University of California, Los Angeles
Department of Statistics & Data Science

Statistics C183/C283 **Instructor: Nicolas Christou**

Value at risk

- The value at risk (VaR) provides a single number to measure the risk of a portfolio.

- VaR is defined as "We are $(1-\alpha)\times\%$ certain that we will not lose more than V dollars in the next n days."

- If we assume that the change in the portfolio value follows a normal distribution, then the 1-day 97% VaR is the 3rd percentile of the distribution of the change in the portfolio.

- Find the 1-day VaR and then the VaR for the next n days is VaR(1-day) $\times \sqrt{n}$. Why?

- If the changes in the value of the portfolio in n successive days are i.i.d. $N(0, \sigma)$, then the sum of the changes follow $N(0, \sigma\sqrt{n})$. Suppose we want $1-\alpha = 0.95$ (95% VaR). Then the 10-day VaR is $-1.645\sigma\sqrt{10}$, but this is the same as VaR(1-day) $\times \sqrt{10}$, because -1.645σ is the 1-day 95% VaR.

- Bank regulators and VaR:
  The Basel Committee on Bank Regulators meets in Basel, Switzerland.
  The 1988 BIS (Bank for International Settlements) Accord (also called The Accord) which was an agreement between regulators and banks.
  The 1996 BIS Amendment calculates a bank's capital using VaR with $n = 10$ and $1-\alpha = 0.99$. The capital required is equal to $k \times$ 10-day VaR $= k \times$ 1-day VaR$\sqrt{10}$. Usually, $k = 3$, therefore the required capital is $3 \times \sqrt{10} = 9.49$ times the 1-day 99% VaR.

- Compute VaR using (i) historical simulations or (ii) a model approach.

Find the unbiased estimator of VaR. Assume that the change in the portfolio $\Delta P \sim N(0, \sigma)$. The unbiased estimator of σ^2 is S^2. Therefore the $1-\alpha$ VaR is $V = zS$, where z is the α percentile of $N(0, 1)$. For example if $1 - \alpha = 0.95$ it follows that $z = -1.645$. Is $E[V] = z\sigma$?

Historical simulations

We use past data on the assets in the portfolio to answer the question: “What might happen tomorrow?” Use data from the last 500 days, starting at day 0, day 1, …, day 499. Today is day 500 and we want to know the change in the portfolio on day 501 (tomorrow).

Day	Asset 1	Asset 2	Asset 3
0	P_{01}	P_{02}	P_{03}
1	P_{11}	P_{12}	P_{13}
2	P_{21}	P_{22}	P_{23}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
499	$P_{499,1}$	$P_{499,2}$	$P_{499,3}$
500	$P_{500,1}$	$P_{500,2}$	$P_{500,3}$
501	?	?	?

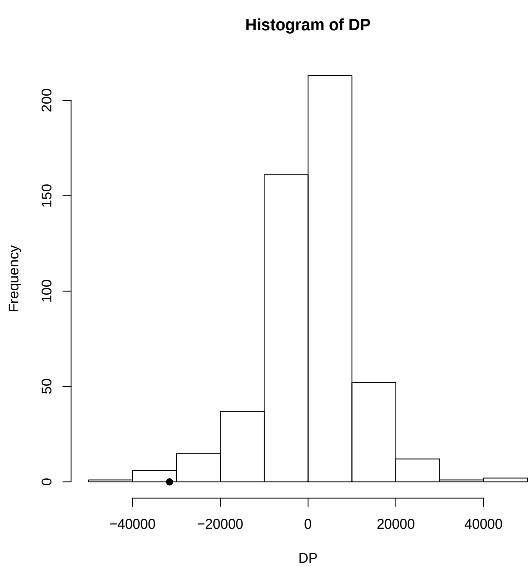
We have 500 scenarios to consider for the next day based on days 0-500. For example, using the first scenario, the value of asset 1 tomorrow will be $P_{500,1} \times \frac{P_{11}}{P_{01}}$ and the change in asset 1 will be $P_{500,1} \times (\frac{P_{11}}{P_{01}} - 1)$.

Similarly, for the second scenario, the change in asset 1 will be $P_{500,1} \times (\frac{P_{21}}{P_{11}} - 1)$.

For the last scenario, the change is asset 1 will be $P_{500,1} \times (\frac{P_{500,1}}{P_{499,1}} - 1)$.

Repeat for asset 2 and asset 3 and finally construct the histogram of the change in the portfolio using the 500 scenarios. The 1-day 99% VaR is the 1st percentile on the histogram (it corresponds to the 5th worst portfolio change) as shown below.

Example:
Suppose there are 5 stocks and we have invested \$200000 in each stock.



Model approach
Consider two assets A and B.

Suppose a portfolio consists of \$10000000 in asset A with daily volatility $\sigma = 0.02$. Therefore the standard deviation of the daily changes will be \$200000. Assume the expected value of the change to be zero. Under normality, compute the 99% 1-day VaR. What is the 10-day 99% VaR?

Suppose a portfolio consists of \$5000000 in asset B with daily volatility $\sigma = 0.01$. Therefore the standard deviation of the daily changes will be \$50000. Assume the expected value of the change to be zero. Under normality, compute the 99% 1-day VaR. What is the 10-day 99% VaR?

Suppose now we have a portfolio that consists of \$10000000 in asset A and \$5000000 in asset B. Assume that the returns on the two assets follow a bivariate normal distribution, which means that linear combinations of the two assets follow a univariate normal distribution with variance $\sigma_{A+B}^2 = \sigma_A^2 + \sigma_B^2 + 2\rho\sigma_A\sigma_B$. Using the data above, we have $\sigma_A = \$200000$ and $\sigma_A = \$50000$. Assume also that $\rho = 0.3$ and that the mean change is zero.

$\sigma_{A+B} =$

Compute the 1-day 99% VaR and then the 10-day 99% VaR.

How would the answer change if assets A and B were perfectly correlated ($\rho = 1$)?

Linear model

Suppose the value of the portfolio is P with n assets in the portfolio. Let r_i be the return on asset i in one day and let a_i be the amount invested in asset i. Then the change is the portfolio is given by $\Delta P = \sum_{i=1}^{n} a_i r_i = \mathbf{a}'\mathbf{r}$. Assume that $\mathbf{r} \sim N_n(\mathbf{0}, \mathbf{\Sigma})$.

It follows that $\Delta P \sim N(0, \sqrt{\mathbf{a}'\mathbf{\Sigma}\mathbf{a}})$ and the 1-day 99% VaR is $2.33 \times \sqrt{\mathbf{a}'\mathbf{\Sigma}\mathbf{a}}$.

Revisit the previous example of a portfolio that consists of the two assets A and B to compute the 1-day 99% VaR.

Linear model for options

Consider an option written on a single stock. The delta of a call option is the change in the call price divided by the change in the stock price: $\delta = \frac{\Delta P}{\Delta S}$ and therefore $\Delta P = \delta \Delta S$ is the portfolio change in one day. If we let $r = \frac{\Delta S}{S}$ be the percentage change in the stock price in one day then $\Delta P = \delta \times S \times r$.

If the portfolio consists of many options written on different stocks we then use $\Delta P = \sum_{i=1}^{n} \delta_i \times S_i \times r_i$. If we let $a_i = \delta_i \times S_i$ then $\Delta P = \sum_{i=1}^{n} a_i r_i = \mathbf{a}'\mathbf{r}$.

Example:

Assume two call options written on two stocks A and B. Let $S_A = \$120, S_B = \$30$, $\delta_1 = 1000, \delta_2 = 20000$. Also, assume $\sigma_A = 0.02, \sigma_B = 0.01, \rho = 0.3$.

Then $\Delta P = 120000 r_1 + 600000 r_2$. As before, assume that $\mathbf{r} \sim N_n(\mathbf{0}, \mathbf{\Sigma})$, therefore ΔP follows a univariate normal distribution.

Simulations based on the linear model
Steps:

- We know the value of the portfolio today.
- Take a random sample from $N_n(\mathbf{0}, \mathbf{\Sigma})$. This will give us one realization of the return vector $\mathbf{r}$.
- Compute $\Delta P = \mathbf{a}'\mathbf{r}$.
- Repeat the previous two steps many times to get many changes in the portfolio.
- Construct the histogram of the simulated changes in the portfolio and locate on the histogram the 1st percentile. This is the 1-day 99% VaR.

Note:
To take a random sample from multivariate normal distribution we use either the Cholesky decomposition or the spectral decomposition of a symmetric positive definite matrix, in this case the variance covariance matrix $\mathbf{\Sigma}$ of the returns vector $\mathbf{r}$.