University of California, Los Angeles
Department of Statistics

Statistics C183/C283 **Instructor: Nicolas Christou**

Lower and upper bounds for the price of a European call and put

A. Lower bound for the price of a European call:

	Time $t=0$	Payoff at time $t=1$	
		$S_1 > E$	$S_1 \leq E$
Portfolio A:			
Buy 1 call	$-C$	$S_1 - E$	0
Cash (lend)	$-\frac{E}{1+r}$	$+E$	$+E$
Total		S_1	E
Portfolio B:			
Buy 1 share	$-S_0$	S_1	S_1

$$c \geq S_0 - \frac{E}{1+r} \quad \text{or} \quad c \geq S_0 - Ee^{-rt}.$$

If it doesn't hold then there is an opportunity for a riskless profit. Here is an example. Suppose $S_0 = \$40, E = \$38, r = 10\%$ per year, and time to expiration is $t = 1$ year. Then the lower bound is: $c \geq 40 - 38e^{-0.10\times 1} = 5.62$.

Suppose there is a European call written on this stock with price $c = \$5$. It is cheaper! How can one make riskless profit?
- Short the stock
- Buy the call

Explain:
How much is the cash inflow at $t = 0$?
How much will it grow in 1 year?

At expiration (in 1 year):
If stock price $S_T > 38$ then ...
If stock price $S_T < 38$ then ...

B. Lower bound for the price of a European put:

	Time $t = 0$	Payoff at time $t = 1$	
		$S_1 \geq E$	$S_1 < E$
Portfolio A:			
Buy 1 put	$-P$	0	$E - S_1$
Buy 1 share	$-S_0$	S_1	S_1
Total		S_1	E
Portfolio B:			
Cash (lend)	$-\frac{E}{1+r}$	$+E$	$+E$

$$p \geq \frac{E}{1+r} - S_0 \quad \text{or} \quad p \geq Ee^{-rt} - S_0$$

If it doesn't hold then there is an opportunity for a riskless profit. Here is an example. Suppose $S_0 = \$40, E = \$43, r = 5\%$ per year, and time to expiration is $t = 0.5$ years. Then the lower bound is: $p \geq 43e^{-0.05\times0.5} - 40 = 1.94$.

Suppose there is a European put written on this stock with price $p = \$1$. It is cheaper! How can one make riskless profit?

- Borrow \$41
- Buy the put and the stock

Explain:
At $t = 0.5$ must pay back the loan
How much?

At expiration (in 6 months):
Stock price $S_T < 43$ then ...
Stock price $S_T > 43$ then ...

C. Upper bound for the price of a European call:

No matter what happens, $C \leq S_0$

If not, there will be an opportunity for a riskless profit by buying the stock and selling the call option. How? Suppose $C > S_0$.

	Time $t = 0$	Payoff at time $t = 1$	
		$S_1 > E$	$S_1 \leq E$
Sell 1 call	C	$E - S_1$	0
Buy 1 stock	$-S_0$	S_1	S_1
Total	$C - S_0$	E	S_1

D. Upper bound for the price of a European put:

No matter what happens, $P \leq \frac{E}{1+r}$.
If not, there will be an opportunity for a riskless profit by selling the put and investing the proceeds at the risk free interest rate. How? Suppose $P > \frac{E}{1+r}$.

	Time $t=0$	Payoff at time $t=1$	
		$S_1 \geq E$	$S_1 < E$
Sell 1 put	$P > \frac{E}{1+r}$	0	$S_1 - E$

Put-call parity

This is an important relationship between the price of a put and the price of the call. A put and the underlying stock can be combined in such a way that they have the same payoff as a call at expiration. Consider the following two portfolios:
Portfolio A: Buy the call and lend an amount of cash equal to $\frac{E}{1+r}$.
Portfolio B: Buy the stock, buy the put.

This is shown on the table below:

	Time $t=0$	Payoff at time $t=1$	
		$S_1 > E$	$S_1 \leq E$
Portfolio A:			
Buy 1 call	$-C$	$S_1 - E$	0
Lend cash	$-\frac{E}{1+r}$	E	E
Total	$-C - \frac{E}{1+r}$	S_1	E

		$S_1 \geq E$	$S_1 < E$
Portfolio B:			
Buy 1 put	$-P$	0	$E - S_1$
Buy 1 stock	$-S_0$	S_1	S_1
Total	$-P - S_0$	S_1	E

$$c + \frac{E}{1+r} = p + S_0 \quad \text{or} \quad c + Ee^{-rt} = p + S_0.$$

If it doesn't hold then there is an opportunity for a riskless profit.

Example 1:
$S_0 = \$30, E = \$28, r = 10\%$ per year, and $t = 3$ months to expiration.
Suppose $c = \$4$ and $p = \$3$.

Let's compute both sides of the put-call parity equation.
$c + Ee^{-rt} = 4 + 28e^{-0.10 \times \frac{3}{12}} = \$31.31$.
$p + S_0 = 3 + 30 = \$33$.

The second portfolio is overpriced compared to the first portfolio. Therefore,

- Short the put and the stock
- Buy the call

Explain:
How much is the cash inflow at $t = 0$?
How much will it grow in 3 months?

At expiration (in 3 months):
If stock price $S_T > 28$ then ...
If stock price $S_T < 28$ then ...

Example 2:
Suppose $S_0 = \$30, E = \$28, r = 10\%$ per year, and $t = 3$ months to expiration.
Suppose $c = \$4$ and $p = \$1$.

Let's compute both sides of the put-call parity equation.
$c + Ee^{-rt} = 4 + 28e^{-0.10\times\frac{3}{12}} = \$31.31$.
$p + S_0 = 3 + 30 = \$31$.

The second portfolio is underrpriced compared to the first portfolio. Therefore,

- Borrow \$31 to buy the put and the stock
- Sell the call

In 3 months we must return $27 \times e^{0.1\frac{3}{12}} = 27.68$.

At expiration (in 3 months):
If stock price $S_T > 28$ then ...
If stock price $S_T < 28$ then ...