

# Bivariate Lasso Distribution Derivation

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## 1 Setting

If  $x \sim \text{MultiLasso}(A, b, c)$  with then it has density given by

$$p(x, a, b, c) = Z^{-1} \exp\left(-\frac{1}{2}x^T A x + b^T x - c\|x\|_1\right)$$

where  $A \in S_d^+$ : positive definite matrix with dimension  $d$ ,  $b \in R^d$ ,  $c > 0$

## 2 Finding Normalizing Constant

$$\begin{aligned} Z(a, b, c) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}x^T A x + \mathbf{b}^T x - c\mathbf{1}^T |x|_1\right] d\mathbf{x} \\ &= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty} \int_{-\infty}^0 \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c[1, -1]^T)x\right] d\mathbf{x} \\ &\quad + \int_{-\infty}^0 \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c[-1, 1]^T)x\right] d\mathbf{x} + \int_{-\infty}^0 \int_{-\infty}^0 \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\ &= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\ &= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A^* x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A^* x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x - (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\ &= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x^T A x - 2(\mathbf{b}^T - c\mathbf{1}^T)x)\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x^T A^* x - 2(b_1 - c, -b_2 - c)^T x)\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x^T A^* x - 2(-b_1 - c, b_2 - c)^T x)\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x^T A x + 2(\mathbf{b}^T + c\mathbf{1}^T)x)\right] d\mathbf{x} \\ &= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_1)^T A(x - \mu_1) + \frac{(A\mu_1)^T A^{-1}(A\mu_1)}{2}\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_2)^T A^*(x - \mu_2) + \frac{(A^*\mu_2)^T A^{*-1}(A^*\mu_2)}{2}\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_3)^T A^*(x - \mu_3) + \frac{(A^*\mu_3)^T A^{-1}(A^*\mu_3)}{2}\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_4)^T A(x - \mu_4) + \frac{(A\mu_4)^T A^{-1}(A\mu_4)}{2}\right] d\mathbf{x} \\ &= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_1)^T \Sigma_1^{-1}(x - \mu_1) + \frac{(A\mu_1)^T \Sigma_1(A\mu_1)}{2}\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_2)^T \Sigma_2^{-1}(x - \mu_2) + \frac{(A^*\mu_2)^T \Sigma_2(A^*\mu_2)}{2}\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_3)^T \Sigma_2^{-1}(x - \mu_3) + \frac{(A^*\mu_3)^T \Sigma_2(A^*\mu_3)}{2}\right] d\mathbf{x} \\ &\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_4)^T \Sigma_1^{-1}(x - \mu_4) + \frac{(A\mu_4)^T \Sigma_1(A\mu_4)}{2}\right] d\mathbf{x} \\ &= 2\pi|\Sigma_1|^{\frac{1}{2}} \left[ \exp\left[\frac{(A\mu_1)^T \Sigma_1(A\mu_1)}{2}\right] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_1, \Sigma_1) d\mathbf{x} + \exp\left[\frac{(A\mu_4)^T \Sigma_1(A\mu_4)}{2}\right] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_4, \Sigma_1) d\mathbf{x} \right] \\ &\quad + 2\pi|\Sigma_2|^{\frac{1}{2}} \left( \exp\left[\frac{(A^*\mu_2)^T \Sigma_2(A^*\mu_2)}{2}\right] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_2, \Sigma_2) d\mathbf{x} + \exp\left[\frac{(A^*\mu_3)^T \Sigma_2(A^*\mu_3)}{2}\right] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_3, \Sigma_2) d\mathbf{x} \right) \\ &= |\Sigma_1| \left( \frac{\int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_1, \Sigma_1) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1})} + \frac{\int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_4, \Sigma_1) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1})} \right) + |\Sigma_2| \left( \frac{\int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_2, \Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_2, \Sigma_2^{-1})} + \frac{\int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_3, \Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_3, \Sigma_2^{-1})} \right) \end{aligned}$$

where  $\mu_1 = A^{-1}(b - c\mathbf{1})^T$ ,  $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$ ,  $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T$ ,  $\mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$   
and  $\Sigma_1 = A^{-1}$ ,  $\Sigma_2 = A^{*-1}$   $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ .

### 3 Find Expectation

Follow similar step as before

$$\begin{aligned}
E[X] &= Z^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \otimes \exp \left[ -\frac{1}{2} x^T A x + \mathbf{b}^T x - c \mathbf{1}^T ||x||_1 \right] d\mathbf{x} \\
&= Z^{-1} \int_0^{\infty} \int_0^{\infty} x \otimes \exp \left[ -\frac{1}{2} x^T A x + (\mathbf{b}^T - c \mathbf{1}^T) x \right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} [1, -1]^T \otimes x \otimes \exp \left[ -\frac{1}{2} x^T A^* \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (b_1 - c, -b_2 - c)^T x \right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} [-1, 1]^T \otimes x \otimes \exp \left[ -\frac{1}{2} x^T A^* \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (-b_1 - c, b_2 - c)^T x \right] d\mathbf{x} \\
&\quad - \int_0^{\infty} \int_0^{\infty} x \otimes \exp \left[ -\frac{1}{2} x^T A x + (\mathbf{b}^T + c \mathbf{1}^T) x \right] d\mathbf{x} \\
&= Z^{-1} [|\Sigma_1| \left( \frac{\int_0^{\infty} \int_0^{\infty} x \otimes \phi_2(x; \mu_1, \Sigma) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1})} - \frac{\int_0^{\infty} \int_0^{\infty} x \otimes \phi_2(x; \mu_4, \Sigma) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1})} \right) \\
&\quad + |\Sigma_2| \left( [1, -1]^T \frac{\int_0^{\infty} \int_0^{\infty} x \otimes \phi_2(x; \mu_2, \Sigma_2) d\mathbf{x}}{\phi_2(A^* \mu_2, \Sigma_2^{-1})} + [-1, 1]^T \frac{\int_0^{\infty} \int_0^{\infty} x \otimes \phi_2(x; \mu_3, \Sigma_2) d\mathbf{x}}{\phi_2(A^* \mu_3, \Sigma_2^{-1})} \right)] \\
&= Z^{-1} [|\Sigma_1| \left( \frac{E[\mathbf{A}] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_1, \Sigma_1) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1})} - \frac{E[D] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_4, \Sigma_1) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1})} \right) \\
&\quad + |\Sigma_2| \left( [1, -1]^T \frac{E[B] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_2, \Sigma_2) d\mathbf{x}}{\phi_2(A^* \mu_2, \Sigma_2^{-1})} + [-1, 1]^T \frac{E[C] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_3, \Sigma_2) d\mathbf{x}}{\phi_2(A^* \mu_3, \Sigma_2^{-1})} \right)]
\end{aligned}$$

where  $\mu_1 = A^{-1}(b - c\mathbf{1})^T$ ,  $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$ ,  $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T$ ,  $\mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$  and  $\Sigma_1 = A^{-1}$ ,  $\Sigma_2 = A^{*-1}$ ,  $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ .  $\mathbf{A} \sim MTN_+(\mu_1, \Sigma_1)$ ,  $B \sim MTN_+(\mu_2, \Sigma_2)$ ,  $C \sim MTN_+(\mu_3, \Sigma_2)$ ,  $D \sim MTN_+(\mu_4, \Sigma_1)$  is denotes the multivariate positively truncated normal distribution.

### 4 Find Covariance Matrix

Follow similar steps as before

$$\begin{aligned}
Cov(X) &= E[XX^T] - E[X]E[X]^T \\
E[XX^T] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xx^T \otimes \exp \left[ -\frac{1}{2} x^T A x + \mathbf{b}^T x - c \mathbf{1}^T ||x||_1 \right] d\mathbf{x} \\
&= Z^{-1} 2\pi |\Sigma|^{\frac{1}{2}} \left[ \exp \left[ \frac{(A\mu_1)^T \Sigma (A\mu_1)}{2} \right] \int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x; \mu_1, \Sigma_1) d\mathbf{x} \right. \\
&\quad + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \exp \left[ \frac{(A^* \mu_2)^T \Sigma (A^* \mu_2)}{2} \right] \int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x; \mu_2, \Sigma_2) d\mathbf{x} \\
&\quad + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \exp \left[ \frac{(A^* \mu_3)^T \Sigma (A^* \mu_3)}{2} \right] \int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x; \mu_3, \Sigma_2) d\mathbf{x} \\
&\quad \left. - \exp \left[ \frac{(A\mu_4)^T \Sigma (A\mu_4)}{2} \right] \int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x; \mu_4, \Sigma_1) d\mathbf{x} \right] \\
&= Z^{-1} |\Sigma| \left[ \frac{\int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x; \mu_1, \Sigma) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma)} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \frac{\int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x; \mu_2, \Sigma) d\mathbf{x}}{\phi_2(A\mu_2, \Sigma)} \right. \\
&\quad \left. + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \frac{\int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x; \mu_3, \Sigma) d\mathbf{x}}{\phi_2(A\mu_3, \Sigma)} - \frac{\int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x; \mu_4, \Sigma) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma)} \right] \\
&= Z^{-1} [|\Sigma_1| \left( \frac{E[\mathbf{A}\mathbf{A}^T] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_1, \Sigma_1) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1})} + \frac{E[D D^T] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_4, \Sigma_1) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1})} \right) \\
&\quad + |\Sigma_2| \left( \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \frac{E[B B^T] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_2, \Sigma_2) d\mathbf{x}}{\phi_2(A^* \mu_2, \Sigma_2^{-1})} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \frac{E[C C^T] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_3, \Sigma_2) d\mathbf{x}}{\phi_2(A^* \mu_3, \Sigma_2^{-1})} \right)]
\end{aligned}$$

where  $\mu_1 = A^{-1}(b - c\mathbf{1})^T$ ,  $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$ ,  $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T$ ,  $\mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$  and  $\Sigma_1 = A^{-1}$ ,  $\Sigma_2 = A^{*-1}$ ,  $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ .  $\mathbf{A} \sim MTN_+(\mu_1, \Sigma_1)$ ,  $B \sim MTN_+(\mu_2, \Sigma_2)$ ,  $C \sim MTN_+(\mu_3, \Sigma_2)$ ,  $D \sim MTN_+(\mu_4, \Sigma_1)$  is denotes the multivariate positively truncated normal distribution.

$$E[AA^T] = Cov(A) - E[A]E[A]^T$$

## 5 Finding marginal distribution

$$f(x_1, x_2) = \frac{1}{Z} \exp(-\frac{1}{2}x^T A x + b^T x - c||x||_1) dx_2$$

$$\begin{aligned} f(x_1) &= \int_{-\infty}^{\infty} f(x_1, x_2) dx_2 \\ &= Z^{-1} \exp(-\frac{1}{2}x^T A x + b^T x - c||x||_1) dx_2 \\ &= Z^{-1} \exp(-0.5a_{11}x_1^2 + b_1x_1 - c|x_1|) \\ &\quad \int_{-\infty}^{\infty} \exp(-\frac{1}{2}[a_{12}a_{21}x_1x_2 + a_{22}x_2^2] + b_2x_2 - c|x_2|) dx_2 \\ &= k \int_{-\infty}^{\infty} \exp[-a_{12}a_{21}x_1x_2 - 0.5a_{22}x_2^2 + b_2x_2 - c|x_2|] dx_2 \\ &= k [\int_0^{\infty} \exp[-0.5a_{12}a_{21}x_1x_2 - 0.5a_{22}x_2^2 + (b_2 - c)x_2] dx_2 \\ &\quad + \int_{-\infty}^0 \exp[-0.5a_{12}a_{21}x_1x_2 - 0.5a_{22}x_2^2 + (b_2 + c)x_2] dx_2] \\ &= k [\int_0^{\infty} \exp[-0.5a_{12}a_{21}x_1x_2 - 0.5a_{22}x_2^2 + (b_2 - c)x_2] dx_2 \\ &\quad + \int_0^{\infty} \exp[0.5a_{12}a_{21}x_1x_2 - 0.5a_{22}x_2^2 - (b_2 + c)x_2] dx_2] \\ &= k [\int_0^{\infty} \exp[-\frac{(x_2 - \mu_1)^2}{2\sigma^2} + \frac{\mu_1^2}{2\sigma^2}] dx_2 + \int_0^{\infty} \exp[-\frac{(x_2 - \mu_2)^2}{2\sigma^2} + \frac{\mu_2^2}{2\sigma^2}] dx_2] \\ &= k\sigma [\frac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)} + \frac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}] \end{aligned}$$

where  $\mu_1 = (\frac{a_{12}a_{21}}{2a_{22}}x_1 + \frac{b_2 - c}{a_{22}})$ ,  $\mu_2 = (\frac{a_{12}a_{21}}{2a_{22}}x_1 - \frac{b_2 + c}{a_{22}})$ ,  $\sigma^2 = 1/a_{22}$ ,  $k = Z^{-1} \exp(-0.5a_{11}x_1^2 + b_1x_1 - c|x_1|)$

$$\begin{aligned} f(x_2) &= \int_{-\infty}^{\infty} f(x_1, x_2) dx_1 \\ &= Z^{-1} \exp(-\frac{1}{2}x^T A x + b^T x - c||x||_1) dx_1 \\ &= Z^{-1} \exp(-0.5a_{22}x_2^2 + b_2x_2 - c|x_2|) \\ &\quad \int_{-\infty}^{\infty} \exp(-\frac{1}{2}[a_{12}a_{21}x_1x_2 + a_{11}x_1^2] + b_1x_1 - c|x_1|) dx_1 \\ &= k \int_{-\infty}^{\infty} \exp[-0.5a_{12}a_{21}x_1x_2 - 0.5a_{11}x_1^2 + b_1x_1 - c|x_1|] dx_1 \\ &= k [\int_0^{\infty} \exp[-0.5a_{12}a_{21}x_1x_2 - 0.5a_{11}x_1^2 + (b_1 - c)x_1] dx_1 \\ &\quad + \int_{-\infty}^0 \exp[-0.5a_{12}a_{21}x_1x_2 - 0.5a_{11}x_1^2 + (b_1 + c)x_1] dx_1] \\ &= k [\int_0^{\infty} \exp[-0.5a_{12}a_{21}x_1x_2 - 0.5a_{11}x_1^2 + (b_1 - c)x_1] dx_1 \\ &\quad + \int_0^{\infty} \exp[-0.5a_{12}a_{21}x_1x_2 - 0.5a_{11}x_1^2 - (b_1 + c)x_1] dx_1] \\ &= k [\int_0^{\infty} \exp[-\frac{(x_1 - \mu_1)^2}{2\sigma^2} + \frac{\mu_1^2}{2\sigma^2}] dx_1 + \int_0^{\infty} \exp[-\frac{(x_1 - \mu_2)^2}{2\sigma^2} + \frac{\mu_2^2}{2\sigma^2}] dx_1] \\ &= k\sigma [\frac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)} + \frac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}] \end{aligned}$$

where  $\mu_1 = (\frac{a_{12}a_{21}}{2a_{11}}x_2 + \frac{b_1 - c}{a_{11}})$ ,  $\mu_2 = (\frac{a_{12}a_{21}}{2a_{11}}x_2 - \frac{b_1 + c}{a_{11}})$ ,  $\sigma^2 = 1/a_{11}$ ,  $k = Z^{-1} \exp(-0.5a_{22}x_2^2 + b_2x_2 - c|x_2|)$