Bivariate Lasso Distribution Derivation

Yuhao Li

February 10, 2023

1 Setting

If $x \sim \text{MultiLasso}(A, b, c)$ with then it has density given by

$$p(x, a, b, c) = Z^{-1} exp(-\frac{1}{2}x^T Ax + b^T x - c||x||_1)$$

where $A \in S_d^+$: positive definite matrix with dimension d, $b \in R^2$, $c > 0$

2 Finding Normalizing Constant

$$\begin{aligned}
Z(a,b,c) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + \mathbf{b}^T x - c\mathbf{1}^T|x|_1\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty}\int_{-\infty}^{0} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c[1,-1]^T)x\right] d\mathbf{x} \\
&\quad + \int_{-\infty}^{0}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c[-1,1]^T)x\right] d\mathbf{x} + \int_{-\infty}^{0}\int_{-\infty}^{0} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T A^* x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T A^* x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}x^T Ax - (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x^T Ax - 2(\mathbf{b}^T - c\mathbf{1}^T)x)\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x^T A^* x - 2(b_1 - c, -b_2 - c)^T x)\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x^T A^* x - 2(-b_1 - c, b_2 - c)^T x)\right] d\mathbf{x} + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x^T Ax + 2(\mathbf{b}^T + c\mathbf{1}^T)x)\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_1)^T A(x-\mu_1) + \tfrac{(A\mu_1)^T A^{-1}(A\mu_1)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_2)^T A^*(x-\mu_2) + \tfrac{(A^*\mu_2)^T A^{*-1}(A^*\mu_2)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_3)^T A^*(x-\mu_3) + \tfrac{(A^*\mu_3)^T A^{-1}(A^*\mu_3)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_4)^T A(x-\mu_4) + \tfrac{(A\mu_4)^T A^{-1}(A\mu_4)]}{2}\right] d\mathbf{x} \\
&= \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1) + \tfrac{(A\mu_1)^T \Sigma_1(A\mu_1)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_2)^T \Sigma_2^{-1}(x-\mu_2) + \tfrac{(A^*\mu_2)^T \Sigma_2(A^*\mu_2)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_3)^T \Sigma_2^{-1}(x-\mu_3) + \tfrac{(A^*\mu_3)^T \Sigma_2(A^*\mu_3)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty}\int_0^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_4)^T \Sigma_1^{-1}(x-\mu_4) + \tfrac{(A\mu_4)^T \Sigma_1(A\mu_4)]}{2}\right] d\mathbf{x} \\
&= 2\pi|\Sigma_1|^{\frac{1}{2}}[\exp\left[\tfrac{(A\mu_1)^T\Sigma_1(A\mu_1)]}{2}\right]\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_1,\Sigma_1)d\mathbf{x} + \exp\left[\tfrac{(A\mu_4)^T\Sigma_1(A\mu_4)]}{2}\right]\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}]) \\
&\quad +2\pi|\Sigma_2|^{\frac{1}{2}}(\exp\left[\tfrac{(A^*\mu_2)^T\Sigma_2(A^*\mu_2)]}{2}\right]\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_2,\Sigma_2)d\mathbf{x} + \exp\left[\tfrac{(A^*\mu_3)^T\Sigma_2(A^*\mu_3)]}{2}\right]\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_3,\Sigma_2)d\mathbf{x} \\
&= |\Sigma_1|(\tfrac{\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_1,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma_1^{-1})} + \tfrac{\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma_1^{-1})}) + |\Sigma_2|(\tfrac{\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_2,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_2,\Sigma_2^{-1})} + \tfrac{\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_3,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_3,\Sigma_2^{-1})} +))
\end{aligned}$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$, $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T \mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

3 Find Expectation

Follow similar step as before

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \otimes \exp\left[-\frac{1}{2}x^T A x + \mathbf{b}^T x - c\mathbf{1}^T ||x||_1\right] d\mathbf{x} \\
&= Z^{-1}[|\Sigma_1|(\frac{\int_0^\infty\int_0^\infty x\otimes\phi_2(x;\mu_1,\Sigma)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma_1^{-1}))} - \frac{\int_0^\infty\int_0^\infty x\otimes\phi_2(x;\mu_4,\Sigma)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma_1^{-1}))}) + |\Sigma_2|[1,-1]^T(\frac{\int_0^\infty\int_0^\infty x\otimes\phi_2(x;\mu_2,\Sigma)d\mathbf{x}}{\phi_2(A^*\mu_2,\Sigma_2^{-1}))} \\
&\quad +[-1,1]^T\frac{\int_0^\infty\int_0^\infty x\otimes\phi_2(x;\mu_3,\Sigma)d\mathbf{x}}{\phi_2(A^*\mu_3,\Sigma_2^{-1})})] \\
&= Z^{-1}[|\Sigma_1|(\frac{E[A]}{\phi_2(A\mu_1,\Sigma_1^{-1}))} - \frac{E[D]}{\phi_2(A\mu_4,\Sigma_1^{-1}))}) + |\Sigma_2|([1,-1]^T\frac{E[B]}{\phi_2(A^*\mu_2,\Sigma_2^{-1}))} \\
&\quad +[-1,1]^T\frac{E[C]}{\phi_2(A^*\mu_3,\Sigma_2^{-1})})]
\end{aligned}
$$

where $\mu_1 = A^{-1}(b-c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1-c,-b_2-c)^T, \mu_3 = A^{*-1}(-b_1-c,b_2-c)^T \mu_4 = A^{-1}(-b-c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A\otimes\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$. $\mathbf{A} \sim MTN_+(\mu_1,\Sigma)$, $B \sim MTN_+(\mu_2,\Sigma)$, $B \sim MTN_+(\mu_3,\Sigma)$, $D \sim MTN_+(\mu_4,\Sigma)$ is denotes the multivariate positively truncated normal distribution.

4 Find Covariance Matrix

Follow similar steps as before

$$Cov(X) = E[XX^T] - E[X]E[X]^T$$

$$
\begin{aligned}
E[XX^T] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xx^T \otimes \exp\left[-\frac{1}{2}x^T A x + \mathbf{b}^T x - c\mathbf{1}^T ||x||_1\right] d\mathbf{x} \\
&= Z^{-1}2\pi|\Sigma|^{\frac{1}{2}}[\exp\left[\frac{(A\mu_1)^T\Sigma(A\mu_1)]}{2}\right]\int_0^\infty\int_0^\infty xx^T\otimes\phi_2(x;\mu_1,\Sigma)d\mathbf{x} \\
&\quad + \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\otimes\exp\left[\frac{(A\mu_2)^T\Sigma(A\mu_2)]}{2}\right]\int_0^\infty\int_0^\infty xx^T\otimes\phi_2(x;\mu_2,\Sigma)d\mathbf{x} \\
&\quad + \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\otimes\exp\left[\frac{(A\mu_3)^T\Sigma(A\mu_3)]}{2}\right]\int_0^\infty\int_0^\infty xx^T\otimes\phi_2(x;\mu_3,\Sigma)d\mathbf{x} \\
&\quad - \exp\left[\frac{(A\mu_4)^T\Sigma(A\mu_4)]}{2}\right]\int_0^\infty\int_0^\infty xx^T\otimes\phi_2(x;\mu_4,\Sigma)d\mathbf{x}] \\
&= Z^{-1}|\Sigma|[\frac{\int_0^\infty\int_0^\infty xx^T\otimes\phi_2(x;\mu_1,\Sigma)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma))} + \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\otimes\frac{\int_0^\infty\int_0^\infty xx^T\otimes\phi_2(x;\mu_2,\Sigma)d\mathbf{x}}{\phi_2(A\mu_2,\Sigma))} \\
&\quad + \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\otimes\frac{\int_0^\infty\int_0^\infty xx^T\otimes\phi_2(x;\mu_3,\Sigma)d\mathbf{x}}{\phi_2(A\mu_3,\Sigma))} - \frac{\int_0^\infty\int_0^\infty xx^T\otimes\phi_2(x;\mu_4,\Sigma)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma))}] \\
&= Z^{-1}|\Sigma|[\frac{E[\mathbf{A}\mathbf{A}^T]}{\phi_2(A\mu_1,\Sigma))} + \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\otimes\frac{E[BB^T]}{\phi_2(A\mu_2,\Sigma))} + \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\otimes\frac{E[CC^T]}{\phi_2(A\mu_3,\Sigma))} - \frac{E[DD^T]}{\phi_2(A\mu_4,\Sigma))}]
\end{aligned}
$$

where $\mu_1 = A^{-1}(b-c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1-c,-b_2-c)^T, \mu_3 = A^{*-1}(-b_1-c,b_2-c)^T \mu_4 = A^{-1}(-b-c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A\otimes\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$. $\mathbf{A} \sim MTN_+(\mu_1,\Sigma)$, $B \sim MTN_+(\mu_2,\Sigma)$, $B \sim MTN_+(\mu_3,\Sigma)$, $D \sim MTN_+(\mu_4,\Sigma)$ is denotes the multivariate positively truncated normal distribution.

$$E[AA^T] = Cov(A) - E[A]E[A]^T$$