# Bivariate Lasso Distribution Derivation

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March 2, 2023

## 1 Setting

If $x \sim \text{MultiLasso}(A, b, c)$ with then it has density given by

$$p(x, a, b, c) = Z^{-1} exp(-\frac{1}{2}x^T Ax + b^T x - c||x||_1)$$

where $A \in S_d^+$: positive definite matrix with dimension d, $b \in R^2$, $c > 0$

## 2 Finding Normalizing Constant

$$\begin{aligned}
Z(a,b,c) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + \mathbf{b}^T x - c\mathbf{1}^T|x|_1\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty}\int_{-\infty}^{0} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c[1,-1]^T)x\right] d\mathbf{x} \\
&\quad + \int_{-\infty}^{0}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c[-1,1]^T)x\right] d\mathbf{x} + \int_{-\infty}^{0}\int_{-\infty}^{0} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T A^* x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T A^* x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax - (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x^T Ax - 2(\mathbf{b}^T - c\mathbf{1}^T)x)\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x^T A^* x - 2(b_1 - c, -b_2 - c)^T x)\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x^T A^* x - 2(-b_1 - c, b_2 - c)^T x)\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x^T Ax + 2(\mathbf{b}^T + c\mathbf{1}^T)x)\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x - \mu_1)^T A(x - \mu_1) + \tfrac{(A\mu_1)^T A^{-1}(A\mu_1)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x - \mu_2)^T A^*(x - \mu_2) + \tfrac{(A^*\mu_2)^T A^{*-1}(A^*\mu_2)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x - \mu_3)^T A^*(x - \mu_3) + \tfrac{(A^*\mu_3)^T A^{-1}(A^*\mu_3)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x - \mu_4)^T A(x - \mu_4) + \tfrac{(A\mu_4)^T A^{-1}(A\mu_4)]}{2}\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x - \mu_1)^T \Sigma_1^{-1}(x - \mu_1) + \tfrac{(A\mu_1)^T \Sigma_1(A\mu_1)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x - \mu_2)^T \Sigma_2^{-1}(x - \mu_2) + \tfrac{(A^*\mu_2)^T \Sigma_2(A^*\mu_2)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x - \mu_3)^T \Sigma_2^{-1}(x - \mu_3) + \tfrac{(A^*\mu_3)^T \Sigma_2(A^*\mu_3)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x - \mu_4)^T \Sigma_1^{-1}(x - \mu_4) + \tfrac{(A\mu_4)^T \Sigma_1(A\mu_4)]}{2}\right] d\mathbf{x} \\
&= 2\pi|\Sigma_1|^{\frac{1}{2}}[\exp\left[\tfrac{(A\mu_1)^T \Sigma_1(A\mu_1)]}{2}\right] \int_0^{\infty}\int_0^{\infty} \phi_2(x; \mu_1, \Sigma_1) d\mathbf{x} + \exp\left[\tfrac{(A\mu_4)^T \Sigma_1(A\mu_4)]}{2}\right] \int_0^{\infty}\int_0^{\infty} \phi_2(x; \mu_4, \Sigma_1) d\mathbf{x}]) \\
&\quad +2\pi|\Sigma_2|^{\frac{1}{2}}(\exp\left[\tfrac{(A^*\mu_2)^T \Sigma_2(A^*\mu_2)]}{2}\right] \int_0^{\infty}\int_0^{\infty} \phi_2(x; \mu_2, \Sigma_2) d\mathbf{x} + \exp\left[\tfrac{(A^*\mu_3)^T \Sigma_2(A^*\mu_3)]}{2}\right] \int_0^{\infty}\int_0^{\infty} \phi_2(x; \mu_3, \Sigma_2) d\mathbf{x} \\
&= |\Sigma_1|\left(\frac{\int_0^{\infty}\int_0^{\infty} \phi_2(x;\mu_1,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1})} + \frac{\int_0^{\infty}\int_0^{\infty} \phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1})}\right) + |\Sigma_2|\left(\frac{\int_0^{\infty}\int_0^{\infty} \phi_2(x;\mu_2,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_2, \Sigma_2^{-1})} + \frac{\int_0^{\infty}\int_0^{\infty} \phi_2(x;\mu_3,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_3, \Sigma_2^{-1})} + \right))
\end{aligned}$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T, \mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T \mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

## 3 Find Expectation

Follow similar step as before

$$
\begin{aligned}
E[X] \quad &= Z^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \otimes \exp\left[-\tfrac{1}{2} x^T A x + \mathbf{b}^T x - c\mathbf{1}^T ||x||_1\right] d\mathbf{x} \\
&= Z^{-1} \int_0^{\infty} \int_0^{\infty} x \otimes \exp\left[-\tfrac{1}{2} x^T A x + (\mathbf{b}^T - c\mathbf{1}^T) x\right] d\mathbf{x} \\
&+ \int_0^{\infty} \int_0^{\infty} [1,-1]^T \otimes x \otimes \exp\left[-\tfrac{1}{2} x^T A^* \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\
&+ \int_0^{\infty} \int_0^{\infty} [-1,1]^T \otimes x \otimes \exp\left[-\tfrac{1}{2} x^T A^* \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} \\
&- \int_0^{\infty} \int_0^{\infty} x \otimes \exp\left[-\tfrac{1}{2} x^T A x + (\mathbf{b}^T + c\mathbf{1}^T) x\right] d\mathbf{x} \\
&= Z^{-1} [|\Sigma_1| ( \tfrac{\int_0^{\infty} \int_0^{\infty} x \otimes \phi_2(x;\mu_1,\Sigma) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1}))} - \tfrac{\int_0^{\infty} \int_0^{\infty} x \otimes \phi_2(x;\mu_4,\Sigma) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1}))}) \\
&+ |\Sigma_2| ([1,-1]^T \tfrac{\int_0^{\infty} \int_0^{\infty} x \otimes \phi_2(x;\mu_2,\Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_2, \Sigma_2^{-1}))} + [-1,1]^T \tfrac{\int_0^{\infty} \int_0^{\infty} x \otimes \phi_2(x;\mu_3,\Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_3, \Sigma_2^{-1})})] \\
&= Z^{-1} [|\Sigma_1| ( \tfrac{E[\mathbf{A}] \int_0^{\infty} \int_0^{\infty} \phi_2(x;\mu_1,\Sigma_1) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1}))} - \tfrac{E[D] \int_0^{\infty} \int_0^{\infty} \phi_2(x;\mu_4,\Sigma_1) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1}))}) \\
&+ |\Sigma_2| ([1,-1]^T \tfrac{E[B] \int_0^{\infty} \int_0^{\infty} \phi_2(x;\mu_2,\Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_2, \Sigma_2^{-1}))} + [-1,1]^T \tfrac{E[C] \int_0^{\infty} \int_0^{\infty} \phi_2(x;\mu_3,\Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_3, \Sigma_2^{-1})})]
\end{aligned}
$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$, $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T \mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. $\mathbf{A} \sim MTN_+(\mu_1, \Sigma_1)$, $B \sim MTN_+(\mu_2, \Sigma_2)$, $C \sim MTN_+(\mu_3, \Sigma_2)$, $D \sim MTN_+(\mu_4, \Sigma_1)$ is denotes the multivariate positively truncated normal distribution.

## 4 Find Covariance Matrix

Follow similar steps as before

$$Cov(X) = E[XX^T] - E[X]E[X]^T$$

$$
\begin{aligned}
E[XX^T] \quad &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xx^T \otimes \exp\left[-\tfrac{1}{2} x^T A x + \mathbf{b}^T x - c\mathbf{1}^T ||x||_1\right] d\mathbf{x} \\
&= Z^{-1} 2\pi |\Sigma|^{\frac{1}{2}} [\exp\left[\tfrac{(A\mu_1)^T \Sigma (A\mu_1)]}{2}\right] \int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x;\mu_1,\Sigma_1) d\mathbf{x} \\
&+ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \exp\left[\tfrac{(A^*\mu_2)^T \Sigma (A^*\mu_2)]}{2}\right] \int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x;\mu_2,\Sigma_2) d\mathbf{x} \\
&+ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \exp\left[\tfrac{(A^*\mu_3)^T \Sigma (A^*\mu_3)]}{2}\right] \int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x;\mu_3,\Sigma_2) d\mathbf{x} \\
&- \exp\left[\tfrac{(A\mu_4)^T \Sigma (A\mu_4)]}{2}\right] \int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x;\mu_4,\Sigma_1) d\mathbf{x}] \\
\\
&= Z^{-1} |\Sigma| [ \tfrac{\int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x;\mu_1,\Sigma) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma))} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \tfrac{\int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x;\mu_2,\Sigma) d\mathbf{x}}{\phi_2(A\mu_2, \Sigma))} \\
&+ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \tfrac{\int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x;\mu_3,\Sigma) d\mathbf{x}}{\phi_2(A\mu_3, \Sigma))} - \tfrac{\int_0^{\infty} \int_0^{\infty} xx^T \otimes \phi_2(x;\mu_4,\Sigma) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma))}] \\
\\
&= Z^{-1} [|\Sigma_1| ( \tfrac{E[\mathbf{A}\mathbf{A}^T] \int_0^{\infty} \int_0^{\infty} \phi_2(x;\mu_1,\Sigma_1) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1}))} + \tfrac{E[DD^T] \int_0^{\infty} \int_0^{\infty} \phi_2(x;\mu_4,\Sigma_1) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1}))}) \\
&+ |\Sigma_2| ( \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \tfrac{E[BB^T] \int_0^{\infty} \int_0^{\infty} \phi_2(x;\mu_2,\Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_2, \Sigma_2^{-1}))} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \tfrac{E[CC^T] \int_0^{\infty} \int_0^{\infty} \phi_2(x;\mu_3,\Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_3, \Sigma_2^{-1})})]
\end{aligned}
$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$, $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T \mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. $\mathbf{A} \sim MTN_+(\mu_1, \Sigma_1)$, $B \sim MTN_+(\mu_2, \Sigma_2)$, $C \sim MTN_+(\mu_3, \Sigma_2)$, $D \sim MTN_+(\mu_4, \Sigma_1)$ is denotes the multivariate positively truncated normal distribution.

$$E[AA^T] = Cov(A) - E[A]E[A]^T$$

# 5 Finding marginal distribution

$$f(x_1,x_2)=\frac{1}{Z}exp(-\frac{1}{2}x^TAx+b^Tx-c||x||_1))dx_2$$

$$\begin{array}{ll}
f(x_1) & =\int_{-\infty}^{\infty}f(x_1,x_2)dx_2 \\
& =Z^{-1}exp(-\frac{1}{2}x^TAx+b^Tx-c||x||_1))dx_2 \\
& =Z^{-1}exp(-0.5a_{11}x_1^2+b_1x_1-c|x_1|) \\
& \int_{-\infty}^{\infty}exp(-\frac{1}{2}[(a_{12}+a_{21})x_1x_2+a_{22}x_2^2]+b_2x_2-c|x_2|]dx_2 \\
& =k\int_{-\infty}^{\infty}exp[-(0.5(a_{12}+a_{21})x_1x_2-0.5a_{22}x_2^2+b_2x_2-c|x_2|]dx_2 \\
& =k[\int_0^{\infty}exp[-0.5(a_{12}+a_{21})x_1x_2-0.5a_{22}x_2^2+(b_2-c)x_2]dx_2 \\
& +\int_{-\infty}^{0}exp[-0.5(a_{12}+a_{21})x_1x_2-0.5a_{22}x_2^2+(b_2+c)x_2]dx_2 \\
& =k[\int_0^{\infty}exp[-0.5(a_{12}+a_{21})x_1x_2-0.5a_{22}x_2^2+(b_2-c)x_2]dx_2 \\
& +\int_0^{\infty}exp[0.5(a_{12}+a_{21})x_1x_2-0.5a_{22}x_2^2-(b_2+c)x_2]dx_2] \\
& =k[\int_0^{\infty}exp[-\frac{(x_2-\mu_1)^2}{2\sigma^2}+\frac{\mu_1^2}{2\sigma^2}]dx_2]+\int_0^{\infty}exp[-\frac{(x_2-\mu_2)^2}{2\sigma^2}+\frac{\mu_2^2}{2\sigma^2}]dx_2] \\
& =k\sigma[\frac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)}+\frac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}]
\end{array}$$

where $\mu_1=(-\frac{a_{12}+a_{21}}{2a_{22}}x_1+\frac{b_2-c}{a_{22}})$, $\mu_2=(\frac{a_{12}+a_{21}}{2a_{22}}x_1-\frac{b_2+c}{a_{22}})$, $\sigma^2=1/a_{22}$, $k=Z^{-1}exp(-0.5a_{11}x_1^2+b_1x_1-c|x_1|)$

$$\begin{array}{ll}
f(x_2) & =\int_{-\infty}^{\infty}f(x_1,x_2)dx_1 \\
& =Z^{-1}exp(-\frac{1}{2}x^TAx+b^Tx-c||x||_1))dx_1 \\
& =Z^{-1}exp(-0.5a_{22}x_2^2+b_2x_2-c|x_2|) \\
& \int_{-\infty}^{\infty}exp(-\frac{1}{2}[a_{12}a_{21}x_1x_2+a_{11}x_1^2]+b_1x_1-c|x_1|]dx_1 \\
& =k\int_{-\infty}^{\infty}exp[-0.5(a_{12}+a_{21})x_1x_2-0.5a_{11}x_1^2+b_1x_1-c|x_1|]dx_1 \\
& =k[\int_0^{\infty}exp[-0.5(a_{12}+a_{21})x_1x_2-0.5a_{11}x_1^2+(b_1-c)x_1]dx_1 \\
& +\int_{-\infty}^{0}exp[-0.5(a_{12}+a_{21})x_1x_2-0.5a_{11}x_1^2+(b_1+c)x_1]dx_1] \\
& =k[\int_0^{\infty}exp[-0.5(a_{12}+a_{21})x_1x_2-0.5a_{11}x_1^2+(b_1-c)x_1]dx_1 \\
& +\int_0^{\infty}exp[0.5(a_{12}+a_{21})x_1x_2-0.5a_{11}x_1^2-(b_1+c)x_1]dx_1] \\
& =k[\int_0^{\infty}exp[-\frac{(x_2-\mu_1)^2}{2\sigma^2}+\frac{\mu_1^2}{2\sigma^2}]dx_1]+\int_0^{\infty}exp[-\frac{(x_2-\mu_2)^2}{2\sigma^2}+\frac{\mu_2^2}{2\sigma^2}]dx_1] \\
& =k\sigma[\frac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)}+\frac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}]
\end{array}$$

where $\mu_1=(-\frac{a_{12}+a_{21}}{2a_{11}}x_2+\frac{b_1-c}{a_{11}})$, $\mu_2=(\frac{a_{12}+a_{21}}{2a_{11}}x_2-\frac{b_1+c}{a_{11}})$, $\sigma^2=1/a_{11}$, $k=Z^{-1}exp(-0.5a_{22}x_2^2+b_2x_2-c|x_2|)$