拓扑群

$$\begin{tikzcd}
\text{群} \arrow[r, "+\text{拓扑}"] \arrow[dr, "+\text{光滑结构}"] \arrow[ddr, "+\text{代数簇}"] \arrow[dddr, "+\text{概型}"'] & \text{拓扑群} \\
& \text{李群} \\
& \text{代数群} \\
& \text{群概型}
\end{tikzcd}$$

定义. 设 G 为一个 Hausdorff top sp, 且 G 为一个群, 且设 G 的群运算都连续 i.e. 下面两个映射

$$m: G \times G \to G \qquad (g, h) \mapsto g \cdot h.$$

$$i: G \to G \qquad g \mapsto g^{-1}$$

连续, 则称 G 为一个拓扑群.

Rmk. G top group. $\forall g \in G$, 定义 $l_g: G \to G$, $h \mapsto gh$. l_g 为一个同胚.

$l_g: \quad G \longrightarrow G \times G \xrightarrow{m} G$

$\qquad h \longmapsto (g, h) \longmapsto g \cdot h.$

$\Rightarrow l_g$ 连续.

$l_{g_1} \circ l_{g_2} = l_{g_1 \cdot g_2}$. $\left(\begin{array}{l} l_{g_1} \circ l_{g_2}(h) = l_{g_1}(\underbrace{l_{g_2}(h)}_{= g_2 \cdot h}) = g_1 (g_2 \cdot h) \\ l_{g_1 \cdot g_2}(h) = (g_1 \cdot g_2) \cdot h \end{array} \right)$

$\Rightarrow l_{g^{-1}} = l_g^{-1}$ $\quad (l_{g^{-1}} \circ l_g = l_e = id_G, \ l_g \circ l_{g^{-1}} = id_G).$

$\Rightarrow l_g: G \to G$ 为同胚.

$\forall g, h \in G$,

$\quad l_{h \cdot g^{-1}}: G \longrightarrow G \quad g \longmapsto h.$

例: $(\mathbb{R}, +)$ 为拓扑群.

例: $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} = \{ e^{i\theta} \mid \theta \in \mathbb{R} \}$ 为一个拓扑群

$\quad S^1 \times S^1 \longrightarrow S^1 \quad (e^{i\alpha}, e^{i\beta}) \longmapsto e^{i(\alpha+\beta)}$

$\quad S^1 \longrightarrow S^1 \quad e^{i\alpha} \longmapsto e^{-i\alpha}.$

例. 任意一个群 G，取离散拓扑.

例. $T = S^1 \times S^1$.

「一般事实：设 G 为 top group, H 为 top group.
则 $G \times H$ 也为拓扑群（练习）」.

例. $GL_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid A \text{ 可逆}\} \subset \mathbb{R}^{n^2}$.

↑ 子空间拓扑

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \longmapsto (a_{11} \cdots a_{1n} \cdots \cdots a_{n1} \cdots a_{nn})$$

$m: GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$. $(A, B) \mapsto A \cdot B$

$i: GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$, $A \mapsto A^{-1}$.

$m = (m_{ij})$. $m_{ij}(A, B) = A \cdot B$ 的第 i 行第 j 列，$1 \le i \le n$, $1 \le j \le n$

$$= \underbrace{\sum_{k=1}^{n} a_{ik} b_{kj}}_{\text{多项式}} \qquad A = (a_{ij}), \; B = (b_{ij}).$$

$\Rightarrow m$ 连续

$i = (i_{kl})$ $i_{kl}(A) = A^{-1}$ 的第 k 行第 l 列 $\Rightarrow i$ 连续.

$$= \frac{1}{\det A} (a_{kl}^*)$$

$\det A \longrightarrow$ 关于 $\{a_{ij}\}$ 的多项式

$GL_n(\mathbb{R})$ 为一个拓扑群

$$\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^*.$$
$$A \longmapsto \det(A).$$

$\det(A)$ ← 多项式 $\Rightarrow$ det 为连续函数.

$$\Rightarrow GL_n(\mathbb{R}) = \underbrace{\det^{-1}((0,+\infty))}_{\text{非空开集}} \amalg \underbrace{\det^{-1}((-\infty,0))}_{\text{非空开集}}$$

$\Rightarrow GL_n(\mathbb{R}^n)$ 不是连通的.

事实: $\det^{-1}((0,+\infty))$、$\det^{-1}((-\infty,0))$ 为 $GL_n(\mathbb{R})$ 的两个连通分支

[c.f. Warner. Foundations of differential manifolds and Lie groups].

例. $O(n) = \{A \in GL_n(\mathbb{R}) \mid A \cdot A^T = I_n\} < GL_n(\mathbb{R})$

$SO(n) = \{A \in O(n) \mid \det A = 1\} < GL_n(\mathbb{R})$

子空间拓扑.

$$\begin{tikzcd}
O(n)\times O(n) \arrow[r] \arrow[d, "连续"'] & O(n) \arrow[d, "连续"] \\
GL_n(\mathbb{R})\times GL_n(\mathbb{R}) \arrow[r, "m"] & GL_n(\mathbb{R})
\end{tikzcd}$$

已知：$GL_n(\mathbb{R})\times GL_n(\mathbb{R}) \xrightarrow{m} GL_n(\mathbb{R})$ 为连续映射.

$\Rightarrow$ $O(n)\times O(n) \longrightarrow O(n)$ 为连续映射

类似地：$O(n) \longrightarrow O(n)$ 为 ....

$\rightsquigarrow$ $O(n)$ 为一个拓扑群，类似地，$SO(n)$也为拓扑群.

下面：要说明 $O(n)$, $SO(n)$ 都为紧的拓扑群.

$$O(n),\ SO(n) \subset \mathbb{R}^{n^2}$$

只需证：$O(n)$, $SO(n)$ 为有界闭集.

$$\forall A \in O(n),\quad A=(a_{ij}),\quad A\cdot A^T = I_n.$$

$$\sum_k a_{ik}^2 = 1 \Rightarrow |a_{ik}| \le 1$$

$\Rightarrow$ $\left.\begin{matrix} O(n) \\ SO(n) \end{matrix}\right\}$ 为有界集

$$O(n) = \{ A \in M_{n\times n}(\mathbb{R}) \mid A\cdot A^T = I_n \}.$$

$f_{ij}: M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$.

$\mathbb{R}^{n^2} \cong M_{n\times n}(\mathbb{R})$

$A = (a_{ij}) \longmapsto A\cdot A^T$ 的第 i 行第 j 列 $= \sum_{k=1}^{n} a_{ik}\cdot a_{jk}$

$\Rightarrow \{f_{ij}\}_{\substack{1\le i\le n\\ 1\le j\le n}}$ 为 n^2 个 $M_{n\times n}(\mathbb{R})$ 上连续函数.

$$O(n) = \left(\bigcap_{j=1}^{n} f_{jj}^{-1}(1)\right) \cap \left(\bigcap_{i\neq j} f_{ij}^{-1}(0)\right)$$

为一个 $\mathbb{R}^{n^2}$ 中闭集.

$\Rightarrow SO(n) = O(n) \cap (\det^{-1}(1))$ 也是 $\mathbb{R}^{n^2}$ 中闭集

$\Rightarrow O(n), SO(n)$ 为紧的拓扑群.

later 证: SO(n) 为连通的.

$$SO(n) = O(n) \cap \underbrace{\left(\det{}^{-1}((0,+\infty))\right)}_{\text{开集}}$$

$\Rightarrow$ $SO(n)$ 又为 $O(n)$ 中开集

$\Rightarrow$ $SO(n)$ 为 $O(n)$ 中既开又闭集.

令 $g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in O(n)$.

$l_g : O(n) \longrightarrow O(n)$ 同胚.

$\underline{l_g(SO(n))}$ 也为 $O(n)$ 中既开又闭集.

$$\Rightarrow \quad O(n) = \underbrace{SO(n)}_{\text{既开又闭集}} \sqcup \underbrace{gSO(n)}_{\text{既开又闭集}}.$$

$\Rightarrow$ $SO(n)$ 与 $gSO(n)$ 为 $O(n)$ 的两个连通分支.

例: $SO(2) \longrightarrow S^1 \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \longmapsto e^{i\theta}$

为一个群同构, 且为同胚

定义：设 G, H 为两个拓扑群，$\varphi: G \longrightarrow H$ 是一个拓扑群同态，if：① φ 为群同态，② φ 为连续映射

若进一步 φ 为同胚，则称 φ 为一个拓扑群同构.

$\Longrightarrow \quad SO(2) \cong S^1$

例. $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$.

设: $\mathbb{H} = \{a + b\vec{i} + c\vec{j} + d\vec{k} \mid a, b, c, d \in \mathbb{R}\}$.

$\cdot : \mathbb{R} \times \mathbb{H} \longrightarrow \mathbb{H}$.

$(\lambda, (a + b\vec{i} + c\vec{j} + d\vec{k})) \longmapsto \lambda \cdot (a + b\vec{i} + c\vec{j} + d\vec{k})$

$:= \lambda a + \lambda b\vec{i} + \lambda c\vec{j} + \lambda d\vec{k}$.

$+ : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$ $\quad \forall X, Y \in \mathbb{H}$

$X + Y :=$ 分量相加.

$(\mathbb{H}, +, \cdot)$ 为一个 $\mathbb{R}$-线性空间.

$\cdot : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$

规定: $\vec{i} \cdot \vec{j} = \vec{k}$, $\vec{j} \cdot \vec{k} = \vec{i}$, $\vec{k} \cdot \vec{i} = \vec{j}$

$\vec{i} \cdot \vec{j} = -\vec{j} \cdot \vec{i}$, $\quad \vec{j} \cdot \vec{k} = -\vec{k} \cdot \vec{j}$, $\quad \vec{k} \cdot \vec{i} = -\vec{i} \cdot \vec{k}$

$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1$

$(a_1 + b_1\vec{i} + c_1\vec{j} + d_1\vec{k}) \cdot (a_2 + b_2\vec{i} + c_2\vec{j} + d_2\vec{k})$

$:=$ 按分配律 + "规定" 乘所得四元数.

$$\mathbb{H}' = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{C} \right\} \subset M_{2\times 2}(\mathbb{C})$$

$\cdot : \mathbb{R} \times \mathbb{H}' \longrightarrow \mathbb{H}'$ $\quad \lambda \cdot \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ =按分量相乘

$+ : \mathbb{H}' \times \mathbb{H}' \longrightarrow \mathbb{H}'$ 矩阵加法

$\cdot : \mathbb{H}' \times \mathbb{H}' \longrightarrow \mathbb{H}'$，矩阵乘法

$\forall \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix} \in \mathbb{H}'$

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \cdot \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix} = \begin{pmatrix} ac - b\bar{d} & ad + b\bar{c} \\ -\bar{b}c - \bar{a}\bar{d} & -\bar{b}d + \bar{a}\bar{c} \end{pmatrix} \in \mathbb{H}'.$$

$(M_{2\times 2}(\mathbb{C}), +, \cdot)$ 为一个 $\mathbb{R}$-代数.

$\Rightarrow (\mathbb{H}', +, \cdot)$ 为一个 $\mathbb{R}$-代数.

$$\varphi : \mathbb{H} \longrightarrow \mathbb{H}'$$
$$(a + b\vec{i} + c\vec{j} + d\vec{k}) \longmapsto \begin{pmatrix} a+bi & c+di \\ -\overline{c+di} & \overline{a+bi} \end{pmatrix}$$

Claim: φ 为一个保持一切运算的双射.

proof of Claim. 只需证 φ 保乘法.

i.e. $\forall\ X, Y \in \mathbb{H}$, $\varphi(X\cdot Y) = \varphi(X)\cdot\varphi(Y)$.

$\Updownarrow$

当 $X, Y \in \{1, \vec{i}, \vec{j}, \vec{k}\}$, $\varphi(X\cdot Y) = \varphi(X)\cdot\varphi(Y)$.

$$\varphi(1) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \varphi(\vec{i}) = \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

$$\varphi(\vec{j}) = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \varphi(\vec{k}) = \begin{pmatrix} & i \\ i & \end{pmatrix}.$$

$$\because\ \varphi(\underbrace{\vec{i}\cdot\vec{j}}_{=\vec{k}}) = \varphi(\vec{i})\cdot\varphi(\vec{j})$$

$$\begin{pmatrix} & i \\ i & \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix}\cdot\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$\because\ \varphi(\underbrace{\vec{i}\cdot\vec{i}}_{=-1}) = \varphi(\vec{i})\cdot\varphi(\vec{i}),$$

$$\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} i & \\ & -i \end{pmatrix}^2$$

\#

$\Rightarrow$ $\mathbb{H}$ 为一个 $\mathbb{R}$-代数

下面：证明：$\left.\begin{array}{l}\mathbb{H}\\ \mathbb{H}'\end{array}\right\}$ 为一个可除代数

i.e. 非零元可逆.

共轭：$\forall\ X = a + b\vec{i} + c\vec{j} + d\vec{k} \in \mathbb{H}$,

$$\overline{X} := a - b\vec{i} - c\vec{j} - d\vec{k}.$$

$$\varphi(\overline{X}) = \begin{pmatrix} a-bi & -c-di \\ c-di & a+bi \end{pmatrix} = \overline{\varphi(X)}^T.$$

$$= \overline{\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}}^T$$

$$= \begin{pmatrix} a-bi & -c-di \\ c-di & a+bi \end{pmatrix}.$$

性质：$\forall X, Y \in \mathbb{H}$, $\overline{X\cdot Y} = \overline{Y}\cdot\overline{X}$.

$\varphi: \mathbb{H} \to \mathbb{H}$.

「Pf. $\varphi(\overline{X\cdot Y}) = \varphi(\overline{Y}\cdot\overline{X})$?

$$\varphi(\overline{X\cdot Y}) = \overline{\varphi(X\cdot Y)}^T$$

$$\varphi(\overline{Y}\cdot\overline{X}) = \varphi(\overline{Y})\cdot\varphi(\overline{X}) = \overline{\varphi(Y)}^T\cdot\overline{\varphi(X)}^T = (\overline{\varphi(X)}\cdot\overline{\varphi(Y)})^T = \overline{\varphi(X)\cdot\varphi(Y)}^T$$

$$= \overline{\varphi(X\cdot Y)}^T.$$ 」

$\forall X \in \mathbb{H}$，考虑 $X \cdot \overline{X}$，

$$\overline{(X \cdot \overline{X})} = \overline{(\overline{X})} \cdot \overline{X} = X \cdot \overline{X}$$

$\Rightarrow X \cdot \overline{X}$ 为一个实数。

设 $X = a + b\vec{i} + c\vec{j} + d\vec{k}$.

$$X \cdot \overline{X} = (a + b\vec{i} + c\vec{j} + d\vec{k})(a - b\vec{i} - c\vec{j} - d\vec{k})$$

$$= a^2 + b^2 + c^2 + d^2 \geq 0$$

定义 X 的模长：$|X| := \sqrt{X \cdot \overline{X}} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

$|X| = 0 \iff X = 0$.

$\forall X \in \mathbb{H}$，$X \neq 0 \Rightarrow |X| \neq 0$.

$$X \cdot \frac{\overline{X}}{|X|^2} = 1. \Rightarrow X \text{ 可逆}.$$

$\Rightarrow \mathbb{H}$ $(\mathbb{H}')$ 为一个 $\mathbb{R}$-可除代数 (Hamilton 四元数代数)

小结：$\mathbb{H}$ ($\mathbb{H}'$) 为 $\mathbb{R}$-可除代数

$\varphi: \mathbb{H} \longrightarrow \mathbb{H}'$ 为一个 $\mathbb{R}$-代数同构.

$\varphi(\overline{X}) = \overline{\varphi(X)}^{T}$, $\forall X \in \mathbb{H}$

$\overline{X \cdot Y} = \overline{Y} \cdot \overline{X}$, $\forall X, Y \in \mathbb{H}$.

$|X|^2 = \det \varphi(X)$.

$\Rightarrow |X \cdot Y| = |X| \cdot |Y|$

$$\begin{array}{ccc} \| & & \| \\ \sqrt{\det \varphi(X \cdot Y)} & & \sqrt{\det \varphi(X) \cdot \det \varphi(Y)}. \\ \| & // & \\ \sqrt{\det(\varphi(X)\,\varphi(Y))} & & \end{array}$$

设 $X = a + b\vec{i} + c\vec{j} + d\vec{k}$, $|X| = \sqrt{a^2 + b^2 + c^2 + d^2}$

$\varphi(X) = \begin{pmatrix} a+bi & c+di \\ -\overline{c+di} & \overline{a+bi} \end{pmatrix}$
$\det \varphi(X) = |a+bi|^2 + |c+di|^2$
$= a^2 + b^2 + c^2 + d^2$
$= |X|^2$.

考虑 $Sp(1) = \{ X \in \mathbb{H} \mid |X| = 1 \}$.

$$= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathbb{H} \;\middle|\; |a|^2 + |b|^2 = 1 \right\}$$

$\because$ $Sp(1) \times Sp(1) \longrightarrow Sp(1)$. $\quad |X \cdot Y| = |X| \cdot |Y| = 1$

$(X, Y) \longmapsto X \cdot Y$ ← 四元数乘法

$\Rightarrow (Sp(1), \cdot)$:

① 结合律

② 有幺元: $1 \in Sp(1)$ $\quad 1 \cdot X = X \cdot 1 = X$.

③ 有逆元: $\forall X \in Sp(1)$, $\quad X^{-1} \in \mathbb{H}$.

$$X^{-1} = \frac{\overline{X}}{|X|^2} = \overline{X}, \Rightarrow X^{-1} \in Sp(1).$$

$\Rightarrow Sp(1)$ 为一个群 (单位四元数群).

$\forall \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in Sp(1)$, $\Longrightarrow \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$.

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \cdot \overline{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}}^T = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \cdot \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} = \begin{pmatrix} |a|^2+|b|^2 & 0 \\ 0 & |b|^2+|a|^2 \end{pmatrix}$$

反之，证：$SU(2) \subset Sp(1)$.

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$.

$$\begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \\ ad - bc = 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} |a|^2 + |b|^2 = 1 = |c|^2 + |d|^2 . \\ a\bar{c} + b\bar{d} = 0 \ (\Leftrightarrow \bar{a}c + \bar{b}d) \\ \underline{ad - bc = 1} \end{cases}$$

$$\bar{a} \cdot ad - \bar{a} \cdot bc = \bar{a} \qquad \Longrightarrow \bar{a} = d .$$

$$\overset{\parallel}{}$$

$$|a|^2 d - (-\bar{b}d) \cdot b = d \cdot (|a|^2 + |b|^2) = d$$

$$\bar{c}\, ad - b \cdot |c|^2 = \bar{c}$$

$$-b \cdot |d|^2 - b \cdot |c|^2 = \bar{c} \Longrightarrow -b = \bar{c} .$$

$$\overset{\parallel}{}$$

$$-b(|d|^2 + |c|^2)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in Sp(1) .$$

结论：$Sp(1) = SU(2)$.

$$Sp(1) \subset \mathbb{H} = \{a + b\vec{i} + c\vec{j} + d\vec{k} \mid a, b, c, d \in \mathbb{R}\} \xrightarrow{\cong} \mathbb{R}^4$$ ℝ-linear

$$a + b\vec{i} + c\vec{j} + d\vec{k} \xmapsto{\tau} (a, b, c, d).$$

赋予子空间拓扑.

$Sp(1)$ 为一个拓扑群.

i.e. $Sp(1) \times Sp(1) \longrightarrow Sp(1)$.

$(X, Y) \longmapsto X \cdot Y$

$Sp(1) \longrightarrow Sp(1) \quad X \longmapsto X^{-1}$

都连续.

留作习题

$\tau: Sp(1) \hookrightarrow \mathbb{R}^4$

$$\tau(Sp(1)) = \{(a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 = 1\}$$

$$= S^3$$

$$Sp(1) \cong S^3 \cong SU(2)$$

Rmk. $\{S^n\}_{n\geq 1}$ 中具有拓扑群结构的，只有：

$$S^1 \text{ 和 } S^3$$

下面：将构造满的拓扑群同态：

$$\Theta: Sp(1) \longrightarrow SO(3).$$

称：$X = a + b\vec{i} + c\vec{j} + d\vec{k} \in \mathbb{H}$，为纯四元数，if $a = 0$

记 $\mathrm{Im}\mathbb{H} = \{\text{纯四元数}\}$

$\forall X \in \mathrm{Im}\mathbb{H}$，$X = x\vec{i} + y\vec{j} + z\vec{k}$.

$$X^2 = (x\vec{i} + y\vec{j} + z\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$= -x^2 - y^2 - z^2$ 为一个非正实数.

Claim：$\mathrm{Im}\mathbb{H} = \{X \in \mathbb{H} \mid X^2 \leq 0\}$

Pf. "$\subset$"

"$\supset$" $\forall X = a + b\vec{i} + c\vec{j} + d\vec{k} \in \mathbb{H}$, $X^2 \leqslant 0$.

$$(a + (b\vec{i} + c\vec{j} + d\vec{k}))^2$$

$$= \underbrace{a^2}_{\in \mathbb{R}} + \underline{2a(b\vec{i} + c\vec{j} + d\vec{k})} + \underbrace{(b\vec{i} + c\vec{j} + d\vec{k})^2}_{\leqslant 0} \leqslant 0.$$

$$\Rightarrow 2a(b\vec{i} + c\vec{j} + d\vec{k}) = 0.$$

若 $a \neq 0$. $\Rightarrow b = c = d = 0$. $\Rightarrow a^2 \leqslant 0 \Rightarrow a = 0$

矛盾.

$\Rightarrow a = 0 \Rightarrow X \in \operatorname{Im}\mathbb{H}$. #

$$\underline{\operatorname{Im}\mathbb{H} = \{b\vec{i} + c\vec{j} + d\vec{k} \mid b, c, d \in \mathbb{R}\}} \overset{\mathbb{R}\text{-linear}}{\cong} \mathbb{R}^3$$

$\forall X \in Sp(1)$, 定义 $\theta_X : \operatorname{Im}\mathbb{H} \longrightarrow \operatorname{Im}\mathbb{H}$.

$$Y \longmapsto X \cdot Y \cdot X^{-1}.$$

$$(X \cdot Y \cdot X^{-1})^2 = X \cdot Y \cdot \overset{1}{\overset{\|}{(X^{-1} \cdot X)}} \cdot Y \cdot X^{-1} = X \cdot \underbrace{Y^2}_{\leqslant 0} \cdot X^{-1} = Y^2 \leqslant 0.$$

$\Rightarrow \theta_X$ is well-defined. θ_X: 为一个 $\mathbb{R}$-线性映射.

(1) $\theta_X \circ \theta_Y = \theta_{X \cdot Y}$ $\quad \forall X, Y \in Sp(1)$.

(2) $\theta_{X^{-1}} = (\theta_X)^{-1}$.

⊢ (1). $\forall Z \in \mathrm{Im}\,\mathbb{H}$. $\theta_X \circ \theta_Y(Z) = \theta_X(\theta_Y(Z))$

$$= \theta_X(Y \cdot Z \cdot Y^{-1}) = X \cdot (Y \cdot Z\, Y^{-1}) \cdot X^{-1}.$$

$$\theta_{X \cdot Y}(Z) = (X \cdot Y) \cdot Z \cdot (X \cdot Y)^{-1}$$

$$= X \cdot Y \cdot Z \cdot Y^{-1} \cdot X^{-1}$$

(2). $\underset{\theta_1.}{\underset{\parallel}{\theta_{X^{-1}} \circ \theta_X}} = id.$ $\qquad \underset{\theta_1.}{\underset{\parallel}{\theta_X \circ \theta_{X^{-1}}}} = id.$

$$\theta_1(Z) = 1 \cdot Z \cdot 1^{-1} = Z.$$

$\boxed{\text{目标：} \ \theta : Sp(1) \to SO(3)}$

有映射： $\theta : Sp(2) \longrightarrow GL_3(\mathbb{R})$

$\theta \longmapsto \theta_X$. （$\theta_X : \mathrm{Im}\,\mathbb{H} \to \mathrm{Im}\,\mathbb{H}$，$\mathrm{Im}\,\mathbb{H}$取基后，在该基下$\theta_X$的矩阵表示。）

(1) $\Longrightarrow$ θ 为一个群同态.

下面要证：$\forall x\in Sp(1)$，$\Theta_x\in SO(3)$.

Step 1. $\underline{\Theta_x\in O(3)}$.

$\Theta_x: \mathrm{Im}\mathbb{H}\to\mathrm{Im}\mathbb{H}$ 在 $\{\vec{i},\vec{j},\vec{k}\}$ 下的矩阵表示.

$\mathrm{Im}\mathbb{H}=\mathbb{R}\vec{i}+\mathbb{R}\vec{j}+\mathbb{R}\vec{k}\cong\mathbb{R}^3$

定义内积：$\langle\cdot,\cdot\rangle: \mathrm{Im}\mathbb{H}\times\mathrm{Im}\mathbb{H}\to\mathbb{R}$.

$\langle(x_1,x_2,x_3),(y_1,y_2,y_3)\rangle=x_1y_1+x_2y_2+x_3y_3$.

$\{\vec{i},\vec{j},\vec{k}\}$ 构成了一组标准正交基.

只要证：$\Theta_x: \mathrm{Im}\mathbb{H}\to\mathrm{Im}\mathbb{H}$ 保内积.

只要证：$\forall Y\in\mathrm{Im}\mathbb{H}$，$\|\Theta_x(Y)\|=\|Y\|$ $\left(=\sqrt{\langle Y,Y\rangle}\right)$

又 $\forall Y\in\mathrm{Im}\mathbb{H}$，$\|Y\|=|Y|$（作为四元数模长）.

只要证：$\underline{|\Theta_x(Y)|=|Y|,\ \forall Y\in\mathrm{Im}\mathbb{H}}$. OK!

$|\Theta_x(Y)|=|x\cdot Y\cdot x^{-1}|=|x|\cdot|Y|\cdot|x^{-1}|=|Y|$

Step 2. 下证 $\Theta(Sp(1)) \subset SO(3)$. $\Rightarrow$ $\det(\Theta(Sp(1))) \subset \{\pm 1\}$.

Step 1 已证: $\Theta(Sp(1)) \subset O(3)$.

$\therefore$ 只要证: $\det(\Theta(Sp(1))) = \{1\}$.

$\det \circ \Theta : Sp(1) \xrightarrow{\boxed{\Theta}} O(3) \xrightarrow{\det} \mathbb{R}$.

也连续. 连续.

注意到 $Sp(1) \cong S^3$ (连通).

$\det \circ \Theta\ (Sp(1))$ 连通.

$\Rightarrow \det(\Theta(Sp(1))) \neq \{\pm 1\}$.

$\Rightarrow \det(\Theta(Sp(1))) = 1$ or -1.

又 $\det(\Theta(1)) = 1 \Rightarrow \det(\Theta(Sp(1))) = \{1\}$

$\Rightarrow$ $\boxed{\Theta : Sp(1) \longrightarrow SO(3)}$

练习: $\Theta : Sp(1) \longrightarrow GL_3(\mathbb{R}) \subset \mathbb{R}^{3^2}$ 连续.

$\Rightarrow \quad \Theta: Sp(1) \longrightarrow SO(3)$ 为拓扑群同态.

Claim: (1) $\ker\Theta = \{\pm 1\}$

(2) $\operatorname{Im}\Theta = SO(3)$ $\cong \mathbb{R}^3$

(1). $\forall X \in \ker\Theta$. $\Rightarrow \forall Y \in \operatorname{Im}\mathbb{H}$. $\Theta_X(Y) = Y$

$\Leftrightarrow$

$$\left\{\begin{array}{l} \boxed{\begin{array}{l}\Theta_X(\vec{i}) = \vec{i} \\ \Theta_X(\vec{j}) = \vec{j}\end{array}} \\ \Theta_X(\vec{k}) = \vec{k} \end{array}\right.$$

$$\Theta_X(\vec{k}) = \Theta_X(\vec{i}\cdot\vec{j}).$$

$$= X\cdot\vec{i}\cdot\vec{j}\cdot X^{-1}$$

$$= (X\cdot\vec{i}\cdot X^{-1})\cdot \underline{X\cdot\vec{j}\cdot X^{-1}} = \vec{i}\cdot\vec{j} = \vec{k}.$$

$$X\cdot\vec{i}\cdot X^{-1} = \vec{i}, \quad X\cdot\vec{j}\cdot X^{-1} = \vec{j}$$

$$\left\{\begin{array}{l} X\cdot\vec{i}\cdot X^{-1} = \vec{i} \\ X\cdot\vec{j}\cdot X^{-1} = \vec{j} \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} X\cdot\vec{i} = \vec{i}\cdot X \\ X\cdot\vec{j} = \vec{j}\cdot X \end{array}\right. \qquad X = a + b\vec{i} + c\vec{j} + d\vec{k}.$$

$$\left\{\begin{array}{l} (a+b\vec{i}+c\vec{j}+d\vec{k})\cdot\vec{i} = \vec{i}\cdot(a+b\vec{i}+c\vec{j}+d\vec{k}). \Rightarrow \left\{\begin{array}{l} c = 0 \\ d = 0 \end{array}\right. \\ \quad - \quad - \quad - \quad - \quad - \quad - \qquad \Rightarrow b = 0. \end{array}\right.$$

$$\Rightarrow X = 1 \text{ or } -1. \Rightarrow \ker\Theta = \{1, -1\}.$$

(2). $\theta: Sp(1) \to SO(3)$ 为一个满射（留作习题）.

「提示：$\{\vec{i}, \vec{j}, \vec{k}\} \subset \mathrm{Im}\mathbb{H}$ 为一组基.

$\theta(x) = \theta_x$ 在 $\{\vec{i}, \vec{j}, \vec{k}\}$ 下的矩阵表示. 」

$\Rightarrow \quad Sp(1)/\{\pm 1\} \cong SO(3)$ (as groups).

希望：$Sp(1)/\{\pm 1\} \cong SO(3)$ (as top groups).

Lemma. 设 G 为一个拓扑群，$H \lhd G$，H 为闭子群，

则 G/H 在商拓扑下构成一个拓扑群.

Pf. 要证：① G/H 是 Hausdorff.

② $\left\{\begin{array}{l} G/H \times G/H \xrightarrow{\bar{m}} G/H \quad (\bar{g}_1, \bar{g}_2) \mapsto \overline{g_1 g_2} \\ G/H \xrightarrow{\bar{i}} G/H, \quad \bar{g} \mapsto \overline{g^{-1}} \end{array}\right.$

连续

先证②: $G/H \times G/H \xrightarrow{\bar{m}} G/H$ 连续.

$$\begin{tikzcd}
G/H \times G/H \arrow[r, "\bar{m}"] & G/H \supset U \\
G \times G \arrow[u, "\pi\times\pi"] \arrow[r, "m"] & G \arrow[u, "\pi"]
\end{tikzcd}$$

只要证: $\forall\, U \underset{open}{\subset} G/H$, $\bar{m}^{-1}(U)$ 为开集.

Claim: $\pi\times\pi$ 为一个开映射.

若 claim 对, 则等价于证 $(\pi\times\pi)^{-1}(\bar{m}^{-1}(U))$ 为开集, 而由图表交换性, 这是显然的.

①: $\pi: G \to G/H$. $\forall\, U \underset{open}{\subset} G$
$\pi(U)$ open $\Longleftrightarrow \pi^{-1}(\pi(U)) \underset{open}{\subset} G$.
只要证 $\pi^{-1}(\pi(U))$ 开, $\pi^{-1}(\pi(U)) = \bigcup_{h\in H} h\cdot U$.
"$\supset$" 显然.
"$\subset$" $\forall\, x \in LHS$, $\pi(x) = \pi(y)$, $y \in U$, $\pi(x\cdot y^{-1}) = \bar{e}$ $\Rightarrow x\cdot y^{-1} \in H \Rightarrow x \in Hy \Rightarrow x \in RHS$

proof of Claim: ① π 为一个开映射,

② 开映射之乘积为开映射.

「$f: X_1 \to Y_1$, $g: X_2 \to Y_2$ open maps.
$f\times g: X_1\times X_2 \to Y_1 \times Y_2$
要证 $f\times g$ 开 只要证: $\forall\, U \underset{open}{\subset} X_1$, $V \underset{open}{\subset} X_2$.
$f\times g(U\times V) \overset{开}{=\!=} f(U)\times g(V)$.」

下证：G/H 是 Hausdorff 空间

Lemma：设 G 为一个拓扑空间，且 G 为一个群，且 G 的群运算连续，则下列命题等价：

(1) G 为一个 Hausdorff 空间。

(2) $\{e\} \subset G$ 为闭子集

Pf. (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1).

$$\varphi: G \times G \longrightarrow G \quad \text{连续}.$$
$$(g, h) \longmapsto g \cdot h^{-1}.$$

$$G \times G \longrightarrow G \times G \xrightarrow{m} G$$
$$(g, h) \longmapsto (g, h^{-1}) \longmapsto g \cdot h^{-1}$$

$\varphi^{-1}(e)$ 为 $G \times G$ 中闭集，

$$\varphi^{-1}(e) = \{(g, g) \mid g \in G\} \quad \#$$

习题：设 X 为 top sp. 则 X Hausdorff $\Leftrightarrow \Delta = \{(x, x) \mid x \in X\} \subset X \times X$ closed.

由引理，只要证：$\underline{\{\bar{e}\} \subset G/H \text{ 为闭集}}$.

$\pi: G \to G/H$ $\Updownarrow$.

$\underline{\pi^{-1}(\bar{e})}$ 为闭集

$\parallel$

H.

$\Rightarrow$ G/H Hausdorff

(1) (2) $\Rightarrow$ G/H 为一个 top group. #.

群同态第一基本定理（拓扑版本）.

设 G、H 为 top. group. $\varphi: G \to H$ 为满的群同态、φ 为粘合映射，则 $\exists!$ $\bar{\varphi}$: $\underline{G/\ker\varphi}$ $\to H$ 为拓扑群同构，使下面图表交换：

$$\begin{tikzcd} & G \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\ G/\ker\varphi \arrow[rr, "\exists!\ \bar{\varphi}"] & & H \end{tikzcd}$$

Pf. $\exists!$ $\bar{\varphi}$ 为群同构使左表交换

$\bar{\varphi}$ 连续显然，只需证 $\bar{\varphi}^{-1}$ 连续，只要证：$\forall U \underset{\text{open}}{\subset} G/\ker\varphi$

$\bar{\varphi}(U)$ 开 $\Leftrightarrow$ $\underline{\varphi^{-1}(\bar{\varphi}(U))}$ 开

$= \pi^{-1}(U)$ #

已知：Θ：$Sp(1) \longrightarrow SO(3)$ 满的拓扑群同态。

$Sp(1) \cong S^3$ 紧，$SO(3)$ Hausdorff

还需验证：Θ 为粘合映射 OK！

$$\Rightarrow Sp(1)/\{\pm 1\} \cong SO(3) \text{ (as topological groups)}$$

$$Sp(1)/\{\pm 1\} = Sp(1)/\sim$$

$a+b\vec{i}+c\vec{j}+d\vec{k}$ 所代表的陪集为：

$$\{a+b\vec{i}+c\vec{j}+d\vec{k},\ -a-b\vec{i}-c\vec{j}-d\vec{k}\}$$

$$a+b\vec{i}+c\vec{j}+d\vec{k} \sim -(a+b\vec{i}+c\vec{j}+d\vec{k}).$$

$$Sp(1) \xrightarrow{\cong} S^3$$

$$a+b\vec{i}+c\vec{j}+d\vec{k} \longmapsto (a,b,c,d)$$

"$\sim$" 诱导 S^3 上的等价关系为：

$$(a,b,c,d) \sim (-a,-b,-c,-d)$$

$$Sp(1)/\sim \ \cong S^3/\text{对径点} \cong \mathbb{R}P^3$$

$$SO(3) \cong \mathbb{RP}^3.$$

总结：

$$\begin{tikzcd}
Sp(1) \cong SU(2) \arrow[d, "\theta"] & \cong & S^3. \arrow[d, "\pi"] \\
SO(3) & \cong & \mathbb{RP}^3.
\end{tikzcd}$$