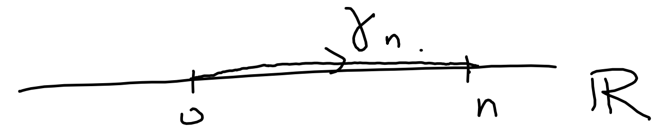
回顾上一次： $\pi_1(S^1, 1) \cong \mathbb{Z}$.

$\pi: \mathbb{R} \longrightarrow S^1 \quad x \longmapsto e^{2\pi i x}$. covering map.

$\varphi: \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$. $n \longmapsto \langle \pi \circ \gamma_n \rangle$.



良好定义. 群同态

① φ 满： 道路提升引理.

② φ 单： 同伦提升引理.

观察： $\mathbb{Z} \curvearrowright \mathbb{R}$. $\forall n \in \mathbb{Z}$, $x \in \mathbb{R}$, $n(x) = x + n$.

$\mathbb{R}/\mathbb{Z} \cong S^1$. π 可视为从 $\mathbb{R}$ 到 $\mathbb{R}/\mathbb{Z}$ 之间的投射.

X: top space, G: group. $G \curvearrowright X$. 考虑 X/G.

$\pi_1(X/G, \bar{x}_0) \cong G$

条件……

希望： $X \longrightarrow X/G$ 为 covering map. Step 1.

定义：$G \curvearrowright X$ 称为 even 的，if：

$\forall x \in X$，$\exists\, x$ 的开邻域 U，s.t. $U \cap g(U) = \phi$，

$\forall g \in G$，$g \neq e$.

Lemma. 设 $G \curvearrowright X$ 为 even 的，则 $X \xrightarrow{\pi} X/G$ 为一个 covering map.

Pf. $\forall \bar{x} \in X/G$，其中 $x \in X$. $\exists\, x$ 的开邻域 U，

s.t. $U \cap g(U) = \phi$，$\forall g \in G$，$g \neq e$.

回顾：$\pi: X \to X/G$ 为开映射.

$\pi(U)$ 为 X/G 中 $\bar{x}$ 的开邻域.

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} g(U).$$

Claim: ① 此为无交并

② $\pi|_{g(U)}: g(U) \xrightarrow{\cong} \pi(U)$

①：If $y \in g(U) \cap g'(U)$，$y = g(x_1) = g'(x_2)$.

$\Rightarrow U \ni x_1 = g^{-1} g'(x_2) \in g^{-1} g'(U) \Rightarrow g^{-1} g' = e \Rightarrow g = g'$.

②
$$\begin{tikzcd} g(U) \arrow[r, "\pi|_{g(U)}"] & \pi(U) \\ U \arrow[u, "g", "\cong"'] \arrow[ur, "\pi|_U"'] & \end{tikzcd}$$

只需证: $\pi|_U : U \to \pi(U)$ 为同胚.

只需证: $\pi|_U : U \to \pi(U)$ 为单射.

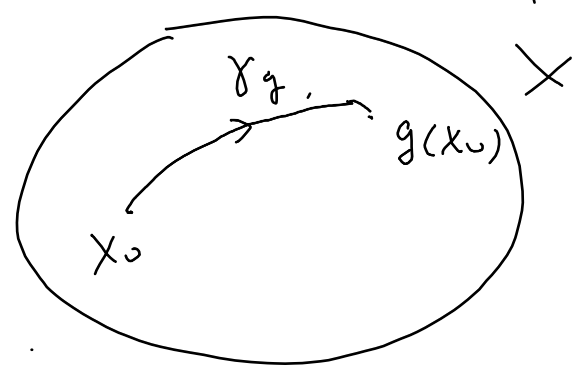
若 $y_1, y_2 \in U$, $\pi(y_1) = \pi(y_2)$.

$\Rightarrow \underset{U \,\ni}{y_1} = \underset{\in\, g(U)}{g(y_2)}$ 对某 $g \in G$ 成立

$\Rightarrow g = e$. $\Rightarrow y_1 = y_2$ #.

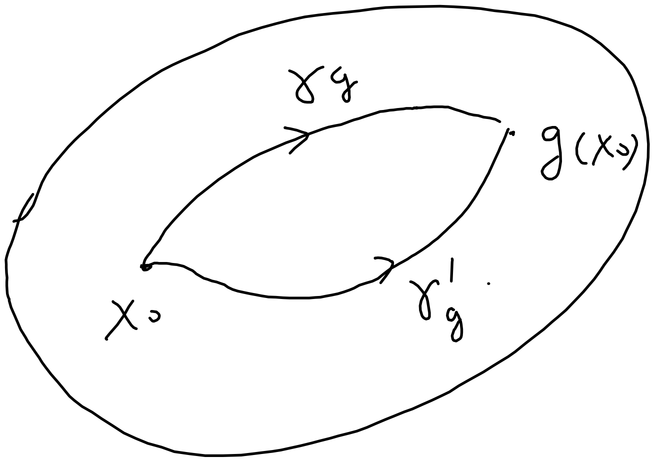
Step 2. $\varphi : G \longrightarrow \pi_1(X/G, \bar{x}_0)$.

$g \longmapsto \langle \pi \circ \gamma_g \rangle$



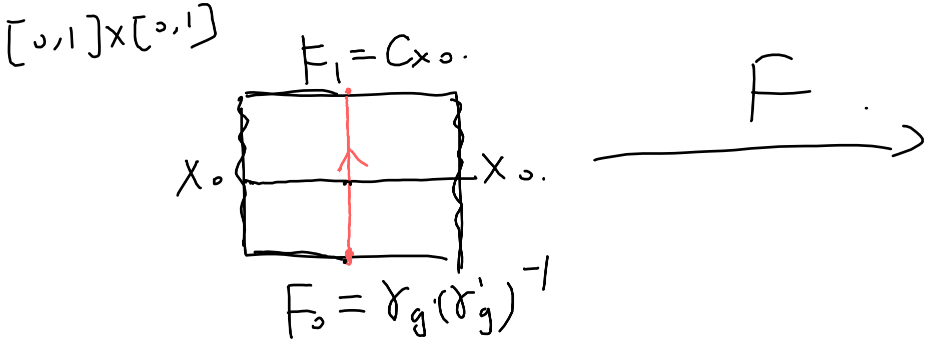
良好定义? 加条件: X单连通.

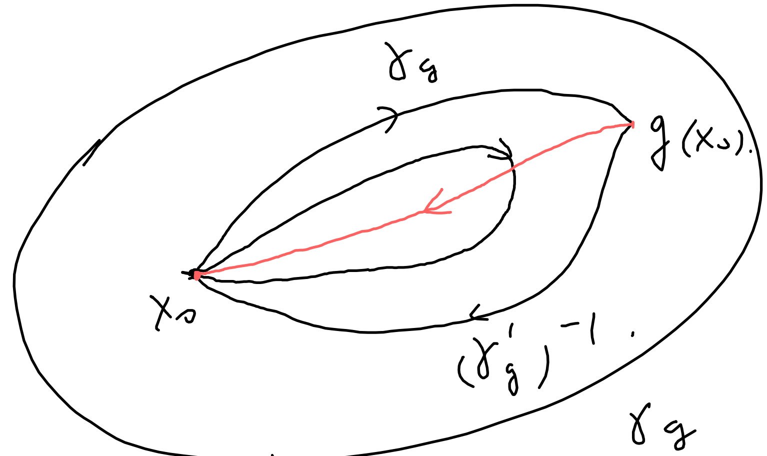
设X单连通.



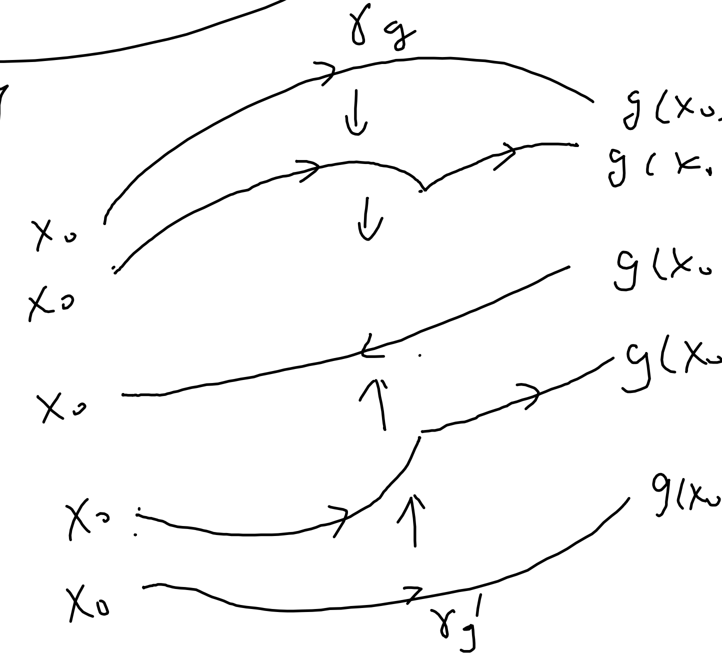
X 单连通：

$\gamma_g \cdot (\gamma'_g)^{-1} \underset{F}{\simeq} c_{x_0}$ rel $\{0,1\}$





我们希望：$\gamma_g \simeq \gamma'_g$ rel $\{0,1\}$



Step 3. φ 为同态.（抄）.

Step 4. φ 为双射.（抄）.

命题：设 X 单连通，G: top group，设 $G \curvearrowright X$ 为 even 的，则 $\pi_1(X/G, \overline{x_0}) \cong G$.

例：$\mathbb{RP}^n = S^n/\mathbb{Z}_2$. $n \geq 2$.

$\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$

$n=1$, $\mathbb{RP}^1 = S^1/\mathbb{Z}_2 = S^1$.

$\pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$.

例：$T = \mathbb{R}^2/\mathbb{Z}\times\mathbb{Z}$. $\pi_1(T) \cong \mathbb{Z}\times\mathbb{Z}$. $(p,q)=1$.

例：$\mathbb{Z}_p \curvearrowright S^3$, $g(z_1,z_2) = (e^{\frac{2\pi i}{p}}z_1, e^{\frac{2\pi i q}{p}}z_2)$.

$S^3 = \{(z_1,z_2) \mid |z_1|^2+|z_2|^2=1\}$

$L(p,q) = S^3/\mathbb{Z}_p \Rightarrow \pi_1(L(p,q)) \cong \mathbb{Z}_p$.

§6. 同伦型 (homotopy type).

定义. 设 X, Y 为 top spaces, 称 X 与 Y 具有相同的同伦型 (或同伦等价), if:

$$\exists\ X \underset{g}{\overset{f}{\rightleftarrows}} Y, \text{ s.t.}$$

$$f\circ g \simeq id_Y, \quad g\circ f \simeq id_X.$$

此时, 称 $\left.\begin{matrix} f \\ g \end{matrix}\right\}$ 为从 $\left\{\begin{matrix} X \\ Y \end{matrix}\right.$ 到 $\left\{\begin{matrix} Y \\ X \end{matrix}\right.$ 的一个同伦等价.

称 $\left.\begin{matrix} g \\ f \end{matrix}\right\}$ 为 $\left.\begin{matrix} f \\ g \end{matrix}\right\}$ 的同伦逆. (homotopy inverse).

命题: "$\simeq$" 为等价关系.

Pf. ① $X \simeq X$.
② $X \simeq Y$, ~~$Y \simeq X$~~ } 显然
③ $X \simeq Y$, $Y \simeq Z$ $\not\Rightarrow$ $X \simeq Z$.

设 $X \underset{g}{\overset{f}{\rightleftarrows}} Y$，$Y \underset{k}{\overset{h}{\rightleftarrows}} Z$ 为同伦等价，

$$X \underset{g\circ k}{\overset{h\circ f}{\rightleftarrows}} Z.$$

Claim： $h\circ f$ 与 $g\circ k$ 互为同伦逆.

$$(h\circ f)\circ(g\circ k) = h\circ\underbrace{(f\circ g)}_{\simeq \mathrm{Id}_Y}\circ k \simeq h\circ \mathrm{Id}_Y\circ k = h\circ k \cong \mathrm{Id}_Z.$$

「Lemma： $X \xrightarrow{F} Y \overset{h_1}{\underset{h_2}{\rightrightarrows}} Z \xrightarrow{G} W$. $h_1\simeq h_2$，则

$$\begin{cases} ① & h_1\circ F \simeq h_2\circ F, \\ ② & G\circ h_1 \simeq G\circ h_2 \end{cases}$$
」.

同理：$(g\circ k)\circ(h\circ f) \simeq \mathrm{Id}_X$.

$\Rightarrow X\simeq Z$. #

例: $X \underset{\text{同胚}}{\cong} Y$, 则 $X \simeq Y$.

例: $C \subset \mathbb{R}^n$, C convex. 则 $C \simeq \{pt\}$.

$$
\begin{tikzcd}
C \arrow[r, bend left, "f"] & \{pt\} \arrow[l, bend left, "g"]
\end{tikzcd}
$$

$x_0 \longleftarrow\!\shortmid\; pt$

$f \circ g = id_{\{pt\}}$.

$g \circ f : C \longrightarrow C$.

$x \longmapsto x_0$.

$g \circ f \underset{\text{直线同伦}}{\simeq} Id_C$.

例: $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$.

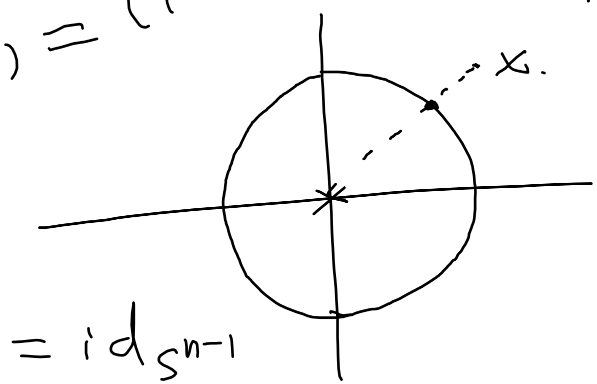
$$
\begin{tikzcd}
S^{n-1} \arrow[r, bend left, "f"] & \mathbb{R}^n \setminus \{0\} \arrow[l, bend left, "g"]
\end{tikzcd}
$$

$x \longmapsto x = f(x)$.

$g(x) = \frac{x}{\|x\|} \longleftarrow\!\shortmid\; x$

$F(x,t) = (1-t)\frac{x}{\|x\|} + t \cdot x$.

$n=2$



$g \circ f = id_{S^{n-1}}$

$f \circ g : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$

$x \longmapsto \frac{x}{\|x\|}$.

$\Rightarrow f \circ g \underset{F}{\simeq} id_{\mathbb{R}^n \setminus \{0\}}$.

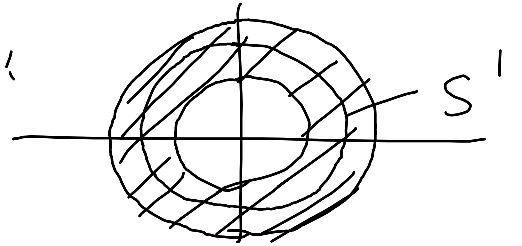
定义：设 X: top sp. $A \subset X$. $r: X \to A$ 连续, 称 r 为一个收缩(retraction), 若 $r|_A = id_A$. (i.e. $r \circ \iota_A = id_A$, 其中 $\iota_A: A \to X$).

定义. $A \subset X$, 称 A 为 X 的一个(强)形变收缩核, 若: 存在同伦: $F: X \times I \to X$, 使:

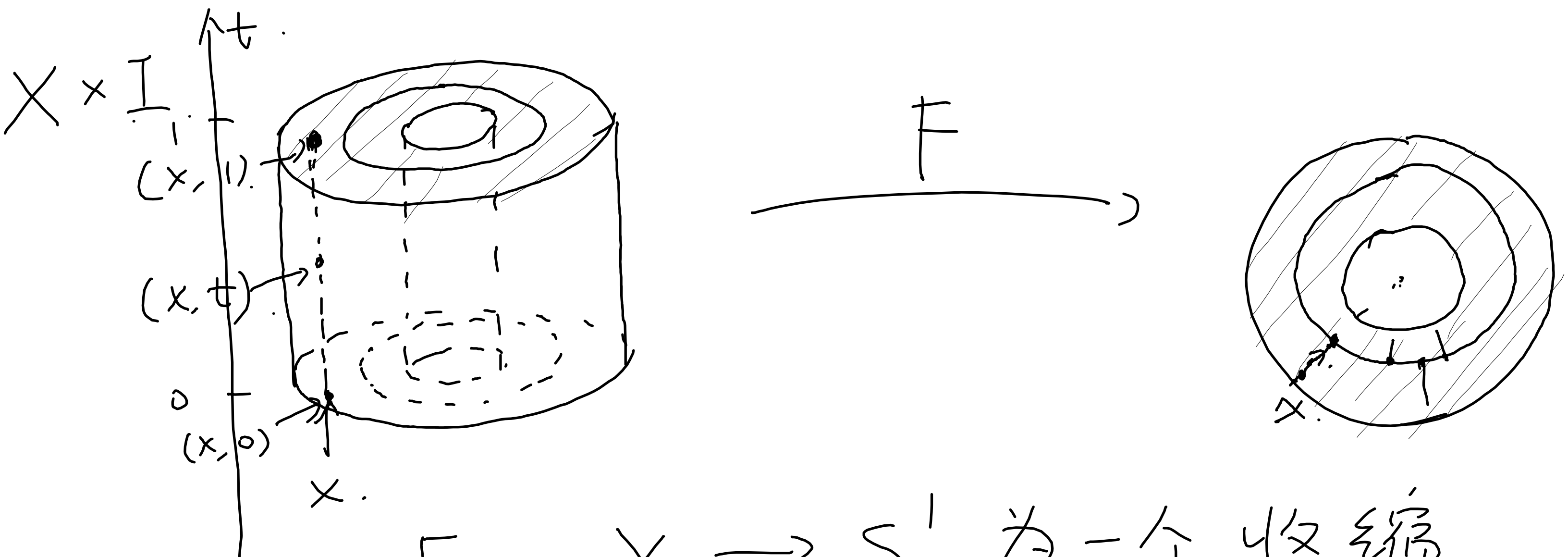
$$\begin{cases} ① \ F_0 = id_X \\ ② \ F_1: X \longrightarrow F_1(X) = A \text{ 为从 } X \text{ 到 } A \text{ 的一个收缩} \\ ③ \ \forall a \in A, \ \forall t \in I, \ F(a,t) = a. \end{cases}$$

$X \underset{g}{\overset{f}{\rightleftarrows}} S^1$

X:



$$F(x,t) = (1-t)x + t\cdot\frac{x}{\|x\|}$$



$F_1: X \longrightarrow S^1$ 为一个收缩.

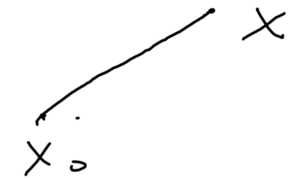
$x \longmapsto \dfrac{x}{\|x\|}$

$\therefore$ S^1 为 X 的(强)形变收缩核.

例: S^{n-1} 为 $\mathbb{R}^n \setminus \{0\}$ 的(强)形变收缩核.

例: $\{x_0\} \subset \mathbb{R}^n$.

$$F(x,t) = x + (x_0 - x)t.$$

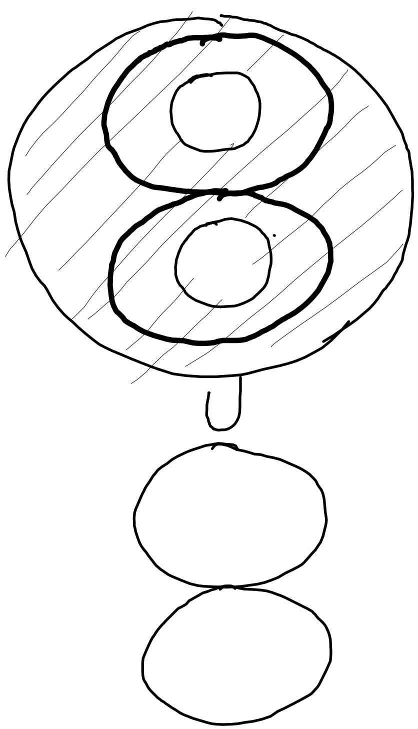


$$F(x,0) = x$$

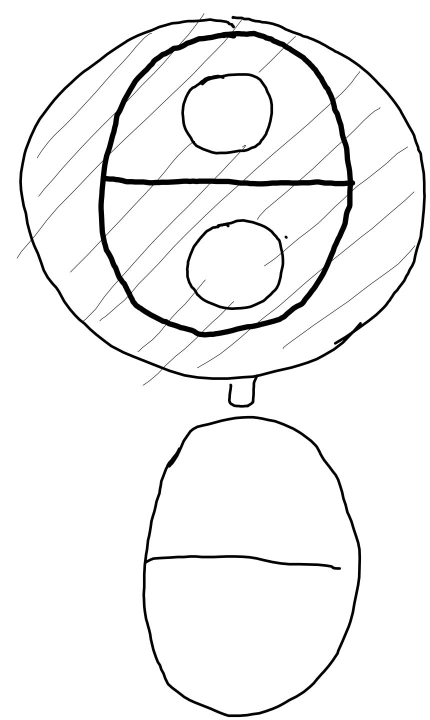
$$F(x,1) = x_0$$

$$F(x_0,t) = x_0$$

例:



$\simeq$



Rmk. 设 $A \subset X$ 为一个形变收缩核, $A \simeq X$.

$$\begin{tikzcd} A \arrow[r, bend left, "i"] & X \arrow[l, bend left, "F_1"] \end{tikzcd}$$

$F_1 \circ i : A \to A$.
$F_1 \circ i = Id_A$.

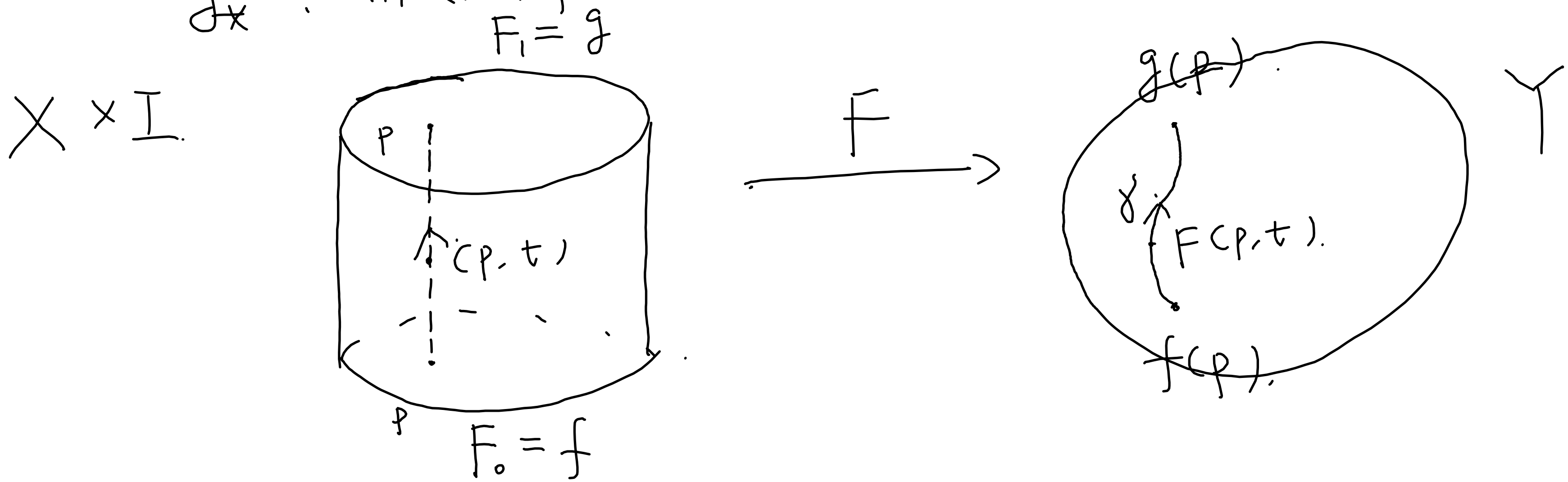
$i \circ F_1 : X \to X$.
$x \mapsto F_1(x)$.
$i \circ F_1 = F_1 \simeq id_X$.

下面要记: $X \simeq Y$. X, Y path conn.
$\pi_1(X) \cong \pi_1(Y)$

Lemma. 设 $f, g: X \longrightarrow Y$, $f \underset{F}{\simeq} g$, 设 $p \in X$.

$f_* : \pi_1(X, p) \longrightarrow \pi_1(Y, f(p))$,

$g_* : \pi_1(X, p) \longrightarrow \pi_1(Y, g(p))$.



$\gamma : [0, 1] \longrightarrow Y$, $\quad t \longmapsto F(p, t)$.

$\gamma_* : \pi_1(Y, f(p)) \longrightarrow \pi_1(Y, g(p))$ $\quad \langle \alpha \rangle \longmapsto \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle$.

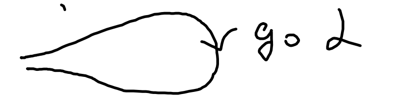
则有交换图表:

$$\begin{tikzcd} \forall \langle \alpha \rangle \in \pi_1(X, p) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(Y, f(p)) \arrow[d, "\gamma_*", "\cong"'] \\ & \pi_1(Y, g(p)). \end{tikzcd}$$

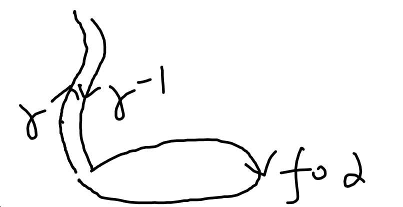
Pf. 要证: $\forall \langle\alpha\rangle \in \pi_1(X,p)$, 有:

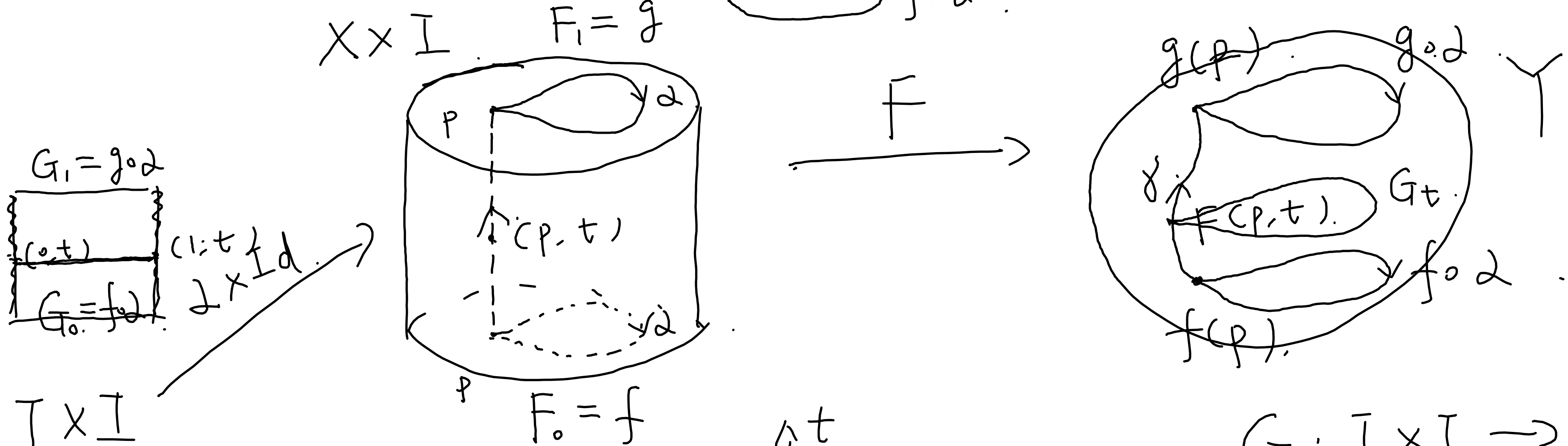
$$\gamma_* \circ f_*(\langle\alpha\rangle) = g_*(\langle\alpha\rangle).$$

$$\gamma_*(\langle f\circ\alpha\rangle) \qquad\qquad \langle g\circ\alpha\rangle.$$



$$\langle \underline{\gamma^{-1}\cdot(f\circ\alpha)\cdot\gamma} \rangle$$

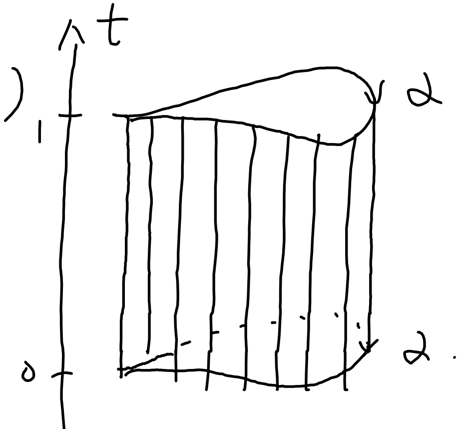


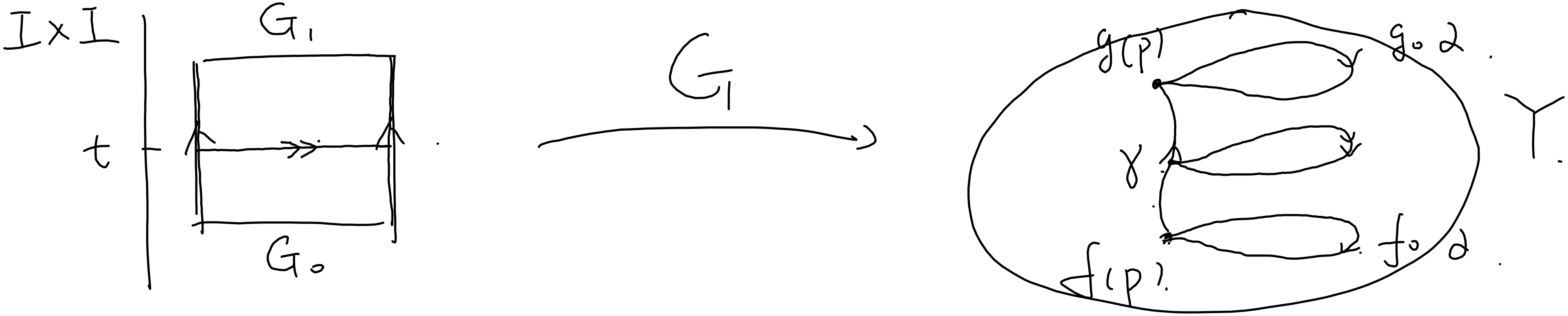


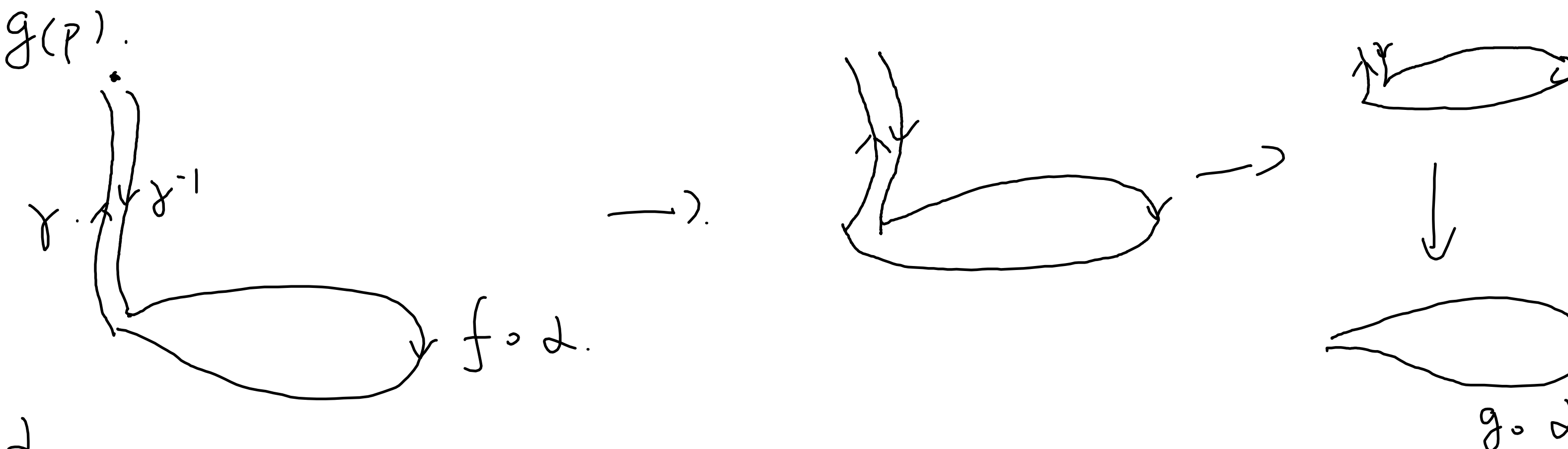
$$(s,t) \longmapsto (\alpha(s),t)$$

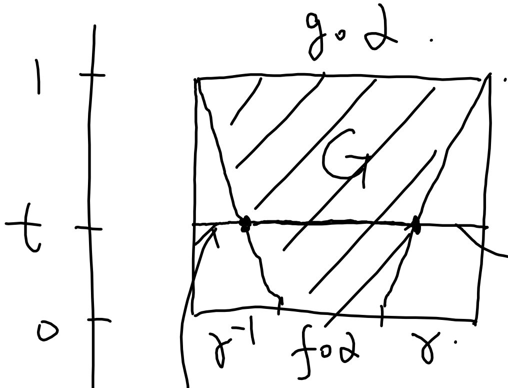
$$G: I\times I \to Y$$

$$G := F\circ(\alpha\times Id)$$









从 $\gamma(t)$ 沿 γ 走到 $g(p)$

从 $g(p)$ 沿 γ^{-1} 走到 $\gamma(t)$,

#

命题：具有相同伦型的两个道路连通空间基本群互相同构.

Pf. 设 $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ 为同伦等价. X, Y path conn.

$q \in Y$

$$\begin{tikzcd} & q \arrow[dl, "g"'] \\ p \arrow[dr] & \\ & q' \end{tikzcd}$$

$f_* : \pi_1(X, p) \to \pi_1(Y, q')$. 下面证：$f_*$ 为同构.

$(f\circ g)_* : \pi_1(Y, q) \to \pi_1(Y, q')$.

由 $f\circ g \simeq Id_Y$，有交换图表：

$$\begin{tikzcd} \pi_1(Y,q) \arrow[r, "(f\circ g)_*"] \arrow[dr, "(Id_Y)_*"'] & \pi_1(Y,q') \arrow[d, "\cong"] \\ & \pi_1(Y,q) \end{tikzcd}$$

$\Rightarrow (f\circ g)_*$ 为群同构.

$(f\circ g)_* = f_* \circ g_*$. $\Rightarrow f_*$ 满

$(g\circ f)_* : \pi_1(X, p) \xrightarrow{(g\circ f)_*} \pi_1(X, g(q'))$. $\Rightarrow (g\circ f)_*$ 为同构 $= g_* \circ f_*$

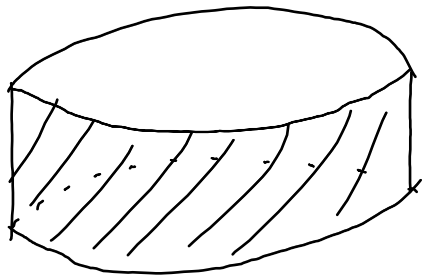
又由 $g\circ f \simeq Id_X$,

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "(g\circ f)_*"] \arrow[dr, "(id_X)_*"'] & \pi_1(X,g(q')) \arrow[d, "\cong"] \\ & \pi_1(X,p) \end{tikzcd}$$

$\Rightarrow f_*$ 单. #

例： $M \simeq S^1$.

$$\pi_1(M) \cong \pi_1(S^1) \cong \mathbb{Z}$$

例： $C \simeq S^1$

$$\Rightarrow \pi_1(C) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

例：$\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) \cong \pi_1(S^1) \cong \mathbb{Z}$.

可缩空间

定义：设 X: top sp. X 称为可缩的，若 $\exists p \in X$, s.t.

$$Id_X \simeq C_p, \text{ 其中 } C_p : X \to X, \ x \mapsto p.$$

命题：① X 可缩的 $\Longleftrightarrow$ ② $\forall p \in X$, $Id_X \simeq c_p$

$\Longleftrightarrow$ ③ $X \simeq \{pt\}$.

Pf. ② $\Rightarrow$ ① 显然.

① $\Rightarrow$ ②:

Lemma: 设 X 可缩, $f, g: Y \to X$, 则 $f \simeq g$.

Pf. 设 $p \in X$, s.t. $Id_X \simeq c_p$.

$$f = Id_X \circ f \simeq c_p \circ f = c_p \circ g \simeq Id_X \circ g = g$$

#

设 $p_0 \in X$, s.t. $Id_X \simeq c_{p_0}$.

$\forall p \in X$, $c_p: X \to X$, $x \mapsto p$.

$\overset{\text{Lemma}}{\Longrightarrow}$ $c_p \simeq Id_X$.

① $\Rightarrow$ ③. 设 $p_0 \in X$, s.t. $Id_X \simeq c_{p_0}$.

$$X \underset{g}{\overset{f}{\rightleftarrows}} \{pt\} \qquad g(pt) = p_0.$$

$g \circ f: X \to X$, $x \mapsto p_0$.

$g \circ f = c_{p_0}$.

③ ⇒ ① 设 $X \underset{g}{\overset{f}{\rightleftarrows}} \{pt\}$ 为同伦等价.

设 $g(pt) = p_0$,

Claim: $C_{p_0} \simeq Id_X$

$g \circ f = C_{p_0} \simeq Id_X$. #

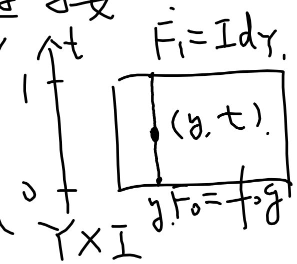
推论: 可缩的空间是单连通的.

Pf. 只要证: 可缩空间~~为~~道路连通.

Lemma: 若 $X \simeq Y$, 则 X path conn. $\Leftrightarrow$ Y path conn.

Pf. 设 X path conn. 设 $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ 为同伦等价,

取 $q \in X$, $p = f(q)$. 只要证: $\forall y \in Y$, y 与 p 可用道路连接. $\forall y \in Y$, y 可与 $f\circ g(y)$ 用道路连接



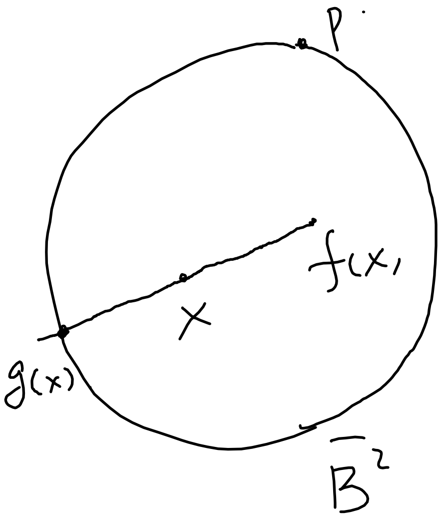
$\xrightarrow{F} Y$) 又 $f\circ g(y)$ 与 $p = f(q)$ 可用道路连接 ($\because$ 同属 $f(X)$) #

#

例：(Brower 不动点定理 ($n=2$ 版本))

定理：设 $\overline{B^n}=\{x\in\mathbb{R}^n \mid |x|\le 1\}$. 则 $\forall$ 连续 $f:\overline{B^n}\to\overline{B^n}$,
必一个不动点.

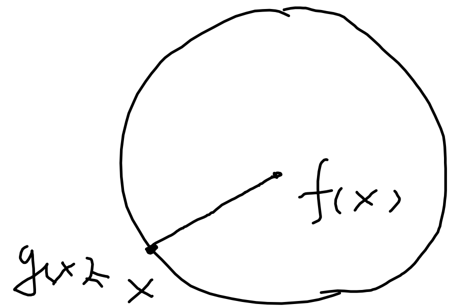
proof of "$n=2$"：$f:\overline{B^2}\to\overline{B^2}$ 连续. 设 $\forall x\in\overline{B^2}$,



有 $f(x)\neq x$. $g:\overline{B^2}\longrightarrow\partial\overline{B^2}\cong S^1$

$x\longmapsto g(x)$

易见 $\begin{cases} g \text{ 连续} \\ g|_{\partial\overline{B^2}}=Id_{\partial\overline{B^2}} \end{cases}$.



$\Rightarrow$ g 为从 $\overline{B^2}$ 到 $\partial\overline{B^2}$ 的收缩映射.

记 $i:\partial\overline{B^2}\to\overline{B^2}$ $x\mapsto x$.

$$g_*\circ i_*$$
$$\parallel$$
$$Id_{\partial\overline{B^2}}=g\circ i:\partial\overline{B^2}\to\partial\overline{B^2}\Rightarrow(g\circ i)_*:\pi_1(\partial\overline{B^2},p)\xrightarrow{\cong}\pi_1(\partial\overline{B^2},p)$$

$$\Rightarrow g_*:\underset{\{e\}}{\underset{\cong}{\pi_1(\overline{B^2},p)}}\to\underset{\mathbb{Z}}{\underset{\cong}{\pi_1(\partial\overline{B^2},p)}}$$ 为满同态. 矛盾.

#

§7. 基本群的计算

X : top space, $X_0, X_1 \subset X$, $\mathrm{int}(X_0) \cup \mathrm{int}(X_1) = X$,

$X_{01} = X_0 \cap X_1$, X_0, X_1, X_{01} 道路连通. $p \in X_{01}$

Van Kampen 定理:

$$\begin{array}{ccc} X_{01} & \xrightarrow{j_0} & X_0 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle i_0} \\ X_1 & \xrightarrow[i_1]{} & X \end{array} \qquad \begin{array}{ccc} \pi_1(X_{01}, p) & \xrightarrow{(j_0)_*} & \pi_1(X_0, p) \\ {\scriptstyle (j_1)_*}\downarrow & & \downarrow{\scriptstyle (i_0)_*} \\ \pi_1(X_1, p) & \xrightarrow[(i_1)_*]{} & \pi_1(X, p) \end{array}$$

$$\pi_1(X, p) \cong \pi_1(X_0, p) *_{\pi_1(X_{01}, p)} \pi_1(X_1, p)$$

$$\pi_1(X_0, p) *_{\pi_1(X_{01}, p)} \pi_1(X_1, p) := \pi_1(X_0, p) * \pi_1(X_1, p) \Big/ \left\{ (j_0)_*(\langle \alpha \rangle)^{-1} \cdot (j_1)_*(\langle \alpha \rangle) \,\middle|\, \langle \alpha \rangle \in \pi_1(X_{01}, p) \right\}$$

Rmk. 若 $\langle \gamma_1 \rangle, \cdots, \langle \gamma_n \rangle$ 为 $\pi_1(X_0, p)$ 的一组生成元

$\langle \beta_1 \rangle, \cdots, \langle \beta_m \rangle$ .. $\pi_1(X_1, p)$ . . . . .

$$\Rightarrow \left\{ (i_0)_*(\langle \gamma_i \rangle), (i_1)_*(\langle \beta_j \rangle) \,\middle|\, \begin{array}{l} 1 \le i \le n \\ 1 \le j \le m \end{array} \right\} \text{ 为 } \pi_1(X, p) \text{ 的一组生成元.}$$

常用的情况：X_1 单连通，$\pi_1(X_1,p)=\{e\}$.

$$\pi_1(X_0,p)*\pi_1(X_1,p)\Big/\overline{\{(j_0)_*(\langle\alpha\rangle)^{-1}\cdot(j_1)_*(\langle\alpha\rangle)\mid\langle\alpha\rangle\in\pi_1(X_{01},p)\}}.$$

$$\pi_1(X_0,p)*\pi_1(X_1,p)\cong\pi_1(X_0,p).$$

$$\Rightarrow\pi_1(X_0,p)*_{\pi_1(X_{01},p)}\pi_1(X_1,p)\cong\pi_1(X_0,p)/N,$$

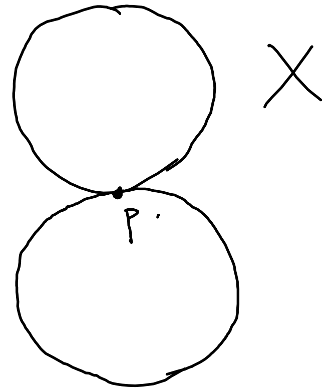
其中 $N=\overline{\{(j_0)_*(\langle\alpha\rangle)\mid\langle\alpha\rangle\in\pi_1(X_{01},p)\}}$.

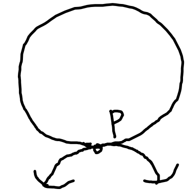
特别地，若 $\pi_1(X_{01},p)=\langle\langle\alpha_1\rangle,\cdots,\langle\alpha_n\rangle\rangle$，其中 $\langle\alpha_i\rangle\in\pi_1(X_{01},p)$,

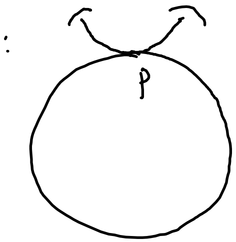
$$N=\overline{\{(j_0)_*(\langle\alpha_i\rangle)\mid i=1,\cdots,n\}}$$

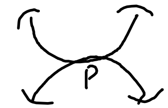
$$\pi_1(X,p)\cong\pi_1(X_0,p)\Big/\overline{\{(j_0)_*(\langle\alpha_1\rangle),\cdots,(j_0)_*(\langle\alpha_n\rangle)\}}.$$

例1. (8字形)



X_0: 

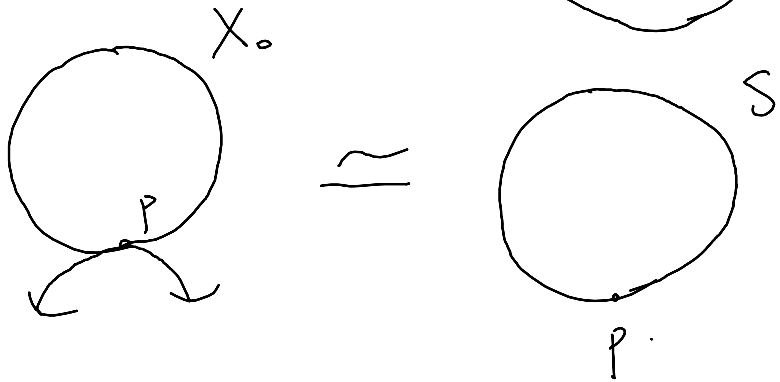
X_1: 

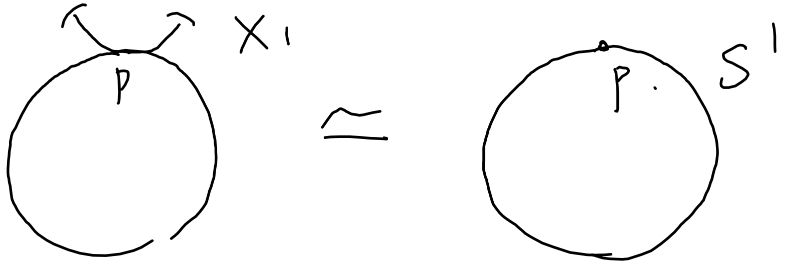
X_{01}: 

$\simeq \{pt\}$.

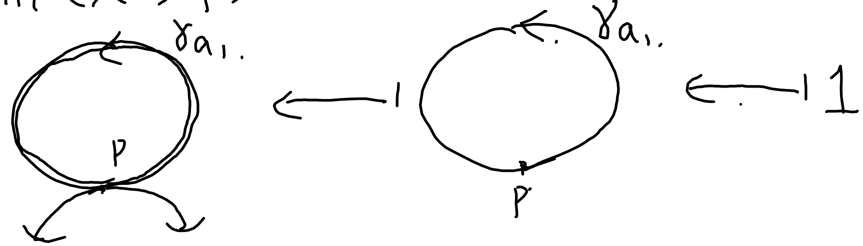
$$\pi_1(X,p) \cong \pi_1(X_0,p) * \pi_1(X_1,p).$$
$$\cong \mathbb{Z} * \mathbb{Z}$$

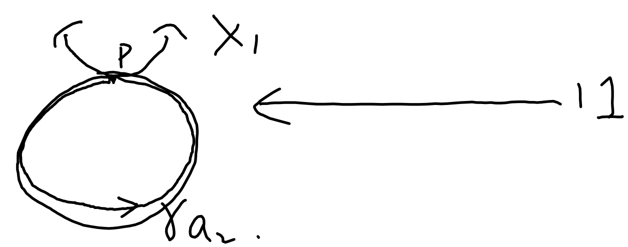
记 1_i 是 $\mathbb{Z} * \mathbb{Z}$ 中第 i 个 $\mathbb{Z}$ 的 1, $i=1,2$.





选定: $\pi_1(X_0,p) \cong \pi_1(S^1,p) \cong \mathbb{Z}$, $\pi_1(X_1,p) \cong \pi_1(S^1,p) \cong \mathbb{Z}$.





$$\begin{tikzcd}
 & \pi_1(X_0, p) \arrow[d, "(i_0)_*"] \arrow[drr, "\iota_1"] \arrow[rr, "\cong"] & & \mathbb{Z} \arrow[d, "\tilde{\iota}_1"] \\
\pi_1(X_1, p) \arrow[r, "(i_1)_*"] \arrow[d, "\cong"'] \arrow[rr, bend right, "\iota_2"'] & \pi_1(X, p) \arrow[r, equal] & \pi_1(X_0, p) * \pi_1(X_1, p) \arrow[r, equal] & \mathbb{Z} * \mathbb{Z} \\
\mathbb{Z} \arrow[urrr, "\tilde{\iota}_2"'] & & &
\end{tikzcd}$$

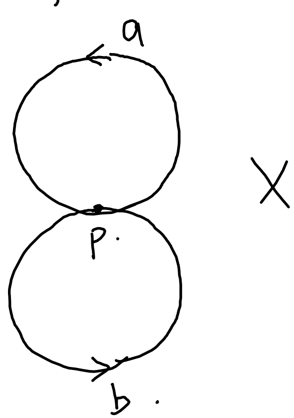
$1 \mapsto 1_1$

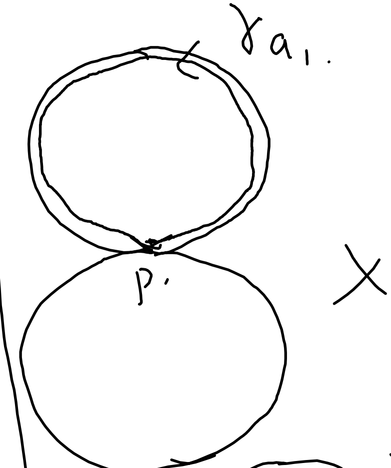
$1 \mapsto 1_2$

$\pi_1(X, p) \cong \mathbb{Z} * \mathbb{Z}$.

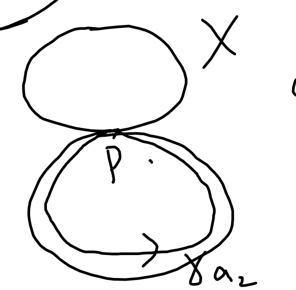
$(i_0)_*(\langle \gamma_{a_1} \rangle)$:

$\pi_1(X, p) \cong \langle a, b \rangle$





$\leftarrow\!\!\shortmid\ 1_1$



$\leftarrow\!\!\shortmid\ 1_2$

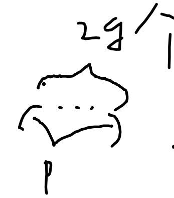
例2. $\bigvee_g S^1$ (wedge).

归纳地：设 $\pi_1(\bigvee_{g-1} S^1, p) \cong \pi_1(S^1,p) * \cdots * \pi_1(S^1,p)$

$\cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{g-1个}$

X_0:

X_{01}:



$\simeq \{pt\}$

X_1:

$$\pi_1(\bigvee_g S^1, p) \cong \pi_1(X_0, p) * \pi_1(X_1, p).$$

$$\cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{g个}$$

$$\pi_1(\bigvee_g S^1, p) \cong \langle a_1, \cdots, a_g \rangle$$

例3. $T = S^1 \times S^1$

之前：$T = \mathbb{R}^2/\mathbb{Z}\times\mathbb{Z}$，$\pi_1(T,\bar{0}) \cong \mathbb{Z}\times\mathbb{Z}$ （方法一）

（方法二）Lemma：$\pi_1(X\times Y,(x_0,y_0)) \cong \pi_1(X,x_0)\times\pi_1(Y,y_0)$.

Pf. $\varphi: \pi_1(X\times Y,(x_0,y_0)) \longrightarrow \pi_1(X,x_0)\times\pi_1(Y,y_0)$.

$\langle\gamma\rangle \longmapsto (\langle p_1\circ\gamma\rangle, \langle p_2\circ\gamma\rangle)$

(well-defined, 群同态.)

$$\begin{tikzcd} & X\times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

$$(x,y) \mapsto x, \quad (x,y) \mapsto y.$$

φ 满：$\forall (\langle\alpha\rangle,\langle\beta\rangle) \in \pi_1(X,x_0)\times\pi_1(Y,y_0)$.

令 $\gamma: [0,1] \longrightarrow X\times Y$ $\in L(X\times Y,(x_0,y_0))$

$t \longmapsto (\alpha(t),\beta(t))$.

$\varphi(\langle\gamma\rangle) = (\langle\alpha\rangle,\langle\beta\rangle)$.

φ 单：若 $\langle\gamma\rangle \in \ker\varphi$，设 $\alpha = p_1\circ\gamma$，$\beta = p_2\circ\gamma$.

$\alpha \underset{F}{\simeq} c_{x_0}$ rel $\{0,1\}$. $\beta \underset{G}{\simeq} c_{y_0}$ rel $\{0,1\}$

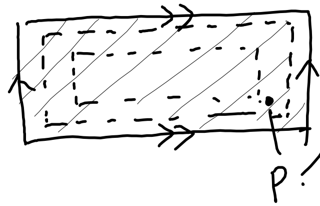
$H: I\times I \longrightarrow X\times Y$ $\Rightarrow \gamma \underset{H}{\simeq} c_{(x_0,y_0)}$ rel$\{0,1\}$.

$(s,t) \longmapsto (F(s,t),G(s,t))$.

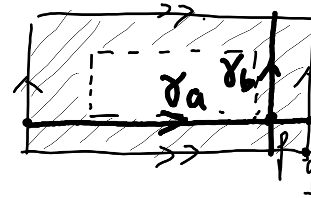
#

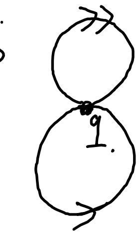
φ 为群同构 $\Longleftarrow$

(方法三). Van Kampen 定理.

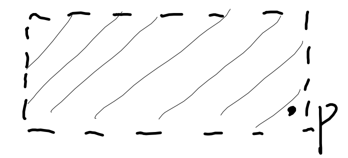
T : 

$/\sim$.

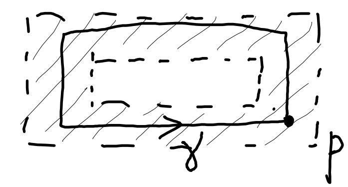
X_0 : 

$/\sim \quad \xrightarrow{r} \simeq$ 

j_0

X_1 : 

单连通

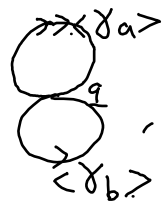
$X_0 \cap X_1 = X_{01}$: 

$\simeq S^1$

$$\pi_1(T,p) \cong \pi_1(X_0,p) *_{\pi_1(X_{01},p)} \pi_1(X_1,p).$$

$$\cong \pi_1(X_0,p) \Big/ \underbrace{\overline{\{(j_0)_*(\langle\alpha\rangle) \mid \langle\alpha\rangle \in \pi_1(X_{01},p)\}}}_{N} \cong \langle a,b\rangle / N'$$

$$= \langle a,b\rangle / \overline{\{b\cdot a^{-1}\cdot b^{-1}\cdot a\}} = \langle a,b \mid b a^{-1} b^{-1} a\rangle = \langle a,b \mid a^{-1}b^{-1} = b^{-1}a^{-1}\rangle = \langle a,b \mid a\cdot b = b\cdot a\rangle \cong \mathbb{Z}\times\mathbb{Z}$$

$\pi_1(X_0,p) \cong \pi_1($ 

$, q) \cong \langle a, b\rangle$

$\triangledown \qquad\qquad\qquad\qquad \triangledown$

$N \longrightarrow N' = \overline{\{c\}}$

$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$

$(j_0)_*(\langle\gamma\rangle) \longmapsto \langle\gamma_b\rangle\cdot\langle\gamma_a\rangle^{-1}\cdot\langle\gamma_b\rangle^{-1}\cdot\langle\gamma_a\rangle \longmapsto c = b\cdot a^{-1}\cdot b^{-1}\cdot a$.

$N = \overline{\{(j_0)_*(\langle\gamma\rangle)\}}$