LEC 1 intro

choose stock; first + second period, will only use second after week 5 but ensure they exist. the data gather deal with incontinuity from split (2-for-1, 3-for-1, 3-for-2)

the highest return portfolio is associated with the highest risk and the lowest uncertainty (risk) is associated with lowest return. Between these two extremes we find "efficient portfolios".

Define return (e.g., 2%) at time t of a stock:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \quad R_t = \frac{P_t + D - P_{t-1}}{P_{t-1}}$$

where P is price (closing price). D are the dividends paid between time t and t − 1 (adjusted price). Usually use monthly returns.

Define the mean and the variance of the returns of stock i:

$$\bar{R}_i = \frac{1}{n}\sum_{t=1}^{n} R_{it},\ \sigma_i^2 = \frac{1}{n-1}\sum_{t=1}^{n}(R_{it} - \bar{R}_i)^2$$

the covariance between the returns of stocks i and j as

$$cov(R_i, R_j) = \sigma_{ij} = \frac{1}{n-1}\sum_{t=1}^{n}(R_{it} - \bar{R}_i)(R_{jt} - \bar{R}_j)$$

Performance of the market: S&P 500 (old ones: DJIA)

Black Monday refers to Monday, October 19, 1987, when stock markets around the world crashed.

LEC2 min risk

2 stocks $\quad ax + \overset{\text{return}}{by} \quad (a+b=1)$

min $\quad Var(ax+by) = a^2\sigma_A^2 + b^2\sigma_B^2 + 2ab\,\sigma_{AB}$

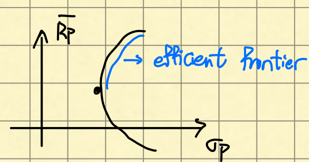
s.t. $\quad a+b=1 \quad b = 1-a$

min $\quad S = a^2\sigma_A^2 + (1-a)^2\sigma_B^2 + 2a(1-a)\sigma_{AB}$

$$\frac{ds}{da} = 0 \Rightarrow a^* = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} \qquad b^* = \frac{\sigma_A^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}$$

$$\bar{R_p} = a\bar{R_A} + b\bar{R_B}$$

$$\sigma_p^2 = a^2\sigma_A^2 + b^2\sigma_B^2 + 2ab\,\sigma_{AB}$$

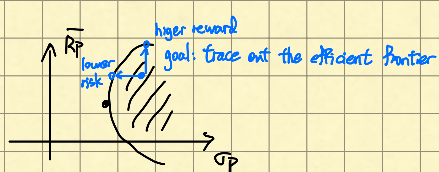


portfolio possibilities curve

3 stocks $\quad \overset{A}{a}x + \overset{B}{b}y + \overset{C}{c}z \quad (a+b+c=1)$

$$\sigma_p^2 = Var(ax+by+cz) = a^2\sigma_A^2 + b^2\sigma_B^2 + c^2\sigma_C^2 + 2ab\sigma_{AB} + 2ac\,\sigma_{AC} + 2bc\,\sigma_{BC}$$

$$\bar{R_p} = a\bar{R_A} + b\bar{R_B} + c\bar{R_C}$$



n stocks, find the composition of the minimum risk profolio

min $\quad \sigma_p^2 = X^T\Sigma X$

s.t. $\quad 1^TX = 1$

using the method of Lagrange multipliers

$$\min \quad S = X^T\Sigma X - 2\lambda(\mathbf{1}^TX - 1)$$

$$\frac{ds}{dX} = 0 = 2\Sigma X - 2\lambda\mathbf{1}$$

$$X = \lambda\Sigma^{-1}\mathbf{1}$$

constraint $1^TX = 1$

$$1 = \mathbf{1}^TX = \lambda\,\mathbf{1}^T\Sigma^{-1}\mathbf{1}$$

$$\lambda = \frac{1}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \qquad X^* = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

mean & variance of min risk profolio

$$\bar{R_p} = X^T\bar{R} = \frac{\mathbf{1}^T\Sigma^{-1}\bar{R}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}$$

matrix/vector decomposition

Let $\theta = \begin{pmatrix}\theta_1 \\ \vdots \\ \theta_n\end{pmatrix}$ and $f(\theta)$ be fn. of θ

$\frac{df(\theta)}{d\theta}$

$$\frac{df(\theta)}{d\theta} = \begin{pmatrix}\frac{df(\theta)}{d\theta_1} \\ \vdots \\ \frac{df(\theta)}{d\theta_n}\end{pmatrix}$$

① Let $f(\theta) = c^T\theta \quad . \quad \frac{df(\theta)}{d\theta} = c$

$$\sigma_p^2 = X^T \Sigma X = \frac{1^T(\Sigma^{-1})^T}{1^T\Sigma^{-1}1} \cdot \Sigma \frac{\Sigma^{-1}1}{1^T\Sigma^{-1}1}$$

② Let $f(\theta) = \underset{1\times p}{\theta^T} \underset{p\times p}{A} \underset{p\times 1}{\theta}$ $\quad \frac{d f(\theta)}{d\theta} = 2A\theta$

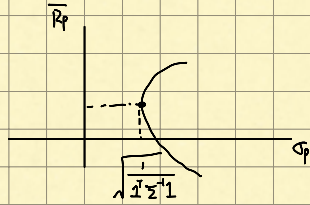
A: symmetric matrix

Σ is covariance. symmetric & pos def

$(\Sigma^{-1})^T = \Sigma^{-1}$

$$= \frac{1^T \Sigma^{-1}\Sigma \Sigma^{-1} 1}{(1^T\Sigma^{-1}1)^2}$$

$$= \frac{1}{1^T\Sigma^{-1}1}$$



$$\Sigma^{-1} = \begin{pmatrix} V_{11} & \cdots & V_{1n} \\ \vdots & \ddots & \\ V_{n1} & & V_{nn} \end{pmatrix} \qquad X_k = \frac{\sum_{j=1}^{n} V_{kj}}{\sum_{i=1}^{n}\sum_{j=1}^{n} V_{ij}}$$

diversified risk

let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be the vector of profolio weights

$$\bar{R} = \begin{pmatrix} \bar{R}_1 \\ \vdots \\ \bar{R}_n \end{pmatrix} \qquad Var(R) = \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & & & \\ \vdots & & & \\ \sigma_{n1} & & & \sigma_n^2 \end{pmatrix}$$

variance covariance matrix

covariance $= n^2 - n = n(n-1)$

$$R_p = X^T R = x_1 R_1 + \cdots + x_n R_n$$

$$\bar{R}_p = X^T \bar{R} = x_1 \bar{R}_1 + \cdots + x_n \bar{R}_n$$

$$\sigma_p^2 = Var(R_p) = Var(X^T R) = X^T \Sigma X$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} = \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n}\sum_{j\neq i}^{n} x_i x_j \sigma_{ij}$$

$$2\sum_{i=1}^{n-1}\sum_{j>i} x_i x_j \sigma_{ij}$$

assume equal allocation $= \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sigma_{ij}$

$$= \frac{\frac{\sum_{i=1}^{n}\sigma_i^2}{n}}{n} + \frac{n-1}{n}\frac{\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sigma_{ij}}{n(n-1)}$$

$$= \frac{\overline{\sigma_i^2}}{n} + \frac{n-1}{n}\overline{\sigma_{ij}}$$

$\sigma_p^2 \simeq \overline{\sigma_{ij}}$ when n is very large $\Rightarrow$ individual risk is diversified away

LEC 3 efficient frontier

Assume 2 stocks A, B

$$\bar{R}_p = a\bar{R}_A + b\bar{R}_B$$

$$\sigma_p^2 = a^2\sigma_A^2 + b^2\sigma_B^2 + 2ab\,\sigma_{AB}$$

$$\underset{[-1,1]}{\rho} = \frac{\sigma_{AB}}{\sigma_A\sigma_B} \longrightarrow \sigma_{AB} = \rho\,\sigma_A\,\sigma_B$$

suppose $\rho = +1$ $\qquad \sigma_p^2 = a^2\sigma_A^2 + b^2\sigma_B^2 + 2ab\,\sigma_{AB}$

in practice. there is a threshold $\rho = 0.9/0.95$ $\qquad 2ab\cdot\rho\sigma_A\sigma_B$

$$= (a\sigma_A + b\sigma_B)^2$$

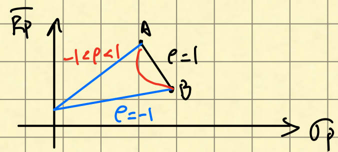
$$\sigma_p = a\sigma_A + (1-a)\sigma_B$$

$$a = \frac{\sigma_p - \sigma_B}{\sigma_A - \sigma_B} \qquad b = \frac{\sigma_A - \sigma_p}{\sigma_A - \sigma_B}$$

$$\bar{R}_p = \frac{\sigma_p - \sigma_B}{\sigma_A - \sigma_B}\bar{R}_A + \frac{\sigma_A - \sigma_p}{\sigma_A - \sigma_B}\bar{R}_B = \frac{\sigma_A \bar{R}_B - \sigma_B \bar{R}_A}{\sigma_A - \sigma_B} + \frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B}\sigma_p$$

suppose $\rho = -1$. $\sigma_p^2 = a^2\sigma_A^2 + b^2\sigma_B^2 - 2ab\,\sigma_A\sigma_B$

$\sigma_p^2 = (a\sigma_A - b\sigma_B)^2$



$$\min \quad \sigma_p^2 = X^T \Sigma X$$

$$\text{s.t.} \quad \sum_{i=1}^{n} X_i \bar{R}_i = E \quad \text{or} \quad \bar{R}^T X = E$$

$$\sum_{i=1}^{n} X_i = 1 \quad \text{or} \quad 1^T X = 1$$

using lagrange multipliers:

$$\min \; S = X^T \Sigma X - 2\lambda_1(\bar{R}^T X - E) - 2\lambda_2(1^T X - 1)$$

$$\frac{ds}{dx} = 2\Sigma X - 2\lambda_1 \bar{R} - 2\lambda_2 1 = 0$$

$$\underline{X = \Sigma^{-1}(\lambda_1 \bar{R} + \lambda_2 1) = \lambda_1 \Sigma^{-1}\bar{R} + \lambda_2 \Sigma^{-1} 1} \quad ①$$

$$X_k = \lambda_1 \sum_{j=1}^{n} v_{kj}\bar{R}_j + \lambda_2 \sum_{j=1}^{n} v_{kj}$$

Find λ_1, λ_2, multiply ① with $\bar{R}^T$

$$\bar{R}^T X = \bar{R}^T \Sigma^{-1}(\lambda_1 \bar{R} + \lambda_2 1)$$

$$E = \lambda_1 \underbrace{\bar{R}^T \Sigma^{-1} \bar{R}}_{B} + \lambda_2 \underbrace{\bar{R}^T \Sigma^{-1} 1}_{A} \quad \text{or} \quad \lambda_1 B + \lambda_2 A = E \quad \text{(page 1853 top of paper)}$$

E_j is $\bar{R}_j$

multiply ① with 1^T

$$\underbrace{1^T X}_{1} = 1^T \Sigma^{-1}(\lambda_1 \bar{R} + \lambda_2 1) \quad \text{or} \quad \lambda_1 A + \lambda_2 C = 1$$

$$C = 1^T \Sigma^{-1} 1$$

$$\begin{pmatrix} B & A \\ A & C \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} E \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} B & A \\ A & C \end{pmatrix}^{-1}\begin{pmatrix} E \\ 1 \end{pmatrix} = \frac{1}{BC - A^2}\begin{pmatrix} C & -A \\ -A & B \end{pmatrix}\begin{pmatrix} E \\ 1 \end{pmatrix}$$

$$\lambda_1 = \frac{CE - A}{BC - A^2} \qquad \lambda_2 = \frac{B - AE}{BC - A^2}$$

$D = BC - A^2 > 0$. (bottom of page 1853)

$$= \frac{CE - A}{D} \qquad = \frac{B - AE}{D}$$

to show that $D > 0$. begin with. $(A\bar{R} - B1)^T \underline{\Sigma^{-1}} (A\bar{R} - B1) > 0$

pos def. $^{-1}$ also pos def

$$(A\bar{R})^T \Sigma^{-1}(A\bar{R}) - (A\bar{R})^T \Sigma^{-1} B1 - (B1)^T \Sigma^{-1} A\bar{R} + (B1)^T \Sigma^{-1} B1 > 0$$

$$A^2 \underbrace{\bar{R}^T \Sigma^{-1} \bar{R}}_{B} - AB \underbrace{\bar{R}^T \Sigma^{-1} 1}_{A} - AB \underbrace{1^T \Sigma^{-1} \bar{R}}_{A} + B^2 \underbrace{1^T \Sigma^{-1} 1}_{C} > 0$$

$$A^2 B - 2A^2 B + B^2 C > 0 \qquad B > 0 \Rightarrow BC - A^2 > 0$$

multiply ① with $X^T\Sigma$

$$X^T\Sigma X = X^T \underbrace{\Sigma\Sigma^{-1}}_{I}(\lambda_1\bar{R}+\lambda_2 1)$$

$$\sigma_p^2 = \lambda_1 \underbrace{X^T\bar{R}}_{E} + \lambda_2 \underbrace{X^T 1}_{1}$$ (page 1854. eq (11))

or $$\sigma_p^2 = \frac{CE-A}{D}E + \frac{B-AE}{D} = \frac{CE^2-2AE+B}{D}$$ (page 1854. eq (12))

(E - σ_p^2 is parabola)

LEC 4

review:

The vector of the weights of the global minimum variance portfolio is given by $\mathbf{x} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}$, where $\mathbf{\Sigma}$ is the $n \times n$ variance covariance matrix of the returns of the n stocks, and $\mathbf{1} = (1, 1, \ldots, 1)'$, ($n \times 1$ vector). This portfolio has expected return $\bar{R}_{pmin} = \frac{A}{C}$ and variance $\sigma^2_{pmin} = \frac{1}{C}$.

$$\sigma_p^2 = \frac{CE^2-2AE+B}{D}$$

$$D = BC - A^2 > 0$$

$$(y = ax^2 + bx + c)$$

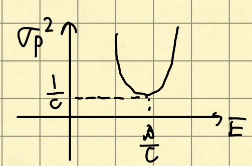
$$\frac{d\sigma_p^2}{dE} = \frac{2CE-2A}{D} = 0$$ then $$E = \frac{A}{C} = \frac{1^T\Sigma^{-1}\bar{R}}{1^T\Sigma^{-1}1}$$

$$\frac{d^2\sigma_p^2}{d^2E} = \frac{2C}{D} > 0 \Leftarrow C = 1^T\Sigma^{-1}1 > 0,\ D > 0$$

$$\sigma_p^2 = \frac{1}{C} = \frac{1}{1^T\Sigma^{-1}1}$$

This is global min risk

then $$\sigma_p^2 = \frac{C\left(\frac{A}{C}\right)^2 - 2A\left(\frac{A}{C}\right) + B}{D} = \frac{1}{C}$$



$$BC - A^2 = D$$

$$B = \frac{A^2}{C} + \frac{D}{C}$$

$$\sigma_p^2 = \frac{CE^2-2AE+B}{D}$$

$$= \frac{CE^2 - 2AE + \frac{A^2}{C} + \frac{D}{C}}{D}$$

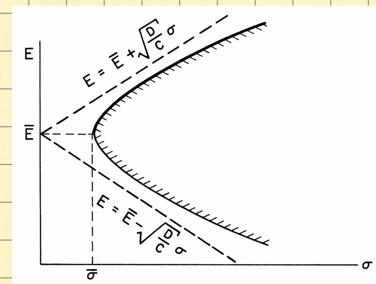
$$= \frac{\left(E^2 - 2\frac{A}{C}E + \frac{A^2}{C^2}\right)C + \frac{D}{C}}{D}$$

$$\sigma_p^2 = \frac{\left(E-\frac{A}{C}\right)^2 C + \frac{D}{C}}{D}$$

$$C\sigma_p^2 = \frac{\left(E-\frac{A}{C}\right)^2 C^2 + D}{D}$$

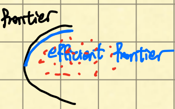
$$\frac{(\sigma-0)^2}{1/C} - \frac{\left(E-\frac{A}{C}\right)^2}{D/C^2} = 1$$ hyperbola

$$\frac{(E-\frac{A}{C})^2}{D/C^2} = C\sigma^2 - 1$$



$$(E-\frac{A}{C})^2 = \frac{D(C\sigma^2-1)}{C^2}$$

$$E = \frac{A}{C} \pm \frac{1}{C}\sqrt{D(C\sigma^2-1)}$$



rand / sum of rands

way to check: plot random X.

they should be at right of frontier

$$E = \frac{A}{C} \pm \frac{1}{C}\sqrt{DC(\sigma^2-\frac{1}{C})}$$

frontier

$$E = E_{min} \pm \frac{1}{C}\sqrt{DC(\sigma^2-\sigma^2_{min})}$$

$+ \rightarrow$ efficient frontier

$$X = \Sigma^{-1}(\lambda_1 \bar{R} + \lambda_2 \mathbb{1})$$

$$\lambda_1 = \frac{CE-A}{D} \qquad \lambda_2 = \frac{B-AE}{D}$$

A, B, C, D are fixed

mutual fund theorem

2 efficient frontier. combine ⇒ also on efficient frontier.

$$X = \lambda_1 \Sigma^{-1}\bar{R} + \lambda_2 \Sigma^{-1}\mathbb{1}$$

$$= \frac{CE-A}{D}\Sigma^{-1}\bar{R} + \frac{B-AE}{D}\Sigma^{-1}\mathbb{1}$$

$$X = \underbrace{\frac{1}{D}(B\Sigma^{-1}\mathbb{1} - A\Sigma^{-1}\bar{R})}_{g} + \underbrace{\frac{1}{D}(C\Sigma^{-1}\bar{R} - A\Sigma^{-1}\mathbb{1})}_{h}E$$

g: fixed for dataset

not depend on E

$$X = g + hE$$

consider 2 values of E: E_1, E_2

then $X_1 = g + hE_1$, $X_2 = g + hE_2$ } combine 2 protfolios to get a new protfolio

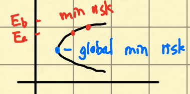
$$aX_1 + bX_2 \quad \text{with } a+b=1$$

$$a(g+hE_1) + (1-a)(g+hE_2) = g + h\underbrace{(aE_1 - aE_2 + E_2)}_{E^*}$$

$a, b,$	$\bar{R}_c$	σ_c
⋮	$a\bar{R}_1 + b\bar{R}_2$	$\sqrt{a^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\sigma_{12}}$

$$= g + hE^*$$

LEC 5



$R_a = g + hE_a$

$R_b = g + hE_b$

$$\text{Cov}(R_a, R_b) = \frac{C}{D}(E_a - \frac{A}{C})(E_b - \frac{A}{C}) + \frac{1}{C}$$

$$= X_a \Sigma X_b$$

$X_A = g + hE_A \qquad X_B = g + hE_B$

$$\sigma_{AB} = X_A^T \Sigma X_B = g^T\Sigma g + g^T\Sigma h E_B + E_A h^T \Sigma g + E_A E_B h^T \Sigma h$$

$$= \underbrace{g^T\Sigma g}_{\frac{B}{D}} + (E_A+E_B)\underbrace{g^T\Sigma h}_{-\frac{A}{D}} + E_A E_B \underbrace{h^T\Sigma h}_{\frac{C}{D}}$$

frontier w/ risk free asset

consider n risky assets

and one risk free asset with expected return R_f

we still minimize $\sigma_p^2 = X^T \Sigma X$. Note: $\mathbb{1}^T X \neq 1$

$X_1 + X_2 + \cdots + X_n + X_{n+1} = 1$

$X_{n+1} = 1 - \sum_{i=1}^{n} X_i$

$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$

$$\min \ \sigma_p^2 = X^T \Sigma X$$
$$\text{s.t.} \ \sum_{i=1}^{n} X_i \bar{R}_i + X_{n+1} \cdot R_F = E$$
$$\text{or} \ \sum_{i=1}^{n} X_i \bar{R}_i - \sum_{i=1}^{n} X_i R_F + R_F = E$$
$$\sum_{i=1}^{n} X_i (\bar{R}_i - R_F) + R_F = E$$

or

$$\min \ \sigma_p^2 = X^T \Sigma X$$
$$\text{s.t.} \ (\bar{R} - R_F \cdot \mathbf{1})^T X + R_F = E \quad \text{or} \quad \bar{R} X^T + (1 - \mathbf{1}^T X) R_F = E$$

Lagrange: $\min \ S = X^T \Sigma X - 2\lambda \left[(\bar{R} - R_F \cdot \mathbf{1})^T X + R_F - E \right]$

$$\frac{dS}{dX} = 2\Sigma X - 2\lambda \left[\bar{R} - R_F \cdot \mathbf{1} \right] = 0$$

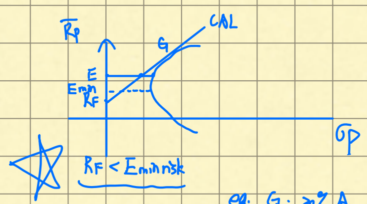
$X = \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1}) \lambda$. next. find λ

multiply both sides $(\bar{R} - R_F \cdot \mathbf{1})^T$

$$(\bar{R} - R_F \cdot \mathbf{1})^T X = (\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1}) \lambda$$

From constraint $(\bar{R} - R_F \cdot \mathbf{1})^T X + R_F = E$

$$(\bar{R} - R_F \cdot \mathbf{1})^T X = E - R_F$$



eg: G : 20% A . 30% B . 50% C

$\begin{cases} 80\% \ G \Rightarrow 16\% A \cdots \\ 20\% \ R_F \end{cases}$

$$\lambda = \frac{E - R_F}{(\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1})}$$

$$X = \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1}) \lambda$$
$$= \frac{(E - R_F) \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1})}{(\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1})}$$

Back to $\sigma_p^2 = X^T \Sigma X$

$$= \frac{(E - R_F)(\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} \Sigma (E - R_F) \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1})}{\left((\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1}) \right)^2}$$

$$\sigma_p^2 = \frac{(E - R_F)^2 (\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1})}{\left((\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1}) \right)^2}$$

$$= \frac{(E - R_F)^2}{(\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1})}$$

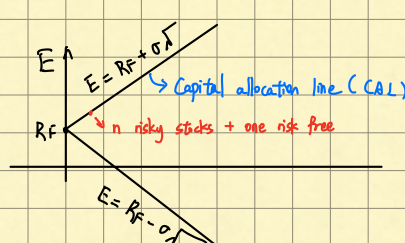
$$(E - R_F)^2 = \sigma_p^2 (\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1})$$

$$E - R_F = \pm \sigma_p \sqrt{(\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1})}$$

$$E = R_F \pm \sigma_p \sqrt{(\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1})}$$

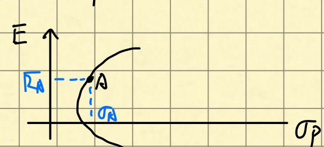
NOTE: $(\bar{R} - R_F \cdot \mathbf{1})^T \Sigma^{-1} (\bar{R} - R_F \cdot \mathbf{1}) = \underbrace{\bar{R}^T \Sigma^{-1} \bar{R}}_{B} - R_F \underbrace{\bar{R}^T \Sigma^{-1} \mathbf{1}}_{A} - R_F \underbrace{\mathbf{1}^T \Sigma^{-1} \bar{R}}_{A} + R_F^2 \underbrace{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}_{C}$

$$E = R_F \pm \sigma_p \sqrt{B - 2 R_F A + R_F^2 C}$$



a different method : tangent to the efficient frontier of n risky stocks

First consider profolio A on the efficient frontier



combine A with the risk free asset to get a new portfolio

let X be the investor's wealth invested in A

$1-X$ $\qquad$ R_F

$$\bar{R}_C = X\bar{R}_A + (1-X)R_F$$

$$\sigma_C^2 = X^2\sigma_A^2$$

$$\sigma_C = X\sigma_A$$

$$X = \frac{\sigma_C}{\sigma_A}$$

$$\bar{R}_C = \frac{\sigma_C}{\sigma_A}\bar{R}_A + \left(1-\frac{\sigma_C}{\sigma_A}\right)R_F$$

$$\bar{R}_C = R_F + \frac{\bar{R}_A - R_F}{\sigma_A}\sigma_C$$

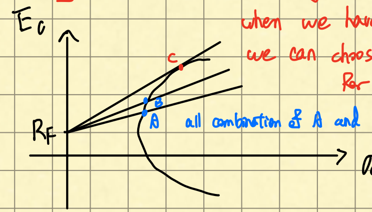
A is arbitray point on efficient frontier

others can choose B, C

$C > B > A$. optimal is always C.

so only one combination of risky assets is optimal when we have risk free assets.

we can choose X and $1-X$ for C and R_F



all combination of A and R_F lies on line

all efficient portfolios on CAL are perfectly correlated

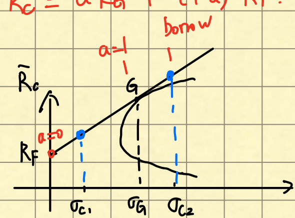
$$R_{p_1} = X_1^T R + (1 - 1^T X_1) R_F$$

$$R_{p_2} = X_2^T R + (1 - 1^T X_2) R_F$$

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2} = \frac{Cov(X_1^T R + const,\ X_2^T R + const)}{\sqrt{Var(X_1^T R + const)}\sqrt{Var(X_2^T R + const)}} = \frac{X_1^T \Sigma X_2}{\sqrt{X_1^T\Sigma X_1}\sqrt{X_2^T\Sigma X_2}} = 1$$

LEC 6

$$\bar{R}_C = a\bar{R}_G + (1-a)R_F.$$



σ_c $(C = G + R_F)$

$$\sigma_C = x\sigma_G$$

$$x = \frac{\sigma_c}{\sigma_G} \quad \begin{matrix} >1 \ (\sigma_{c_1} \text{ land money}) \\ <1 \ (\sigma_{c_2} \text{ borrow money}) \end{matrix}$$

$$\bar{R}_C = R_F + \frac{\bar{R}_P - R_F}{\sigma_P}\sigma_c$$

R_F

max $\theta = \dfrac{\bar{R}_P - R_F}{\sigma_P}$ $\to$ slope of

s.t. $\sum X_i = 1$

or max $\theta = \dfrac{\bar{R}_P - \overbrace{\sum X_i}^{=1} R_F}{\sigma_P}$ or max $\theta = \dfrac{\sum X_i\bar{R}_i - \sum X_i R_F}{(X^T\Sigma X)^{\frac{1}{2}}}$ or max $\theta = \dfrac{\sum(\bar{R}_i - R_F)X_i}{(X^T\Sigma X)^{\frac{1}{2}}}$

$$\max \quad \theta = \frac{(\bar{R} - R_F\cdot\mathbb{1})^T X}{(X^T\Sigma X)^{\frac{1}{2}}}$$

$$\frac{\partial\theta}{\partial X} = \frac{(\bar{R} - R_F\cdot\mathbb{1})(X^T\Sigma X)^{\frac{1}{2}} - \frac{1}{2}(X^T\Sigma X)^{-\frac{1}{2}}2\Sigma X(\bar{R} - R_F\mathbb{1})^T X}{X^T\Sigma X} = 0$$

$$(\bar{R} - R_F\cdot\mathbb{1})(X^T\Sigma X)^{\frac{1}{2}} - (X^T\Sigma X)^{-\frac{1}{2}}\Sigma X(\bar{R} - R_F\mathbb{1})^T X = 0$$

multiply both by $(X^T\Sigma X)^{\frac{1}{2}}$

$$(\bar{R} - R_F\cdot\mathbb{1})\sigma_P^2 - \Sigma X(\bar{R} - R_F\mathbb{1})^T X = 0 \to (\bar{R} - R_F\cdot\mathbb{1})\sigma_P^2 = \Sigma X\lambda\sigma_P^2$$

let $\lambda = \dfrac{(\bar{R} - R_F\mathbb{1})^T X}{\sigma_P^2}$

$$(\bar{R} - R_F\mathbb{1})^T X = \lambda\sigma_P^2$$

$$\bar{R} - R_F\cdot\mathbb{1} = \Sigma\underbrace{X\lambda}$$

$$Z = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}\lambda$$

Aside $(\bar{R} - R_F 1) = \begin{pmatrix} \bar{R}_1 - R_F \\ \bar{R}_2 - R_F \\ \cdots \\ \bar{R}_n - R_F \end{pmatrix}$ axces return

$$\bar{R} - R_F \cdot 1 = \Sigma Z$$

$$Z = \Sigma^{-1}(\bar{R} - R_F \cdot 1)$$

we need X

find λ

$$\lambda X = \Sigma^{-1}(\bar{R} - R_F \cdot 1)$$

multiply 1^T

$$\lambda 1^T X = 1^T \Sigma^{-1}(\bar{R} - R_F \cdot 1)$$

$1^T X = 1$

$$\lambda = 1^T \Sigma^{-1}(\bar{R} - R_F \cdot 1) = 1^T Z$$

Finally, $X = \frac{\Sigma^{-1}(\bar{R} - R_F \cdot 1)}{\lambda} = \frac{\Sigma^{-1}(\bar{R} - R_F \cdot 1)}{1^T \Sigma^{-1}(\bar{R} - R_F \cdot 1)} = \frac{Z}{1^T Z}$

this is the composition of protfolio G (tangency point to the efficient frontier)

invest 100% into risky assets

$$a \underbrace{\bar{R}_G}_{X^T \bar{R}} + (1-a) R_F = 3\%$$

can solve a

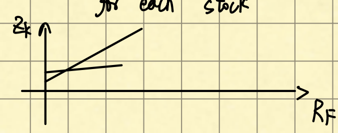
$$Z = \Sigma^{-1}(\bar{R} - R_F \cdot 1)$$

$$Z = \Sigma^{-1}\bar{R} - R_F \Sigma^{-1} 1$$

$$Z = \sum_{j=1} V_{kj} \bar{R}_j - R_F \sum_{j=1}^{n} V_{kj}$$

$$\overbrace{Z_k}^{\lambda X_k} = C_{0k} - C_{1k} R_F$$

C_{0k} and C_{1k} are unique and fixed for each stock



LEC 7

Review

① global min risk protfolio $X_{min} = \frac{\Sigma^{-1} 1}{1^T \Sigma^{-1} 1}$ (short sales allowed. no R_F)

do not allow short sales ⇒ $X_i \geq 0$.
allow short sales can achieve larger return.
but extra return comes with larger risk
efficient frontier no finite upper bound

② min risk protfolio for a given E $\quad X = \Sigma^{-1}(\lambda_1 R + \lambda_2 1)$. $\lambda_1 \lambda_2$ defined in E

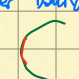
$$X = g + hE$$

③ hyperbola. $E = E_{min} \pm \frac{1}{C}\sqrt{DC(\sigma^2 - \sigma_{min}^2)}$

$E_{min} = X_{min}^T R$

$\sigma_{min}^2 = \frac{1}{C}$

$\sigma_{min}^2 = X_{min}^T \Sigma X_{min}$

short sales allowed ⇒ 

short sales not allowed ⇏

④ mutual fund theorem
① 2E
② many Es
trace frontier

$X_A = g + hE_a$, $\quad \bar{R}_A = X_A^T R_A$, $\sigma_A^2 = X_A^T \Sigma X_A$

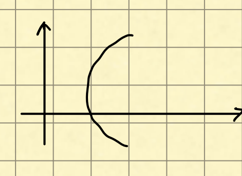
$X_B = g + hE_b$, $\quad \bar{R}_B = X_B^T R_B$, $\sigma_B^2 = X_B^T \Sigma X_B$

same

$\sigma_{AB} = X_A^T \Sigma X_B$ (short sales allowed. have R_F)

$\sigma_P = \sqrt{\frac{CE^2 - 2AE + B}{D}}$

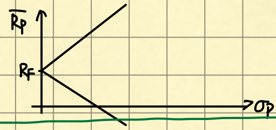
$(a+b=1)$

a b $\quad R_P = a\bar{R}_A + b\bar{R}_B \quad \sigma_P = \sqrt{a^2\sigma_A^2 + b^2\sigma_B^2 + 2ab\sigma_{AB}}$ ⇒ 

⋮ ⋮ ⋮ ⋮

③ Risk free asset exist

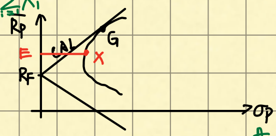
$$X = \frac{(E - R_F)\,\Sigma^{-1}(\bar{R} - R_F \mathbf{1})}{(\bar{R} - R_F\mathbf{1})\,\Sigma^{-1}(\bar{R} - R_F\mathbf{1})}$$



shape (n)

$X_{n+1} = 1 - \sum_{i=1}^{n} X_i$

④ Tangent



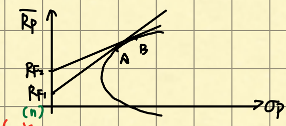
$$Z = \Sigma^{-1}(\bar{R} - R_F \cdot \mathbf{1})$$

$$X_G = \frac{Z}{\mathbf{1}^T Z} = \frac{\Sigma^{-1}(\bar{R} - R_F\cdot\mathbf{1})}{\mathbf{1}^T\Sigma^{-1}(\bar{R} - R_F\cdot\mathbf{1})}$$

(n)

$1 = \sum_{i=1}^{n} X_{G_i}$

$\mathbf{1}^T X_G = 1 \Rightarrow$ 0.6 G, 0.4 RF $\Rightarrow \underset{(n)}{X} = 0.6 \cdot \underset{(n)}{X_G}$



trace frontier ⑤ 2 RFs

$$Z_A = \Sigma^{-1}(\bar{R} - R_{F_1}\cdot\mathbf{1}) \to X_A = \frac{Z_A}{\mathbf{1}^T Z_A} \to \bar{R}_A \;\; \sigma_A^2$$

0% invested to risk free assets

$\sigma_{AB} \Rightarrow$ can trace out efficient frontier

$$Z_B = \Sigma^{-1}(\bar{R} - R_{F_2}\cdot\mathbf{1}) \to X_B = \frac{Z_B}{\mathbf{1}^T Z_B} \to \bar{R}_B \;\; \sigma_B^2$$

Single Index Model (SIM)

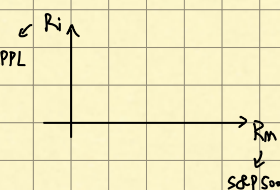
$R_i = a_i + \beta_i R_m$

Let $a_i = \alpha_i + \varepsilon_i$

with $E\,\varepsilon_i = 0$

$Var(\varepsilon_i) = \sigma_{\varepsilon i}^2$

Single Index Model (SIM) $R_i = \alpha_i + \beta_i R_m + \varepsilon_i$



$R_i = \alpha_i + \beta_i R_m + \varepsilon_i$

- R_i ↓ return of stock i
- R_m ↓ market return (random)
- ε_i ↓ error term

assumption: R_m, ε_i uncorrelated. $Cov(R_m, \varepsilon_i) = 0$

$\varepsilon_i, \varepsilon_j$ uncorrelated $Cov(\varepsilon_i, \varepsilon_j) = 0$

the only association between 2 stocks is due to the market

Notation: $\bar{R}_m = E(R_m)$ market/index

$\sigma_m^2 = Var(R_m)$

LEC 8

$R_p = X^T R = \sum X_i R_i$

Aside: $\bar{R}_i = \alpha_i + \beta_i \bar{R}_m$

$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2$ constant has zero cov with everything

$$\begin{aligned}\sigma_{ij} = Cov(R_i, R_j) &= Cov(\alpha_i + \beta_i R_m + \varepsilon_i,\ \alpha_j + \beta_j R_m + \varepsilon_j)\\ &= Cov(\beta_i R_m + \varepsilon_i,\ \beta_j R_m + \varepsilon_j)\\ &= \beta_i\beta_j\, cov(R_m, R_m) + \beta_i \underbrace{Cov(R_m, \varepsilon_j)}_{0} + \beta_j \underbrace{Cov(\varepsilon_i, R_m)}_{0} + \underbrace{Cov(\varepsilon_i, \varepsilon_j)}_{0}\\ &= \beta_i\beta_j\sigma_m^2\end{aligned}$$

mean & variance of the portfolio using SIM

$$R_p = X^T R = \sum X_i R_i$$

$$\bar{R}_p = \sum X_i \bar{R}_i = \sum X_i(\alpha_i + \beta_i \bar{R}_m)$$

$$\bar{R}_p = \sum X_i \alpha_i + \bar{R}_m \sum X_i \beta_i$$

or $\bar{R}_p = \alpha_p + \beta_p \bar{R}_m$

$$\sigma_p^2 = X^T\Sigma X = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\, Cov(R_i, R_j)$$

$$= \sum_{i=1}^{n} X_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j\neq i}^{n} X_i X_j \sigma_{ij} = \sum_{i=1}^{n} X_i^2(\beta_i^2\sigma_m^2 + \sigma_{\varepsilon i}^2) + \sum_{i=1}^{n}\sum_{j\neq i}^{n} X_i X_j \beta_i\beta_j\sigma_m^2$$

$$= \sigma_m^2\sum_{i=1}^{n} X_i^2\beta_i^2 + \sum X_i^2\sigma_{\varepsilon i}^2 + \sigma_m^2\sum_{i=1}^{n}\sum_{j\neq i}^{n} X_i X_j\beta_i\beta_j$$

$$= \sum_{i=1}^{n} X_i^2\sigma_{\varepsilon i}^2 + \sigma_m^2\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j \beta_i\beta_j$$

$$= \sum_{i=1}^{n} X_i^2\sigma_{\varepsilon i}^2 + \sigma_m^2\left(\sum_{i=1}^{n} X_i\beta_i\right)\left(\sum_{j=1}^{n} X_j\beta_j\right)$$

Finally $\sigma_p^2 = \sum_{i=1}^{n} X_i^2\sigma_{\varepsilon i}^2 + \beta_p^2\sigma_m^2$

Assume equal allocation: $X_i = \frac{1}{n}$, $i = 1 \cdots n$

$$\sigma_p^2 \simeq \beta_p^2\sigma_m^2$$

$$\begin{cases} \sigma_p^2 = \beta_i^2\sigma_m^2 + \sigma_{\varepsilon i}^2 \\ \text{diversifiable / non-market / unsystematic risk} \Rightarrow \sigma_{\varepsilon i}^2 \\ \text{non-diversifiable / market / systematic risk} \Rightarrow \beta_i^2\sigma_m^2 \end{cases}$$

suppose $n=3$ $\quad \bar{R} = \begin{pmatrix} \alpha_1 + \beta_1 \bar{R}_m \\ \alpha_2 + \beta_2 \bar{R}_m \\ \alpha_3 + \beta_3 \bar{R}_m \end{pmatrix}$

$$\Sigma = \begin{pmatrix} \beta_1^2\sigma_m^2 + \sigma_{\varepsilon 1}^2, & \beta_1\beta_2\sigma_m^2, & \beta_1\beta_3\sigma_m^2 \\ \beta_2\beta_1\sigma_m^2, & \beta_2^2\sigma_m^2 + \sigma_{\varepsilon 2}^2, & \beta_2\beta_3\sigma_m^2 \\ \beta_3\beta_1\sigma_m^2, & \beta_3\beta_2\sigma_m^2, & \beta_3^2\sigma_m^2 + \sigma_{\varepsilon 3}^2 \end{pmatrix}$$

Σ_n R: $\quad B = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \quad \sigma_m^2 BB^T + \text{diag}(\varepsilon)$

no model $\Rightarrow$ $n=250$ stocks. $\to$ 31625 mean / var / cov $\left(250 + 250 + \frac{250 \cdot 249}{2}\right)$

SIM $\Rightarrow$ $n=250$ stocks. $\to$ 752 $\left(250\,\alpha + 250\,\beta + 250\,\sigma_{\varepsilon i}^2 + \bar{R}_m + \bar{\sigma}_m\right)$

Estmation: $\quad R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}$. $t = 1, 2 \cdots m$ (m months)

$\Sigma R_{mt} R_{it} - \underbrace{\bar{R}_i}_{\frac{1}{m}\Sigma R_{it}} \Sigma R_{mt} - \underbrace{\bar{R}_m}_{\frac{1}{m}\Sigma R_{mt}} \Sigma R_{it} + m \underbrace{\bar{R}_i}_{\frac{1}{m}\Sigma R_{it}} \underbrace{\bar{R}_m}_{\frac{1}{m}\Sigma R_{mt}}$

$$\hat{\beta}_i = \frac{\sum_{t=1}^m (R_{mt} - \bar{R}_m)(R_{it} - \bar{R}_i)}{\sum_{t=1}^m (R_{mt} - \bar{R}_m)^2} = \frac{\Sigma R_{mt} R_{it} - \frac{1}{m}(\Sigma R_{mt})(\Sigma R_{it})}{\Sigma R_{mt}^2 - \frac{(\Sigma R_{mt})^2}{m}}$$

$\Sigma R_{mt}^2 - 2\underbrace{\bar{R}_m}_{\frac{1}{m}\Sigma R_{mt}} \Sigma R_{mt} + m \underbrace{\bar{R}_m^2}_{(\frac{1}{m}\Sigma R_{mt})^2}$

$$\hat{\beta}_i = \frac{\Sigma (R_{mt} - \bar{R}_m) R_{it} \left(- \bar{R}_i \cdot \underbrace{\Sigma (R_{mt} - \bar{R}_m)}_{0}\right)}{\Sigma (R_{mt} - \bar{R}_m)^2} \quad \text{or} \quad \hat{\beta}_i = \frac{\text{Cov}(R_i, R_m)}{\sigma_m^2}$$

min $\quad S = \sum_{t=1}^m (R_{it} - \alpha_i - \beta_i R_{mt})^2$

① $\frac{\partial S}{\partial \alpha_i} = -2 \sum_{t=1}^m (R_{it} - \alpha_i - \beta_i R_{mt}) = 0$

$\sum_{t=1}^m R_{it} - \beta_i \sum_{t=1}^m R_{mt} = m \cdot \alpha_i$

$\alpha_i = \bar{R}_i - \beta_i \bar{R}_m$

② $\frac{\partial S}{\partial \beta_i} = \cancel{-2} \sum_{t=1}^m (R_{it} - \alpha_i - \beta_i R_{mt}) R_{mt} = 0$

$\sum_{t=1}^m (R_{it} - \bar{R}_i - \beta_i \bar{R}_m - \beta_i R_{mt}) R_{mt} = 0$

$\sum_{t=1}^m (R_{it} - \bar{R}_i) R_{mt} - \beta_i \sum_{t=1}^m (R_{mt} - \bar{R}_m) R_{mt} = 0$

$\sum_{t=1}^m (R_{it} - \bar{R}_i)(R_{mt} - \bar{R}_m + \bar{R}_m) \qquad \beta_i \sum_{t=1}^m \left[(R_{mt} - \bar{R}_m)^2 + \bar{R}_m \underbrace{(R_{mt} - \bar{R}_m)}_{0}\right]$

$\sum_{t=1}^m (R_{it} - \bar{R}_i)(R_{mt} - \bar{R}_m) + \underbrace{\sum_{t=1}^m (R_{it} - \bar{R}_i)}_{0} \bar{R}_m$

$$\beta_i = \frac{\sum_{t=1}^m (R_{it} - \bar{R}_i)(R_{mt} - \bar{R}_m)}{\sum_{t=1}^m (R_{mt} - \bar{R}_m)^2} = \frac{\text{Cov}(R_i, R_m)}{\text{Var}(R_m)}$$

project 31×31

$$\begin{pmatrix} \overset{\text{market}}{\sigma_m^2} & \overbrace{\sigma_{1m} \cdots \sigma_{30\,m}}^{\text{divide 30 stocks}} \\ & \\ & \end{pmatrix} \Rightarrow \text{30×1 vector of } \beta_i$$

$\hat{\alpha}_i = \bar{R}_i - \hat{\beta}_i \bar{R}_m$

the fitted line: $\quad \hat{R}_{it} = \hat{\alpha}_{it} + \hat{\beta}_i R_{mt}$

residuals: $\quad e_{it} = R_{it} - \hat{R}_{it}$

properties: ① $\sum_{t=1}^m e_{it} = 0$

② $\sum_{t=1}^m e_{it} R_{mt} = 0$

③ $\sum_{t=1}^m e_{it} \hat{R}_{it} = 0 \; = \hat{\alpha}_i \Sigma e_{it} + \hat{\beta}_i \Sigma e_{it} R_{mt}$

LEC 9

$$\text{var}(\hat{\beta}_i) = \frac{\sigma_{\varepsilon i}^2}{\sum_t (R_{mt} - \bar{R}_m)^2}$$

$$\hat{\beta}_i = \frac{\sum_{t=1}^m (R_{mt} - \bar{R}_m)(R_{it} - \bar{R}_i)}{\sum_{t=1}^m (R_{mt} - \bar{R}_m)^2} = \frac{\Sigma (R_{mt} - \bar{R}_m)\left((\alpha_i + \beta_i R_{mt} + \varepsilon_{it}) - (\alpha_i + \beta_i \bar{R}_m + \bar{\varepsilon}_i)\right)}{\Sigma (R_{mt} - \bar{R}_m)^2} \qquad \bar{\varepsilon}_i = \frac{1}{m} \Sigma \varepsilon_{it}$$

$$= \frac{\Sigma (R_{mt} - \bar{R}_m)\left(\beta_i (R_{mt} - \bar{R}_m) + (\varepsilon_{it} - \bar{\varepsilon}_i)\right)}{\Sigma (R_{mt} - \bar{R}_m)^2} = \frac{\beta_i \Sigma (R_{mt} - \bar{R}_m)^2 + \Sigma (R_{mt} - \bar{R}_m)\varepsilon_{it} - \bar{\varepsilon}_i \overbrace{\Sigma (R_{mt} - \bar{R}_m)}^{0}}{\Sigma (R_{mt} - \bar{R}_m)^2}$$

$$= \beta_i + \frac{\Sigma (R_{mt} - \bar{R}_m)\varepsilon_{it} \text{ - random}}{\Sigma (R_{mt} - \bar{R}_m)^2}$$

$$\text{Var}(\hat{\beta}_i) = \frac{\sigma_{\varepsilon i}^2 \cancel{\Sigma (R_{mt} - \bar{R}_m)^2}}{\left(\Sigma (R_{mt} - \bar{R}_m)^2\right)^{\cancel{2}}}$$

$$Var(\hat{\alpha}_i) = \sigma_{\varepsilon i}^2 \left(\frac{1}{m} + \frac{\bar{R}_m^2}{\sum_t (R_{mt} - \bar{R}_m)^2} \right)$$

$$\hat{\alpha}_i = \bar{R}_i - \hat{\beta}_i \bar{R}_m = \alpha_i + \beta_i \bar{R}_m + \bar{\varepsilon}_i - \hat{\beta}_i \bar{R}_m$$

$$\hat{\alpha}_i - \alpha_i = \bar{\varepsilon}_i - \bar{R}_m (\hat{\beta}_i - \beta_i)$$

$$Var(\alpha_i) = Var(\bar{\varepsilon}_i) + \bar{R}_m^2 Var(\beta_i) - 2\bar{R}_m Cov(\bar{\varepsilon}_i, \hat{\beta}_i - \beta_i)$$

$\bar{\varepsilon}_i = \frac{1}{m}\sum \varepsilon_{it}$ 　　 $Cov(\bar{\varepsilon}_i, \hat{\beta}_i - \beta_i) = 0$

$$= \frac{1}{m^2} \sum Var(\varepsilon_{it})$$

$$= \frac{1}{m^2} \cdot m \cdot \varepsilon_i^2$$

$$\sum (R_{it} - \bar{R}_i)^2 = \sum_t (R_{it} - \hat{R}_{it})^2 + \sum_t (\hat{R}_{it} - \bar{R}_i)^2$$

$$= \sum_t e_{it}^2 + \hat{\beta}_i^2 \sum_t (R_{mt} - \bar{R}_m)^2$$

$$SST = SSE + SSR$$

Sum square total/error/regression

Note: $\hat{R}_{it} = \underbrace{\hat{\alpha}_i}_{\bar{R}_i - \hat{\beta}_i \bar{R}_m} + \hat{\beta}_i R_{mt} = \bar{R}_i + \hat{\beta}_i (R_{mt} - \bar{R}_m)$

R^2 (coeff of determination) $= \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$ 　 $0 \le R^2 \le 1$

usually 30% ~ 40% in SIM 　　 perfect fit

$w = R^2 \sim BETA\left(\frac{1}{2}, \frac{1}{2}(m-2)\right)$, see handout #18

Note: $y \sim BETA(\alpha, \beta)$

$$f(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}$$

$B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} dy$

Beta fn

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

and gamma fn: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$

$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$

if α is integer. $\Gamma(\alpha) = (\alpha-1)!$

$$Cov(\hat{\alpha}_i, \hat{\beta}_i) = -\frac{\sigma_{\varepsilon i}^2 \bar{R}_m}{\sum (R_{mt} - \bar{R}_m)^2}$$

$$Cov(\bar{\varepsilon}_i - \bar{R}_m(\hat{\beta}_i - \beta), \hat{\beta}_i - \beta) = \underbrace{Cov(\bar{\varepsilon}_i, \hat{\beta}_i - \beta)}_{0} - \bar{R}_m Var(\hat{\beta}_i - \beta)$$

MML (max. likelihood)

$\varepsilon_{it} \sim N(0, \sigma_{\varepsilon it})$

$$L = (2\pi\sigma_{\varepsilon i}^2)^{-\frac{m}{2}} e^{-\frac{1}{2\sigma_{\varepsilon i}^2}\sum (R_{it} - \alpha_i - \beta_i R_{mt})^2}$$

$R_{it} \sim N(\alpha_i + \beta_i R_{mt}, \sigma_{\varepsilon i})$

$$f(R_{it}) = \frac{1}{\sigma_{\varepsilon it}\sqrt{2\pi}} e^{-\frac{1}{2\sigma_{\varepsilon it}^2}(R_{it} - \alpha_i - \beta_i R_m)^2}$$

PDF

(skip $\hat{\alpha}_i$ $\hat{\beta}_i$. the same)

$X \sim N(\mu, \sigma)$

PDF $f(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$L(\mu,\sigma|x_1) = f(x_1|\mu,\sigma)$

$L(\mu,\sigma|x_1 \cdots x_n) = \prod_{i=1}^n f(x_i|\mu,\sigma)$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum (x_i - \mu)^2\right)$$

max log likelihood $\ln L = -\frac{m}{2}\ln(2\pi\sigma_{\varepsilon it}^2) - \frac{1}{2\sigma_{\varepsilon i}^2}\sum (R_{it} - \alpha_i - \beta_i R_{mt})^2$

MLE of $\sigma_{\varepsilon i}^2$ 　 $\frac{d \ln L}{d\sigma_{\varepsilon i}^2} = -\frac{m}{2\sigma_{\varepsilon i}^2} + \frac{1}{2\sigma_{\varepsilon i}^4}\sum (R_{it} - \alpha_i - \beta_i R_{mt})^2 = 0$

$$\hat{\sigma}_{\varepsilon i}^2 = \frac{\Sigma(R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{mt})^2}{m} = \frac{\Sigma e_{it}^2}{m}$$ not unbiased

instead use $\hat{\sigma}_{\varepsilon i}^2 = \frac{\Sigma e_{it}^2}{m-2}$

c. Estimate of $\sigma_{\epsilon_i}^2$ (variance of random error term associated with stock i):

$$\hat{\sigma}_{\epsilon_i}^2 = \frac{\sum_{t=1}^m e_{it}^2}{m-2} = \frac{\sum_{t=1}^m (R_{it} - \hat{\alpha}_i - \hat{\beta}_i R_{mt})^2}{m-2}.$$

m observasions

R_{it} 2 parameters $\Rightarrow$ 2 degree of freedom

in simple regression $r^2 = R^2$. $r = \frac{Cov(R_i, R_m)}{\sqrt{Var(R_i)}\sqrt{Var(R_m)}}$. $-1 \le r \le 1$

r: corr between stock & market

R^2: coeff of determination from regression

e. Correlation between stock i and stock j:

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} = \frac{\beta_i \beta_j \sigma_m^2}{\sigma_i \sigma_j}.$$

f. Correlation between stock i and market:

$$\rho_{im} = \beta_i \frac{\sigma_m}{\sigma_i} \Rightarrow \beta_i = \rho_{im} \frac{\sigma_i}{\sigma_m}.$$

centered SIM $\quad R_{it} = \gamma_i + \beta_i \underbrace{(R_{mt} - \bar{R}_m)}_{Z_{mt}} + \varepsilon_{it}$

(non centered model $R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}$)

$$\hat{\beta}_i = \frac{\Sigma(Z_{mt} - \bar{Z}_m) R_{it}}{\Sigma(Z_{mt} - \bar{Z}_m)^2} = \frac{\Sigma Z_{mt} R_{it}}{\Sigma Z_{mt}^2} = \frac{\Sigma(R_{mt} - \bar{R}_m) R_{it}}{\Sigma(R_{mt} - \bar{R}_m)^2}$$

but $\bar{Z}_m = \frac{\Sigma(R_{mt} - R_m)}{m} = 0$ $\quad$ ($\to 0$)

And $\hat{\gamma}_i = \bar{R}_i - \hat{\beta}_i \bar{Z}_m$

$\hat{\gamma}_i = \bar{R}_i$

β adjust
Blume method

original paper: handout #20

$\beta_p = \Sigma x_i \beta_i$

$x_i = \frac{1}{n}$. $i = 1 .. n$

$\beta_p = \bar{\beta}$

skip Section B. The Equilibrium Approach

2017 - 22 $\quad$ 2022 - 27

History $\quad$ Forcast

↓

estimate β $\Rightarrow$ unadjusted β.

$\Rightarrow$ adjusted β

Table 2. corr between β in 7/26 ~ 6/33

and β in 7/33 ~ 6/40

β is equal allocation of 1, 2, 4, 7 ... 50 stocks

LEC 10

β of stock is measue of risk ($\beta > 1$ risky)

$\frac{\bar{R}_i - R_F}{\beta_i}$ $\quad$ $\frac{\text{return}}{\text{risk}}$

unadjusted β

β (2020 - 25) ⟶ predict ? (2025 - 30)

Blume method:

H_1 (2015 - 20) $\quad$ H_2 (2020 - 25) $\quad$ F (2025 - 30)

① compute the betas in H_1 and H_2

② regress the betas in H_2 on the betas in H_1

$\underline{\beta_{i2}} = \hat{C}_0 + \hat{C}_1 \beta_{i1}$

③ $\hat{\beta}_{iF} = \hat{C}_0 + \hat{C}_1 \underbrace{\beta_{i2}}_{\text{actual } \beta \text{ from } H_2}$

PRESS

prediction sum of square

$$PRESS_1 = \frac{1}{m} \sum_{j=1}^{m} (A_j - P_j)^2_{\text{unadjusted}}$$

m: # stock; A_j: actual β (estimated β); P_j: forcast β

$$PRESS_2 = \frac{1}{m} \sum_{j=1}^{m} (A_j - P_j)^2_{\text{adjusted}}$$

y	X_1	X_2		y	X_1	X_3
1						
2						
⋮						
100						

$$PRESS = \frac{1}{100} \sum_{i=1}^{100} (\underbrace{y_i}_{\text{observed}} - \underbrace{\hat{y}_{(i)}}_{\text{predicted}})^2$$

y_1 from 2-100

y_2 1 + 3-100

cross-validation

S_p divided by m. not $m-1$; i index stock. j index time; original paper: handout # 26

$$PRESS = \frac{1}{m} \sum_{j=1}^{m} (A_j - P_j)^2 \quad \text{can be adjusted or unadjusted}$$

$$= \frac{1}{m} \sum_{j=1}^{m} (A_j - \bar{A} - (P_j - \bar{P}) + \bar{A} - \bar{P})^2$$

$$= \frac{1}{m} \sum (A_j - \bar{A})^2 + \frac{1}{m} \sum (P_j - \bar{P})^2 + \overset{\frac{1}{m} \cdot m}{} (\bar{A} - \bar{P})^2$$

$$- \frac{2}{m} (\bar{A} - \bar{P}) \underbrace{\sum (A_j - \bar{A})}_{0 \cdot = \sum X_i - \bar{X}} - \frac{2}{m} (\bar{A} - \bar{P}) \underbrace{\sum (P_j - \bar{P})}_{0} - \frac{2}{m} \sum (A_j - \bar{A})(P_j - \bar{P})$$

$$= \underset{\text{var}}{S_A^2} + \underset{\text{var}}{S_p^2} + (\bar{A} - \bar{P})^2 - \underbrace{\frac{2}{m} \sum (A_j - \bar{A})(P_j - \bar{P})}_{2\, cov(A \cdot P)}$$

$$= 2 \hat{\beta}_i \cdot \Sigma (P_j - \bar{P})^2 = 2\hat{\beta}_i \cdot S_p^2 \qquad \left(\frac{1}{m} \Sigma (A_j - \bar{A})(P_j - \bar{P}) = \beta_1 S_p^2 \right)$$

$$= (\bar{A} - \bar{P})^2 + S_p^2 - \beta_1 S_p^2 + S_A^2 - \beta_1 S_p^2 \qquad \left(R_{AP} = \frac{\beta_i^2 S_p^2}{S_A^2} \right)$$

$$= \underbrace{(\bar{A} - \bar{P})^2}_{\text{bias}} + \underbrace{(1 - \beta_1)^2 S_p^2}_{\text{inefficiency. adjust } \beta \text{ help this}} + \underbrace{(1 - R_{AP})^2 S_A^2}_{\text{random error}}$$

β_1: slope; R_{AP}: coeff of determination

Aside:

$$R_{AP}^2 = \frac{SSR}{SST} = \frac{\hat{\beta}_i^2 \, \Sigma (P_j - \bar{P})^2}{\Sigma (A_j - \bar{A})^2} = \hat{\beta}_i^2 \frac{S_p^2}{S_A^2}$$

$$\hat{\beta}_i = \frac{\Sigma (A_j - \bar{A})(P_j - \bar{P}) / m}{\Sigma (P_j - \bar{P}) / m} \qquad \frac{Cov(A \cdot P)}{Var(P)}$$

LEC 11

β adjust Vasicek method

original paper: handout #20

$\underbrace{\quad H_1 \quad}$ $\underbrace{\quad P \quad}$

$\left.\begin{matrix}\hat{\beta}_1 \\ \vdots \\ \hat{\beta}_n\end{matrix}\right\} \begin{matrix}\bar{\beta} \\ Var(\beta)\end{matrix}$

$$Var(\hat{\beta}_i) = \frac{\hat{\sigma}_{\varepsilon i}^2}{\Sigma (R_{mt} - \bar{R}_m)^2}$$

$$\text{Vasicek: } \hat{\beta}_{i\,adjust} = \frac{Var(\hat{\beta}_i)}{Var(\beta) + Var(\hat{\beta}_i)} \bar{\beta} + \frac{Var(\beta)}{Var(\beta) + Var(\hat{\beta}_i)} \hat{\beta}_i$$

the $\hat{\beta}_i$ to adjust

handout #23 (also see #24)

method comparison

Elton, Gruber, and Urich (1978) compared the following 4 models in terms of their ability to predict the correlation matrix of n stocks:

a. The historical correlation matrix.

b. The correlation matrix based on the unadjusted betas.

c. The correlation matrix based on the Blume's technique.

d. The correlation matrix based on the Vasicek's technique.

They found that the historical matrix perform the poorest predictions. Therefore, the single-index model not only reduces the amount of data input, but also produces better estimates of the variance-covaiance matrix. However, the comparison of the three beta techniques was more difficult, but in some cases the Blume technique was the winner, while in some other the Vasicek's technique was the winner.

CCM > SIM adjusted (Blume / Vasicek) > unadjusted / historical Σ

$$z = \Sigma^{-1} (\bar{R} - R_f \cdot \mathbb{1}) \qquad X_G = \frac{z}{\mathbb{1}^T z}$$

if we replace Σ with SIM/CCM estimated Σ.
the X_G we get is extremely close to use ground truth Σ
more at handout #27

short sales not allow
risk free asset exist
w/SIM

$\begin{cases}\text{short sales allow} \\ \text{risk free asset exist}\end{cases} \Rightarrow \max \ \theta = \frac{\bar{R}_p - R_f}{\sigma_p} \quad s.t. \ \Sigma x_i = 1$

$\begin{cases}\text{short sales not allow} \to x_i \geq 0 \\ \text{risk free asset exist}\end{cases}$ see handout #29 for Kuhn-Tucker condition

$$\bar{R} - R_f \cdot \mathbb{1} = \Sigma z$$

$$\begin{pmatrix} \bar{R}_1 - R_f \\ \vdots \\ \bar{R}_i - R_f \\ \vdots \\ \bar{R}_n - R_f \end{pmatrix} = \begin{pmatrix} \\ \sigma_{i1} & \sigma_{i2} & \cdots & \sigma_{in} \\ \\ \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$\bar{R}_i - R_f = z_1 \sigma_{i1} + z_2 \sigma_{i2} + \cdots + z_i \sigma_i^2 + \cdots + z_n \sigma_{in}$$
$$= z_i \sigma_i^2 + \sum_{j \neq i}^{n} z_j \sigma_{ij}$$

$$\text{SIM} \begin{cases} \sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2 \\ \sigma_{ij} = \beta_i \beta_j \sigma_m \end{cases}$$

$$\bar{R}_i - R_f = z_i (\beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2) + \sigma_m^2 \beta_i \sum_{j \neq i}^{n} z_j \beta_j$$
$$= z_i \sigma_{\varepsilon i}^2 + \sigma_m^2 \beta_i \sum_{j=1}^{n} z_j \beta_j$$

solve for z_i: $$z_i = \frac{\bar{R}_i - R_f}{\sigma_{\varepsilon i}^2} - \sigma_m^2 \frac{\beta_i}{\sigma_{\varepsilon i}^2} \sum_{j=1}^{n} z_j \beta_j$$

or $$z_i = \frac{\beta_i}{\sigma_{\varepsilon i}^2} \left(\underbrace{\frac{\bar{R}_i - R_f}{\beta_i}}_{\text{excess return to } \beta \text{ ratio}} - \underbrace{\sigma_m^2 \sum_{j=1}^{n} z_j \beta_j}_{\text{common for all stocks}} \right)$$

① we know everything except $\Sigma z_j \beta_j$

$\sigma_{\varepsilon i}^2 > 0$

for now assume $\underbrace{\beta_i > 0}$

it's rare to get $\beta_i < 0$ in real data

next, need to find $\sum_{j=1}^{n} z_j \beta_j$: multiply ① by β_i and sum $i = 1 \cdots n$.

$$\sum_{i=1}^{n} z_i \beta_j = \sum_{i=1}^{n} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\varepsilon i}^2} - \sigma_m^2 \sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_{\varepsilon i}^2} \sum_{j=1}^{n} z_j \beta_j$$

$$\sum z_j \beta_j \left(1 + \sigma_m^2 \sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_{\varepsilon i}^2}\right) = \sum \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\varepsilon i}^2}$$

$$\sum z_j \beta_j = \frac{\sum \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\varepsilon i}^2}}{1 + \sigma_m^2 \sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_{\varepsilon i}^2}}$$

Finally, $$C^* = \sigma_m^2 \sum z_j \beta_j = \frac{\sigma_m^2 \sum \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\varepsilon i}^2}}{1 + \sigma_m^2 \sum_{i=1}^{n} \frac{\beta_i^2}{\sigma_{\varepsilon i}^2}}$$

LEC 12

if $\frac{\bar{R}_i - R_f}{\beta_i} > C^* \to z_i > 0$. $x_i > 0$. stock is held long

$<$ $\quad$ $<$ $\quad$ $x_i < 0$ $\quad$ short

only when short sales allow

$z_j = \lambda x_j$, $\quad \lambda = \frac{\sum x_i(\bar{R}_i - R_f)}{\sigma_p^2} = \frac{\bar{R}_p - R_f}{\sigma_p^2}$ $\quad$ we cannot use this to compute C^*

$C^* = \sigma_m^2 \sum_{j=1}^{n} z_j \beta_j = \sigma_m^2 \lambda \sum_{j=1}^{n} x_j \beta_j = \sigma_m^2 \frac{\bar{R}_p - R_f}{\sigma_p^2} \beta_p \cdot \frac{\beta_i}{\beta_i} = \frac{\beta_i \beta_p \sigma_m^2}{\sigma_p^2} \cdot \frac{\bar{R}_p - R_f}{\beta_i} = \beta_{ip} \cdot \frac{\bar{R}_p - R_f}{\beta_i}$ because $\beta_p = \sum x_i \beta_i$, x_i are unknown

a stock enter if $\frac{\bar{R}_i - R_f}{\beta_i} > C^* = \beta_{ip} \cdot \frac{\bar{R}_p - R_f}{\beta_i}$, or $\bar{R}_i - R_f > \beta_{ip}(\bar{R}_p - R_f)$, or excess return > expected excess return

a. **Step 1:** By regressing the returns of each stock on the returns of the market obtain for each stock: $\hat{\beta}, \hat{\alpha}, \hat{\sigma_\epsilon^2}$ and construct the table below:

Stock i	$\hat{\alpha}_i$	$\hat{\beta}_i$	$\bar{R}_i$	$\hat{\sigma^2_{\epsilon i}}$	$\frac{R_i - R_f}{\hat{\beta}_i}$
IBM					
GOOGLE					
⋮					

b. **Step 2:** Sort the table above based on the excess return to beta ratio:

$$\frac{\bar{R}_i - R_f}{\beta_i}$$

c. **Step 3:** Create 5 columns to the right of the sorted table as follows:

Stock i	$\hat{\alpha}_i$	$\hat{\beta}_i$	$\bar{R}_i$	$\hat{\sigma^2_{\epsilon i}}$	$\frac{R_i - R_f}{\hat{\beta}_i}$	$\frac{(\bar{R}_i - R_f)\hat{\beta}_i}{\sigma^2_{\epsilon i}}$	$\sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\hat{\beta}_j}{\sigma^2_{\epsilon j}}$	$\frac{\hat{\beta}_i^2}{\sigma^2_{\epsilon i}}$	$\sum_{j=1}^{i} \frac{\hat{\beta}_j^2}{\sigma^2_{\epsilon j}}$	C_i
						k_1	k_1	l_1	l_1	C_1
						k_2	$k_1 + k_2$	l_2	$l_1 + l_2$	C_2
						k_3	$k_1 + k_2 + k_3$	l_3	$l_1 + l_2 + l_3$	C_3
						⋮	⋮	⋮	⋮	⋮
						k_n	$k_1 + k_2 + \cdots + k_n$	l_n	$l_1 + l_2 + \cdots + l_n$	C_n

Note: Compute all the $C_i, i = 1, \cdots, n$ (last column) as follows:

$$C_i = \frac{\sigma_m^2 \sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\beta_j}{\sigma^2_{\epsilon j}}}{1 + \sigma_m^2 \sum_{j=1}^{i} \frac{\beta_j^2}{\sigma^2_{\epsilon j}}} = \frac{\sigma_m^2 \times \text{COL2}}{1 + \sigma_m^2 \times \text{COL4}}$$

Once the $C_i's$ are calculated we find the C^* as follows:

If short sales are allowed, C^* is the last element in the last column.
If short sales are not allowed, C^* is the element in the last column for which $\frac{\bar{R}_i - R_f}{\beta_i} > C^*$. or the maximum C_i

In both cases the $z_i's$ are computed as follows

$$z_i = \frac{\beta_i}{\sigma^2_{\epsilon i}} \left(\frac{\bar{R}_i - R_f}{\beta_i} - C^* \right)$$

and the $x_i's$

$$x_i = \frac{z_i}{\sum_{i=1}^{n} z_i}$$

see handout #31.32.33 for examples

Constant correlation model (CCM)

common ρ

$$\rho = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & & \\ \vdots & & \ddots & \\ \rho_{n1} & & & 1 \end{pmatrix}$$

$$\bar{\rho} = \frac{\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\right) - n}{n(n-1)}$$

$\sigma_{ij} = \bar{\rho}\sigma_i\sigma_j$ $\quad \Sigma = \begin{pmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \; \sigma_n^2 \end{pmatrix}$

$$z = \Sigma^{-1}(\bar{R} - R_f \cdot \mathbf{1})$$

$$\bar{R} - R_f \cdot \mathbf{1} = \Sigma z$$

$$\bar{R}_i - R_f = z_1\sigma_{i1} + \cdots + z_i\sigma_i^2 + \cdots + z_n\sigma_{in}$$

$$= z_i\sigma_i^2 + \sum_{j\neq i}^{n} z_j\sigma_{ij} + z_i\rho\sigma_i^2 - z_i\rho\sigma_i^2$$

$$\bar{R}_i - R_f = z_i(1-\rho)\sigma_i^2 + \rho\sum_{j=1}^{n} z_j\sigma_i\sigma_j \quad (\text{NOTE: } \sigma_{ij} = \rho\sigma_i\sigma_j)$$

solve for z_i, $\quad z_i = \dfrac{\bar{R}_i - R_f}{(1-\rho)\sigma_i^2} - \dfrac{\rho\sigma_i}{(1-\rho)\sigma_i^2}\sum_{j=1}^{n} z_j\sigma_j$

$$= \frac{1}{(1-\rho)\sigma_i}\left(\underbrace{\frac{\bar{R}_i - R_f}{\sigma_i}}_{\text{excess return to std ratio}} - \underbrace{\rho\sum_{j=1}^{n} z_j\sigma_j}_{C^* \text{ (common for all stocks)}}\right) \quad ①$$

next, need to find $\sum z_i\sigma_i$, multiply ① by σ_i and sum $i=1\cdots n$

$$\sum z_i\sigma_i = \frac{1}{1-\rho}\sum_{i=1}^{n}\frac{(R_i - R_f)}{\sigma_i} - \frac{\rho}{1-\rho}\, n \cdot \sum_{j=1}^{n} z_j\sigma_j$$

$$\sum z_j\sigma_j\left(1 + \frac{n\rho}{1-\rho}\right) = \frac{1}{1-\rho}\sum\frac{R_i - R_f}{\sigma_i}$$

Finally, $\quad \sum z_j\sigma_j = \dfrac{\frac{1}{1-\rho}\sum\frac{R_i - R_f}{\sigma_i}}{1 + \frac{n\rho}{1-\rho}}$

or $\quad \sum z_j\sigma_j = \dfrac{\sum\frac{R_i - R_f}{\sigma_i}}{1 - \rho + n\rho}$

and $\quad C^* = \rho\sum z_j\sigma_j = \dfrac{\rho\sum\frac{R_i - R_f}{\sigma_i}}{1 - \rho + n\rho}$

a. **Step 1:** Compute the historical mean return, standard deviation for each stock. You will also need the correlation coefficients for all pairs of stocks (step 2). Construct the table below:

Stock i	$\bar{R}_i$	$\bar{R}_i - R_f$	σ_i	$\frac{\bar{R}_i - R_f}{\sigma_i}$
IBM				
GOOGLE				
$\vdots$				

b. **Step 2:** Sort the table above based on the excess return to standard deviation ratio:

$$\frac{\bar{R}_i - R_f}{\sigma_i}$$

c. **Step 3:** Create 3 columns to the right of the sorted table as follows:

Stock i	$\bar{R}_i$	$\bar{R}_i - R_f$	σ_i	$\frac{\bar{R}_i - R_f}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	$\sum_{j=1}^{i}\frac{\bar{R}_j - R_f}{\sigma_j}$	C_i

Note: ρ is the average correlation. It is equal to:

$$\rho = \frac{\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\rho_{ij}}{n(n-1)}$$

Note: Compute all the $C_i, i = 1, \cdots, n$ (last column) as follows:

$$C_i = \frac{\rho}{1-\rho+i\rho}\sum_{j=1}^{i}\frac{\bar{R}_j - R_f}{\sigma_j} = COL1 \times COL2.$$

Once the $C_i's$ are calculated we find the C^* as follows:

If short sales are allowed, C^* is the last element in the last column.
If short sales are not allowed, C^* is the element in the last column for which $\frac{\bar{R}_i - R_f}{\sigma_i} > C^*$.

In both cases the $z_i's$ are computed as follows

$$z_i = \frac{1}{(1-\rho)\sigma_i}\left[\frac{\bar{R}_i - R_f}{\sigma_i} - C^*\right]$$

and the $x_i's$

$$x_i = \frac{z_i}{\sum_{i=1}^{n} z_i}$$

see handout #34, 35 for examples

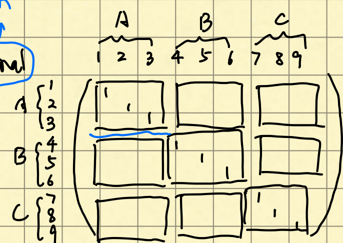
LEC13 multi-group model (MGM)

original paper for multi-group/index model: handout #36

suppose: 3 $\underbrace{\text{groups}}_{\text{eg. tech, entity, ...}}$ x 3 stocks (MGM requires at least 2 stocks in each group, don't have to have same stocks in each group)

from R

original $\rho =$



new $\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}$

avg ρ in A·A except 1 on diagonal

avg ρ in A·B (all 9 elements)

$$\begin{cases} \sigma_{12} = \rho_{11}\sigma_1\sigma_2 \\ \sigma_{14} = \rho_{12}\sigma_1\sigma_4 \\ \sigma_{28} = \rho_{13}\sigma_2\sigma_8 \end{cases} \Rightarrow \text{construct } \Sigma$$

$\underline{Z = \Sigma^{-1}(\bar{R} - R_f \cdot 1)}$ approach 1 to solve for Z

approach 2 to solve for Z (similar to C^* cut off point method)

assume 2 groups and 2 stocks in each group

stock 1, 2 in group A. 3, 4 in group B

$\bar{R}_1 - R_f = z_1\sigma_1^2 + z_2\sigma_{12} + z_3\sigma_{13} + z_4\sigma_{14}$ ①

$\bar{R}_2 - R_f = z_1\sigma_{21} + z_2\sigma_2^2 + z_3\sigma_{23} + z_4\sigma_{24}$ ②

$\bar{R}_3 - R_f = z_1\sigma_{31} + z_2\sigma_{32} + z_3\sigma_3^2 + z_4\sigma_{34}$ ③

$\bar{R}_4 - R_f = z_1\sigma_{41} + z_2\sigma_{42} + z_3\sigma_{43} + z_4\sigma_4^2$ ④

examine ① w/ MGM

$$\bar{R}_1 - R_f = z_1\sigma_1^2 + z_2\rho_{11}\sigma_1\sigma_2 + z_3\rho_{12}\sigma_1\sigma_3 + z_4\rho_{12}\sigma_1\sigma_4 \pm z_1\rho_{11}\sigma_1^2$$

$$= z_1(1-\rho_{11})\sigma_1^2 + \sigma_1\left[\rho_{11}(\underbrace{z_1\sigma_1 + z_2\sigma_2}_{\phi_1}) + \rho_{12}(\underbrace{z_3\sigma_3 + z_4\sigma_4}_{\phi_2})\right]$$

$$\bar{R}_1 - R_f = z_1(1-\rho_{11})\sigma_1^2 + \sigma_1\sum_{g=1}^{2}\rho_{1g}\phi_g$$

solve for z_1. $z_1 = \frac{\bar{R}_1 - R_f}{(1-\rho_{11})\sigma_1^2} - \frac{\sigma_1}{(1-\rho_{11})\sigma_1^2}\sum_{g=1}^{2}\rho_{1g}\phi_g$

or $z_1 = \frac{1}{(1-\rho_{11})\sigma_1}\left[\frac{\bar{R}_1 - R_f}{\sigma_1} - \underbrace{\sum_{g=1}^{2}\rho_{1g}\phi_g}_{C_A^*}\right]$ } we know everything except ϕ_g

similarly, $z_2 = \frac{1}{(1-\rho_{11})\sigma_2}\left[\frac{\bar{R}_2 - R_f}{\sigma_2} - \sum_{g=1}^{2} \rho_{1g}\phi_g\right]$

LEC 14

$$z_1\sigma_1 = \frac{\bar{R}_1 - R_f}{(1-\rho_{11})\sigma_1} - \frac{1}{1-\rho_{11}}\sum_{g=1}^{2}\rho_{1g}\phi_g$$

$$z_2\sigma_2 = \frac{\bar{R}_2 - R_f}{(1-\rho_{11})\sigma_2} - \frac{1}{1-\rho_{11}}\sum_{g=1}^{2}\rho_{1g}\phi_g$$

$$\phi_1 = \sum_{i\in A}\frac{(\bar{R}_i - R_f)}{(1-\rho_{11})\sigma_i} - \frac{2}{1-\rho_{11}}\sum_{g=1}^{2}\rho_{1g}\phi_g$$

stock 1&2

similarly using stock 3 & 4 $\quad \phi_2 = \sum_{i\in B}\frac{(\bar{R}_i - R_f)}{(1-\rho_{22})\sigma_i} - \frac{2}{1-\rho_{22}}\sum_{g=1}^{2}\rho_{2g}\phi_g$

solve for $\phi_1\ \phi_2$:

$$\begin{pmatrix} 1+\frac{2\rho_{11}}{1-\rho_{11}} & \frac{2\rho_{12}}{1-\rho_{11}} \\ \frac{2\rho_{21}}{1-\rho_{22}} & 1+\frac{2\rho_{22}}{1-\rho_{22}} \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\sum_{i\in A}\frac{(\bar{R}_i - R_f)}{(1-\rho_{11})\sigma_i} \\ \sum_{i\in B}\frac{(\bar{R}_i - R_f)}{(1-\rho_{22})\sigma_i}\end{pmatrix}$$

$$A\phi = C \qquad \phi = A^{-1}C$$

generalize: suppose 3 groups with n_1, n_2, n_3 stocks. find $A\phi = C$

$$\begin{pmatrix} 1+\frac{n_1\rho_{11}}{1-\rho_{11}}, & \frac{n_1\rho_{12}}{1-\rho_{11}}, & \frac{n_1\rho_{13}}{1-\rho_{11}} \\ \frac{n_2\rho_{21}}{1-\rho_{22}}, & 1+\frac{n_2\rho_{22}}{1-\rho_{22}}, & \frac{n_2\rho_{23}}{1-\rho_{22}} \\ \frac{n_3\rho_{21}}{1-\rho_{33}}, & \frac{n_3\rho_{33}}{1-\rho_{33}}, & 1+\frac{n_3\rho_{33}}{1-\rho_{33}} \end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix}\sum_{i\in A}\frac{(\bar{R}_i - R_f)}{(1-\rho_{11})\sigma_i} \\ \sum_{i\in B}\frac{(\bar{R}_i - R_f)}{(1-\rho_{22})\sigma_i} \\ \sum_{i\in C}\frac{(\bar{R}_i - R_f)}{(1-\rho_{33})\sigma_i}\end{pmatrix}$$

see handout #37 for an example

multi-index model $\quad$ index is industry / group (no prj because no index return)

Stocks are grouped by industry. The multi-index model used here gives a diagonal form of the covariance matrix between stocks. The assumptions for this model is that stocks are linearly related to the group index (industry) and the industry is linearly related to the market index. Here is the model:

$$\begin{aligned} R_i &= \alpha_i + \beta_i I_j + \epsilon_i \\ I_j &= \gamma_j + b_j R_m + c_j \end{aligned}$$

with,

$$\begin{aligned} &E(\epsilon_i\epsilon_k) = 0 && i = 1,\ldots,n,\ k = 1,\ldots,n,\ i \neq k \\ &E(c_j c_l) = 0 && j = 1,\ldots,p,\ l = 1,\ldots,p,\ j \neq l \\ &E(\epsilon_i c_j) = 0 && i = 1,\ldots,n,\ j = 1,\ldots,p \end{aligned}$$

where,

R_i	Return of stock i
I_j	Return of index j
R_m	Return of the market
α_i, β_i	Parameters associated with stock i
γ_j, b_j	Parameters associated with index j
ϵ_i	Error term with mean zero and variance $\sigma^2_{\epsilon_i}$
c_j	Error term with mean zero and variance $\sigma^2_{c_i}$

Using these assumptions we get the following.
Variance of the return of stock i:

$$\begin{aligned} \sigma_i^2 &= \beta_i^2\sigma_j^2 + \sigma^2_{\epsilon_i} \\ \text{But } \sigma_j^2 &= b_j^2\sigma_m^2 + \sigma^2_{c_j} \\ \text{Therefore, } \sigma_i^2 &= \beta_i^2(b_j^2\sigma_m^2 + \sigma^2_{c_j}) + \sigma^2_{\epsilon_i} \end{aligned}$$

$$\underbrace{Cov(R_i, R_k)}_{\text{same industry}} = Cov(\alpha_i + \beta_i I_j + \epsilon_i,\ \alpha_k + \beta_k I_j + \epsilon_k) = \beta_i\beta_k\sigma_j^2 = \beta_i\beta_k(b_j^2\sigma_m^2 + \sigma_{c_j}^2)$$

$$\underbrace{Cov(R_i, R_k)}_{\text{diff industry}} = Cov(\alpha_i + \beta_i I_j + \epsilon_i,\ \alpha_k + \beta_k I_l + \epsilon_k) = \beta_i\beta_k Cov(I_j, I_l) = \sigma_m^2\beta_i\beta_k b_j b_l$$

Note: $Cov(I_j, I_l) = Cov(\gamma_j + b_j R_m + c_j,\ \gamma_l + b_l R_m + c_l) = \sigma_m^2 b_j b_l$

LEC 15

Assume two stocks and two industries (two per industry). The solution as always is given by the following system of equations:

$$\bar{R}_1 - R_f = z_1\sigma_1^2 + z_2\sigma_{12} + z_3\sigma_{13} + z_4\sigma_{14} \tag{4}$$
$$\bar{R}_2 - R_f = z_1\sigma_{21} + z_2\sigma_2^2 + z_3\sigma_{23} + z_4\sigma_{24} \tag{5}$$
$$\bar{R}_3 - R_f = z_1\sigma_{31} + z_2\sigma_{32} + z_3\sigma_3^2 + z_4\sigma_{34} \tag{6}$$
$$\bar{R}_4 - R_f = z_1\sigma_{41} + z_2\sigma_{42} + z_3\sigma_{43} + z_4\sigma_4^2 \tag{7}$$

Let's examine equation (4) and see how it can be written using equations (1), (2), and (3).

$$\begin{aligned}\bar{R}_1 - R_f &= z_1\left(\beta_1^2[b_1^2\sigma_m^2 + \sigma_{c_1}^2] + \sigma_{\epsilon_1}^2\right) + z_2\left(\beta_1\beta_2[b_1^2\sigma_m^2 + \sigma_{c_1}^2]\right)\\ &+ z_3\left(\beta_1\beta_3 b_1 b_2\sigma_m^2\right) + z_4\left(\beta_1\beta_4 b_1 b_2 \sigma_m^2\right)\end{aligned}$$

Rearrange

$$\begin{aligned}\bar{R}_1 - R_f &= z_1\sigma_{\epsilon_1}^2 + \beta_1\left[z_1\beta_1 b_1^2\sigma_m^2 + z_2\beta_2 b_1^2\sigma_m^2 + z_1\beta_1\sigma_{c_1}^2 + z_2\beta_2\sigma_{c_1}^2\right]\\ &+ \beta_1\left[z_3\beta_3 b_1 b_2\sigma_m^2 + z_4\beta_4 b_1 b_2\sigma_m^2\right]\end{aligned}$$

Or

$$\bar{R}_1 - R_f = z_1\sigma_{\epsilon_1}^2 + \beta_1\left[b_1^2\sigma_m^2(z_1\beta_1 + z_2\beta_2) + \sigma_{c_1}^2(z_1\beta_1 + z_2\beta_2)\right] + \beta_1\left[b_1 b_2\sigma_m^2(z_3\beta_3 + z_4\beta_4)\right]$$

If we let $\Phi_1 = z_1\beta_1 + z_2\beta_2$ and $\Phi_2 = z_3\beta_3 + z_4\beta_4$ we get:

$$\bar{R}_1 - R_f = z_1\sigma_{\epsilon_1}^2 + \beta_1\left[(\sigma_{c_1}^2 + b_1\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2\right]$$

Now solve for z_1:

$$z_1 = \frac{\beta_1}{\sigma_{\epsilon_1}^2}\left[\frac{\bar{R}_1 - R_f}{\beta_1} - \left[\underbrace{(\sigma_{c_1}^2 + b_1^2\sigma_m^2)}_{Var(I_1)}\Phi_1 + \underbrace{b_1 b_2\sigma_m^2}_{Cov(I_1,I_2)}\Phi_2\right]\right] \tag{8}$$

Similarly, stock 2 will give the following expression:

$$z_2 = \frac{\beta_2}{\sigma_{\epsilon_2}^2}\left[\frac{\bar{R}_2 - R_f}{\beta_2} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2\right]\right] \tag{9}$$

These expressions look similar to the single index model solution (see class notes). But now the C^* cut-off point is a more complicated expression and of course it is the same for stocks in the same industry. In our example the C^* for industry 1 is equal to:

$$C_1^* = (\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2$$

If we know C_1^* it will be easy to compute the $z_i's$ and from there the $x_i's$. In order to find C_1^* we need to find Φ_1 and Φ_2.

Multiply (8) by β_1 and (9) by β_2:

$$z_1\beta_1 = \frac{\beta_1^2}{\sigma_{\epsilon_1}^2}\left[\frac{\bar{R}_1 - R_f}{\beta_1} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2\right]\right] \tag{10}$$

and

$$z_2\beta_2 = \frac{\beta_2^2}{\sigma_{\epsilon_2}^2}\left[\frac{\bar{R}_2 - R_f}{\beta_2} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2\right]\right] \tag{11}$$

To produce Φ_1 on the left hand side add (10) and (11):

$$\begin{aligned}\sum_{i=1}^{2}\frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon_i}^2} &= \Phi_1 + \frac{\beta_1^2}{\sigma_{\epsilon_1}^2}\left[\sigma_{c_1}^2 + b_1^2\sigma_m^2\right]\Phi_1 + \frac{\beta_1^2}{\sigma_{\epsilon_1}^2}b_1 b_2\sigma_m^2\Phi_2\\ &+ \frac{\beta_2^2}{\sigma_{\epsilon_2}^2}\left[\sigma_{c_1}^2 + b_1^2\sigma_m^2\right]\Phi_1 + \frac{\beta_2^2}{\sigma_{\epsilon_2}^2}b_1 b_2\sigma_m^2\Phi_2\end{aligned}$$

Or

$$\begin{aligned}\sum_{i=1}^{2}\frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon_i}^2} &= \Phi_1\left[1 + \frac{\beta_1^2}{\sigma_{\epsilon_1}^2}\left[\sigma_{c_1}^2 + b_1^2\sigma_m^2\right] + \frac{\beta_2^2}{\sigma_{\epsilon_2}^2}\left[\sigma_{c_1}^2 + b_1^2\sigma_m^2\right]\right]\\ &+ \Phi_2\left[\frac{\beta_1^2}{\sigma_{\epsilon_1}^2}b_1 b_2\sigma_m^2 + \frac{\beta_2^2}{\sigma_{\epsilon_2}^2}b_1 b_2\sigma_m^2\right]\end{aligned} \tag{12}$$

Similarly, for stocks 3 and 4 that belong in industry 2 we get:

$$\begin{aligned}\sum_{i=3}^{4}\frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon_i}^2} &= \Phi_1\left[\frac{\beta_3^2}{\sigma_{\epsilon_3}^2}b_1 b_2\sigma_m^2 + \frac{\beta_4^2}{\sigma_{\epsilon_4}^2}b_1 b_2\sigma_m^2\right]\\ &+ \Phi_2\left[1 + \frac{\beta_3^2}{\sigma_{\epsilon_3}^2}\left[\sigma_{c_2}^2 + b_2^2\sigma_m^2\right] + \frac{\beta_4^2}{\sigma_{\epsilon_4}^2}\left[\sigma_{c_2}^2 + b_2^2\sigma_m^2\right]\right]\end{aligned} \tag{13}$$

The system above can be written in vector and matrix form as $\boldsymbol{M\Phi} = \boldsymbol{R}$ and therefore: $\boldsymbol{\Phi} = \boldsymbol{M^{-1}R}$. The dimensions of the matrix $\boldsymbol{M}$ in our example are 2×2 (two industries).

industry 1

$1 + Var(I_1) \cdot \sum_{i \in A} \frac{\beta_i^2}{\sigma_{\varepsilon_i}^2}$

Multi-index system $Cov(I_1, I_2) \sum_{i \in B} \frac{\beta_i^2}{\sigma_{\varepsilon_i}^2}$

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_1 \\ \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\beta_1^2}{\sigma_{\epsilon_1}^2}\left[\sigma_{c_1}^2 + b_1^2\sigma_m^2\right] + \frac{\beta_2^2}{\sigma_{\epsilon_2}^2}\left[\sigma_{c_1}^2 + b_1^2\sigma_m^2\right] & \left[\frac{\beta_1^2 b_1 b_2}{\sigma_{\epsilon_1}^2} + \frac{\beta_2^2 b_1 b_2}{\sigma_{\epsilon_2}^2}\right]\sigma_m^2 \\ \\ \underbrace{\left[\frac{\beta_3^2 b_1 b_2}{\sigma_{\epsilon_3}^2} + \frac{\beta_4^2 b_1 b_2}{\sigma_{\epsilon_4}^2}\right]\sigma_m^2}_{Cov(I_1, I_2) \sum_{i \in B} \frac{\beta_i^2}{\sigma_{\varepsilon_i}^2}} & \underbrace{1 + \frac{\beta_3^2}{\sigma_{\epsilon_3}^2}\left[\sigma_{c_2}^2 + b_2^2\sigma_m^2\right] + \frac{\beta_4^2}{\sigma_{\epsilon_4}^2}\left[\sigma_{c_2}^2 + b_2^2\sigma_m^2\right]}_{1 + Var(I_2) \sum_{i \in B} \frac{\beta_i^2}{\sigma_{\varepsilon_i}^2}} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{2} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon_i}^2} \\ \\ \sum_{i=3}^{4} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon_i}^2} \end{pmatrix}$$

Once Φ_1 and Φ_2 are obtained we can compute the $z_i's$ (see equations (8) and (9)).

$$z_1 = \frac{\beta_1}{\sigma_{\epsilon_1}^2}\left[\frac{\bar{R}_1 - R_f}{\beta_1} - \left[\overbrace{(\sigma_{c_1}^2 + b_1^2\sigma_m^2)}^{Var(I_1)}\Phi_1 + \overbrace{b_1 b_2 \sigma_m^2}^{Cov(I_1, I_2)}\Phi_2\right]\right]$$

$$z_2 = \frac{\beta_2}{\sigma_{\epsilon_2}^2}\left[\frac{\bar{R}_2 - R_f}{\beta_2} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2\right]\right]$$

$$z_3 = \frac{\beta_3}{\sigma_{\epsilon_3}^2}\left[\frac{\bar{R}_3 - R_f}{\beta_3} - \left[(\sigma_{c_2}^2 + b_2^2\sigma_m^2)\Phi_2 + b_1 b_2 \sigma_m^2 \Phi_1\right]\right]$$

$$z_4 = \frac{\beta_4}{\sigma_{\epsilon_4}^2}\left[\frac{\bar{R}_4 - R_f}{\beta_4} - \left[(\sigma_{c_2}^2 + b_2^2\sigma_m^2)\Phi_2 + b_1 b_2 \sigma_m^2 \Phi_1\right]\right]$$

Write the system in vector and matrix form when there are three industries with three stocks in each industry.

partial regression — original paper. handout #41, 42. partial regression simplify the construction of Cov matrix. because Cov between 2 indices is 0

suppose we have I_1, I_2, I_3, I_4 indices (inflation, market, ...)

$R_{it} = \alpha_i + \beta_{i1} I_{1t} + \beta_{i2} I_{2t} + \beta_{i3} I_{3t} + \beta_{i4} I_{4t} + \varepsilon_{it}$. indicies are not orthogonal.

we can transform the model so that the indices are orthogonal (simplify Cov)

① begin with $R_i = \alpha_i + \beta_{i1} I_1 + \varepsilon_i$

② next regress I_2 on I_1 and compute the residuals e_2^*

$I_2 = \gamma_0 + \gamma_1 I_1 + c_i$

$e_2^* = I_2 - (\hat{\gamma}_0 + \hat{\gamma}_1 I_1)$

by least square property. e_2^* is orthogonal to I_1

add now e_2^* in the model above $R_i = \alpha_i + \beta_{i1} I_1 + \beta_{i2} e_{2i}^* + \varepsilon_i$

$Cov(I_1, e_2^*) = 0$ (orthogonal)

③ next regress I_3 on I_1 and e_2^* and compute the residual e_3^*

→ e_3^* orthogonal to I_2

e_3^* orthogonal to e_2^*

new model: $R_i = \alpha_i + \beta_{i1} I_1 + \beta_{i2} e_{2i}^* + \beta_{i3} e_{3i}^* + \varepsilon_i$

④ next regress I_4 on I_1, e_2^*, e_3^*. and compute the residual e_4^*

Final model: $R_i = \alpha_i + \beta_{i1} I_1 + \beta_{i2} e_{2i}^* + \beta_{i3} e_{3i}^* + \beta_{i4} e_{4i}^* + \varepsilon_i$

estimation using partial regression.

assume 4 indices I_1, I_2, I_3, I_4

$R_i = \alpha_i + \beta_{i1} I_1 + \beta_{i2} I_2 + \beta_{i3} I_3 + \beta_{i4} I_4 + \varepsilon_i$ multiple regression

estimate β_4 using partial regression (proof in looc)

1. regress R_i on I_1, I_2, I_3 and compute the residuals R_i^*
2. regress I_4 on I_1, I_2, I_3 and compute the residuals I_4^*

there is alway intercepts in the model

3. to estimate β_4, regress R_i^* on I_4^*, get $\hat{\beta}_4$ — simple regression

or: skip step 1, regress R_i on I_4^* get the same $\hat{\beta}_4$

same

LEC 16

e.g. SIM: $R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}$

$$\hat{\beta}_i = \frac{\Sigma (R_{mt} - \bar{R}_m)(R_{it} - \bar{R}_i)}{\Sigma (R_{mt} - \bar{R}_m)^2}$$ (original estimation method)

estimate β_i using partial regression

① regress R_{it} on 1 — and compute residual $R_{it}^* = R_{it} - \bar{R}_i$

$R_{it} = \gamma_0 + \varepsilon_{it}$

↓

$\hat{\gamma}_0 = \bar{R}_i$

② regress R_{mt} on 1 — and compute residual $R_{mt}^* = R_{mt} - \bar{R}_m$

$R_{mt} = \delta_0 + \eta_{it}$

↓

$\hat{\delta}_0 = \bar{R}_m$

③ regress R_{it}^* on R_{mt}^* to estimate β_i

$R_{it}^* = \theta_0 + \theta_1 \cdot R_{mt}^* + \zeta_{it}$ $\quad \Sigma_i (X_i - \bar{X}) = 0$

$$\hat{\theta}_1 = \frac{\Sigma (R_{it}^* - \bar{R}_m^*)(R_{it}^* - \bar{R}_i^*)}{\Sigma (R_{mt}^* - \bar{R}_m^*)^2}$$ → same as original estimation method

with more predictors. we can also show partial regression is same as original estimation method

$R_{it} = \alpha_i + \beta_{i1} I_1 + \beta_{i2} I_2 + \beta_{i3} I_3 + \beta_{i4} I_4 + \varepsilon_{it}$

estimate $\beta_{i1}\ \beta_{i2}\ \beta_{i3}\ \beta_{i4}$ using partial regression

① regress R_{it} on 1 → $R_{it}^* = R_{it} - \bar{R}_i$

② regress each I_j, $j = 1,2,3,4$ on 1 and compute the residual $I_{jt}^* = I_{jt} - \bar{I}_j$, $j = 1,2,3,4$ } 5 regressions

③ regress R_{it}^* on $I_{1t}^*\ I_{2t}^*\ I_{3t}^*\ I_{4t}^*$ to estimate $\beta_{i1}\ \beta_{i2}\ \beta_{i3}\ \beta_{i4}$

(multiple regression)

partial correlation

we have $R_i = \alpha_i + \beta_{i1} I_1 + \beta_{i2} I_2 + \beta_{i3} I_3 + \varepsilon_{it}$

add a new index I_4, use partial regression to estimate β_{i4}

① regress R_{it} on $I_{1t}, I_{2t}, I_{3t} \to R_{it}^*$

② regress I_4 on $I_{1t}, I_{2t}, I_{3t} \to I_{4t}^*$

③ regress R_{it}^* on I_{4t}^* to estimate β_{i4}

to decide whether to add I_4, we calculate partial correlation (add predictor always decrease SSE, increase R^2)

$r^2_{R_i I_4 | I_1 I_2 I_3}$: partial correlation between R_i & I_4 after removing linear effects of I_1, I_2, I_3

3 methods to get $r^2_{R_i I_4 | I_1 I_2 I_3}$:

method 1 ① regress R_{it} on $I_1, I_2, I_3 \to R_{it}^*$

② regress I_{4t} on $I_1, I_2, I_3 \to I_{4t}^*$

$$r^2_{R_{it}, I_4 | I_1, I_2, I_3} = \frac{Cov^2(I_{4t}^*, R_{it}^*)}{Var(I_{4t}^*) \cdot Var(R_{it}^*)} = \frac{(I_{4t}^{*T} R_{it}^*)^2}{(I_{4t}^{*T} I_{4t}^*)(R_{it}^{*T} R_{it}^*)}$$

note: $Cov(x,y) = \frac{1}{n-1} \Sigma (x_i - \bar{x})(y_i - \bar{y})$

$Var(x) = \frac{1}{n-1} \Sigma (x_i - \bar{x})^2$

method 2 use error sum of squares (SSE)

$R_{it} = \alpha_i + \beta_{i1} I_{1t} + \beta_{i2} I_{2t} + \beta_{i3} I_{3t} + \beta_{i4} I_{4t} + \varepsilon_{it}$ long regression

$R_{it} = \alpha_i + \beta_{i1} I_{1t} + \beta_{i2} I_{2t} + \beta_{i3} I_{3t} + \varepsilon_{it}$ short regression

$$r^2_{R_{it}, I_4 | I_1, I_2, I_3} = \frac{SSE(short) - SSE(long)}{SSE(short)}$$

$r^2_{R_{it}, I_4 | I_1, I_2, I_3}$

> 0

add predictor, SSE always ↓

method 3 use the t statistics for testing from the long regression. $H_0: \beta_4 = 0$, $H_a: \beta_4 \neq 0$

$$r^2_{R_{it}, I_4 | I_1, I_2, I_3} = \frac{t_4^2}{t_4^2 + n - k - 1}$$

k: # predictors, here $k=4$

n: # months

similarly

$$r^2_{R_{it}, I_2 | I_1, I_3, I_4} = \frac{t_2^2}{t_2^2 + n - k - 1}$$

methods to assess profolio performance

PLOT

$$\begin{array}{c|c} & R_1 \ R_2 \ \cdots \ R_{30} \\ \hline \begin{array}{c} M_1 \\ \vdots \\ M_{45} \end{array} & \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ \vdots \\ x_{30} \end{pmatrix} = \begin{pmatrix} R_{P_1} \\ R_{P_2} \\ \vdots \\ R_{P45} \end{pmatrix}$$

before_money $\underbrace{(1+R_{P_1})(1+R_{P_2}) \cdots (1+R_{45})}_{\text{cumprod in R}}$ = after_money

number that measure the growth of profolio

$R_{PA} = \frac{\Sigma R_{Pi}}{45}$, $(1+R_{P_1})(1+R_{P_2}) \cdots (1+R_{45}) \neq (1+R_{PA})^{45}$

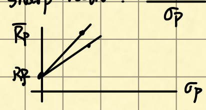
$<$

instead use geometric average $(1+R_{P_1})(1+R_{P_2}) \cdots (1+R_{45}) = (1+R_{PG})^{45}$

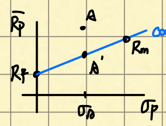
$\to R_{PG} = ((1+R_{P_1})(1+R_{P_2}) \cdots (1+R_{45}))^{1/45} - 1$

other 4 evaluation methods

① sharp ratio: $\frac{\bar{R}_P - R_f}{\sigma_P}$



② differential excess return when risk is measured by std $= \overline{R_A} - \overline{R_{A'}}$



$\overline{R_c} = R_f + \frac{\overline{R_m} - R_f}{\sigma_m}\sigma_c$

naive investment

$\overline{R_{A'}} = R_f + \frac{\overline{R_m} - R_f}{\sigma_m}\sigma_{A'}$

③ excess return to non-diversifiable risk (treynor measure): $\frac{\overline{R_A} - R_f}{\beta_A}$

we move to $\overline{R_P}$ vs beta space

$\begin{cases} \text{suppose } w\% \text{ is invested in profolio A} \\ \text{and } (1-w)\% \text{ in } R_f \end{cases}$

$\overline{R_c} = w\overline{R_A} + (1-w)R_f$

$= w\sum x_i \overline{R_i} + (1-w)R_f$

assume SIM

$\overline{R_c} = w\sum x_i(\alpha_i + \beta_i\overline{R_m}) + (1-w)R_f$

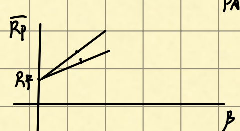
$= w\left[\sum x_i \alpha_i + \overline{R_m}\sum x_i\beta_i\right] + (1-w)R_f$

$= w\alpha_A + w\beta_A\overline{R_m} + (1-w)R_f$

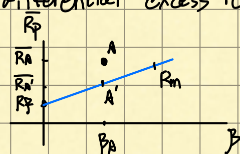
$\beta_c = w\beta_A \qquad w = \frac{\beta_c}{\beta_A}$

then $\overline{R_c} = \frac{\beta_c}{\beta_A}\overline{R_A} + \left(1 - \frac{\beta_c}{\beta_A}\right)R_f$

$= R_f + \frac{\overline{R_A} - R_f}{\beta_A}\beta_c$



④ differential excess return when risk is measured by β (Jensen differencial performance index): $\overline{R_A} - \overline{R_{A'}}$

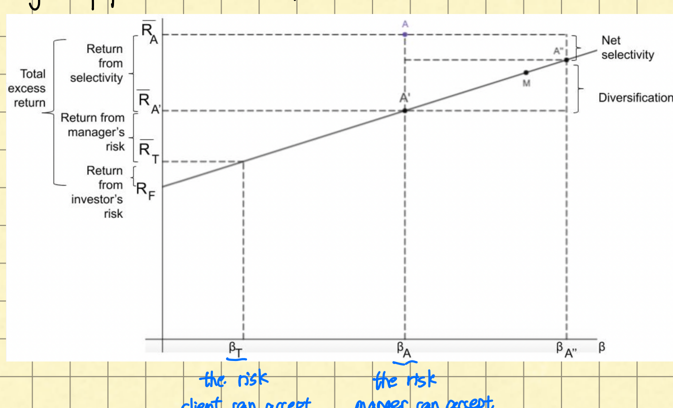


$\overline{R_c} = R_f + \frac{\overline{R_m} - R_f}{\beta_m}\cdot\beta_c, \quad \beta_m = 1 \quad \Rightarrow \overline{R_c} = R_f + (\overline{R_m} - R_f)\cdot\beta_c$

$\Rightarrow \overline{R_{A'}} = R_f + (\overline{R_m} - R_f)\beta_A$

decomposition of overall performance

original paper: handout #39



A'': combine M and R_f to replicate risk of profolio A

total risk of profolio A: $\sigma_A^2 = \beta_A^2\sigma_m^2 + \sum x_i^2\sigma_{\varepsilon i}^2$

$\beta_{A''}^2\sigma_m^2 = \beta_A^2\sigma_m^2 + \sum x_i^2\sigma_{\varepsilon i}^2$

$\Rightarrow \beta_{A''} = \sqrt{\frac{\beta_A^2\sigma_m^2 + \sum x_i^2\sigma_{\varepsilon i}^2}{\sigma_m^2}}$

overall performance $\overline{R_A} - R_f$
- return from selectivity $\overline{R_A} - \overline{R_{A'}}$
  - net selectivity $\overline{R_A} - \overline{R_{A''}}$
  - diversification $\overline{R_{A''}} - \overline{R_{A'}}$
- return from taking risk $\overline{R_{A'}} - R_f$
  - return from manger's risk $\overline{R_{A'}} - \overline{R_T}$
  - return from investor's risk $\overline{R_T} - R_f$

LEC 17 option intro

Option is contract between { seller / issuer / writer ; buyer / holder }

{ call option : buyer { pay premium / option price C ; get the right to buy a stock at strike price / exercise price E before expiration date / maturity date }
{ put option : buyer { pay premium / option price P ; get the right to sell a stock at strike price / exercise price before expiration date / maturity date }

{ European option : can be exercised only at expiration date
{ American option : can be exercised any time before expiration date

$t=0$, $S_0=50$, $E=50$

$t=1$, $S_1=70 \Rightarrow$ call : make \$20 ; put : lose \$20

Stock options mechanics: we just use one share for math simplicity

1. Options are normally traded in units of 100 shares. The price of the option is on a per share basis. Therefore, if the price of an option is priced at \$0.50, the total premium for that option would be \$50 ($0.50 \times 100 =$ \$50.)
2. Stock options are on a January, February, or March cycle. Stocks are randomly assigned in one of these three cycles. For example, IBM is on a January cycle.
3. Stock options expired on the Saturday immediately following the third Friday of the expiration month.

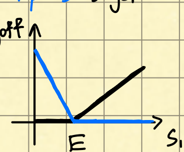
payoff : the money you get at expiration (not including premium)

Options - Examples

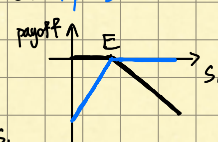
Holder of a call option ($E=\$40$)	Payoff at expiration if $S_1=\$38$	0 . buyer do nothing . let the option expire (it's cheaper to buy from market at \$38)
Writer of a call option ($E=\$45$)	Payoff at expiration if $S_1=\$50$	−5 buyer exercise call option , seller sell a stock worth \$50 at \$45 (must buy at \$50 and sell at \$45 according to contract)
Holder of a call option ($E=\$60$)	Payoff at expiration if $S_1=\$63$	3 buyer exercise call option , buyer buy a stock worth \$50 at \$45 (can sell at \$50 immediately)
Writer of a call option ($E=\$50$)	Payoff at expiration if $S_1=\$48$	0
Holder of a put option ($E=\$40$)	Payoff at expiration if $S_1=\$38$	2
Writer of a put option ($E=\$45$)	Payoff at expiration if $S_1=\$50$	0
Holder of a put option ($E=\$60$)	Payoff at expiration if $S_1=\$63$	0
Writer of a put option ($E=\$50$)	Payoff at expiration if $S_1=\$48$	−2

$E=50$. $C/P=5$. call buyer's payoff : $\max(S_1-E, 0)$
profit : $\max(S_1-E, 0) - 5$
call seller's payoff : $\min(E-S_1, 0)$
profit : $\min(E-S_1, 0) + 5$
put buyer's payoff : $\max(E-S_1, 0)$
profit : $\max(E-S_1, 0) - 5$
put seller's payoff : $\min(S_1-E, 0)$
profit : $\min(S_1-E, 0) + 5$

call's / put's buyer



call's / put's seller



lower/upper bound for European call/put premium

① lower bound for European call premium

	Time $t=0$	Payoff at time $t=1$ $S_1 > E$	$S_1 \le E$
Portfolio A:		buy at E. sell at S_1	let the option expire
Buy 1 call	$-C$	$S_1 - E$	0
Cash (lend)	$\left[-\frac{E}{1+r}\right]$	$+E$	$+E$
Total	present value, e.g. bank deposit; risk free	S_1	E
Portfolio B:			
Buy 1 share	$-S_0$	S_1	S_1

$(1+r)^6 = 1 + 0.05$

6 months get 0.05%

A is at least as good as B. thus at $t=0$. A should be more expensive. (otherwise there is riskless profit)

$$c \geq S_0 - \frac{E}{1+r} \quad \text{or} \quad c \geq S_0 - Ee^{-rt}.$$

e.g. Suppose $S_0 = \$40, E = \$38, r = 10\%$ per year, and time to expiration is $t = 1$ year.
Then the lower bound is: $c \geq 40 - 38e^{-0.10\times1} = 5.62$.

Suppose there is a European call written on this stock with price $c = \$5$. It is cheaper!
How can one make riskless profit?

- Short the stock (short means borrow a stock to sell now, and promise to return later)
- Buy the call

Explain:
How much is the cash inflow at $t = 0$? $40-5=35$
How much will it grow in 1 year? $35 \cdot e^{0.1\cdot1} = 38.68$
At expiration (in 1 year):
If stock price $S_T > 38$ then ... exercise call option. buy at 38 to return the stock. make \$0.68
If stock price S_T 36 < 38 then ... do not exercise call option. buy at 36 to return the stock. make \$2.68

LEC 18

② lower bound for European put premium

	Time $t=0$	Payoff at time $t=1$	
		$S_1 \geq E$	$S_1 < E$
Portfolio A:			
Buy 1 put	$-P$	0	$E - S_1$
Buy 1 share	$-S_0$	S_1	S_1
Total		S_1	E
Portfolio B:			
Cash (lend)	$-\frac{E}{1+r}$	$+E$	$+E$

A is at least as good as B.

$$p \geq \frac{E}{1+r} - S_0 \quad \text{or} \quad p \geq Ee^{-rt} - S_0$$

e.g. Suppose $S_0 = \$40, E = \$43, r = 5\%$ per year, and time to expiration is $t = 0.5$ years.
Then the lower bound is: $p \geq 43e^{-0.05\times0.5} - 40 = 1.94$.

Suppose there is a European put written on this stock with price $p = \$1$. It is cheaper!
How can one make riskless profit?

- Borrow \$41
- Buy the put and the stock

Explain:
At $t = 0.5$ must pay back the loan 6 months later. need pay back $41 \cdot e^{0.05\cdot\frac{6}{12}} = 42.04$
How much?

At expiration (in 6 months):
Stock price $S_T < 43$ then ... exercise the put. sell at 43. make 0.96 profit
Stock price S_T 45 > 43 then ... sell at 45. make 2.96 profit

③ upper bound for European call premium

No matter what happens, $C \leq S_0$

If not, there will be an opportunity for a riskless profit by buying the stock and selling the call option. How? Suppose $C > S_0$.

	Time $t=0$	Payoff at time $t=1$	
		$S_1 > E$	$S_1 \leq E$
Sell 1 call	C	$E - S_1$	0
Buy 1 stock	$-S_0$	S_1	S_1
Total	$C - S_0 > 0$	E	S_1
profit		$C-S_0+E$	$C-S_0+S_1$

the seller can make riskless profit

④ upper bound for European put premium

No matter what happens, $P \leq \frac{E}{1+r}$.
If not, there will be an opportunity for a riskless profit by selling the put and investing the proceeds at the risk free interest rate. How? Suppose $P > \frac{E}{1+r}$.

	Time $t=0$	Payoff at time $t=1$	
		$S_1 \geq E$	$S_1 < E$
Sell 1 put	$P > \frac{E}{1+r}$	0	$S_1 - E$
profit		$P(1+r)$	$P(1+r)+S_1-E > 0$

$P > \frac{E}{1+r}$
$P(1+r) - E > 0$

put-call parity

relationship between put premium and call premium
put & stock can be combined in a way that give some payoff as call

Portfolio A: Buy the call and lend an amount of cash equal to $\frac{E}{1+r}$.
Portfolio B: Buy the stock, buy the put.

	Time $t=0$	Payoff at time $t=1$	
		$S_1 > E$	$S_1 \leq E$
Portfolio A:			
Buy 1 call	$-C$	$S_1 - E$	0
Lend cash	$-\frac{E}{1+r}$	E	E
Total	$-C - \frac{E}{1+r}$	S_1	E

		$S_1 \geq E$	$S_1 < E$
Portfolio B:			
Buy 1 put	$-P$	0	$E - S_1$
Buy 1 stock	$-S_0$	S_1	S_1
Total	$-P - S_0$	S_1	E

$$c + \frac{E}{1+r} = p + S_0 \quad \text{or} \quad c + Ee^{-rt} = p + S_0.$$

if this doesn't hold. there is opportunity for riskless profit

e.g.1. $S_0 = \$30, E = \$28, r = 10\%$ per year, and $t = 3$ months to expiration.
Suppose $c = \$4$ and $p = \$3$.

Let's compute both sides of the put-call parity equation.
$c + Ee^{-rt} = 4 + 28e^{-0.10 \times \frac{3}{12}} = \$31.31$.
$p + S_0 = 3 + 30 = \$33$.

The second portfolio is overpriced compared to the first portfolio. Therefore,

- Short the put and the stock
- Buy the call

{ short the stock. short = borrow and sell
short the put. short = sell (sign the contract)

Explain:
How much is the cash inflow at $t = 0$? $30 + 3 - 4 = 29$
How much will it grow in 3 months? $29 \cdot e^{0.1 \frac{3}{12}} = 29.73$

At expiration (in 3 months):
If stock price $S_T > 28$ then ... we exercise call option. (we are call buyer). buy stock at 28. return the stock. make \$1.73
If stock price $S_T < 28$ then ... the put buyer exercise put option (we are put seller). buy stock at 28. return the stock. make \$1.73

e.g.2. Suppose $S_0 = \$30, E = \$28, r = 10\%$ per year, and $t = 3$ months to expiration.
Suppose $c = \$4$ and $p = \$1$.

Let's compute both sides of the put-call parity equation.
$c + Ee^{-rt} = 4 + 28e^{-0.10 \times \frac{3}{12}} = \$31.31$.
$p + S_0 = 3 + 30 = \$31$.

The second portfolio is underrpriced compared to the first portfolio. Therefore,

- Borrow \$31 to buy the put and the stock $-30-1$
- Sell the call $+4$

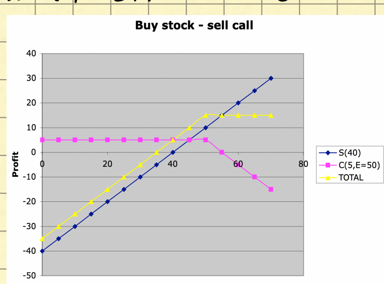
$\} -27$

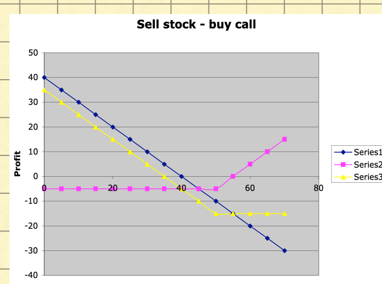
In 3 months we must return $\underline{27} \times e^{0.1 \frac{3}{12}} = 27.68$.

At expiration (in 3 months):
If stock price $S_T > 28$ then call buyer exercise call (we are call seller). sell stock at 28. cover loan \$27.68. make \$0.32
If stock price $S_T < 28$ then ... we exercise put (we are put buyer). sell stock at 28. cover loan \$27.68. make \$0.32

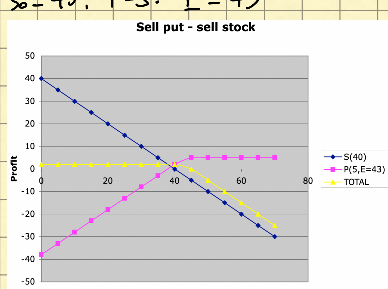
LEC 19 call/put + stock

$S_0 = 40$, $C/P = 5$. $E = 50$

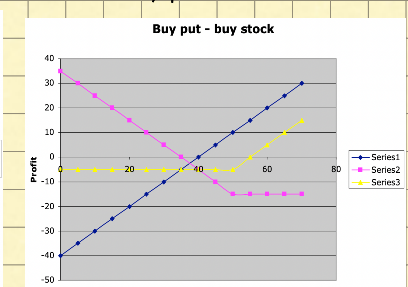




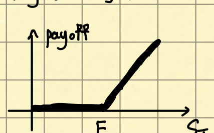
$S_0 = 40$, $P = 5$. $E = 43$



$S_0 = 40$, $P = 15$. $E = 50$

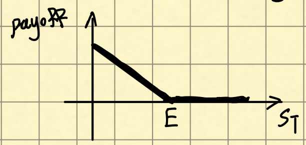


① buy put
buy stock } $C + Ee^{-rt} = P + S_0$
same as buy call + cash

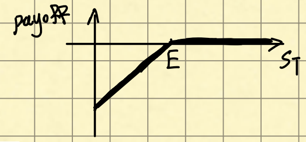


② sell put
sell stock } same as sell call − cash

③ buy call }
sell stock } $C - S = P - Ee^{-rt}$ same as buy put − cash



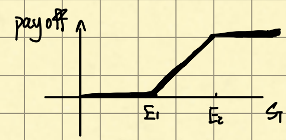
④ "writing a covered call"
sell call }
buy stock } $S - C = Ee^{-rt} - P$



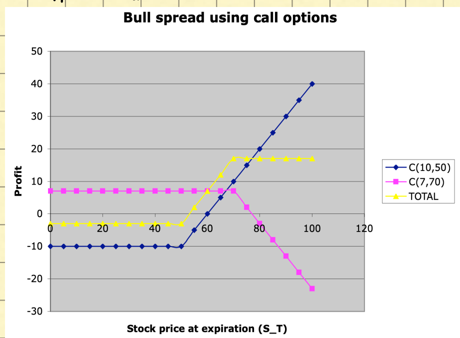
spread:
≥ 2 calls/puts
bull spread

Bull spread:
Buy one call with exercise price E_1
Sell one call with exercise price E_2
Note: $E_1 < E_2$

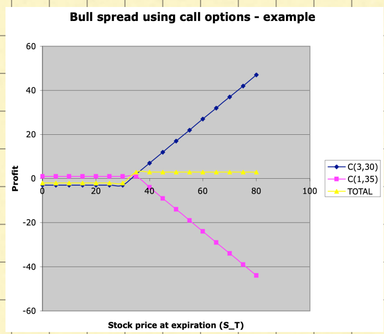
Stock price at expiration	Payoff from long call with exercise price E_1	Payoff from short call with exercise price E_2	Total payoff
$S_T < E_1$	0	0	0
$E_1 < S_T < E_2$	$S_T - E_1$	0	$S_T - E_1$
$S_T > E_2$	$S_T - E_1$	$-(S_T - E_2)$	$E_2 - E_1$



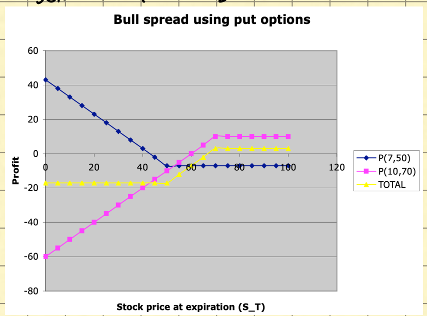
e.g. 1. buy $C_1 = 10$. $E_1 = 50$
sell $C_2 = 7$. $E_2 = 70$



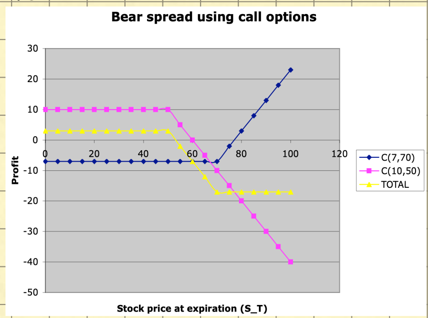
e.g. 2. buy $C_1 = 3$. $E_1 = 30$
sell $C_2 = 1$. $E_2 = 35$



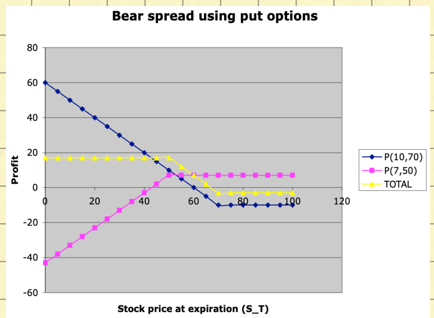
e.g. 3. buy $P_1 = 7$. $E_1 = 50$
sell $P_2 = 10$. $E_2 = 70$



e.g. 4. buy $C_1 = 7$. $E_1 = 70$
sell $C_2 = 10$. $E_2 = 50$

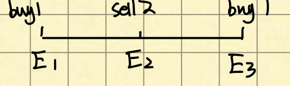


e.g. 5. buy $P_1 = 10$. $E_1 = 70$
sell $P_2 = 7$. $E_2 = 50$



LEC 20 butterfly spread

① butterfly spread using call option

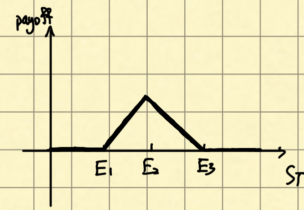


$E_2 = \frac{E_1 + E_3}{2}$

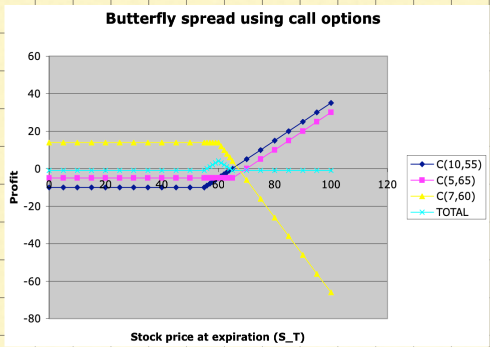
Buy one call with exercise price E_1
Buy one call with exercise price E_3
Sell two calls with exercise price E_2 (halfway between E_1 and $E3$)

Stock price at expiration	Payoff from long call with exercise price E_1	Payoff from two short calls with exercise price E_2	Payoff from long call with exercise price E_3	Total payoff
$S_T < E_1$	0	0	0	0
$E_1 < S_T < E_2$	$S_T - E_1$	0	0	$S_T - E_1$
$E_2 < S_T < E_3$	$S_T - E_1$	$-2(S_T - E_2)$	0	$E_3 - S_T$
$S_T > E_3$	$S_T - E_1$	$-2(S_T - E_2)$	$S_T - E_3$	0

$E_3 - S_T \Leftarrow 2E_2 - E_1 - S_T,\ E_2 = \frac{E_1 + E_3}{2}$



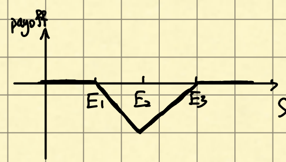
e.g.
buy (1) $C_1 = 10$ $E_1 = 55$
sell (2) $C_2 = 5$ $E_2 = 65$
buy (1) $C_3 = 7$ $E_3 = 60$



② reverse butterfly spread using call option

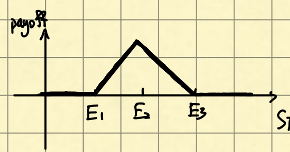
sell 1 buy 2 sell 1
E_1 E_2 E_3

	E_1	E_2	E_3	total
$S_T < E_1$	0	0	0	0
$E_1 < S_T < E_2$	$-(S_T - E_1)$	0	0	$E_1 - S_T$
$E_2 < S_T < E_3$	$-(S_T - E_1)$	$2(S_T - E_2)$	0	$S_T - E_3$
$S_T > E_3$	$-(S_T - E_1)$	$2(S_T - E_2)$	$-(S_T - E_3)$	0



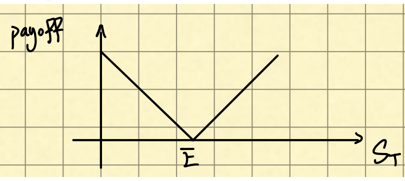
③ butterfly spread using put option

	E_1	E_2	E_3	total
$S_T < E_1$	$E_1 - S_T$	$-2(E_2 - S_T)$	$E_3 - S_T$	0
$E_1 < S_T < E_2$	0	$-2(E_2 - S_T)$	$E_3 - S_T$	$S_T - E_1$
$E_2 < S_T < E_3$	0	0	$E_3 - S_T$	$E_3 - S_T$
$S_T > E_3$	0	0	0	0

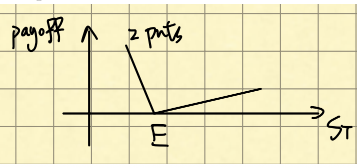


Combinations

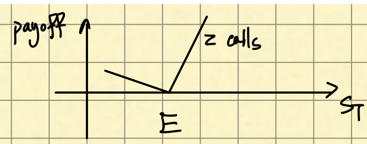
1. Straddle: Buy a call and a put with the same exercise price and the same time to expiration.



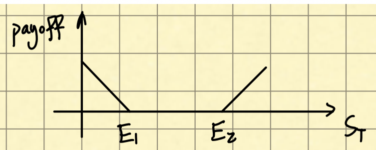
2. Strip: Buy one call and two puts with the same exercise price and the same time to expiration.



3. Strap: Buy two calls and one put with the same exercise price and the same time to expiration.



4. Strangle: Buy a call and a put with the same time to expiration but different exercise prices.



lower/upper bound for European call/put premium
put-call parity w/dividend

lower bound for European call premium when stock pays dividends

		t=0	t=1: $S_1>E$	$S_1\le E$
A.	buy call	$-C$	S_1-E	0
	lend cash	$-D-Ee^{-rt}$	$E+D^*$	$E+D^*$
			S_1+D^*	D^*+E
B.	buy stock	$-S_0$	S_1+D^*	S_1+D^*

D ———— D^*
0 ———— T

same

$E > S_1$

at expiration
A is at least as good as B
at time 0 we should have
$-C-D-Ee^{-rt} \le -S_0$
$C \ge S_0 - D - Ee^{-rt}$

D is present value of dividend pay.

LEC 21

lower bound for European put premium when stock pays dividends

		t=0	t=1: $S_T>E$	$S_T\le E$
A.	buy put	$-P$	0	$E-S_T$
	buy stock	$-S_0$	S_T+D^*	S_T+D^*
			S_T+D^*	D^*+E
B.	cash	$-D-Ee^{-rt}$	D^*+E	D^*+E

A is at least as good as B
$-P-S_0 \le -D-Ee^{-rt}$
$P \ge D+Ee^{-rt}-S_0$

Put-Call parity when stock pays dividends

		t=0	t=1: $S_T>E$	$S_T\le E$
A	Buy call	$-C$	S_T-E	0
	cash	$-D-Ee^{-rt}$	D^*+E	D^*+E
			S_T+D^*	D^*+E
B	buy put	$-P$	0	$E-S_T$
	buy stock	$-S_0$	S_T+D^*	S_T+D^*
			S_T+D^*	D^*+E

At expiration, A = B, at t=0, we should have $C+D+Ee^{-rt} = P+S_0$

e.g. consider a European call and put on a same stock with $S_0=19$, $E=20$, $r=10\%$ per year, $p=3$, $c=3$. a dividend of \$1 is expected in 1 month. Identify the riskless opportunity and the present value

profit at expiration discounted back to now use risk-free interest rate

$C=3$
$P=3$

$S_0=19$ at $t=0$; \$1 at $t=1$ month; $E=20$ at $t=3$ month

$C+D+Ee^{-rt} = P+S_0$

$3 + 1\cdot e^{-0.1\frac{1}{12}} + 20e^{-0.1\frac{3}{12}} - 19 = 4.5$

the put is underpriced

strategy: buy put -3; buy stock -19; sell call $+3$ — need to borrow 19, invester must return in 3 month $19 \times e^{0.1\frac{3}{12}} = 19.5$

at expiration. if $S_T > 20$, call will be exercised, sell stock at 20
if $S_T < 20$, put will be exercised, sell stock at 20
$20 - 19.5 = 0.5$

the present value of this riskless strategy $1\cdot e^{-0.1\frac{1}{12}} + 0.5 \cdot e^{-0.1\cdot\frac{3}{12}}$

American option

Parameters affecting option prices. Suppose the parameters listed next increase.

Parameter	European call	European put	American call	American put
Stock price S_0 ↑	up	down	up	down
Exercise price E ↑	down	up	down	up
Volatility σ ↑	up	up	up	up
Dividends paid ↑	down	up	down	up
Interest rate ↑	down	up	down	up

It is never optimal to exercise early an American call when the stock does not pay dividends.
We have seen that the lower bound for a European call is $c \geq S_0 - Ee^{-rt}$. The same holds for an American call, $C \geq S_0 - Ee^{-rt}$. If at a certain point in time before expiration $S > E$ there is an opportunity for early exercise to make a profit of $S - E$. For example if $E = \$50$ and $S = \$60$ we can make a profit of \$10. However, looking at the lower bound we see that since $r > 0$ it follows that $C > S - E$, therefore it is better not to exercise the American call option early.

$S_1 = 60$ $\quad E = S_0$

$t_0 = 0$ $\quad t_1 = 2$ month $\quad t_2 = 3$ month

$C_1 \geq S_1 - Ee^{-r(t_2 - t_1)} > S_1 - E$

price of call > payoff of exercise call

It can be optimal to exercise an American put early when the stock does not pay dividends for a sufficiently low stock price.

$S_1 = 50$ $\quad E = 60$

$t_0 = 0$ $\quad t_1 = 2$ month $\quad t_2 = 3$ month

$P_1 \geq Ee^{-r(t_2 - t_1)} - S_1 < E - S_1$

the put-call parity holds only for European option : $c + Ee^{-rt} = p + S_0$
For American options. we can find bounds for $C - P$: $a \leq C - P \leq b$

$P \geq p$ or $P \geq c + Ee^{-rt} - S_0$

the stock does not pay dividends, $C = c$

$P \geq C + Ee^{-rt} - S_0$ or $C - P \leq S_0 - Ee^{-rt}$

for the lower bound of $C - P$ consider the following

		t=0	t=1 $S_T > E$	$S_T \leq E$
A	buy call	$-c$	$S_T - E$	0
	cash	$-E$	Ee^{rt}	Ee^{rt}
			$S_T - E + Ee^{rt}$	Ee^{rt}
B	buy put	$-P$	0	$E - S_T$
	buy stock	$-S_0$	S_T	S_T
			S_T	E

case 1 : no early exercise of American put
at expiration. B is worth $\max(S_T, E)$
A is worth $\max(S_T - E, 0) + Ee^{rt}$ or $\max(S_T, E) - E + Ee^{rt}$
$A \geq B$

case 2 : put option is exercised early at $t < T$, thus $S_T \leq E$
at t, B is worth E
A is worth $E \cdot e^{rt}$
$A \geq B$

0 $\quad$ t $\quad$ T

in all 2 cases. A is better than B, thus $-C - E \leq -P - S_0$
$C - P \geq S_0 - E$

$\Rightarrow S_0 - E \leq C - P \leq S_0 - Ee^{-rt}$

American option w/ dividend

$S_0 - E - D \leq C - P \leq S_0 - Ee^{-rt}$
right side : when pay dividend. $C \downarrow$ $P \uparrow$. thus $C - P \leq S_0 - Ee^{-rt}$ still holds
left side :

		t=0	t=1 $S_T > E$	$S_T \leq E$
A	buy call	$-c$	$S_T - E$	0
	cash	$-E-D$	$D^* + Ee^{rt}$	$D^* + Ee^{rt}$
			$D^* + S_T - E + Ee^{rt}$	$D^* + Ee^{rt}$
B	buy put	$-P$	0	$E - S_T$
	buy stock	$-S_0$	$S_T + D^*$	$S_T + D^*$
			$S_T + D^*$	$E + D^*$

same. $A \geq B$ in 2 cases
$-C - D - E \leq -P - S_0$
$C - P \geq S_0 - D - E$

LEC22 binomial model 1-step original paper : handout #48

one step first $t=3$ month. $E=50$. $r=10\%$ per year.
find price of call option at $t=0$ using no arbitrage argument

$S_0=48 <$ $S_T=55$, $S_T=45$

	$t=0$	$t=3$ months $S_T=55$	$S_T=45$
write one call	$+C$	-5	0
buy α shares	-48α	55α	45α

$$-5+55\alpha = 45\alpha$$

$\alpha=\frac{1}{2}$. $\underline{\text{hedge ratio}}$ in practice people update every week

$$-5+55\cdot\frac{1}{2} = 22.5$$

the payoff $=22.5$ at expiration is riskless regardless of $S_T=55$ or 45

$$-C+48\cdot\frac{1}{2} = 22.5\cdot e^{-0.1\frac{3}{12}}$$

$$C=2.0555$$

the price for put can be obtained using put-call parity $P+S_0 = C+Ee^{-rt}$

$$P = 2.0555+50e^{-0.1\frac{3}{12}}-48 = 2.821$$

in general :

	$t=0$	$t=T$ $S_T=uS_0$	$S_T=dS_0$
write 1 call	$+C$	$-C_u$	$-C_d$
buy α share	$-\alpha S_0$	αuS_0	αdS_0

$S_0 <$ $uS_0 <$ u^2S_0, udS_0; dS_0

$u=e^{\sigma\sqrt{\frac{t}{n}}}$ volatility/std, t years, n steps $\frac{t}{n}$ is length of one binomial period

$d=\frac{1}{u}$

$u>0$: upper, $d<0$: down, $d=\frac{1}{u}$

$$-C_u+\alpha uS_0 = -C_d+\alpha dS_0$$

$$\alpha = \frac{C_u-C_d}{uS_0-dS_0}$$

$C_u=\max(S_T-E,0)$, where $S_T=uS_0$ C_u Price of call at $t=1$ if stock price increases

$C_d=\max(S_T-E,0)$, where $S_T=dS_0$ C_d Price of call at $t=1$ if stock price decreases

$$\underbrace{-C+\alpha S_0}_{t=0} = (\underbrace{-C_u+\alpha uS_0}_{t=1})e^{-rt} \quad \text{per period}$$

$$C=\alpha S_0-(\alpha uS_0-C_u)e^{-rt}$$

$$=(\alpha S_0e^{rt}-\alpha uS_0+C_u)e^{-rt}$$

$$=\left[\frac{C_u-C_d}{uS_0-dS_0}S_0e^{rt}-\frac{C_u-C_d}{uS_0-dS_0}uS_0+C_u\right]e^{-rt}$$

$$=\left[\frac{C_u-C_d}{u-d}e^{rt}-\frac{C_u-C_d}{u-d}u+\frac{C_u(u-d)}{u-d}\right]e^{-rt}$$

$$C=\left[\frac{e^{rt}-d}{u-d}C_u+\frac{u-e^{rt}}{u-d}C_d\right]e^{-rt}$$

let $p=\frac{e^{rt}-d}{u-d}$ be risk-neutral prob (prob of up movement)

$1-p=\frac{u-e^{rt}}{u-d}$ be prob of down movement

$$C=\underbrace{(C_up+C_d(1-p))}_{\text{expected payoff at expiration}}\underbrace{e^{-rt}}_{\text{discounted at time } t=0}$$

$\Rightarrow$ the price of a European call with one period to expiration

$S_0 <$
uS_0 $C_u=\max(uS_0-E,0)$ p
dS_0 $C_d=\max(dS_0-E,0)$ $1-p$

for the stock price $E[S_T]=uS_0p+dS_0(1-p)$

$$=uS_0\frac{e^{rt}-d}{u-d}+dS_0\frac{u-e^{rt}}{u-d}$$

$$=\cdots=S_0e^{rt}$$

Back to the numerical example, $t = 3$ month. $E = 50$. $r = 10\%$ per year. $S_0 = 48$ < $S_T = 55$ / $S_T = 45$. find p.

Besides using no arbitrage argument, we can also use risk-neutral valuation:

$$E[S_T] = 55p + 45(1-p) = 48e^{0.1\frac{3}{12}}$$

$$p = \frac{48e^{0.1\frac{3}{12}} - 45}{10} = 0.4215$$

$$C = (5 \cdot p + 0(1-p))\, e^{-0.1\frac{3}{12}} = 2.0555$$

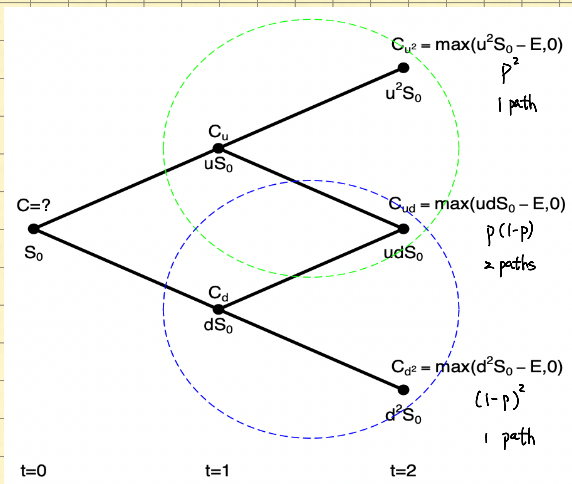
e.g. 2 $E = 50$ $C_u = 2.5$ $C_d = 0$. $r = 0.01$ $S_0 = 50$ < $S_1 = \underbrace{1.05}_{u}(50) = 52.5$ / $S_1 = \underbrace{0.95}_{d}(50) = 47.5$

$$p = \frac{1+r-d}{u-d} = \frac{1.01 - 0.95}{1.05 - 0.95} = 0.6$$

$$C = \frac{2.5p + 0(1-p)}{1+r} = \frac{2.5 \times 0.6}{1.01} = 1.4851$$ using risk-neutral valuation

verify using no arbitrage argument that C is also 1.4851

LEC 23 binomial 2-step



Let's examine the upper branch (top circle) of this two-step binomial tree:

	Flows at $t=1$	Flows at $t=2$	
Action		$S_2 = u^2S_0$	$S_2 = udS_0$
Write 1 call	C_u	$-C_{u^2}$	$-C_{ud}$
Buy α shares of stock	$-\alpha uS_0$	αu^2S_0	αudS_0

This will be a hedged portfolio if $-C_{u^2} + \alpha u^2S_0 = -C_{ud} + \alpha udS_0$, and solving for α we get $\alpha = \frac{C_{u^2} - C_{ud}}{u^2S_0 - udS_0}$. Since at $t=2$ the payoff is riskless the portfolio that we constructed at time $t=1$ must have earned the risk free interest rate, i.e. $(-C_u + \alpha uS_0)(1+r) = \alpha udS_0 - C_{ud}$. Solve for C_u to get:

$$C_u = \frac{C_{u^2}p + C_{ud}(1-p)}{1+r}. \quad (2)$$

Similarly, we examine the lower branch (bottom circle) of the two-step binomial tree:

	Flows at $t=1$	Flows at $t=2$	
Action		$S_2 = udS_0$	$S_2 = d^2S_0$
Write 1 call	C_d	$-C_{ud}$	$-C_{d^2}$
Buy α shares of stock	$-\alpha dS_0$	αudS_0	αd^2S_0

This will be a hedged portfolio if $-C_{ud} + \alpha udS_0 = -C_{d^2} + \alpha d^2S_0$, and solving for α we get $\alpha = \frac{C_{ud} - C_{d^2}}{udS_0 - d^2S_0}$. Since at $t=2$ the payoff is riskless the portfolio that we constructed at time $t=1$ must have earned the risk free interest rate, i.e. $(\alpha dS_0 - C_u)(1+r) = -C_{d^2} + \alpha d^2S_0$. Solve for C_d to get:

$$C_d = \frac{C_{ud}p + C_{d^2}(1-p)}{1+r}. \quad (3)$$

$$C = \frac{C_u p + C_d(1-p)}{1+r} \quad (1)$$

Using equations (2) and (3) we update equation (1):

$$C = \frac{\frac{C_{u^2}p + C_{ud}(1-p)}{1+r}p + \frac{C_{ud}p + C_{d^2}(1-p)}{1+r}(1-p)}{1+r}$$

or

$$C = \frac{C_{u^2}p^2 + 2C_{ud}p(1-p) + C_{d^2}(1-p)^2}{(1+r)^2}.$$

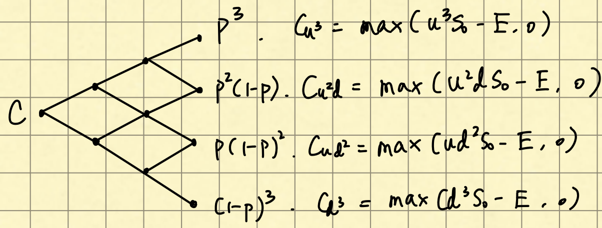
This is the price of the call with two periods to expiration.

Complete the three-step binomial tree and express C as a function of $C_{u^3}, C_{d^3}, C_{u^2d}, C_{ud^2}, p, (1-p), r$.

binomial n-step

binomial coeff. # paths

$$C = \frac{C_{u^3}p^3 + 3C_{u^2d}\,p^2(1-p) + 3C_{ud^2}\,p(1-p)^2 + C_{d^3}(1-p)^3}{(1+r)^2} = \frac{\sum_{j=0}^{3}\binom{3}{j} C_{u^j d^{3-j}}\, p^j (1-p)^{3-j}}{(1+r)^3}$$



LEC24

In general if we divide the time to expiration into n periods we get

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} C_{u^j d^{n-j}}}{(1+r)^n}, \text{ where } C_{u^j d^{n-j}} = \max(u^j d^{n-j} S_0 - E, 0).$$

This expression can be simplified because the call is not always in the money at the end of the nth period. For the call to be in the money we want a minimum of k up movements of the stock. Therefore if we find k the summation will simply begin from $j = k$. The call is in the money as long as $u^k d^{n-k} S_0 - E > 0$ and solving for k we get $k = \frac{log(\frac{E}{d^n S_0})}{log(\frac{u}{d})}$. And the price of the call is equal to:

$k = 2.2$. round up to 3 for call (round down to 2 for put)

$$\begin{aligned}
C &= \frac{\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} C_{u^j d^{n-j}}}{(1+r)^n} = \frac{\sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} C_{u^j d^{n-j}}}{(1+r)^n} \\
C &= \frac{\sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \left(u^j d^{n-j} S_0 - E\right)}{(1+r)^n} \\
C &= S_0 \left[\sum_{j=k}^{n} \frac{n!}{j!(n-j)!} \frac{(pu)^j [(1-p)d]^{n-j}}{(1+r)^n} \right] - \frac{E}{(1+r)^n} \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}
\end{aligned}$$

Aside: If we let $p' = \frac{pu}{1+r}$ then $\frac{(pu)^j [(1-p)d]^{n-j}}{(1+r)^n} = p'(1-p')^{n-j}$ and the price of the call is equal to:

$$\begin{aligned}
C &= S_0 \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p'^j (1-p')^{n-j} - \frac{E}{(1+r)^n} \sum_{j=k}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \\
C &= S_0 P(X \geq k) - \frac{E}{(1+r)^n} P(Y \geq k), \text{ where } X \sim b(n, p'), \text{and } Y \sim b(n, p).
\end{aligned}$$

Find an expiration for the price of a European put using the binomial model

use put-call parity $p + S_0 = c + Ee^{-rt}$

$$\begin{aligned}
p &= c + Ee^{-rt} - S_0 \\
&= S_0 P(X \geq k) - Ee^{-rt} P(Y \geq k) + Ee^{-rt} - S_0 \\
&= Ee^{-rt} (1 - P(Y \geq k)) - S_0 (1 - P(X \geq k)) \\
p &= Ee^{-rt} P(Y < k) - S_0 P(X < k)
\end{aligned}$$

e.g.

Using the binomial option pricing model find the price of a European call if $S_0 = \$30, E = \$29, \sigma = 0.30, r = 0.05$, with 73 days to expiration ($\frac{1}{5}$ of a year), and $n = 5$ periods (five-step binomial tree).

per year

5 periods = 73 days

$$\begin{aligned}
u &= e^{\sigma\sqrt{\frac{t}{n}}} = exp(0.30\sqrt{\frac{0.2}{5}}) = 1.061837. \\
d &= e^{-\sigma\sqrt{\frac{t}{n}}} = \frac{1}{u} = 0.941764.
\end{aligned}$$

per period

$$\begin{aligned}
r_p &= (1.05)^{\frac{1}{25}} - 1 = 0.001954. \\
p &= \frac{1 + r_p - d}{u - d} = 0.50128. \\
p' &= \frac{pu}{1 + r_p} = 0.53124. \\
k &= \frac{log(\frac{E}{d^n S_0})}{log(\frac{u}{d})} = 2.22 \Rightarrow k = 3.
\end{aligned}$$

method 1:

$$C = 30P(X \geq 3) - \frac{29}{1 + 0.001954)^5} P(Y \geq 3) = 2.325.$$

method 2

$$\frac{11.496 \cdot p^5 + 5\,(6.917)\, p^4 (1-p) + 10\,(2.855)\, p^3 (1-p)^2}{(1+r_p)^5}$$

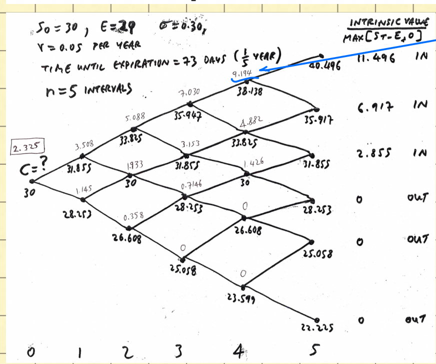
Note: $X \sim b(5, 0.53124)$ and $Y \sim b(5, 0.50128)$.

The R command:

method 3. step by step

$$\frac{11.496 \cdot p + 6.917\,(1-p)}{1 + r_p} = 9.194$$

```
30*pbinom(2,5,0.53124, lower.tail=FALSE) -
(29/(1+.001954)^5)*pbinom(2, 5, 0.50128, lower.tail=FALSE)
```



put & American put examples . see handout # 50

LEC25 stochastic process

$t=0$ now — T $\begin{cases} \text{discrete time} \\ \text{continuous time} \end{cases}$

markov process : only current value of r.v. matters to predict future

wiener process : a r.v. z follow a wiener process if the change in z is given by interval Δt

① $\Delta z_t = \varepsilon\sqrt{\Delta t}$ where $\varepsilon \sim N(0,1) \Rightarrow \Delta z_t \sim N(0, \sqrt{\Delta t})$

② Δz_1, Δz_2 are independent in two intervals $\Delta t_1, \Delta t_2$ (day 1 & day 2 changes are independent)

$z(0)$ Δt_1, Δt_2, ... Δt_n $z(T)$; 0 ... T $(\Delta t_1 = \Delta t_2 = 1 \text{ day})$

$$\Delta z = z(T) - z(0)$$
$$= \Delta z_1 + \Delta z_2 + \cdots + \Delta z_n$$
$$= \varepsilon_1\sqrt{\Delta t_1} + \varepsilon_2\sqrt{\Delta t_2} + \cdots + \varepsilon_n\sqrt{\Delta t_n}$$

e.g. Suppose $S_0 = 50$. $E = 1$ year

(a) find the distri of the stock price in 1 year

S_0 S_1; 0 1 $\quad S_1 = S_0 + \Delta S_1 = 50 + \varepsilon_1\sqrt{\Delta t_1} \sim N(50, 1)$

(b) $t = 2$ years $\quad S_2 = S_0 + \Delta S_1 + \Delta S_2$

S_0 S_1 S_2; ΔS_1 ΔS_2; 0 1 2 $\quad = 50 + \varepsilon_1\sqrt{\Delta t_1} + \varepsilon_2\sqrt{\Delta t_2} \sim N(50, \sqrt{2})$

(c) in 5 years $\quad S_5 \sim N(50, \sqrt{5})$

time↑ std↑ mean stay same

generalized wiener process → mean change over time

A r.v. X follows a generalized wiener process if the change in X is given by $\Delta X = a\Delta t + b\varepsilon\sqrt{\Delta t}$

$\left.\begin{array}{l} a \text{ drift rate} \\ b^2 \text{ Variance} \end{array}\right\}$ per Δt

e.g. $S_0 = 50$. $\underbrace{a = 20. \; b = 30}_{\text{per year}}$ $\quad b$ → std. $\quad$ variance $= b^2 = 900$

(a) in 1 year, $S_1 = S_0 + \Delta S_1 = 50 + 20\Delta t_1 + 30\varepsilon\sqrt{\Delta t_1} \sim N(70, 30)$

(b) in 2 years, $S_2 = S_0 + \Delta S_1 + \Delta S_2 = 50 + 20\Delta t_1 + 30\varepsilon_1\sqrt{\Delta t_1} + 20\Delta t_2 + 30\varepsilon_2\sqrt{\Delta t_2} \sim N(90, 30\sqrt{2})$

(sum of independent normal is normal)

model for stock prices $\quad \frac{\Delta S}{S} = \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t} \sim N(\mu\Delta t, \sigma\sqrt{\Delta t})$

$$\Delta S = \mu S\Delta t + \sigma S\varepsilon\sqrt{\Delta t}$$

S follows a generalized wiener process with drift rate μS and variance $\sigma^2 S^2$ $\Big\}$ per Δt

e.g.

The current price of a stock is $S_0 = \$100$. The expected return is $\mu = 0.10$ per year, and the standard deviation of the return is $\sigma = 0.20$ (also per year).

2. Find the distribution of the change in S divided by S (distribution of $\frac{\Delta S}{S}$).

$$\frac{\Delta S}{S} = \mu\Delta T + \sigma\varepsilon\sqrt{\Delta t}$$
$$\frac{\Delta S}{S} \sim N(0.1\Delta t, 0.2\sqrt{\Delta t})$$

3. Divide the year in weekly intervals and find the distribution of $\frac{\Delta S}{S}$ at the end of each weekly interval.

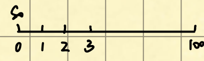
$$\frac{\Delta S}{S} \sim N\left(0.1\frac{1}{52}, 0.2\frac{1}{\sqrt{52}}\right)$$

52 weeks

4. Repeat (3) by assuming daily intervals.

52 → 365 days / 250 trading days

e.g.

simulation of the path of a stock  $\Delta t = 0.01$ year (3.65 days)

$\underbrace{\mu = 0.15. \quad \sigma = 0.2}$

markov also needs to estimate from historical data

$\Delta S = 0.15 \cdot S \cdot 0.01 + 0.2 \cdot S \cdot \varepsilon \cdot \sqrt{0.01}$

$\Delta S_1 = 0.15 \cdot S_0 \cdot 0.01 + 0.2 \cdot S_0 \cdot \underline{\text{random normal (1)}} \sqrt{0.01}$

can be negative

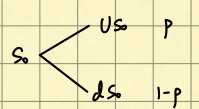
$S_1 = S_0 + \Delta S_1$

$\Delta S_2 = 0.15 \cdot S_1 \cdot 0.01 + 0.2 \cdot S_1 \cdot \text{random normal (1)} \sqrt{0.01}$

$S_2 = S_1 + \Delta S_2$

binomial model determination of U, d, p

$U = e^{\sigma\sqrt{\frac{t}{n}}} = e^{\sigma\sqrt{dt}}$



in risk return world we have

$E[S_T] = S_0\, e^{rdt} = US_0 p + d S_0 (1-p)$

first result $p = \dfrac{e^{rdt} - d}{u - d}$ and $1 - p = \dfrac{u - e^{rdt}}{u - d}$

for the variance : $Var(y) = Eg^2 - (Ey)^2$

from generalized wiener process

$\underline{U^2 S_0^2 p + d^2 S_0^2 (1-p)} - \underline{S_0^2 e^{2rdt}} = \sigma^2 S_0^2 (dt)$ square of $\varepsilon\sqrt{\Delta t}$

$U^2 p + d^2 (1-p) - e^{2rdt} = \sigma^2 dt$

find U, d

$$\frac{e^{rdt} - d}{U - d} u^2 + \frac{U - e^{rdt}}{U - d} d^2 - e^{2rdt} = \sigma^2 dt$$

$$\frac{e^{rdt}(u^2 - d^2) - ud(u-d)}{u-d} - e^{2rdt} = \sigma^2 dt$$

$$e^{rdt}(u+d) - e^{2rdt} - ud = \sigma^2 dt \quad (\text{assume } d = \tfrac{1}{u} \rightarrow ud = 1)$$

use series expansion of $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots$

ignore terms involving dt^2 because they are too small $e^{rdt} = 1 + \frac{rdt}{1!}$

$e^{2rdt} = 1 + \frac{2rdt}{1!}$

conclusion
$$\begin{cases} U = e^{\sigma\sqrt{dt}} = 1 + \dfrac{\sigma\sqrt{dt}}{1!} + \dfrac{\sigma^2 dt}{2!} + \cdots \\ d = e^{-\sigma\sqrt{dt}} = 1 - \dfrac{\sigma\sqrt{dt}}{1!} + \dfrac{\sigma^2 dt}{2!} - \cdots \end{cases}$$

$(1 + rdt)(u+d) - (1 + 2rdt) - 1 = \sigma^2 dt$

left $= (1+rdt)(1 + \cancel{\sigma\sqrt{dt}} + \frac{\sigma^2 dt}{2} + 1 - \cancel{\sigma\sqrt{dt}} + \frac{\sigma^2 dt}{2}) - (1 + 2rdt) - 1$

$= (1 + rdt)(2 + \sigma^2 dt) - (1 + 2rdt) - 1$

$= \cancel{2} + \sigma^2 dt + \cancel{2rdt} + \cancel{\sigma^2 r (dt)^2} - \cancel{1} - \cancel{2rdt} \cancel{-1}$

$= \sigma^2 dt$

LEC 26 ITO process

generalized wiener process: $\Delta x = a \cdot \Delta t + b \cdot \varepsilon\sqrt{\Delta t}$

ITO process: $\Delta x = a(x,t) \cdot \Delta t + b(x,t) \cdot \varepsilon\sqrt{\Delta t}$. a, b are fn of x & t

taylor $\Delta G \approx \frac{\partial G}{\partial x}\underset{\sim}{\Delta x} + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\underset{\sim}{(\Delta x)^2} + \frac{\partial^2 G}{\partial x \partial t}\underset{\sim}{\Delta x \Delta t} + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}(\Delta t)^2$

$\Delta x = a\Delta t + b\Delta z$

only keep up to order Δt. $(E[(\Delta z)^2] = \Delta t)$

$\Delta G \approx \frac{\partial G}{\partial x}(a\Delta t + b\Delta z) + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 (\Delta z)^2$

ITO lemma (derivative chain rule of r.v.). G is a fn of x & t

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2\right) dt + \frac{\partial G}{\partial x} b \underbrace{\varepsilon\sqrt{dt}}_{\Delta z}$$

$\Rightarrow$ G follows a generalized wiener process with $\begin{cases} \text{drift rate } \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2\right) \text{ per } dt \\ \text{variance } \left(\frac{\partial G}{\partial x}\right)^2 b^2 \end{cases}$

stock price model $\Delta S = \mu S \Delta t + \sigma S \Delta z = \mu S \Delta t + \sigma S \varepsilon\sqrt{\Delta t}$ $(a \rightarrow \mu S, b \rightarrow \sigma S)$

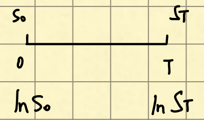
let G be a fn of x & t

$$dG = \Big(\underbrace{\frac{dG}{dS}}_{\frac{1}{S}} \mu \cancel{S} + \underbrace{\frac{dG}{dt}}_{0} + \frac{1}{2}\underbrace{\frac{d^2 G}{dS^2}}_{-\frac{1}{S^2}} \sigma^2 \cancel{S^2}\Big) dt + \underbrace{\frac{dG}{dS}}_{\frac{1}{S}} \sigma \cancel{S} \varepsilon\sqrt{dt}$$

let $G = \ln S$, $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2} = \frac{1}{S^2}$, $\frac{\partial G}{\partial t} = 0$

$$\Rightarrow dG = \underbrace{(\mu + \frac{\sigma^2}{2})}_{\text{fixed}} dt + \sigma \underbrace{\varepsilon}_{\sim N(0,1)} \sqrt{dt}$$

G follows a generalized wiener process with $\begin{cases} \text{drift rate} & \mu - \frac{\sigma^2}{2} \\ \text{variance} & \sigma^2 \end{cases}$



from t (now) to T. $dt = T - t$. examine distri of $\Delta G = \ln(S_T) - \ln(S_0)$

$$\Delta G = \ln S_T - \ln S_0 \sim N\left((\mu - \tfrac{1}{2}\sigma^2) dt, \sigma\sqrt{dt}\right)$$

S_0 is known $\Rightarrow \ln S_T \sim N(\ln S_0 + (\mu - \frac{\sigma^2}{2}) dt, \sigma\sqrt{dt})$

e.g. Let $S = \$40$, and $\mu = 0.16, \sigma = 0.20$ per year.

a. Find the distribution of lnS_T in 6 months.

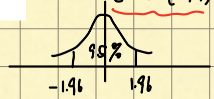
b. Find a, b such that $P(a < S_T < b) = 0.95$.

a. Using the previous result:

$$ln(S_T) \sim N\left(ln(40) + (0.16 - \frac{0.20^2}{2})0.5,\ \ 0.2\sqrt{0.5}\right)$$

$$ln(S_T) \sim N(3.759,\ \ 0.141)$$

$Z \sim N(0,1)$



b. $P(a < S_T < b) = 0.95$

$P(\ln a < \ln S_T < \ln b) = 0.95$

$$P\left(\frac{\ln a - \overbrace{(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)dt)}^{\text{mean} = 3.759}}{\underbrace{\sigma\sqrt{dt}}_{\text{std} = 0.141}} < Z < \frac{\ln b - (\ln S_0 + (\mu - \frac{1}{2}\sigma^2)dt)}{\sigma\sqrt{dt}}\right) = 0.95$$

standardize

$= -1.96$

$$\frac{\ln b - \ln S_0 - (\mu - \frac{1}{2}\sigma^2)dt}{\sigma\sqrt{dt}} = 1.96$$

$-1.96 = \frac{ln(a) - 3.759}{0.141}$ and $a = 32.55$,

$1.96 = \frac{ln(b) - 3.759}{0.141}$ and $b = 56.56$.

$\ln b = \ln S_0 + (\mu - \frac{1}{2}\sigma^2)dt + 1.96\sigma\sqrt{dt}$

$b = e^{(\cdots)} = S_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)dt + 1.96\sigma\sqrt{dt}}$

$a = S_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)dt - 1.96\sigma\sqrt{dt}}$

mean & var of stock price at T

moment generating fn. let X be an r.v., $M_X(\underbrace{t^*}_{\text{just notation}}) = Ee^{t^*X} = \begin{cases} \sum_x e^{t^*x} p(x) \\ \int_x e^{t^*x} f(x)\, dx \end{cases}$

First moment $EX = M_X'(0) = \left.\frac{dM_X(t^*)}{dt^*}\right|_{t^*=0}$

second moment $EX^2 = M_X''(0)$

e.g.1. $X \sim$ binomial (n, p)

$$M_X(t^*) = Ee^{t^*x} = \sum_{x=0}^{n} e^{t^*x} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^{t^*})^x (1-p)^{n-x} = (pe^{t^*} + 1 - p)^n$$

e.g.2. $X \sim$ Poisson (λ)

$M_X(t^*) = e^{\lambda(e^{t^*} - 1)}$

e.g.3. $X \sim \Gamma(\alpha, \beta)$

$M_X(t^*) = (1 - \beta t^*)^{-\alpha}$

e.g.4. $Z \sim N(0,1)$

$M_Z(t^*) = e^{\frac{1}{2}t^{*2}}$

let $X \sim N(\mu, \sigma)$ $M_X(t^*) = e^{t^*\mu + \frac{1}{2}t^{*2}\sigma^2}$

our r.v. $\ln S_T \sim N(\underbrace{\ln S_0 + (\mu - \frac{\sigma^2}{2}) dt}_{\mu}, \underbrace{\sigma\sqrt{dt}}_{\sigma})$ $dt = T - t$

$$M_{\ln S_T}(t^*) \overset{\text{Def}}{=} Ee^{t^* \ln S_T} = Ee^{\ln S_T^{t^*}} = ES_T^{t^*}$$

$$\overset{\text{normal}}{=} e^{t^*[\ln S_0 + (\mu - \frac{1}{2}\sigma^2)dt] + \frac{1}{2}t^{*2}\sigma^2 dt}$$

to find ES_T set $t^* = 1$ to get $ES_T = e^{\ln S_0 + (\mu - \frac{1}{2}\sigma^2)dt + \frac{1}{2}\sigma^2 dt}$

$= e^{\ln S_0 + \mu dt}$

$= S_0 e^{\mu dt}$ ← looks like risk-neutral valuation $S_0 e^{r dt}$

to find $Var(S_T)$, use $Var(S_T) = ES_T^2 - (ES_T)^2$

find ES_T^2 by setting $t^* = 2$. $ES_T^2 = S_0^2 e^{2\mu dt + \sigma^2 dt}$

$Var(S_T) = S_0^2 e^{2\mu dt}(e^{\sigma^2 dt} - 1)$

e.g.

A stock has a current price \$20, and $\mu = 0.20, \sigma = 0.40$ per year. Find its expected price and variance in 1 year from now if the stock price follows the lognormal distribution.

$$E(S_T) = 20e^{0.20(1-0)} = 24.43$$

$$var(S_T) = 20^2 e^{2(0.20)(1-0)}(e^{0.40^2(1-0)} - 1) = 103.54$$

LEC 27

estimate volatility σ $\quad \ln \frac{S_T}{S_0} \sim N\left((\mu - \frac{1}{2}\sigma^2)dt,\ \sigma\sqrt{dt}\right)$

most recent data

S_1	S_1/S_2	$u_1 = \ln\frac{S_1}{S_2}$
S_2	S_2/S_3	$u_2 = \ln\frac{S_2}{S_3}$
S_3	S_3/S_4	$u_3 = \ln\frac{S_3}{S_4}$
$\vdots$	$\vdots$	$\vdots$

$\rightarrow$ compute the sample std s to estimate $\sigma\sqrt{dt}$

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^n (u_i - \bar{u})^2} = \sqrt{\frac{1}{n-1}\left(\sum_{i=1}^n u_i^2 - \frac{(\sum_{i=1}^n u_i)^2}{n}\right)}.$$

$\hat{\sigma}\sqrt{dt} = s$

$\hat{\sigma} = \frac{s}{\sqrt{dt}} \overset{\text{daily data}}{=\!=} \frac{s}{\sqrt{252}}$

annual volatility $\quad$ trading days per year

e.g. $\sum_{i=1}^{59} u_i = -0.06481782$ and $\sum_{i=1}^{59} u_i^2 = 0.01203149$ and

$$s = \sqrt{\frac{1}{58}\left(0.01203149 - \frac{(-0.06481782)^2}{59}\right)} = \sqrt{0.0002062117} = 0.01436007.$$

Therefore the annual volatility is $\hat{\sigma} = \sqrt{252} \times 0.01436007 = 0.227959$. Therefore the annual volatility is $\sigma = 22.8\%$.

Black-Scholes-Merton model

European call option is a fn of S & t, let call price be C as a fn of S & t

$$dC = \left(\frac{dC}{dS}\mu S + \frac{dC}{dt} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \frac{dC}{dS}\sigma S \varepsilon\sqrt{dt}$$

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\varepsilon\sqrt{dt}$$

ITO lemma

create a riskless protfolio

write 1 call
buy n shares $\Big\}$ $-C + nS = \Pi$

in dt this protfolio changes to $-dC + n\,dS = d\Pi$

$$-\left(\frac{dC}{dS}\mu S + \frac{dC}{dt} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt - \frac{dC}{dS}\sigma S\varepsilon\sqrt{dt} + n\left(\mu S dt + \sigma S\varepsilon\sqrt{dt}\right) = d\Pi$$

$$\left(\underbrace{-\frac{dC}{dS}\mu S}_{-n\mu S} - \frac{dC}{dt} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + n\mu S\right)dt + \left(n - \frac{dC}{dS}\right)\sigma S\varepsilon\sqrt{dt} = d\Pi$$

this will be a riskless protfolio if we eliminate ε which is the only random term. we set $n = \frac{dC}{dS}$

a riskless protfolio must be equal to $r\Pi dt$

$$\left(-\frac{dC}{dt} - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt = r\left(-C + \frac{dC}{dS}S\right)dt$$

$$\frac{dC}{dt} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + rS\frac{dC}{dS} - rC = 0 \quad \text{(differencial equation)}$$

B-S-M option pricing formula $\quad C = S_0\,\phi(d_1) - Ee^{-r(T-t)}\phi(d_2)$

where $\phi(d_1) = P(Z \le d_1)$

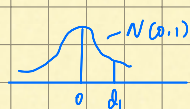
$\phi(d_2) = P(Z \le d_2)$

cdf of $Z \sim N(0,1)$

$t \quad T$

$dt = T - t$

where $d_1 = \dfrac{\ln\frac{S_0}{E} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$



$d_2 = d_1 - \sigma\sqrt{T-t} = \dfrac{\ln\frac{S_0}{E} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

use put-call parity to find the price of an European put

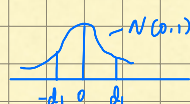
$P + S_0 = C + Ee^{-r(T-t)}$

$P = C + Ee^{-r(T-t)} - S_0$

$= S_0\,\phi(d_1) - Ee^{-r(T-t)}\phi(d_2) + Ee^{-r(T-t)} - S_0$

$= Ee^{-r(T-t)}(1 - \phi(d_2)) - S_0(1 - \phi(d_1))$

$= Ee^{-r(T-t)}\phi(-d_2) - S_0\,\phi(-d_1)$



$1 - \phi(d_1) = \phi(-d_1)$

properties: ① suppose S_0 is very large. then d_1, d_2 are large and $\phi(d_1) \approx 1 \quad \phi(d_2) \approx 1$ $\Big]$ $C = S - Ee^{-r(T-t)}$

② suppose $\sigma \to 0$. $S_T = S_0 e^{r(T-t)}$

therefore payoff $= \max(S_0 \cdot e^{r(T-t)} - E, 0)$

and $C = e^{-r(T-t)} \max(S_0 e^{r(T-t)} - E, 0)$

$= \max(S_0 - E e^{-r(T-t)}, 0)$

consider: $S_0 > E e^{-r(T-t)}$ $(S_T > E)$

$\ln S_0 > \ln E - r(T-t)$

$\ln \frac{S_0}{E} + r(T-t) > 0$

$\rightarrow \left.\begin{array}{l} d_1 \rightarrow \infty \\ d_2 \rightarrow \infty \end{array}\right\} \begin{array}{l} \phi(d_1) \simeq 1 \\ \phi(d_2) \simeq 1 \end{array} \rightarrow C = S_0 - E e^{-r(T-t)}$

or: $S_0 < E e^{-r(T-t)}$ $(S_T < E)$

$\ln S_0 < \ln E - r(T-t)$

$\ln \frac{S_0}{E} + r(T-t) < 0$

$\rightarrow \left.\begin{array}{l} d_1 \rightarrow -\infty \\ d_2 \rightarrow -\infty \end{array}\right\} \begin{array}{l} \phi(d_1) = 0 \\ \phi(d_2) = 0 \end{array} \rightarrow C = 0$

$\longrightarrow C = \max(S_0 - E e^{-r(T-t)}, 0)$

e.g. use B-S-M to find price of European call. $S_0 = 30$. $E = 29$. days to expiration is 40, annual std $\sigma = 0.3$, risk free $r = 0.05$

$t = 40/365 = 0.1096$ year

$d_1 = 0.446$. $d_2 = 0.347$

$C = 1.83$

previous simulations $\quad ds = \mu s dt + \sigma S \varepsilon \sqrt{dt}$

$S \quad S_1 \quad S_2$

$S_1 = S_0 + dS_1$

$S_2 = S_1 + dS_2$

use log-normal prob of stock prices

$\ln S_T - \ln S = (\mu - \frac{1}{2}\sigma^2) dt + \sigma \varepsilon \sqrt{dt}$

$\ln \frac{S_T}{S_0} = (\mu - \frac{1}{2}\sigma^2) dt + \sigma \varepsilon \sqrt{dt}$

$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2) dt + \sigma \varepsilon \sqrt{dt}}$

$S_1 = S_0 e^{(\ )}$

$S_2 = S_1 e^{(\ )}$

risk-neutral valuation

① assume $\mu = r$

② find the expected payoff at S_T

③ discount the expected payoff at time 0 using r

e.g. Assume that a non-dividend-paying stock has an expected return of μ and volatility of σ. A financial institution has just announced that it will trade a security that pays off a dollar amount equal to $ln(S_T)$ at time T, where S_T denotes the value of the stock price at time T. Answer the following questions:

a. Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price at time T.

b. Confirm that your price satisfies the Black-Scholes-Merton differential equation.

a. $\ln S_T - \ln S = (\mu - \frac{1}{2}\sigma^2) dt + \sigma \varepsilon \sqrt{dt}$

$E \ln S_T = \ln S_0 + (\mu - \frac{1}{2}\sigma^2) dt$

price $= V = e^{-r(T-t)} \left[\ln S_0 + (r - \frac{1}{2}\sigma^2) dt \right]$

b. show $\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 + rS \frac{\partial V}{\partial S} - rV = 0$

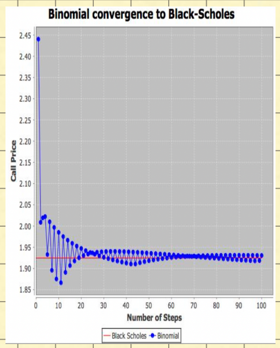
other examples see handout #59

binomial model converge to B-S-M

The binomial formula converges to the Black-Scholes formula when the number of periods n is large. In the example below we value the call option using the binomial formula for different values of n and also using the Black-Scholes formula. We then plot the value of the call (from binomial) against the number of periods n. The value of the call using Black-Scholes remains the same regardless of n. The data used for this example are:

$$S_0 = \$48,\ E = \$50,\ R_f = 0.05,\ \sigma = 0.30,\ \text{Days to expiration} = 73.$$

Using the Statistics Online Computational Resource (SOCR) at `http://www.socr.ucla.edu` we find the results on the next page.



LEC 28 implied volatility

One of the most important uses of the Black-Scholes-Merton model is the calculation of implied volatilities. These are the volatilities implied by the option prices observed in the market. Given the price of a call option, the implied volatility can be computed from the Black-Scholes formula. However σ cannot be expressed as a function of S_0, E, r, t, c and therefore a numerical method must be employed:

given $C = S_0\,\phi(d_1) - Ee^{-r(T-t)}\,\phi(d_2)$

$f(\sigma) = S_0\,\phi(d_1) - Ee^{-r(T-t)}\,\phi(d_2) - C$. solve for σ

a. By trial and error. Begin with some value of σ and compute c using the Black-Scholes model. If the price of c is too low (compare to the market price) increase σ and iterate the procedure until the value of c in the market is found. Note: the price of the call increases with volatility.

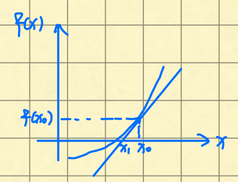
say $C = \$10$

begin with $\sigma_0 = 0.25 \rightarrow C = \$13$

update $\sigma_1 = 0.15 \rightarrow C = \$9$

update $\sigma_2 = 0.18 \rightarrow C = \$9.5$

b. newton - raphson, solve for σ. find the root of $f(\sigma)$



$f'(x_0) = \dfrac{f(x_0)}{x_0 - x_1}$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

- - -

converge (stop when $|x_i - x_{i+1}| < 0.001$)

$$f(\sigma) = S_0\,\phi(d_1) - Ee^{-r(T-t)}\,\phi(d_2) - C = 0$$

$$d_1 = \frac{\ln\frac{S_0}{E} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

begin with σ_0. $\sigma_1 = \sigma_0 - \dfrac{f(\sigma_0)}{f'(\sigma_0)}$

$$\sigma_2 = \sigma_1 - \frac{f(\sigma_1)}{f'(\sigma_1)}$$

$\vdots$

$$f'(\sigma) = \frac{d f(\sigma)}{d\sigma} = S_0\,\underbrace{\phi'(d_1)}_{\text{cdf}}\,\frac{dd_1}{d\sigma} - Ee^{-r(T-t)}\,\phi'(d_2)\,\frac{dd_2}{d\sigma}$$

$$= S_0\,\underbrace{f(d_1)}_{\text{pdf of standard normal}} \cdot \frac{dd_1}{d\sigma} - Ee^{-r(T-t)}\,f(d_2)\,\frac{dd_2}{d\sigma}$$

$$f(d_1) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}d_1^2}. \quad f(d_2) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}d_2^2}$$

dividend discount model

suppose we know the amount and timing of the dividend

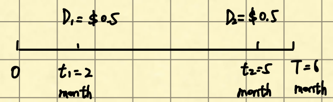
ex-dividend date : dividend paid date. on this date the stock price declines by the dividend amount

dividend discount model for European call options

$S_0' = S_0 - $ [Sum of present value of all dividends paid during option lifetime]

then use the B-S-M to price the option

e.g.



$S_0 = 40$. $E = 40$. $r = 0.09$. $\sigma = 0.3$

present value of dividend $= 0.5 \cdot e^{-0.09 \cdot \frac{2}{12}} + 0.5\, e^{-0.09 \cdot \frac{5}{12}} = 0.9742$

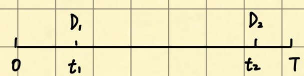
$S_0' = 40 - 0.9742 = 39.0258$

$$d_1 = \frac{\ln(S_0'/E) + (r + \frac{1}{2}\sigma^2) \cdot \underbrace{(T-t)}_{\frac{6}{12}}}{0.3 \cdot \underbrace{\sqrt{T-t}}_{\frac{6}{12}}} = 0.202. \rightarrow \phi(d_1) = 0.58$$

$d_2 = d_1 - \sigma\sqrt{T-t} = -0.0101$. $\phi(d_2) = 0.496$

$C = 39.0258 \cdot 0.58 - 40 \cdot e^{-0.09 \frac{6}{12}} \cdot 0.496 = 3.67$

American call option



$S(t_2) - D_2 - Ee^{-r(T-t_2)} > S(t_2) - E$

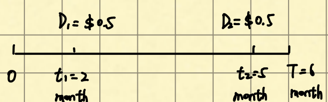
exercise at t_2. pay E

exercise at T. present value at t_2 : $Ee^{-r(T-t_2)}$

early exercise at t_2 if $D_2 > \underline{E(1-e^{-r(T-t_2)})}$

early exercise at t_1 if $D_1 > E(1-e^{-r\underbrace{(t_2-t_1)}})$ potentially we may exercise at t_2

e.g.



$S_0 = 40$. $E = 40$. $r = 0.09$. $\sigma = 0.3$

$40 \cdot [1 - e^{-0.09 \frac{3}{12}}] = 0.89$ $\quad \frac{3}{12}$ is time from t_1 to t_2

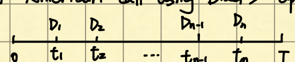
$D_1 = 0.5 < 0.89$. it's never optimal to exercise before t_1

$40 \cdot [1 - e^{-0.09 \frac{1}{12}}] = 0.3$ $\quad \frac{1}{12}$ is time from t_2 to T

$D_2 = 0.5 > 0.3$

the call will be exercised at t_2

price of American call using Black's approximation



use dividend discount model to compute

(a) C_1 using $S^*_{0,n} = S_0 - \sum_{i=1}^{n} \underbrace{PV}_{\text{present value}}(D_i)$

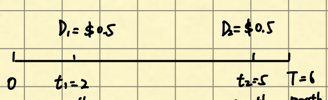
$C_1 = S^*_{0,n}\, \phi(d_1) - Ee^{-rT} \phi(d_2)$ $\quad C_1$ assume expiration at T

(b) C_2 using $S^*_{0,n-1} = S_0 - \sum_{i=1}^{n-1} PV(D_i)$

$C_2 = S^*_{0,n-1}\, \phi(d_1) - Ee^{-rt_n} \phi(d_2)$ $\quad C_2$ assume expiration at t_n

Finally $C = \max(C_1, C_2)$

e.g.



$S_0 = 40$. $E = 40$. $r = 0.09$. $\sigma = 0.3$

$C_1 = 3.67$ (same as before)

compute C_2: $S_0^* = 40 - 0.5 \cdot e^{-0.09 \frac{2}{12}} = 39.5074$

$$d_1 = \frac{\ln(S^*/E) + (0.09 + \frac{1}{2} 0.3^2) \frac{5}{12}}{0.3 \sqrt{\frac{5}{12}}} = 0.2265 \qquad \phi(d_1) = 0.5896$$

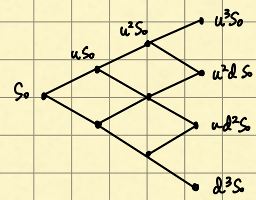
$d_2 = d_1 - \sigma\sqrt{t_2} = 0.0329$ $\qquad \phi(d_2) = 0.5131$

$C_2 = S_0^* \phi(d_1) - 40 \cdot e^{-0.09 \frac{5}{12}} \phi(d_2) = 3.52$

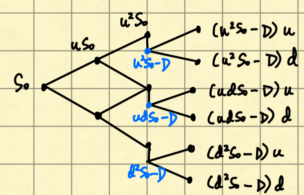
$C = \max(C_1, C_2) = 3.67$

binomial model w/ dividends

consider a 3-step binomial tree without dividends

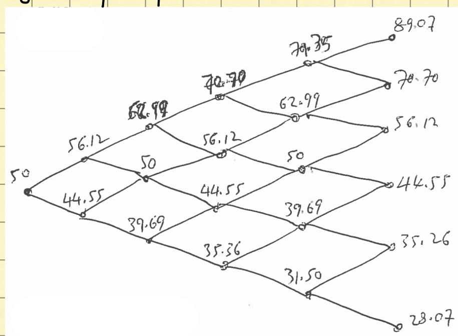


suppose stock pays a single dividend between step 1 and step 2



we can simplify this by adding the PV of the dividend to all nodes before the ex-dividend date

e.g. European put



$S_0 = 50$. $E = 50$. $r = 10\%$. $\sigma = 0.4$

$T = 5$ month $= \frac{5}{12} = 0.4167$. $n = 5$

$u = e^{\sigma\sqrt{t/n}} = 1.1224$

$d = e^{-\sigma\sqrt{t/n}} = 0.8909$

$p = \dfrac{e^{r\Delta t} - d}{u-d} = 0.5073$. $1-p = 0.4927$

using the binomial option pricing model we get $P = \$4.49$

suppose now $S_0 = 52$. others are same. stock pays a single dividend of \$2.06 in 3.5 month

PV of dividend $= 2.06 \cdot e^{-0.1 \cdot \frac{3.5}{12}} = 2$

use now $S_0' = 52 - 2 = 50$.

construct the same binomial tree as before

add the PV of the dividend to all nodes before the ex-dividend date. other nodes stay the same

THEREFORE, $S_0 = 50 + 2 = 52$

NODE 1: $56.12 + 2.06\, e^{-0.10 \frac{2.5}{12}} = 58.14$

NODE 2: $44.55 + 2.06\, e^{-0.10 \frac{2.5}{12}} = 46.56$

NODE 3: $62.99 + 2.06\, e^{-0.10 \frac{1.5}{12}} = 65.02$

⋮

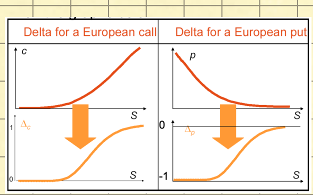
NODE 9: $35.36 + 2.06\, e^{-0.10 \frac{0.5}{12}} = 37.41$

LEC 29 the greeks

measure the sensitivity of options to one determinants (holding others constant) partial derivates

accurate only for small changes

① Delta $\Delta = \dfrac{\partial c}{\partial S}$

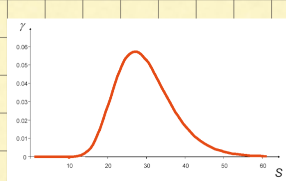


for call. $\Delta_c = \phi(d_1)$

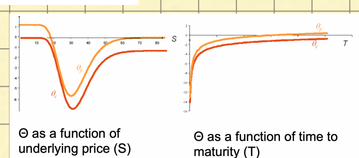
for put. $\Delta_p = \phi(d_1) - 1$

$\Delta c > 0$. $\Delta p < 0$

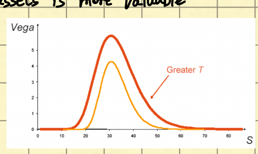
② Gamma $\Gamma = \dfrac{\partial^2 c}{\partial S^2}$. is greatest for options that are at-the-money stock price = strike price



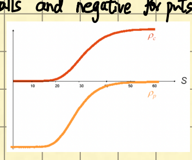
③ Theta $\theta = \dfrac{\partial c}{\partial t}$. usually negative since an option becomes less valuable as time passes



④ VEGA $= \dfrac{\partial c}{\partial \sigma}$. positive because an option on more volatile assets is more valuable



⑤ Rho $= \dfrac{\partial c}{\partial r}$. positive for calls and negative for puts



dynamic delta hedging

suppose $S_0 = 49$, $r = 0.05$, $T = 20$ weeks, $E = 50$, $\sigma = 0.2$, $\mu = 0.13$

a financial institution has sold a call option that involves 100,000 shares with price \$300,000

B-S-M model price $C = \$240,000$

The hedge is to be adjusted weekly during lifetime of the call (20 weeks)

① do nothing. works well when $S_T < S_0$

what if $S_T = \$60$, loss: \$1,000,000

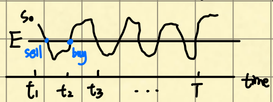
② adopt a "covered" position, buy 100,000 shares, works well when $S_T > 50$

what if $S_T = 40$, loss: $\$900,000 = (49-40) \times 100,000$

③ stop-loss strategy:

idea: we should have the stock at expiration if $S_T > 50$. (need to deliver call option)

not have the stock if $S_T < 50$. (stop losing money when stock prices goes down)



in practice it may not hit E. (50.5 → 49.5). purchase at $E + \varepsilon$. sell at $E - \varepsilon$

④ delta hedging $\Delta = \frac{\partial c}{\partial S} = 0.7$

when the stock increase by \$1, the call increases by 0.7

e.g. call is \$10, $n = 2000$ } buy $2000 \times 0.7 = 1400$ shares

suppose stock increases by \$1, profit is \$1400

from call option: $2000 \cdot 10 - 2000 \cdot 10.7 = -\$1400$

Initially (at the beginning of the option lifetime) Delta (Δ) is calculated using:

$$d_1 = \frac{ln(\frac{S_0}{E}) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = \frac{ln(\frac{49}{50}) + (0.05 + \frac{1}{2}0.20^2)0.3846}{0.20\sqrt{0.3846}} = 0.05417375.$$

Therefore, $\Delta = \Phi(d_1) = \Phi(0.05417375) = 0.522$. This means that the institution must purchase $100000 \times 0.522 = 52200$ shares. Since each share cost \$49, the institution must borrow $52200 \times 49 = \$2557800$ to buy them. In the meantime, during week 1 of the life of the option the interest cost will be $2557800 \times (exp(0.05\frac{1}{52}) - 1) \approx \$2500$. All these calculations are shown on the first row of the table below.

At the end of week 1, suppose that the price of the stock is \$48.12. The new Δ ratio will be:

$$d_1 = \frac{ln(\frac{S_0}{E}) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = \frac{ln(\frac{48.12}{50}) + (0.05 + \frac{1}{2}0.20^2)\frac{19}{52}}{0.20\sqrt{\frac{19}{52}}} = -0.1054492.$$

Therefore, $\Delta = \Phi(d_1) = \Phi(-0.1054492) = 0.458$. For a hedge position the institution must have $0.458 \times 100000 = 45800$ shares, but since they already have 52200 shares they must sell $52200 - 45800 = 6400$ shares of the underlying stock. From selling 6400 shares the institution will receive $6400 \times 48.12 \approx \$308000$. The total cost by the end of week 2 will be: $\$2557800 + \$2500 - \$308000 = \$2252300$.

week 3: $d_1 = \frac{\ln \frac{50.25}{50} + (0.05 + \frac{1}{2} 0.2^2)\frac{17}{52}}{0.2 \cdot \sqrt{\frac{17}{52}}} = 0.2457$

$\Phi(d_1) = 0.596$

need to have $100,000 \cdot 0.596 = 59600$

from previous weeks. we have $52200 - 6400 - 5800 = 40,000$

need to purchase 19,600 more shares

Example 1: The complete table of delta hedging (option closes in the money):

Week	Stock price	Delta	Shares purchased or (sold)	Cost of shares purchased or sold	Total cost including interest	Interest cost
0	49.00	0.522	52200	-2557800	2557800	2500
1	48.12	0.458	−(6400)	+308000	2252300	2200
2	47.37	0.400	(5800)	+274700	1979800	1900
3	50.25	0.596	19600	-984900	2966600	2900
4	51.75	0.693	9700	-502000	3471500	3300
5	53.12	0.774	8100	-430300	3905100	3800
6	53.00	0.771	(300)	+15900	3893000	3700
7	51.87	0.706	(6500)	+337200	3559500	3400
8	51.38	0.674	(3200)	+164400	3398500	3300
9	53.00	0.787	11300	-598900	4000700	3800
10	49.88	0.550	(23700)	+1182200	2822300	2700
11	48.50	0.413	(13700)	+664400	2160600	2100
12	49.88	0.542	12900	-643500	2806200	2700
13	50.37	0.591	4900	-246800	3055700	2900
14	52.13	0.768	17700	-922700	3981300	3800
15	51.88	0.759	(900)	+46700	3938400	3800
16	52.87	0.865	10600	-560400	4502600	4300
17	54.87	0.978	11300	-620000	5126900	4900
18	54.62	0.990	1200	-65500	5197300	5000
19	55.87	1.000	1000	-55900	5258200	5100
20	57.25	1.000	0	0	5263300	

Example 2: Option closes out of the money:

Week	Stock price	Delta	Shares purchased or (sold)	Cost of shares purchased or sold	Total cost including interest	Interest cost
0	49.00	0.522	52200	-2557800	2557800	2500
1	49.75					
2	52.00		to be filled			
3	50.00					
4	48.38					
5	48.25					
6	48.75					
7	49.63					
8	48.25					
9	48.25					
10	51.12					
11	51.50					
12	49.88					
13	49.88					
14	48.75					
15	47.50					
16	48.00					
17	46.25					
18	48.13					
19	46.63	0.007	(17600)	+820700	290000	300
20	48.12	0.000	(700)	+33700	256600	