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Statistics C183/C283 **Instructor: Nicolas Christou**

Results

Consider a portfolio consisting of n risky assets. When short sales allowed, the efficient frontier of all feasible portfolios which can be constructed from these n assets is defined as the locus of feasible portfolios that have the smallest variance for a prescribed expected return E is determined by solving the problem

$$\begin{array}{ll} \text{min} & \frac{1}{2}\mathbf{x}'\mathbf{\Sigma}\mathbf{x} \\ \text{subject to} & \bar{\mathbf{R}}'\mathbf{x} = E \\ \text{and} & \mathbf{1}'\mathbf{x} = 1 \end{array}$$

In the paper "An Analytic Derivation of the Efficient Portfolio Frontier," by Robert Merton the vector of the weights of the minimum variance portfolio is given by $\mathbf{x} = \mathbf{\Sigma}^{-1}[\lambda_1\bar{\mathbf{R}} + \lambda_2\mathbf{1}]$.

a. The vector $\mathbf{x}$ of the weights of the minimum variance portfolio can also be expressed as $\mathbf{x} = \mathbf{g} + \mathbf{h}E$, where $\mathbf{g}$ and $\mathbf{h}$ are $n \times 1$ vectors, given by

$$\begin{aligned} \mathbf{g} &= \frac{1}{D}\left[B\mathbf{\Sigma^{-1}1} - A\mathbf{\Sigma^{-1}}\bar{\mathbf{R}}\right] \\ \mathbf{h} &= \frac{1}{D}\left[C\mathbf{\Sigma^{-1}}\bar{\mathbf{R}} - A\mathbf{\Sigma^{-1}1}\right]. \end{aligned}$$

b. The frontier (and therefore the efficient frontier) can be generated from any two minimum variance portfolios. This is called the "mutual fund theorem". It follows that combinations of efficient portfolios are also efficient portfolios.

c. Consider two portfolios a, b on the efficient frontier (other than the minimum risk portfolio). It can be shown that the covariance between the returns of the two portfolios is given by

$$\mathrm{cov}(R_a, R_b) = \frac{C}{D}\left(E_a - \frac{A}{C}\right)\left(E_b - \frac{A}{C}\right) + \frac{1}{C}.$$

d. The vector of the weights of the global minimum variance portfolio is given by $\mathbf{x} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}$, where $\boldsymbol{\Sigma}$ is the $n \times n$ variance covariance matrix of the returns of the n stocks, and $\mathbf{1} = (1, 1, \ldots, 1)'$, ($n \times 1$ vector). This portfolio has expected return $\bar{R}_{pmin} = \frac{A}{C}$ and variance $\sigma^2_{pmin} = \frac{1}{C}$.

e. Find the covariance of the return of the global minimum variance portfolio with any asset or other portfolio.