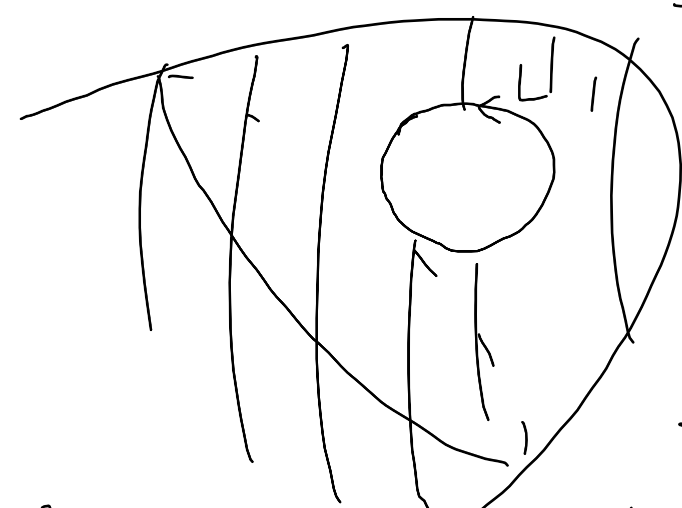
X, Y: top sp. $A \subset X$, $f: A \to Y$ 连续.

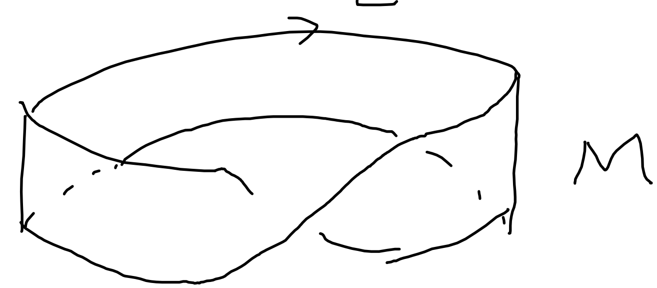
$\boxed{X \sim_f Y} \longrightarrow \boxed{Y \cup_f X}$

上一次：安 crosscap.

$S \setminus D$

↑ 开圆盘

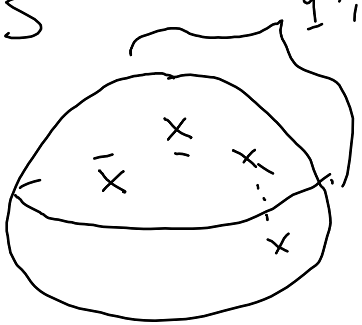




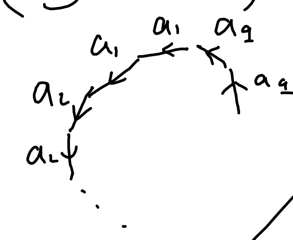
选取一个同胚 $\varphi: L' \to L$.

$(S \setminus D) \cup_\varphi M$. 已证：所得空间与 φ 的选取无关.

S^2: q 个 crosscap.

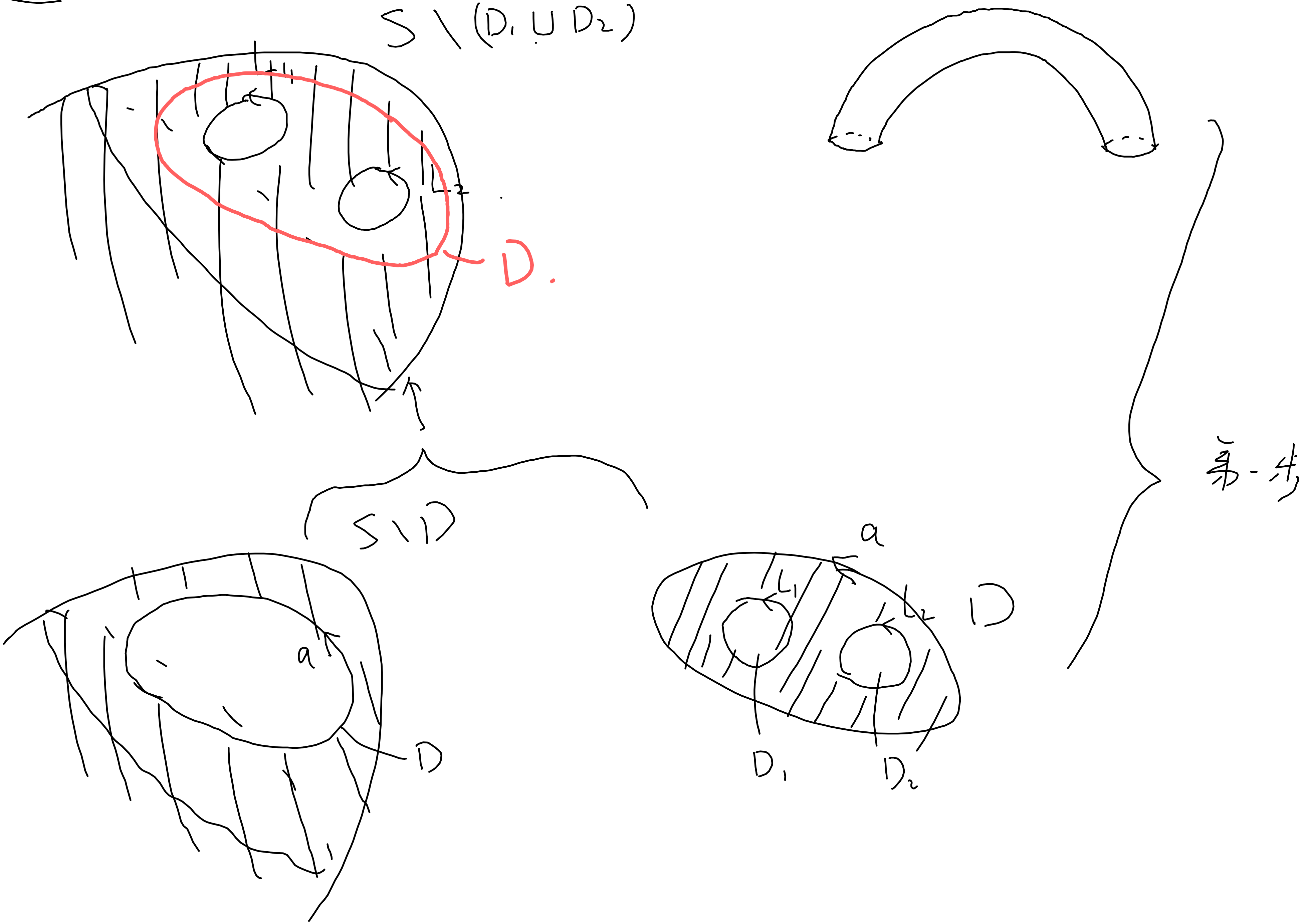


$\cong$

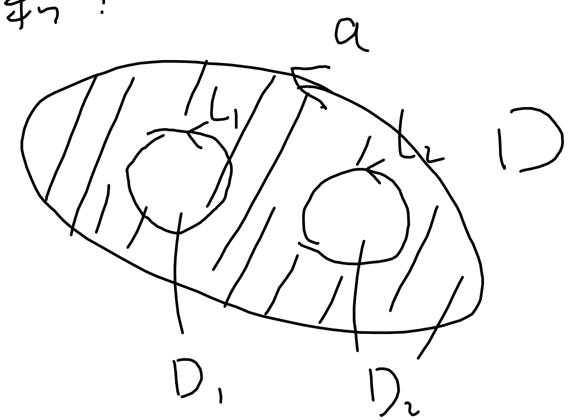


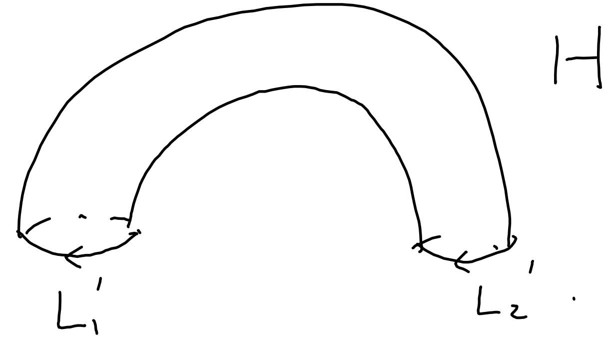
(2-q 边形)

这一次：粘 handles.



第一步，粘：



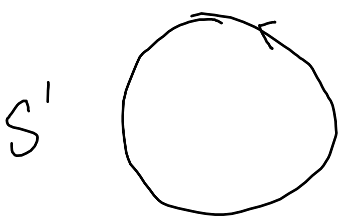


选取：同胚：$\varphi_1 : L_1' \longrightarrow L_1$，　$\varphi_1 \sqcup \varphi_2 : L_1' \sqcup L_2'$

$\varphi_2 : L_2' \longrightarrow L_2$.　$\longrightarrow L_1 \sqcup L_2$.

$$(D \setminus (D_1 \cup D_2)) \bigcup_{\varphi_1 \sqcup \varphi_2} H .$$

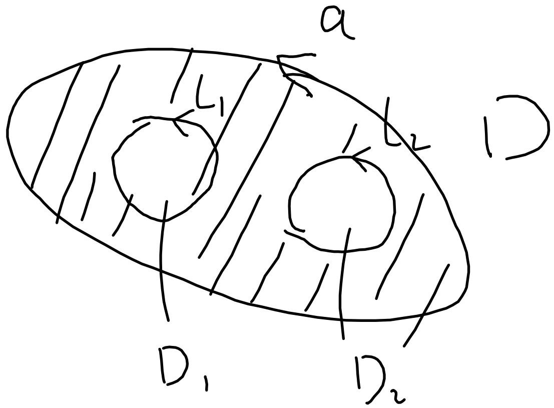
说法：设 $\varphi : S^1 \longrightarrow S^1$ 为一个同胚，S^1 规定一个方向，

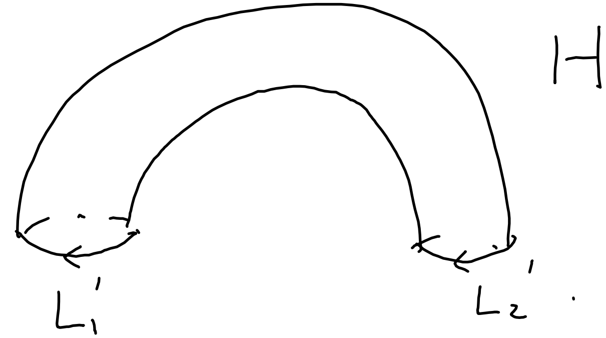


，称 φ 是保定向的，若当 $x \in S^1$ 沿 S^1 的方向运动时，$\varphi(x)$ 也沿已规定的 S^1 的方向运动，反之，称 φ 是反定向的。

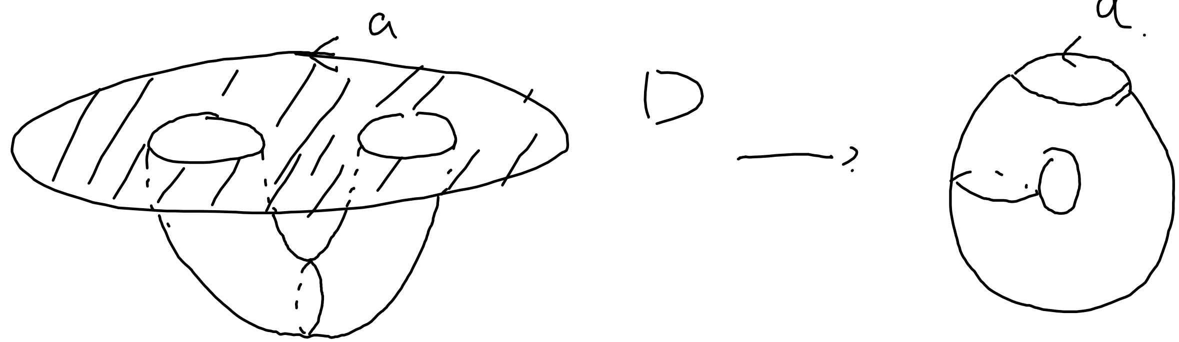
分成四种情况：① φ_1, φ_2 保定向，② φ_1, φ_2 反定向

③ φ_1 保定向，φ_2 反定向 ④ φ_1 反定向，φ_2 保反向





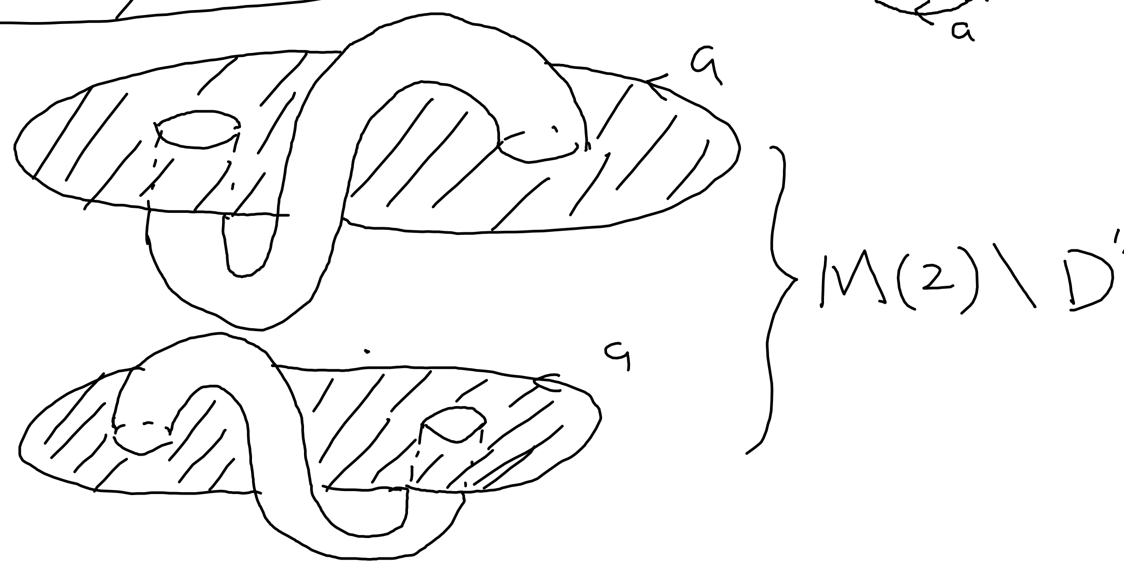
① φ_1、φ_2 保定向.

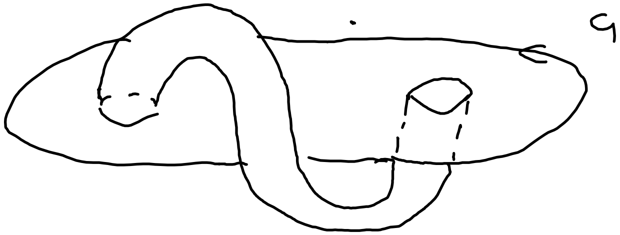


② φ_1、φ_2 反定向：

③ φ_1 保定向，φ_2 反定向：

④ φ_1 反定向，φ_2 保定向：

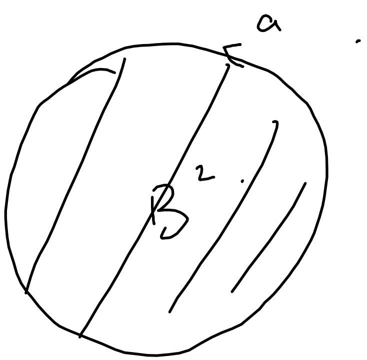


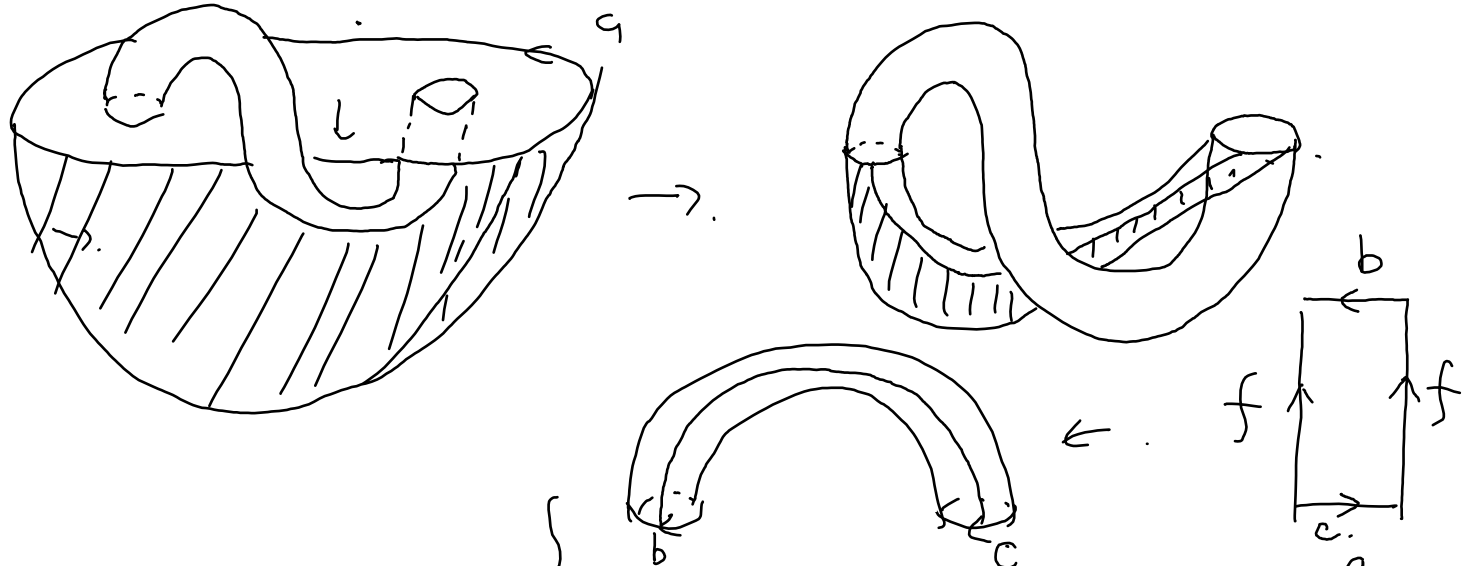


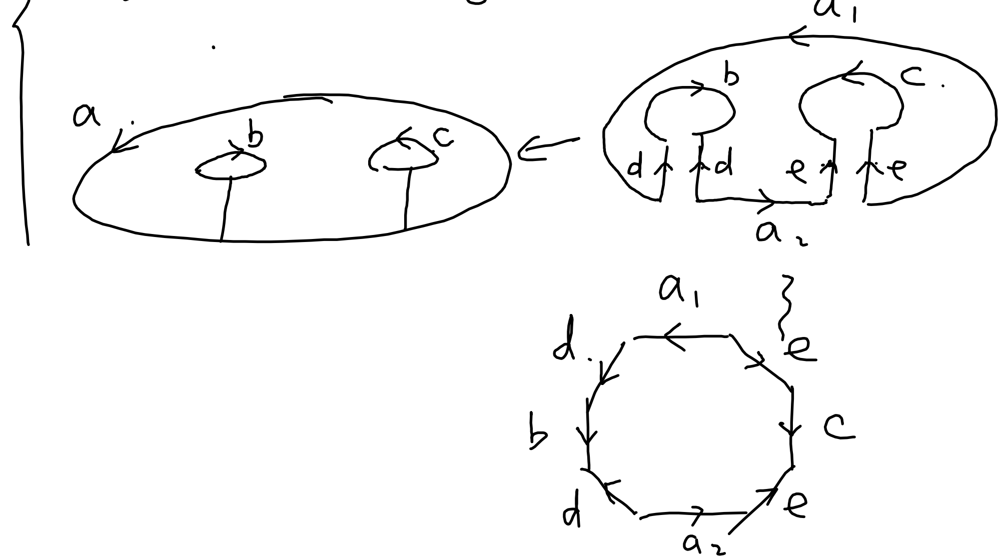
$\cong M(2) \setminus D''$. D'' 开圆盘 .

Klein 瓶

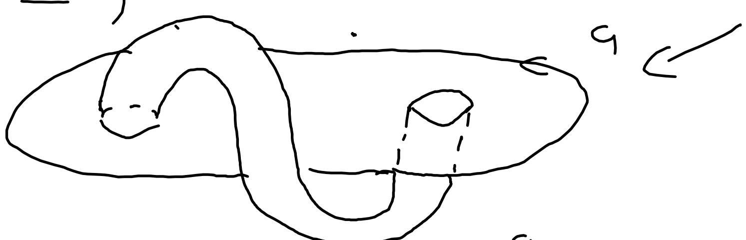
(看法一) .

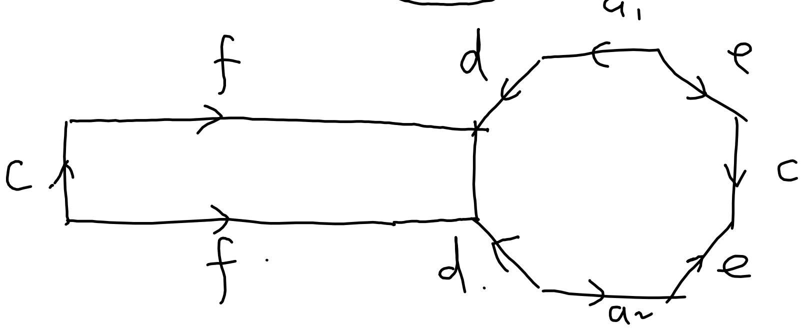


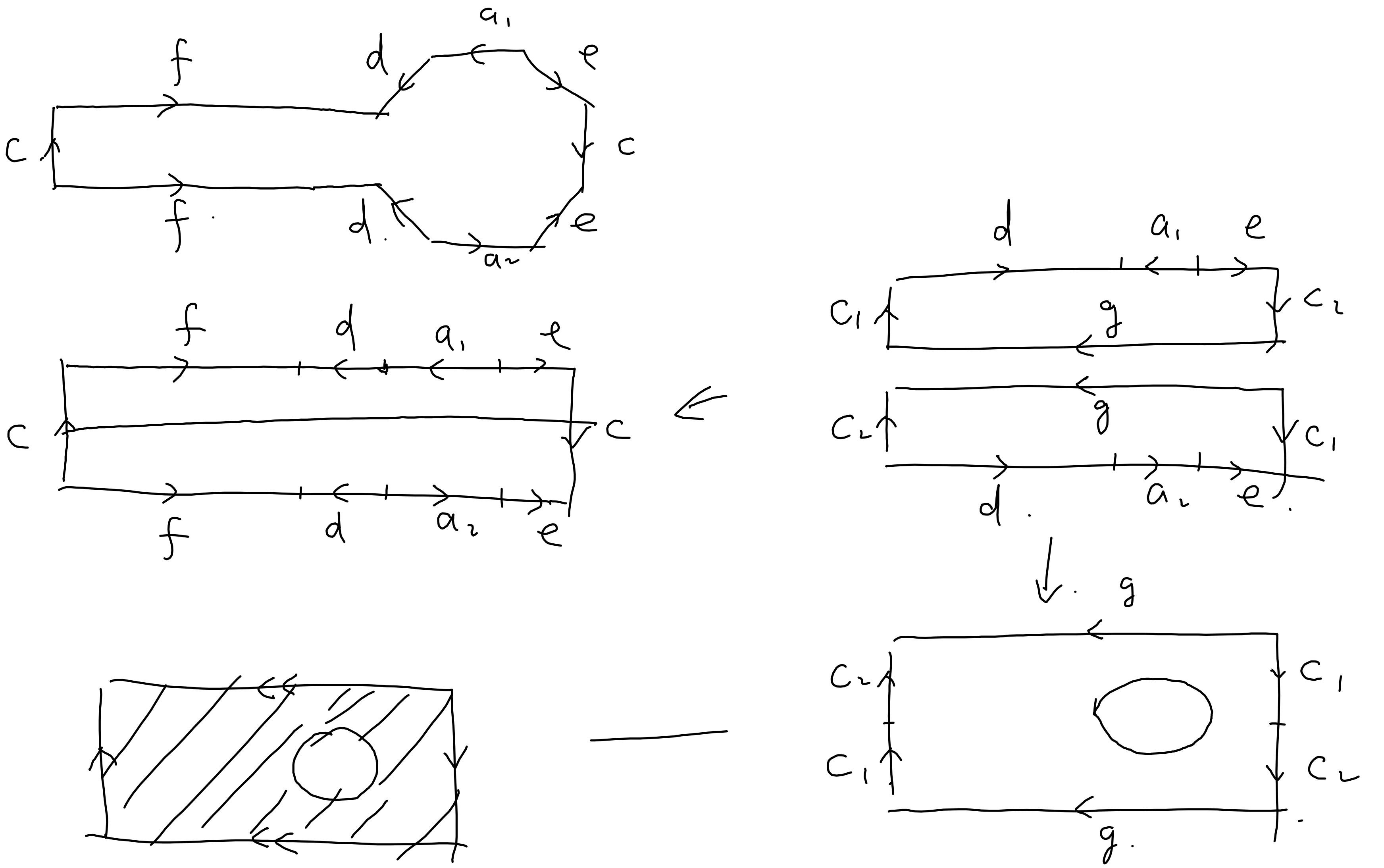




(看法二)







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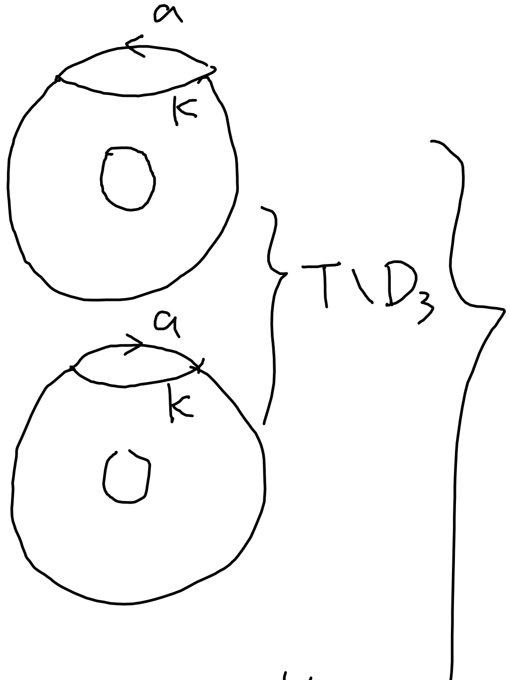
第二步：

(i)

(ii)



(iii) $M(2)\setminus D''$

沿 $\vec{a}$

粘

(i) 选择同胚 $\varphi_1: K \to K'$，保(i)所示定向，

考虑 $(S\setminus D)\cup_{\varphi_1}(T\setminus D_3)$

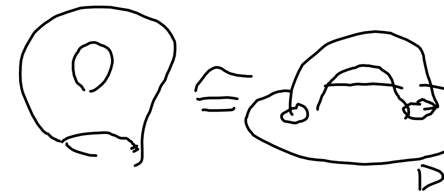
(ii) ……φ_2……(ii)……

……$(S\setminus D)\cup_{\varphi_2}(T\setminus D_3)$

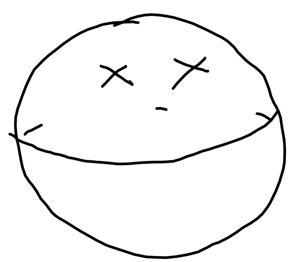
习题：仿照粘Möbius带时的证明，证明 $(S\setminus D)\cup_{\varphi_1}(T\setminus D_3)$

$\cong (S\setminus D)\cup_{\varphi_2}(T\setminus D_3)$

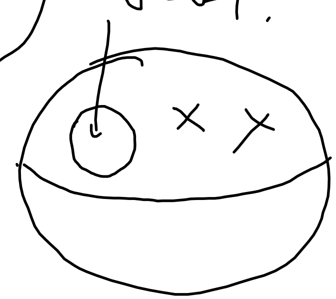
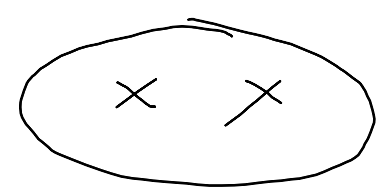
小结：(i)(ii)所得空间同胚于：

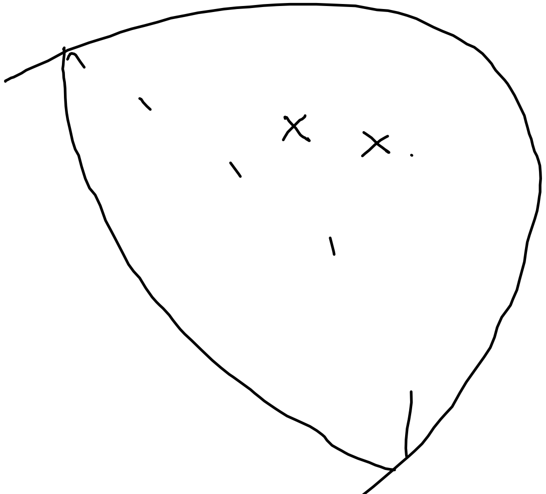


(iii)

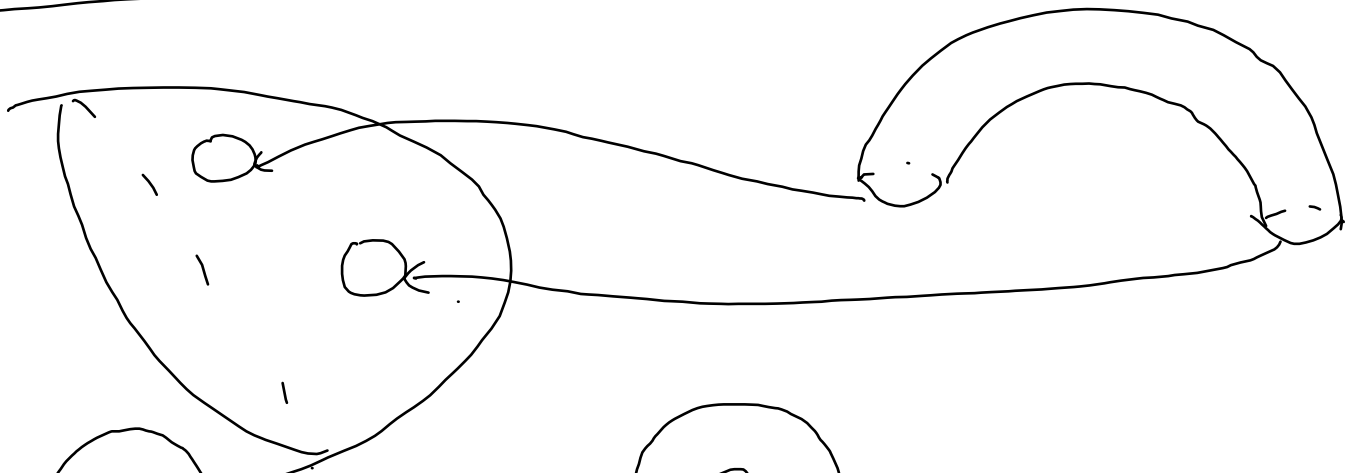
M(2) $\cong$ 

挖掉

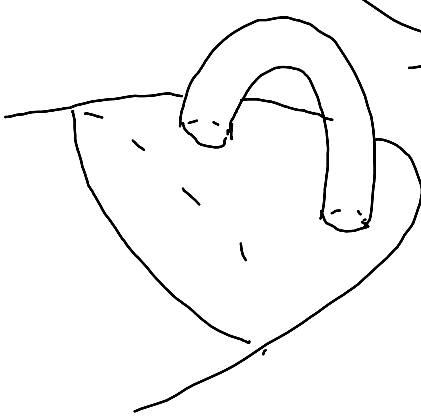
M(2)\D'' $\cong$  $\cong$ 



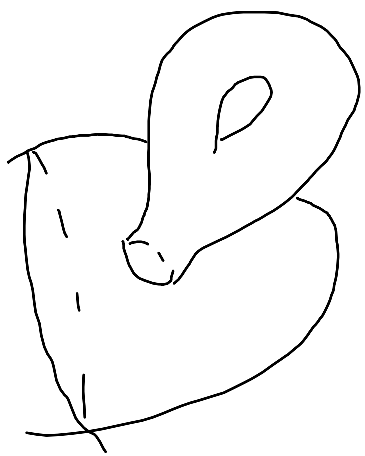
总结：



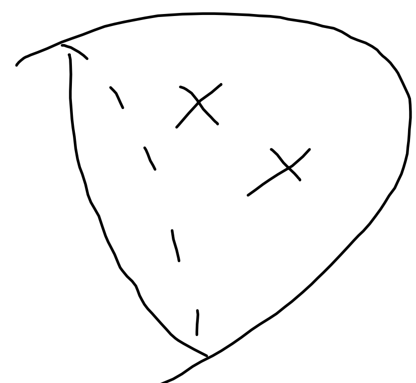
(i)(ii)



$\simeq$



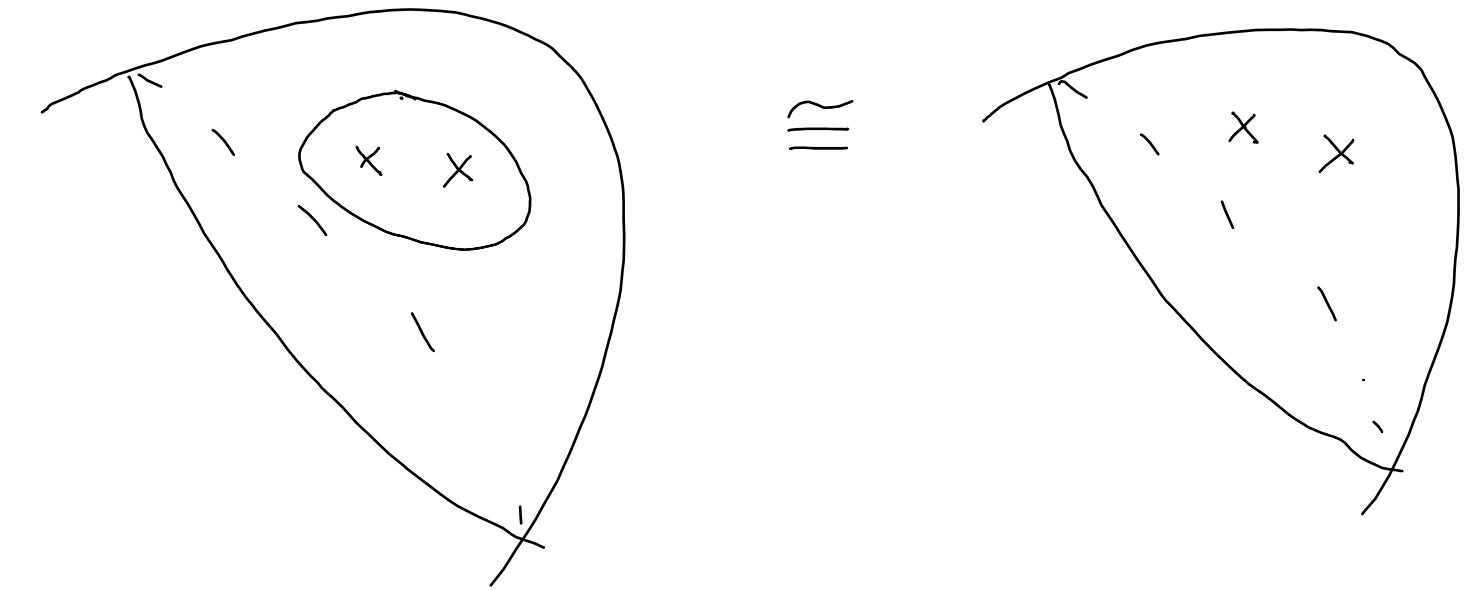
(iii)
(iv).



真正粘把柄的粘法应是 (i) (ii)

两种粘法等价

称为粘一个 handle.



真正粘把柄的方法应该是①、②

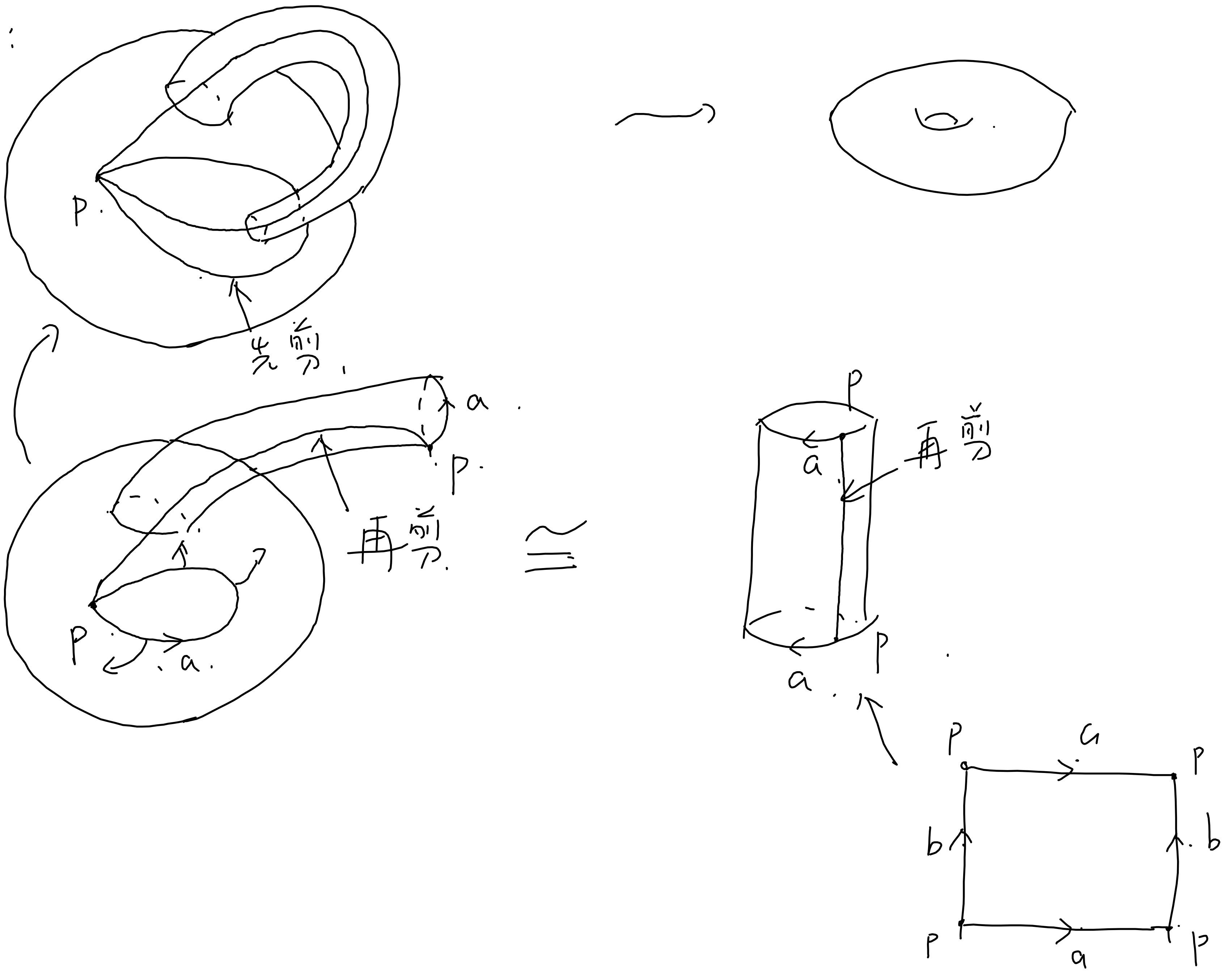
两种粘法等价

称为粘一个handle

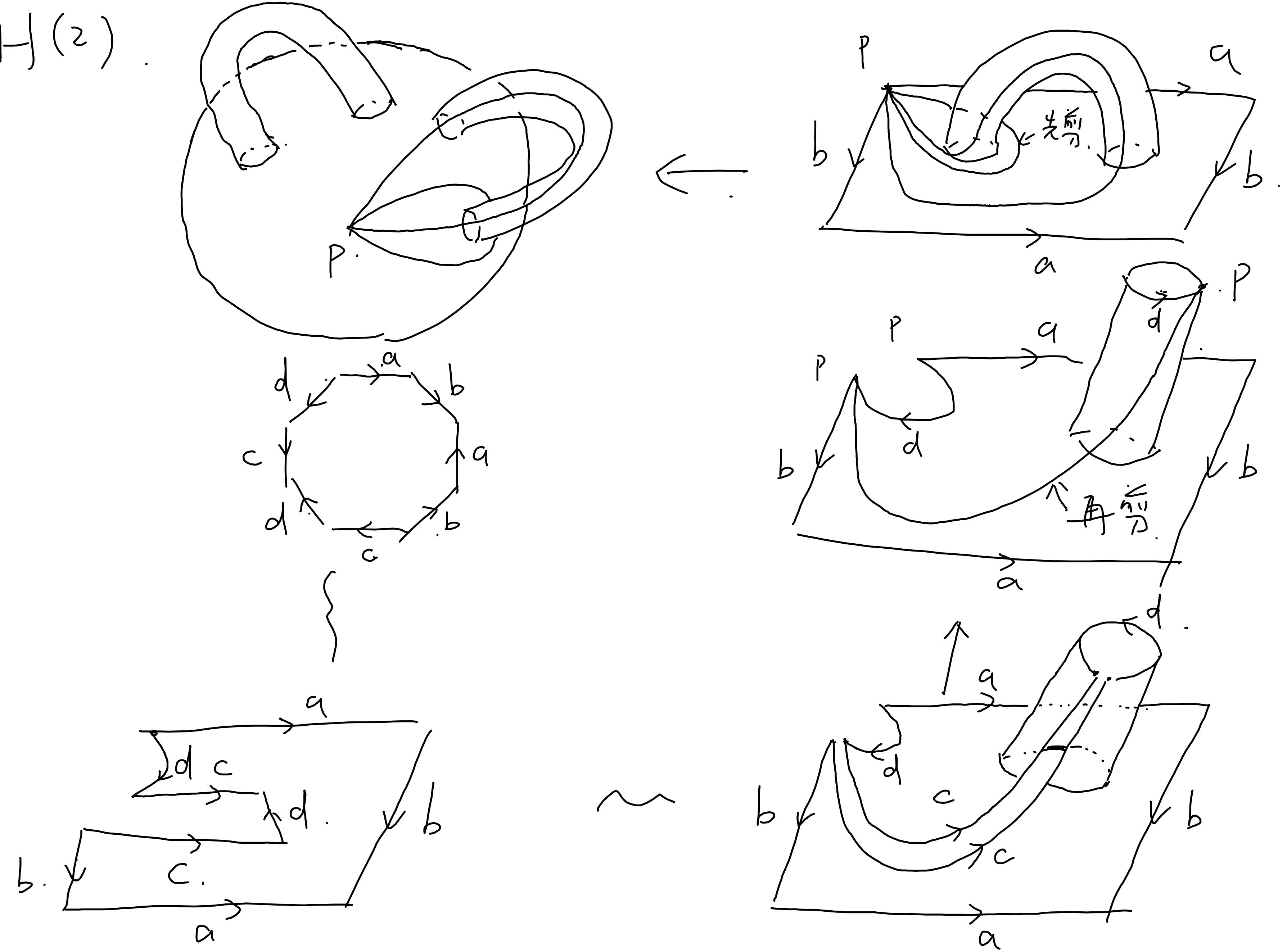
记$H(g)$为在球面上粘g个handles所得拓扑空间.

下面还需验证$H(g)$只与粘的handles的个数有关.

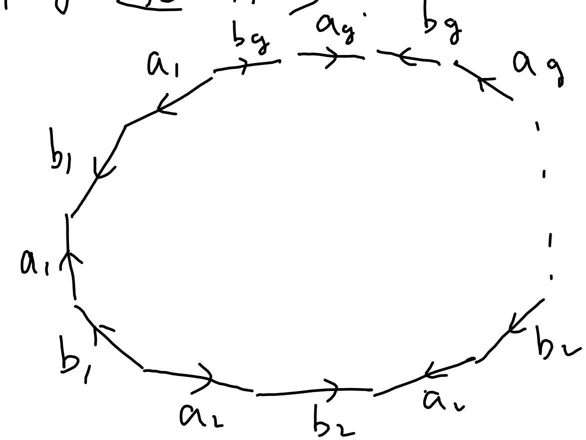
H(1) :



H(2).



$H(g)$: $4g$ 边形

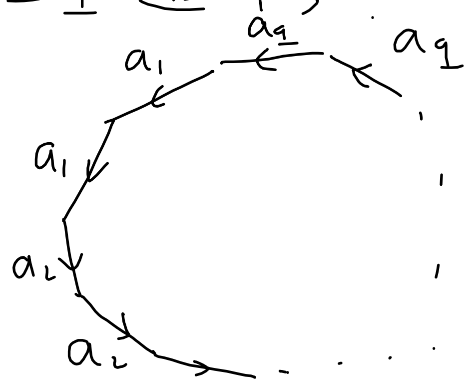


: $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$.

$H(g)$: 亏格为 g 的定向曲面.

surface symbol

$M(q)$: $2q$ 边形



: $a_1a_1a_2a_2\cdots a_qa_q$.

定理: S^2, $H(g)$, $M(q)$, $g,q\geq 1$ 为所有的互不同胚的闭曲面.

① S^2, $H(g)$, $M(q)$, $g,q\geq 1$ 互不同胚.

② $\forall$ 闭曲面 X, X 必同胚于 S^2, $H(g)$, $M(q)$, $g,q\geq 1$ 中的一个.

定义：设 X 为一个 Hausdorff 空间，称 X 为一个 n 维(无边)(拓扑)流形(manifold)，if：

(△) $\forall p \in X$，$\exists p$ 的邻域 U_p，s.t. $\exists$ 同胚：

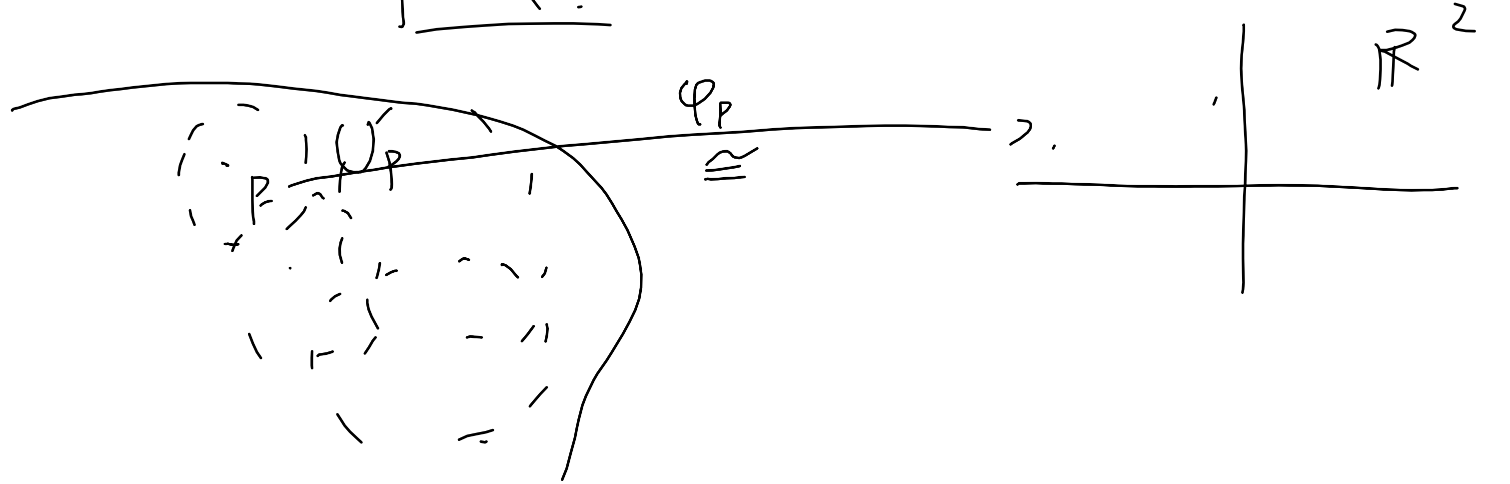
$$\varphi_p : U_p \to \mathbb{R}^n .$$

2 维拓扑流形：(拓扑)曲面.

闭曲面：紧致，连通的，无边的曲面.

manifold < many / fold.

$n=2$.



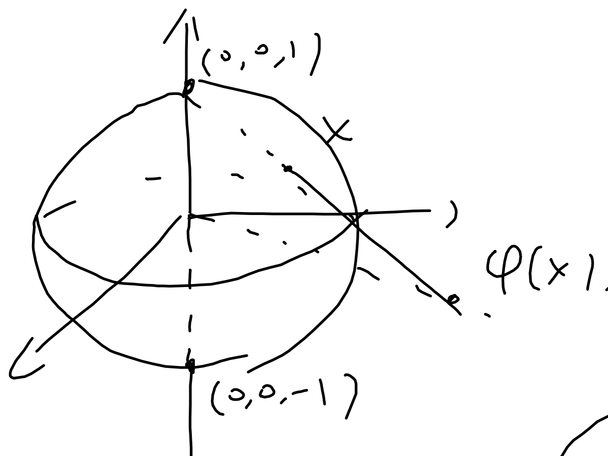
练习：等价定义.

(△) 可换为：

(△')：$\forall p \in X$，$\exists p$ 的邻域 U_p，以及 U_p 到 $\mathbb{R}^n$ 的一个开子集 $\hat{U}_p$ 的同胚，记为 $\varphi_p: U_p \to \hat{U}_p$.

定义，设 M 为一个拓扑流形，$U \underset{\text{open}}{\subseteq} M$，同胚 $\varphi: U \to \hat{U}$ $\left(\underset{\mathbb{R}^n}{\cap}\right)$ 称为 M 的一个局部坐标卡.

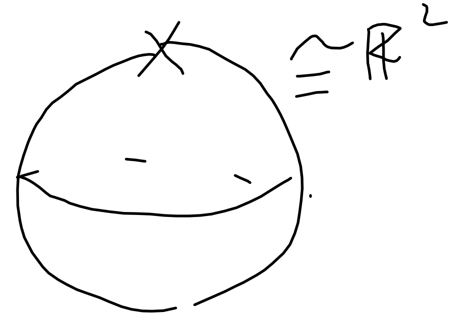
例：S^2.

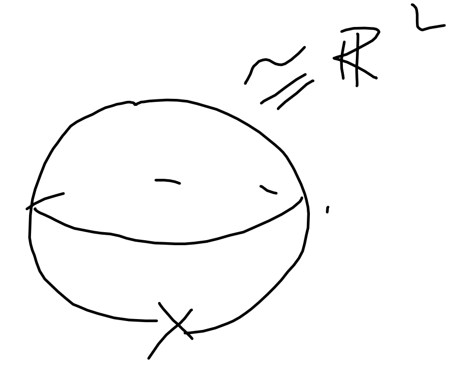


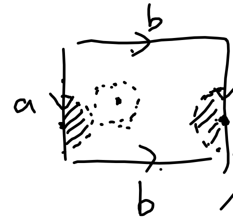
$\varphi: S^2 \setminus \{(0,0,1)\} \xrightarrow{\cong} \mathbb{R}^2$

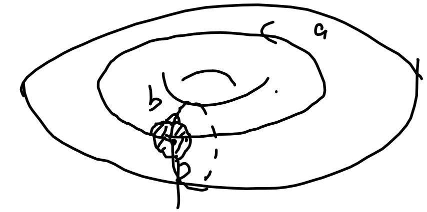
$x \longmapsto \varphi(x)$

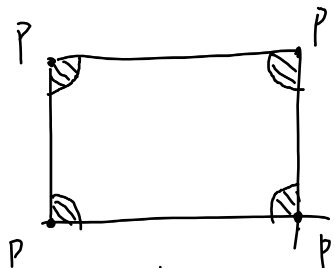
$\psi: S^2 \setminus \{(0,0,-1)\} \xrightarrow{\cong} \mathbb{R}^2$.

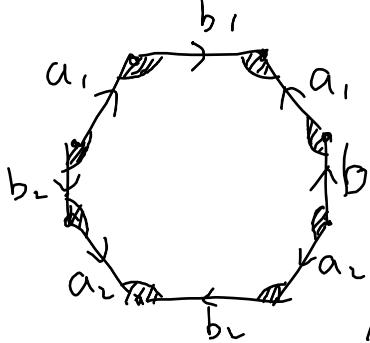




例: $H(1) =$ 

$/\sim \;\cong$ 



$H(2)$: 

$/\sim$

$$\boxed{\frac{H(g)}{M(g)}}$$

CW 复形 (J.H.C. Whitehead).

记号: $B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$, $n \geq 1$, $B^0 = \{pt\}$.

B^n 称为: 开的 n-胞腔 (n-cell)

与 B^n 同胚的拓扑空间都称为 n-cell.

$\overline{B}^n = B^n$ 的闭包

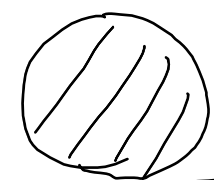
例. S^n.

$$S^n = B^0 \cup_f \overline{B}^n$$

$$f: \partial \overline{B}^n \longrightarrow B^0.$$

$$\partial \overline{B}^n \cong (S^{n-1}), \quad x \longmapsto pt.$$

$n=2$, $\overline{B}^2$



$\partial \overline{B}^2 = S^1$

$$\boxed{S^n = e^0 \amalg e^n}$$

记自然投射为 $\pi: B^0 \amalg \overline{B}^n \longrightarrow S^n$

$\pi|_{B^0}: B^0 \xrightarrow{\cong} \pi(B^0)$ $\pi|_{B^n}: B^n \xrightarrow{\cong} \pi(B^n)$.

记 $e^0 = \pi(B^0)$, $e^n = \pi(B^n)$. $\Rightarrow$ e^0 为 0-cell

e^n 为 n-cell.

⌐ $n=2$.

$\cdot\ e^0$. e^2.

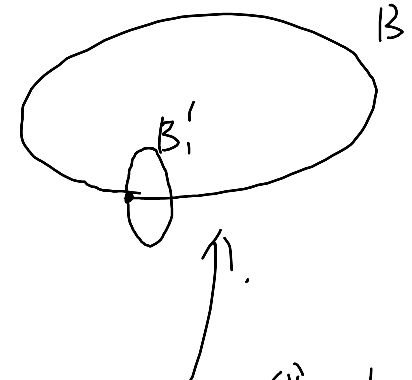
$\cong$ $\cong$

B^1. B^2.

$S^2 = e^0 \amalg e^2$. ⌋

例. $T = S^1 \times S^1$

torus.



T 可按如下步骤粘得：

1) 从 $B^0 = \{pt\}$ 出发，记 $X^0 = B^0$

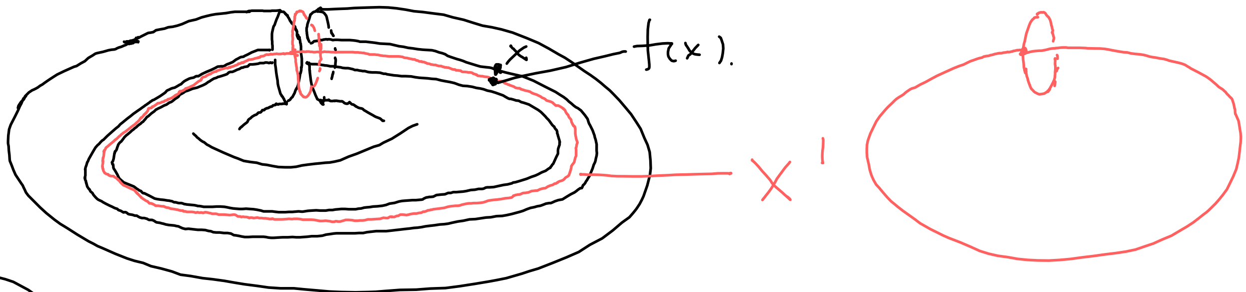
2) 往 B^0 上粘两个 1-cells: B_1^1, B_2^1, by: 得出(空间)，记为 X^1

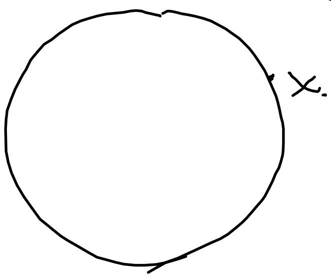
$$f_1^1 : \partial \bar{B}_1^1 \longrightarrow B^0 \qquad f_2^1 : \partial \bar{B}_2^1 \longrightarrow B^0$$
$$x \longmapsto pt \qquad\qquad x \longmapsto pt.$$

3) 往 X^1 上 粘 2-cell B^2 by

$$f: \partial B^2 \longrightarrow X^1$$
$$x \longmapsto f(x)$$

$\rightsquigarrow$ 得到 $X^2 = T$





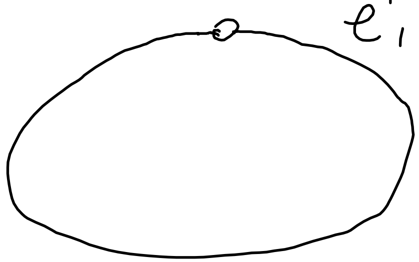
$\pi|_{B^2}$, $\pi|_{B^1_1}$, $\pi|_{B^1_2}$, $\pi|_{B^0}$ 同胚

$\big\downarrow$ 像

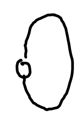
e^2, e^1_1, e^1_2, e^0.

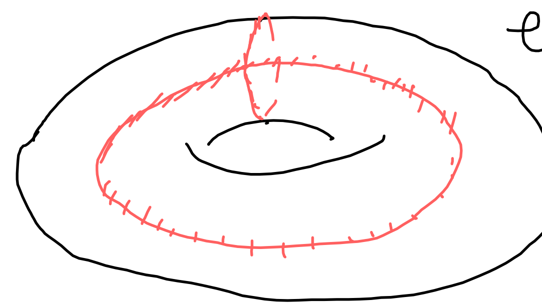
$$T = e^0 \sqcup e^1_1 \sqcup e^1_2 \sqcup e^2.$$

e^0.

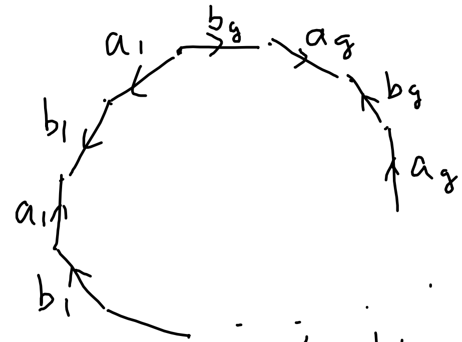

e^1_1

e^1_2.




e^2

例：$H(g)$，

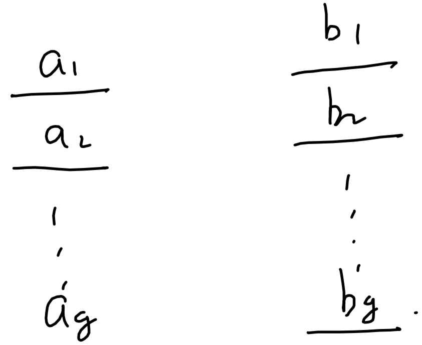


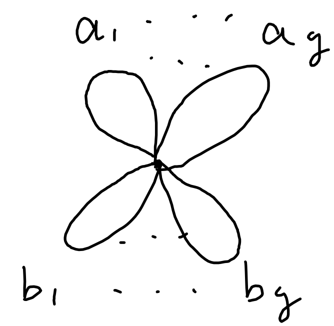
$4g$-边形

$H(g)$ 可按如下程序粘贴：

1) 从 $X^0 = B^0$ 出发.

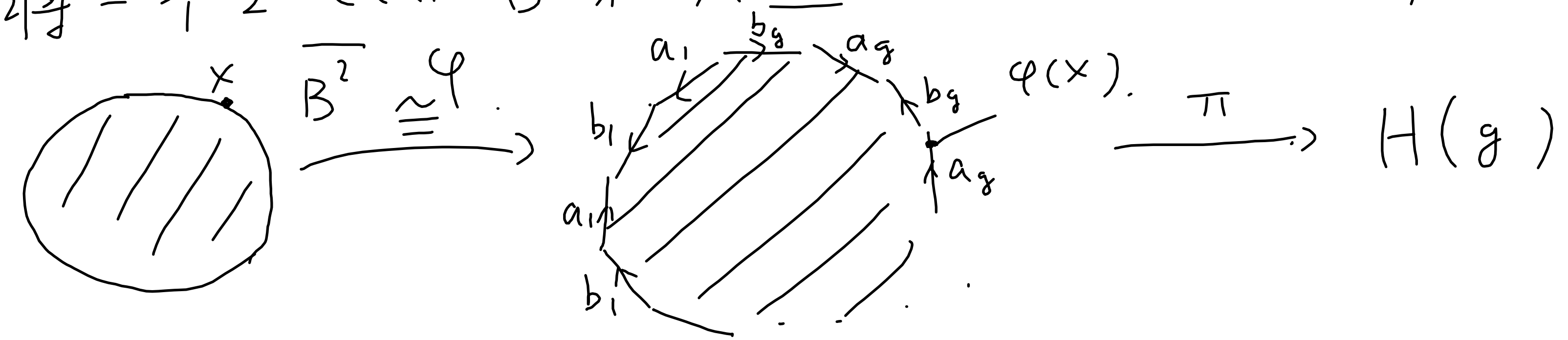
2) 将 $2g$ 个 1-cells 粘到 X^0 上.





$\rightsquigarrow X^1$

3) 将一个 2-cell B^2 粘 X^1 上 $\rightsquigarrow H(g)$



$\bar{B}^2$ 粘到 X^1 上 by:

$$f: \partial\bar{B}^2 \longrightarrow X^1.$$
$$x \longmapsto \pi(\varphi(x)).$$

$\rightsquigarrow X^1 \cup_f \bar{B}^2 = X^2 \cong H(g).$

记 $\tilde{\pi}: B^0 \coprod_{i=1}^{g} \overset{a_i}{\longrightarrow} \coprod_{i=1}^{g} \overset{b_i}{\longrightarrow} \coprod \bar{B}^2 \longrightarrow H(g)$

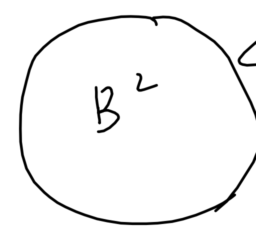
注意: $\tilde{\pi}|_{B^0}$, $\tilde{\pi}|_{\mathring{a}_i}$, $\tilde{\pi}|_{\mathring{b}_i}$, $\tilde{\pi}|_{B^2}$ 为同胚.

$\rightsquigarrow$ 给

e^0, e^1_i, e^1_{i+g}, e^2.

$$H(g) = e^0 \coprod \underbrace{e^1_1 \coprod \cdots \coprod e^1_{2g}}_{2g\text{个 } 1\text{-cells}} \coprod e^2.$$

例: $\mathbb{RP}^n = \bar{B}^n / _{\partial\bar{B}^n \ni x \sim -x}$



粘对径点.

$\partial\bar{B}^n = S^{n-1}$. $\quad \partial\bar{B}^n / _{x\sim -x} \cong \mathbb{RP}^{n-1}$.

$$\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_f \bar{B}^n$$

$f: \partial \bar{B}^n \longrightarrow \mathbb{R}P^{n-1}$ 为商映射

$\tilde{\pi}: \mathbb{R}P^{n-1} \sqcup \bar{B}^n \longrightarrow \mathbb{R}P^n$. 自然投射.

$\tilde{\pi}|_{B^n}$ 为同胚. $e^n := \tilde{\pi}(B^n)$.

$$\Rightarrow \mathbb{R}P^n = \mathbb{R}P^{n-1} \sqcup e^n$$
$$= \mathbb{R}P^{n-2} \sqcup e^{n-1} \sqcup e^n.$$
$$= \cdots\cdots = e^0 \sqcup e^1 \sqcup \cdots \sqcup e^n.$$

例: $\mathbb{C}P^n$, $\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{(0,\cdots,0)\})/\sim$.

$x \sim y \Leftrightarrow \exists \lambda \in \mathbb{C}^*$, s.t. $x = \lambda \cdot y$

$$\mathbb{C}P^n = e^0 \sqcup e^2 \sqcup \cdots \sqcup e^{2n}$$

定义. (有限 CW-复形 (胞腔复形)). CW complex

cell complex.

若 top. sp. X 可由有限步如下操作得到：（0-骨架）

1). 从一些离散点集 X^0 出发.（X^0 称为 0-skeleton）.

2). 往 X^0 上粘一些 1-cells（通过 $g_\alpha^1: \partial \bar{B}_\alpha^1 \to X^0$ 粘）.

得到：X^1.（称为 1-skeleton）.

3) 归纳地，若已得到 (n-1)-skeleton X^{n-1}，再往上粘若干 n-cells.（通过 $g_\alpha^n: \partial \bar{B}_\alpha^n \to X^{n-1}$ 粘）.

则称 X 为一个 CW-复形

记 e_α^k 为 B_α^k 在 X 中像. 则 $X = \coprod_{k,\alpha} e_\alpha^k$，称为 X 的胞腔分解.

定理（Morse）. 每个紧致的光滑流形都有有限 CW-复形的同伦型（homotopy type）.