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Statistics C183/C283 **Instructor: Nicolas Christou**

Short sales not allowed, risk free asset exists
Single index model - Ranking stocks

The calculation of optimal portfolios is simplified by using the single index model to rank securities based on the excess return to beta ratio defined as follows:

$$\text{Excess return to beta} = \frac{\bar{R}_i - R_f}{\beta_i}.$$

After stocks are ranked using the above ratio the optimum portfolio consists of investing in all stocks for which the excess return to beta is greater than the cut-off point C^*. This cut-off rate is computed as follows:

$$C^* = \frac{\sigma_m^2 \sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\beta_j}{\sigma_{\epsilon j}^2}}{1 + \sigma_m^2 \sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{\epsilon j}^2}}.$$

where

$\bar{R}_j$	Expected return on stock j.
R_f	Return on a riskless asset.
β_j	Change in the rate of return of stock j associated with a 1% change in the market return.
σ_m^2	Variance in the market index .
$\sigma_{\epsilon j}^2$	Variance of the error term. Also known as unsystematic risk.

To find C^* we compute all C_i's using portfolios that consist with the first ranked stock, the first and second ranked stocks, the first, second, and third ranked stock etc. We know we have found the cut-off poit C^* when all stocks used in calculating C_i satisfy:

$$\frac{\bar{R}_i - R_f}{\beta_i} > C^*.$$

Once C^* is found, we know that the optimum portfolio consists of the first i stocks which satisfy the above inequality. To find the proportion of funds invested in each of these stocks we use:

$$z_i = \frac{\beta_i}{\sigma_{\epsilon i}^2}\left(\frac{\bar{R}_i - R_f}{\beta_i} - C^*\right)$$

Therefore $x_i = \frac{z_i}{\sum_{i=1}^{n} z_i}$, where n is equal to the number of stocks consisting the optimum portfolio. Below an example with 16 stocks is presented. Monthly returns on the stocks listed below were selected from January 1980 to February 2001. Using the single index model we estimate the mean return of each stock. The results of the simple regressions for each stock against the market index (DJIA) are summarized in the next table. Also for these data the expected return of the market index is $\bar{R}_m = 0.01082$ and its variance is $\sigma_m^2 = 0.00192$. Assume $R_f = 0.005$

Stock i	$\hat{\alpha}_i$	$\hat{\beta}_i$	$\bar{R}_i$	$\sigma^2_{\epsilon i}$	$\frac{R_i - R_f}{\hat{\beta}_i}$
Consolidated Edison	0.01226	0.22198	0.01466	0.00297	0.04352
Merck & Co.	0.01065	0.79296	0.01923	0.00365	0.01794
Coca Cola Co.	0.01045	0.77550	0.01884	0.00337	0.01785
Johnson & Johnson	0.00855	0.84097	0.01765	0.00331	0.01504
Pepsi	0.00773	0.80803	0.01647	0.00354	0.01419
Texaco	0.00637	0.57284	0.01257	0.00396	0.01322
General Electric	0.00735	1.09940	0.01925	0.00174	0.01296
Ford Motor	0.00717	1.12085	0.01930	0.00509	0.01276
Procter & Gamble	0.00523	0.75287	0.01338	0.00316	0.01113
Citigroup	0.00502	1.33139	0.01943	0.00555	0.01084
MN Mining & Man. Co.	0.00382	0.88206	0.01336	0.00189	0.00948
Exxon Mobil	0.00387	0.63036	0.01069	0.00163	0.00903
Alcoa Inc.	0.00177	1.21786	0.01494	0.00505	0.00816
Boeing Co.	0.00087	1.09064	0.01267	0.00524	0.00703
Ibm	-0.00036	0.95736	0.01000	0.00437	0.00522
Xerox Corp.	-0.00857	1.10821	0.00342	0.00876	-0.00142

Using the previous table we compute the entries in the next table. The last column contains the C_i's.

Stock i	$\frac{\bar{R}_i - R_f}{\beta_i}$	$\frac{(\bar{R}_i - R_f)\beta_i}{\sigma^2_{\epsilon i}}$	$\sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\beta_j}{\sigma^2_{\epsilon j}}$	$\frac{\beta_i^2}{\sigma^2_{\epsilon i}}$	$\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma^2_{\epsilon j}}$	C_i
Consolidated Edison	0.04352	0.72225	0.72225	16.597	16.597	0.00135
Merck & Co.	0.01794	3.09081	3.81306	172.270	188.867	0.00538
Coca Cola Co.	0.01785	3.18560	6.99866	178.457	367.324	0.00788
Johnson & Johnson	0.01504	3.21456	10.21323	213.665	580.988	0.00927
Pepsi	0.01419	2.61795	12.83117	184.439	765.427	0.00998
Texaco	0.01322	1.09543	13.92660	82.865	848.292	0.01018
General Electric	0.01296	9.00346	22.93006	694.644	1542.936	0.01111
Ford Motor	0.01276	3.14906	26.07912	246.818	1789.754	0.01129
Procter & Gamble	0.01113	1.99621	28.07533	179.371	1969.125	0.01128
Citigroup	0.01084	3.46146	31.53679	319.387	2288.513	0.01123
MN Mining & Man. Co.	0.00948	3.90113	35.43792	411.656	2700.169	0.01100
Exxon Mobil	0.00903	2.20096	37.63888	243.775	2943.944	0.01086
Alcoa Inc.	0.00816	2.39780	40.03667	293.700	3237.644	0.01065
Boeing Co.	0.00703	1.59642	41.63309	227.003	3464.647	0.01045
Ibm	0.00522	1.09563	42.72872	209.734	3674.381	0.01019
Xerox Corp.	-0.00142	-0.19927	42.52945	140.197	3814.578	0.00981

We find from the previous table that $C^* = 0.01129$. Therefore the optimum portfolio consists of the first 8 ranked stocks. In solving this problem there is no need to fill in all the enries of the previous table. The reason is simple: Once we find the cut-off point C^* we can ignore all stocks that are ranked below the last stock included in the optimum portfolio.
We first find the values of z_i's:

$$z_1 = \frac{\beta_1}{\sigma^2_{\epsilon 1}} \left(\frac{\bar{R}_1 - R_f}{\beta_1} - C^* \right) = \frac{0.22198}{0.00297}(0.04352 - 0.01129) = 2.409.$$

Similarly $z_2 = 1.445$, $z_3 = 1.510$, $z_4 = 0.953$, $z_5 = 0.662$, $z_6 = 0.279$, $z_7 = 1.055$, and $z_8 = 0.324$. The sum of the z_i's is $\sum_{i=1}^{8} z_i = 8.636$. Therefore $x_1 = \frac{2.409}{8.636} = 0.28$, $x_2 = \frac{1.445}{8.636} = 0.17$, $x_3 = \frac{1.510}{8.636} = 0.17$, $x_4 = \frac{0.953}{8.636} = 0.11$, $x_5 = \frac{0.662}{8.636} = 0.08$, $x_6 = \frac{0.279}{8.636} = 0.03$, $x_7 = \frac{1.055}{8.636} = 0.12$, and $x_8 = \frac{0.324}{8.636} = 0.04$. We conclude that the optimum portfolio consists of 28% Consolidated Edison, 17% Merck, 17% Coca Cola, 11% Johnson & Johnson, 8% Pepsi, 3% Texaco, 12% General Electric, and 4% Ford Motor stocks.