Rutherford Scattering with Levitating Targets

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You know the drill.... calculating classical long range scattering rate for a DM nugget off a levitating nanosphere target through the exchange of a (nearly) massless mediator

I. PRELIMINARIES

We are interested in calculating the rate of classical long range DM scattering off levitating nanospheres. We assume that the potential between the DM χ and the target T is given by

$$V(r) = \frac{g_V g_D}{4\pi r} e^{-mr}, \quad g_D \equiv N_D y_D, \quad g_V \equiv N_V y_V, \tag{1}$$

where we allow for the possibility that both the DM and the target respectively contain $N_{D,V}$ constituent particles that individually couple to the new mediator with strengths y_D and y_V. Here we work in natural units and follow Peskin conventions where the 4π is part of the Yukawa interaction.

For this potential, the differential scattering cross section can be written [**?**]

$$\frac{d\sigma}{dE_R} = \frac{8\pi\alpha_V\alpha_D m_T}{(2m_T E_R + m^2)^2}\frac{1}{v^2}\,|F(q)|^2 \quad , \tag{2}$$

where v is the DM's lab frame velocity, E_R is the kinetic energy of the recoiling target, $q^2 = 2m_T E_R$ is the momentum transfer, m_T is the target particle's mass, and F is the form factor that accounts for the compositeness of the target.

II. INTERACTION RATE

For a single sensor the differential interaction rate can be written

$$\frac{dR}{dE_R} = \frac{\rho_\chi}{m_\chi}\int_{v_{\rm min}(E_R)}^{v_{\rm esc}} d^3v f(v) v \frac{d\sigma}{dE_R}, \tag{3}$$

since $d\sigma/dE_R \propto v^{-2}$ we can separate the particle physics content from the velocity integration using Eq. (2)

$$\frac{dR}{dE_R} = \frac{\rho_\chi}{m_\chi}\frac{8\pi\alpha_V\alpha_D m_T}{(2m_T E_R + m^2)^2}\int_{v_{\rm min}(E_R)}^{v_{\rm esc}} d^3v \frac{f(v)}{v}, \tag{4}$$

where $v_{\rm esc} \approx 550$ km/sec is the Galactic halo escape velocity and we define the inverse mean speed

$$\eta(E_R) \equiv \int_{v_{\rm min}(E_R)}^{v_{\rm esc}} d^3v \frac{f(v)}{v} \;, \quad v_{\rm min}(E_R) = \sqrt{\frac{m_T E_R}{2\mu_{\chi T}^2}} \tag{5}$$

and $v_{\rm min}(E_R)$ is the minimum DM speed required for a target recoil energy of E_R, so we have

$$\frac{dR}{dE_R} = \frac{\rho_\chi}{m_\chi}\frac{8\pi\alpha_V\alpha_D m_T}{(2m_T E_R + m^2)^2}\eta(E_R), \tag{6}$$

which is valid for a point particle in a Yukawa potential and recovers the Rutherford cross section in the $m \to 0$ limit. Since we are always in the $q \gg m$ limit for the threshold impulse $q \sim 75$ MeV, we can simplify this

$$\frac{dR}{dE_R} = \frac{\rho_\chi}{m_\chi}\frac{2\pi\alpha_V\alpha_D}{m_T E_R^2}\eta(E_R), \tag{7}$$

which yields a rate per sensor. The total observed yield is thus

$$N_{\rm sig} = N_S \frac{\rho_\chi}{m_\chi}\frac{2\pi\alpha_V\alpha_D}{m_T}\int_{E_*}^{\infty}\frac{dE_R}{E_R^2}\eta(E_R), \tag{8}$$

III. FINITE DENSITY SPHERE

A. Poisson Method

Following the approach Dave's notebook, we generalize this to the case of "soft" sphere scattering where the sensor is a sphere of radius R with constant mass density ρ. Poisson's equation for a massive force carrier ϕ generalizes to

$$(\nabla^2 - m_\phi^2)V(r) = \left(\partial_r^2 V + \frac{2}{r}\partial_r - m_\phi^2\right)V(r) = -\rho\,\Theta(R-r). \tag{9}$$

outside the sphere whose center defines the origin of our coordinates, the potential is

$$V(r>R) = c_1\frac{e^{-mr}}{r} \tag{10}$$

where c_1 is an arbitrary constant and. Inside the sphere the general solution for constant ρ is

$$V(r \le R) = \frac{\rho}{m^2} + c_2\frac{e^{+mr}}{2r} + c_3\frac{e^{-mr}}{2r} \tag{11}$$

where the latter two terms satisfy the homogeneous Poisson equation $(\nabla^2 + m^2)V = 0$ and we choose $c_1 = -c_2$ to enforce a finite potential at the origin, which yields

$$V(r \le R) = \frac{\rho}{m^2} + c_2\frac{\sinh(mr)}{r} \tag{12}$$

and enforcing continuity of potential and derivatives at $r = R$ yields an interior solution

$$V(r<R) = \frac{3Q}{4\pi(mR)^3}\left(m - \frac{1+mR}{r[1+\coth(mR)]}\frac{\sinh(mr)}{\sinh(mR)}\right) \tag{13}$$

and an exterior solution

$$V(r>R) = \frac{3Q}{4\pi(mR)^3}\left[mR\cosh(mR) - \sinh(mR)\right]\frac{e^{-mr}}{r} \tag{14}$$

where in the limit $R \to 0$ or $r \to \infty$, the prefactor

$$\frac{3}{(mR)^3}\left[mR\cosh(mR) - \sinh(mR)\right] \to 1, \quad V(r \gg R) \to \frac{Qe^{-mr}}{4\pi r}. \tag{15}$$

which is surprising because, unlike for a Coulomb potential, here you feel the fuzziness of the sphere even on the outside.

B. Green's Function Method

The same potentials can be obtained by directly integrating the Green's function

$$V(r) = \frac{1}{4\pi}\int d^3r'\rho(\vec{r}\,')\frac{e^{-m|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \tag{16}$$

for a constant density we can define coordinates with $\vec{r} = r\hat{z}$ so we can do the azimuthal integral and for $r > R$

$$\begin{aligned}
V(r) &= \frac{\rho}{2}\int_0^R dr' r'^2 \int_{-1}^1 dx \frac{e^{-m\sqrt{r^2+r'^2-2rr'x}}}{\sqrt{r^2+r'^2-2rr'x}} \\
&= \frac{\rho}{2}\int_0^R dr' r'^2 \frac{e^{-m\sqrt{(r-r')^2}} - e^{-m\sqrt{(r+r')^2}}}{mrr'} \\
&= \rho\int_0^R dr' r'^2 \frac{e^{-m(r-r')} - e^{-m(r+r')}}{2mrr'} \qquad (\ r > r' \to \text{take + root}) \\
&= \frac{\rho\, e^{-mr}}{r}\int_0^R dr' r'^2 \frac{\sinh mr'}{mr'} \\
&= \frac{\rho\, e^{-mr}}{m^3 r}\Big[(mR)\cosh mR - \sinh mR\Big]
\end{aligned} \tag{17}$$

so replacing the constant density $\rho = 3Q/4\pi R^3$ we find

$$V(r) = \frac{Qe^{-mr}}{4\pi r}\frac{3}{(mR)^3}\Big[(mR)\cosh mR - \sinh mR\Big] \tag{18}$$

where we recover the point particle Yukawa potential result in the $R \to 0$ limit.

IV. GENERALIZED CROSS SECTION

Our goal here is to take the finite sphere potential described above and derive the differential cross section for this potential, which generalizes the Rutherford scattering setup.

A. Kinematics and Jacobians

In the heavy target limit $m_\chi \ll M_T$, the scattered particle loses only a small fraction of its incident kinetic energy, so momentum transfer can be written

$$q = |\vec{p}_i - \vec{p}_f| = \sqrt{p_f^2 + p_i^2 - 2p_i p_f \cos\theta} \approx p\sqrt{2(1-\cos\theta)} = 2p\sin\frac{\theta}{2} \quad \Longrightarrow \theta(q) = 2\sin^{-1}\frac{q}{2p}, \tag{19}$$

where θ is the scattering angle and we have approximated $p_i \approx p_f \equiv p$ for the magnitudes of the incident and outgoing DM momenta. For our problem this is a good approximation because the energy lost by the DM and transferred to the target is

$$\Delta E = \sqrt{q^2 + M^2} - M \approx \frac{q^2}{2M} \simeq 5\times 10^{-9}\,\text{eV}\left(\frac{q}{75\,\text{MeV}}\right)^2\left(\frac{\text{ng}}{M}\right), \tag{20}$$

so this is an excellent approximation for our small momentum transfers and macroscopic masses since ng $\sim 10^{14}$ GeV. For classical scattering, the impact parameter uniquely determines the scattering angle and by conservation of particle number, the annulus through which particles are incident with impact parameter b is

$$d\sigma = 2\pi b db \quad , \quad d\Omega = 2\pi \sin\theta d\theta \quad , \quad \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{21}$$

where Ω is the solid angle, so our goal is to determine the relationship between impact parameter and scattering angle and relate this back to the momentum transfer in Eq. (19) by using

$$q^2 = 2p^2(1-\cos\theta) \to 2qdq = 2p^2 d\cos\theta \to d\cos\theta = \frac{q}{p^2}dq \tag{22}$$

which we can substitute into the general formula for a differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{2\pi d\cos\theta} = \frac{p^2}{2\pi q}\frac{d\sigma}{dq} \quad \to \quad \frac{d\sigma}{dq} = \frac{2\pi q}{p^2}\frac{d\sigma}{d\Omega}, \tag{23}$$

so all we need now is to calculate the $\theta(q)$ relationship using standard classical mechanics.

B. Scattering Angle

The general formalism for scattering in a central potential $V(r)$ applies to our situation, where

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) \quad , \quad \ell \equiv mr^2\dot{\theta} = \text{const.} \tag{24}$$

so the integral of the motion is

$$\dot{r} = \sqrt{E - \frac{\ell^2}{2mr^2} - V(r)} \quad \to \quad \frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \frac{\ell}{mr^2\sqrt{E - \frac{\ell^2}{2mr^2} - V(r)}} \tag{25}$$

so changing variables to $u = 1/r^2$, rearranging terms, and integrating, we obtain the angle as a function of radial position

$$\theta(r) = \theta_0 + \int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r')}{\ell^2} - \frac{1}{r'^2}}} \tag{26}$$

with initial condition θ_0 and r_0. For the scattering setup, $r_0 = \infty$, $\theta_0 = 0$ and we can change variables to

$$\theta(u) = \theta_0 - \int_0^u \frac{du'}{\sqrt{\frac{2mE}{\ell^2} - \frac{2mV(u')}{\ell^2} - u'^2}} \, . \tag{27}$$

To relate this to the incident impact parameter, we note that the angular momentum at $r = \infty$ satisfies

$$\ell = mvb = b\sqrt{2mE} \quad \to \quad \ell^2 = 2mEb^2 \tag{28}$$

so the integral simplifies further

$$\theta(u) = \int_0^u \frac{du'}{\sqrt{\frac{1}{b^2} - \frac{V(u')}{Eb^2} - u'^2}} = \int_0^u \frac{b\,du'}{\sqrt{1 - \frac{V(u')}{E} - b^2u'^2}} \tag{29}$$

and can exploty the symmetry about the point of nearest approach $r_m = 1/u_m$ to express the asymptotic scattering angle $\theta_\infty = \pi - 2\psi$, where ψ is the angle between the beamline and the vector to the incident particle at the point of nearest approach. Thus, we have

$$\theta(u) = \pi - 2\psi = \pi - 2\int_0^{u_m} \frac{b\,du}{\sqrt{1 - \frac{V(u)}{E} - b^2u^2}} \tag{30}$$

where $u_m = 1/r_m$ is defined using the orbital equation

$$\frac{d\theta}{du} = \frac{b}{\sqrt{1 - b^2u^2 - \frac{V(u)}{E}}} \quad \to \quad \left.\frac{du}{d\theta}\right|_{u=u_m} = 0, \tag{31}$$

so we have a simple expression for u_m

$$1 - b^2u_m^2 - \frac{V(u_m)}{E} = 0, \tag{32}$$

which we may have to solve numerically depending on the potential, but this is just some function of conserved kinematic invariants E and ℓ (and ℓ is secretly b anyway), so we formally have $\theta(b, E)$ which governs

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta} \quad , \quad \frac{d\sigma}{dq} = \frac{2\pi q}{p^2}\frac{d\sigma}{d\Omega} \tag{33}$$

so we just need the functional form of the scattering angle and we know the recoil distribution

V. ARE THE NUMBERS REASONABLE?

For the $q \sim 75$ MeV momentum transfer regime, we're safely in the $q \gg m$ limit, where m is the mass of the mediator. Thus, we can get some intuition for the problem in the Rutherford limit $m \to 0$ to estimate the size of the impact parameter that gives such a momentum transfer. Let $V(r) = k/r = ku$ so that

$$\psi(u) = \int_0^{u_m} \frac{b\,du}{\sqrt{1 - \frac{ku}{E} - b^2u^2}} = \int_0^{x_m} \frac{dx}{\sqrt{1 - x^2 - \alpha x}} \tag{34}$$

where $x = bu, \alpha = k/(Eb)$ and point of closest approach satisfies

$$1 - x_m^2 - \alpha x_m = 0, \tag{35}$$

so integration yields

$$\theta = 2\sin^2\left(\frac{\alpha}{\sqrt{4+\alpha^2}}\right) \tag{36}$$

which gives a relation between the momentum transfer q and the impact parameter b through

$$b^2 = \frac{k^2}{4E^2}\cot^2\left(\frac{\theta}{2}\right) = \frac{k^2}{4E^2}\frac{1-\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \frac{k^2}{4E^2}\left(\frac{4p^2}{q^2}-1\right) \tag{37}$$

and since $q \ll p$, we can approximate this as

$$b \approx \frac{k}{E}\frac{p}{q} \sim 10^{-4}\mu\text{m}\left(\frac{k}{1}\right)\left(\frac{10^{-3}c}{v}\right)\left(\frac{75\,\text{MeV}}{q}\right) \tag{38}$$

where we have used $p = m_\chi v$, $E = p^2/2m_\chi = m_\chi v^2/2$, and set $k = (N_T y_T)(N_\chi y_\chi)/4\pi = 1$ which saturates the numerator of the Coulomb potential. It seems generically true that we need *tiny* impact parameters to deliver a ~ 75 MeV momentum transfer to the target