§2. 基本群

设 X 为 top space, $\forall\, p_1, p_2 \in X$, 记:

$$P(p_1, p_2) = \{\gamma: [0,1] \to X \mid \gamma \text{ 连续}, \gamma(0) = p_1, \gamma(1) = p_2\}$$

若 $p_1 = p_2 = p$, $L(X, p) = P(p, p)$

上一次: $\forall\, \gamma_1, \gamma_2 \in P(p_1, p_2)$,

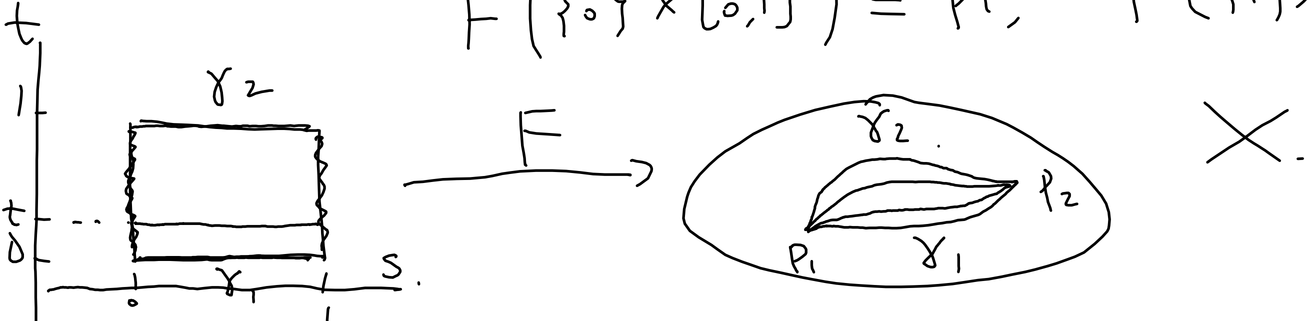
" $\gamma_1 \underset{F}{\simeq} \gamma_2 \text{ rel } \{0, 1\}$ "

$\iff \exists\, F: [0,1] \times [0,1] \to X$ 连续 (记 $F_t: [0,1] \to X$, $s \mapsto F(s,t)$)

$(s, t) \longmapsto F(s,t)$.

$F_0 = \gamma_1$, $F_1 = \gamma_2$

$F(\{0\} \times [0,1]) = p_1$, $F(\{1\} \times [0,1]) = p_2$.



定义 $P(p_1,p_2)$ 上的等价关系 " $\sim$ " :

$$\gamma_1 \sim \gamma_2 \iff \gamma_1 \simeq \gamma_2 \ \text{rel} \ \{0,1\}.$$

商集: $P(p_1,p_2)/\sim \ = \{ \langle \gamma \rangle \mid \gamma \in P(p_1,p_2) \}.$

$\forall \gamma \in P(p_1,p_2)$, 记 $\langle \gamma \rangle$ 为 γ 所代表的等价类 (同伦类).

特别地, $p_1 = p_2 = p$, $L(X,p)/\sim \ \overset{\text{记为}}{=} \ \underline{\pi_1(X,p)}.$

$$\pi_1(X,p) = \left\{ \langle \gamma \rangle \;\middle|\; \begin{array}{l} \gamma : [0,1] \to X, \gamma \text{ 连续} \\ \gamma(0) = \gamma(1) = p \end{array} \right\}$$ ($\langle \gamma \rangle$ 为 γ 的 rel $\{0,1\}$ 的同伦类).

下面: 将定义 $\cdot$: $\pi_1(X,p) \times \pi_1(X,p) \to \pi_1(X,p)$,

使 $(\pi_1(X,p), \cdot)$ 为一个群.

$$\boxed{\alpha \cdot \beta(s) := \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}}$$

① " $\cdot$ " 之定义:

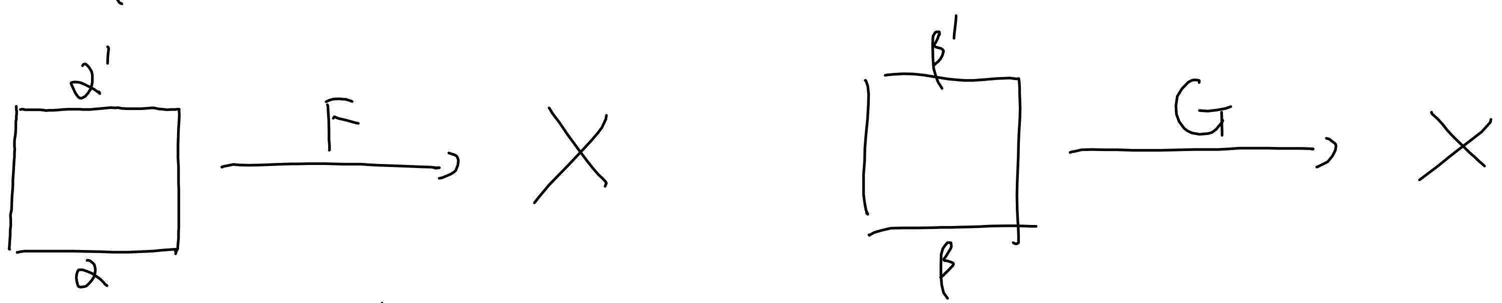
$\forall \langle \alpha \rangle, \langle \beta \rangle \in \pi_1(X,p)$, $\langle \alpha \rangle \cdot \langle \beta \rangle := \langle \alpha \cdot \beta \rangle$

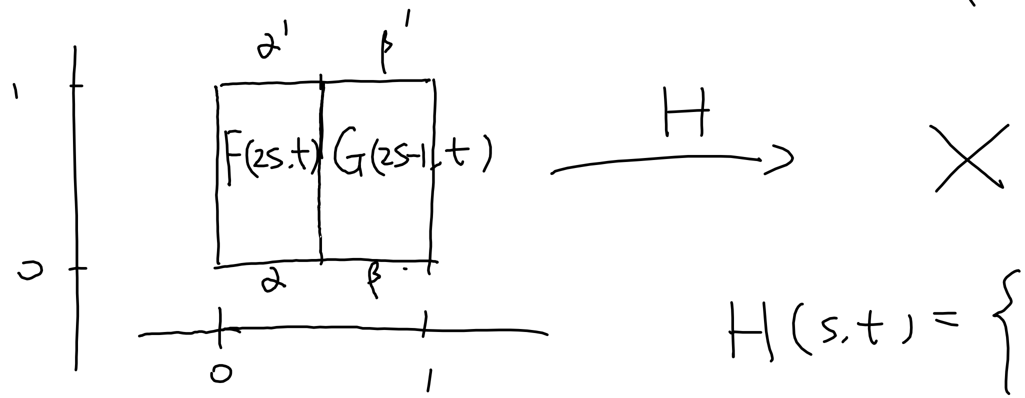
well-defined: 设 $\langle \alpha' \rangle = \langle \alpha \rangle$, $\langle \beta' \rangle = \langle \beta \rangle$, 要证:

$\langle \alpha'\cdot\beta'\rangle = \langle \alpha\cdot\beta\rangle$.

i.e. 设 $\alpha \underset{F}{\simeq} \alpha'$ rel $\{0,1\}$, $\beta \underset{G}{\simeq} \beta'$ rel $\{0,1\}$.

要证 $\alpha\cdot\beta \simeq \alpha'\cdot\beta'$ rel $\{0,1\}$.





$$H(s,t)=\begin{cases}F(2s,t), & 0\le s\le \frac{1}{2}\\ G(2s-1,t), & \frac{1}{2}\le s\le 1\end{cases}$$

② 验证: $(\pi_1(X,p),\cdot)$ 为一个群.

「$(G,\cdot)$ 称为一个群, 若:

① $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ $\forall a,b,c\in G$

② $\exists e\in G$ (恒元), s.t. $\forall a\in G$, $e\cdot a = a$.

③ $\forall a\in G$, $\exists b\in G$, s.t. $b\cdot a = e$ 」

(i) 结合律：

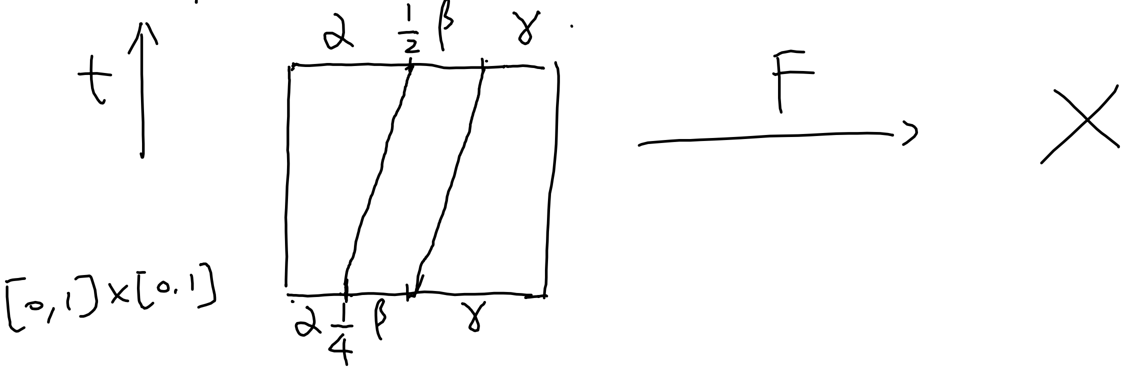
$\forall \langle\alpha\rangle, \langle\beta\rangle, \langle\gamma\rangle \in \pi_1(X,p)$，要证：

$$(\langle\alpha\rangle\cdot\langle\beta\rangle)\cdot\langle\gamma\rangle = \langle\alpha\rangle\cdot(\langle\beta\rangle\cdot\langle\gamma\rangle).$$

$$(\langle\alpha\rangle\cdot\langle\beta\rangle)\cdot\langle\gamma\rangle = \langle\alpha\cdot\beta\rangle\cdot\langle\gamma\rangle = \langle(\alpha\cdot\beta)\cdot\gamma\rangle$$

$$\langle\alpha\rangle\cdot(\langle\beta\rangle\cdot\langle\gamma\rangle) = \langle\alpha\rangle\cdot\langle\beta\cdot\gamma\rangle = \langle\alpha\cdot(\beta\cdot\gamma)\rangle.$$

即证：$\underline{(\alpha\cdot\beta)\cdot\gamma} \simeq \alpha\cdot(\beta\cdot\gamma) \quad \text{rel}\ \{0,1\}$



$$(\alpha\cdot\beta)\cdot\gamma(s) = \begin{cases} \alpha\cdot\beta(2s), & 0\le s\le \frac{1}{2} \\ \gamma(2s-1), & \frac{1}{2}\le s\le 1 \end{cases}$$

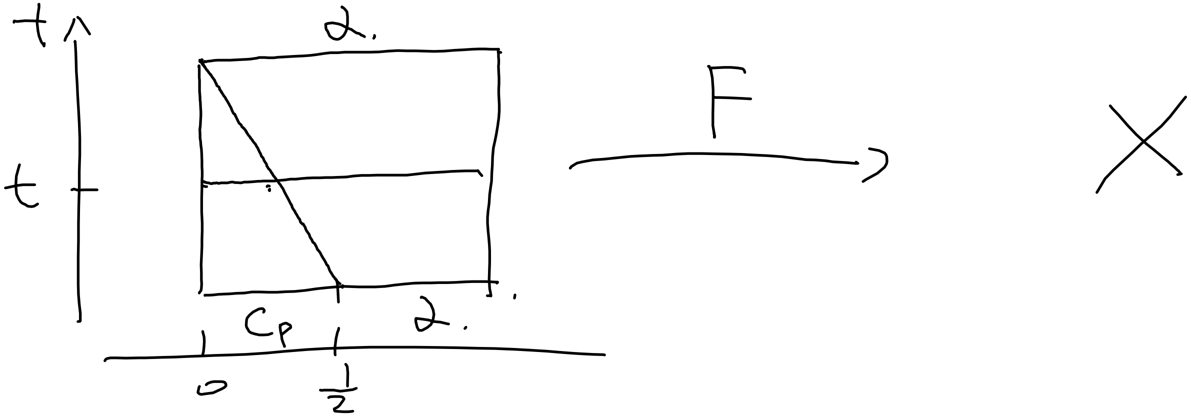
$$\alpha\cdot\beta(2s) = \begin{cases} \alpha(4s), & \underline{0\le 2s\le \frac{1}{2}} \\ \beta(4s-1), & \frac{1}{2}\le 2s\le 1 \end{cases}$$

(ii) 有幺元。

定义 $c_p: [0,1] \to X \quad t \mapsto p$ （常道路）

则 $\langle c_p\rangle$ 为幺元，只需证：$\forall \langle\alpha\rangle \in \pi_1(X,p)$，$\langle c_p\rangle\cdot\langle\alpha\rangle = \langle\alpha\rangle$.

i.e. $c_p \cdot \alpha \simeq \alpha \quad \text{rel } \{0,1\}$.



(iii) $\forall \langle \alpha \rangle \in \pi_1(X, p)$,

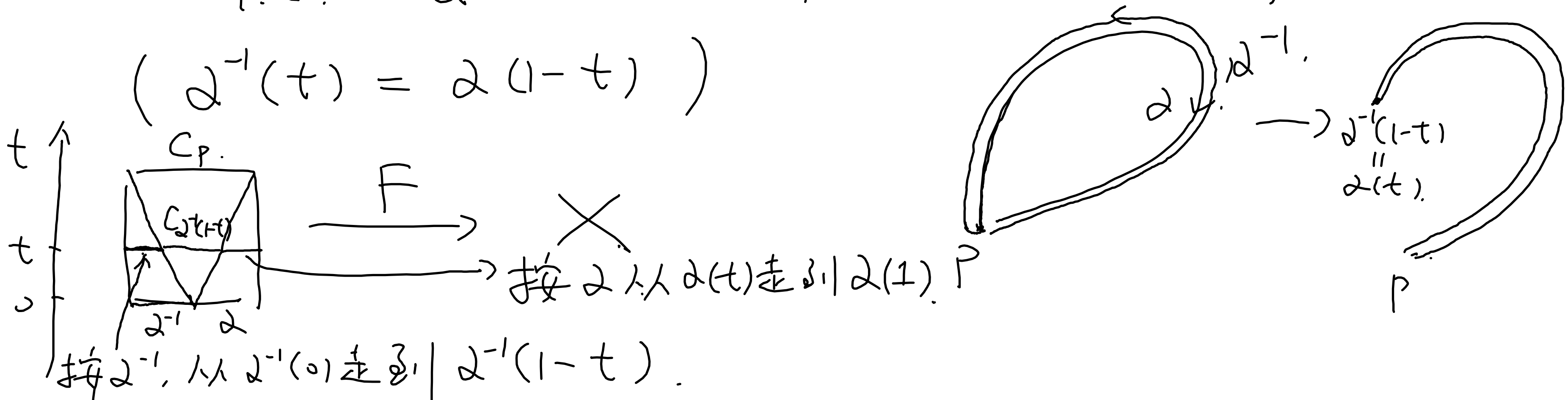
Claim: $\langle \alpha^{-1} \rangle$ 就是 $\langle \alpha \rangle$ 的左逆.

证: 即要证: $\langle \alpha^{-1} \rangle \cdot \langle \alpha \rangle = \langle c_p \rangle$

$$\langle \alpha^{-1} \rangle \cdot \langle \alpha \rangle = \langle \alpha^{-1} \cdot \alpha \rangle$$

i.e. $\alpha^{-1} \cdot \alpha \simeq c_p \quad \text{rel } \{0,1\}$. 时刻 t.

$(\alpha^{-1}(t) = \alpha(1-t))$



综上：$(\pi_1(X,p),\cdot)$ 为一个群 #.

$\pi_1(X,p)$ 称为 X 上以 p 为基点的基本群.
(fundamental group).

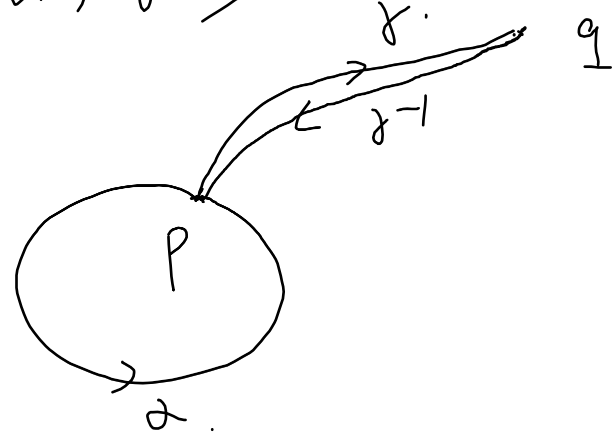
命题1 若 X 为道路连通空间，$\forall p, q \in X$，有：

$$\pi_1(X,p) \cong \pi_1(X,q) \quad (\text{群同构}).$$

Pf $\forall p, q \in X$，取 $\gamma: [0,1] \to X$，$\gamma(0) = p$，$\gamma(1) = q$.

定义 $\gamma_*: \pi_1(X,p) \longrightarrow \pi_1(X,q)$.

$$\langle \alpha \rangle \longmapsto \langle (\gamma^{-1} \alpha) \cdot \gamma \rangle$$

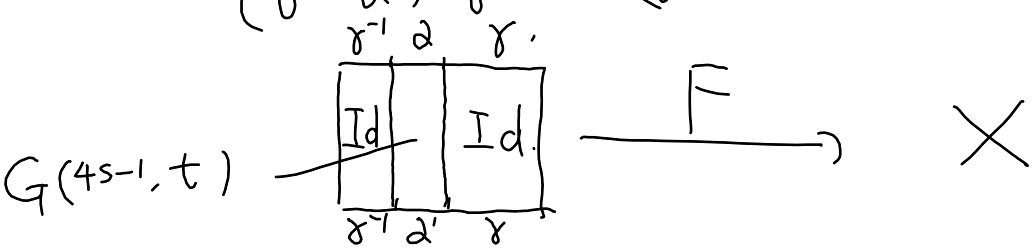


「γ_* 是良好定义的：

若 $\langle \alpha' \rangle = \langle \alpha \rangle$，要证：

设 $\alpha' \underset{G}{\simeq} \alpha$ rel$\{0,1\}$

$$(\gamma^{-1} \alpha') \cdot \gamma \simeq (\gamma^{-1} \alpha) \cdot \gamma \quad \text{rel } \{0,1\}$$



」

下证：$\gamma_*: \pi_1(X,p) \longrightarrow \pi_1(X,q)$ 为群同构.

$\langle\alpha\rangle \longmapsto \langle(\gamma^{-1}\cdot\alpha)\cdot\gamma\rangle$

即证：① γ_* 为双射

② γ_* 为群同态.

① Claim：$(\gamma^{-1})_*: \pi_1(X,q) \longrightarrow \pi_1(X,p)$. 即为 γ_* 之逆

i.e. $\underline{(\gamma^{-1})_*\circ\gamma_* = id}$, $\gamma_*\circ(\gamma^{-1})_* = id$.

只证：$(\gamma^{-1})_*\circ\gamma_* = id$.

即证：$\forall \langle\alpha\rangle \in \pi_1(X,q)$,

$(\gamma^{-1})_*\circ\gamma_*(\langle\alpha\rangle) = \langle\alpha\rangle$.

$\qquad\qquad\parallel$

$(\gamma^{-1})_*(\underbrace{\gamma_*(\langle\alpha\rangle)}_{\langle(\gamma^{-1}\cdot\alpha)\cdot\gamma\rangle}) = \langle((\gamma^{-1})^{-1}\cdot((\gamma^{-1}\cdot\alpha)\cdot\gamma))\cdot\gamma^{-1}\rangle$.

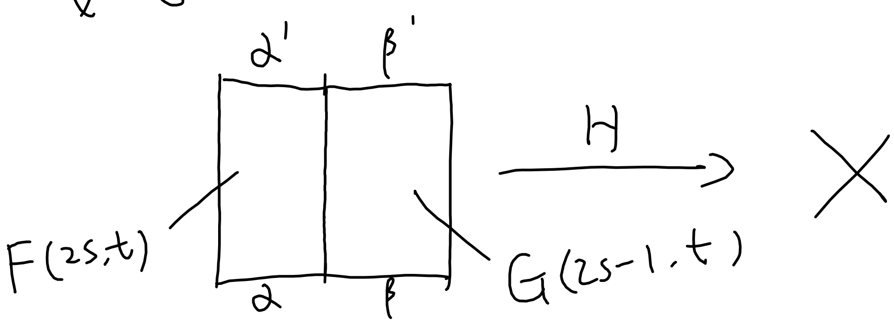
$= \langle(\gamma\cdot((\gamma^{-1}\cdot\alpha)\cdot\gamma))\cdot\gamma^{-1}\rangle$

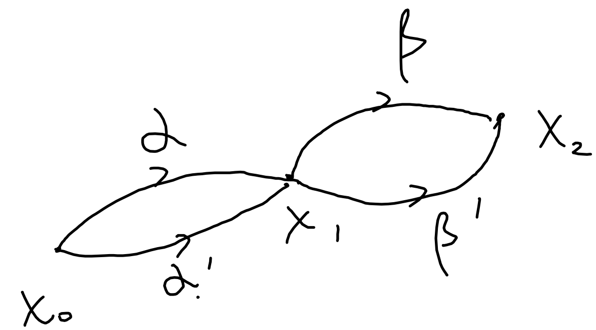
即证: $(\gamma\cdot((\gamma^{-1}\cdot\alpha)\cdot\gamma))\cdot\gamma^{-1} \simeq \alpha$ rel $\{0,1\}$. (△)

Lemma 1. 设 $x_0, x_1, x_2 \in X$, $\alpha, \alpha' \in P(x_0, x_1)$, $\beta, \beta' \in P(x_1, x_2)$. 若 $\langle\alpha\rangle = \langle\alpha'\rangle$, $\langle\beta\rangle = \langle\beta'\rangle$, 则 $\langle\alpha\cdot\beta\rangle = \langle\alpha'\cdot\beta'\rangle$.

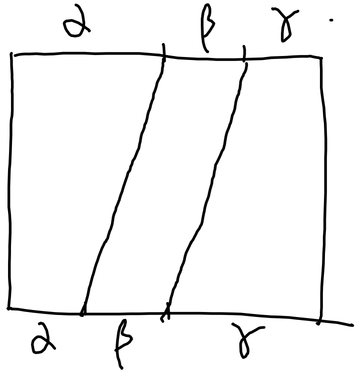
设 $\alpha \underset{F}{\simeq} \alpha'$ rel $\{0,1\}$, $\beta \underset{G}{\simeq} \beta'$ rel $\{0,1\}$.

要证: $\alpha\cdot\beta \simeq \alpha'\cdot\beta'$ rel $\{0,1\}$.





Lemma 2. 设 $x_0, x_1, x_2, x_3 \in X$, $\alpha \in P(x_0, x_1)$, $\beta \in P(x_1, x_2)$, $\gamma \in P(x_2, x_3)$, 则 $\langle(\alpha\cdot\beta)\cdot\gamma\rangle = \langle\alpha\cdot(\beta\cdot\gamma)\rangle$.

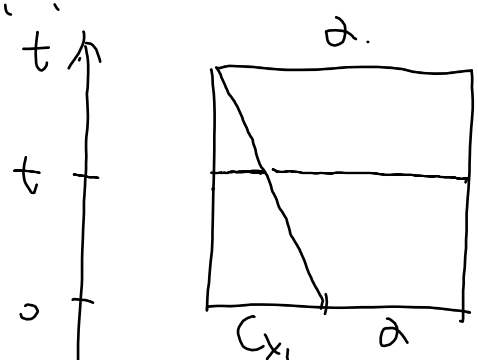


Lemma 3 设 $C_{x_0}: [0,1] \to X$, $t \mapsto x_0$, 则: $\forall \langle \alpha \rangle \in P(x_1, x_2)$

设 X 为 top. space, $x_0 \in X$.

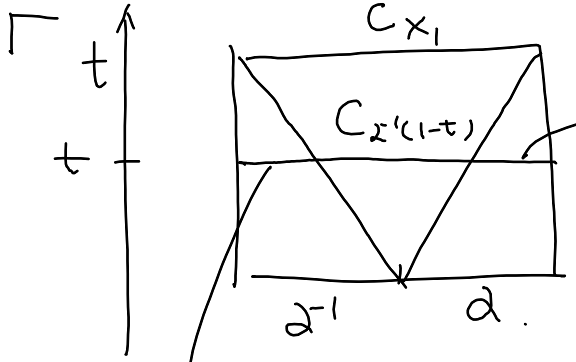
$$\langle C_{x_1} \cdot \alpha \rangle = \langle \alpha \rangle = \langle \alpha \cdot C_{x_2} \rangle$$

证: 只证: $C_{x_1} \cdot \alpha \simeq \alpha$ rel $\{0,1\}$



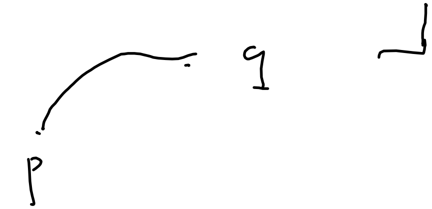
Lemma 4. $\forall \langle \alpha \rangle \in P(x_1, x_2)$, $\langle \alpha^{-1} \cdot \alpha \rangle = \langle C_{x_2} \rangle$

$\langle \alpha \cdot \alpha^{-1} \rangle = \langle C_{x_1} \rangle$.



按 α 走，从 $\alpha^{-1}(1-t)$ 到 $\alpha(1)$
$\qquad\qquad\quad \parallel$
$\qquad\qquad \alpha(t)$

按 α^{-1} 走，从 $\alpha^{-1}(0)$ 到 $\alpha^{-1}(1-t)$.



下证 (△)：$\left(\gamma\cdot\left((\gamma^{-1}\cdot\alpha)\cdot\gamma\right)\right)\cdot\gamma^{-1}$

$$\gamma\cdot\left((\gamma^{-1}\cdot\alpha)\cdot\gamma\right) \overset{\text{Lemma 2}}{\simeq} \underline{\left(\gamma\cdot(\gamma^{-1}\cdot\alpha)\right)}\cdot\gamma$$

$$\left.\begin{aligned} &\gamma\cdot(\gamma^{-1}\cdot\alpha) \overset{\text{Lemma 2}}{\simeq} (\gamma\cdot\gamma^{-1})\cdot\alpha. \\ &\gamma\cdot\gamma^{-1} \overset{\text{Lemma 4}}{\simeq} C_p \end{aligned}\right\} \overset{\text{Lemma 1}}{\Longrightarrow} (\gamma\cdot\gamma^{-1})\cdot\alpha \simeq C_p\cdot\alpha \overset{\text{Lemma 3}}{\simeq} \alpha$$

$$\Rightarrow \gamma\cdot(\gamma^{-1}\cdot\alpha) \simeq \alpha \quad \text{rel } \{0,1\}$$

$$\text{Lemma 1} \Rightarrow \left(\gamma\cdot(\gamma^{-1}\cdot\alpha)\right)\cdot\gamma \simeq \alpha\cdot\gamma \simeq \gamma\cdot\left((\gamma^{-1}\cdot\alpha)\cdot\gamma\right)$$

$$\text{Lemma 1} \Rightarrow \left(\gamma\cdot\left((\gamma^{-1}\cdot\alpha)\cdot\gamma\right)\right)\cdot\gamma^{-1} \simeq (\alpha\cdot\gamma)\cdot\gamma^{-1} \overset{\text{Lemma 2}}{\simeq} \alpha\cdot(\gamma\cdot\gamma^{-1})$$

$$\simeq \alpha.$$

$\therefore \quad \gamma_*: \pi_1(X,p) \longrightarrow \pi_1(X,q)$ 为双射.

$\pi_1(X,p) \longrightarrow \pi_1(X,q)$ 为群同态.

② $\gamma_*: \pi_1(X,p) \longrightarrow \pi_1(X,q)$

$\forall \langle \alpha \rangle, \langle \beta \rangle \in \pi_1(X,p)$.

要证: $\gamma_*(\langle \alpha \rangle \cdot \langle \beta \rangle) = \gamma_*(\langle \alpha \rangle) \cdot \gamma_*(\langle \beta \rangle)$.

$$\gamma_*(\langle \alpha \rangle \cdot \langle \beta \rangle) = \gamma_*(\langle \alpha \cdot \beta \rangle) = \langle \gamma^{-1} \cdot \alpha \cdot \beta \cdot \gamma \rangle.$$

$$\gamma_*(\langle \alpha \rangle) \cdot \gamma_*(\langle \beta \rangle) = \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle \cdot \langle \gamma^{-1} \cdot \beta \cdot \gamma \rangle = \langle \gamma^{-1} \cdot \alpha \cdot \gamma \cdot \gamma^{-1} \cdot \beta \cdot \gamma \rangle = \langle \gamma^{-1} \cdot \alpha \cdot \beta \cdot \gamma \rangle.$$

$$\langle \gamma^{-1} \cdot \alpha \cdot \beta \cdot \gamma \rangle = \langle \gamma^{-1} \cdot \alpha \cdot \beta \cdot \gamma \rangle.$$

综上: γ_* 为群同构. #.

Rmk. 设 X 为道路连通空间, 则 $\pi_1(X,p)$ 的同构类与 p 之选取无关, 将此群的同构类记为 $\pi_1(X)$.

Fundamental groupoid.

定义. 设 $\mathcal{C}$ 为一个范畴. 称 $\mathcal{C}$ 为一个 groupoid, 若 $\mathcal{C}$ 中任意态射均为同构.

Rmk $\forall$ top sp. X, 上述构造其实给出了一个 groupoid. 记为 $\Pi(X)$.

$\Pi(X)$ 之构造:

$$\mathrm{Ob}(\Pi(X)) = X$$

$\mathrm{Mor}(\Pi(X))$: $\forall\, p, q \in \mathrm{Ob}(\Pi(X)) = X$

$$\mathrm{Hom}_{\Pi(X)}(p, q) := P(p, q)/\sim$$

定义: $\circ : \mathrm{Hom}_{\Pi(X)}(p, q) \times \mathrm{Hom}_{\Pi(X)}(q, r) \longrightarrow \mathrm{Hom}_{\Pi(X)}(p, r)$

$$(\langle \alpha \rangle, \langle \beta \rangle) \longmapsto \langle \beta \rangle \circ \langle \alpha \rangle := \langle \alpha \cdot \beta \rangle$$

(良好定义: 若 $\alpha' \simeq \alpha$, $\beta' \simeq \beta$, $\alpha \cdot \beta \simeq \alpha' \cdot \beta'$ (Lemma 1))

(1) 有恒等态射.

$$\forall\, p \in Ob(\Pi(X)) = X.$$

定义 $1_p \in Hom_{\Pi(X)}(p, p)$ 为: $1_p := \langle c_p \rangle$.

$$\forall\, \langle\alpha\rangle \in Hom_{\Pi(X)}(p, q).$$

$$LHS = \langle c_p \cdot \alpha \rangle \overset{\text{Lemma 3}}{=\!=} \langle \alpha \rangle.$$

$$\begin{cases} \underline{\langle\alpha\rangle \circ 1_p = \langle\alpha\rangle.} \\ 1_q \circ \langle\alpha\rangle = \langle\alpha\rangle. \end{cases}$$

(2) 结合律:

$$p \xrightarrow{\langle\alpha\rangle} q \xrightarrow{\langle\beta\rangle} r \xrightarrow{\langle\gamma\rangle} s$$

要验证: $\langle\gamma\rangle \circ (\langle\beta\rangle \circ \langle\alpha\rangle) = (\langle\gamma\rangle \circ \langle\beta\rangle) \circ \langle\alpha\rangle$

$$\langle\gamma\rangle \circ (\langle\beta\rangle \circ \langle\alpha\rangle) = \langle\gamma\rangle \circ \langle\alpha \cdot \beta\rangle = \langle(\alpha\cdot\beta)\cdot\gamma\rangle \overset{\text{Lemma 2}}{=\!=} \langle\alpha\cdot(\beta\cdot\gamma)\rangle = \langle\beta\cdot\gamma\rangle \circ \langle\alpha\rangle = (\langle\gamma\rangle \circ \langle\beta\rangle) \circ \langle\alpha\rangle$$

综上. Lemma 1、2、3 $\Longrightarrow$ $\Pi(X)$ 为一个范畴.

由 Lemma 4. $\forall \langle \alpha \rangle \in \mathrm{Hom}_{\Pi(X)}(p, q)$.

$$\begin{cases} \langle \alpha^{-1} \cdot \alpha \rangle = \langle c_q \rangle \\ \langle \alpha \cdot \alpha^{-1} \rangle = \langle c_p \rangle \end{cases}$$

1_q

$\langle \alpha \rangle \circ \langle \alpha^{-1} \rangle$

$$\Rightarrow \begin{cases} \langle \alpha \rangle \circ \langle \alpha^{-1} \rangle = 1_q \\ \langle \alpha^{-1} \rangle \circ \langle \alpha \rangle = 1_p \end{cases} \Rightarrow \langle \alpha \rangle \text{为同构}.$$

$\Rightarrow$ $\Pi(X)$ 为一个 groupoid.

groupoid $\Pi(X)$ 称为 X 的 Fundamental groupoid.

Rmk. 设 $\mathcal{C}$ 为一个 groupoid, 则 $\forall X \in Ob(\mathcal{C})$,

在 $\mathrm{Hom}_{\mathcal{C}}(X, X)$ 定义:

$\cdot : \mathrm{Hom}_{\mathcal{C}}(X, X) \times \mathrm{Hom}_{\mathcal{C}}(X, X) \to \mathrm{Hom}_{\mathcal{C}}(X, X) \quad (f, g) \mapsto f \cdot g := g \circ f$.

则 $(\mathrm{Hom}_{\mathcal{C}}(x,x),\cdot)$ 构成一个群，称为 $\mathcal{C}$ 在 x 处的自同构群，or isotropy group. 记为 $\mathrm{Aut}_{\mathcal{C}}(x)$.

练习：$\forall x,y\in \mathrm{Ob}(\mathcal{C})$，if $\exists f\in \mathrm{Hom}_{\mathcal{C}}(x,y)$，

则 f 诱导了 $f_*: \mathrm{Aut}_{\mathcal{C}}(x)\xrightarrow{\cong}\mathrm{Aut}_{\mathcal{C}}(y)$.

（为群同构）.

Rmk. $\forall p\in X=\mathrm{Ob}(\Pi(X))$.

$\mathrm{Aut}_{\Pi(X)}(p) = \pi_1(X,p)$. （作为群）.

$\| $ set theoretically

$\mathrm{Hom}_{\Pi(X)}(p,p)$

$\|$

$L(X,p)/\sim$.

$\forall \langle\alpha\rangle,\langle\beta\rangle\in \mathrm{Aut}_{\Pi(X)}(p)$

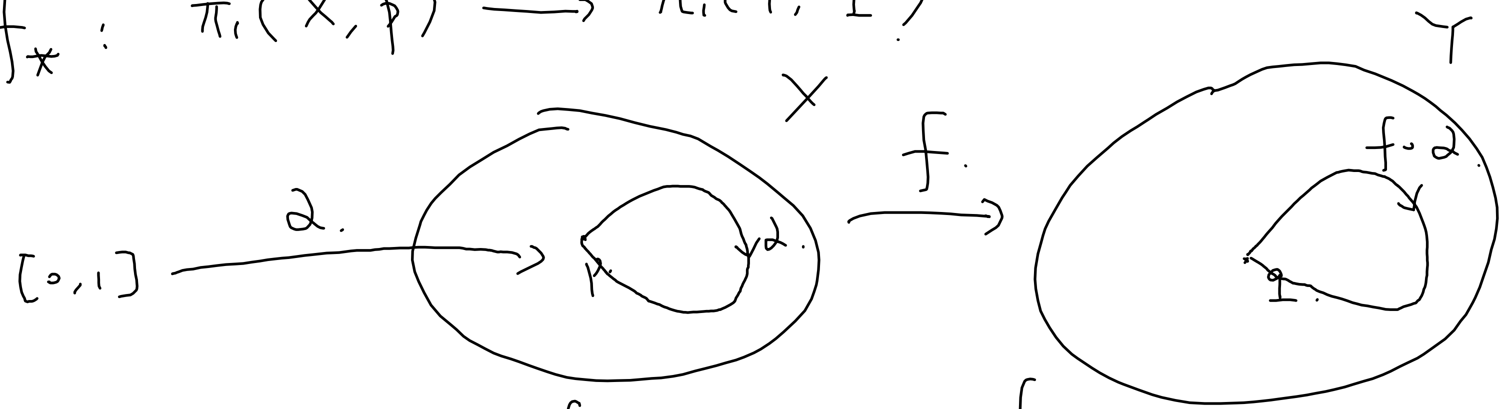
$\langle\alpha\rangle\cdot\langle\beta\rangle = \langle\beta\rangle\circ\langle\alpha\rangle$.

$\uparrow$ 自同构群乘法 $\qquad \| \; \langle\alpha\cdot\beta\rangle$

告诉你：命题1其实是 groupoid 的一般性质.

设 X, Y: top spaces, $f: X \to Y$ 连续, $p \in X$, $q = f(p)$

定义: $f_*: \pi_1(X, p) \to \pi_1(Y, q)$



$\forall \langle \alpha \rangle \in \pi_1(X, p)$, $f_*(\langle \alpha \rangle) := \langle f \circ \alpha \rangle$.

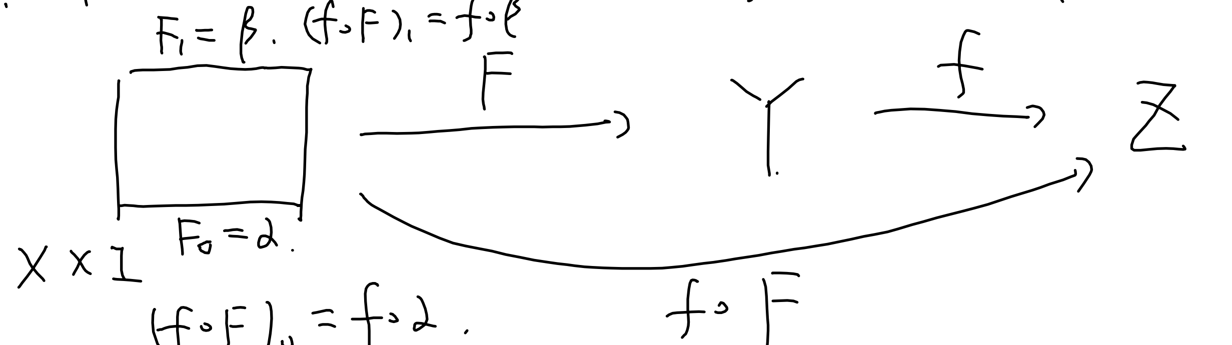
「良好定义: 若 $\langle \alpha' \rangle = \langle \alpha \rangle$, 需验证 $\langle f \circ \alpha' \rangle = \langle f \circ \alpha \rangle$.

i.e. 若 $\alpha' \simeq \alpha \ \mathrm{rel}\{0,1\}$, 要证 $f \circ \alpha' \simeq f \circ \alpha \ \mathrm{rel}\{0,1\}$.

Lemma. 设 $\alpha, \beta: X \to Y$ 连续, $\alpha \underset{F}{\simeq} \beta$, 其中 $F: X \times I \to Y$ 连续,

$f: Y \to Z$ 连续, 则 $f \circ \alpha \simeq f \circ \beta$.

Pf.



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」

Claim: $f_*: \pi_1(X,p) \to \pi_1(Y,q)$ 为群同态.

「 $\forall \langle\alpha\rangle, \langle\beta\rangle \in \pi_1(X,p)$,

要证: $f_*(\langle\alpha\rangle\cdot\langle\beta\rangle) = f_*(\langle\alpha\rangle)\cdot f_*(\langle\beta\rangle)$

$$f_*(\langle\alpha\rangle\cdot\langle\beta\rangle) = f_*(\langle\alpha\cdot\beta\rangle) = \langle f\circ(\alpha\cdot\beta)\rangle$$

$$f_*(\langle\alpha\rangle)\cdot f_*(\langle\beta\rangle) = \langle f\circ\alpha\rangle\cdot\langle f\circ\beta\rangle = \langle (f\circ\alpha)\cdot(f\circ\beta)\rangle$$

$$\underline{\langle f\circ(\alpha\cdot\beta)\rangle} = \underline{\langle (f\circ\alpha)\cdot(f\circ\beta)\rangle}$$

「 $\forall s \in [0,1]$

$$f\circ(\alpha\cdot\beta)(s) = f((\alpha\cdot\beta)(s)) = \begin{cases} f(\alpha(2s)), & 0 \le s \le \frac{1}{2} \\ f(\beta(2s-1)), & \frac{1}{2} \le s \le 1 \end{cases}$$

$$(f\circ\alpha)\cdot(f\circ\beta)(s) = \begin{cases} f\circ\alpha(2s), & 0 \le s \le \frac{1}{2} \\ f\circ\beta(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$ 」 」

Lemma. 若 $X \xrightarrow{f} Y \xrightarrow{g} Z$, $p \in X$, $q = f(p) \in Y$, $r = g(q) \in Z$,

则 $(g\circ f)_* = g_* \circ f_*$.

「
$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "f_*"] \arrow[rr, bend right, "(g\circ f)_*"'] & \pi_1(Y,q) \arrow[r, "g_*"] & \pi_1(Z,r) \end{tikzcd}$$
」

Pf. 要证：$\forall \langle\alpha\rangle \in \pi_1(X,p)$，有

$$(g\circ f)_*(\langle\alpha\rangle) = g_*\circ f_*(\langle\alpha\rangle).$$

$$(g\circ f)_*(\langle\alpha\rangle) = \langle (g\circ f)\circ\alpha\rangle$$

$$g_*\circ f_*(\langle\alpha\rangle) = g_*(f_*(\langle\alpha\rangle)) = g_*(\langle f\circ\alpha\rangle) = \langle g\circ(f\circ\alpha)\rangle$$ #.

推论：设 X，Y 为道路连通空间，若 X 与 Y 是同胚的，则

$$\pi_1(X) \cong \pi_1(Y).$$

Pf. 设 $f: X\to Y$，同胚，记 $g=f^{-1}: Y\to X$.

$p\in X$，$q=f(p)$.　　$f_*: \pi_1(X,p)\to\pi_1(Y,q)$.

$g_*: \pi_1(Y,q)\to\pi_1(X,p)$.

$$f_*\circ g_* = (f\circ g)_* = (id_Y)_* = id$$

$$g_*\circ f_* = (g\circ f)_* = (id_X)_* = id.$$ #.

推论：设 X 和 Y 为道路连通空间，若 $\pi_1(X)\ncong\pi_1(Y)$，则 $X\ncong Y$.

$$\begin{array}{ccc} (X, p) & \longmapsto & \pi_1(X, p) \\ \downarrow f & & \downarrow f_* \\ (Y, q) & \longmapsto & \pi_1(Y, q) \end{array}$$

(Functor)

定义 设 $\mathcal{C}$、$\mathcal{D}$ 为两个范畴，一个从 $\mathcal{C}$ 到 $\mathcal{D}$ 的函子 F 是指如下数据：

① $F: \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ （映射）

② $F: \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ （映射）.

满足：$\forall X, Y \in \mathrm{Ob}(\mathcal{C})$,

$$F(\mathrm{Hom}_{\mathcal{C}}(X, Y)) < \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

满足条件：(1) $\forall X \in \mathrm{Ob}(\mathcal{C})$, $F(1_X) = 1_{F(X)}$.

(2) $\forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$F(g \circ f) = F(g) \circ F(f).$$

练习：
若 f 为 $\mathcal{C}$ 中同构，则 $F(f)$ 为 $\mathcal{D}$ 中同构.

Rmk. 记 $\underline{\mathrm{Top}}^{\circ}$ 为带点的拓扑空间范畴.

Top° : $\mathrm{Ob}(\mathrm{Top}^{\circ}) = \{(X, p) \mid X \text{ 为 top sp. } p \in X\}$

$\forall (X, p), (Y, q) \in \mathrm{Ob}(\mathrm{Top}^{\circ})$

$\mathrm{Hom}((X,p),(Y,q)) := \{f: X \to Y \mid f(p) = q, f \text{ 连续}\}$

$\circ$: 取映射复合.

基本群之构造其实构造了函子:

$$\pi_1 : \mathrm{Top}^{\circ} \longrightarrow \mathrm{Grp}.$$

. on objects:

$\forall (X, p) \in \mathrm{Ob}(\mathrm{Top}^{\circ})$, $\pi_1(X, p) =$ 以 p 为基点的基本群.

. on Morphisms: $\forall (X,p), (Y,q) \in \mathrm{Ob}(\mathrm{Top}^{\circ})$.

$\forall f : (X, p) \to (Y, q)$, $\pi_1(f) = f_* : \pi_1(X,p) \longrightarrow \pi_1(Y, q)$

- $\pi_1(1_{(X,p)}) = id$.
- 保态射复合.

$\forall\ (X,p) \xrightarrow{f} (Y,q) \xrightarrow{g} (Z,r)$.

要验证: $\pi_1(g\circ f) = \pi_1(g)\circ\pi_1(f)$

$$\begin{matrix} \pi_1(g\circ f) & = & \pi_1(g)\circ\pi_1(f) \\ \| & & \| \\ (g\circ f)_* & \underset{\text{Lemma}}{=\!=} & g_*\circ f_* \end{matrix}$$

Fundamental groupoid 之构造其实给出了函子:

$$\Pi: \mathcal{T}op \longrightarrow \mathrm{Grpd}.$$

<u>Grpd</u> (Groupoid 范畴) 之定义:

(i) $Ob(\mathrm{Grpd})$: groupoids.

(ii) $Mor(\mathrm{Grpd})$: groupoid 之间的函子.

(iii) $\circ$: $\forall\ \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in Ob(\mathrm{Grpd})$, $\mathcal{X} \xrightarrow{F} \mathcal{Y} \xrightarrow{G} \mathcal{Z}$.

$G\circ F$ 定义为: ($G\circ F: \mathcal{X} \to \mathcal{Z}$ 为一个函子)

要证(i)(ii)(iii) 构成一个范畴, 需证: $\begin{cases} \text{① 有恒等态射} \\ \text{② “}\circ\text{” 满足结合律} \end{cases}$

①: $\forall$ groupoid $\mathcal{X}$

$\boxed{1_{\mathcal{X}}: \mathcal{X} \to \mathcal{X}}$.

$1_{\mathcal{X}}(X) = X$, $\forall X \in Ob(\mathcal{X})$

$1_{\mathcal{X}}(f) = f$, $\forall f \in Mor(\mathcal{X})$

- on objects: $\left(\mathcal{X} \xrightarrow{F} \mathcal{Y} \xrightarrow{G} \mathcal{Z} \right.$, 定义 $\left. G\circ F : \mathcal{X} \to \mathcal{Z} \right)$

  函子

  $\forall X \in \mathrm{Ob}(\mathcal{X})$,

  定义 $G\circ F(X) := G(F(X))$.

- on morphisms:

  $\forall X, Y \in \mathrm{Ob}(\mathcal{X})$, $f: X \to Y$,

  定义: $G\circ F(f) := G(F(f))$.

- $G\circ F(1_X) = 1_{G\circ F(X)}$ $\quad \left(G\circ F(1_X) = G(\underbrace{F(1_X)}_{\substack{\| \\ 1_{F(X)}}}) \right)$

  $$\begin{array}{c} 1_{G(F(X))} \\ \| \\ G(F(1_X)) \end{array}$$

- 保态射复合.

  设 $X_1, X_2, X_3 \in \mathrm{Ob}(\mathcal{X})$

  $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$

  要证: $G\circ F(g\circ f) = G\circ F(g) \circ G\circ F(f)$.

  $$\begin{array}{ccc} G\circ F(g\circ f) & & G\circ F(g)\circ G\circ F(f) \\ \| & & \| \\ G(F(g\circ f)) & = & G(F(g))\circ G(F(f)). \\ & & \| \\ & & G(F(g)\circ F(f)), \end{array}$$

下面：构造 $\Pi: \mathcal{T}op \longrightarrow \mathcal{G}rpd$ （函子）.

(1) On objects:

$\forall X \in Ob(\mathcal{T}op)$, $X \longmapsto \Pi(X)$ （X 的 fundamental groupoid）

(2) On morphisms:

$\forall X, Y \in Ob(\mathcal{T}op)$, $f: X \longrightarrow Y$.

定义 $\Pi(f): \Pi(X) \longrightarrow \Pi(Y)$ （是一个函子）.

- on objects:

  $\forall p \in Ob(\Pi(X)) = X$, $\Pi(f)(p) = f(p)$.

  i.e. $\Pi(f): \underset{X}{\underset{\parallel}{Ob(\Pi(X))}} \rightarrow \underset{Y}{\underset{\parallel}{Ob(\Pi(Y))}}$

  定义为 f.

- on morphisms:

  $\forall \langle \alpha \rangle \in Hom_{\Pi(X)}(p, q)$, 定义 $\Pi(f)(\langle \alpha \rangle)$

  $:= \langle f \circ \alpha \rangle \in Hom_{\Pi(Y)}(f(p), f(q))$ （良好定义的）

「验证π满足的两个条件：

(1) $\Pi(f)(\underbrace{\langle c_p \rangle}_{= 1_p}) = \langle \underbrace{f \circ c_p}_{= c_{f(p)}} \rangle = \langle c_{f(p)} \rangle = 1_{f(p)}$

(2) 保态射之复合. $\forall p, q, r \in X = Ob(\Pi(X))$

$$p \xrightarrow{\langle \alpha \rangle} q \xrightarrow{\langle \beta \rangle} r.$$

要证：$\Pi(f)(\langle \beta \rangle \circ \langle \alpha \rangle) = \Pi(f)(\langle \beta \rangle) \circ \Pi(f)(\langle \alpha \rangle)$

$\Pi(f)(\langle \beta \rangle \circ \langle \alpha \rangle) = \Pi(f)(\langle \alpha \cdot \beta \rangle) = \langle \underline{f \circ (\alpha \cdot \beta)} \rangle$

$\Pi(f)(\langle \beta \rangle) \circ \Pi(f)(\langle \alpha \rangle) = \langle f \circ \beta \rangle \circ \langle f \circ \alpha \rangle = \langle \underline{(f \circ \alpha) \cdot (f \circ \beta)} \rangle.$

$\langle f \circ (\alpha \cdot \beta) \rangle = \langle (f \circ \alpha) \cdot (f \circ \beta) \rangle$ 」

下面验证：① $\Pi(1_X) = 1_{\Pi(X)}$

② $\forall X, Y, Z \in Ob(\mathcal{T}_{op})$, $X \xrightarrow{f} Y \xrightarrow{g} Z$.

则 $\Pi(g \circ f) = \Pi(g) \circ \Pi(f)$!

① $\forall X \in Ob(\mathcal{T}op)$, $\Pi(1_X) = 1_{\Pi(X)}$.

$\Pi(1_X) : \Pi(X) \longrightarrow \Pi(X)$.

objects: $p \longmapsto 1_X(p) = p$

morphism: $p \xrightarrow{\langle \alpha \rangle} q \longmapsto \langle 1_X \circ \alpha \rangle = \langle \alpha \rangle$.

② 设 $X, Y, Z \in Ob(\mathcal{T}op)$, $X \xrightarrow{f} Y \xrightarrow{g} Z$.

要证: $\underline{\Pi(g \circ f) = \Pi(g) \circ \Pi(f)}$ (作为函子的等式).

- on objects:

$\Pi(g \circ f) : \Pi(X) \longrightarrow \Pi(Z)$.

$\Pi(X) \xrightarrow{\Pi(f)} \Pi(Y) \xrightarrow{\Pi(g)} \Pi(Z)$.

$\forall p \in Ob(\Pi(X)) = X$, $\Pi(g \circ f)(p) = \Pi(g) \circ \Pi(f)(p)$

$$\Pi(g \circ f)(p) = g \circ f(p).$$

$$\Pi(g) \circ \Pi(f)(p) = \Pi(g)(\Pi(f)(p)) = \Pi(g)(f(p)) = g(f(p))$$

- on morphisms:

$$\Pi(g\circ f): \Pi(X) \longrightarrow \Pi(Z).$$

$$\Pi(X) \xrightarrow{\Pi(f)} \Pi(Y) \xrightarrow{\Pi(g)} \Pi(Z).$$

$$\forall \langle\alpha\rangle \in \mathrm{Hom}_{\Pi(X)}(p, q). \qquad p \xrightarrow{\alpha} q$$

$$\underline{\Pi(g\circ f)}(\langle\alpha\rangle) = \langle\underline{(g\circ f)\circ\alpha}\rangle$$

$$\underline{\Pi(g)\circ\Pi(f)}(\langle\alpha\rangle) = \Pi(g)\big(\underbrace{\Pi(f)(\langle\alpha\rangle)}_{=\langle f\circ\alpha\rangle}\big)$$

$$= \langle \underline{g\circ(f\circ\alpha)} \rangle.$$

总结:

$$\boxed{\begin{array}{lll} \pi_1: & \mathbf{Top}^{\circ} \longrightarrow \mathbf{Grp}. & \\ \Pi: & \mathbf{Top} \longrightarrow \mathbf{Grpd} & \end{array} \Bigg\} \text{为函子}}$$