt) = 0$, if $k>0$.

$\Rightarrow H_k(X^i) = 0$, $\forall\, 0 \leq i < k$. 由此(a)得证.

下证(b).

Case 1. X 为有限维 CW 复形.

设 $X = X^m$, $X^m \setminus X^{m-1} \neq \phi$.

要证: $H_k(X^n) \longrightarrow H_k(X^m)$ 为 $\begin{cases} 同构, & \text{if } k<n \\ 满同态, & \text{if } k=n \end{cases}$.

$(n \leq m)$.

① $k<n$, $(k<n\leq m)$. 由序列立得结论.

② $k=n$, 由序列立得结论.

Case 2. X 为无限维 CW 复形.

要证: $H_k(X^n) \longrightarrow H_k(X)$ 为 $\begin{cases} 同构, & \text{if } k<n \\ 满同态, & \text{if } k=n \end{cases}$

要证: $\begin{cases} ① H_k(X^n) \longrightarrow H_k(X) \text{ 为单同态, if } k<n. \\ ② H_k(X^n) \longrightarrow H_k(X) \text{ 为满同态, if } k\leq n. \end{cases}$

① $\forall \bar{c} \in H_k(X^n)$, 其中 c 为 X^n 中的一个 k-cycle.

设 $\bar{c}=0$ in $H_k(X)$. i.e. $c = \partial e$, for $\underbrace{\text{some } (k+1)\text{-chain}}_{X中的}$ e 成立

由上一节结论 $\Rightarrow \exists m \gg 0$, s.t. c 与 e 为 X^m 中的 chain.

$\Rightarrow \bar{c}=0$ in $H_k(X^m)$, 同时 $H_k(X^n) \overset{\cong}{\longrightarrow} H_k(X^m)$ $\bar{c} \longmapsto \bar{c}=0$

$\Rightarrow \bar{c} = 0$ in $H_k(X^n) \Rightarrow H_k(X^n) \to H_k(X)$ 为单同态.

② 要证: $H_k(X^n) \xrightarrow{i_*} H_k(X)$ 为满同态, $\forall\, k \le n$.

$\forall\, \bar{c} \in H_k(X)$, 其 c 为 X 中的 k-cycle.

设 $c \in X^m$, 其中 $m \gg 0$. 由 Case 1.

$$\begin{tikzcd}
H_k(X^n) \arrow[r, "\text{满同态}"] \arrow[dr, "i_*"'] & H_k(X^m) \ni \bar{c} \arrow[d] \\
\bar{e} \mapsto \bar{c} & H_k(X) \ni \bar{c}
\end{tikzcd}$$

#

知道更多下图表之信息:

$$\begin{tikzcd}
 & & & 0 & \\
0 = H_n(X^{n-1}) \arrow[dr] & & H_n(X^{n+1}) \arrow[ur] \arrow[rr, "\cong"] & & H_n(X) \\
 & H_n(X^n) \arrow[ur, "\text{满}"] \arrow[dr, "j_n"] & & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] &
\end{tikzcd}$$

看出: $H_n(X) \cong H_n(X^n) / \operatorname{Im} \partial_{n+1}$.

(要干的事: 证: $H_n(X) \cong H_n^{CW}(X)$)

$$\begin{tikzcd}
H_n(X^n) \arrow[rr, "j_n"] \arrow[dr, "j_n"'] & \circlearrowleft & H_n(X^n, X^{n-1}) \\
& \operatorname{Ker} d_n \arrow[ur] \arrow[dr] & \\
& & \operatorname{Ker} d_n / \operatorname{Im} d_{n+1} = H_n^{CW}(X)
\end{tikzcd}$$

记由上表定义的从 $H_n(X^n)$ 到 $H_n^{CW}(X)$ 的群同态为 π_n.

又注意：$\operatorname{Ker}\pi_n \supset \operatorname{Im}\partial_{n+1}$. $\because j_n \circ \partial_{n+1} = d_{n+1}$.

$\therefore$ π_n 诱导了：$\overline{\pi_n}: H_n(X^n)/\operatorname{Im}\partial_{n+1} \longrightarrow H_n^{CW}(X)$.

$$H_n(X^n)/\operatorname{Im}\partial_{n+1} \cong H_n(X).$$

下证：① $\overline{\pi_n}$ 为单
② $\overline{\pi_n}$ 为满. } 用图表追踪立得..

$$
\begin{tikzcd}[column sep=small]
 & & & & 0 & \\
 & 0=H_n(X^{n-1}) \arrow[dr] & & H_n(X^{n+1}) \arrow[ur] \arrow[r, "\cong"] & H_n(X) & \\
 & & H_n(X^n) \ni x \arrow[ur, "\text{满}"] \arrow[dr, "j_n"] & & & \\
H_{n+1}(X^{n+1},X^n) \arrow[urr, "\partial_{n+1}"] \arrow[rrr, "d_{n+1}"] & & & H_n(X^n,X^{n-1}) \ni y \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1},X^{n-2}) \\
 & & & & H_{n-1}(X^{n-1}) \ni \partial_n(y)=0 \arrow[ur, "j_{n-1}"] & \\
 & & & 0=H_{n-1}(X^{n-2}) \arrow[ur] & &
\end{tikzcd}
$$

$y \mapsto 0$, $\partial_n(y) \mapsto 0$

#

小结: $H_n(X) \cong H_n^{CW}(X)$. 其中 X 为 CW-复形.

$$H_n^{CW}(X) \cong \ker d_n / \operatorname{Im} d_{n+1}$$

$$\underbrace{d_n}_{\text{这次}} : \bigoplus_{\alpha\in I_n}\mathbb{Z}e_\alpha^n \longrightarrow \bigoplus_{\alpha\in I_{n-1}}\mathbb{Z}e_\alpha^{n-1}$$

$\alpha \in I_n$, $\beta \in I_{n-1}$

这次 显式表达 $\mathbb{Z}e_\alpha^n$ $\qquad$ $\mathbb{Z}e_\beta^{n-1}$

回顾代数：

设 G_α，$\alpha\in I$，为一族 Abel 群，$\bigoplus_{\alpha\in I} G_\alpha$.

$$G_\alpha \xrightarrow{\iota_\alpha} \bigoplus_{\alpha\in I} G_\alpha$$

$$\bigoplus_{\alpha\in I} G_\alpha \xrightarrow{\pi_\alpha} G_\alpha$$

满足

① $\pi_\beta\circ\iota_\alpha = \begin{cases} 0, & \text{if } \alpha\neq\beta \\ \mathrm{Id}_{G_\alpha}, & \text{if } \alpha=\beta. \end{cases}$

② $\sum_{\alpha\in I}\iota_\alpha\circ\pi_\alpha = \mathrm{Id}_{\bigoplus_{\alpha\in I} G_\alpha}$

$$\begin{array}{ccc}
\bigoplus_{\alpha\in I_n}\mathbb{Z}e_\alpha^n & \cong & \tilde{H}_n\left(\bigvee_{\alpha\in I_n} S_\alpha^n\right) \\
\iota_\alpha^n \uparrow\ \downarrow \pi_\alpha^n & & \uparrow\downarrow \\
\mathbb{Z}e_\alpha^n & \cong & \tilde{H}_n(S_\alpha^n)
\end{array}$$

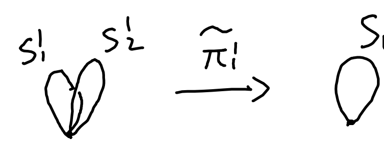
→ 几何实现.

Lemma. 有下面的交换图表：

$$
\begin{tikzcd}
\bigoplus_{\alpha\in I_n}\mathbb{Z}e_\alpha^n \arrow[r, equal, "\textcircled{1}\ \sim"] & \widetilde{H}_n\Big(\bigvee_{\alpha\in I_n}S_\alpha^n\Big) \\
\mathbb{Z}e_\alpha^n \arrow[u, "i_\alpha^n"] \arrow[r, equal, "\textcircled{2}\ \sim"] \arrow[ur, phantom, "\circlearrowleft"] & \widetilde{H}_n(S_\alpha^n) \arrow[u, "(\tilde{i}_\alpha^n)_*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bigoplus_{\alpha\in I_n}\mathbb{Z}e_\alpha^n \arrow[r, equal, "\textcircled{1}"] \arrow[d, "\pi_\alpha^n"'] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{H}_n\Big(\bigvee_{\alpha\in I_n}S_\alpha^n\Big) \arrow[d, "(\tilde{\pi}_\alpha^n)_*"] \\
\mathbb{Z}e_\alpha^n \arrow[r, equal, "\textcircled{2}"] & \widetilde{H}_n(S_\alpha^n)
\end{tikzcd}
$$

其中 $\tilde{i}_\alpha^n : S_\alpha^n \longrightarrow \bigvee_{\alpha\in I_n} S_\alpha^n$ 为典则嵌入.

$\tilde{\pi}_\alpha^n : \bigvee_{\alpha\in I_n} S_\alpha^n \longrightarrow S_\alpha^n$ 为典则收缩.



其中②为选定的一个群同构, $\forall \alpha \in I_n$.

①为由②所决定的群同构.

$$
\widetilde{H}_n\Big(\bigvee_{\alpha\in I_n} S_\alpha^n\Big) \cong H_n\Big(\coprod_{\alpha\in I_n} S_\alpha^n, \coprod_{\alpha\in I_n}\{x_\alpha\}\Big) \cong \bigoplus_{\alpha\in I_n} H_n(S_\alpha^n, x_\alpha) \cong \bigoplus_{\alpha\in I_n}\widetilde{H}_n(S_\alpha^n) \overset{\text{由②}}{\cong} \bigoplus_{\alpha\in I_n}\mathbb{Z}e_\alpha^n .
$$

Pf 「Lemma. 设 X_α, $\alpha\in I$, 为一族 top. sp., $x_\alpha \in X_\alpha$, $\alpha\in I$. 记 $i_\alpha : (X_\alpha, x_\alpha) \longrightarrow (\coprod_\alpha X_\alpha, \coprod_\alpha\{x_\alpha\})$ 为典则嵌入, $\pi_\alpha : (\coprod_\alpha X_\alpha, \coprod_\alpha\{x_\alpha\}) \longrightarrow (X_\alpha, x_\alpha)$. 则

① $(i_\alpha)_* : H_n(X_\alpha, x_\alpha) \longrightarrow H_n(\coprod_\alpha X_\alpha, \coprod_\alpha\{x_\alpha\})$ 所诱导的 $\bigoplus_\alpha H_n(X_\alpha,x_\alpha) \to H_n(\coprod_\alpha X_\alpha, \coprod_\alpha\{x_\alpha\})$ 为群同构.

② $(\pi_\alpha)_* : H_n(\coprod_\alpha X_\alpha, \coprod_\alpha\{x_\alpha\}) \longrightarrow H_n(X_\alpha, x_\alpha)$, 满足：
$$
\begin{cases}
(\pi_\beta)_* \circ (i_\alpha)_* = \begin{cases} 0, & \text{if } \alpha\neq\beta \\ \mathrm{Id}_{H_n(X_\alpha,x_\alpha)}, & \text{if } \alpha=\beta \end{cases} \\
\sum (i_\alpha)_* \circ (\pi_\alpha)_* = \mathrm{Id}
\end{cases}
$$
」

注意

$$\begin{tikzcd}
\tilde{H}_n(\bigvee_\alpha S^n_\alpha) \arrow[r, "\cong"] \arrow[dr, phantom, "(a)\ \circlearrowleft"] & H_n\left(\coprod_\alpha S^n_\alpha / \coprod_\alpha\{x_\alpha\},\ \coprod_\alpha\{x_\alpha\}/\coprod_\alpha\{x_\alpha\}\right) \arrow[dr, phantom, "\circlearrowleft\ (b)"] & H_n\left(\coprod_\alpha S^n_\alpha,\ \coprod_\alpha\{x_\alpha\}\right) \arrow[l, "\cong"', "(\text{quotient})_*"] \\
\tilde{H}_n(S^n_\alpha) \arrow[u, "(\tilde{i}_\alpha)_*"] \arrow[r, "\cong"] & H_n(S^n_\alpha, x_\alpha) \arrow[u, "(\text{inclusion})_*"] \arrow[ur] &
\end{tikzcd}$$

为交换图表

$$\begin{tikzcd}
\tilde{H}_n(\bigvee_\alpha S^n_\alpha) \arrow[r, "\cong"] \arrow[d, "(\tilde{\pi}^n_\alpha)_*"] \arrow[dr, phantom, "(a)\ \circlearrowleft"] & H_n\left(\coprod_\alpha S^n_\alpha / \coprod_\alpha\{x_\alpha\},\ \coprod_\alpha\{x_\alpha\}/\coprod_\alpha\{x_\alpha\}\right) \arrow[d, "(\text{quotient})_*"] \arrow[dr, phantom, "\circlearrowleft\ (b)"] & H_n\left(\coprod_\alpha S^n_\alpha,\ \coprod_\alpha\{x_\alpha\}\right) \arrow[l, "\cong"', "(\text{quotient})_*"] \arrow[dl] \\
\tilde{H}_n(S^n_\alpha) \arrow[r, "\cong"] & H_n(S^n_\alpha, x_\alpha) &
\end{tikzcd}$$

为交换图表

(a) 因为都是长正合列之间态射的局部

(b) 因为定义.

写好一点:

$$\begin{tikzcd}
\tilde{H}_n(\bigvee_\alpha S^n_\alpha) \arrow[r, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(\coprod_\alpha S^n_\alpha, \coprod_\alpha\{x_\alpha\}) \arrow[r, "f", "\cong"', "\text{由②}"] \arrow[dr, phantom, "\circlearrowleft\ \text{由构造}"] & \bigoplus_\alpha \mathbb{Z} e^n_\alpha \\
\tilde{H}_n(S^n_\alpha) \arrow[u, "(\tilde{i}_\alpha)_*"] \arrow[r, phantom, "\cong"] & H_n(S^n_\alpha, x_\alpha) \arrow[u, "i_\alpha"] \arrow[r, "g", "\cong"'] & \mathbb{Z} e^n_\alpha \arrow[u, "i^n_\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{H}_n(\bigvee_\alpha S^n_\alpha) \arrow[r, phantom, "\cong"] \arrow[d, "(\tilde{\pi}^n_\alpha)_*"] \arrow[dr, phantom, "\circlearrowleft"] & H_n(\coprod_\alpha S^n_\alpha, \coprod_\alpha\{x_\alpha\}) \arrow[r, "f", "\cong"', "\text{由②}"] \arrow[d, "p_\alpha"] \arrow[dr, phantom, "(※)"] & \bigoplus_\alpha \mathbb{Z} e^n_\alpha \arrow[d, "\pi^n_\alpha"] \\
\tilde{H}_n(S^n_\alpha) \arrow[r, phantom, "\cong"] & H_n(S^n_\alpha, x_\alpha) \arrow[r, "g", "\cong"'] & \mathbb{Z} e^n_\alpha
\end{tikzcd}$$

「要证(※)交换, i.e. $g\circ p_\alpha = \pi^n_\alpha \circ f$.

$\sum_\alpha i_\alpha \circ p_\alpha = Id$

又 $\pi^n_\alpha \circ f \circ Id = \pi^n_\alpha \circ f \circ \sum_\beta i_\beta \circ p_\beta$

$= \pi^n_\alpha \circ \sum_\beta i^n_\beta \circ g \circ p_\beta$

$= Id_{\mathbb{Z} e^n_\alpha} \circ g \circ p_\alpha = g \circ p_\alpha$.」

#

$$
\begin{tikzcd}
\tilde{H}_n\left(\bigvee_{\alpha\in I_n} S^n_\alpha\right) \arrow[r, phantom, "\cong"] \arrow[d, shift left, "(\tilde{\pi}^n_\alpha)_*"] & \bigoplus_{\alpha\in I_n} \mathbb{Z}e^n_\alpha \arrow[d, shift left] \\
\tilde{H}_n(S^n_\alpha) \arrow[r, phantom, "\cong"] \arrow[u, shift left, "(\tilde{\iota}^n_\alpha)_*"] & \mathbb{Z}e^n_\alpha \arrow[u, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z}e^n_\alpha \arrow[d] & \mathbb{Z}e^{n-1}_\beta \\
\bigoplus_{\alpha\in I_n} \mathbb{Z}e^n_\alpha \arrow[r] & \bigoplus_{\beta\in I_{n-1}} \mathbb{Z}e^{n-1}_\beta \arrow[u]
\end{tikzcd}
\quad (n\geq 2)
$$

回到我们的目标：计算：d_n：

$$
\begin{tikzcd}
\bigoplus_{\alpha\in I_n} \mathbb{Z}e^n_\alpha \arrow[r] \arrow[d, phantom, "\cong"] & \bigoplus_{\beta\in I_{n-1}} \mathbb{Z}e^{n-1}_\beta \arrow[d, phantom, "\cong"] \\
H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[dr, "\partial_n"'] & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & H_n(X^{n-1}) \arrow[u, "j_{n-1}"']
\end{tikzcd}
$$

固定同胚 $B^n_\alpha/\partial B^n_\alpha \cong S^n_\alpha$, $\forall \alpha\in I$ $\xrightarrow{\text{诱导}}$ $X^n/X^{n-1} \cong \bigvee_{\alpha\in I_n} S^n_\alpha$.

$$
\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[r, phantom, "\cong"] & \tilde{H}_n(X^n/X^{n-1}) \arrow[r, phantom, "\cong"] & \tilde{H}_n\left(\bigvee_{\alpha\in I_n} S^n_\alpha\right) \arrow[r, phantom, "\cong"] & \bigoplus_{\alpha\in I_n} \mathbb{Z}e^n_\alpha \\
H_n(B^n_\alpha, \partial B^n_\alpha) \arrow[r, phantom, "\cong"] \arrow[u, "(\Phi^n_\alpha)_*"] & \tilde{H}_n(B^n_\alpha/\partial B^n_\alpha) \arrow[r, phantom, "\cong"] \arrow[u, "(\Phi^n_\alpha)_*"] & \tilde{H}_n(S^n_\alpha) \arrow[r, phantom, "\cong"] \arrow[u, "(\iota^n_\alpha)_*"] & \mathbb{Z}e^n_\alpha \arrow[u, "\iota^n_\alpha"]
\end{tikzcd}
$$

Φ^n_α: $B^n_\alpha \to X^{n-1} \sqcup_{\alpha\in I_n} B^n_\alpha \to X^n$

$$
\begin{tikzcd}
H_n(X^n, X^{n-1}) \arrow[r, phantom, "\cong"] & \tilde{H}_n(X^n/X^{n-1}) \arrow[r, phantom, "\cong"] \arrow[d, "\text{多模一点}"] & \tilde{H}_n\left(\bigvee_{\alpha\in I_n} S^n_\alpha\right) \arrow[r, phantom, "\cong"] \arrow[d, "(\tilde{\pi}^n_\alpha)_*"] & \bigoplus_{\alpha\in I_n} \mathbb{Z}e^n_\alpha \arrow[d, "\pi^n_\alpha"] \\
 & \tilde{H}_n(X^n/(X^n\setminus e^n_\alpha)) \arrow[r, phantom, "\cong"] & \tilde{H}_n(S^n_\alpha) \arrow[r, phantom, "\cong"] & \mathbb{Z}e^n_\alpha
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z}e_\alpha^n \arrow[dd] \arrow[dr, equals, "\cong"] & & & & \widetilde{H}_{n-1}(S_\beta^{n-1}) \arrow[r, equals, "\cong"] \arrow[d, equals, "\cong"'] & \mathbb{Z}e_\beta^{n-1} \\
& H_n(B_\alpha^n, \partial B_\alpha^n) \arrow[ddd, "(\Phi_\alpha^n)_*"'] \arrow[dr, "\cong"] & & & \widetilde{H}_{n-1}(X^{n-1}/(X^{n-1}\setminus e_\beta^{n-1})) & \\
\bigoplus_{\alpha\in I_n} \mathbb{Z}e_\alpha^n \arrow[rrrrr, "\text{多模一点}"] \arrow[ddr, equals, "\cong"'] & & \widetilde{H}_{n-1}(\partial B_\alpha^n) \arrow[ddd, "(\Phi_\alpha^n|_{\partial B_\alpha^n})_*"] & & & \bigoplus_{\beta\in I_{n-1}} \mathbb{Z}e_\beta^{n-1} \arrow[uu] \\
& & & & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[uu] \arrow[d, equals, "\cong"] & \\
& H_n(X^n, X^{n-1}) \arrow[dr, "\partial_n"] \arrow[ddr, "\partial_n"'] & & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[uur, equals, "\cong"'] & \\
& & \widetilde{H}_{n-1}(X^{n-1}) \arrow[uuuurr, "(\text{quotient})_*"] \arrow[uurr, "(\text{quotient})_*"] \arrow[urr, "j_{n-1}"] & & & \\
& & H_{n-1}(X^{n-1}) \arrow[uurr, "j_{n-1}"'] & & &
\end{tikzcd}
$$

整理一下：

$$\begin{array}{ccc}
\bigoplus_{\alpha\in I_n}\mathbb{Z}e_\alpha^n & \xrightarrow{\quad d_n \quad} & \bigoplus_{\beta\in I_{n-1}}\mathbb{Z}e_\beta^{n-1} \\
\uparrow \iota_\alpha^n & & \downarrow \pi_\beta^{n-1} \\
e_\alpha^n \in \mathbb{Z}e_\alpha^n & & \mathbb{Z}e_\beta^{n-1} \ni e_\beta^{n-1} \\
\parallel\!\wr & & \parallel\!\wr \\
[e_\alpha^n] \in \tilde{H}_{n-1}(\partial B_\alpha^n) & \xrightarrow{(m_\beta^{n-1}\circ \Phi_\alpha^n|_{\partial B_\alpha^n})_*} & \tilde{H}_{n-1}(S_\beta^{n-1}) \ni [e_\beta^{n-1}]
\end{array}$$

其中，$\Phi_\alpha^n : B_\alpha^n \to X^n$ 为 characteristic map.

$m_\beta^{n-1} : X^{n-1} \longrightarrow X^{n-1}/(X^{n-1}\setminus e_\beta^{n-1}) \cong S_\beta^{n-1}$.

$\Rightarrow \exists\, d_{\alpha\beta}^{n-1} \in \mathbb{Z}$, s.t. $(m_\beta^{n-1}\circ \Phi_\alpha^n|_{\partial B_\alpha^n})_*([e_\alpha^n]) = d_{\alpha\beta}^{n-1}[e_\beta^{n-1}]$. $(n\geq 1)$

$\Rightarrow d_n(e_\alpha^n) = \sum_{\beta\in I_{n-1}} d_{\alpha\beta}^{n-1} e_\beta^{n-1}$, $n\geq 2$.

注：closure-finiteness $\Rightarrow$ 右边为有限和

$\Rightarrow$ $\Phi_\alpha^n|_{\partial B_\alpha^n}$ 之像只与有限个开胞腔相交.

$\Rightarrow$ $m_\beta^{n-1}\circ\Phi_\alpha^n|_{\partial B_\alpha^n}$ 只对有限个 β 可能为满射.

Lemma：若 $f: S^n \to S^n$ 为不满连续映射，则 $f_*: H_n(S^n)\to H_n(S^n)$ 必为零.

Pf. 取 $p\in S^n\setminus \mathrm{Im} f$. $S^n \xrightarrow{f} S^n\setminus\{p\} \hookrightarrow S^n$（$S^n\setminus\{p\}$ 可缩） #.

再看 d_1:

$$
\begin{tikzcd}
d_1: \bigoplus\limits_{\alpha\in I_1}\mathbb{Z}e^1_\alpha \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \bigoplus\limits_{\alpha\in I_0}\mathbb{Z}e^0_\alpha \arrow[d, phantom, "\cong" rotate=90] \\
H_1(X^1,X^0) \arrow[r, "\partial_1"] & H_0(X^0) \cong S_0(X^0)\cancel{/\mathrm{Im}(S_1(X^1)\to S_0(X^0))}
\end{tikzcd}
$$

$$S_0(X^0) \cong \bigoplus_{\alpha\in I_0}\mathbb{Z}e^0_\alpha$$

$$
\begin{tikzcd}
 & \tilde{H}_0(\partial B^1_\alpha) \cong \{np_0-np_1 \mid n\in\mathbb{Z}\} \arrow[dr, hook] & \subset \ \cong \mathbb{Z}p_0\oplus\mathbb{Z}p_1 & \mathbb{Z}e^0_\beta \\
\mathbb{Z}e^1_\alpha \arrow[r, "\cong"] \arrow[d] & H_1(B^1_\alpha,\partial B^1_\alpha) \arrow[u, "\cong"] \arrow[r] \arrow[d, "(\Phi^1_\alpha)_*"] & H_0(\partial B^1_\alpha) \ni np_0+mp_1 \arrow[d, "(\Phi^1_\alpha|_{\partial B^1_\alpha})_*"] & \\
\bigoplus\limits_{\alpha\in I_1}\mathbb{Z}e^1_\alpha \arrow[rr, "d_1"] \arrow[dr, "\cong"] & & \arrow[r] & \bigoplus\limits_{\alpha\in I_0}\mathbb{Z}e^0_\alpha \arrow[uu] \\
 & H_1(X^1,X^0) \arrow[r, "\partial_1"] & H_0(X^0) & \\
 & & n\Phi^1_\alpha(p_0)+m\Phi^1_\alpha(p_1) &
\end{tikzcd}
$$

e^0_β 之前系数

$\partial B^1_\alpha = \{p_0, p_1\}$

因此: $d_1 e^1_\alpha = \sum_{\beta\in I_0} d^0_{\alpha\beta}\, e^0_\beta$, 其中 $d^0_{\alpha\beta}$ 为将 $\{np_0-np_1 \mid n\in\mathbb{Z}\}$ 中对应于 e^1_α 的生成元 (p_0-p_1 or p_1-p_0) 经由映射 $np_0+mp_1 \longmapsto n\Phi^1_\alpha(p_0)+m\Phi^1_\alpha(p_1) \longmapsto$ 为 e^0_β 之前系数所得之整数.

Rmk. 若 X^0 为单点集, 则 $d_1 = 0$.

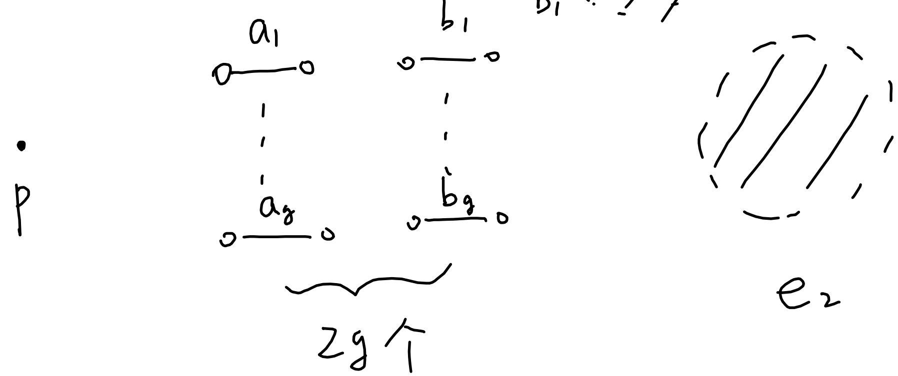
例: $\mathbb{C}P^n \cong e_0 \amalg e_2 \amalg \cdots \amalg e_{2n}$.

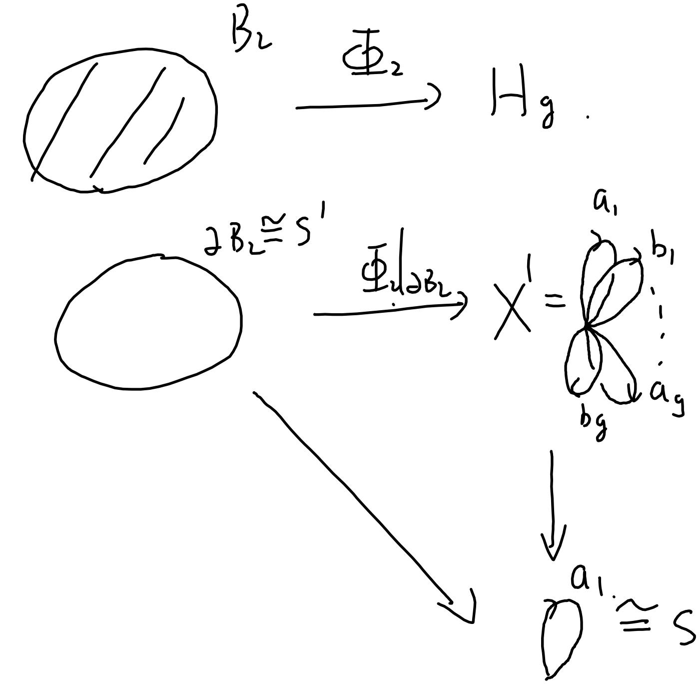
$$0 \to \mathbb{Z}e_{2n} \to 0 \to \mathbb{Z}e_{2n-2} \to 0 \to \cdots \to 0 \to \mathbb{Z}e_2 \to 0 \to \mathbb{Z}e_0$$

$$\Rightarrow H_k^{CW}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \cdots, 2n \\ 0, & \text{else} \end{cases}$$

$$\begin{array}{c} \| \wr \\ H_k(\mathbb{C}P^n). \end{array}$$

例: $H_n(H_g)$. H_g :





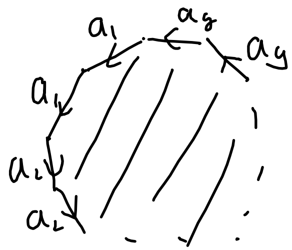
$d_1 = 0$, $\Rightarrow H_0(H_g) \cong \mathbb{Z}p/0 \cong \mathbb{Z}$.

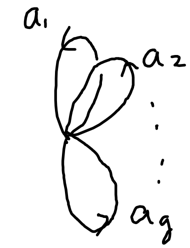
$$H_1(H_g) = \ker d_1/\mathrm{Im}\, d_2 = \left(\bigoplus_{i=1}^{g} \mathbb{Z}a_i \oplus \bigoplus_{i=1}^{g} \mathbb{Z}b_i\right)/\mathrm{Im}\, d_2 \cong \mathbb{Z}^{2g}$$

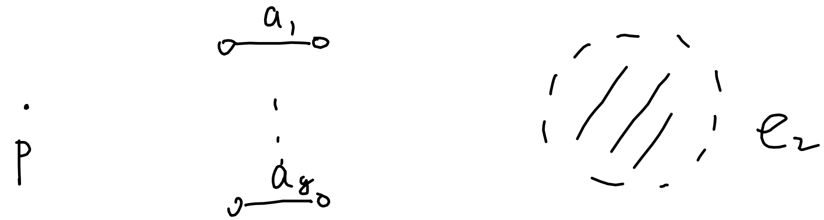
$$d_2 e_2 = n_1 a_1 + \cdots + n_g a_g + m_1 b_1 + \cdots + m_g b_g = 0 \Rightarrow d_2 = 0$$

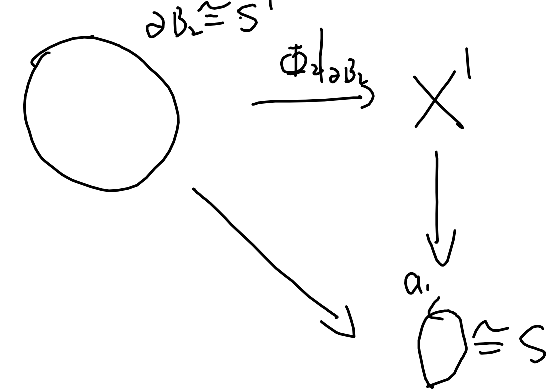
$$H_2(H_g) = \ker d_2 = \mathbb{Z}e_2 \cong \mathbb{Z}.$$

例: $H_n(M_g)$,



$X^1 =$ 





由 Rmk $\Rightarrow d_1=0 \Rightarrow H_0(M_g)\cong\mathbb{Z}$.

$H_1(M_g)\cong \ker d_1/\mathrm{Im}\,d_2 = \bigoplus_{i=1}^{g}\mathbb{Z}a_i/\mathrm{Im}\,d_2$.

$d_2 e_2 = n_1a_1+\cdots+n_g a_g = 2(a_1+\cdots+a_g)$. $\Rightarrow \mathrm{Im}\,d_2=\mathbb{Z}\,2(a_1+\cdots+a_g)$

「Lemma: 对于 $f: S^1\to S^1$, $z\mapsto z^k$, 其中 $S^1=\{z\in\mathbb{C}\mid |z|=1\}$, $f_*: H_1(S^1)\to H_1(S^1)$

记 $[S^1]$ 为 $H_1(S^1)$ 的生成元, 则 $f_*[S^1]=k[S^1]$ (Hatcher)」

$\therefore H_1(M_g)\cong \mathbb{Z}^g/\mathbb{Z}(2,\cdots,2)\cong \underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{(g-1)个}\oplus\mathbb{Z}/2\mathbb{Z}$.

$H_2(M_g)=\ker d_2=0$.