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Statistics C183/C283 **Instructor: Nicolas Christou**

Single index model - short sales not allowed
Risk free asset exists
Kuhn-Tucker conditions

If we assume short sales then we can simply maximize the slope and find the tangent to the efficient frontier subject to the constraint $\sum_{i=1}^{N} x_1 = 1$

$$\max \, \theta = \frac{\bar{R}_G - R_f}{\sigma_G}$$

To find the $x_i's$ we take derivatives w.r.t. each x_i set them equal to zero and solve $\cdots$

$$\frac{d\theta}{dx_i} = z_i\sigma_i^2 + \sum_{j \neq i}^{N} z_j \sigma_{ij} = 0, \; i = 1, \cdots N$$

or

$$\bar{R}_i - R_f = z_i\sigma_i^2 + \sum_{j \neq i}^{N} z_j \sigma_{ij}, \; i = 1, \cdots N$$

If short sales are not allowed we have an extra set of constraints $x_i \geq 0$. We still take the derivative w.r.t. each x_i but now if the maximum occurs at $x_i < 0$ then it is not feasible for our problem. Then $\frac{d\theta}{dx_i} < 0$. But if the maximum occurs at a positive x_i then $\frac{d\theta}{dx_i} = 0$ (see figure below). To summarize

$$\frac{d\theta}{dx_i} \leq 0$$

which can me written as equality

$$\frac{d\theta}{dx_i} + u_i = 0, \text{ this is the first Kuhn-Tucker condition.}$$

About u_i: If the maximum occurs at a positive x_i then $u_i = 0$. If the maximum occurs at $x_i = 0$ then $\frac{d\theta}{dx_i} < 0$ and therefore $u_i > 0$. This is the second Kuhn-Tucker condition and can be written as

$$\begin{aligned} x_i u_i &= 0 \\ x_i &\geq 0 \\ u_i &\geq 0 \end{aligned}$$

Now the system of equations with the Kuhn-Tucker conditions will be:

$$\begin{aligned} \bar{R}_i - R_f &= z_i\sigma_i^2 + \sum_{j \neq i}^{N} z_j \sigma_{ij} - u_i, \; i = 1, \cdots N. \\ z_i u_i &= 0, \; i = 1, \cdots N. \\ z_i &\geq 0, \; i \cdots N. \\ u_i &\geq 0, \; i \cdots N. \end{aligned}$$

If the single index model is assumed then

$$\sigma_{ij} = \beta_i\beta_j\sigma_m^2, \text{ and } \sigma_i^2 = \beta_i^2\sigma_m^2 + \sigma_{\epsilon_i}^2$$

If we substitute this into the first Kuhn-Tucker condition we get

$$\bar{R}_i - R_f = z_i(\beta_i^2 \sigma_m^2 + \sigma_{\epsilon_i}^2) + \sum_{j \neq i}^{N} z_j \beta_i \beta_j \sigma_m^2 - u_i, \ i = 1, \cdots N$$

If we combine the terms on the right side we get

$$\bar{R}_i - R_f = z_i \sigma_{\epsilon_i}^2 + \sum_{j=1}^{N} z_j \beta_i \beta_j \sigma_m^2 - u_i, \ i = 1, \cdots N$$

Suppose now that k out of N securities will be included in the optimum portfolio. For those that are not included $z_i = 0$ and the summation in the previous expression will concern only the securities in the set of k securities

$$\bar{R}_i - R_f = z_i \sigma_{\epsilon_i}^2 + \sum_{j \epsilon k} z_j \beta_i \beta_j \sigma_m^2 - u_i, \ i = 1, \cdots N$$

But then for those securities that have positive z_i the corresponding $u_i = 0$. We can write now

$$\bar{R}_i - R_f = z_i \sigma_{\epsilon_i}^2 + \sum_{j \epsilon k} z_j \beta_i \beta_j \sigma_m^2, \ \text{for } j \epsilon k$$

Solve for z_i

$$z_i = \frac{\beta_i}{\sigma_{\epsilon_i}^2} \left[\frac{\bar{R}_i - R_f}{\beta_i} - \sigma_m^2 \sum_{j \epsilon k} z_j \beta_j \right], \text{for } i \epsilon k \tag{1}$$

We need an expression of $\sum z_j \beta_j$. If we multiply (1) by β_i and some over the set of k securities we get

$$\sum_{i \epsilon k} z_i \beta_i = \sum_{i \epsilon k} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon_i}^2} - \sigma_m^2 \sum_{i \epsilon k} \frac{\beta_i^2}{\sigma_{\epsilon_i}^2} \sum_{j \epsilon k} z_j \beta_j$$

Solve for $\sum z_j \beta_j$ to get:

$$\sum_{j \epsilon k} z_j \beta_j = \frac{\sum_{j \epsilon k} \frac{(\bar{R}_i - R_f)\beta_j}{\sigma_{\epsilon_i}^2}}{1 + \sigma_m^2 \sum_{j \epsilon k} \frac{\beta_j^2}{\sigma_{\epsilon_j}^2}}$$

Expression (1) can be written as:

$$z_i = \frac{\beta_i}{\sigma_{\epsilon_i}^2} \left[\frac{\bar{R}_i - R_f}{\beta_i} - C^* \right]$$

where

$$C^* = \sigma_m^2 \sum_{j \epsilon k} z_j \beta_j = \sigma_m^2 \frac{\sum_{j \epsilon k} \frac{(\bar{R}_i - R_f)\beta_j}{\sigma_{\epsilon_i}^2}}{1 + \sigma_m^2 \sum_{j \epsilon k} \frac{\beta_j^2}{\sigma_{\epsilon_j}^2}}$$