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Statistics C183/C283 **Instructor: Nicolas Christou**

Ito's lemma, lognormal property of stock prices
Black-Scholes-Merton Model

From *Options Futures and Other Derivatives* by John Hull, Prentice Hall 6th Edition, 2006.

A. Ito process:

Earlier we defined the generalized Wiener process where the change in the underlying variable has a drift rate a per Δt and variance b^2 per Δt, i.e. $\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t}$. Now suppose the drift rate and variance rate are both functions of x and t. Then Ito process is defined as $\Delta x = a(x,t)\Delta t + b(x,t)\Delta z = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$.

B. Ito's lemma:

Ito's lemma gives a derivative chain rule of random variables. Suppose G is a function of x and t. Ito's lemma states that

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\Delta z$$

or

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\epsilon\sqrt{dt}$$

Therefore G follows a generalized Wiener process with drift rate $\left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)$ per dt and variance $\left(\frac{\partial G}{\partial x}\right)^2 b^2$ per dt.

Now, apply Ito's lemma to the model for stock prices:
Earlier we have seen that $\Delta S = \mu S\Delta t + \sigma S\Delta Z = \mu S\Delta t + \sigma S\epsilon\sqrt{\Delta t}$. Now let G be a function of (S,t). Ito's lemma states that G follows the generalized Wiener process as follows:

C. The lognormal property of stock prices:

Let $G = lnS$, and let us apply Ito's lemma to this function (here G is only function of S). Therefore

$$\frac{\partial G}{\partial S} = \qquad \frac{\partial^2 G}{\partial S^2} = \qquad \frac{\partial G}{\partial t} =$$

And the change in G will be:

$$dG = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\epsilon\sqrt{dt}$$

Therefore, G follows the generalized Wiener process with drift rate $\mu - \frac{\sigma^2}{2}$ and variance σ^2. Now, examine the change in G from time t (now) to time T, so $\Delta t = T - t$. This can be expressed as: $\Delta G = ln(S_T) - ln(S_0)$. What is the distribution of this change?

But S_0 is known, it is the price of the stock at time $t = 0$ (now). Therefore, the distribution of $ln(S_T)$ is:

Example:
Let $S = \$40$, and $\mu = 0.16, \sigma = 0.20$ per year.

a. Find the distribution of lnS_T in 6 months.
b. Find a, b such that $P(a < S_T < b) = 0.95$.

Answers:

a. Using the previous result:

$$\begin{aligned} ln(S_T) &\sim N\left(ln(40) + (0.16 - \frac{0.20^2}{2})0.5, \quad 0.2\sqrt{0.5}\right) \\ ln(S_T) &\sim N(3.759, \quad 0.141) \end{aligned}$$

b. We are using here the fact that $ln(S_T)$ follows a normal distribution.

$$\begin{aligned} P\left(a < S_T < b\right) &= 0.95 \\ P\left(ln(a) < ln(S_T) < ln(b)\right) &= 0.95 \\ P\left(\frac{ln(a) - 3.759}{0.141} < Z < \frac{ln(b) - 3.759}{0.141}\right) &= 0.95 \end{aligned}$$

Therefore,
$-1.96 = \frac{ln(a)-3.759}{0.141}$ and $a = 32.55$,
$1.96 = \frac{ln(b)-3.759}{0.141}$ and $b = 56.56$.

Mean and variance of the stock price at time T:
Use moment generating functions to find $E(S_T)$ and $var(S_T)$. Reminder: The mgf of a normal random variable X with mean μ and standard deviation σ is

$$M_X(t^*) = E(e^{t^*X}) = e^{\mu t^* + \frac{1}{2}\sigma^2 t^{*2}}$$

Here we use t^* so that we don't confuse it with t which stands for time. In our case the random variable is $Y = ln(S_T)$. Therefore the mgf of Y is:

$$M_Y(t^*) = E(e^{t^*Y}) = E(e^{t^* lnS_T}) = E(e^{lnS_T{}^{t^*}}) = E(S_T)^{t^*} \tag{1}$$

But in page 2 we found that:

$$Y = lnS_T \sim N\left(lnS + (\mu - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}\right),$$

which means that

$$M_Y(t^*) = e^{\left(lnS + (\mu - \frac{\sigma^2}{2})(T-t)\right)t^* + \frac{1}{2}\sigma^2(T-t)t^{*2}} \tag{2}$$

Equations (1) and (2) are equal:

$$M_Y(t^*) = E(S_T)^{t^*} = e^{\left(lnS + (\mu - \frac{\sigma^2}{2})(T-t)\right)t^* + \frac{1}{2}\sigma^2(T-t)t^{*2}}$$

Therefore when $t^* = 1$ we will get $E(S_T)$ and when $t^* = 2$ we will get $E(S_T^2)$.

$$E(S_T) = e^{lnS}e^{\mu(T-t)} = Se^{\mu(T-t)}$$

and

$$E(S_T^2) = S^2 e^{2\mu(T-t) + \sigma^2(T-t)}$$

Now combining $E(S_T)$ and $E(S_T^2)$ we can find the variance of S_T:

$$var(S_T) = E(S_T^2) - (E(S_T))^2 = s^2 e^{2\mu(T-t)}\left[e^{\sigma^2(T-t)} - 1\right]$$

Example:
A stock has a current price \$20, and $\mu = 0.20, \sigma = 0.40$ per year. Find its expected price and variance in 1 year from now if the stock price follows the lognormal distribution.

Answer:

$$\begin{aligned} E(S_T) &= 20e^{0.20(1-0)} = 24.43 \\ var(S_T) &= 20^2 e^{2(0.20)(1-0)}(e^{0.40^2(1-0)} - 1) = 103.54 \end{aligned}$$

Estimation of volatility $\boldsymbol{\sigma}$:
Using the lognormal distribution of stock prices and Ito's lemma we found that:

$$ln(S_T) - ln(S_0) \sim N\left((\mu - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}\right)$$

or

$$ln\frac{S_T}{S_0} \sim N\left((\mu - \frac{\sigma^2}{2})(T-t), \sigma\sqrt{T-t}\right)$$

Now we can collect data for the past 60 days and compute the following:

a. $\frac{S_i}{S_{i-1}}$, $\frac{S_{i-1}}{S_{i-2}}$, $\frac{S_{i-2}}{S_{i-3}}$, etc.

b. $u_1 = ln\frac{S_i}{S_{i-1}}$, $u_2 = ln\frac{S_{i-1}}{S_{i-2}}$, $u_3 = ln\frac{S_{i-2}}{S_{i-3}}$, etc.

c. Compute the standard deviation of $u_1, u_2, u_3, \ldots$
$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2} = \sqrt{\frac{1}{n-1}\left(\sum_{i=1}^{n} u_i^2 - \frac{(\sum_{i=1}^{n} u_i)^2}{n}\right)}$.

d. This sample standard deviation estimates $\sigma\sqrt{T-t}$.
Therefore, $\hat{\sigma} = \frac{s}{\sqrt{T-t}}$. If we have daily data, then $\hat{\sigma} = \sqrt{252}s$, where, 252 is the number of trading days.

For the data on the next page we have:
$\sum_{i=1}^{59} u_i = -0.06481782$ and $\sum_{i=1}^{59} u_i^2 = 0.01203149$ and

$$s = \sqrt{\frac{1}{58}\left(0.01203149 - \frac{(-0.06481782)^2}{59}\right)} = \sqrt{0.0002062117} = 0.01436007.$$

Therefore the annual volatility is $\hat{\sigma} = \sqrt{252} \times 0.01436007 = 0.227959$. Therefore the annual volatility is $\sigma = 22.8\%$.

Estimation of volatility σ. Estimate annual volatility for `AAPL`:

Read daily data from 2016-03-10 to 2016-05-19:

	Date	Open	High	Low	Close	Volume	pi	pi[-1]	pi/pi[-1]	u=ln(pi/pi[-1])
1	2016-05-19	94.64	94.64	93.57	94.20	30342700	94.20000	94.56000	0.9961929	-3.814383e-03
2	2016-05-18	94.16	95.21	93.89	94.56	41923100	94.56000	93.49000	1.0114451	1.138008e-02
3	2016-05-17	94.55	94.70	93.01	93.49	46507400	93.49000	93.88000	0.9958458	-4.162882e-03
4	2016-05-16	92.39	94.39	91.65	93.88	61140600	93.88000	90.52000	1.0371189	3.644655e-02
5	2016-05-13	90.00	91.67	90.00	90.52	44188200	90.52000	90.34000	1.0019925	1.990502e-03
6	2016-05-12	92.72	92.78	89.47	90.34	76109800	90.34000	92.51000	0.9765430	-2.373648e-02
7	2016-05-11	93.48	93.57	92.46	92.51	28539900	92.51000	93.42000	0.9902591	-9.788665e-03
8	2016-05-10	93.33	93.57	92.11	93.42	33592500	93.42000	92.79000	1.0067895	6.766548e-03
9	2016-05-09	93.00	93.77	92.59	92.79	32855300	92.79000	92.72000	1.0007550	7.546763e-04
10	2016-05-06	93.37	93.45	91.85	92.72	43458200	92.72000	93.24000	0.9944230	-5.592583e-03
11	2016-05-05	94.00	94.07	92.68	93.24	35890500	93.24000	93.62000	0.9959410	-4.067265e-03
12	2016-05-04	95.20	95.90	93.82	94.19	41025500	93.62000	94.60401	0.9895987	-1.045580e-02
13	2016-05-03	94.20	95.74	93.68	95.18	56831300	94.60401	93.07333	1.0164460	1.631220e-02
14	2016-05-02	93.97	94.08	92.40	93.64	48160100	93.07333	93.17272	0.9989332	-1.067330e-03
15	2016-04-29	93.99	94.72	92.51	93.74	68531500	93.17272	94.25613	0.9885057	-1.156087e-02
16	2016-04-28	97.61	97.88	94.25	94.83	82242700	94.25613	97.22803	0.9694337	-3.104321e-02
17	2016-04-27	96.00	98.71	95.68	97.82	114602100	97.22803	103.71851	0.9374221	-6.462157e-02
18	2016-04-26	103.91	105.30	103.91	104.35	56016200	103.71851	104.44410	0.9930529	-6.971367e-03
19	2016-04-25	105.00	105.65	104.51	105.08	28031600	104.44410	105.04047	0.9943225	-5.693676e-03
20	2016-04-22	105.01	106.48	104.62	105.68	33683100	105.04047	105.32871	0.9972634	-2.740384e-03
21	2016-04-21	106.93	106.93	105.52	105.97	31552500	105.32871	106.48169	0.9891721	-1.088698e-02
22	2016-04-20	106.64	108.09	106.06	107.13	30611000	106.48169	106.26303	1.0020577	2.055629e-03
23	2016-04-19	107.88	108.00	106.23	106.91	32384900	106.26303	106.82958	0.9946967	-5.317419e-03
24	2016-04-18	108.89	108.95	106.94	107.48	60821500	106.82958	109.18523	0.9784252	-2.181097e-02
25	2016-04-15	112.11	112.30	109.73	109.85	46939000	109.18523	111.42161	0.9799286	-2.027553e-02
26	2016-04-14	111.62	112.39	111.33	112.10	25473900	111.42161	111.36198	1.0005355	5.353536e-04
27	2016-04-13	110.80	112.34	110.80	112.04	33257300	111.36198	109.77166	1.0144875	1.438355e-02
28	2016-04-12	109.34	110.50	108.66	110.44	27232300	109.77166	108.36025	1.0130252	1.294109e-02
29	2016-04-11	108.97	110.61	108.83	109.02	29407500	108.36025	108.00244	1.0033130	3.307542e-03
30	2016-04-08	108.91	109.77	108.17	108.66	23581700	108.00244	107.88316	1.0011056	1.105002e-03
31	2016-04-07	109.95	110.42	108.12	108.54	31801900	107.88316	110.28851	0.9781903	-2.205100e-02
32	2016-04-06	110.23	110.98	109.20	110.96	26404100	110.28851	109.14547	1.0104727	1.041820e-02
33	2016-04-05	109.51	110.73	109.42	109.81	26578700	109.14547	110.44755	0.9882109	-1.185915e-02
34	2016-04-04	110.42	112.19	110.27	111.12	37356200	110.44755	109.32438	1.0102737	1.022129e-02
35	2016-04-01	108.78	110.00	108.20	109.99	25874000	109.32438	108.33043	1.0091752	9.133314e-03
36	2016-03-31	109.72	109.90	108.88	108.99	25888400	108.33043	108.89698	0.9947974	-5.216204e-03
37	2016-03-30	108.65	110.42	108.60	109.56	45601100	108.89698	107.02836	1.0174591	1.730845e-02
38	2016-03-29	104.89	107.79	104.88	107.68	31190100	107.02836	104.55343	1.0236714	2.339560e-02
39	2016-03-28	106.00	106.19	105.06	105.19	19411400	104.55343	105.03053	0.9954576	-4.552751e-03
40	2016-03-24	105.47	106.25	104.89	105.67	26133000	105.03053	105.48774	0.9956657	-4.343715e-03
41	2016-03-23	106.48	107.07	105.90	106.13	25703500	105.48774	106.07417	0.9944715	-5.543857e-03
42	2016-03-22	105.25	107.29	105.21	106.72	32444400	106.07417	105.26908	1.0076480	7.618875e-03
43	2016-03-21	105.93	107.65	105.14	105.91	35502700	105.26908	105.27901	0.9999056	-9.436324e-05
44	2016-03-18	106.34	106.50	105.19	105.92	44205200	105.27901	105.15974	1.0011342	1.133527e-03
45	2016-03-17	105.52	106.47	104.96	105.80	34420700	105.15974	105.32871	0.9983958	-1.605495e-03
46	2016-03-16	104.61	106.31	104.59	105.97	38303500	105.32871	103.94712	1.0132913	1.320370e-02
47	2016-03-15	103.96	105.18	103.85	104.58	40067700	103.94712	101.89959	1.0200937	1.989447e-02
48	2016-03-14	101.91	102.91	101.78	102.52	25076100	101.89959	101.64117	1.0025425	2.539257e-03
49	2016-03-11	102.24	102.28	101.50	102.26	27408200	101.64117	100.55776	1.0107740	1.071636e-02
50	2016-03-10	101.41	102.24	100.15	101.17	33513600	100.55776	100.50807	1.0004944	4.942959e-04

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sum(u)
[1] -0.06481782

sum(u^2)
[1] 0.01203149
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D. Black-Scholes-Merton model:

A call option is a function of S (stock price) and t (time). Let C be the price of the call option. Then from Ito's lemma we have:

$$dC = \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S}\sigma S \epsilon \sqrt{dt} \tag{3}$$

Similar to the binomial option pricing model we want to create a riskless portfolio by

- Buying the call.
- Sell n shares of the stock per call.

Then, the portfolio at time 0 is:

$$C - nS = \Pi$$

This portfolio will change from time t to time $t + dt$ as follows:

$$dC - ndS = d\Pi \tag{4}$$

But, dc is given by (3) and also S follows generalized Wiener process, that is:

$$dS = \mu S dt + \sigma S \epsilon \sqrt{dt}$$

Therefore (4) is:

$$d\Pi = \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S}\sigma S \epsilon \sqrt{dt} - n\left(\mu S dt + \sigma S \epsilon \sqrt{dt}\right)$$

or

$$d\Pi = \left(\frac{\partial C}{\partial S} - n \right) \sigma S \epsilon \sqrt{dt} + \left(\frac{\partial C}{\partial S}\mu S + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - n\mu S \right) dt$$

This will be a riskless portfolio if we eliminate the term involving ϵ (the only random component) from the above expression. So if we choose $n = \frac{\partial C}{\partial S}$ then

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt$$

One more step: Since this is a riskless portfolio it must earn the risk free rate during time dt:

$$r\Pi dt = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \right) dt$$

Also, $\Pi = C - nS$ and $n = \frac{\partial C}{\partial S}$. Putting all these together we get the Black-Scholes differential equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + rS\frac{\partial C}{\partial S} - rC = 0$$

The solution of this differential equation gives the Black-Scholes-Merton option pricing formula:

The value C of a European call option at time $t = 0$ is:

$$C = S_0\Phi(d_1) - \frac{E}{e^{rt}}\Phi(d_2)$$

$$d_1 = \frac{ln(\frac{S_0}{E}) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

$$d_2 = \frac{ln(\frac{S_0}{E}) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$

S_0	Price of the stock at time $t = 0$
E	Exercise price at expiration
r	Continuously compounded risk-free interest
σ	Annual standard deviation of the returns of the stock
t	Time to expiration in years
$\Phi(d_i)$	Cumulative probability at d_i of the standard normal distribution $N(0, 1)$, that is, $\Phi(d_i) = P(Z \leq d_i)$

Example:
Use the Black-Scholes-Merton option pricing formula to find the price of the European call if $S_0 = \$30, E = \$29$, days to expiration $t = 40$, annual standard deviation $\sigma = 0.30$, and continuously compounded risk-free interest rate $r = 0.05$.