Bivariate Lasso Distribution Derivation

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1 Setting

If $x \sim \text{MultiLasso}(A, b, c)$ with then it has density given by

$$p(x, a, b, c) = Z^{-1} exp(-\frac{1}{2}x^T A x + b^T x - c||x||_1)$$

where $A \in S_d^+$: positive definite matrix with dimension d, $b \in R^2$, $c > 0$

2 Finding Normalizing Constant

$$
\begin{aligned}
Z(a, b, c) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}x^T A x + \mathbf{b}^T x - c\mathbf{1}^T |x|_1\right] d\mathbf{x} \\
&= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty} \int_{-\infty}^0 \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c[1, -1]^T)x\right] d\mathbf{x} \\
&\quad + \int_{-\infty}^0 \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c[-1, 1]^T)x\right] d\mathbf{x} + \int_{-\infty}^0 \int_{-\infty}^0 \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A^* x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A^* x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}x^T A x - (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x^T A x - 2(\mathbf{b}^T - c\mathbf{1}^T)x)\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x^T A^* x - 2(b_1 - c, -b_2 - c)^T x)\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x^T A^* x - 2(-b_1 - c, b_2 - c)^T x)\right] d\mathbf{x} + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x^T A x + 2(\mathbf{b}^T + c\mathbf{1}^T)x)\right] d\mathbf{x} \\
&= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_1)^T A(x - \mu_1) + \frac{(A\mu_1)^T A^{-1}(A\mu_1)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_2)^T A^*(x - \mu_2) + \frac{(A^*\mu_2)^T A^{*-1}(A^*\mu_2)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_3)^T A^*(x - \mu_3) + \frac{(A^*\mu_3)^T A^{-1}(A^*\mu_3)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_4)^T A(x - \mu_4) + \frac{(A\mu_4)^T A^{-1}(A\mu_4)]}{2}\right] d\mathbf{x} \\
&= \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_1)^T \Sigma_1^{-1}(x - \mu_1) + \frac{(A\mu_1)^T \Sigma_1(A\mu_1)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_2)^T \Sigma_2^{-1}(x - \mu_2) + \frac{(A^*\mu_2)^T \Sigma_2(A^*\mu_2)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_3)^T \Sigma_2^{-1}(x - \mu_3) + \frac{(A^*\mu_3)^T \Sigma_2(A^*\mu_3)]}{2}\right] d\mathbf{x} \\
&\quad + \int_0^{\infty} \int_0^{\infty} \exp\left[-\frac{1}{2}(x - \mu_4)^T \Sigma_1^{-1}(x - \mu_4) + \frac{(A\mu_4)^T \Sigma_1(A\mu_4)]}{2}\right] d\mathbf{x} \\
&= 2\pi|\Sigma_1|^{\frac{1}{2}}[\exp\left[\frac{(A\mu_1)^T \Sigma_1(A\mu_1)]}{2}\right] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_1, \Sigma_1)d\mathbf{x} + \exp\left[\frac{(A\mu_4)^T \Sigma_1(A\mu_4)]}{2}\right] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_4, \Sigma_1)d\mathbf{x}]) \\
&\quad + 2\pi|\Sigma_2|^{\frac{1}{2}}(\exp\left[\frac{(A^*\mu_2)^T \Sigma_2(A^*\mu_2)]}{2}\right] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_2, \Sigma_2)d\mathbf{x} + \exp\left[\frac{(A^*\mu_3)^T \Sigma_2(A^*\mu_3)]}{2}\right] \int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_3, \Sigma_2)d\mathbf{x} \\
&= |\Sigma_1|(\frac{\int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_1, \Sigma_1)d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1})} + \frac{\int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_4, \Sigma_1)d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1})}) + |\Sigma_2|(\frac{\int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_2, \Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_2, \Sigma_2^{-1})} + \frac{\int_0^{\infty} \int_0^{\infty} \phi_2(x; \mu_3, \Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_3, \Sigma_2^{-1})} +))
\end{aligned}
$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$, $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T \mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$
and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

3 Find Expectation

Follow similar step as before

$$
\begin{aligned}
E[X] \quad &= Z^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \otimes \exp\left[-\tfrac{1}{2}x^T A x + \mathbf{b}^T x - c\mathbf{1}^T ||x||_1\right] d\mathbf{x} \\
&= Z^{-1} \int_{0}^{\infty} \int_{0}^{\infty} x \otimes \exp\left[-\tfrac{1}{2}x^T A x + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} \\
&+ \int_{0}^{\infty} \int_{0}^{\infty} [1,-1]^T \otimes x \otimes \exp\left[-\tfrac{1}{2}x^T A^* \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\
&+ \int_{0}^{\infty} \int_{0}^{\infty} [-1,1]^T \otimes x \otimes \exp\left[-\tfrac{1}{2}x^T A^* \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} \\
&- \int_{0}^{\infty} \int_{0}^{\infty} x \otimes \exp\left[-\tfrac{1}{2}x^T A x + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= Z^{-1}[|\Sigma_1|(\tfrac{\int_{0}^{\infty}\int_{0}^{\infty} x \otimes \phi_2(x;\mu_1,\Sigma)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma_1^{-1})} - \tfrac{\int_{0}^{\infty}\int_{0}^{\infty} x \otimes \phi_2(x;\mu_4,\Sigma)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma_1^{-1})}) \\
&+ |\Sigma_2|([1,-1]^T \tfrac{\int_{0}^{\infty}\int_{0}^{\infty} x \otimes \phi_2(x;\mu_2,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_2,\Sigma_2^{-1})} + [-1,1]^T \tfrac{\int_{0}^{\infty}\int_{0}^{\infty} x \otimes \phi_2(x;\mu_3,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_3,\Sigma_2^{-1})})] \\
&= Z^{-1}[|\Sigma_1|(\tfrac{E[\mathbf{A}]\int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_1,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma_1^{-1})} - \tfrac{E[D]\int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma_1^{-1})}) \\
&+ |\Sigma_2|([1,-1]^T \tfrac{E[B]\int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_2,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_2,\Sigma_2^{-1})} + [-1,1]^T \tfrac{E[C]\int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_3,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_3,\Sigma_2^{-1})})]
\end{aligned}
$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$, $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T$ $\mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. $\mathbf{A} \sim MTN_+(\mu_1, \Sigma_1)$, $B \sim MTN_+(\mu_2, \Sigma_2)$, $C \sim MTN_+(\mu_3, \Sigma_2)$, $D \sim MTN_+(\mu_4, \Sigma_1)$ is denotes the multivariate positively truncated normal distribution.

4 Find Covariance Matrix

Follow similar steps as before

$$
Cov(X) = E[XX^T] - E[X]E[X]^T
$$

$$
\begin{aligned}
E[XX^T] \quad &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xx^T \otimes \exp\left[-\tfrac{1}{2}x^T A x + \mathbf{b}^T x - c\mathbf{1}^T ||x||_1\right] d\mathbf{x} \\
&= Z^{-1} 2\pi |\Sigma|^{\frac{1}{2}} [\exp\left[\tfrac{(A\mu_1)^T \Sigma (A\mu_1)]}{2}\right] \int_{0}^{\infty} \int_{0}^{\infty} xx^T \otimes \phi_2(x;\mu_1,\Sigma_1)d\mathbf{x} \\
&+ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \exp\left[\tfrac{(A^*\mu_2)^T \Sigma (A^*\mu_2)]}{2}\right] \int_{0}^{\infty} \int_{0}^{\infty} xx^T \otimes \phi_2(x;\mu_2,\Sigma_2)d\mathbf{x} \\
&+ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \exp\left[\tfrac{(A^*\mu_3)^T \Sigma (A^*\mu_3)]}{2}\right] \int_{0}^{\infty} \int_{0}^{\infty} xx^T \otimes \phi_2(x;\mu_3,\Sigma_2)d\mathbf{x} \\
&- \exp\left[\tfrac{(A\mu_4)^T \Sigma (A\mu_4)]}{2}\right] \int_{0}^{\infty} \int_{0}^{\infty} xx^T \otimes \phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}] \\
&= Z^{-1}|\Sigma|[\tfrac{\int_{0}^{\infty}\int_{0}^{\infty} xx^T \otimes \phi_2(x;\mu_1,\Sigma)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma)} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \tfrac{\int_{0}^{\infty}\int_{0}^{\infty} xx^T \otimes \phi_2(x;\mu_2,\Sigma)d\mathbf{x}}{\phi_2(A\mu_2,\Sigma)} \\
&+ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \tfrac{\int_{0}^{\infty}\int_{0}^{\infty} xx^T \otimes \phi_2(x;\mu_3,\Sigma)d\mathbf{x}}{\phi_2(A\mu_3,\Sigma)} - \tfrac{\int_{0}^{\infty}\int_{0}^{\infty} xx^T \otimes \phi_2(x;\mu_4,\Sigma)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma)}] \\
&= Z^{-1}[|\Sigma_1|(\tfrac{E[\mathbf{A}\mathbf{A}^T]\int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_1,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma_1^{-1})} + \tfrac{E[DD^T]\int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma_1^{-1})}) \\
&+ |\Sigma_2|(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \tfrac{E[BB^T]\int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_2,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_2,\Sigma_2^{-1})} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \tfrac{E[CC^T]\int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_3,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_3,\Sigma_2^{-1})})]
\end{aligned}
$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$, $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T$ $\mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. $\mathbf{A} \sim MTN_+(\mu_1, \Sigma_1)$, $B \sim MTN_+(\mu_2, \Sigma_2)$, $C \sim MTN_+(\mu_3, \Sigma_2)$, $D \sim MTN_+(\mu_4, \Sigma_1)$ is denotes the multivariate positively truncated normal distribution.

$$
E[AA^T] = Cov(A) - E[A]E[A]^T
$$

5 Finding marginal distribution

$$f(x_1, x_2) = \frac{1}{Z} exp(-\frac{1}{2} x^T A x + b^T x - c||x||_1)) dx_2$$

$$
\begin{aligned}
f(x_1) &= \int_{-\infty}^{\infty} f(x_1, x_2) dx_2 \\
&= Z^{-1} exp(-\frac{1}{2} x^T A x + b^T x - c||x||_1)) dx_2 \\
&= Z^{-1} exp(-0.5 a_{11} x_1^2 + b_1 x_1 - c|x_1|) \\
&\quad \int_{-\infty}^{\infty} exp(-\frac{1}{2}[(a_{12} + a_{21}) x_1 x_2 + a_{22} x_2^2] + b_2 x_2 - c|x_2|] dx_2 \\
&= k \int_{-\infty}^{\infty} exp[-(0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{22} x_2^2 + b_2 x_2 - c|x_2|] dx_2 \\
&= k[\int_0^{\infty} exp[-0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{22} x_2^2 + (b_2 - c) x_2] dx_2 \\
&\quad + \int_{-\infty}^0 exp[-0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{22} x_2^2 + (b_2 + c) x_2] dx_2 \\
&= k[\int_0^{\infty} exp[-0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{22} x_2^2 + (b_2 - c) x_2] dx_2 \\
&\quad + \int_0^{\infty} exp[0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{22} x_2^2 - (b_2 + c) x_2] dx_2] \\
&= k[\int_0^{\infty} exp[-\frac{(x_2 - \mu_1)^2}{2\sigma^2} + \frac{\mu_1^2}{2\sigma^2}] dx_2] + \int_0^{\infty} exp[-\frac{(x_2 - \mu_2)^2}{2\sigma^2} + \frac{\mu_2^2}{2\sigma^2}] dx_2] \\
&= k\sigma[\frac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)} + \frac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}]
\end{aligned}
$$

where $\mu_1 = (-\frac{a_{12}+a_{21}}{2a_{22}} x_1 + \frac{b_2 - c}{a_{22}})$, $\mu_2 = (\frac{a_{12}+a_{21}}{2a_{22}} x_1 - \frac{b_2 + c}{a_{22}})$, $\sigma^2 = 1/a_{22}$, $k = Z^{-1} exp(-0.5 a_{11} x_1^2 + b_1 x_1 - c|x_1|)$

$$
\begin{aligned}
f(x_2) &= \int_{-\infty}^{\infty} f(x_1, x_2) dx_1 \\
&= Z^{-1} exp(-\frac{1}{2} x^T A x + b^T x - c||x||_1)) dx_1 \\
&= Z^{-1} exp(-0.5 a_{22} x_2^2 + b_2 x_2 - c|x_2|) \\
&\quad \int_{-\infty}^{\infty} exp(-\frac{1}{2}[a_{12} a_{21} x_1 x_2 + a_{11} x_1^2] + b_1 x_1 - c|x_1|] dx_1 \\
&= k \int_{-\infty}^{\infty} exp[-0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{11} x_1^2 + b_1 x_1 - c|x_1|] dx_1 \\
&= k[\int_0^{\infty} exp[-0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{11} x_1^2 + (b_1 - c) x_1] dx_1 \\
&\quad + \int_{-\infty}^0 exp[-0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{11} x_1^2 + (b_1 + c) x_1] dx_1] \\
&= k[\int_0^{\infty} exp[-0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{11} x_1^2 + (b_1 - c) x_1] dx_1 \\
&\quad + \int_0^{\infty} exp[0.5(a_{12} + a_{21}) x_1 x_2 - 0.5 a_{11} x_1^2 - (b_1 + c) x_1] dx_1] \\
&= k[\int_0^{\infty} exp[-\frac{(x_2 - \mu_1)^2}{2\sigma^2} + \frac{\mu_1^2}{2\sigma^2}] dx_1] + \int_0^{\infty} exp[-\frac{(x_2 - \mu_2)^2}{2\sigma^2} + \frac{\mu_2^2}{2\sigma^2}] dx_1] \\
&= k\sigma[\frac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)} + \frac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}]
\end{aligned}
$$

where $\mu_1 = (-\frac{a_{12}+a_{21}}{2a_{11}} x_2 + \frac{b_1 - c}{a_{11}})$, $\mu_2 = (\frac{a_{12}+a_{21}}{2a_{11}} x_2 - \frac{b_1 + c}{a_{11}})$, $\sigma^2 = 1/a_{11}$, $k = Z^{-1} exp(-0.5 a_{22} x_2^2 + b_2 x_2 - c|x_2|)$