§2. 一点同调代数.

回顾：

$\varphi: A \to B$ 为 Abel 群群同态.

$\ker\varphi = \{x\in A \mid \varphi(x)=0\}$. $\mathrm{Im}\,\varphi = \{\varphi(x) \mid x\in A\}$.

$\mathrm{coker}\,\varphi := B/\mathrm{Im}\,\varphi$, $\mathrm{coIm}\,\varphi := A/\ker\varphi$.

考虑一列 Abel 群群同态：

$$\cdots\cdots \longrightarrow A_{i-1} \xrightarrow{\phi_{i-1}} A_i \xrightarrow{\phi_i} A_{i+1} \longrightarrow \cdots \qquad (*)$$

称 $(*)$ 是在 A_i 处正合的 (exact)，若 $\mathrm{Im}\,\phi_{i-1} = \ker\phi_i$.

称 $(*)$ 是一个正合列 (exact sequence)，若 $\forall i$，$(*)$ 在 A_i 处正合.

规定：对于 $A_0 \to A_1 \to A_2 \to \cdots\cdots$ (1)，称 (1) 为一个正合列若 $\forall i \geq 1$，(1) 在 A_i 处正合.

对于 $\cdots\cdots \to A_{-2} \to A_{-1} \to A_0$ (2)，称 (2) 为一个正合列，若 $\forall i \leq -1$，(2) 在 A_i 处正合.

Rmk. 记 0 为平凡 Abel 群，则：

$A \xrightarrow{\phi} B \to 0$ 为正合列 $\Longleftrightarrow$ ϕ 为满同态

$0 \to A \xrightarrow{\phi} B$ 为正合列 $\Longleftrightarrow$ ϕ 为单同态.

定义. 若 $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ 正合，则称之为一个短正合列 (short exact sequence).

Rmk. 设 $B \xrightarrow{\psi} C$ 为一个满同态，$K = \ker\psi$. 记 $K \xrightarrow{i} B$ 为包含映射. 于是 $0 \to K \xrightarrow{i} B \xrightarrow{\psi} C \to 0$ 为一个短正合列.

反过来，设 $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ 为一个短正合列，

$\Rightarrow$ ϕ 为单同态，ψ 为满同态，$\phi(A) = \ker\psi$.

$\Rightarrow$ $\phi: A \to \ker\psi$ 为群同构.

例. 设 A, B 为 Abel 群，则

$$0 \to A \xrightarrow{i} A \oplus B \xrightarrow{pr_2} B \to 0 \quad \text{为短正合列.}$$
$$a \longmapsto (a, 0)$$
$$(a, b) \longmapsto b.$$

例：$0 \to \mathbb{Z} \xrightarrow{\varphi_n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$， $n \in \mathbb{Z}_{\geq 1}$.

$k \mapsto k \cdot n$.

为短正合列.

定义：设 $\cdots \to A_i \xrightarrow{\varphi_i} A_{i+1} \to \cdots$ ($A_\bullet$) 为两列 Abel 群同态.

$\cdots \to B_i \xrightarrow{\psi_i} B_{i+1} \to \cdots$ ($B_\bullet$)

从 $A_\bullet$ 到 $B_\bullet$ 的一个态射 $\alpha: A_\bullet \to B_\bullet$ 是指一系列群同态

$\alpha_i: A_i \to B_i$，$\forall i \in \mathbb{Z}$，使下列图表交换：

$$\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, "\varphi_i"] \arrow[d, "\alpha_i"] & A_{i+1} \arrow[r] \arrow[d, "\alpha_{i+1}"] & \cdots \\
\cdots \arrow[r] & B_i \arrow[r, "\psi_i"] & B_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

若 $\forall i$，α_i 为群同构，则称 $\alpha: A_\bullet \to B_\bullet$ 为一个同构.

定义：设 $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ 为一个短正合列，称它是分裂的 (split)，

若 $\exists$ 群同构 $\theta: B \to A \oplus C$，使：

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "="] & B \arrow[r, "\psi"] \arrow[d, "\theta"] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$
为两个短正合列之同构.

Rmk. 设 $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ 为分裂的短正合列, θ 同上.

令 $i: A \to A \oplus C$, $a \mapsto (a, 0)$, $j: C \to A \oplus C$, $c \mapsto (0, c)$ 为典则嵌入. 则

$A \oplus C = i(A) \oplus j(C)$ (内直和).

$\theta^{-1}: A \oplus C \xrightarrow{\cong} B$

$\Rightarrow B = \theta^{-1}(i(A)) \oplus \theta^{-1}(j(C))$

$\theta^{-1}(i(A)) = \theta^{-1} \circ i(A) = \phi(A)$.

$\Rightarrow B = \phi(A) \oplus \underbrace{\theta^{-1}(j(C))}_{D}$.

$\psi|_D: D \to C$ 为群同构.

"$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ split" $\Longrightarrow$ "$B = \phi(A) \oplus D$ (内直和), 且 $\psi|_D: D \cong C$".

反之, 若 $B = \phi(A) \oplus D$, 其中 $\psi|_D: D \xrightarrow{\cong} C$.

定义 $\theta: B \longrightarrow A \oplus C$，则

$$\phi(a)+d \longmapsto (a, \psi(d))$$

$(a \in A, d \in D)$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "="] & B \arrow[r, "\psi"] \arrow[d, "\theta"] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$\phi(a)+d \longmapsto \psi(d)$

$(a, \psi(d)) \longmapsto \psi(d)$.

交换。

因此：对于短正合列 $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ $(*)$，

"$(*)$ split" $\Longleftrightarrow$ "$B = \phi(A) \oplus D$，其中 $\psi|_D : D \xrightarrow{\cong} C$"。

Rmk. 设 $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ 为 split 短正合列，则上述 D 不唯一。

例：$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{pr_2} \mathbb{Z} \to 0$　　$\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus j(\mathbb{Z})$

$n \longmapsto (n, 0)$

$(n, m) \longmapsto m$.　　$j: \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$

$n \longmapsto (0, n)$

同时，若令 $D = \{(n, n) \mid n \in \mathbb{Z}\} < \mathbb{Z} \oplus \mathbb{Z}$. $\mathbb{Z} \oplus \mathbb{Z} = i(\mathbb{Z}) \oplus D$ ⌟

例.(不分裂的短正合列).

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0 \quad 不分裂.$$

「若分裂,则

$$\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}. \quad 矛盾 」$$

$$k \longleftarrow (0, 1)$$

Lemma (splitting lemma). 设 $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ 为短正合列,

则下列等价:

(1) 该短正合列分裂.

(2) $\exists p: B \to A$, s.t. $p \circ \phi = Id_A$.

(3) $\exists q: C \to B$, s.t. $\psi \circ q = Id_C$.

证明: (1) $\Rightarrow$ (2).

设 $\theta: B \xrightarrow{\cong} A \oplus C$ 为使下面图表交换的同构:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "="] & B \arrow[r, "\psi"] \arrow[d, "\theta"] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & A \arrow[r, "i"] & A \oplus C \arrow[r] \arrow[l, bend left, "pr_1"] & C \arrow[r] \arrow[l, bend left, "j"] & 0
\end{tikzcd}$$

$p = pr_1 \circ \theta$.

$p \circ \phi = pr_1 \circ \theta \circ \phi = pr_1 \circ i$

$= Id_A$

(1) $\Rightarrow$ (3): Θ 同上.

$$\begin{tikzcd}
\Theta^{-1}(j(c)) \in B \arrow[r, "\psi"] \arrow[d, "\Theta"'] & C \arrow[d, equal] \\
A \oplus C \arrow[r, "pr_2"] & C \arrow[l, bend left, "j"]
\end{tikzcd}$$

令 $q = \Theta^{-1} \circ j$.

看 $\psi \circ q \overset{?}{=} Id_C$.

$\forall c \in C$, $\psi \circ q(c) = \psi(\Theta^{-1}(j(c)))$

$$= pr_2 \circ \Theta(\Theta^{-1}(j(c))),$$

$$= pr_2(j(c)) = c.$$

因此 $\psi \circ q = Id_C$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi"] & B \arrow[r, "\psi"] \arrow[l, bend left, "p"] & C \arrow[r] & 0
\end{tikzcd}$$

b

$p \circ \phi = id_A$.

$p(b - \phi(p(b)))$

$= p(b) - p \circ \phi(p(b))$

$= p(b) - p(b) = 0$

(2) $\Rightarrow$ (1).

Claim: $B = \phi(A) \oplus \ker p$.

pf of claim:

$\forall b \in B$, $\phi(p(b)) \in \phi(A)$.

$$b = \underbrace{\phi(p(b))}_{\in \phi(A)} + \underbrace{(b - \phi(p(b)))}_{\in \ker p}.$$

设 $b = \phi(a_1) + d_1 = \phi(a_2) + d_2$, $a_1, a_1 \in A$, $d_1, d_2 \in \ker p$.

$\phi(a_1 - a_2) = d_2 - d_1$. $\Rightarrow p(\phi(a_1 - a_2)) = 0 \Rightarrow a_1 = a_2$, $d_1 = d_2$ $\lrcorner$

Claim: $\psi|_{\ker p}: \ker p \longrightarrow C$ 为群同构.

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(3) $\Rightarrow$ (1).

$$0 \to A \xrightarrow{\phi} B \underset{q}{\overset{\psi}{\rightleftarrows}} C \to 0$$

$$\psi \circ q = \mathrm{Id}_C$$

Claim: $B = \phi(A) \oplus q(C)$.

⌈pf of claim:

$\forall b \in B$, $b = \underbrace{(b - q(\psi(b)))}_{\in \ker\psi = \operatorname{Im}\phi} + \underbrace{q(\psi(b))}_{\in q(C)}$

$\Rightarrow B = \phi(A) + q(C)$.

类似可证: $B = \phi(A) \oplus q(C)$ (留做习题). #

Lemma (Snake lemma). 考虑 Abel 群范畴中的交换图表:

$$\begin{array}{ccccccc} & & L' & \xrightarrow{\phi} & L & \xrightarrow{\psi} & L'' & \longrightarrow 0 \\ & & \downarrow u' & & \downarrow u & & \downarrow u'' & \\ 0 & \longrightarrow & M' & \xrightarrow{\varepsilon} & M & \xrightarrow{\theta} & M'' & \end{array} \quad (*)$$

其中两行均为正合列, 记 $\ker u' \to \ker u \to \ker u''$,

$\mathrm{coker}\,u' \to \mathrm{coker}\,u \to \mathrm{coker}\,u''$ 为自然群同态，则有连接同态 $\delta: \ker u'' \to \mathrm{coker}\,u'$，使下面的同态序列为正合列：

$$\ker u' \to \ker u \to \ker u'' \xrightarrow{\delta} \mathrm{coker}\,u' \to \mathrm{coker}\,u \to \mathrm{coker}\,u''$$

"snake 之解释"：

$$\begin{tikzcd}
 & \ker u' \arrow[r] \arrow[d] & \ker u \arrow[r] \arrow[d] & \ker u'' \arrow[d] \arrow[ddll, bend left, "\delta"] & \\
 & L' \arrow[r] \arrow[d, "u'"] & L \arrow[r] \arrow[d, "u"] & L'' \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[d] & \\
 & \mathrm{coker}\,u' \arrow[r] & \mathrm{coker}\,u \arrow[r] & \mathrm{coker}\,u'' &
\end{tikzcd}$$

证明：δ 之定义：

$\forall x \in \ker u''$，取 $y \in L$，s.t. $\psi(y) = x$.

由图表交换，$\theta(u(y)) = u''(\psi(y))$
$= u''(x) = 0$

$\Longrightarrow u(y) \in \ker\theta = \mathrm{Im}\,\varepsilon$，$\Rightarrow \exists z \in M'$，s.t. $\varepsilon(z) = u(y)$.

$$\begin{tikzcd}
 & L' \arrow[r, "\varphi"] \arrow[d] & L \arrow[r, "\psi"] \arrow[d] & L'' \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & M' \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & M \arrow[r, "\theta"] & M'' & \\
 & \mathrm{coker}\,u' & & &
\end{tikzcd}$$

$y \mapsto x$，$z \mapsto u(y) \mapsto 0$，$\pi(z)$

定义 $\delta(x) = \pi(z)$.

$$\begin{tikzcd}
 & L' \arrow[r, "\phi"] \arrow[d, "u'"] & L \arrow[r, "\psi"] \arrow[d, "u"] & L'' \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & M' \arrow[r, "\varepsilon"] \arrow[d, "\pi"] & M \arrow[r, "\theta"] & M'' & \\
 & \operatorname{coker} u' & & &
\end{tikzcd}$$

$y \mapsto x \in L''$, $\quad y \mapsto u(y) \mapsto 0$, $\quad z \mapsto u(y)$

$\pi(z) =: \delta(x)$

「δ 是良好定义的：

设另有 $y' \in L$, s.t. $\psi(y') = x$, 另有 $z' \in M'$, s.t. $\varepsilon(z') = u(y')$.

要证. $\pi(z) = \pi(z')$. 即证: $z - z' \in \ker \pi = \operatorname{Im} u'$.

即证: $z - z' \in \operatorname{Im} u'$.

换言之, 要证: $\exists w \in L'$, s.t. $z - z' = u'(w)$.

注意: $y - y' \in \ker \psi = \operatorname{Im} \phi$, $\Rightarrow \exists w \in L'$, s.t. $\phi(w) = y - y'$.

Claim: 这个 w 满足 $u'(w) = z - z'$.

「pf of claim: $\varepsilon(u'(w) - (z - z')) = u(\phi(w)) - (u(y) - u(y')) = 0$,

$\Longrightarrow u'(w) - (z - z') \in \operatorname{Ker}\varepsilon = 0.$

$\Longrightarrow u'(w) = z - z'$ $\lrcorner$

$$\begin{tikzcd}
 & L' \arrow[r, "\phi"] \arrow[d] & L \arrow[r, "\psi"] \arrow[d] & L'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, "\varepsilon"] \arrow[d] & M \arrow[r, "\theta"] & M'' & \\
 & \operatorname{coker} u' & & &
\end{tikzcd}$$

$y_1+y_2 \longmapsto x_1+x_2$

$y_i \longmapsto x_i$

$z_i \longmapsto u(y_i)$

$z_i \longmapsto \pi(z_i)$

下证 δ 为~~解~~同态.

$\forall\ x_1, x_2 \in \operatorname{Ker} u''$.

设 $\psi(y_i) = x_i$, $\varepsilon(z_i) = u(y_i)$.

则 $\delta(x_i) = \pi(z_i)$.

再算 $\delta(x_1 + x_2)$.

注意: $\psi(y_1 + y_2) = x_1 + x_2$

$\varepsilon(z_1 + z_2) = u(y_1 + y_2)$.

$\therefore\ \delta(x_1 + x_2) = \pi(z_1 + z_2)$.

显然有: $\delta(x_1) + \delta(x_2) = \delta(x_1 + x_2)$.

下证正合性:

(i). 在 $\ker u$ 处的正合性： $\ker u' \xrightarrow{\phi|_{\ker u'}} \ker u \xrightarrow{\psi|_{\ker u}} \ker u''$.

$\psi \circ \phi = 0$.

$\Rightarrow \operatorname{Im} \phi|_{\ker u'} \subset \ker \psi|_{\ker u}$.

∴ 只要证： $\ker \psi|_{\ker u} \subset \operatorname{Im} \phi|_{\ker u'}$.

$\forall\, x \in \ker \psi|_{\ker u}$.

$$\begin{tikzcd}
L' \arrow[r, "\phi"] \arrow[d, "u'"] & L \arrow[r, "\psi"] \arrow[d, "u"] & L'' \arrow[r] \arrow[d, "u''"] & 0 \\
y \arrow[r, mapsto] & x \arrow[r, mapsto] & 0 & \\
0 \arrow[r] & M' \arrow[r, "\varepsilon"] & M \arrow[r, "\theta"] & M''
\end{tikzcd}$$

∵ $\psi(x) = 0$, $\Rightarrow \exists\, y \in L'$, s.t. $\phi(y) = x$.

$\varepsilon(u'(y)) = u(\phi(y)) = u(x) = 0$.

$\Rightarrow u'(y) \in \ker \varepsilon = 0 \Rightarrow u'(y) = 0 \Rightarrow y \in \ker u'$.

(ii) 在 $\ker u''$ 处的正合性： $\ker u \xrightarrow{\psi|_{\ker u}} \ker u'' \xrightarrow{\delta} \operatorname{coker} u'$.

要证： $\ker \delta = \operatorname{Im} \psi|_{\ker u}$.

先证： $\operatorname{Im} \psi|_{\ker u} \subset \ker \delta$.

$\forall\, x \in \operatorname{Im} \psi|_{\ker u}$.

不妨设 $x = \psi(y)$，其中 $y \in \ker u$.

由 δ 之定义，$\delta(x) = \pi(0) = 0$.

$\Rightarrow x \in \ker \delta$.

$$\begin{tikzcd}
L' \arrow[r, "\phi"] \arrow[d] & L \arrow[r, "\psi"] \arrow[d] & L'' \arrow[r] \arrow[d] & 0 \\
 & y \arrow[r, mapsto] \arrow[d, mapsto] & x & \\
0 \arrow[r] & M' \arrow[r, "\varepsilon"] & M \arrow[r, "\theta"] & M'' \\
 & 0 \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

再证：$\ker \delta \subset \operatorname{Im} \psi|_{\ker u}$.

$\forall\, x \in \ker \delta$.

由 δ 之定义，不妨设：

$\psi(y) = x$，$\varepsilon(z) = u(y)$，

$u'(w) = z$，$w \in L'$.

$$\begin{tikzcd}
L' \arrow[r, "\phi"] \arrow[d, "u'"'] & L \arrow[r, "\psi"] \arrow[d, "u"'] & L'' \arrow[r] \arrow[d] & 0 \\
w \arrow[d, mapsto] & y \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto] & \\
0 \arrow[r] & M' \arrow[r, "\varepsilon"] & M \arrow[r] & M'' \\
z \arrow[r, mapsto] & u(y) & 0 &
\end{tikzcd}$$

考虑：$y - \phi(w)$.

$$\psi(y - \phi(w)) = \psi(y) - \psi(\phi(w)) = x.$$

$$u(y - \phi(w)) = u(y) - u(\phi(w)) = u(y) - \varepsilon(u'(w))$$
$$= u(y) - \varepsilon(z) = 0.$$

$\therefore\ y - \phi(w) \in \ker u$，$\Rightarrow x \in \operatorname{Im} \psi|_{\ker u}$.

小结: $\ker u' \to \ker u \to \ker u'' \xrightarrow{\delta} \operatorname{coker} u' \to \operatorname{coker} u \to \operatorname{coker} u''$

已证: 正合　　　　正合性留作习题.　　　　#

Lemma (five lemma). 设有Abel范畴中的交换图表:

$$\begin{tikzcd}
L^1 \arrow[r] \arrow[d, "u^1"'] & L^2 \arrow[r] \arrow[d, "u^2"'] & L^3 \arrow[r] \arrow[d, "u^3"'] & L^4 \arrow[r] \arrow[d, "u^4"'] & L^5 \arrow[d, "u^5"'] \\
M^1 \arrow[r] & M^2 \arrow[r] & M^3 \arrow[r] & M^4 \arrow[r] & M^5
\end{tikzcd}$$

假设两行均为正合列, 则: u^1, u^2, u^4, u^5为同构 $\Rightarrow$ u^3为同构.

证明: 令 $Z^i = \ker(L^i \to L^{i+1})$, $\tilde{Z}^i = \ker(M^i \to M^{i+1})$, $i=1,\cdots,4$

$$\begin{tikzcd}
0 \arrow[r] & L^2/Z^2 \arrow[r] \arrow[d, "\overline{u^2}"'] & L^3 \arrow[r] \arrow[d, "u^3"] & Z^4 \arrow[r] \arrow[d, "\tilde{u}^4"] & 0 \\
0 \arrow[r] & M^2/\tilde{Z}^2 \arrow[r] & M^3 \arrow[r] & \tilde{Z}^4 \arrow[r] & 0
\end{tikzcd}$$

对之用Snake lemma.

$\ker \overline{u^2} \to \underline{\ker u^3} \to \ker \tilde{u}^4 \xrightarrow{\delta} \operatorname{coker} \overline{u^2} \to \underline{\operatorname{coker} u^3} \to \operatorname{coker} \tilde{u}^4$

注意 $\overline{u^2}$ 与 $\widehat{u^4}$ 均为同构.

⌜

$$\begin{tikzcd}
 & L^1 \arrow[r, "u^1"] \arrow[d, "f"'] & M^1 \arrow[r] \arrow[d, "g"'] & 0 \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & L^2 \arrow[r, "u^2"] & M^2 \arrow[r] & 0 &
\end{tikzcd}$$

$$\ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h = 0.$$

$\ker h = 0$.

$$\Rightarrow \operatorname{coker} f \cong \operatorname{coker} g$$

$\operatorname{coker} f = L^2/Z^2$, $\operatorname{coker} g = M^2/\tilde{Z}^2$

对

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & L^4 \arrow[r] \arrow[d] & M^4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] & L^5 \arrow[r] & M^5 &
\end{tikzcd}$$

用 snake lemma, $\Longrightarrow \widehat{u^4}$ 为同构 ⌟.

因而

$$\ker \overline{u^2} \to \underline{\ker u^3} \to \ker \widehat{u^4} \xrightarrow{\delta} \operatorname{coker} \overline{u^2} \to \underline{\operatorname{coker} u^3} \to \operatorname{coker} \widehat{u^4}$$

变为: $0 \to \ker u^3 \to 0 \xrightarrow{\delta} 0 \to \operatorname{coker} u^3 \to 0$. $\Rightarrow \ker u^3 = 0$, $\operatorname{coker} u^3 = 0$ #

定义：一个在Abel群范畴中的链复形 (chain complex) 是：

指一族Abel族 C_p, $p \in \mathbb{Z}$, 以及群同态 $\partial_p : C_p \to C_{p-1}$,

$p \in \mathbb{Z}$. 满足 $\partial_{p-1} \circ \partial_p = 0$, $\forall p \in \mathbb{Z}$. 经常记之为：

$$\cdots \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots$$

缩写：$(C_\bullet, \partial_\bullet)$ 或 $C_\bullet$ 或 C

定义. 设 C 为 chain complex. 记：

$$Z_p C := \operatorname{Ker} \partial_p \quad (p\text{-cycles})$$

$$B_p C := \operatorname{Im} \partial_{p+1} \quad (p\text{-boundaries}).$$

$$H_p(C_\bullet) := \operatorname{Ker} \partial_p / \operatorname{Im} \partial_{p+1} \quad (C\text{的第}p\text{个同调群}).$$

例：设 K 为 simplicial complex, 令 $C_p = C_p(K)$, $\partial_p : C_p(K) \to C_{p-1}(K)$

为边界算子，$(C_\bullet, \partial_\bullet)$ 为一个 chain complex.

$\rightsquigarrow$ K 的单纯同调.

定义：设 $C_\bullet$, $D_\bullet$ 为 chain complexes，从 $C_\bullet$ 到 $D_\bullet$ 的一个态射 $u: C_\bullet \to D_\bullet$ 为一族同态 $u_i: C_i \to D_i$, $\forall i$，使下面图表交换：

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_p & \xrightarrow{\partial_p} & C_{p-1} & \longrightarrow \cdots \\
& \downarrow{u_p} & & \downarrow{u_{p-1}} & \\
\cdots \longrightarrow & D_p & \xrightarrow{\partial_p} & D_{p-1} & \longrightarrow \cdots
\end{array}$$

Rmk. 设 $u: C_\bullet \to D_\bullet$ 为一个态射，则 $\forall p$，u 诱导了群同态：$(u_*)_p: H_p(C) \to H_p(D)$.

$\ulcorner$ $u_p(Z_pC) \subset Z_pD$.

$u_p|_{Z_pC}: Z_pC \to Z_pD$. $\pi: Z_pD \to H_p(D)$

$\pi \circ u_p|_{Z_pC}: Z_pC \to H_p(D)$.

注意到：$u_p(B_pC) \subset B_pD$.

$$
\begin{tikzcd}
C_{p+1} \ni y \arrow[r] \arrow[d] & C_p \ni x = \partial_{p+1}(y) \arrow[d, "u_p"] \\
D_{p+1} \arrow[r, "\partial_{p+1}"] & D_p
\end{tikzcd}
$$

$$
\begin{tikzcd}
y \arrow[r, mapsto] \arrow[d, mapsto] & x = \partial_{p+1}(y) \arrow[d, mapsto] \\
u_{p+1}(y) \arrow[r, mapsto] & \partial_{p+1}(u_{p+1}(y)) \in B_p D
\end{tikzcd}
$$

$\Rightarrow B_p C \subset \pi \circ u_p|_{Z_p C}$.

$\therefore$ $\pi \circ u_p|_{Z_p C}$ 诱导了: $Z_p C / B_p C \longrightarrow H_p(D)$

$$Z_p C / B_p C = H_p(C).$$

Rmk. 设 $C_\cdot \xrightarrow{u} D_\cdot \xrightarrow{v} E_\cdot$ 为链复形之态射, 则

$$(v_*)_p \circ (u_*)_p = ((v \circ u)_*)_p : H_p(C) \to H_p(E), \ \forall p.$$

定义: 称 $0 \to L'_\cdot \xrightarrow{u} L_\cdot \xrightarrow{v} L''_\cdot \to 0$ 为链复形范畴中的短正合列, 若 $\forall i$, $0 \to L'_i \xrightarrow{u_i} L_i \xrightarrow{v_i} L''_i \to 0$ 为Abel群范畴中的短正合列.

$$\begin{tikzcd}
0 \arrow[r] & L'_\bullet \arrow[r, "u"] & L_\bullet \arrow[r, "v"] & L''_\bullet \arrow[r] & 0 \\
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & L'_i \arrow[r, "u_i"] \arrow[d] & L_i \arrow[r, "v_i"] \arrow[d] & L''_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'_{i-1} \arrow[r, "u_{i-1}"] \arrow[d] & L_{i-1} \arrow[r, "v_{i-1}"] \arrow[d] & L''_{i-1} \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

Lemma.（短正合列诱导长正合列）.

设 $0 \to L'_\bullet \xrightarrow{u} L_\bullet \xrightarrow{v} L''_\bullet \to 0$ 为短正合列，则存在连接同态 $\delta_i : H_i(L''_\bullet) \longrightarrow H_{i-1}(L'_\bullet)$，$\forall i$，使下面的同态序列为正合列：

$$\begin{tikzcd}
\cdots \arrow[r] & H_i(L') \arrow[r, "(u_*)_i"] & H_i(L) \arrow[r, "(v_*)_i"] & H_i(L') \arrow[dll, "\delta_i"'] \\
& H_{i-1}(L') \arrow[r, "(u_*)_{i-1}"] & H_{i-1}(L) \arrow[r, "(v_*)_{i-1}"] & H_{i-1}(L') \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
L'_i \arrow[r] \arrow[d] & Z_{i-1}L' \\
L'_i/B_iL' \arrow[r] & Z_{i-1}L'
\end{tikzcd}$$

〃

证明：$\forall i$，有交换图表：

$$\begin{tikzcd}
& L'_i/B_iL' \arrow[r] \arrow[d] & L_i/B_iL \arrow[r] \arrow[d] & L''_i/B_iL'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{i-1}L' \arrow[r] & Z_{i-1}L \arrow[r] & Z_{i-1}L''
\end{tikzcd}$$

$$L'_\bullet \xrightarrow{u} L_\bullet \xrightarrow{v} L''_\bullet$$

$$L'_i \xrightarrow{u_i} L_i \longrightarrow L_i/B_iL$$

上面图表每行均正合、

「对
$$\begin{array}{ccccccccc} 0 & \to & L'_{i+1} & \to & L_{i+1} & \to & L''_{i+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L'_i & \to & L_i & \to & L''_i & \to & 0 \end{array}$$
应用 snake lemma.

$$\Rightarrow Z_{i+1}L' \to Z_{i+1}L \to Z_{i+1}L'' \to L'_i/B_iL' \to L_i/B_iL \to L''_i/B_iL''$$
」

因此可对:
$$\begin{array}{ccccccccc} & & L'_i/B_iL' & \to & L_i/B_iL & \to & L''_i/B_iL'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Z_{i-1}L' & \to & Z_{i-1}L & \to & Z_{i-1}L'' & & \end{array}$$

使用 snake lemma:

$$H_i(L') \to H_i(L) \to H_i(L'') \xrightarrow{\delta} H_{i-1}(L') \to H_{i-1}(L) \to H_{i-1}(L'').$$

对所有 $i\in\mathbb{Z}$ 做上述事情, 证毕 #

$$
\begin{tikzcd}
 & \ker u' \arrow[r] \arrow[d] & \ker u \arrow[r] \arrow[d] & \ker u'' \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
 & L' \arrow[r] \arrow[d, "u'"] & L \arrow[r] \arrow[d, "u"] & L'' \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[d] & \\
 & \operatorname{coker} u' \arrow[r] & \operatorname{coker} u \arrow[r] & \operatorname{coker} u'' &
\end{tikzcd}
$$