A novel algorithm for the Bayesian Lasso: A local approximation adjustment approach

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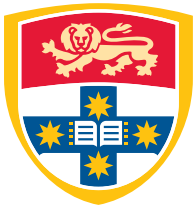
A thesis submitted in partial fulfillment of

the requirements for the degree of

Bachelor of Science(Honour)(Data Science)

School of Mathematics and Statistics



THE UNIVERSITY OF SYDNEY

June 2023

Statement of originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Yuhao Li

Abstract

Least Absolute Shrinkage and Selection Operator penalized regression, with an abbreviation of Lasso penalized regression, is regarded as a core statistical technique for simultaneous coefficient estimation and model selection. The idea of Lasso is to add the additional l_1 norm penalty function to the objective function that have sum of squared residual only for ordinary regression, generating and eliminating sparse coefficient to achieve efficient and interpretable model selection.

By considering the Lasso problem from a Bayesian perspective, the Bayesian Lasso model uses a double-exponential prior to model the variation of inferential quantity and to obtain interval estimates of coefficients. Moreover, the Bayesian framework offers an automatic tuning process for the tuning parameter λ, which controls the strength of penalization. This process replaces the time-consuming n-fold cross-validation technique used in the ordinary Lasso.

On the other hand, obtaining the posterior of the Bayesian Lasso model involves using Monte Carlo Markov Chain methods, such as Gibbs sampler for exact posterior distribution. Even though MCMC is famous for its oracle property such as generating exact target samples if Markov Chain is run for enough iterations, it is slow and has a high computational cost.

Meanwhile, variational approximation, as a deterministic approximation algorithm for intractable posterior distribution, has been applied prevalently for fast Approximate Bayesian Inference(ABI) among the Bayesian Statistical community. It is also a faster alternative to Monte Carlo methods such as MCMC.

The concept of variational approximation involves proposing a set of known densities, and subsequently identifying the density within that set that best approximates the target. The level of approximation is evaluated using Kullback-Leibler (KL) divergence.

Nevertheless, variational approximation unlike the MCMC methods does not guarantee to approximate the exact target density. It can only find a density close to the target which means the approximation accuracy might be a pivotal concern and it fails to provide

reliable estimates of posterior variances

In order to address the slow speed issue of obtaining posterior distribution of the Bayesian Lasso problem, new alternative ABI methods would be explored, especially for deterministic algorithm such as variational Bayes.

This thesis presents two recently discovered distributions, the univariate and multivariate lasso distributions. These distributions can be used to assist in approximating marginal distributions. Additionally, the thesis introduces two highly efficient and precise variational approximation algorithms and their application in solving the Bayesian Lasso regression problem.

The first method involves matching with a marginal univariate lasso distribution by updating the global parameter for each variable per iteration. Additionally, we propose another algorithm that matches a local bivariate lasso distribution to update the global parameter for each pair of variables per iteration. This algorithm successfully addresses the issue that arise when an initial diagonal covariance matrix is assigned.

To verify the efficiency and accuracy of our algorithm, we conducted numerous experiments using real-world datasets such as the Hitters Dataset and evaluated it using several metrics such as l_1 accuracy and running speed.

Acknowledgements

Firstly, I would like to express my deep thanks to my supervisor: A/Prof: John Ormerod and Co-supervisor Doctor Mohammad Javad Davoudabadi, for their constant guidance and patience throughout this year. I would not make it so far without them. Secondly, I would like to thank my friends, my family, my mother, father. The entire honour year has been full of challenges not only in the researching a novel area, but also the heavy coursework workload. I could not have overcome them without your support behind the scenes. Finally, I would like to thank the University of Sydney, for providing me a chance to study and research in this school. It is an unforgettable experience.

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Chapter 1

Introduction

1.1 Background and Motivation

Introduction of Lasso Problem

The Least Absolute Selection and Shrinkage Operator (Lasso) regression proposed by Tibshirani (1996) belongs to one of the shrinkage methods. As one of the most traditional shrinkage methods, Lasso regression has been proven in Statistical Community over the years.

The Lasso serves two purposes. One is the estimation of regression parameter and the other is to the effective shrinking of the coefficients to achieve variable selection purpose, which is also the fundamental difference of Lasso with other methods. The Lasso regression is helpful particularly for high-dimensional because of its sparsity nature.

Explain purpose of Lasso The linear regression model is

$$y = \mu 1_n + X\beta + \epsilon. \tag{1.1}$$

where β is a $p * 1$ vector of regression coefficient, y is a $n * 1$ vector of response variable, X is a $n * p$ matrix of covariates, μ is a $n * 1$ vector of population mean, and ϵ is the model uncertainty which is distributed normally with mean 0 and variance σ^2

The Least square estimator suggest the sum of square of the difference between estimated response variable and true response variable should be used as loss function as described in Equation (1.2)

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} (\tilde{y} - X\beta)^T (\tilde{y} - X\beta). \tag{1.2}$$

The tuning parameter λ controls the strength of the penalty.

$$\hat{\beta}_{lasso} = \underset{\beta}{\operatorname{argmin}} (\tilde{y} - X\beta)^T (\tilde{y} - X\beta) + \lambda ||\beta||_1, \tag{1.3}$$

where $\lambda \geq 0$. $\tilde{y} = y - \bar{y}\mathbf{1}_n$. Larger penalization leads to more sparse solution of regression coefficient due to a square constraint set results from L_1 penalty function, resulting in implicit variable selection of parameter. This can generate a higher prediction accuracy as well as an more interpretable model since we could drop the estimated regression coefficient that has 0 and state they have weak effect for prediction according to Tibshirani (1996). However, due to non-existence of derivative of absolute value of regression coefficient β, alternative improved algorithm have been purposed and deployed such as least angle regression (LARS), iterative soft-thresholding, subgradient method, and iteratively reweighted least square (IRLS) by Efron et al. (2004), Beck and Teboulle (2009), Zhang and Zeng (2005) and Friedman et al. (2010).

Bayesian Lasso One drawback of the ordinary lasso is that it cannot properly capture the variation of inferential quantity. To address this issue, Tibshirani (1996) suggested extending the lasso estimate under the Bayesian framework. This can be achieved by assigning an independent and identically distributed Laplacian prior from Equation(1.4) and combining it with the likelihood form in Equation (1.5) to obtain the posterior mode Equation(1.6).

$$\hat{\beta}_{lasso} = \underset{\beta}{\operatorname{argmax}} P(\beta|y, \sigma^2, \tau), \quad \tau = \frac{\lambda}{2\sigma^2} \tag{1.4}$$

$$f(\beta_j) = (\frac{\tau}{2}) \exp(-\tau|\beta_j|), \tag{1.5}$$

$$p(y|\beta, \sigma^2) = N(y|X\beta, \sigma^2 I_n). \tag{1.6}$$

Introduction of Bayesian Lasso Problem (Why Bayesian Lasso?) Nevertheless, there is no tractable integration form for the Bayesian Lasso posterior until Park and Casella (2008) further explore the Lasso model under the setting of Bayesian framework. The choice of a conditional Laplace prior distribution over the regression coefficient β conditioning by standard error σ^2 is added to the Lasso penalty formulation in the frequentist framework. This would ensure the unimodality of full posterior distribution. Based on closed form of tractable posterior distribution, a three-step Gibbs sampler is proposed to draw samples from Bayes Lasso posterior distribution, which can be utilized for further inference of parameter of interest. There are several benefits for using the Bayesian Lasso model. Firstly, it has easier implementation than the traditional Lasso, although it demands more computation of conditional distribution form is demanded. Secondly, we can generate

Bayesian credible intervals simultaneously for the parameters in a model, allowing for the modeling of uncertainty and guiding variable selection. Thirdly, Park and Casella (2008) also state that the Bayesian Lasso model could be a potential solution for addressing the issue of attaining optimal tuning parameter λ. We can achieve this by using the marginal maximum likelihood method along with a suitable hyperprior, such as a gamma prior on the square of the tuning parameter (λ^2). It could support a more stable automatic tuning process of choosing the most appropriate tuning parameter λ for the Lasso problem. This is in contrast to the inefficient K-fold cross-validation approach under the ordinary Lasso model, which is time-consuming and computationally demanding. Lastly, the three-step Bayesian Lasso Gibbs Sampler proposed by Park and Casella (2008) would yield an exact posterior distribution that can be sampled given exact form of conditional distribution of each model parameter conditioning on rest of the other parameters. Theoretically, Khare and Hobert (2013) demonstrated a Bayesian Lasso Gibbs sampler version of central limit theorem (CLT), indicating Bayesian Lasso Gibbs sampler satisfies geometric ergodicity for any values of sample size $n \geq 3$ and an arbitary number of regression coefficient p, data matrix X, tuning parameter λ. This means, the Bayesian lasso Gibbs sampler is able to achieve asymptotically uncertainty of posterior estimation. To address this issue, Rajaratnam and Sparks (2015) invented a reduced step Gibbs sampler instead, successfully accelerating the sampling procedure.

Approxiamte Bayesian Inference Intuition The Review of Bayesian inference intuition can be referred from Section 2.1, but challenges of Bayesian inference motivates the approximate Bayesian inference method thereafter. To be specific, Bishop (2006) states three main challenges of obtaining posterior distribution. Firstly, the dimension of target parameter might be high, which results in heavy computational cost for estimating posterior distribution. Secondly, it takes a long time to converge to the target posterior distribution. Thirdly, the computation of the posterior mean parameter $\int_\theta \theta p(\theta|\mathcal{D})$ has high probability without having a simple calculation, resulting in the fact that there might not exist a closed form analytical solution for integration. Much efforts have been paid over the years, there are two main types of sampling approach that are effective currently, which are stochastic sampling algorithms and deterministic approximation algorithms.

Introduction of ABI method There are two genres of ABI methods, which include stochastic approaches such as MCMC, where an exact result can be obtained if infinite computational resource is assigned. Another category lies in deterministic approaches, which provide a faster substitution compared to stochastic approximation approaches. As stated above, an approximate inference methods such as MCMC are used for posterior distribution estimation. The need for approximate Bayesian inference methods arises from the difficulty and impracticality of estimating the exact posterior distribution parameters.

Challenges of Bayesian Lasso model Three-step Gibbs sampler belongs to the class of MCMC algorithms, which produces samples demonstrates gradually decreasing correlated samples as the burn-in period is increased. Burn-in period refers to the initial period during which the sample is discarded or ignored to eliminate any potential biases or inconsistencies in the measurement process. The purpose of the burn-in period is to ensure that the measurements are stable and consistent before beginning the actual analysis. The three-step Gibbs sampler is time-consuming and computational challenging, given the fact that it normally requires a long time to converge inside the interval of an acceptable tolerance.

Approximation Algorithm: Deterministic type & VI. As mentioned before, due to the limitations of stochastic algorithms, alternative methods such as deterministic Variational Inference (VI) have become popular due to their fast speed and simple computation. Numerous algorithms have been designed and utilized widely such as VB, Expectation Propagation algorithms etc. A common variation is the Coordinate Ascent Variational Inference approach produced by Blei et al. (2003), which assumes that the approximation originates from an analytically tractable class of distribution Q. Afterwards it attempts to search for the distribution from this family that is closest to the target posterior distribution with some discrepancy metric, such as the Kullback-Leibler divergence. An optimization based system in Equation (1.7) is established by iteratively updating variational parameter with an appropriate optimization algorithm. A traditional optimization algorithm can be Coordinate Ascent that could obtain approximated posterior distribution in the family of Q, while the most common choice is the Normal Distribution due to its simple form and adaptability to other distribution. To further illustrate the intuition, Figure 1.1 pro-

vides further explanation of aforementioned intuition, goal and procedure of Variational Inference.

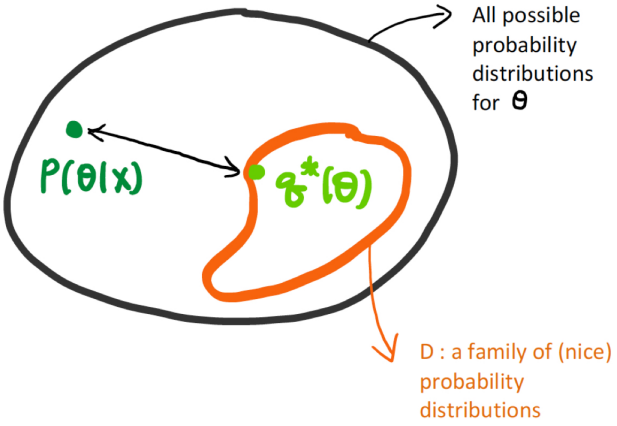

Figure 1.1: Variational Inference intuition, where X is data $\mathcal{D}$, $\mathcal{D}$ is equivalent to Q defined above

$$q^*(\theta) = \underset{q_\theta \in Q}{\text{argmin}} \text{KL}(q(\theta)||p(\theta|\mathcal{D})) := \int q(\theta) \log(\frac{q(\theta)}{p(\theta|\mathcal{D})}) d\theta \tag{1.7}$$

$$\text{KL}(q||p(.|\mathcal{D})) = -\int q(\theta) log(\frac{p(\theta)p(\mathcal{D}|\theta)}{q(\theta)}) d\theta + \log p(\mathcal{D}). \tag{1.8}$$

In addition, the exact form of KL divergence can be found in Equation (1.8) In practice, minimizing the KL divergence from Equation 1.7 is difficult due to its complexity, and therefore it is often converted into an equivalent formulation in Equation 1.9 that maximizes the lower bound of $\log(p(y))$. This lower bound is known as the Evidence Lower Bound (ELBO), and can be defined by Equation 1.9).

$$q^*(\theta) = \underset{q_\theta \in Q}{\text{argmax}} \text{ ELBO}(q(\theta)), \tag{1.9}$$

$$\begin{aligned} \text{ELBO}(q(\theta)) &= \int q(\theta) \log(\frac{p(\theta)p(\mathcal{D}|\theta)}{q(\theta)} d\theta = \mathbb{E}_{q(\theta)} \log(\frac{p(\theta)p(\mathcal{D}|\theta)}{q(\theta)}). \\ &= \mathbb{E}_{q(\theta)}[\log p(\theta, \mathcal{D})] - \mathbb{E}_{q(\theta)}[\log[q(\theta)]] \\ &= \mathbb{E}_{q(\theta)}[\log p(\theta, \mathcal{D})] - \text{KL}(q(\theta)||p(\theta)) \end{aligned} \tag{1.10}$$

Mean Field Variational Bayes, History and Introduction

The most traditional Variational Inference algorithm is known as MFVB motivated by mean-field theory in statistical physics, as proposed by Parisi and Shankar (1988). The algorithm assumes that the approximated distribution is a product of independent parameter distributions from set Q, as described in Equation 1.11, assuming there are k sub-parameters of the target parameter θ.

$$q(\theta) = \prod_{i=1}^{k} q_i(\theta) \tag{1.11}$$

The MFVBs have been adapted and developed over the last two decades, especially in mixture modelling and probabilistic graphical modelling. It sometimes provides efficient variable selection as well, We will introduce more about the algebra of MFVB in Chapter 2. **Advantages of Variational Inference** One is that Variational Inference algorithm shows a descent computation cost and scalability, and thus Variational inference can be adaptive when the amount of data is huge. For instance, if there exists a billion image that requires to be fitted into probabilistic machine learning model, then exact precision method such as MCMC will be computataional demanding, while Variational Inference would sacrifice tiny accuracy with hundreds of time faster speed as return. Secondly, the descent time-efficiency of Variational Inference becomes another significant factor why it is popular, given the fact it only involves updating variational parameters iteratively until convergence, as opposed to MCMC that produce correlated samples that limit the ideal behavior of MCMC algorithm.

Drawbacks of VI Meanwhile, disadvantages of Variational Bayes include inexact approximation result under some scenarios, although it could capture marginal density. For example, it is suggested by Blei et al. (2017), that Variational Inference algorithm might underestimate the covariance between parameter of interest, if inter-parameter correlation is strong. It tends to ignore the correlation between parameters, results in unideal behavior as a resullt. Figure 1.2 further demonstrates this phenomenon, the ture overall posterior of x_2 and x_1 have a exploded correlation with a eclipse-shaped density, while a circled-shape mean-field approximation is established instead due to its product density family limitation. We will expand properties and derivation of Variational Inference more in subsection 2.5.

Overall, Variational Inference has proven its effectiveness in distinct application fields

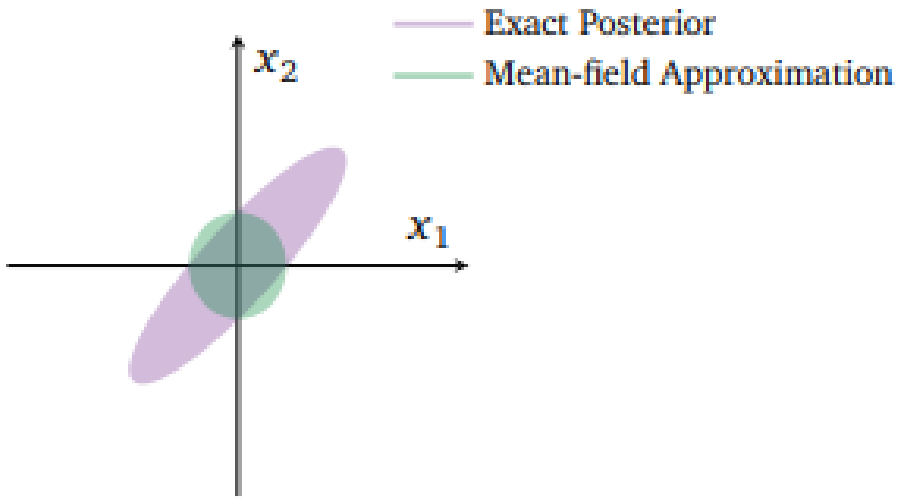

Figure 1.2: Visualization of Mean-Field Variational Approximation compared with exact posterior when correlation is large

such as speech recognition and document retrieval in natural language processing, computer vision etc. Despite small disadvantages of Variational Inference, the potential of variational approximation haven't been fully discovered yet by researchers, its ability to provide reliable posterior estimate is invaluable in the future in the era of big data and deep learning nowadays.

Motivation This study is motivated by the desire to improve the approximation accuracy of Variational Bayes and to take advantage of the oracle property of Bayesian Lasso regression coefficient estimation for variable selection and standard error estimation, there is an increasing demand for fast approximate inference. We would like to design new VI based algorithms for the Bayesian Lasso regression problem, for the purpose of obtaining Bayesian lasso posterior distribution in a much faster and more accurate manner. We fed the initial values of the mean μ and covariance Σ and wrote out the marginal likelihood form of the regression coefficient β_j for the j_{th} variable. Using this information, we invented a new distribution called the univariate Lasso distribution to match the marginal likelihood, represented by $\log(p(\mathcal{D})|\beta_j)$, we have invented a new distribution called univariate lasso distribution for matching the marginal likelihood. Mixing each marginal likelihood with a gaussian approximation. We have demonstrated that our algorithm's approximation accu-

racy surpasses that of every existing algorithm, including MFVBs. Even though the speed of our algorithm is slightly slower than MFVB, the approximation accuracy illustrates a small gap between exact estimation from MCMC, with hundred times faster time complexity. Nevertheless, there is a drawback of this method, given the fact that the global covariance matrix would remain diagonal if the initial global covariance matrix is diagonal. To remedy this issue, we have also purposed another algorithm based on marginal likelihood estimation by a bivariate Lasso distribution. Instead of updating corresponding mean, covariance matrix for each variable in each iteration, the marginal likelihood of each pair of variables would be matched, so that further generalize our algorithm. Our conclusions are the univariate Lasso algorithm is faster with a lower accuracy while bivariate Lasso algorithm is slower with a higher accuracy, since it updates each pair of variables at a time resulting in $\binom{p}{2}$ of unique parirs. We will show the full intuition and idea later in Chapter 3. By utilizing and fitting a both univariate and bivariate Lasso distribution to each of the marginal distribution, an improved estimate for global gaussian approximation can be obtained as defined in Equation (1.12). Finally, we will show our experiment result in Chapter 4 using various accuracy metrics.

Our contribution have been listed in the following subsection 1.2.

$$q^*(\theta) \approx N(\mu^*, \Sigma^*) \tag{1.12}$$

1.2 Contribution

Our main contribution could be concluded as the following parts:

- Introduction of univariate and multivariate Lasso distribution
- Derivation of properties for Univariate Lasso distribution, such as the expectation, variance, cumulative density function form etc.
- Derivation of properties for multivariate Lasso distribution such as the expectation, variance, cumulative density Function form etc.
- Implementation of univariate Lasso distribution and multivariate Lasso distribution property in R.

- Design of two new VI approaches based on local approximation by univariate lasso distribution and multivariate lasso distribution respectively.
- Conduct of experiment to testify two algorithms under dataset by several evaluation metrics for approximation accuracy such as Hitters dataset etc.

1.3 Thesis Organization

The thesis is organized as follows. Chapter 1 briefly illustrates the motivation and background of the Lasso problem, Bayesian Lasso Problem, and Approximate Bayesian Inference, with a specific focus on determinstic Variational Approximation. Chapter 2 briefly reviews and explains the details of the methods in previous work such as the Lasso problem, Approximate Bayesian Inference algorithm, MCMC, Bayesian Expectation Maximization algorithm and their variants and MFVB. We present our main methodology of variational algorithm in Chapter 3, followed by a comprehensive experiment for testing the effectiveness of algorithm in Chapter 4.

Chapter 2

Literature Review

2.1 Bayesian Inference Paradigm

Why Bayesian Bayesian inference approaches offer numerous advantages in the statistical community and application areas, especially in situations where data is scarce. In cases where effective data is extremely rare and insufficient, an appropriate prior choice can provide significant benefits, particularly in medical problems. Unlike frequentist inference approaches that treat parameter estimates as fixed values, Bayesian inference approaches regard parameter estimates as random variables that have probability distributions. This unique feature allows for interval estimates and error variances to be generated, providing a more comprehensive understanding of uncertainty and increasing confidence in interpreting parameter estimates. By incorporating prior knowledge and probability distributions, Bayesian Inference approaches offer a more flexible and intuitive framework for statistical analysis.

Bayesian Inference Intuition Bayesian inference approach stems from the Bayes rule, which is defined as Equation (2.1) based on theory developed by Beech et al. (1959). Suppose θ is our model parameter of interest, $\mathcal{D}$ is data, then $p(\theta)$ is known as prior distribution, which offers pre-existing knowledge or information about θ. Posterior distribution $p(\theta|\mathcal{D})$ refers to the likelihood conditioning on the data $\mathcal{D}$. Incorporating information from current data and prior knowledge, posterior distribution can be then inferred and simplified to Equation (2.2), as the value of $p(\mathcal{D})$ is constant and insignificant for determining the overall posterior distribution.

$$p(\theta|\mathcal{D}) = \frac{p(\mathcal{D}|\theta)p(\theta)}{p(\mathcal{D})}, \tag{2.1}$$

$$p(\theta|\mathcal{D}) \propto p(\mathcal{D}|\theta)p(\theta), \tag{2.2}$$

2.2 Least Absolute Shrinkage and Selection Operator (LASSO) penalized regression

2.2.1 Lasso penalty formulation

The constraint form of lasso can be shown by Equation 2.3, where $t \geq 0$ is denoted as a tuning term, regression coefficient is β, $||\beta||_1$ is the l_1 norm of beta, $||y - X\beta||_2$ is the l_2 norm: of residual value, data matrix is X, response variable is y. The estimation for lasso estimate $\hat{\beta}_{lasso}$ is defined by Equation 2.3.

$$\hat{\beta}_{lasso} = \underset{\beta}{\operatorname{argmin}} ||y - X\beta||_2, s.t. ||\beta||_1 \leq t, t \geq 0. \tag{2.3}$$

In order to transform constraint form of lasso to penalty form, Lagranage multiplier method, as a pivotal technique from transforming a constraint optimization system into an unconstrained penalty formulation of system has been used. The Lagrangian function for constrained Lasso Regression is constructed by Equation 2.4

$$\mathcal{L}(\beta, \lambda) = ||y - X\beta||_2 + \lambda||\beta||_1 - \lambda t, \lambda \geq 0 \tag{2.4}$$

Since the objective function contains a quadratic term $||y - X\beta||_2$ with a linear term $\lambda||\beta||_1 - \lambda t)$, leading to a convex optimization problem. Due to strong duality theorem in convex optimization system, therefore the penalty formulation of lasso regression can be deduced as Equation Equation 1.3, is equivalent to constraint form Equation 2.3 after ignoring the unaffected constant $-\lambda t$.

Graphical demonstration of the lasso for Equation 2.3 and Equation 1.3 can also be found on the left hand side of the Figure 2.1, where the squared constraint set is drawn, in addition to the contour line of regression coefficient. The penalty term λ controls the strength of the penalization in Lasso regression. When the value of λ is set higher, a more sparse solution is facilitated. This forces the estimated coefficients to lie closer to the axis of each parameter, as shown in 2.1. As a result, Lasso regression coefficients are more likely to intersect with the corners of the squared constraint set, leading to the occurrence of sparse estimated regression coefficients. By encouraging sparsity, Lasso regression provides a useful tool for variable selection and reducing model complexity, leading to more interpretable and

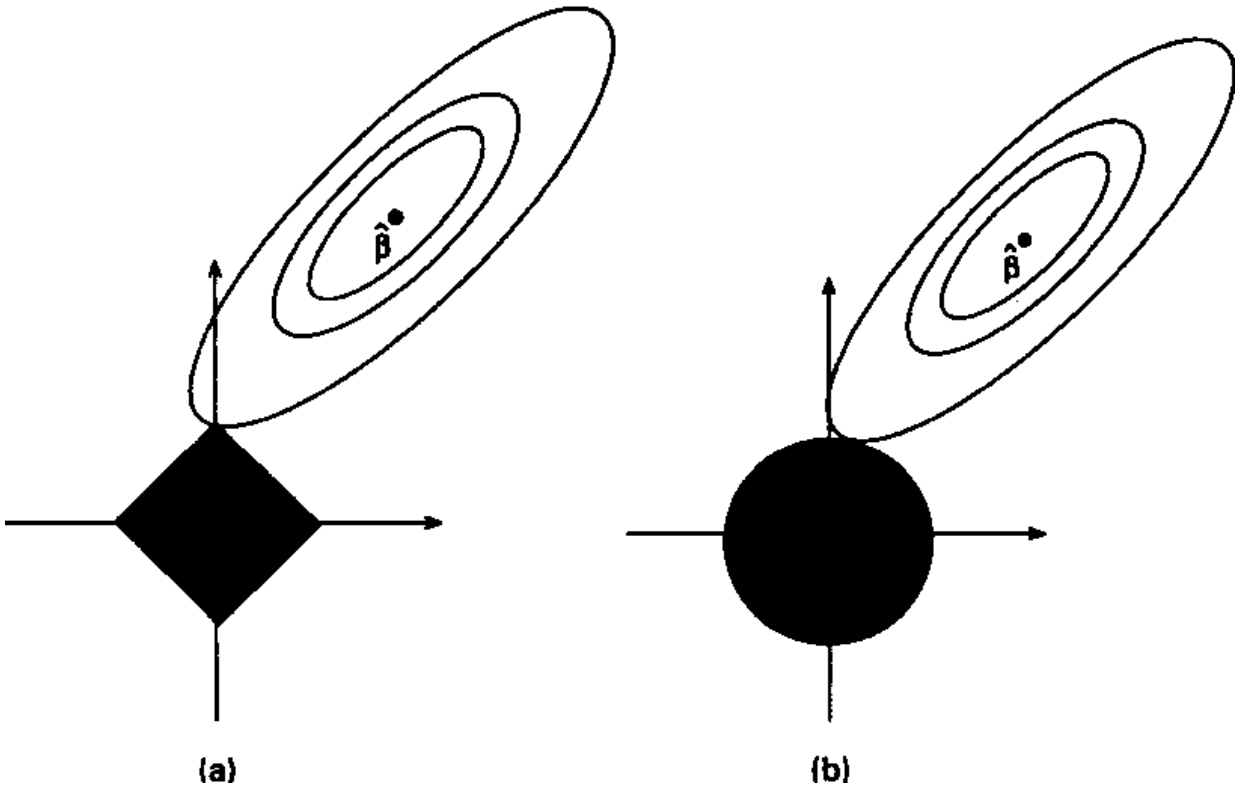

Figure 2.1: Graphical comparison between lasso regression and ridge regression

generalizable models. On the other hand, ridge regression uses a l_2 penalty to estimate the regression coefficients, but it tends to gain a non-sparse solution due to circled constraint set for β.

2.3 Bayesian Lasso

2.3.1 Bayesian Lasso model

Park and Casella (2008) proposed an alternative formula for the conditional Laplace prior, which takes the form of Equation 1.5 and is expanded in Equation 2.5. This approach offers a Bayesian interpretation of the Lasso penalty and provides a framework for incorporating prior information into the variable selection process. By using the conditional Laplace prior, the Bayesian Lasso regression model can be tuned to strike a balance between sparsity and estimation accuracy, resulting in a more robust and interpretable model.

$$\pi(\beta) = \prod_{j=1}^{p} \frac{\lambda}{2} e^{-\lambda|\beta_j|} \tag{2.5}$$

For a given variance, the mode of posterior form in Equation 2.6 is consistent with the estimate of lasso equation in Equation 1.3, but it will hinder the bayesian interpretation, inference and variable selection since the bayesian predictive distribution make future inference via a posterior mean instead of posterior mode. In addition, if a variance is unknown, the posterior will be a multimodal distribution, the derivation has been provided by the appendix from Park and Casella (2008).

$$\pi(\beta, \sigma^2|\tilde{y}) \propto \pi(\sigma^2)(\sigma^2)^{-(n-1)/2}\exp(\frac{1}{2\sigma^2}(\tilde{y} - X\beta)^T(\tilde{y} - X\beta) - \lambda\sum_{j=1}^{p}|\beta_j|) \tag{2.6}$$

$$\pi(\beta|\sigma^2) = \prod_{j=1}^{p}\frac{\lambda}{2\sqrt{\sigma^2}}e^{-\lambda|\beta_j|/\sqrt{\sigma^2}} \tag{2.7}$$

To remedy this issue, a conditional Laplacian prior from 2.7 with respect to Equation(2.5) has been designed, ensuring the unimodality of the posterior for β, and the current prior with respect to β, σ^2 can be written as

$$\pi(\beta, \sigma^2) \propto \pi(\sigma^2)\prod_{j=1}^{p}\frac{\lambda}{2\sqrt{\sigma^2}}e^{-\lambda|\beta_j|/\sqrt{\sigma^2}}, \tag{2.8}$$

which can result in the unimodal joint posterior distribution $\pi(\beta, \sigma^2|\tilde{y})$ of β and $\sigma^2 > 0$ under the new prior 2.8, given an improper prior selection for $\pi(\sigma^2) = \frac{1}{\sigma^2}$ and $\lambda \geq 0$. Additionally, an additional latent variable τ is introduced as a scale mixture of Gaussians for reformulation of conditional prior 2.7 as 2.9, which can be regarded as corresponding weight assigned to each regression coefficient. If τ_j goes to 0 then the corresponding regression coefficient will be shrunk towards zero accordingly.

$$\frac{\alpha}{2}e^{-\alpha|z|} = \int_0^\infty \frac{1}{\sqrt{2\pi s}}e^{-z^2/(2s)}\frac{\alpha^2}{2}e^{-\alpha^2 s/2}ds, \alpha > 0 \tag{2.9}$$

Finally, the hierarchical bayesian Lasso model functional form can be written as Equation (2.10).

$$\begin{aligned}
y|\mu, X, \beta, \sigma^2 &\sim N_n(\mu + X\beta, \sigma^2 I) \\
\beta|\tau_1^2, ..., \tau_p^2 &\sim N_p(0, \sigma^2 D_\tau) \\
D_\tau &= diag(\tau_1^2, ..., \tau_p^2) \\
\tau_1^2.., \tau_p^2 &\sim \prod_{j=1}^{p}\frac{\lambda^2}{2}e^{-\lambda^2\tau_j^2/2}d\tau_j^2, \tau_1^2, .., \tau_j^2 > 0 \\
\sigma^2 &\sim \pi(\sigma^2) = 1/\sigma^2, \sigma^2 > 0
\end{aligned} \tag{2.10}$$

.

2.3.2 Bayesian Lasso Gibbs Sampler

Gibbs Sampler

Geman and Geman (1984) introduced a special case of the Metropolis-Hastings algorithm called the Gibbs sampler. As a Markov Chain Monte Carlo sampling algorithm, it can be used for efficient sampling of any probability density function, given the posterior form from the corresponding conditional distribution. In each iteration, each parameter of interest is sampled once by conditional distribution for the current iteration. After running the chain for long enough iterations, it returns posterior distribution samples with descent approximation accuracy after discarding samples in burn-in period. Notice that, the functional form of conditional distribution given any other parameter of interest has to be acquired. In this study, we need to have the marginal distributions of (β, σ^2, τ):

$$p(\beta|\mathcal{D}, \sigma^2, \tau^2)$$
$$p(\sigma^2|\mathcal{D}, \beta, \tau^2)$$
$$p(\tau|\mathcal{D}, \beta, \sigma^2)$$

After getting functional form and ignoring the normalizing constant, we can infer their category of probability distribution for each expression, and we can sample from the corresponding distribution.

Initial Setting: In the initial setting we consider that: λ is fixed, and σ^2 has a a Gamma distribution with parameter a and b: $\pi(\sigma^2) = \frac{b^a}{\Gamma(a)}(\sigma^2)^{-a-1}e^{-b^2/\sigma^2}, \sigma^2 > 0, a > 0, b > 0$
Our first step is to write the joint distribution: $p(\beta, \tau^2, \sigma^2, \mathcal{D})$

Joint distributional form: Given Equation 2.10, we can write the joint distribution as

$$\begin{aligned} p(\tilde{y}, \beta, \tau^2, \sigma^2) &= p(\tilde{y}|\beta, \sigma^2, \tau)p(\sigma^2)\prod_{j=1}^{p} p(\beta|\sigma^2, \tau_j)p(\tau_j^2) \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}}e^{\frac{-(\tilde{y}-X\beta)^T(\tilde{y}-X\beta)}{2\sigma^2}}\frac{b^a}{\Gamma(a)}(\sigma^2)^{-a-1}e^{-b^2/\sigma^2}\prod_{j=1}^{p}\frac{1}{(2\sigma^2\tau_j^2)^{1/2}}e^{-\frac{-1}{2\sigma^2\tau_j^2}\beta_j^2}\frac{\lambda^2}{2}e^{-\lambda^2\tau_j^2/2} \end{aligned} \tag{2.11}$$

Full conditional distribution of β:

$$p(\beta|\tilde{y},\tau,\sigma^2) \propto p(\tilde{y},\beta,\tau,\sigma^2) \tag{2.12}$$

Recognizing the term without β as constant, the conditional distribution of β can be simplified to

$$\begin{aligned} p(\beta|\tilde{y},\sigma^2,\tau^2) &\propto \exp(\frac{\beta^T X^T X\beta - 2y^T X\beta + \lambda^2\beta^T A^{-1}\beta}{-2\sigma^2}), A = diag(\tau) \\ &= \exp(-\frac{1}{2}\beta^T(\frac{X^T X + \lambda^2 A^{-1}}{-2\sigma^2})\beta + \frac{\tilde{y}^T X\beta}{\sigma^2}) \\ &\sim \text{MVN}(\mu^*,\Sigma^*) \\ \text{where } \mu^* = (X^T X + \lambda^2 A^{-1})^{-1}X^T y, &\Sigma^* = (X^T X + \lambda^2 A^{-1})^{-1}\sigma^2 \end{aligned} \tag{2.13}$$

So we can sample β from a multivariate normal distribution with its corresponding mean and variance.

Full conditional distribution of σ^2:

$$\begin{aligned} &p(\sigma^2|\tilde{y},\beta,\tau^2) \propto p(\tilde{y},\beta,\tau^2,\sigma^2) \\ &= (\sigma^2)^{-\frac{n}{2}-\frac{p}{2}-a-1}\exp(-\frac{1}{2\sigma^2}(\tilde{y}-X\beta)^T(\tilde{y}-X\beta) + \frac{1}{2\sigma^2}\beta^T D_\tau\beta + \frac{b}{\sigma^2}). \\ &\sim \text{Inverse-Gamma}(\alpha^*,\beta^*) \\ \text{where } \alpha^* = \frac{n}{2} + \frac{p}{2} + a, &\beta^* = (\tilde{y}-X\beta)^T(\tilde{y}-X\beta)/2 + \beta^T D_\tau\beta/2 + b \end{aligned} \tag{2.14}$$

Full conditional distribution of τ_j^2:

$$\begin{aligned} p(\tau_j^2|\tilde{y},\beta,\sigma^2) \propto p(\tilde{y},\beta,\tau^2,\sigma^2) &= \frac{1}{\sqrt{\frac{2\pi\sigma^2\tau_j}{\lambda^2}}}\exp(-\frac{-\beta_j^2\lambda^2}{2\sigma^2\tau_j})\exp(-\frac{1}{2}\tau_j)] \\ &\sim GIG(a^*,b^*,p^*) \\ \text{where GIG is generalized inverse gaussian distribution with parameters } a^* = 1, b^* &= \frac{\beta_j^2\lambda^2}{\sigma^2}, p = \frac{1}{2} \end{aligned} \tag{2.15}$$

Summary The gibbs sampler can be established by the following algorithm.

Automatic selection of the penalty parameter λ: Common choice of penalty parameter λ in the non-bayesian paradigm involves cross-validation approach, which is time-consuming and computational challenging especially for large datasets. Park and

Algorithm 1 Gibbs Sampler for the Bayesian Lasso

Given $\lambda^2 > 0, \tau^{(1)} = \mathbf{1_n}, \sigma^{2(1)} = 1, t = 1$ ▷ Initial Setting

while $t \leq 10^5$ **do**

  Sampling $\beta^{(t+1)} \sim \text{MVN}((X^TX + \lambda^2 A^{-1})^{-1}X^Ty, (X^TX + \lambda^2 A^{-1})^{-1}\sigma^2)$ ▷ Generate sample β

  Sampling $\sigma^{2(t+1)} \sim IG(\frac{n}{2} + \frac{p}{2} + a, \frac{||y - X\beta||_2^2}{2} + \frac{\lambda^2 \sum_j \beta_j^2}{2\tau_j} + b)$ ▷ Generate sample σ^2

  for j=1,...,p **do**

    Sampling $\tau_j^{2(t+1)} \sim GIG(1, \frac{\beta_j^2 \lambda^2}{\sigma^2}, 1/2)$ ▷ Generate sample τ_j

  $t \leftarrow t + 1$

return β, σ^2, τ^2

Casella (2008) set a hyperprior to λ^2 as 2.16, instaed of λ to facilitate conjugacy. According to Park and Casella (2008), there are some additional notification of choosing prior, which involves: firstly, to avoid mixing issue, the prior distribution for λ^2 should reach zero asymptotically with a descent speed as λ^2 goes to infinity, Secondly, the density at maximum likelihood estimate should be assigned with enough probability density with a overall flat distribution.

$$\pi(\lambda^2) = \frac{\delta^\gamma}{\Gamma(\gamma)}(\lambda^2)^{\gamma-1}e^{-\delta\lambda^2}, \text{for } \delta > 0, \gamma > 0, \lambda^2 > 0 \tag{2.16}$$

The penalty parameter is the extent of penalization of non-zero coefficient, which is also a compromise between model simplicity and fitting capability to data in the frequentist lasso setting. According to the posterior form of τ_j, λ controls the shape of generalized inverse-Gaussian posterior distribution of τ_j as shown before.

To obtain the posterior form of λ, we need to incorpoate a proper hyperprior distribution to the joint distribution $p(y, \beta, \sigma^2, \tau)$. First, assuming the prior of λ^2 is with shape and rate parameters θ and γ, respectively.

$$\begin{aligned} p(\lambda^2|\tilde{y}, \beta, \sigma^2, \tau^2) &\propto p(\tilde{y}, \beta, \tau^2, \sigma^2, \lambda) \\ &= (\prod_{j=1}^{p} \frac{\lambda^2}{2} e^{-\lambda^2\tau_j^2/2})(\lambda^2)^{\gamma-1}e^{-\delta\lambda^2} \\ &= (\lambda^2)^{p+\gamma-1}e^{-\lambda^2(\frac{1}{2}\sum_{j=1}^{p}\tau_j^2+\delta)} \end{aligned} \tag{2.17}$$

Thus, the posterior distribution of λ^2 is still following a Gamma distribution, with a shape

parameter $p+\gamma-1$ and rate parameter $\sum_{j=1}^{p}\tau_j^2+\delta$. The λ^2 can be sampled by 2.17, based on using an augmented Gibbs sampler.

2.4 Expectation Maximization

Even though the posterior distribution can be sampled by Gibbs sampler in the last subsection, the sparsity nature of the Bayesian lasso is not captured by posterior mean given by Gibbs Sampler. The posterior mode calculated by Bayesian Expectation Maximization, however, could capture the posterior mode and preserve the sparsity feature of the basic lasso.

2.4.1 Classical Expectation Maximization

The expectation maximization (EM) algorithm was proposed by Dempster et al. (1977). It is an iterative approach for seeking the maximum likelihood estimate of parameters for probabilistic models that have missing data or latent variables. The application of EM algorithm includes the inference of the parameters of the Gaussian Mixture model etc. The EM algorithm involves two main steps, which are E-steps and M-steps. Suppose Z is the set of latent variable, X is the set of entire set of observed variables, θ is parameter. t refers to the step during iteration, $\log(P(X,Z|\theta)))$ refers to the complete log-likelihood of data, and $\log(P(X|\theta)))$ refers to the incomplete log-likelihood of data without considering the hidden variables.

E-steps

By calculating the posterior distribution of the hidden variable given by the observed data and current parameter estimates, the purpose of this step is to compute the expectation of the latent variables by observed data, which is equivalent to calculate the expected value of the complete log-likelihood given the current parameter estimation and observed data. Mathematically, the E-step involves calculating the expectation of the complete data log-likelihood with respect to the conditional distribution of the hidden data given the observed

data and current parameter estimates:

$$Q(\theta, \theta^{(t-1)}) = E_{Z|X,\theta^{(t-1)}}[\log(P(X, Z|\theta))]. \tag{2.18}$$

Overall, the purpose of this step is to use the observed data to estimate and update the values of the missing data

M-steps

The purpose of this step is to update the parameters that could maximize the expected complete data log-likelihood generated by the E-step, according to the current estimates

$$\theta^{(t)} = \underset{\theta}{\operatorname{argmax}}\, Q(\theta, \theta^{(t-1)}). \tag{2.19}$$

The algorithm runs until the difference between $\theta^{(t)}$ and $\theta^{(t-1)}$ is within an acceptable tolerance. The advantages and disadvantages of EM algorithm are as following: EM algorithm guarantees the increase of likelihood for each iteration according to Dempster et al. (1977), which enable EM algorithm becoming a greedy algorithm. However, it might suffer from slow convergence speed, sensitivity to initial parameter value, and the convergence to a local optima if there are multiple local optimas in the optimisation error surface.

2.4.2 Bayesian Expectation Maximization

The Bayesian EM algorithm incorporates the idea of EM algorithm and Bayesian inference for estimation of the probabilistic model when the data has missing or hidden values. As opposed to the traditional EM approach, the Bayesian EM approach incorporates prior knowledge of the parameter, for the estimation of the posterior mode $p(\theta|\mathcal{D})$, considering the prior distribution as $p(\theta)$, the Bayes rule can be written in the log scale

$$\ln p(\theta|\mathcal{D}) = \ln(p(\mathcal{D}|\theta)) + \ln(p(\theta)) - \ln(p(\mathcal{D})) \tag{2.20}$$

We can then further expand Equation 2.20 to Equation 2.21

$$\ln p(\theta|\mathcal{D}) = Q(\theta, \theta^{(old)}) + \mathrm{KL}(q||p(Z|X)) + \ln(p(\theta)) - \ln(p(\mathcal{D})) \tag{2.21}$$

, where $\mathrm{KL}(P||Q)$ is defined by Equation 1.8

2.4.3 Bayesian EM for Bayesian Lasso model

In order to deploy Bayesian EM algorithm to the Bayesian Lasso model for attaining the posterior mode, our purpose is to iteratively calcualte

$$\theta_1^{(t+1)} = \underset{\theta_1}{\text{argmax}}[E_{\theta_2|\tilde{y},\theta_1^{(t)}}[\log p(y,\theta_1,\theta_2)]]. \tag{2.22}$$

E-step

Using the same notation as before, firstly, the complete log-likelihood can be written as Equation 2.23

$$\log(p(\theta_1,\theta_2,\tilde{y})) \propto -\frac{n+p}{2}\log(\sigma^2) - \frac{||\tilde{y}-X\beta||_2^2}{2\sigma^2} - \frac{1}{2\sigma^2}\beta^T E_{\theta_2|\tilde{y},\theta_1^{(t)}}[D_\tau]\beta - \frac{b}{\sigma^2} \tag{2.23}$$

given $\theta_1 = (\beta, \sigma^2)$ as set of observed variables, $\theta_2 = \tau^2 = (\tau_1^2, .., \tau_j^2)$ as set of latent variables, y is response variable.

$$E_{\theta_2|\tilde{y},\theta_1^{(t)}}[\log p(y,\theta_1,\theta_2)] = -\frac{n}{2}\log(\sigma^{2(t)}) - \frac{||y-X\beta^{(t)}||_2^2}{2\sigma^{2(t)}} - E_{\theta_2|\tilde{y},\theta_1^{(t)}}[\sum_{j=1}^{p}\frac{\lambda^2\beta_j^2}{2\sigma^2\tau_j^2}] - (a+1)\log(\sigma^{2(t)})) - \frac{b}{\sigma^{2(t)}}. \tag{2.24}$$

Next, we need to take expectation of hidden variables: $E_{\theta_2|\tilde{y},\theta_1^{(t)}}[\sum_{j=1}^{p}\frac{\lambda^2\beta_j^2}{2\sigma^2\tau_j^2}]$. After extracting constant with respect to θ_2, required formulation is $E_{\theta_2|\tilde{y},\theta_1^{(t)}}[\frac{1}{\tau_j}]$. Given the fact that $\tau_j^2|\sigma^2,\tilde{y},\beta \sim GIG(1, \frac{\beta_j^2\lambda^2}{\sigma^2}, \frac{1}{2})$ and the special property of Generalized Inverse Gaussian distribution that if $X \sim GIG(a,b,p)$, then $\frac{1}{X} \sim GIG(b,a,-p)$, the distribution of $\frac{1}{\tau_j^2}$ can be rearranged to $\frac{1}{\tau_j^2}|\sigma^2,\tilde{y},\beta \sim GIG(\frac{\beta_j^2\lambda^2}{\sigma^2}, 1, -\frac{1}{2})$. However, taking expectation with respect to Generalized Gaussian Distribution is still complicated and require advanced mathematical operation and functional properties such as modified Bessel function. Thus, we can continue converting the distribution into a Inverse Gaussian distribution family, which render $\frac{1}{\tau_j^2}|\sigma^2,\theta_1,\tilde{y} \sim \text{InverseGaussian}(b^{-\frac{1}{2}}, 1)$. Rewriting Equation 2.24 can lead to the final conditional expectation form can be written as:

$$Q(\theta,\theta^{(t)}) = -(\frac{n}{2}+\frac{p}{2}+a+1)log(\sigma^2) - \frac{b}{\sigma^2} - \frac{||y-X\beta||_2^2}{2\sigma^2} - \frac{\lambda^2}{2\sigma^2}\sum_{j=1}^{p}(\beta_j^2 E[\frac{1}{\tau_j^2}]),\ E[\frac{1}{\tau_j^2}] = \frac{\sigma^{(t)}}{|\beta_j^{(t)}|\lambda} \tag{2.25}$$

During the iteration, the $E[\frac{1}{\tau_j^2}]$ will be iteratively updated according to the updated $\beta^{(t)}$ and $\sigma^{2(t)}$.

M-step

In order to maximize the expectation of complete log likelihood, taking derivative with respect to each target variable and set them to 0 respectively provides a closed-form solution for updating observed parameter repeatedly:

$$\frac{\partial Q}{\partial \beta} = -\frac{1}{2\sigma^2}(-X^T y + 2X^T X\beta) - \frac{\lambda^2}{2\sigma^2}X^T X\beta = 0. \tag{2.26}$$

Rearranging the Equation 2.26, the updated formula for $\beta^{(t)}$ can be written as

$$\beta^{(t)} = (X^T X + \lambda^2 A)^{-1} X^T y, \text{where } A = \text{diag}(\frac{\sigma^{(t-1)}}{|\beta^{(t-1)}|\lambda}). \tag{2.27}$$

Similarly, set $\frac{\partial Q}{\partial \sigma^2} = 0$:

$$\frac{\partial Q}{\partial \sigma^2} = -\frac{(n+p+2a+2)}{2\sigma^2} + \frac{4b + 2||y - X\beta||_2^2 2 + \lambda^2(\beta^T A\beta)}{4\sigma^4} = 0 \tag{2.28}$$

Rearranging the Equation 2.28, the updated formula for $\sigma^{2(t)}$ can be written as:

$$\sigma^{2(t)} = \frac{||y - X\beta^{(t)}||_2^2 + \lambda^2(\beta^{T(t)} A\beta^{(t)}) + 2b}{n+p+2a+2}. \tag{2.29}$$

Algorithm 2 Bayesian Expectation Maximization algorithm for the Bayesian Lasso

1: Given initial value $\theta_1^{(0)} = (\beta^{(0)}.\sigma^{2(0)})$, $\theta_2^0 = \mathbf{1}_p$, $t = 1$
2: **while** $||\theta_1^{(t)} - \theta_1^{(t-1)}||_2^2 < \epsilon$ **do**
3: $\quad \beta^{(t)} = (X^T X + \lambda^2 A)^{-1} X^T y$, where $A = \text{diag}(\frac{\sigma^{(t-1)}}{|\beta^{(t-1)}|\lambda})$ ▷ Update β
4: $\quad \sigma^{2(t)} = \frac{||y - X\beta^{(t)}||_2^2 + \lambda^2(\beta^{T(t)} A\beta^{(t)}) + 2b}{n+p+2a+2}$ ▷ Update σ^2
5: $\quad A = diag(\frac{\sigma^{(t)}}{|\beta_j^{(t)}|\lambda})$ ▷ Estimate expectation of hidden variable $E[\frac{1}{\tau_j^2}]$
6: $\quad t \leftarrow t + 1$
7: return $\theta_1^{(t)}$

After completing the iteration process of Bayesian Lasso, the posterior mode of Bayesian Lasso posterior distribution can be extracted from β generated by Bayesian Lasso algorithm, as a posterior model retaining variable selection nature.

2.5 Variational Inference

2.5.1 Introduction

One of the core challenges of statistician in a Bayesian setting is to approximate over-complex probability density function in a fast and efficient manner. VI severs as an effective alternative to MCMC algorithm especially for large datasets as mentioned in the previous chapter. By addressing an optimization-based system, it is possible to fit a proxy that accurately represents the posterior distribution. As the indispensable foundation of our proposed method, the purpose of this section is to provide detailed derivation and mathematical reasoning behind variational inference according to the detailed variational inference overview from Blei et al. (2017) and Bishop (2006).

2.5.2 KL divergence and Evidence Lower Bound(ELBO)

The purpose of the variational inference is to find a candidate approxiamation $Q(\theta) \in Q$ after specifying a specific family of posterior distribution Q that minimize the KL divergence to the exact posterior distribution as shown in Equation 1.7 for each subelement of parameter θ. The complexity of finding optimal distribution relies heavily on the complexity of Q. Nevertheless, due to difficulty of computing marginal logarithm evidence $p(\mathcal{D}) = \int_\theta P(\mathcal{D}, \theta) d\theta$, as well as implicity dependency nature of $p(\mathcal{D})$ to KL divergence as explained in Equation 2.30, additional conversion is required for further processing this optimization system, which transform Equation 1.7 to Equation 1.9.

$$\begin{aligned} \mathrm{KL}(q(\theta||p(\theta|\mathcal{D}))) &= \mathbb{E}_{q(\theta)}[\log(q(\theta))] - \mathbb{E}_{q(\theta)}[\log(p(\theta|\mathcal{D}))] \\ &= \mathbb{E}_{q(\theta)}[\log(q(\theta))] - \mathbb{E}_{q(\theta)}[\log(p(\theta, \mathcal{D}))] + \log[p(\mathcal{D})]. \end{aligned} \tag{2.30}$$

KL divergence

KL-divergence defined by Equation 2.30 is a distance metric for measuring the discrepancy of two probability distributions. This metric has several theoretical properties includes non-negativity, and asymmetric property of $\mathrm{KL}(q||p)$ and $\mathrm{KL}(p||q)$.

ELBO

The definition of the Evidence Lower bound is defined by Equation 1.10, which is equivalent to the negative KL divergence despite of adding constant $p(\mathcal{D})$ with respect to $q(\theta)$. Apart from the equivalence of optimization system, the explanation of why it is called "Evidence lower bound" can be shown in Equation 2.31.

$$\log(p(\mathcal{D})) = \mathrm{KL}(q(\theta)||p(\theta|\mathcal{D})) + \mathrm{ELBO}(q(\theta)), \tag{2.31}$$

Given the fact that KL(.|.) is greater than 0, this explains log evidence is bounded below by $ELBO$, i.e: $\log(p(\mathcal{D})) \geq ELBO(q(\theta))$.

2.5.3 Mean-Field Variational Family

Parisi and Shankar (1988) proposed statistical mean-field theory. The mean-field variational family is the most common choice that is easier to optimize with a tractable solution form. The mean field variational family represents a group of probability distributions employed in variational inference. Its objective is to estimate intricate, infeasible distributions, such as the genuine posterior distribution within a Bayesian model, by using more tractable and simpler distributions. The generic member of mean field variational family is as described in Equation 1.11, assuming mutual independence of each target parameter θ_j, and the joint distribution is factorized as a product of individual distributions. Finally, no further assumption has been arranged to each individual distribution $q(\theta_i)$. The two dimensional visualization of mean-field variational family can be observed from Figure 1.2, while the contour of mean-field variational family member forms a concentric contour line over the optimization surface.

2.5.4 Coordinate Ascent Variational Inference (CAVI)

Coordinate Ascent Variational Inference (CAVI) algorithm is the most frequently used optimization algorithm to find an optimum $q^*(\theta)$ that maximizes the ELBO, which is preferred for its simplicity and computational efficiency. The main intuition behind coordinate ascent is to optimize the function with respect to one variable at a time while keeping the

other variables fixed. This is done iteratively until convergence is achieved. This is done iteratively until convergence is achieved. The convergence is achieved when either the difference between the current function value and the previous function value is less than a predefined threshold, or when the maximum number of iterations is reached. Similarly, the CAVI works by iteratively maximizing each component of $q(\theta)$: $q_i(\theta_i)$, while maintaining other factor of distribution unchanged, $q_{-i}(\theta_{-i})$, enforcing the final distribution can achieve a local optimum of the ELBO. Small changes regarding stopping criterion has also been proposed, where the stopping criterion has been transformed from difference of function value into difference of ELBO. Finally, we want to make note that MFVB belongs to a broader category of variational inference approaches, which can also serve as a purpose in the frequentist setting for maximum likelihood estimation. The CAVI algorithm is equivalent to MFVB algorithm in our setting.

Derivation

Substituting the family of factorized target distribution $\prod_i q_i$, the ELBO can be rewritten as:

$$\begin{aligned}
\text{ELBO}(q(\theta)) &= \int q(\theta) \log(\frac{p(\theta)p(\mathcal{D}|\theta)}{q(\theta)}) d\theta \\
&= \int \prod_i q_i(\theta_i)[\log[p(\mathcal{D},\theta) - \sum_i \log(q_i(\theta_i))]] d\theta \\
&= \int q_j(\theta_j)[\int \log p(\mathcal{D},\theta) \prod_{i\neq j} q_i d\theta_j] d\theta_j - \int q_j(\theta_j) \log[q_j(\theta_j)] d\theta_j + const \\
&= \int q_j(\theta_j) \log[\tilde{p}(\mathcal{D},\theta_j)] - \int q_j(\theta_j) \log[q_j(\theta_j)] d\theta_j
\end{aligned} \tag{2.32}$$

where $\log[\tilde{p}(\mathcal{D},\theta_j)] = \mathbb{E}_{i\neq j}[\log[p(\mathcal{D},\theta)]] + const$,
$\mathbb{E}_{i\neq j}[\log[p(\mathcal{D},\theta)]] = \int \log[p(\mathcal{D},\theta)] \prod_{i\neq j} q_j(\theta_j) d\theta_j$ assuming optimizing the jth posterior parameter θ_j. Identifying the 2.32 is negative KL-divergence between $q_j(\theta_j)$ and $\tilde{p}(\mathcal{D},\theta_j)$, the KL-divergence can be minimized proposed distribution is close enough to the target distribution. As a consequence, the optimal $q^*(\theta_i)$ incidence of local maximum is achieved when $q_j(\theta_j) = \tilde{p}(\mathcal{D},\theta_j)$. Thus, the general optimum distribution can be written in the following equation,

$$\log[q_j^*(\theta_j)] = \mathbb{E}_{i\neq j}[\log p(\mathcal{D},\theta)] + const \tag{2.33}$$

Recovering from log scale and removing additional constant by normalization of $q_j^*(\theta_j)$, the optimum $q_j^*(\theta_j)$ can be written as:

$$q_j^*(\theta_j) = \frac{\mathbb{E}_{i\neq j}[\log p(\mathcal{D}, \theta)]}{\int \mathbb{E}_{i\neq j}[\log p(\mathcal{D}, \theta) d\theta_j} \propto \mathbb{E}_{i\neq j}[\log p(\mathcal{D}, \theta)] \tag{2.34}$$

The algorithm then iteratively replaces $q_j(\theta_j)$ using the current estimation for all of other factors, and convergence critera is break when differences of ELBOs changes less than a predefined tolerance. Theoretically, it is proved that the convergence of CAVI algorithm is guaranteed given the fact that the optimization problem is convex Boyd and Vandenberghe (2004). The following pseudo-algorithm demonstrates the entire procedure for CAVI.

Algorithm 3 Coordinate Ascent Variational Inference (CAVI)

Input: $p(\mathcal{D}, \theta)$, data $\mathcal{D}$, Initialize Variational parameters for each $q_j(\theta_j)$
while ELBO has not converged **do**
    for j=1,...,p **do**
        $q_j(\theta_j) \propto \mathbb{E}_{i\neq j}[\log p(\mathcal{D}, \theta)]$
    Compute $ELBO(q(\theta)) = \mathbb{E}[\log[p(\mathcal{D}, \theta)]] + \mathbb{E}[\log[q(\theta)]]$
return $q(\theta)$

Similar to MFVB, CAVI has a similiar updating rule, while the only modification is that it converts the stopping criterion from using consecutive difference of ELBO into variation in the parameter θ during two consecutive.

2.5.5 MFVB for Bayesian Lasso

We now propose MFVB algorithm for the Bayesian Lasso problem. Our goal is to attain Bayesian Lasso posterior approximation in a faster and efficient manner than that of gibbs sampler under the framework of VI. Note in this section, slight modification has been imposed to the definition of τ_j. $a_j \sim \text{Gamma}(1, 1/2)$ for $1 \leq j \leq p$, abandoning the dependence of λ to τ to simplify the derivation and algorithm. The assumption of tractable distribution family Q can be written as the based on MFVB:

$$q(\theta) = q(\beta)q(\sigma^2)\prod_{i=1}^{p} q(a_j). \tag{2.35}$$

Derivation

In addition, we would like to derive the optimal variational distribution for each parameter: ie. $q^*_\beta(\beta)$, $q^*_{\sigma^2}(\sigma^2)$ according to Equation 2.34. We consider the parameters of a_j distribution are fixed, therefore, no hyper-priors are required.

The following derivation is based on the conditional posterior distributional form and expectation with respect to corresponding standard distribution β, $q^*_\beta(\beta)$ can be derived by:

$$\begin{aligned} q^*_\beta(\beta) &\propto \exp[\mathbb{E}_{-\beta}[\log(p(\beta, y, \mathbf{a}, \sigma^2))]] \\ &\sim MVN(((X^TX + \lambda^2(A))^{-1}X^Ty, (X^TX + \lambda^2 A))^{-1}\mathbb{E}_q(\frac{1}{\sigma^2})) \end{aligned} \tag{2.36}$$

where $A = diag(\mathbf{a})$, $\mathbb{E}_{\sigma^2}(\frac{1}{\sigma^2}) = \frac{\tilde{b}}{\tilde{a}}$.

Also, $q^*_{\sigma^2}(\sigma^2)$ can be derived by the following equation. The following derivation are based on the known conditional posterior distributional form and expectation with respect to corresponding standard distribution.

$$\begin{aligned} q^*_{\sigma^2}(\sigma^2) &\propto \exp[\mathbb{E}_{-\sigma^2}[\log(p(\sigma^2, y, \mathbf{a}, \beta))]] \\ &\sim \text{InverseGamma}(\frac{n+p}{2}, \frac{1}{2}\mathbb{E}_q||y - X\beta||^2 + \frac{\lambda}{2}\sum_{j=1}^{p}\mathbb{E}_q(\beta^T A\beta) \end{aligned} \tag{2.37}$$

Therefore, the update Procedure of MFVB for the Bayesian Lasso can be written as following:

- Update Procedure of MFVB for the Bayesian Lasso
  - $Q = X^TX + \lambda^2 A$, where $A = \text{diag}(\tau^2)$.
  - The update for beta leads to

$$\widetilde{\mu} = Q^{-1}X^Ty \quad \text{and} \quad \widetilde{\Sigma} = \mathbb{E}_q\left[\frac{1}{\sigma^2}\right]^{-1}Q^{-1}$$

  - The update for σ^2 leads to

$$\widetilde{a} = \frac{n+p}{2}, \quad \text{and} \quad \widetilde{b} = \frac{E_q||y - X\beta||^2 + \lambda^2\mathbb{E}_q[\beta^T A\beta]}{2}.$$

The aforementioned update procedure for MFVB estimates the parameters of the posterior distributions of β and σ^2 with a descent speed, even though the posterior variance is usually underestimated due to the mean-field restriction for the approximated density.

Chapter 3

Methodlogy

3.1 Introduction

The main intuition behind the method is to adjust MFVB posterior parameter estimate result for the posterior of regression coefficients β: $\tilde{\mu}$ and $\tilde{\Sigma}$. Continuing utilizing mean field variational family $q(\theta) = \prod_i q(\theta_i)$ assumption and Gaussian Approximation: $q^*(\theta) \sim N(\tilde{\mu}, \tilde{\Sigma})$ to approximate the global Bayesian Lasso Posterior $p(\theta|\mathcal{D})$, with the correction of local parameter information that the marginal likelihood approximately coincides with a lasso distribution if a Laplace prior is assigned. As a consequence, the goal is to seek the mathematical expression of each of the local mean parameter μ_j^*, local variance Σ_{jj}^* from a lasso distribution locally, a global mean $\tilde{\mu}$ and global variance $\tilde{\Sigma}$ of $q^*(\theta)$ for Gaussian Approximation can be updated by iteratively to correct global parameter estimate by local parameter expression, for each β_j.

3.2 Basic Setting for the Bayesian Lasso Problem

Firstly, the Bayesian Lasso Posterior approximation can be restricted by the mean field variational family:

$$p(\beta, \sigma^2|\mathcal{D}) \approx q(\beta, \sigma^2) = q(\beta)q(\sigma^2) \tag{3.1}$$

Under the same setting in Variational Inference in Mean-Field-Variational-Bayes by Ormerod and Wand (2010): the parameter of interest θ can be divided up into two parts θ_1: β_j current variable and $\theta_2 : \beta_{-j}$, other variables. The marginal log-likelihood of θ_1 can be divided up into the ELBO part and the KL divergence part by the following derivation:

$$\log(\mathcal{D}, \theta_1) = \mathbb{E}_{q(\theta_2|\theta_1)}[\log(\frac{p(\mathcal{D}, \theta_1, \theta_2)}{q(\theta_2|\theta_1)})] + KL(q(\theta_2|\theta_1), p(\theta_2|\mathcal{D}, \theta_1)) \tag{3.2}$$

Since the KL divergence is greater than 0, the marginal log-likelihood of θ_1 and $\mathcal{D}$ has a more tractable Evidence Lower Bound:

$$\log(\mathcal{D}, \theta_1) \geq \mathbb{E}_{q(\theta_2|\theta_1)}[\log(\frac{p(\mathcal{D}, \theta_1, \theta_2)}{q(\theta_2|\theta_1)})] \tag{3.3}$$

When $q(\theta_2|\theta_1) = p(\theta_2|\mathcal{D}, \theta_1)$ then

$$\log(\mathcal{D}, \theta_1) = \mathbb{E}_{p(\theta_2|\mathcal{D},\theta_1)}[\log p(\mathcal{D}, \theta_2, \theta_1)]$$

In Bayesian Lasso: $\theta = (\beta, \sigma^2)$, however, the update for σ will not be discussed while we will only discuss an approach for updating β, assuming $q(\beta) \sim N(\mu, \Sigma)$. Thus, the conditional distribution $q(\beta_{-j}|\beta_j)$ for any j_{th} variable can be derived by the fact that $q(\beta_{-j}|\beta_j) \propto q(\beta)$, resulting another multivariate normal distribution with dimension of $p-1$ as shown in Equation 3.4

$$q(\beta_{-j}|\beta_j) = N_{p-1}(\mu_{-j} + \Sigma_{-j,j}\Sigma_{j,j}^{-1}(\beta_j - \mu_j), \Sigma_{-j,j}\Sigma_{-j,-j}^{-1}\Sigma_{j,j}) \tag{3.4}$$

With the mean field restriction in Equation 3.1, the result from Equation 3.4, the fact that the products of log density $Y|\beta, \sigma^2$, $\beta|\sigma^2, \lambda$ and a Laplacian prior for $\beta|\sigma^2, \lambda$, the estimated marginal log likelihood for each β_j after taking expectation with respect to $q(\theta)$:

$$\begin{aligned}
\log p(\mathcal{D}, \beta_j) &= \mathbb{E}_{\beta_{-j},\sigma^2|\mathcal{D},\beta_j} \log(p(\beta_j|\mathcal{D}, \beta_{-j}, \sigma^2)) \\
&\approx \mathbb{E}_{q(\beta_{-j}|\beta_j)q(\sigma^2)} \log(p(\beta_j|\mathcal{D}, \beta_{-j}, \sigma^2)) \\
&\propto \mathbb{E}_{q(\beta_{-j}|\beta_j)q(\sigma^2)}[-\frac{||X_j||_2^2}{2\sigma^2}\beta_j^2 + \frac{X_j^T(y - X_{-j}\beta_{-j})}{\sigma^2}\beta_j - \frac{\lambda}{\sigma}|\beta_j|] \\
&= \frac{\tilde{a}}{\tilde{b}}(y - X_{-j}s)\beta_j - \frac{\tilde{a}}{2\tilde{b}}(X_j^T X_j + X_j^T X_{-j}t)\beta_j^2 - \frac{\lambda\Gamma(\tilde{a} + 1/2)}{\Gamma(\tilde{a})\sqrt{\tilde{b}}}|\beta_j|.
\end{aligned} \tag{3.5}$$

where $s = \mu_{-j} - \Sigma_{-j,j}\Sigma_{j,j}^{-1}\mu_j$ and $t = \Sigma_{-j,j}\Sigma_{j,j}^{-1}$, $\tilde{a}$ and $\tilde{b}$ are posterior parameters for σ^2, μ, Σ are posterior parameters for β. One of the fundamental improvements between our method and MFVB is that our method captures the correlation between θ_2 and θ_1. The expectation with respect to $q(\theta_2|\theta_1)$ is performed. To clarify, consider the case for MFVB, where expectation with respect to $q(\theta)$ is performed that assumes the independence of the parameter θ. It is also clear to observe from the fact that our method degrades to the MFVB method when $t = 0$, it results in $s = \mu_{-j}$.

Before continuing to present the methodology, the introduction of lasso distribution is essential for local approximation correction.

3.3 Lasso distribution

Equation 3.5 can be matched to a univariate lasso distribution as a local approximation of Bayesian Lasso posterior. In addition, the joint likelihood of a pair of variables can also be matched by a bivariate lasso distribution.

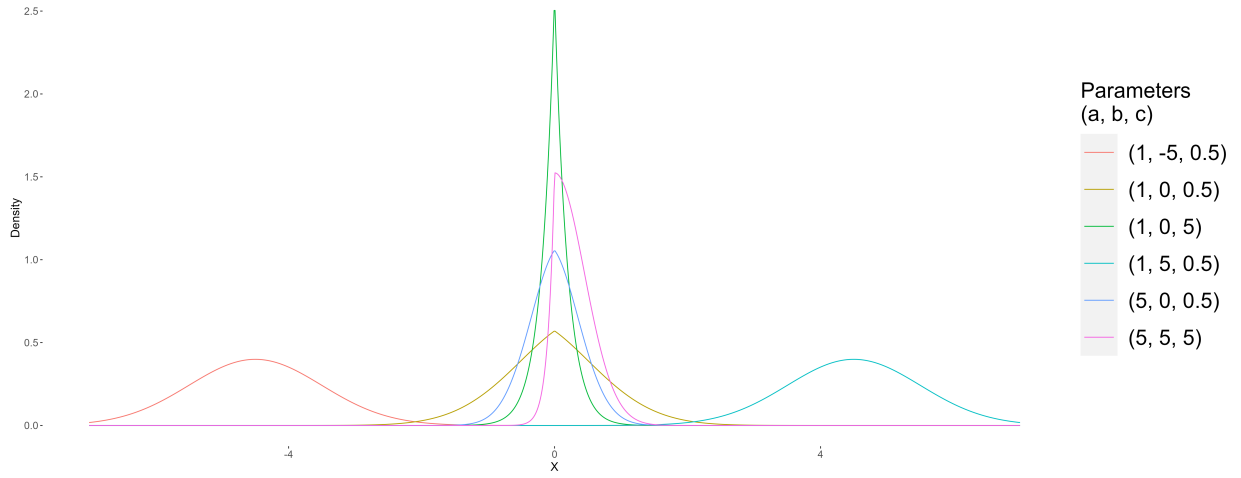

Figure 3.1: Visualization of the Univariate Lasso Distribution PDF for different parameter settings

As shown in Figure 3.1, it demonstrates the shape and location of the univariate lasso distribution with different parameter settings. The orange line depicts the lasso distribution when $(a, b, c) = (1, 5, -0.5)$ respectively, the yellow line depicts the lasso distribution when $(a, b, c) = (1, 0, 0.5)$, the green line depicts the lasso distribution when $(a, b, c) = (1, 0, 0.5)$, the sky blue one depicts the lasso distribution when $(a, b, c) = (5, 0, 0.5)$, the blue line depicts the lasso distribution when $(a, b, c) = (1, 5, 0.5)$, the blue line depicts the lasso distribution when $(a, b, c) = (1, 0, 5)$, the pink line depicts the lasso distribution when $(a, b, c) = (5, 5, 5)$. From the yellow and dark-blue line, we can observe that parameter A control the size of the curvature of the tuning point, larger a implies a smoother tuning point. From the red line, yellow line, and sky-blue line, we can observe that changing b will move the location of the curve. Larger b will move the graph further to the right. From the yellow line and green line, c controls the sharpness of the curve, a larger c implies distribution with smaller variance and a sharper turning point.

3.3.1 Univariate Lasso Distribution

If $x \sim$ Lasso(a,b,c), then the probability density function can be written as:

$$p(x, a, b, c) = Z^{-1} \exp(-\frac{1}{2}ax^2 + bx - c|x|) \tag{3.6}$$

where $a \geq 0, b \in \mathbb{R}, c \geq 0$, there are also certain restrictions to certain parameter settings:

- a and c can't be 0 at the same time
- When $a = 0$, lasso distribution will become an asymmetric Laplace distribution
- When $c = 0$, lasso distribution will become a normal distribution

The probability density function of univariate lasso distribution can be divided up into four components:

- A normalization constant Z, to enable the integration of the probability density function to be 1.
- A quadratic term ax square to control the curvature of the curve.
- A linear term bx to control the location of the curve.
- An absolute term $c|x|$ to control the sharpness of the turning point.

Certain properties of a probability distribution can also be formed to demonstrate the effectiveness, in our algorithm normalizing constant Z, expectation $\mathbb{E}(x)$, second moment $\mathbb{E}(x^2)$ and variance $\mathbb{V}(x)$ are necessary.

Basic Property

Derivation of normalizing constant

The normalizing constant Z can be written as a function of a, b, c.

$$\begin{aligned}
Z(a,b,c) &= \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}ax^2 + bx - c|x|\right] dx \\
&= \int_{0}^{\infty} \exp\left[-\tfrac{1}{2}ax^2 + (b-c)x\right] dx + \int_{-\infty}^{0} \exp\left[-\tfrac{1}{2}ax^2 + (b+c)x\right] dx \\
&= \int_{0}^{\infty} \exp\left[-\tfrac{1}{2}ax^2 + (b-c)x\right] dx + \int_{0}^{\infty} \exp\left[-\tfrac{1}{2}ay^2 - (b+c)y\right] dy \\
&= \int_{0}^{\infty} \exp\left[-\tfrac{(x-\mu_1)^2}{2\sigma^2} + \tfrac{\mu_1^2}{2\sigma^2}\right] dx + \int_{0}^{\infty} \exp\left[-\tfrac{(x-\mu_2)^2}{2\sigma^2} + \tfrac{\mu_2^2}{2\sigma^2}\right] dy \\
&= \sqrt{2\pi\sigma^2}\left[\exp\left\{\tfrac{\mu_1^2}{2\sigma^2}\right\}\int_{0}^{\infty}\phi(x;\mu_1,\sigma^2)dx + \exp\left\{\tfrac{\mu_2^2}{2\sigma^2}\right\}\int_{0}^{\infty}\phi(y;\mu_2,\sigma^2)dy\right] \\
&= \sqrt{2\pi\sigma^2}\left[\exp\left\{\tfrac{\mu_1^2}{2\sigma^2}\right\}\{1-\Phi(-\mu_1/\sigma)\} + \exp\left\{\tfrac{\mu_2^2}{2\sigma^2}\right\}\{1-\Phi(-\mu_2/\sigma)\}\right] \\
&= \sqrt{2\pi\sigma^2}\left[\exp\left(\tfrac{\mu_1^2}{2\sigma^2}\right)\Phi\left(\tfrac{\mu_1}{\sigma}\right) + \exp\left(\tfrac{\mu_2^2}{2\sigma^2}\right)\Phi\left(\tfrac{\mu_2}{\sigma}\right)\right] \\
&= \sigma\left[\tfrac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)} + \tfrac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}\right]
\end{aligned}$$

Derivation of Moments

Note, the expectation is the first moment, and variance of lasso distribution can be computed by the property $\mathbb{V}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

$$\begin{aligned}
E(x^r) &= Z^{-1}\int_{-\infty}^{\infty} x^r \exp\left[-\tfrac{1}{2}ax^2 + bx - c|x|\right] dx \\
&= Z^{-1}\int_{0}^{\infty} x^r \exp\left[-\tfrac{1}{2}ax^2 + (b-c)x\right] dx + \int_{-\infty}^{0} x^r \exp\left[-\tfrac{1}{2}ax^2 + (b+c)x\right] dx \\
&= Z^{-1}\int_{0}^{\infty} x^r \exp\left[-\tfrac{1}{2}ax^2 + (b-c)x\right] dx + (-1)^r\int_{0}^{\infty} y^r \exp\left[-\tfrac{1}{2}ay^2 - (b+c)y\right] dy \\
&= Z^{-1}\sqrt{2\pi\sigma^2}\exp\left(\tfrac{\mu_1^2}{2\sigma^2}\right)\int_{0}^{\infty} x^r\phi(x;\mu_1,\sigma^2)dx \\
&\quad + (-1)^r\sqrt{2\pi\sigma^2}\exp\left(\tfrac{\mu_2^2}{2\sigma^2}\right)\int_{0}^{\infty} y^r\phi(y;\mu_2,\sigma^2)dy \\
&= \tfrac{\sigma}{Z}\left[\tfrac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)}\tfrac{\int_0^\infty x^r\phi(x;\mu_1,\sigma^2)dx}{\Phi(\mu_1/\sigma)} + (-1)^r\tfrac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}\tfrac{\int_0^\infty y^r\phi(y;\mu_2,\sigma^2)dy}{\Phi(\mu_2/\sigma)}\right] \\
&= \tfrac{\sigma}{Z}\left[\tfrac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)}\mathbb{E}(A^r) + (-1)^r\tfrac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}\mathbb{E}(B^r)\right]
\end{aligned} \tag{3.7}$$

where $A \sim TN_+(\mu_1, \sigma^2)$, $B \sim TN_+(\mu_2, \sigma^2)$ and TN_+ is denotes the positively truncated normal distribution; $\mu_1 = (b-c)/a$, $\mu_2 = -(c+b)/a$ and $\sigma^2 = 1/a$. Note that

$$\mathbb{E}(A) = \mu_1 + \frac{\sigma\phi(\mu_1/\sigma)}{\Phi(\mu_1/\sigma)} = \mu_1 + \sigma\zeta_1(\mu_1/\sigma)$$

and

$$\mathbb{V}(A) = \sigma^2\left[1 + \zeta_2(\mu_1/\sigma)\right]$$

where $\zeta_k(x) = d^k \log\Phi(x)/dx^k$, $\zeta_1(t) = \phi(t)/\Phi(t)$, $\zeta_2(t) = -t\,\zeta_1(t) - \zeta_1(t)^2$. Here $\zeta_1(x)$ is the inverse Mills ratio which too needs to be treated with care. Hence,

$$\mathbb{E}(A^2) = \mathbb{V}(A) + \mathbb{E}(A)^2 = \sigma^2\left[1 + \zeta_2(\mu_1/\sigma)\right] + \left[\mu_1 + \sigma\zeta_1(\mu_1/\sigma)\right]^2$$

We now have sufficient information to calculate the moments of the Lasso distribution. We also have sufficient information to implement a VB approximation.

3.3.2 Bivariate Lasso Distribution

If $\mathbf{x} \sim \text{Bilasso}(A, b, c)$ with then it has density given by

$$p(\mathbf{x}) = Z^{-1}\exp(-\frac{1}{2}\mathbf{x}^T A\mathbf{x} + b^T\mathbf{x} - c||\mathbf{x}||_1) \tag{3.8}$$

where $A \in S_d^+$: positive definite matrix with dimension d, $b \in \mathbb{R}^2$, $c \geq 0$

Derivation of Normalizing Constant

The normalizing constant can be calculated via integrate the unnormalized probability density function.

$$
\begin{aligned}
Z(a,b,c) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}x^TAx + \mathbf{b}^Tx - c\mathbf{1}^T|x|_1\right] d\mathbf{x} \\
&= \int_{0}^{\infty}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}x^TAx + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} \\
&\quad + \int_{0}^{\infty}\int_{-\infty}^{0} \exp\left[-\tfrac{1}{2}x^TAx + (\mathbf{b}^T - c[1,-1]^T)x\right] d\mathbf{x} \\
&\quad + \int_{-\infty}^{0}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}x^TAx + (\mathbf{b}^T - c[-1,1]^T)x\right] d\mathbf{x} \\
&\quad + \int_{-\infty}^{0}\int_{-\infty}^{0} \exp\left[-\tfrac{1}{2}x^TAx + (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_{0}^{\infty}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}x^TAx + (\mathbf{b}^T - c\mathbf{1}^T)x\right] d\mathbf{x} \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}x^TA^*x + (b_1 - c, -b_2 - c)^Tx\right] d\mathbf{x} \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}x^TA^*x + (-b_1 - c, b_2 - c)^Tx\right] d\mathbf{x} \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}x^TAx - (\mathbf{b}^T + c\mathbf{1}^T)x\right] d\mathbf{x} \\
&= \int_{0}^{\infty}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1) + \tfrac{(A\mu_1)^T\Sigma_1(A\mu_1)]}{2}\right] d\mathbf{x} \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2) + \tfrac{(A^*\mu_2)^T\Sigma_2(A^*\mu_2)]}{2}\right] d\mathbf{x} \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_3)^T\Sigma_2^{-1}(x-\mu_3) + \tfrac{(A^*\mu_3)^T\Sigma_2(A^*\mu_3)]}{2}\right] d\mathbf{x} \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} \exp\left[-\tfrac{1}{2}(x-\mu_4)^T\Sigma_1^{-1}(x-\mu_4) + \tfrac{(A\mu_4)^T\Sigma_1(A\mu_4)]}{2}\right] d\mathbf{x} \\
&= 2\pi|\Sigma_1|^{\frac{1}{2}}[\exp\left[\tfrac{(A\mu_1)^T\Sigma_1(A\mu_1)]}{2}\right] \int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_1,\Sigma_1)d\mathbf{x} \\
&\quad + \exp\left[\tfrac{(A\mu_4)^T\Sigma_1(A\mu_4)]}{2}\right] \int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}]) \\
&\quad + 2\pi|\Sigma_2|^{\frac{1}{2}}(\exp\left[\tfrac{(A^*\mu_2)^T\Sigma_2(A^*\mu_2)]}{2}\right] \int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_2,\Sigma_2)d\mathbf{x} \\
&\quad + \exp\left[\tfrac{(A^*\mu_3)^T\Sigma_2(A^*\mu_3)]}{2}\right] \int_{0}^{\infty}\int_{0}^{\infty} \phi_2(x;\mu_3,\Sigma_2)d\mathbf{x}) \\
&= |\Sigma_1|(\tfrac{\int_0^\infty\int_0^\infty \phi_2(x;\mu_1,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma_1^{-1})} + \tfrac{\int_0^\infty\int_0^\infty \phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma_1^{-1})}) \\
&\quad + |\Sigma_2|(\tfrac{\int_0^\infty\int_0^\infty \phi_2(x;\mu_2,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_2,\Sigma_2^{-1})} + \tfrac{\int_0^\infty\int_0^\infty \phi_2(x;\mu_3,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_3,\Sigma_2^{-1})} +))
\end{aligned}
$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T$, $\mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T$, $\mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \odot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Derivation of Expectation

The expectation of lasso distribution can be derived by the expectation property: $\mathbb{E}[X] = \int x f(x) dx$.

$$
\begin{aligned}
\mathbb{E}[X] &= Z^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \odot \exp\left[-\tfrac{1}{2} x^T A x + \mathbf{b}^T x - c\mathbf{1}^T ||x||_1\right] d\mathbf{x} \\
&= Z^{-1} \int_{0}^{\infty} \int_{0}^{\infty} x \odot \exp\left[-\tfrac{1}{2} x^T A x + (\mathbf{b}^T - c\mathbf{1}^T) x\right] d\mathbf{x} \\
&+ \int_{0}^{\infty} \int_{0}^{\infty} [1,-1]^T \odot x \odot \exp\left[-\tfrac{1}{2} x^T A^* \odot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (b_1 - c, -b_2 - c)^T x\right] d\mathbf{x} \\
&+ \int_{0}^{\infty} \int_{0}^{\infty} [-1,1]^T \odot x \odot \exp\left[-\tfrac{1}{2} x^T A^* \odot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + (-b_1 - c, b_2 - c)^T x\right] d\mathbf{x} \\
&- \int_{0}^{\infty} \int_{0}^{\infty} x \odot \exp\left[-\tfrac{1}{2} x^T A x + (\mathbf{b}^T + c\mathbf{1}^T) x\right] d\mathbf{x} \\
&= Z^{-1} [|\Sigma_1| (\tfrac{\int_0^\infty \int_0^\infty x \odot \phi_2(x; \mu_1, \Sigma) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1}))} - \tfrac{\int_0^\infty \int_0^\infty x \odot \phi_2(x; \mu_4, \Sigma) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1}))}) \\
&+ |\Sigma_2| ([1,-1]^T \tfrac{\int_0^\infty \int_0^\infty x \odot \phi_2(x; \mu_2, \Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_2, \Sigma_2^{-1}))} + [-1,1]^T \tfrac{\int_0^\infty \int_0^\infty x \odot \phi_2(x; \mu_3, \Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_3, \Sigma_2^{-1})})] \\
&= Z^{-1} [|\Sigma_1| (\tfrac{\mathbb{E}[\mathbf{A}] \int_0^\infty \int_0^\infty \phi_2(x; \mu_1, \Sigma_1) d\mathbf{x}}{\phi_2(A\mu_1, \Sigma_1^{-1}))} - \tfrac{\mathbb{E}[D] \int_0^\infty \int_0^\infty \phi_2(x; \mu_4, \Sigma_1) d\mathbf{x}}{\phi_2(A\mu_4, \Sigma_1^{-1}))}) \\
&+ |\Sigma_2| ([1,-1]^T \tfrac{\mathbb{E}[B] \int_0^\infty \int_0^\infty \phi_2(x; \mu_2, \Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_2, \Sigma_2^{-1}))} + [-1,1]^T \tfrac{\mathbb{E}[C] \int_0^\infty \int_0^\infty \phi_2(x; \mu_3, \Sigma_2) d\mathbf{x}}{\phi_2(A^*\mu_3, \Sigma_2^{-1})})]
\end{aligned}
$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T, \mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T \mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \odot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. $\mathbf{A} \sim MTN_+(\mu_1, \Sigma_1)$, $B \sim MTN_+(\mu_2, \Sigma_2)$, $C \sim MTN_+(\mu_3, \Sigma_2)$, $D \sim MTN_+(\mu_4, \Sigma_1)$ is denotes the multivariate positively truncated normal distribution.

Derivation of Covariance Matrix

The second moment of lasso distribution can be derived by the expectation property: $\mathbb{E}[XX^T] = \int xx^T f(x) dx$.

$$\mathrm{Cov}(X) = \mathbb{E}[XX^T] - \mathbb{E}[X]\mathbb{E}[X]^T$$

$$
\begin{aligned}
\mathbb{E}[XX^T] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xx^T \odot \exp\left[-\tfrac{1}{2}x^TAx + \mathbf{b}^Tx - c\mathbf{1}^T||x||_1\right] d\mathbf{x} \\
&= Z^{-1}2\pi|\Sigma|^{\frac{1}{2}}[\exp\left[\tfrac{(A\mu_1)^T\Sigma(A\mu_1)]}{2}\right]\int_0^{\infty}\int_0^{\infty} xx^T \odot \phi_2(x;\mu_1,\Sigma_1)d\mathbf{x} \\
&\quad + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \odot \exp\left[\tfrac{(A^*\mu_2)^T\Sigma(A^*\mu_2)]}{2}\right]\int_0^{\infty}\int_0^{\infty} xx^T \odot \phi_2(x;\mu_2,\Sigma_2)d\mathbf{x} \\
&\quad + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \odot \exp\left[\tfrac{(A^*\mu_3)^T\Sigma(A^*\mu_3)]}{2}\right]\int_0^{\infty}\int_0^{\infty} xx^T \odot \phi_2(x;\mu_3,\Sigma_2)d\mathbf{x} \\
&\quad - \exp\left[\tfrac{(A\mu_4)^T\Sigma(A\mu_4)]}{2}\right]\int_0^{\infty}\int_0^{\infty} xx^T \odot \phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}] \\
&= Z^{-1}|\Sigma|[\tfrac{\int_0^{\infty}\int_0^{\infty} xx^T\odot\phi_2(x;\mu_1,\Sigma)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma))} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \odot \tfrac{\int_0^{\infty}\int_0^{\infty} xx^T\odot\phi_2(x;\mu_2,\Sigma)d\mathbf{x}}{\phi_2(A\mu_2,\Sigma))} \\
&\quad + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \odot \tfrac{\int_0^{\infty}\int_0^{\infty} xx^T\odot\phi_2(x;\mu_3,\Sigma)d\mathbf{x}}{\phi_2(A\mu_3,\Sigma))} - \tfrac{\int_0^{\infty}\int_0^{\infty} xx^T\odot\phi_2(x;\mu_4,\Sigma)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma))}] \\
&= Z^{-1}[|\Sigma_1|(\tfrac{\mathbb{E}[\mathbf{A}\mathbf{A}^T]\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_1,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_1,\Sigma_1^{-1}))} + \tfrac{\mathbb{E}[DD^T]\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_4,\Sigma_1)d\mathbf{x}}{\phi_2(A\mu_4,\Sigma_1^{-1}))}) \\
&\quad + |\Sigma_2|(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \odot \tfrac{\mathbb{E}[BB^T]\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_2,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_2,\Sigma_2^{-1}))} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \odot \tfrac{\mathbb{E}[CC^T]\int_0^{\infty}\int_0^{\infty}\phi_2(x;\mu_3,\Sigma_2)d\mathbf{x}}{\phi_2(A^*\mu_3,\Sigma_2^{-1})})]
\end{aligned}
$$

where $\mu_1 = A^{-1}(b - c\mathbf{1})^T$, $\mu_2 = A^{*-1}(b_1 - c, -b_2 - c)^T, \mu_3 = A^{*-1}(-b_1 - c, b_2 - c)^T \mu_4 = A^{-1}(-b - c\mathbf{1}^T)^T$ and $\Sigma_1 = A^{-1}$, $\Sigma_2 = A^{*-1}$ $A^* = A \odot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. $\mathbf{A} \sim MTN_+(\mu_1, \Sigma_1)$, $B \sim MTN_+(\mu_2, \Sigma_2)$, $C \sim MTN_+(\mu_3, \Sigma_2)$, $D \sim MTN_+(\mu_4, \Sigma_1)$ is denotes the multivariate positively truncated normal distribution. In addition, the second moment of $\mathbb{E}[AA^T]$ and $\mathbb{E}[BB^T]$ can be derived similary from the variance property in multivariate function:

$$\mathbb{E}[AA^T] = \mathrm{Cov}(A) - \mathbb{E}[A]\mathbb{E}[A]^T$$

Derivation of marginal distribution

The marginal distribution for the bivariate lasso distribution is useful for evaluating l_1 norm approximation accuracy for comparison and visualization for approximation density. marginal distribution can be derived by $f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_2$ and $f(x_2) =$

$\int_{-\infty}^{\infty} f(x_1, x_2)dx_1$ respectively.

$$
\begin{aligned}
f(x_1) &= \int_{-\infty}^{\infty} f(x_1, x_2)dx_2 \\
&= Z^{-1}\exp(-\tfrac{1}{2}x^TAx + b^Tx - c||x||_1))dx_2 \\
&= Z^{-1}\exp(-0.5a_{11}x_1^2 + b_1x_1 - c|x_1|) \\
&\int_{-\infty}^{\infty}\exp(-\tfrac{1}{2}[(a_{12}+a_{21})x_1x_2 + a_{22}x_2^2] + b_2x_2 - c|x_2|]dx_2 \\
&= k\int_{-\infty}^{\infty}\exp[-(0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{22}x_2^2 + b_2x_2 - c|x_2|]dx_2 \\
&= k[\int_0^{\infty}\exp[-0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{22}x_2^2 + (b_2-c)x_2]dx_2 \\
&+ \int_{-\infty}^{0}\exp[-0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{22}x_2^2 + (b_2+c)x_2]dx_2 \\
&= k[\int_0^{\infty}\exp[-0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{22}x_2^2 + (b_2-c)x_2]dx_2 \\
&+ \int_0^{\infty}\exp[0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{22}x_2^2 - (b_2+c)x_2]dx_2] \\
&= k[\int_0^{\infty}\exp[-\tfrac{(x_2-\mu_1)^2}{2\sigma^2} + \tfrac{\mu_1^2}{2\sigma^2}]dx_2] + \int_0^{\infty}\exp[-\tfrac{(x_2-\mu_2)^2}{2\sigma^2} + \tfrac{\mu_2^2}{2\sigma^2}]dx_2] \\
&= k\sigma[\tfrac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)} + \tfrac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}]
\end{aligned}
$$

where $\mu_1 = (-\frac{a_{12}+a_{21}}{2a_{22}}x_1 + \frac{b_2-c}{a_{22}})$, $\mu_2 = (\frac{a_{12}+a_{21}}{2a_{22}}x_1 - \frac{b_2+c}{a_{22}})$, $\sigma^2 = 1/a_{22}$, $k = Z^{-1}\exp(-0.5a_{11}x_1^2 + b_1x_1 - c|x_1|)$

$$
\begin{aligned}
f(x_2) &= \int_{-\infty}^{\infty} f(x_1, x_2)dx_1 \\
&= Z^{-1}\exp(-\tfrac{1}{2}x^TAx + b^Tx - c||x||_1))dx_1 \\
&= Z^{-1}\exp(-0.5a_{22}x_2^2 + b_2x_2 - c|x_2|) \\
&\int_{-\infty}^{\infty}\exp(-\tfrac{1}{2}[a_{12}a_{21}x_1x_2 + a_{11}x_1^2] + b_1x_1 - c|x_1|]dx_1 \\
&= k\int_{-\infty}^{\infty}\exp[-0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{11}x_1^2 + b_1x_1 - c|x_1|]dx_1 \\
&= k[\int_0^{\infty}\exp[-0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{11}x_1^2 + (b_1-c)x_1]dx_1 \\
&+ \int_{-\infty}^{0}\exp[-0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{11}x_1^2 + (b_1+c)x_1]dx_1] \\
&= k[\int_0^{\infty}\exp[-0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{11}x_1^2 + (b_1-c)x_1]dx_1 \\
&+ \int_0^{\infty}\exp[0.5(a_{12}+a_{21})x_1x_2 - 0.5a_{11}x_1^2 - (b_1+c)x_1]dx_1] \\
&= k[\int_0^{\infty}\exp[-\tfrac{(x_2-\mu_1)^2}{2\sigma^2} + \tfrac{\mu_1^2}{2\sigma^2}]dx_1] + \int_0^{\infty}\exp[-\tfrac{(x_2-\mu_2)^2}{2\sigma^2} + \tfrac{\mu_2^2}{2\sigma^2}]dx_1] \\
&= k\sigma[\tfrac{\Phi(\mu_1/\sigma)}{\phi(\mu_1/\sigma)} + \tfrac{\Phi(\mu_2/\sigma)}{\phi(\mu_2/\sigma)}]
\end{aligned}
$$

where $\mu_1 = (-\frac{a_{12}+a_{21}}{2a_{11}}x_2 + \frac{b_1-c}{a_{11}})$, $\mu_2 = (\frac{a_{12}+a_{21}}{2a_{11}}x_2 - \frac{b_1+c}{a_{11}})$, $\sigma^2 = 1/a_{11}$, $k = Z^{-1}\exp(-0.5a_{22}x_2^2 + b_2x_2 - c|x_2|)$

3.4 Local-Global Algorithm

3.4.1 Univariate local global algorithm

Continuing from Equation 3.5, it is evident to observe that $p(\beta_j|\mathcal{D}) \propto p(\beta_j, \mathcal{D}) \sim Lasso(\frac{\tilde{a}}{\tilde{b}}(y - X_{-j}s), \frac{\tilde{a}}{2\tilde{b}}(X_j^T X_j + X_j^T X_{-j}t), \frac{\lambda\Gamma(\tilde{a}+1/2)}{\Gamma(\tilde{a})\sqrt{\tilde{b}}})$.

The local approximation of mean μ_j^* and variance Σ_{jj}^* can be obtained by the expression Equation 3.7 with given Lasso parameter a, b, c. A marginal normal approximation to the conditional distribution $p(\theta_j|\mathcal{D})$: $q^*(\theta_j) \approx N(\mu_j^*, \Sigma_{jj}^*)$. The optimal distribution can be updated via Equation 3.9

$$q^*(\theta) = q(\theta_{-j}|\theta_j)\phi(\theta_j; \mu_j^*, \Sigma_{jj}^*) \tag{3.9}$$

Since, both $q(\theta_{-j}|\theta_j)$ and $q(\theta_j)$ are Normal distributions, the joint distribution will also be a normal distribution as well. Additionally, the marginal mean and variance of the joint distribution for θ_j will be μ_j^* and Σ_{jj}^* The derivation for $\tilde{\mu}$ has been shown below using the property of the conditional expectation trick. It is known that $q(\theta_2|\theta_1)$ and $q(\theta_1)$ are Gaussian. Hence, their joint distribution will also be Gaussian. The marginal mean and variance of the joint distribution for θ_1 will be μ_1^* and Σ_{11}^*. Suppose the mean and covariance of the joint distribution are $\widetilde{\mu}$ and $\widetilde{\Sigma}$ respectively. Then $\widetilde{\mu}_1 = \mu_1^*$, and $\widetilde{\Sigma}_{11} = \Sigma_{11}^*$. The rest of the derivation is to determine $\widetilde{\mu}_2$, $\widetilde{\Sigma}_{22}$ and $\widetilde{\Sigma}_{12}$.

We have

$$\begin{aligned}
\mathbb{E}[\theta_2] &= \mathbb{E}[\mathbb{E}(\theta_2 \mid \theta_1)] \\
&= \mathbb{E}[\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\theta_1 - \mu_1)] \\
&= \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\mu_1^* - \mu_1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{Cov}(\theta_2) &= \mathbb{E}[\mathrm{Cov}(\theta_2 \mid \theta_1)] + \mathrm{Cov}[\mathbb{E}(\theta_2 \mid \theta_1)] \\
&= \mathbb{E}(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}) + \mathrm{Cov}[\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\theta_1 - \mu_1)] \\
&= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} + \Sigma_{21}\Sigma_{11}^{-1}\mathrm{Cov}(\theta_1)\Sigma_{11}^{-1}\Sigma_{12} \\
&= \Sigma_{22} + \Sigma_{21}(\Sigma_{11}^{-1}\Sigma_{11}^*\Sigma_{11}^{-1} - \Sigma_{11}^{-1})\Sigma_{12}.
\end{aligned}$$

Lastly,

$$\begin{aligned}
\text{Cov}(\theta_1, \theta_2) &= \mathbb{E}[(\theta_1 - \mathbb{E}(\theta_1))(\theta_2 - \mathbb{E}(\theta_2))^T] \\
&= \left[\mathbb{E}\left\{(\theta_1 - \mathbb{E}(\theta_1))(\theta_2 - \mathbb{E}(\theta_2))^T \mid \theta_1\right\}\right] \\
&= \mathbb{E}\left[(\theta_1 - \mu_1^*)\left(\Sigma_{21}\Sigma_{11}^{-1}(\theta_1 - \mu_1^*)\right)^T\right] \\
&= \mathbb{E}\left[(\theta_1 - \mu_1^*)(\theta_1 - \mu_1^*)^T\right]\Sigma_{11}^{-1}\Sigma_{12} \\
&= \Sigma_{11}^*\Sigma_{11}^{-1}\Sigma_{12}.
\end{aligned}$$

Hence, $q^*(\theta) = N(\widetilde{\mu}, \widetilde{\Sigma})$ where

$$\widetilde{\mu} = \begin{bmatrix} \mu_1^* \\ \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\mu_1^* - \mu_1) \end{bmatrix}$$

The derivation for $\tilde{\Sigma}$ has been shown below using the property of the conditional covariance:

$$\widetilde{\Sigma} = \begin{bmatrix} \Sigma_{11}^* & \Sigma_{11}^*\Sigma_{11}^{-1}\Sigma_{12} \\ \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{11}^* & \Sigma_{22} + \Sigma_{21}\Sigma_{11}^{-1}(\Sigma_{11}^* - \Sigma_{11})\Sigma_{11}^{-1}\Sigma_{12} \end{bmatrix}.$$

Hence, $q^*(\beta) = N(\widetilde{\mu}, \widetilde{\Sigma})$ at the time when updating variable β_j is:

$$\widetilde{\mu} = \begin{bmatrix} \mu_j^* \\ \mu_{-j} + \Sigma_{-j,j}\Sigma_{jj}^{-1}\left(\mu_j^* - \mu_j\right) \end{bmatrix} \tag{3.10}$$

and

$$\widetilde{\Sigma} = \begin{bmatrix} \Sigma_{jj}^* & \Sigma_{jj}^*\Sigma_{jj}^{-1}\Sigma_{j-j} \\ \Sigma_{-jj}\Sigma_{jj}^{-1}\Sigma_{jj}^* & \Sigma_{-j-j} + \Sigma_{-jj}\Sigma_{jj}^{-1}(\Sigma_{jj}^* - \Sigma_{jj})\Sigma_{jj}^{-1}\Sigma_{j-j} \end{bmatrix}. \tag{3.11}$$

Algorithm 4 Univariate-Local-Global-Algorithm

Input: data X, normalized response variable y, parameter from MFVB $(\tilde{a}, \tilde{b}, \tilde{\mu}, \tilde{\Sigma})$, Penalizing parameter: λ

while $\tilde{\mu}$ is changing less than ϵ **do**

  for $j = 1$ to p **do**

    $a = \frac{\tilde{a}}{\tilde{b}}(X^T X)_{j,j} + (X^T X)_{j,-j}\tilde{\Sigma}_{-j,j}\tilde{\Sigma}_{j,j}^{-1}$ ▷ Obtain Lasso parameter

    $b = \frac{\tilde{a}}{\tilde{b}} X_j(y - X_{-j}(\tilde{\mu}_{-j} - \tilde{\Sigma}_{-j,j}\tilde{\Sigma}_{j,j}^{-1}\tilde{\mu}_j))$ ▷ Obtain Lasso parameter

    $c = \lambda(\exp(\Gamma(\tilde{a} + 0.5) - \Gamma(\tilde{a}) - 0.5\log(\tilde{b})))$ ▷ Obtain Lasso parameter

    $\tilde{\mu_j} = elasso(a, b, c) = \mu_j^*$ ▷ Replace by Local mean

    $\tilde{\mu_{-j}} = \tilde{\mu}_{-j} + \tilde{\Sigma}_{-j,j}\tilde{\Sigma}_{jj}^{-1}(\tilde{\mu}_j^* - \tilde{\mu}_j)$ ▷ Update Global Mean

    $\tilde{\Sigma}_{j,j} = vlasso(a, b, c) = \Sigma_{jj}^*$ ▷ Replace by Updated Local Covariance

    $\tilde{\Sigma}_{j,-j} = \Sigma_{jj}^*\tilde{\Sigma}_{jj}^{-1}\tilde{\Sigma}_{j,-j}$ ▷ Update Global Covariance

    $\tilde{\Sigma}_{-j,j} = \tilde{\Sigma_{j,-j}}^T$ ▷ Update Global Covariance

    $\tilde{\Sigma}_{-j,-j} = \tilde{\Sigma}_{-j,-j} + \tilde{\Sigma}_{-j,j}\tilde{\Sigma}_{j,j}^{-1}(\Sigma_{j,j}^* - \tilde{\Sigma}_{j,j})\tilde{\Sigma}_{j,j}^{-1}\tilde{\Sigma}_{j,-j}$ ▷ Update Global Covariance

return $\tilde{\mu}, \tilde{\Sigma}$

The following bullet points summarized the procedure of our method.

- Our target: $p(\theta|\mathcal{D}) = p(\beta, \sigma^2|\mathcal{D})$, where a normal distribution is used to approximate, with global parameters $\tilde{\mu}, \tilde{\Sigma}$. The initial estimate of $\tilde{\mu}$ and Σ has been obtained via MFVB. Define $\beta = (\beta_j, \beta_{-j})$, separated by a target variable and the rest of the variables. μ_j^*, Σ_{jj}^* as local parameter

- Compute the mean of lasso distribution and variance of the lasso distribution : *elasso* and *vlasso* function in the aforementioned pseudo algorithm, with the current a, b, c: $q^*(\theta_1) \approx N(\mu_j^*, \Sigma_{jj}^*)$

- Then we can use the update expression $q^*(\beta) = q(\beta_{-j}|\beta_j)\phi(\beta_j; \mu_j^*, \Sigma_{jj}^*)$ to adjust the global mean and global covariance of $q(\beta)$

- Iterates through each variable β_j and update $\tilde{\mu}$ and $\tilde{\Sigma}$ each time by the derived expression.

3.4.2 Bivariate local global algorithm

Apart from the Univariate algorithm that matches each variable of interest into a univariate Lasso distribution, each pair of regression coefficients can be matched simultaneously by a bivariate Lasso distribution as mentioned in subsection 3.3.2. The number of index pairs can be calculated while the combination arithmetic. The total number of index pairs is $\binom{p}{2}$. For example, if $p = 3$ then index pair $\mathcal{I}$: can be $(1, 2)$, $(1, 3)$ and $(2, 3)$. The main intuition is similar to Equation 3.2, but the log likelihood of each index pair $\mathcal{I}$ now can be written as:

$$\begin{aligned} \log p(D, \beta_{\mathcal{I}}) &= \mathbb{E}_{\beta_{-\mathcal{I}}, \sigma^2 | \mathcal{D}, \beta_{\mathcal{I}}} \log p(\beta_{\mathcal{I}} \mid \mathcal{D}, \beta_{-\mathcal{I}}, \sigma^2) \\ &\approx \mathbb{E}_{q(\beta_{-\mathcal{I}} | \beta_{\mathcal{I}}) q(\sigma^2)} \log p(\beta_{\mathcal{I}} \mid \mathcal{D}, \beta_{-\mathcal{I}}, \sigma^2) \end{aligned} . \tag{3.12}$$

The log-likelihood of index pair can be written as the following:

$$\begin{aligned} \log p(\beta_{\mathcal{I}} \mid \mathcal{D}, \beta_{-I}, \sigma^2) &= -\tfrac{1}{2\sigma^2} \beta_{\mathcal{I}}^T X_{\mathcal{I}}^T X_{\mathcal{I}} \beta_{\mathcal{I}} - \frac{\lambda}{\sigma} \|\beta_{\mathcal{I}}\|_1 \\ &\quad + \frac{1}{2\sigma^2} (y - X_{-\mathcal{I}} \beta_{-\mathcal{I}})^T X_{\mathcal{I}} \beta_{\mathcal{I}} + \frac{1}{2\sigma^2} \beta_{\mathcal{I}} X_{\mathcal{I}}^T (y - X_{-\mathcal{I}} \beta_{-\mathcal{I}}) \end{aligned} . \tag{3.13}$$

For the purpose of enabling $\beta_{-\mathcal{I}}$ to be symmetric for preventing generating a non-symmetric, the log-likelihood of index pair $\mathcal{I}$ can be remodified as the following:

$$\begin{aligned} \log p(\mathcal{D}, \beta_{\mathcal{I}}) &= \tfrac{\widetilde{a}}{\widetilde{b}} (y - X_{-\mathcal{I}} s)^T X_{\mathcal{I}} \beta_{\mathcal{I}} - \tfrac{\lambda \Gamma(\widetilde{a} + 1/2)}{\Gamma(\widetilde{a}) \sqrt{\widetilde{b}}} |\beta_{\mathcal{I}}\|_1 \\ &\quad - \tfrac{\widetilde{a}}{2\widetilde{b}} \beta_{\mathcal{I}}^T \left(X_{\mathcal{I}}^T X_{\mathcal{I}} + \tfrac{1}{2} X_{\mathcal{I}}^T X_{-\mathcal{I}} T + \tfrac{1}{2} T X_{-\mathcal{I}}^T X_{\mathcal{I}} \right) \beta_{\mathcal{I}} \end{aligned} . \tag{3.14}$$

Then the marginal log-likelihood can be matched to a bivariate lasso distribution:

$$\beta_{\mathcal{I}} \mid \mathcal{D} \overset{\text{approx.}}{\sim} \text{BiLasso} \left(\frac{\widetilde{a}}{\widetilde{b}} \left(X_{\mathcal{I}}^T X_{\mathcal{I}} + \tfrac{1}{2} X_{\mathcal{I}}^T X_{-j} T + \tfrac{1}{2} T^T X_{-j}^T X_{\mathcal{I}} \right), \frac{\widetilde{a}}{\widetilde{b}} X_{\mathcal{I}}^T (y - X_{-\mathcal{I}} s), \frac{\lambda \Gamma(\widetilde{a} + 1/2)}{\Gamma(\widetilde{a}) \sqrt{\widetilde{b}}} \right) . \tag{3.15}$$

To propagate to correct the global mean and variance parameter, note the conditional distribution of $q(\beta_{-\mathcal{I}} \mid \beta_{\mathcal{I}})$ can be written as the following:

$$\begin{aligned} q(\beta_{-\mathcal{I}} \mid \beta_{\mathcal{I}}) &= N_{p-|\mathcal{I}|} \left(\mu_{-\mathcal{I}} + \Sigma_{-\mathcal{I},\mathcal{I}} \Sigma_{\mathcal{I},\mathcal{I}}^{-1} (\beta_{\mathcal{I}} - \mu_{\mathcal{I}}), \Sigma_{-\mathcal{I},-\mathcal{I}} - \Sigma_{-\mathcal{I},\mathcal{I}} \Sigma_{\mathcal{I},\mathcal{I}}^{-1} \Sigma_{\mathcal{I},-\mathcal{I}} \right)) \\ &= N_{p-|\mathcal{I}|} \left(s + T \beta_{\mathcal{I}}, \Sigma_{-\mathcal{I},-\mathcal{I}} - \Sigma_{-\mathcal{I},\mathcal{I}} \Sigma_{\mathcal{I},\mathcal{I}}^{-1} \Sigma_{\mathcal{I},-\mathcal{I}} \right)) \end{aligned} \tag{3.16}$$

Using Equation 3.16 and propagate equation from Equation 3.9, the update formula for $\tilde{\mu}$ and $\tilde{\Sigma}$ can be updated by the same equation as the univariate case from Equation 3.10 and Equation 3.11 with the substitution of subscript j to index pair $\mathcal{I}$.

Algorithm 5 Bivariate-Local-Global-Algorithm

Input: data X, normalized response variable y, parameter from MFVB $(\tilde{a}, \tilde{b}, \tilde{\mu}, \tilde{\Sigma})$, Penalizing parameter: λ

while $\tilde{\mu}$ is changing less than ϵ **do**

  for $\mathcal{I} = (1,1)$ to $(p-1, p)$ **do**

    $a = \frac{\tilde{a}}{\tilde{b}} \left(X_{\mathcal{I}}^T X_{\mathcal{I}} + \frac{1}{2} X_{\mathcal{I}}^T X_{-j} T + \frac{1}{2} T^T X_{-j}^T X_{\mathcal{I}} \right)$ ▷ Obtain Lasso parameter

    $b = \frac{\tilde{a}}{\tilde{b}} X_{\mathcal{I}}^T \left(y - X_{-\mathcal{I}} s \right)$ ▷ Obtain Lasso parameter

    $c = \lambda(\exp(\Gamma(\tilde{a} + 0.5) - \Gamma(\tilde{a}) - 0.5 \log(\tilde{b})))$ ▷ Obtain Lasso parameter

    $\tilde{\mu_{\mathcal{I}}} = Bielasso(a, b, c) = \mu_{\mathcal{I}}^*$ ▷ Replace by Local mean

    $\tilde{\mu_{-\mathcal{I}}} = \tilde{\mu}_{-\mathcal{I}} + \tilde{\Sigma}_{-\mathcal{I},\mathcal{I}} \tilde{\Sigma}_{\mathcal{I}\mathcal{I}}^{-1} (\tilde{\mu}_{\mathcal{I}}^* - \tilde{\mu}_{\mathcal{I}})$ ▷ Update Global Mean

    $\tilde{\Sigma}_{\mathcal{I},\mathcal{I}} = Bivlasso(a, b, c) = \Sigma_{\mathcal{I}\mathcal{I}}^*$ ▷ Replace by Updated Local Covariance

    $\tilde{\Sigma}_{\mathcal{I},-\mathcal{I}} = \Sigma_{\mathcal{I}\mathcal{I}}^* \tilde{\Sigma}_{\mathcal{I}\mathcal{I}}^{-1} \tilde{\Sigma}_{\mathcal{I},-\mathcal{I}}$ ▷ Update Global Covariance

    $\tilde{\Sigma}_{-\mathcal{I},\mathcal{I}} = \tilde{\Sigma_{\mathcal{I},-\mathcal{I}}}^T$ ▷ Update Global Covariance

    $\tilde{\Sigma}_{-\mathcal{I},-\mathcal{I}} = \tilde{\Sigma}_{-\mathcal{I},-\mathcal{I}} + \tilde{\Sigma}_{-\mathcal{I},\mathcal{I}} \tilde{\Sigma}_{\mathcal{I},\mathcal{I}}^{-1} (\Sigma_{\mathcal{I},\mathcal{I}}^* - \tilde{\Sigma}_{\mathcal{I},\mathcal{I}}) \tilde{\Sigma}_{\mathcal{I},\mathcal{I}}^{-1} \tilde{\Sigma}_{\mathcal{I},-\mathcal{I}}$ ▷ Update Global Covariance

return $\tilde{\mu}, \tilde{\Sigma}$

The following bullet points summarized the procedure of Bivariate Local-Global Algorithm

- Our target: $p(\theta|\mathcal{D}) = p(\beta, \sigma^2|\mathcal{D})$, where a normal distribution is used to approximate, with global parameters $\tilde{\mu}, \tilde{\Sigma}$. The initial estimate of $\tilde{\mu}$ and Σ has been obtained via MFVB. Define $\theta = (\beta_j, \beta_{-j})$, separated by a target variable and the rest of the variables. μ_j^*, Σ_{jj}^* as local parameter
- Compute the mean of lasso distribution and variance of **bivariate** lasso distribution : *bielasso* and *bivlasso* function in the aforementioned pseudo algorithm, with the current a, b, c: $q^*(\beta_j) \approx N(\mu_j^*, \Sigma_{jj}^*)$
- Then we can use the update expression $q^*(\theta) = q(\beta_{-j}|\beta_j)\phi(\beta_j; \mu_j^*, \Sigma_{jj}^*)$ to adjust the global mean and global covariance of $q(\beta)$
- Iterates through each variable θ_j and update $\tilde{\mu}$ and $\tilde{\Sigma}$ each time by the derived expression.

Chapter 4

Experiment Result and Analysis

4.1 Experimental Setting

4.1.1 Evaluation metric

To test the effectiveness of our algorithms, several evaluation metrics would be used:

- l_1 norm accuracy

$$l_1(f, g) = \int |f(x) - g(x)|dx \tag{4.1}$$

$$\text{Acc}(f, g) = 1 - \frac{1}{2} l_1(f, g) \tag{4.2}$$

- Running Speed: Total number of times (second) used for generating posterior density

The l_1 norm is a common choice for comparing Probability Density, with a specific focus on the preciseness of the center of the distribution. The reason for calculating l_1 norm accuracy can be attributed to the fact that the approximation correctness for the tail distribution will contribute less than that of the center distribution, which is our main goal for approximation for posterior distribution. Moreover, the execution time when generating posterior parameters should also be examined to provide a better comparison from the perspective of the time complexity with Local-Global Algorithm, MFVB, and MCMC. It is measured in seconds.

4.1.2 Experimental datasets

The following bullet points demonstrate the dataset description, which includes the introduction to the purpose of the dataset, and a number of predictors and number of samples respectively.

- **Hitters**:
  - Type: Baseball statistics dataset
  - Predictors (p): 20, Samples (n): 263
  - Description: Contains baseball player statistics, including performance measures and salary information.
  - Further description: High correlation between predictors
- **Kakadu**:
  - Type: Environmental dataset
  - Predictors (p): 22, Samples (n): 1828
  - Description: Relates environmental factors to the abundance of amphibians in Kakadu National Park, Australia.
  - Further description: approximately normal distribution with a large number of samples, no strong correlation between predictors
- **Bodyfat**:
  - Type: Human body measurements dataset
  - Predictors (p): 15, Samples (n): 250
  - Description: Contains measurements of various body parts for a sample of individuals, such as weight, height, and circumferences.
  - Further description: approximately normal distribution, no strong correlation between predictors
- **Prostate**:
  - Type: Medical dataset
  - Predictors (p): 8, Samples (n): 97
  - Description: Prostate cancer data with clinical measurements and Logarithm of Prostate Specific Antigen (lpsa).

  - Further description: approximately normal distribution, no strong correlation between predictors

- **Credit**:
  - Type: Credit scoring dataset
  - Predictors(p): 11, Samples (n): 400
  - Description: Contains information about loan applicants, such as credit history, employment, and demographics, to assess creditworthiness.
  - Further description: approximately normal distribution, no strong correlation between predictors

- **Eyedata**:
  - Type: Medical dataset
  - Predictors (p): 200, Samples (n): 120
  - Description: Contains measurements related to glaucoma patients, such as intraocular pressure and visual acuity, to assess the effectiveness of treatment options.
  - Further description: More predictors than the number of samples

In particular, the Hitters dataset and Eyedata dataset should be gained more attention, since Hitters has a large correlation between predictors while Eyedata has more predictors than a number of samples meaning Hitters and Eyedata are harder to approximate. On the other hand, if the approximation accuracy and approximation density can achieve higher results than MFVB, then the success of Local-Global Algorithm can also achieve success on other datasets that are easier to approximate. We will show the experimental result with a specific focus on Hitters and Eyedata compared with results on other datasets in the later subsection.

4.2 Experimental Result

4.2.1 Approximation Density Visualization

The following six plots demonstrate part of the approximated density plots on each of the datasets. Due to the large number of predictors in each dataset, we will present two density plots for each dataset. The best density approximation of the Local-Global algorithm with MCMC and the worst density approximation of the Local-Global algorithm is the most deviated compared with MCMC. The best plot will be placed on the left of each plot, and the worst plot will be arranged on the right of each plot. As for the density on Kakadu, Prostate, Bodyfat, and Credit dataset, there is no significant deviation between the density plot generated from MFVB, LG-Global, and LG-Local with MCMC. Almost all of the predictors in these datasets demonstrate a roughly normal density. This will not significantly leads to a considerable decrease in approximation accuracy for MFVB, as opposed to LG-local and LG-Global.

As we can see from the density plot from this particular predictor in Figure 4.1, the plot

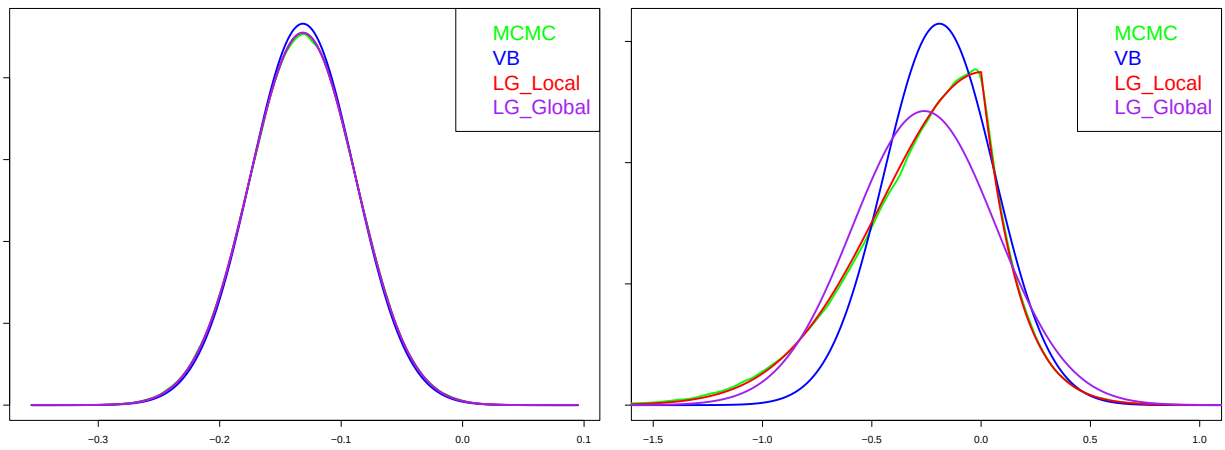

Figure 4.1: Part of Approximation Density for Hitters dataset; Left: best case, Right: worst case

on the left-hand side overlaps significantly with MCMC, indicating high accuracy of the approximation performance. By the worst density plot, MFVB tends to deviate from the left-skewed MCMC (gold standard), especially when the actual distribution has a sharp

tuning point, the local and global algorithms, especially for the local approximation line, however, can approximate well to the gold standard in this scenario. This is mainly due to the absolute term $c|x|$ in the lasso distribution formula, accommodating various potential density functions.

By Figure 4.2, the best density plot of Local-Global algorithms overlaps with MCMC and

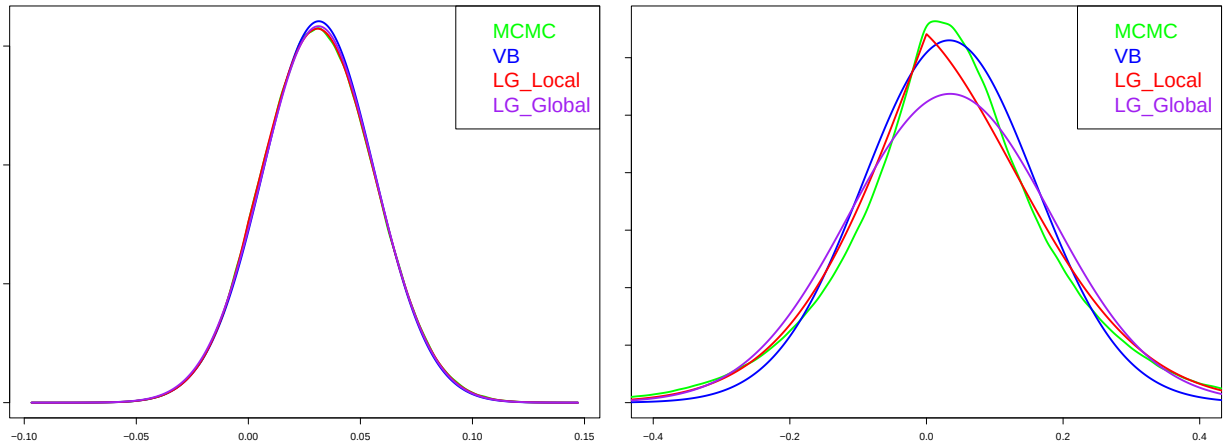

Figure 4.2: Part of Approximation Density for Kakadu dataset; Left: best case, Right: worst case

MFVB, which is similar to Figure 4.1. For the worst case, LG-local still demonstrates the best approximation performance under this case. Even though the gold standard demonstrates a density with high variance, LG-Local fit a density with similar variance even though a sharp tuning point exists.

Figure 4.3, the best density plot of Local-Global algorithms overlaps with MCMC and MFVB considerably as well since the dataset itself demonstrates asymptotically normal data distribution, which is similar to the aforementioned case due to asymptotically normal distributed data distribution. LG-local still demonstrates the best approximation performance under the worst case according to the right of Figure 4.3. The LG-Local line is highly consistent with the density plot generated by the gold standard from the green line.

In the best case in Figure 4.4, MFVB overestimates the posterior variance slightly, while LG-local and LG-global demonstrate high robustness against target distribution. For the worst case, although LG-Global underestimates the posterior variance moderately, LG-local demonstrates high consistency again with MCMC.

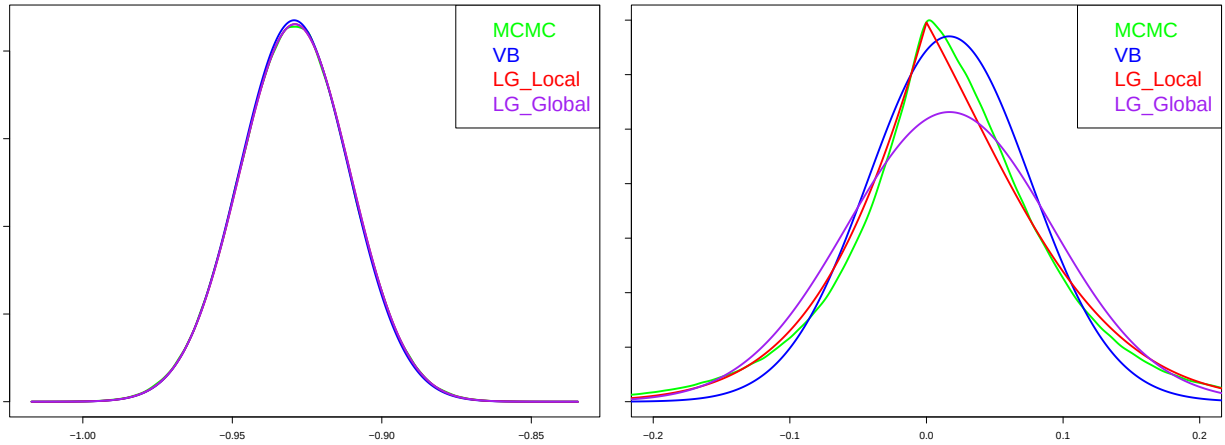

Figure 4.3: Part of Approximation Density for Bodyfat dataset; Left: best case, Right: worst case

Similarly as before, the Local-Global approximation overlap with MFVB and MCMC as shown on the left of the Figure 4.5. For the worst-case plot, MFVB demonstrates an overestimation of the posterior variance, while LG-local and LG-global also significantly coincide with the MCMC density plot.

The left of the Figure 4.6 has shown the largest distinction in the best scenario. Due to the reason that for most of the predictors on the Eyedata, the actual distribution is not asymptotically normal by MCMC plot, the optimal approximation case shows a laplacian-like distribution. This can directly lead to inaccurate results for MFVB and LG-Global. For the worst case on the right of Figure 4.6, the marginal LG-local approximation underestimates the data distribution slightly, but the overall performance is still the best among all the three plots. MFVB approximation again overestimates the posterior variance considerably. Although the normal-distributed assumption exists for LG-Global, LG-Global tends to correct the overestimated posterior variance of the regression coefficient, compared with MFVB.

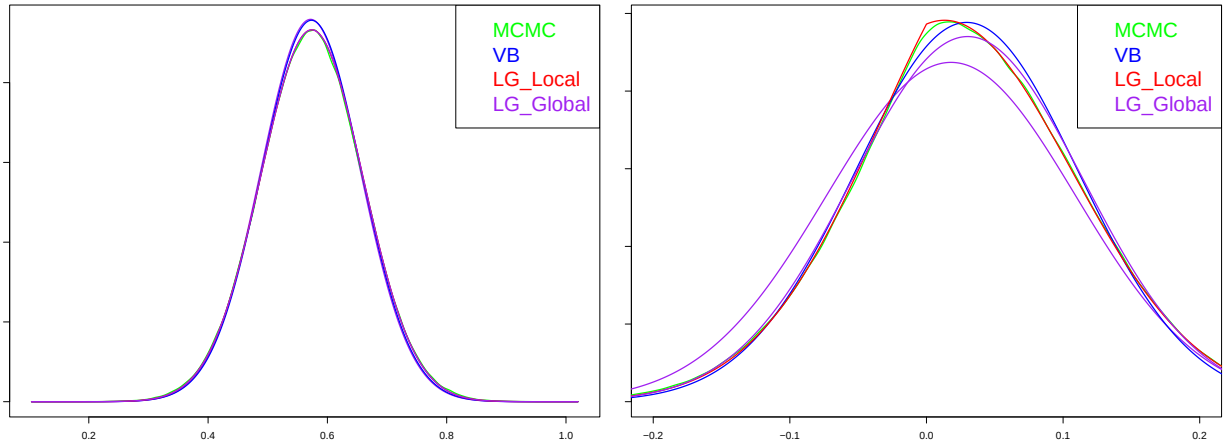

Figure 4.4: Part of Approximation Density for Prostate dataset; Left: best case, Right: worst case

4.2.2 Approximation Accuracy Result

The following six tables demonstrate the experiment results for the approximation accuracy of 3 different algorithms, where LG-Local represents the marginal approximation of the Local-Global Algorithm by lasso distribution, LG-Global represents the global approximation of the Local-Global Algorithm by a univariate normal distribution with a particular global mean at j_{th} component and the variance is used by the j_{th} row and j_{th} column of the $\tilde{\Sigma}$. Each row illustrates a distinct quantile of approximation accuracy, such as minimum approximation accuracy achieved by each algorithm, maximum approximation accuracy via each algorithm, etc. LG-Local represents the local approximation accuracy for our algorithm and LG-Global represents the global approximation accuracy in our algorithm. VB represents the Mean-Field-Variational-Bayes algorithm and MCMC represents the Monte Carlo Markov Chain method, as a gold standard that achieves 100% accuracy for each approximation density.

Table 4.1 illustrates the approximation result on the Hitters dataset, the global approximation of the Local-Global algorithm is superior to the benchmark approach MFVB from the perspective of all the metrics by about 1 to 5 percent. In addition, the time used for running the Local-Global algorithm is 0.17s, which is 0.03s slower than the MFVB. It is expected since MFVB only involves the global parameter approximation, without including

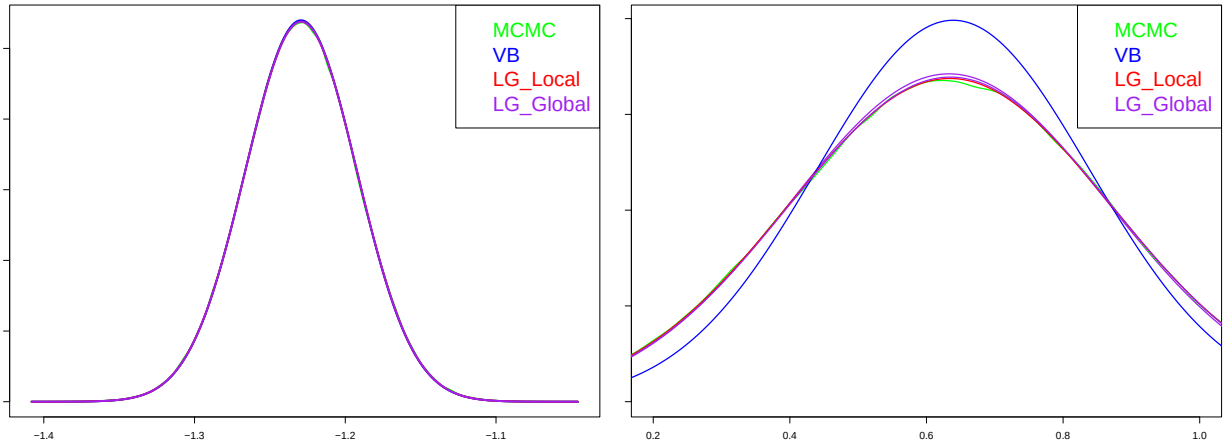

Figure 4.5: Part of Approximation Density for Credit dataset; Left: best case, Right: worst case

Algorithm	MCMC	VB	LG_Local	LG_Global
Min.	100	86.8	97.3	89.3
1st Qu.	100	92.1	99.2	97.0
Median	100	95.7	99.6	97.4
Mean	100	94.2	99.3	97.1
3rd Qu.	100	97.4	99.7	99.0
Max.	100	98.7	99.8	99.7
Run Time(s)	453.75	0.17	0.17	0.17

Table 4.1: Experiment Result on Hitters dataset

the process of local approximation. Meanwhile, both of the methods are 400 times faster than MCMC. The performance of the Local-Global algorithm has proven to be better in approximation accuracy in this particular dataset regarded as the hardest among all of the datasets without a significant increase of running speed.

Table 4.2 illustrates the approximation result on the Kakadu dataset. Even though the

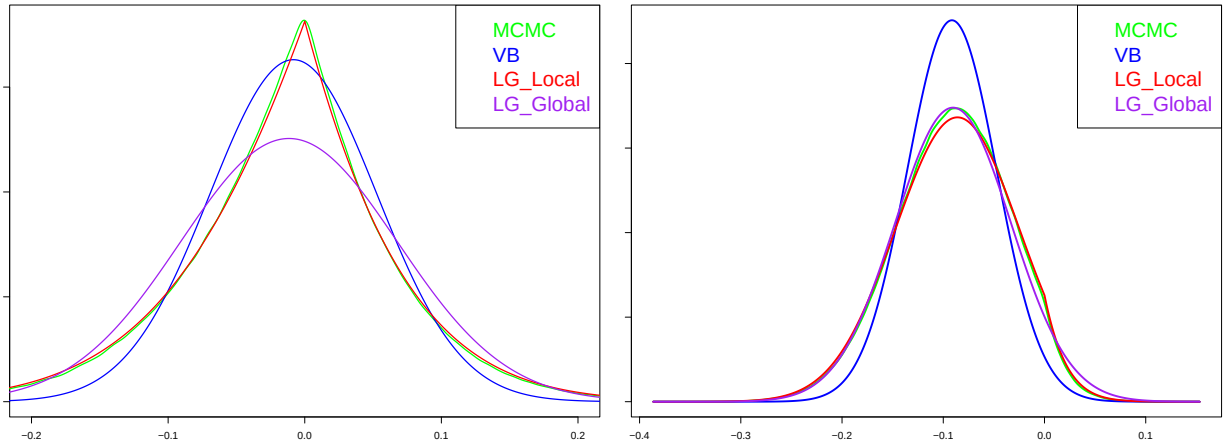

Figure 4.6: Part of Approximation Density for Eyedata dataset; Left: best case from 52nd predictor, Right: worst case from 200th predictor

Algorithm	MCMC	VB	LG_Local	LG_Global
Min.	100	93.2	95.9	92.2
1st Qu.	100	98.9	99.8	99.2
Median	100	99.2	99.8	99.4
Mean	100	98.6	99.4	98.8
3rd Qu.	100	99.3	99.8	99.8
Max.	100	99.7	99.8	99.8
Run Time(s)	6696.56	0.14	0.19	0.19

Table 4.2: Experiment Result on Kakadu dataset

minimum global approximation of the Local-Global algorithm is lower than MFVB, all other metrics are slightly higher than MFVB. For the local approximation of LG-Local, there are one to two percentage increases compared with MFVB. In addition, the execution time for the Local-Global is 0.05s shorter than that of MFVB(0.14s), while both methods achieve 6000 times faster speed than that of MCMC, this is because the numb er of samples

in Kakadu is high and MCMC requires a longer running time to sample. Nevertheless, the performance of the MFVB method and Local-Global algorithm would not degrade severely since the data distribution is approximately normal.

Table 4.3 illustrates the approximation result on Bodyfat dataset. Even though the min-

Algorithm	MCMC	VB	LG_Local	LG_Global
Min.	100	91.3	96.3	90.7
1st Qu.	100	97.0	99.6	97.6
Median	100	98.0	99.7	98.4
Mean	100	97.0	99.2	97.2
3rd Qu.	100	98.4	99.7	98.6
Max.	100	99.3	99.7	99.7
Run Time(s)	398.59	0.14	0.17	0.17

Table 4.3: Experiment Result on bodyfat dataset

imum global approximation of the Local-Global algorithm is lower than MFVB similar to Kakadu, all other metrics are slightly higher than MFVB. For the local approximation of LG-Local, there is a 2 to 5 percent increase compared with MFVB. Apart from that, the execution time for the Local-Global is 0.03s shorter than that of MFVB(0.14s), while both methods achieve 300 times faster speed than that of MCMC. The data distribution on the Bodyfat dataset is similar to that of the Kakadu dataset, which means there is no drastic difference between MFVB and the Local-Global algorithm.

Table 4.4 illustrates the approximation result on the Prostate dataset. The global approximation of the Local-Global algorithm is slightly higher than MFVB similar to Kakadu, Bodyfat. The local approximation accuracy of LG-Local is higher than both global approximation and MFVB. Apart from that, the execution time for the Local-Global is 0.01s shorter than that of MFVB(0.11s), and both methods achieve 300 times faster speed than

Algorithm	MCMC	VB	LG_Local	LG_Global
Min.	100	96.9	99.5	97.4
1st Qu.	100	97.2	99.5	97.9
Median	100	97.6	99.6	98.9
Mean	100	97.5	99.6	98.7
3rd Qu.	100	97.7	99.6	99.5
Max.	100	98.4	99.6	99.6
Run Time(s)	336.31	0.11	0.12	0.12

Table 4.4: Experiment Result on Prostate dataset

that of MCMC. The data distribution on the Prostate dataset is similar to that of Bodyfat and Kakadu with an approximately normal distribution, which means there is no dramatic distinction between MFVB and Local-Global algorithm.

Table 4.5 demonstrates the approximation result on the Credit dataset. The global approximation of the Local-Global algorithm is slightly higher than MFVB, although the minimum value of MFVB is roughly 8% lower than LG-Global. The approximation accuracy of LG-global is similar to LG-local, achieving above 99%. Also, the execution time for the Local-Global is 0.11s, and it is only 0.01s slower than Local-Global Algorithm. Both methods achieve 300 times faster speed than that of MCMC. Likewise, the fact that Prostate, Credit, Bodyfat, and Kakadu have roughly normal distribution properties directly leads to a similar result as before.

Table 4.6 demonstrates the approximation result on the Eyedata dataset. The global approximation of the Local-Global algorithm is abundantly higher than MFVB. The local approximation accuracy of LG-Local is also drastically higher than both global approximation and MFVB. Also, the execution time for the Local-Global is 0.51s shorter than that of MFVB(1.21s), and both methods achieve 18000 times faster speed than that of

Algorithm	MCMC	VB	LG_Local	LG_Global
Min.	100	91.8	99.5	99.3
1st Qu.	100	98.3	99.7	99.5
Median	100	99.4	99.8	99.5
Mean	100	97.9	99.7	99.6
3rd Qu.	100	99.5	99.8	99.7
Max.	100	99.7	99.8	99.8
Run Time(s)	359.92	0.1	0.11	0.11

Table 4.5: Experiment Result on Credit dataset

Algorithm	MCMC	VB	LG_Local	LG_Global
Min.	100	78.4	97.3	86.1
1st Qu.	100	86.9	98.6	90.4
Median	100	90.3	98.7	91.3
Mean	100	88.9	98.7	91.8
3rd Qu.	100	91.2	98.8	92.3
Max.	100	93.1	99.1	99.2
Run Time(s)	18144.7	1.21	1.72	1.72

Table 4.6: Experiment Result on Eyedata dataset

MCMC. The special notification to the Eyedata is a high dimensional sparsity of the data distribution. The curse of dimensionality interferes with the approximation, especially for the MCMC, which requires more steps for convergence. In this case, although MFVB is

slightly faster, the Local-Global algorithm demonstrates a more robust performance from the perspective of approximation accuracy.

Chapter 5

Discussion and Conclusion

5.1 Discussion

From the experiment result such as approximation accuracy and approximation density plot in the last chapter. The following five bullet points mention the noticeable phenomenon in the experiment.

- MFVB tends to produce a density with less variance, making it more concentrated at the center.
- The marginal posterior distribution from the Local-Global Algorithm distribution is the most accurate one that can be adapted to various predictor distributions other than the normal distribution
- The global posterior distribution from Local-Global Algorithm distribution is more accurate compared with MFVB.
- The running time for MFVB is slower than Local-Global algorithms for both local approximation and global approximation
- Local-Global Algorithm is highly accurate even when there is a high correlation between predictors according to the result from the Hitters dataset.
- Local-Global Algorithm is highly accurate even when there are more predictors than a number of samples according to the result from the Eyedata dataset.

Firstly,

Secondly, this phenomenon is mainly due to the adaptability of the function form of univariate lasso distribution that can capture the sharp point in the curve. Thirdly, this is mainly due to the addition of the local adjustment by capturing the correlation between β_j

and β_{-j}. Therefore, the global approximation can be more exact than MFVB. Thirdly, due to an extra procedure for the local approximation process and calculation of moment for Lasso distribution, the Local-Global algorithm is slightly slower than MFVB, but with a moderate amount, and both algorithms are faster than MCMC. Lastly, the fourth and fifth bullet points mention the performance of the Hitters dataset and Eyedata dataset. They can contribute to the adjustment and adaptability of Local-Global algorithms, leading to better approximation results.

5.2 Limitation

There are still some improvements in our work.

- Automatic choice of λ is still obtained by Gibbs Sampling.
- The Univariate Local-Global algorithm can't deal with the case when initial covariance is a diagonal matrix.

The first limitation is that the current choice for λ is from the three-step Gibbs Sampler from Park and Casella (2008). By Park and Casella (2008), if the λ^2 instead of λ is treated as a random variable assigning a diffuse hyper prior distribution such as the gamma prior distribution, then an optimal λ posterior distribution can be derived due to conjugacy. The fully conditional posterior distributions of λ^2 can be with a rate parameter and shape parameter. Gibbs sampler can use the full conditional posterior form to seek the λ samples. The posterior median for λ can be a reliable estimation for optimal λ used afterward. Although the optimal lambda behavior is desired, the execution time for obtaining posterior estimates is as long as sampling other posterior estimates for β and σ^2.

Finally, if the initial covariance $\tilde{Sigma}$ is diagonal, by our update formula, it will remain as a diagonal matrix until the last iteration, which enables the univariate local-global algorithm non-generalizable. For instance, considering the update formula for $\tilde{Sigma}$ if the initial $\tilde{\Sigma}$ is a diagonal covariance matrix, then due to the manner of Local-Global algorithm that update $\tilde{\beta}$ and $\tilde{\Sigma}$ one variable at a time by Equation 3.11. The dimension for Σ^*_{jj} is a scalar value greater than 0 since it is obtained by the lasso variance function, the $\Sigma_{-j,-j}$

will remain a diagonal matrix with dimension $p-1$, $\Sigma_{-j,j} = \Sigma_{j,-j}^T$ can be $\mathbf{0}$ since $\tilde{\Sigma}$ is diagonal. Therefore, the resulting update formula by Equation 3.11 will be diagonal at the end. This can directly lead to issues if the actual Σ is a non-diagonal matrix, causing the discrepancy.

5.3 Future Work

To better resolve the limitation mentioned before, several improvements can be explored and completed in the future.

- Propose a Bivariate-Local-Global Algorithm to address the problem when the initial covariance is a diagonal matrix
- Derive the update formula of σ^2

Firstly, since the update for each iteration is via a pair of variables, the covariance matrix of each variable will be updated several times for each pair. Secondly, our Local-Global algorithm can only obtain the posterior distribution for the regression coefficient β only, while the temporary assumption of the independence between the σ^2 and β is limited. As a matter of fact, an alternative update formula to derive Σ should be derived and explored in the future.

5.4 Conclusion

In conclusion, we have proposed a novel algorithm for the Bayesian Lasso: a local approximation correction approach that could capture the correlation between the distribution of the target variable and the distribution of other variables. In addition, due to the matching with the invented lasso distribution, the local approximation is more precise, causing significant improvement over the MFVB approximation. The main goal of the future work will be the successful implementation of the Bivariatei-Local-Global algorithm, as well as a valid derivation formula for the update of covariance distribution $q(\sigma^2)$

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