CAPM TESTING

LET $\underset{\sim}{Z}_t$ BE $n \times 1$ VECTOR OF EXCESS RETURNS FOR n STOCKS AT TIME $t = 1, 2, \ldots, T$

FOR EXAMPLE $Z_{1t} = R_{1t} - R_F$ EXCESS RETURN FOR STOCK 1 AT TIME t

$Z_{2t} = R_{2t} - R_F$ EXCESS RETURN FOR STOCK 2 AT TIME t

$\vdots$

$Z_{nt} = R_{nt} - R_F$ EXCESS RETURN FOR STOCK n AT TIME t

FOR THE SAME TIME t THE CORRESPONDING MARKET EXCESS RETURN IS $Z_{mt} = R_{mt} - R_F$

WRITE THE ABOVE AS A REGRESSION MODEL FIRST ASSET BY ASSET AND THEN USING VECTOR FORM:

$$\left.\begin{array}{lll}
\text{STOCK } 1 \leftarrow & Z_{1t} & = \alpha_1 + \beta_1 Z_{mt} + \varepsilon_{1t} \\
\text{STOCK } 2 & Z_{2t} & = \alpha_2 + \beta_2 Z_{mt} + \varepsilon_{2t} \\
\vdots & \vdots & \quad \vdots \qquad \vdots \qquad\quad \vdots \\
\text{STOCK } n & Z_{nt} & = \alpha_n + \beta_n \underset{\substack{\uparrow \\ \text{COMMON}}}{Z_{mt}} + \varepsilon_{nt}
\end{array}\right\} \quad \underset{\sim}{Z}_t = \underset{\sim}{\alpha} + \underset{\sim}{\beta} Z_{mt} + \underset{\sim}{\varepsilon}_t$$

ASSUMPTIONS:

$$E(\underset{\sim}{\varepsilon}_t) = 0$$

$$VAR[\underset{\sim}{\varepsilon}_t] = \underset{\sim}{\Sigma}$$

$$E[Z_{mt}] = \mu_m, \quad VAR(Z_{mt}) = \sigma_m^2$$

$$COV[Z_{mt}, \varepsilon_t] = 0$$

ASSUME MULTIVARIATE NORMAL DISTRIBUTION.

WRITE THE PDF OF $\underset{\sim}{z}_t$ GIVEN z_{mt}

$$f(\underset{\sim}{z}_t) = \frac{1}{(2\pi)^{n/2}} |\underset{\sim}{\Sigma}|^{-\frac{1}{2}} e^{-\frac{1}{2}(\underset{\sim}{z}_t - \underset{\sim}{\alpha} - \underset{\sim}{\beta} z_{mt})' \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{z}_t - \underset{\sim}{\alpha} - \underset{\sim}{\beta} z_{mt})}$$

THEREFORE THE JOINT PROBABILITY DENSITY FUNCTION FOR $t = 1, 2, \ldots, T$ IS

$$f(\underset{\sim}{z}_1, \underset{\sim}{z}_2, \ldots, \underset{\sim}{z}_T) = \prod_{t=1}^{T} f(z_t)$$

$$= \prod_{t=1}^{T} \frac{1}{(2\pi)^{n/2}} |\underset{\sim}{\Sigma}|^{-\frac{1}{2}} e^{-\frac{1}{2}(\underset{\sim}{z}_t - \underset{\sim}{\alpha} - \underset{\sim}{\beta} z_{mt})' \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{z}_t - \underset{\sim}{\alpha} - \underset{\sim}{\beta} z_{mt})}$$

WRITE THE LOG LIKELIHOOD FUNCTION:

$$\ln L = -\frac{nT}{2} \ln 2\pi - \frac{T}{2} \ln |\underset{\sim}{\Sigma}| - \frac{1}{2} \sum_{t=1}^{T} (\underset{\sim}{z}_t - \underset{\sim}{\alpha} - \underset{\sim}{\beta} z_{mt})' \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{z}_t - \underset{\sim}{\alpha} - \underset{\sim}{\beta} z_{mt})$$

THE MAXIMUM LIKELIHOOD ESTIMATORS ARE

$$\hat{\underset{\sim}{\alpha}} = \hat{\underset{\sim}{\mu}} - \hat{\underset{\sim}{\beta}} \hat{\mu}_m \quad \text{WHERE} \quad \hat{\underset{\sim}{\mu}} = \frac{1}{T} \sum_{t=1}^{T} \underset{\sim}{z}_t$$

$$\hat{\mu}_m = \frac{1}{T} \sum_{t=1}^{T} z_{mt}$$

$$\hat{\underset{\sim}{\beta}} = \frac{\sum_{t=1}^{T} (\underset{\sim}{z}_t - \hat{\underset{\sim}{\mu}})(z_{mt} - \hat{\mu}_m)}{\sum_{t=1}^{T} (z_{mt} - \hat{\mu}_m)^2}$$

$$\hat{\underset{\sim}{\Sigma}} = \frac{1}{T} \sum_{t=1}^{T} (\underset{\sim}{z}_t - \hat{\underset{\sim}{\alpha}} - \hat{\underset{\sim}{\beta}} z_{mt})(\underset{\sim}{z}_t - \hat{\underset{\sim}{\alpha}} - \hat{\underset{\sim}{\beta}} z_{mt})'$$

(SAMPLE VARIANCE COVARIANCE MATRIX)

DISTRIBUTIONS :

$$\hat{\underset{\sim}{\alpha}} \sim N_n\left(\underset{\sim}{\alpha}, \frac{1}{T}\left(1+\frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right)\underset{\sim}{\Sigma}\right)$$

NOTE :

$$\hat{\sigma}_M^2 = \frac{1}{T}\sum\left(z_{mt}-\hat{\mu}_m\right)^2$$

$$\hat{\underset{\sim}{\beta}} \sim N_n\left(\underset{\sim}{\beta}, \frac{1}{T}\left[\frac{1}{\hat{\sigma}_m^2}\right]\underset{\sim}{\Sigma}\right)$$

$$T\hat{\underset{\sim}{\Sigma}} \sim \underset{\text{DISTRIBUTION}}{\text{WISHART}}\left(T-2, \underset{\sim}{\Sigma}\right)$$

THE WISHART DISTRIBUTION IS A GENERALIZATION OF THE CHI SQUARE DISTRIBUTION.

ALSO, $\text{COV}\left(\hat{\underset{\sim}{\alpha}}, \hat{\underset{\sim}{\beta}}\right) = -\frac{1}{T}\left[\frac{\hat{\mu}_m}{\hat{\sigma}_m^2}\right]\underset{\sim}{\Sigma}$

AND $\hat{\underset{\sim}{\Sigma}}$ IS INDEPENDENT OF $\hat{\underset{\sim}{\alpha}}$ AND $\hat{\underset{\sim}{\beta}}$.

RECALL THAT CAPM STATES THAT

$$\bar{R}_K - R_F = \hat{\beta}_K\left(\bar{R}_M - R_F\right)$$

THEREFORE, HERE WE WANT TO TEST

$$H_0: \alpha = 0$$
$$H_a: \alpha \neq 0$$

IF H_0 IS NOT REJECTED THEN CAPM HOLDS.

IF $\underset{\sim}{\Sigma}$ IS KNOWN THEN

$$\hat{\underset{\sim}{\alpha}}' \left[\operatorname{VAR}(\hat{\underset{\sim}{\alpha}})\right]^{-1} \hat{\underset{\sim}{\alpha}} \sim \chi^2_n$$

OR AFTER SUBSTITUTING $\operatorname{VAR}(\hat{\underset{\sim}{\alpha}})$ WE GET:

$$T\left[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right] \hat{\underset{\sim}{\alpha}}' \underset{\sim}{\Sigma}^{-1} \hat{\underset{\sim}{\alpha}} \sim \chi^2_n$$

BUT $\underset{\sim}{\Sigma}$ IS UNKNOWN. WE CAN REPLACE $\underset{\sim}{\Sigma}$ WITH $\hat{\underset{\sim}{\Sigma}}$ AND THEN ASYMPTOTICALLY WE STILL GET A χ^2_n DISTRIBUTION.

HOWEVER, WE DO NOT NEED TO USE AN ASYMPTOTIC RESULT. INSTEAD WE USE THE FOLLOWING THEOREM THAT INVOLVES THE WISHART DISTRIBUTION:

THEOREM: LET $\underset{\sim}{X}$ BE AN $m \times 1$ RANDOM VECTOR THAT FOLLOWS $N_m(\underset{\sim}{0}, \Omega)$, AND LET THE $m \times m$ RANDOM MATRIX BE DISTRIBUTED AS $\text{WISHART}_m(n, \Omega)$. IF $\underset{\sim}{X}, \underset{\sim}{A}$ ARE INDEPENDENT

IT FOLLOWS THAT $\frac{n-m+1}{m} \underset{\sim}{X}' \underset{\sim}{A} \underset{\sim}{X} \sim F_{m, n-m+1}$

TO APPLY THE THEOREM ABOVE

SET $\underset{\sim}{x} = \left[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right]^{-1/2} \hat{\underset{\sim}{\alpha}}$ AND $\underset{\sim}{A} = T\hat{\underset{\sim}{\Sigma}}$,

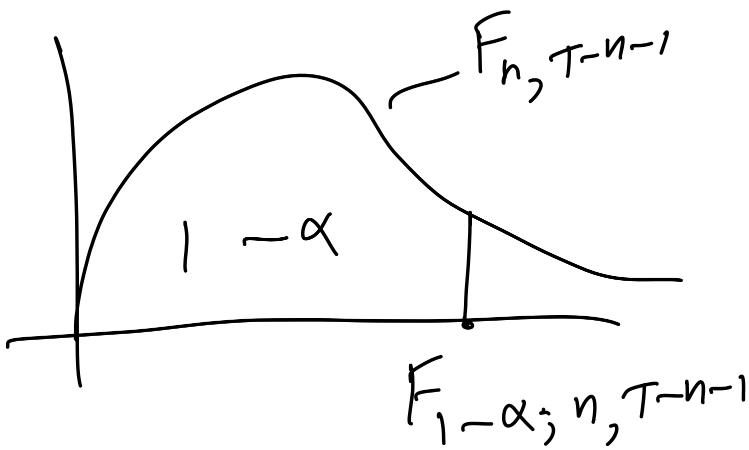
$m = n$, $n = T-2$

TO GET

$$\frac{T-n-1}{n}\left[1 + \frac{\hat{\mu}_m^2}{\hat{\sigma}_m^2}\right]^{-1} \hat{\underset{\sim}{\alpha}}' \hat{\underset{\sim}{\Sigma}}^{-1} \hat{\underset{\sim}{\alpha}} \sim F_{n, T-n-1}$$

REJECT H_0 IF

$$F > F_{1-\alpha;\, n,\, T-n-1}$$



NON - STANDARD FORM OF CAPM

ASSUME NO RISK FREE ASSET.
WE NEED TO FIND THE SO CALLED ZERO-BETA PORTFOLIO.

RESULT 1 : A COMBINATION OF TWO FRONTIER PORTFOLIOS IS ALSO ON THE FRONTIER

RESULT 2: THE EXPECTED RETURN OF ANY ASSET CAN BE EXPRESSED AS A LINEAR COMBINATION OF THE EXPECTED RETURN ON ANY TWO EFFICIENT PORTFOLIOS P AND Q AS FOLLOWS:

$$\bar{R}_K - \bar{R}_Q = (\bar{R}_P - \bar{R}_Q) \frac{Cov(R_i, R_P) - Cov(R_P, R_Q)}{\sigma_P^2 - Cov(R_P, R_Q)}$$

RESULT 3 : EVERY EFFICIENT PORTFOLIO (NOT THE GLOBAL MINIMUM RISK PORTFOLIO) HAS A COMPANION PORTFOLIO ON THE INEFFICIENT HALF OF THE FRONTIER WITH WHICH IT IS UNCORRELATED. THIS PORTFOLIO IS THE ZERO-BETA PORTFOLIO.

SUPPOSE WE CHOOSE P TO BE THE MARKET PORTFOLIO AND Q TO BE THE ZERO-BETA PORTFOLIO WITH EXPECTED RETURN $\bar{R}_Z$.
THEN THE EQUATION ABOVE IS

$$\bar{R}_K - \bar{R}_Z = (\bar{R}_M - \bar{R}_Z) \frac{Cov(R_K, R_M)}{\sigma_M^2}$$

NOTE: $Cov(R_M, R_Z) = 0$

OR $\bar{R}_K - \bar{R}_Z = \hat{\beta}_K (\bar{R}_M - \bar{R}_Z)$

HOW DO WE FIND THE ZERO-BETA PORTFOLIO?

WE HAVE SEEN THAT THE COVARIANCE BETWEEN TWO MINIMUM VARIANCE PORTFOLIOS IS GIVEN BY

$$\frac{C}{D}\left[E_a - \frac{A}{C}\right]\left[E_b - \frac{A}{C}\right] + \frac{1}{C}.$$

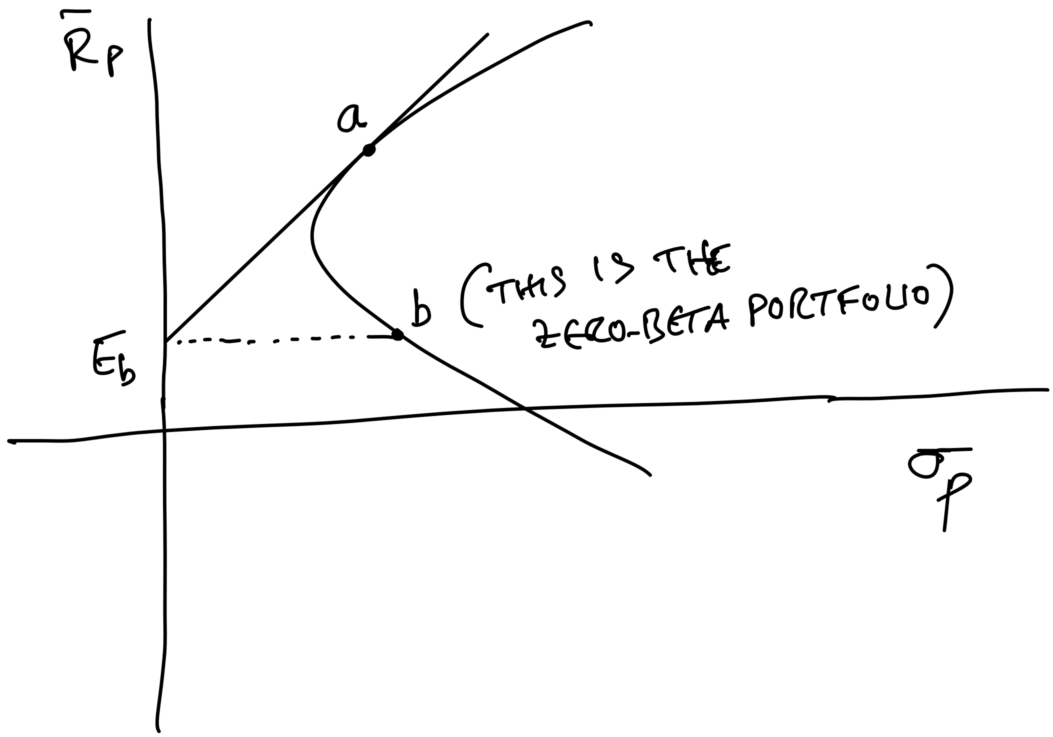
SUPPOSE PORTFOLIO a IS AN EFFICIENT PORTFOLIO. FIND E_b THAT MAKES THE COVARIANCE ZERO.

$$\frac{C}{D}\left[E_a - \frac{A}{C}\right]\left[E_b - \frac{A}{C}\right] + \frac{1}{C} = 0$$

IT FOLLOWS THAT

$$E_b = \frac{A}{C} - \frac{D}{C^2\left[E_a - \frac{A}{C}\right]} \qquad \begin{array}{l} D > 0 \\ C > 0 \\ E_a - \frac{A}{C} > 0 \end{array}$$

THEREFORE $E_b < \frac{A}{C}$ WHICH MEANS THAT PORTFOLIO b IS LOCATED ON THE INEFFICIENT HALF OF THE FRONTIER, AS SHOWN ON THE NEXT PAGE.



EVERY EFFICIENT PORTFOLIO HAS A COMPANION ZERO-BETA PORTFOLIO.

THE TWO PORTFOLIOS ARE UNCORRELATED.