上一次：

X (conn. & locally path conn. & semilocally simply conn.)

$\rightsquigarrow$ universal covering $p: \tilde{X} \to X$ 总存在.

例：$\mathbb{R} \xrightarrow{\pi} S^1$. $s \mapsto e^{2\pi i s}$.

例：$Sp(1) (= SU(2)) \xrightarrow{\varphi} SO(3)$ φ 为一个 univ. covering.

$Spin(n) \xrightarrow{\varphi} SO(n)$, $n \geq 3$. 为 univ. covering.

命题4. X (conn. & locally path conn & semilocally simply conn.), 则其 universal covering $p: \tilde{X} \to X$ 为一个 Galois covering.

Lemma 4. (一般提升引理) 设 $p: Y \to X$ 为一个 covering, $f: Z \to X$. $y_0 \in Y$, $z_0 \in Z$, $p(y_0) = f(z_0) = x_0 \in X$, 设 Z 道路连通且局部道路连通, $f_*(\pi_1(Z, z_0)) \subset p_* \pi_1(Y, y_0)$. 则存在唯一的

f 的提升 $\tilde{f}: Z \to Y$, s.t. $\tilde{f}(z_0) = y_0$.

proof. $\forall z \in Z$, 选取 $\gamma_z \in P(z_0, z)$.

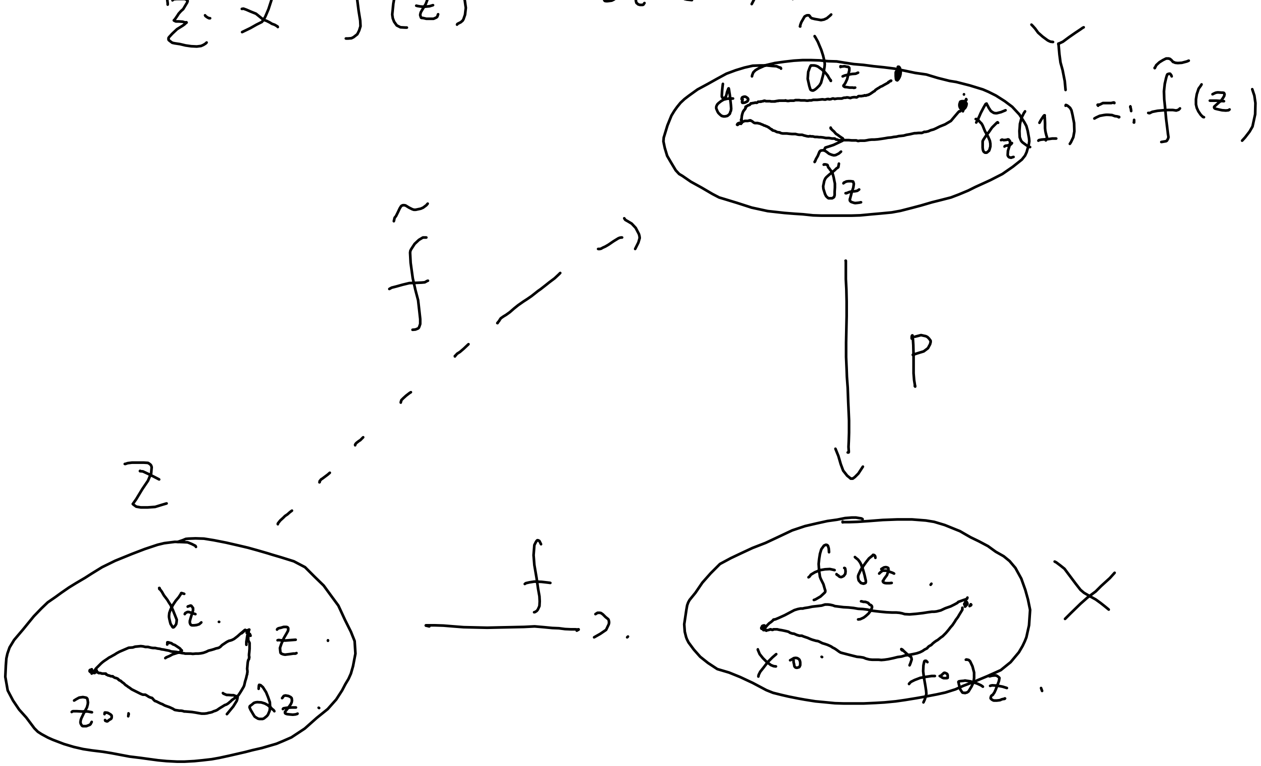
则 $f \circ \gamma_z \in P(x_0, f(z))$.

记 $\tilde{\gamma}_z$ 为 $f \circ \gamma_z$ 的从 y_0 出发的提升.

定义 $\tilde{f}(z) := \tilde{\gamma}_z(1)$.

$$\begin{tikzcd} & Y \ni y_0 \arrow[d, "p"] \\ z_0 \in Z \arrow[ur, dashed, "\exists!\, \tilde{f}"] \arrow[r, "f"] & X \ni x_0 \end{tikzcd}$$

conn. & locally path conn.



$f_* \pi_1(Z, z_0) \subset p_* \pi_1(Y, y_0)$.

$f_*: \pi_1(Z, z_0) \to \pi_1(X, x_0)$

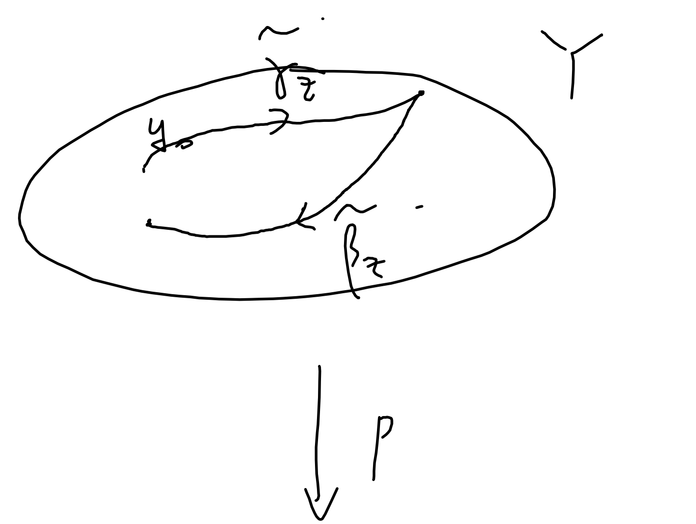
$p_*: \pi_1(Y, y_0) \to \pi_1(X, x_0)$.

下验证 $\tilde f(z)$ 不依赖于 γ_z 之选取：

再取一条 $\alpha_z \in P(z_0, z)$. 令 $\tilde\alpha_z$ 为 $f\circ\alpha_z$ 的从 y_0 出发的提升，只要证： $\tilde\alpha_z(1) = \tilde\gamma_z(1)$.

令 $\tilde\beta_z$ 为 $(f\circ\alpha_z)^{-1}$ 的从 $\tilde\gamma_z(1)$ 出发的提升，只需证：

$\tilde\beta_z(1) = y_0$ 即可.



$\langle (f\circ\gamma_z)\cdot(f\circ\alpha_z^{-1})\rangle \in f_*\pi_1(Z, z_0) \subset p_*\pi_1(Y, y_0)$.

$\Rightarrow \exists\, \delta \in L(Y, y_0)$, s.t.

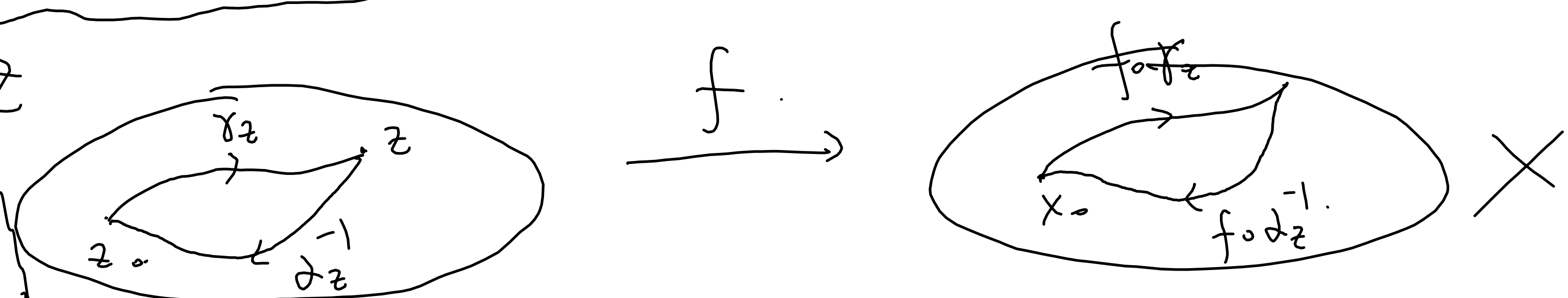
$\underbrace{(f\circ\gamma_z)\cdot(f\circ\alpha_z^{-1})}_{\parallel} \simeq p\circ\delta$ rel $\{0,1\}$.

$p\circ(\tilde\gamma_z\cdot\tilde\beta_z)$

由同伦提升引理，

$\delta \simeq \tilde\gamma_z\cdot\tilde\beta_z$ rel $\{0,1\}$.

$\Rightarrow \tilde\beta_z(1) = y_0$ #

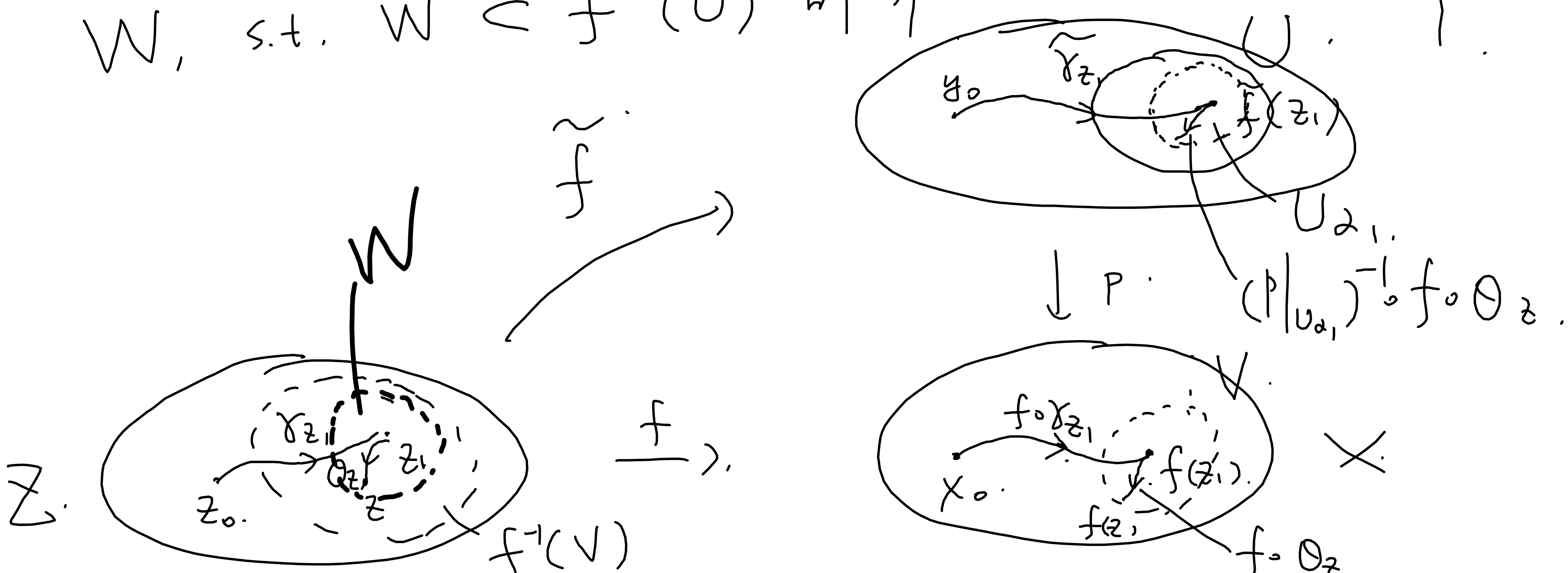


因此 $\tilde{f}: Z \to Y$ 是良好定义的，且显然有：

$$p \circ \tilde{f} = f.$$

只剩验证：$\tilde{f}$ 为连续映射.

$\forall U \underset{open}{\subset} Y$，只要证 $\forall z_1 \in \tilde{f}^{-1}(U)$，$\exists$ z_1 的开邻域 W，s.t. $W \subset \tilde{f}^{-1}(U)$ 即可



$\because$ p 为 covering，$\therefore$ $\exists$ $f(z_1)$ 的开邻域 V，s.t. $p^{-1}(V) = \coprod_{\alpha \in I} U_\alpha$.

其中 $U_\alpha \underset{open}{\subset} Y$，且 $p|_{U_\alpha}: U_\alpha \xrightarrow{\cong} V$，设 $\tilde{f}(z_1) \in U_{\alpha_1}$，通过收缩

V，不妨设 $U_{\alpha_1} \subset U$.

由于 f 连续，$f^{-1}(V)$ 为 z_1 的开邻域，又由 Z locally path conn.

$\exists$ z_1 的 path conn. 开邻域 W，s.t. $z_1 \in W \subset f^{-1}(V)$.

Claim: $\tilde{f}(W) \subset U_{\alpha_1}$.

$\because$ W path conn. $\forall z \in W$，$\exists\, \theta_z \in P(z_1, z)$.

$$\gamma_{z_1} \cdot \theta_z \in P(z_0, z).$$

$$f\circ(\gamma_{z_1} \cdot \theta_z) \in P(x_0, f(z)).$$

记 $\widetilde{f\circ(\gamma_{z_1} \cdot \theta_z)}$ 为从 y_0 出发的提升.

$$\tilde{f}(z) = \widetilde{f\circ(\gamma_{z_1} \cdot \theta_z)}(1).$$

又 $\widetilde{f\circ(\gamma_{z_1} \cdot \theta_z)} = \tilde{\gamma}_{z_1} \cdot ((p|_{U_{\alpha_1}})^{-1} \circ f \circ \theta_z)$

$\therefore$ $\tilde{f}(z) = (p|_{U_{\alpha_1}})^{-1} \circ f \circ \theta_z(1) = (p|_{U_{\alpha_1}})^{-1}(\underbrace{f(z)}_{\in V}) \in U_{\alpha_1}$.

#

命题4之证明：

设 $\tilde{X} \xrightarrow{p} X$ 为 univ. covering，由 X 的条件 $\Rightarrow$ $\tilde{X}$ conn. & locally path conn. 只要证：$\forall$ $\tilde{X}$ 中的两个点 $\tilde{x}_1, \tilde{x}_2$，

若 $p(\tilde{x}_1) = p(\tilde{x}_2)$，则存在 $\varphi \in Deck(\tilde{X}/X)$，s.t. $\varphi(\tilde{x}_1) = \tilde{x}_2$.

对：

$$\begin{tikzcd} & \tilde{X} \ni \tilde{x}_2 \arrow[d, "p"] \\ \tilde{x}_1 \in \tilde{X} \arrow[r, "p"] & X \ni p(\tilde{x}_1) \end{tikzcd}$$

$p_* \pi_1(\tilde{X}, \tilde{x}_1) \subset p_* \pi_1(\tilde{X}, \tilde{x}_2)$.

使用一般提升引理，就找到 $\varphi: \tilde{X} \to \tilde{X}$，$\varphi(\tilde{x}_1) = \tilde{x}_2$，

且 $p \circ \varphi = p$.

$\psi: \tilde{X} \to \tilde{X}$，$\psi(\tilde{x}_2) = \tilde{x}_1$，

且 $p \circ \psi = p$.

$$\begin{tikzcd} \tilde{x}_1 \in \tilde{X} \arrow[rr, bend left, "\varphi"] \arrow[dr] & & \tilde{X} \ni \tilde{x}_2 \arrow[ll, bend left, "\psi"] \arrow[dl] \\ & X \ni p(\tilde{x}_1) & \end{tikzcd}$$

只要证：$\psi \circ \varphi = id$，$\varphi \circ \psi = id$.

$\psi\circ\varphi$ 使下面图表交换：

$$
\begin{tikzcd}
 & \tilde{X} \ni \tilde{x}_1 \arrow[d, "p"] \\
\tilde{x}_1 \in \tilde{X} \arrow[ur, "\psi\circ\varphi"] \arrow[r, "p"] & X \ni p(\tilde{x}_1)
\end{tikzcd}
$$

$\therefore$ $\psi\circ\varphi$ 是把 $\tilde{x}_1$ 映到 $\tilde{x}_1$ 的 p 的提升.

又 $Id_{\tilde{X}}$ ....

由提升唯一性引理，$\Rightarrow \psi\circ\varphi = Id_{\tilde{X}}$. #

推论：设 X conn. & locally path conn. & semilocally simply conn.，则 X 的 univ. covering 存在且在差一个等价的意义下唯一.

Pf. 设 $\tilde{X}_1 \xrightarrow{p_1} X$, $\tilde{X}_2 \xrightarrow{p_2} X$ 为 universal covering.

$$
\begin{tikzcd}
 & \tilde{X}_2 \ni \tilde{x}_2 \arrow[d, "p_2"] \arrow[dl, bend left, "\psi"] \\
\tilde{x}_1 \in \tilde{X}_1 \arrow[ur, bend left, "\varphi"] \arrow[r, "p_1"] & X \ni p_1(\tilde{x}_1)
\end{tikzcd}
$$

$\Rightarrow \varphi\circ\psi = id_{\tilde{X}_2}$, $\psi\circ\varphi = id_{\tilde{X}_1}$.

#

设 X: conn. & locally path conn. & semilocally simply conn.

$\forall\, q: Y \to X$ 为一个 covering, Y 连通的.

$$\begin{tikzcd} & Y \ni y \arrow[d, "q"] \\ \tilde{x} \in \tilde{X} \arrow[ur, "f"] \arrow[r, "p"] & X \ni x \end{tikzcd}$$

$p_* \pi_1(\tilde{X}, \tilde{x}) \subset q_* \pi_1(Y, y)$.

之前已证: f 一定为 Galois 覆盖.

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[rr, bend left] \arrow[dr, "p"'] & Y \arrow[r, "\cong"] \arrow[d, "q"] & \tilde{X}/\mathrm{Deck}(\tilde{X}/Y) \arrow[dl, "\bar{p}"] \\ & X & \end{tikzcd}$$

首先: 计算 $\mathrm{Deck}(\tilde{X}/X)$.

$\mathrm{Deck}(\tilde{X}/X) \circlearrowright \tilde{X}$ even. $\quad p: \tilde{X} \to X$ 为 Galois covering

$$
\begin{tikzcd}
\tilde{X} \arrow[r, "\pi"] \arrow[dr, "p"'] & \tilde{X}/\mathrm{Deck}(\tilde{X}/X) \arrow[d, "\cong"] \\
 & X
\end{tikzcd}
$$

$\therefore \pi_1(X, x) \cong \mathrm{Deck}(\tilde{X}/X)$.

仔细看该同构.

回忆: 选定 $\tilde{x} \in p^{-1}(x)$. $\quad \varphi: \mathrm{Deck}(\tilde{X}/X) \longrightarrow \pi_1(X, x)$.

$$g \longmapsto \langle p \circ \gamma_g \rangle$$

其中 γ_g: $\tilde{X}$ 中的以 $\tilde{x}$ 为起点，以 $g(\tilde{x})$ 为终点的道路.

Rmk. (φ 依赖于 $\tilde{x} \in p^{-1}(x)$ 的方式)

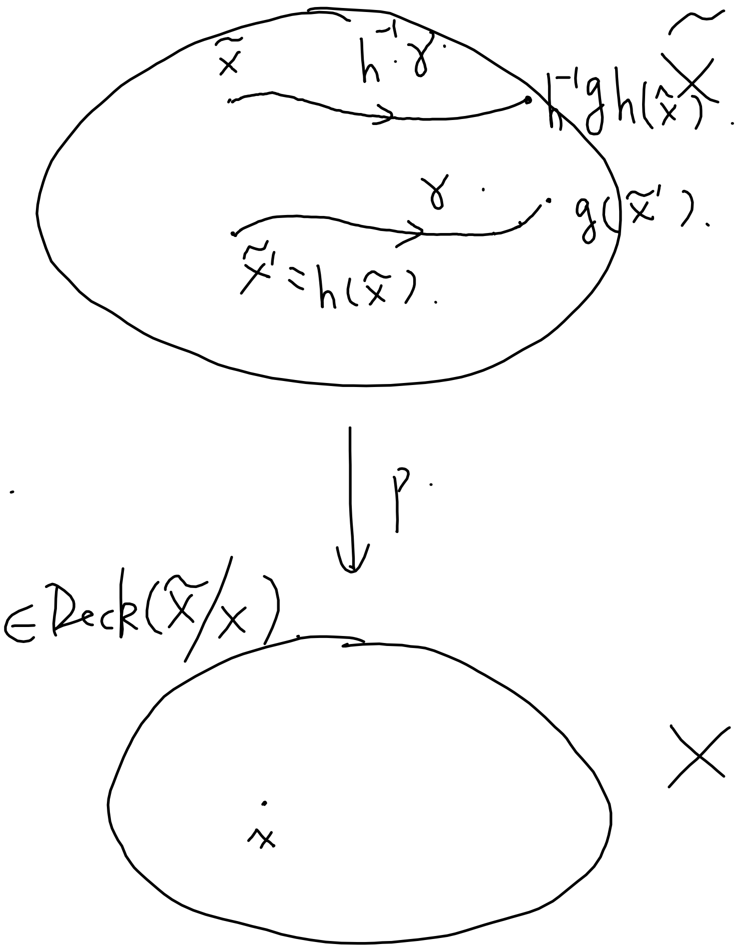
记 $\varphi_{\tilde{x}}$: $\mathrm{Deck}(\tilde{X}/X) \longrightarrow \pi_1(X, x)$, $\quad g \longmapsto \langle p \circ \gamma_g \rangle$.

其中 γ_g: $\tilde{X}$ 中的以 $\tilde{x}$ 为起点，以 $g(\tilde{x})$ 为终点的道路.

称之为以 $\tilde{x}$ 为基点的同构.

再取 $\tilde{x}' \in p^{-1}(x)$，$\exists\, h \in \mathrm{Deck}(\tilde{X}/X)$, s.t. $h(\tilde{x}) = \tilde{x}'$.

想看 $\varphi_{\tilde{x}}$ 与 $\varphi_{h(\tilde{x})}$ 之关系：



$$\varphi_{h(\tilde{x})}(g) = \langle p\circ\gamma\rangle$$
$$= \langle p\circ h^{-1}\gamma\rangle$$
$$= \varphi_{\tilde{x}}(h^{-1}gh)$$

$\therefore\ \varphi_{h(\tilde{x})}(g) = \varphi_{\tilde{x}}(h^{-1}gh)$, $\forall g \in \mathrm{Deck}(\tilde{X}/X)$

下面：固定一个 $\tilde{x} \in p^{-1}(x)$. $\varphi_{\tilde{x}}: \mathrm{Deck}(\tilde{X}/X) \xrightarrow{\cong} \pi_1(X, x)$

$\psi_{\tilde{x}} = \varphi_{\tilde{x}}^{-1}: \pi_1(X, x) \longrightarrow \mathrm{Deck}(\tilde{X}/X)$.

$\forall \langle\gamma\rangle \in \pi_1(X, x)$. 记 $\tilde{\gamma}$ 为 γ 的以 $\tilde{x}$ 为起点的提升.

$\tilde{\gamma}(1), \tilde{\gamma}(0) \in p^{-1}(x)$.

$\Rightarrow \exists!\ g \in Deck(\tilde{X}/X)$, s.t. $g(\tilde{\gamma}(0)) = \tilde{\gamma}(1)$.

$\Rightarrow \tilde{\gamma} \in P(\tilde{x}, g(\tilde{x}))$. $\Rightarrow \langle p\circ\tilde{\gamma} \rangle = \varphi_{\tilde{x}}(g)$

$$\langle p\circ\tilde{\gamma}\rangle = \langle \gamma \rangle.$$

$\Rightarrow \psi_{\tilde{x}}(\langle\gamma\rangle) = g$.

$$\psi_{\tilde{x}} : \pi_1(X, x) \longrightarrow Deck(\tilde{X}/X)$$

$\langle\gamma\rangle \longmapsto$ 那个把 $\tilde{\gamma}(0)$ 映到 $\tilde{\gamma}(1)$ 的覆盖变换.

命题5. 设 $p: \tilde{X} \to X$ 为万有覆盖，其中 X conn. & locally path conn. & semilocally simply conn.，则 p 的中间覆盖的等价类 1-1 对应于 $Deck(\tilde{X}/X)$ 的子群，并且进一步，在同构 $\varphi_{\tilde{x}}, \psi_{\tilde{x}}$ 下，p 的中间覆盖的等价类 1-1 对应于 $\pi_1(X,x)$ 的子群. 且在此对应下，中间覆盖
$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & X \end{tikzcd}$$
对应到的 $\pi_1(X,x)$ 的子群

为 $g_*\pi_1(Y,y)$，其中 $y=f(\tilde{x})$.

$$\begin{tikzcd} \tilde{x}\in\tilde{X} \arrow[r, "f"] \arrow[dr, "p"'] & y\in Y \arrow[d, "g"] \\ & x\in X \end{tikzcd}$$

证明："$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$" $\rightsquigarrow$

$$\begin{tikzcd} \mathrm{Deck}(\tilde{X}/Y) \arrow[d] & \left(< \mathrm{Deck}(\tilde{X}/X)\right) \arrow[d, "\varphi_{\tilde{x}}"] \\ \varphi_{\tilde{x}}(\mathrm{Deck}(\tilde{X}/Y)) & < \pi_1(X,x) \end{tikzcd}$$

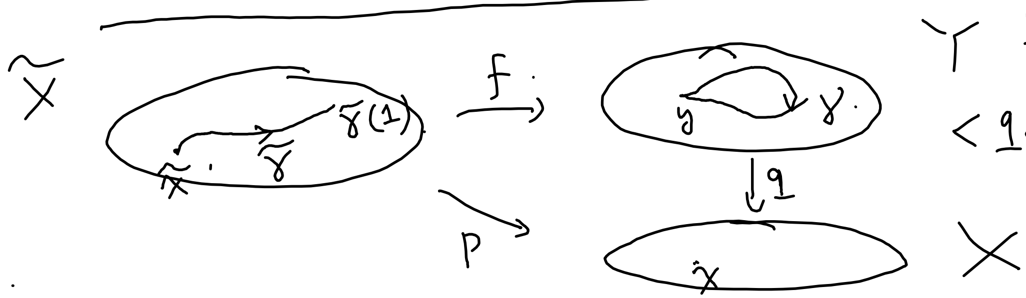
只要证：$\varphi_{\tilde{x}}(\mathrm{Deck}(\tilde{X}/Y)) = g_*\pi_1(Y,y)$.

注意到：$\mathrm{Deck}(\tilde{X}/Y) \cong \pi_1(Y,y)$.

选取 $\varepsilon_{\tilde{x}}:\pi_1(Y,y)\xrightarrow{\cong}\mathrm{Deck}(\tilde{X}/Y)$ 为(对应于 $\tilde{X}\to Y$ 的)以 $\tilde{x}$ 为基点的那个同构.

$$\begin{tikzcd} \pi_1(Y,y) \arrow[r, "\varepsilon_{\tilde{x}}"] \arrow[drr, bend right, "g_*"'] & \mathrm{Deck}(\tilde{X}/Y) < & \mathrm{Deck}(\tilde{X}/X) \arrow[d, "\varphi_{\tilde{x}}"] \\ & & \pi_1(X,x) \end{tikzcd}$$

只要证：$\varphi_{\tilde{x}}\circ\varepsilon_{\tilde{x}} = g_*$.



$g_*(\langle\gamma\rangle)$

$\parallel$

$\langle g\circ\gamma\rangle$

$\parallel$

$\varepsilon_{\tilde{x}}(\langle\gamma\rangle)(\tilde{x}) = \tilde{\gamma}(1)$.

$\varphi_{\tilde{x}}(\varepsilon_{\tilde{x}}(\langle\gamma\rangle)) = \langle p\circ \tilde{x}\overset{\alpha}{\longrightarrow}\varepsilon_{\tilde{x}}(\langle\gamma\rangle)(\tilde{x})\rangle$

#.

下面：给出 X 上的 covering 的分类（X：conn. & locally path conn. & semilocally simply conn.

记 $\mathrm{Cov}(X) = X$ 的连通覆盖全体

$M(\tilde{X}/X) =$ "$\tilde{X} \xrightarrow{p} X$" 的中间覆盖全体

$$= \left\{ \begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & X \end{tikzcd} \;\middle|\; f, q \text{ 均为 covering}, \; q \circ f = p \right\}.$$

有遗忘映射：$F: M(\tilde{X}/X) \longrightarrow \mathrm{Cov}(X)$

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & X \end{tikzcd} \longmapsto \begin{tikzcd} Y \arrow[d, "q"] \\ X \end{tikzcd}.$$

F 为一个满射（一般提升引理）.

记 "$\sim$" 为 $\mathrm{Cov}(X)$ 上面的由覆盖等价定义的等价关系.

$Y_1 \xrightarrow{p_1} X$, $Y_2 \xrightarrow{p_2} X$. $p_1 \sim p_2 \iff \exists$ 同胚 $\varphi: Y_1 \to Y_2$.

s.t. $$\begin{tikzcd} Y_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & \arrow[d, phantom, "\circlearrowleft"] & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

记 "$\sim$" 为 $M(\tilde{X}/X)$ 上面的之前定义的等价关系.

$$\begin{tikzcd} \tilde{X} \arrow[r, "f_1"] \arrow[dr, "p"'] & Y_1 \arrow[d, "q_1"] \\ & X \end{tikzcd} \sim \begin{tikzcd} \tilde{X} \arrow[r, "f_2"] \arrow[dr, "p"'] & Y_2 \arrow[d, "q_2"] \\ & X \end{tikzcd}$$

$\Longleftrightarrow$ ∃同胚 $\varphi: Y_1 \to Y_2$, s.t. 图表交换：

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f_2"] \arrow[r, "f_1"'] \arrow[dr, "p"'] & Y_1 \arrow[r, "\varphi"] \arrow[d, "q_1"] & Y_2 \arrow[dl, "q_2"] \\ & X & \end{tikzcd}$$

F 诱导了满射：$M(\tilde{X}/X)/\sim \xrightarrow{\bar{F}} \mathrm{Cov}(X)/\sim$.

又由命题4. $M(\tilde{X}/X)/\sim \underset{g}{\overset{f}{\rightleftarrows}} \{H \mid H < \pi_1(X, x)\}$ (1:1).

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[dr, "p"'] & \tilde{X}/\psi_{\tilde{x}}(H) \arrow[d, "q"] \\ & X \end{tikzcd} \longleftarrow\!\shortmid \; H.$$

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[dr] & Y \arrow[d, "q"] \\ & X \end{tikzcd} \quad y = f(\tilde{x}). \quad \longmapsto \; q_* \pi_1(Y, y).$$

$$\xi = F\circ g : \{H \mid H < \pi_1(X,x)\} \longrightarrow Cov(X)/\sim \quad \text{为满射}.$$

$$\begin{tikzcd} H \arrow[r, mapsto] & \tilde{X}/\varphi_x(H) \arrow[d, "q"] \\ & X \end{tikzcd}$$

定义 $\{H \mid H < \pi_1(X,x)\}$ 上的等价关系"$\sim$":

$$H_1 \sim H_2 \iff \exists\, g \in \pi_1(X,x),\ \text{s.t. } gH_1g^{-1} = H_2.$$

记 $C(\pi_1(X,x)) = \{H \mid H < \pi_1(X,x)\}/\sim$. 则 $C(\pi_1(X,x))$ 为 $\pi_1(X,x)$ 的子群的共轭类.

ξ 诱导了 $\bar{\xi} : C(\pi_1(X,x)) \longrightarrow Cov(X)/\sim$. 使图表交换.

$$\begin{tikzcd} \{H \mid H < \pi_1(X,x)\} \arrow[r, "\xi"] \arrow[d] & Cov(X)/\sim \\ C(\pi_1(X,x)) \arrow[ur, "\bar{\xi}"'] & \end{tikzcd}$$

$\forall\, \bar{H} \in C(\pi_1(X,x))$

定义 $\bar{\xi}(\bar{H}) = \xi(H)$.

需验证：若 $H_1, H_2 < \pi_1(X, x)$，且 $\exists \langle\gamma\rangle \in \pi_1(X, x)$，

s.t. $\underline{H_1 = \langle\gamma\rangle H_2 \langle\gamma\rangle^{-1}}$，则 $\xi(H_1) = \xi(H_2)$

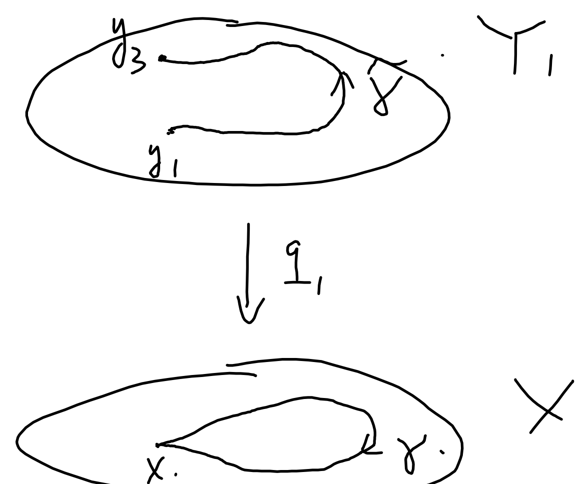
设 $H_1 \rightsquigarrow$

$$\begin{tikzcd} \tilde{X} \arrow[r, "f_1"] \arrow[dr] & \tilde{X}/\psi_{\tilde{x}}(H_1) \ni y_1 = f_1(\tilde{x}) \arrow[d, "g_1"] \\ & X \end{tikzcd} \rightsquigarrow \begin{tikzcd} \tilde{X}/\psi_{\tilde{x}}(H_1) = Y_1 \ni y_1 \arrow[d, "g_1"] \\ X \end{tikzcd}$$

$H_2 \rightsquigarrow$

$$\begin{tikzcd} \tilde{X} \arrow[r, "f_2"] \arrow[dr] & \tilde{X}/\psi_{\tilde{x}}(H_2) \ni y_2 = f_2(\tilde{x}) \arrow[d, "g_2"] \\ & X \end{tikzcd} \rightsquigarrow \begin{tikzcd} \tilde{X}/\psi_{\tilde{x}}(H_2) = Y_2 \ni y_2 \arrow[d, "g_2"] \\ X \end{tikzcd}$$

$(g_1)_* \pi_1(Y_1, y_1) = H_1$，　$(g_2)_* \pi_1(Y_2, y_2) = H_2$.



取 $\tilde{\gamma}$ 为 γ 的以 y_1 为基点的提升，

回忆：$\tilde{\gamma}_* : \pi_1(Y_1, y_1) \xrightarrow{\cong} \pi_1(Y_1, y_3)$

$$\langle\alpha\rangle \longmapsto \langle\tilde{\gamma}^{-1} \cdot \alpha \cdot \tilde{\gamma}\rangle.$$

$\Rightarrow \langle\tilde{\gamma}\rangle^{-1} \cdot \pi_1(Y_1, y_1) \cdot \langle\tilde{\gamma}\rangle = \pi_1(Y_1, y_3)$.

$\Rightarrow \langle\gamma\rangle^{-1} \cdot H_1 \cdot \langle\gamma\rangle = g_* \pi_1(Y_1, y_3) = H_2$.

对：

$$\begin{tikzcd} & Y_1 \ni y_3 \arrow[d, "g_1"] \\ y_2 \in Y_2 \arrow[r, "g_2"] & X \ni x \end{tikzcd}$$

$\left((g_2)_* \pi_1(Y_2, y_2) = H_2 = (g_1)_* (\pi_1(Y_1, y_3)) \right.$

应用一般提升引理 $\Rightarrow$ 同胚 $g: Y_2 \to Y_1$, $g(y_2) = y_1$,

且 $g_1 \circ g = g_2$.

$\Rightarrow$ "$Y_1 \xrightarrow{g_1} X$" $\sim$ "$Y_2 \xrightarrow{g_2} X$".

$\therefore$ Ξ: $C(\pi_1(X, x)) \longrightarrow Cov(X)/\sim$ 是良好定义的.

定理2.(覆盖分类定理) 设 X: conn. & locally path conn. & semi-locally simply conn., 则 Ξ: $C(\pi_1(X, x)) \longrightarrow Cov(X)/\sim$ 为

$$\overline{H} \longmapsto \text{“}\tilde{X}/_{\text{Ker}(H)} \xrightarrow{g} X\text{”}$$

一个双射, 且在此对应下, X 上的{Galois covering 全体}

1-1 对应于 $\pi_1(X, x)$ 的正规子群全体.

证明：只要证：ξ 为一个单射.

即证：$\forall H_1, H_2 < \pi_1(X, x)$，若 $\xi(H_1) = \xi(H_2)$，则

$H_1 \sim H_2$.

下证之：设 $Y_i = \tilde{X}/\psi_{\tilde{X}}(H_i)$，记相关映射如下：

$$\begin{tikzcd} \tilde{X} \arrow[r, "f_i"] \arrow[dr, "p"'] & Y_i \ni y_i = f_i(\tilde{x}) \arrow[d, "g_i"] \\ & X \end{tikzcd}$$

"$\xi(H_i) = \begin{array}{c} Y_i \\ \downarrow g_i \\ X \end{array}$"

$\xi(H_1) = \xi(H_2) \Rightarrow \exists\, g: Y_1 \to Y_2$ 为同胚，使图表交换：

$$\begin{tikzcd} y_1 \in Y_1 \arrow[rr, "g"] \arrow[dr, "g_1"'] & & Y_2 \ni y_2 \arrow[dl, "g_2"] \\ & X & \end{tikzcd}$$

$Y_2 \ni y_3 = g(y_1)$

$(g_1)_* \pi_1(Y_1, y_1) = H_1$

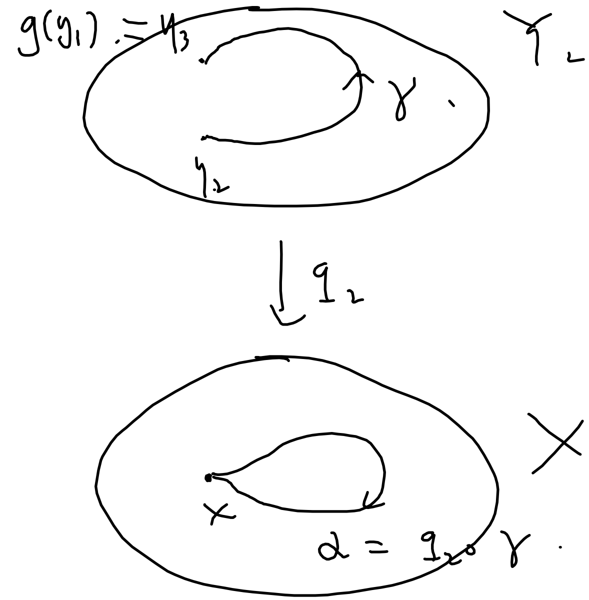
$(g_2)_* \pi_1(Y_2, y_2) = H_2$.

$g_*: \pi_1(Y_1, y_1) \xrightarrow{\cong} \pi_1(Y_2, y_3)$.

$\underbrace{(g_2)_*(g_*)}_{= (g_1)_*} \left(\pi_1(Y_1, y_1)\right) = (g_2)_* \left(\pi_1(Y_2, y_3)\right) = H_1$.

$(q_2)_* \pi_1(Y_2, y_3) = H_1$

$(q_2)_* \pi_1(Y_2, y_2) = H_2$.



$q_2(y_3) = q_2(y_2)$

$= x$.

$\langle\gamma\rangle^{-1} \cdot \pi_1(Y_2, y_2) \cdot \langle\gamma\rangle = \pi_1(Y_2, y_3)$

$\Rightarrow \langle\alpha\rangle^{-1} \cdot H_2 \cdot \underbrace{\langle\alpha\rangle}_{\in \pi_1(X, x)} = H_1$.

$\Rightarrow H_1 \sim H_2$.

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