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Statistics C183/C283 **Instructor: Nicolas Christou**

Exercise

From *Options Futures and Other Derivatives* by John Hull, Prentice Hall 6th Edition, 2006.

The Black-Scholes-Merton formula for the value C of a European call option at time t and expiration time at time T is given by

$$C = S_0\Phi(d_1) - \frac{E}{e^{r(T-t)}}\Phi(d_2)$$

$$d_1 = \frac{ln(\frac{S_0}{E}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{ln(\frac{S_0}{E}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

Answer the following questions:

1. Find $\Phi'(d_1)$.
2. Show that $S_0\Phi'(d_1) = \frac{E}{e^{r(T-t)}}\Phi'(d_2)$.
3. Find $\frac{\partial d_1}{\partial S}$ and $\frac{\partial d_2}{\partial S}$.
4. Show that
$$\frac{\partial C}{\partial t} = -rEe^{-r(T-t)}\Phi(d_2) - S_0\Phi'(d_1)\frac{\sigma}{2\sqrt{T-t}}.$$
5. Show that $\frac{\partial C}{\partial S} = \Phi(d_1)$.
6. Show that C satisfies the Black-Scholes-Merton differential equation.
7. Show that C satisfies the boundary conditions for a European call option, $C = max[S-E, 0]$ as $t \to T$.

Exercise 2
Assume that a non-dividend-paying stock has an expected return of μ and volatility of σ. A financial institution has just announced that it will trade a security that pays off a dollar amount equal to $ln(S_T)$ at time T, where S_T denotes the value of the stock price at time T. Answer the following questions:

a. Use risk-neutral valuation to calculate the price of the security at time t in terms of the stock price at time T.
b. Confirm that your price satisfies the Black-Scholes-Merton differential equation.

Answers

1. Since $\Phi(d_i)$ is the cumulative probability that a standard normal random variable is less than d_1, i.e., $P(Z \le d_1)$ it follows that $\Phi'(d_1) = \frac{\partial \Phi(d_1)}{\partial d_1} = f(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{ln(\frac{S_0}{E})+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)}$.

2. Because
$$d_2 = \frac{ln(\frac{S_0}{E}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},$$
it follows that $d_1 = d_2 + \sigma\sqrt{T-t}$. Therefore,
$$\begin{aligned}
\Phi'(d_1) &= \frac{\partial \Phi(d_1)}{\partial d_1} = f(d_1) = f(d_2 + \sigma\sqrt{T-t}) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_2+\sigma\sqrt{T-t})^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_2^2+2d_2\sigma\sqrt{T-t}+\sigma^2(T-t))} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \times e^{-d_2\sigma\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)} = \Phi'(d_2) \times e^{-\frac{1}{2}d_2^2} \times e^{-d_2\sigma\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)} \\
\text{but} \\
d_2 &= \frac{ln(\frac{S_0}{E}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \text{therefore} \\
\Phi'(d_1) &= \Phi'(d_2) \times e^{-\sigma\left[\frac{ln(\frac{S_0}{E})+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right]\sqrt{T-t}-\frac{1}{2}\sigma^2(T-t)} \\
&= \Phi'(d_2) \times e^{-ln\frac{S_0}{E}-r(T-t)} = \Phi'(d_2) \times e^{ln\frac{E}{S_0}-r(T-t)} \\
&= \Phi'(d_2) \times \frac{E}{S_0} e^{-r(T-t)}. \text{ It follows that} \\
S_0\Phi'(d_1) &= Ee^{-r(T-t)}\Phi'(d_2).
\end{aligned}$$

3. Use $d_1 = \frac{ln(\frac{S_0}{E})+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and write it as $d_1 = \frac{lnS_0+lnE+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.
Therefore, $\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$. Similarly, because $d_2 = d_1 - \sigma\sqrt{T-t}$, it follows that $\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$.

4. Use the B-S-M formula:
$$\begin{aligned}
C &= S_0\Phi(d_1) - \frac{E}{e^{r(T-t)}}\Phi(d_2) \\
\frac{\partial C}{\partial t} &= S\Phi'(d_1)\frac{\partial d_1}{\partial t} - \frac{rE\Phi(d_2)}{e^{r(T-t)}} - \frac{E}{e^{r(T-t)}}\Phi'(d_2)\frac{\partial d_2}{\partial t} \\
\text{From (2) } S_0\Phi'(d_1) &= Ee^{-r(T-t)}\Phi'(d_2), therefore \\
\frac{\partial C}{\partial t} &= S\Phi'(d_1)\frac{\partial d_1}{\partial t} - \frac{rE\Phi'(d_2)}{e^{r(T-t)}} - S\Phi'(d_1)\frac{\partial d_2}{\partial t} \\
&= S\Phi'(d_1)(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}) - \frac{rE\Phi'(d_2)}{e^{r(T-t)}}. \\
\text{But } d_1 - d_2 &= \sigma\sqrt{T-t}, \text{which means } \frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = -\frac{\sigma}{\sqrt{T-t}}. \\
\text{Finally, } \frac{\partial C}{\partial t} &= -rEe^{-r(T-t)}\Phi(d_2) - S_0\Phi'(d_1)\frac{\sigma}{2\sqrt{T-t}}, \text{which is a decreasing function of } t.
\end{aligned}$$

5. This is the hedge ratio $\frac{\partial C}{\partial S} = \Phi(d_1)$. Again, begin with the formula for C.

$$\begin{aligned}
C &= S_0\Phi(d_1) - \frac{E}{e^{r(T-t)}}\Phi(d_2) \\
\frac{\partial C}{\partial S} &= \Phi(d_1) + S\Phi'(d_1)\frac{\partial d_1}{\partial S} - \frac{E}{e^{r(T-t)}}\Phi'(d_2)\frac{\partial d_2}{\partial S} \\
\text{From (2) } S_0\Phi'(d_1) &= \frac{E}{e^{r(T-t)}}\Phi'(d_2) \\
\text{From (3) } \frac{\partial d_1}{\partial S} &= \frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}. \\
\text{Therefore } \frac{\partial C}{\partial S} &= \Phi(d_1) + \frac{E}{e^{r(T-t)}}\Phi'(d_2)\frac{1}{S\sigma\sqrt{T-t}} - \frac{E}{e^{r(T-t)}}\Phi'(d_2)\frac{1}{S\sigma\sqrt{T-t}} = \Phi(d_1).
\end{aligned}$$

6. C satisfies the B-S-M formula.
From (5) and (3) it follows that $\frac{\partial^2 C}{\partial S^2} = \Phi'(d_1)\frac{\partial d_1}{\partial S} = \Phi'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$. Therefore,

$$\frac{\partial C}{\partial t} \text{ (from (3))} + rS\frac{\partial C}{\partial S} \text{ (from (5))} + \frac{1}{2}S^2\frac{\partial^2 C}{\partial S^2} = rC.$$

7. Examine what happens as $t \to T$.
If $S > E$ then $d_1 \to \infty$ and $d_2 \to \infty$ and therefore $\Phi(d_1) \to 1$ and $\Phi(d_2) \to 1$.
In this case $C \to S - E$.

If $S < E$ then $d_1 \to -\infty$ and $d_2 \to -\infty$ and therefore $\Phi(d_1) \to 0$ and $\Phi(d_2) \to 0$.
Now, $C \to 0$.

We see that as $t \to T$, $C \to \max(S - E, 0)$.