$\vec{i}$ $\vec{j}$ $\vec{k}$. basis 基

span. ⇒ all linear combination

$a\vec{i}+b\vec{j}$ span. 二维空间. ⇒ linear independent 线性无关

$a\cdot\vec{v}+b\cdot\vec{w}$ span 一条线 ⇒ linear dependent

$\vec{v} \parallel \vec{w}$

linear transformation $\begin{cases}\text{网格线 直, 平行, 等距} \\ \text{原点不变}\end{cases}$ ⇒ 仅考虑 $\vec{i}$ $\vec{j}$ 变换

$$\left.\begin{aligned}\vec{i}=\begin{bmatrix}1\\0\end{bmatrix} &\Rightarrow \vec{i'}=\begin{bmatrix}1\\-2\end{bmatrix} \\ \vec{j}=\begin{bmatrix}0\\1\end{bmatrix} &\Rightarrow \vec{j'}=\begin{bmatrix}3\\0\end{bmatrix}\end{aligned}\right\} \Rightarrow \begin{bmatrix}1&3\\-2&0\end{bmatrix}$$

linear transformation　原向量

$$\vec{v}=\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow \vec{v'} = x\cdot\vec{i'}+y\cdot\vec{j'}$$

$$= x\cdot\begin{bmatrix}-1\\2\end{bmatrix}+y\cdot\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}1x+3y\\-2x+0y\end{bmatrix} = \begin{bmatrix}1&3\\-2&0\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{array}{cc} i' & j' \\ \end{array}$$
$$\begin{bmatrix}0&-1\\1&0\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow \text{逆时针}90^\circ$$

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow \text{shear}$$

$$\begin{bmatrix}2&-2\\1&-1\end{bmatrix} \quad \text{二维空间} \Rightarrow \text{一条线}$$

Linearly dependent columns. 列线性相关

矩阵乘法 ⇒ 复合变化

先 rotate 再 shear　　shear (Rotate ( )) = Composition

$$\underset{②\text{shear}}{\begin{bmatrix}1&1\\0&1\end{bmatrix}} \underset{①\text{rotate}}{\begin{bmatrix}0&-1\\1&0\end{bmatrix}} = \begin{bmatrix}1&-1\\1&0\end{bmatrix}$$

又 $\begin{bmatrix}0&2\\1&0\end{bmatrix}\begin{bmatrix}1&-2\\1&0\end{bmatrix} \Rightarrow \begin{bmatrix}2&0\\1&-2\end{bmatrix}$

$\vec{i}$ $\begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}0&2\\1&0\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix} = 1\cdot\begin{bmatrix}0\\1\end{bmatrix}+1\cdot\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$

$\vec{j}$ $\begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}-2\\0\end{bmatrix} \to \begin{bmatrix}0&2\\1&0\end{bmatrix}\cdot\begin{bmatrix}-2\\0\end{bmatrix} = -2\cdot\begin{bmatrix}0\\1\end{bmatrix}+0\cdot\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}0\\-2\end{bmatrix}$

$\therefore AB \neq BA$　　$A(BC)=(AB)C$

三维 $\begin{array}{ccc} i' & j' & k' \end{array}$
$$\begin{bmatrix}1&1&1\\0&1&0\\-1&0&1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = x\cdot i'+y\cdot j'+z\cdot k'$$

$$\begin{bmatrix} & ② & \end{bmatrix}\begin{bmatrix} & ① & \end{bmatrix}$$ CG / Robotics

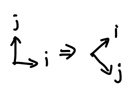
determinant 行列式.

观察 $\vec{i}$ 和 $\vec{j}$ 组成的正方形的变化, 可知 linear tran 是 strech out / squish in space

如 $\begin{bmatrix}3&2\\0&2\end{bmatrix}$. 使 1×1 正方形面积扩大6倍

$\det\left(\begin{bmatrix}3&2\\0&2\end{bmatrix}\right)=6$

det = 0, 将平面 squish to 线/点

det = -2. invert the orientation of the space 

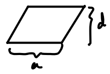
且面积扩大 |det| = 2倍

三维 i j k 组成 parallelepiped 平行六面体

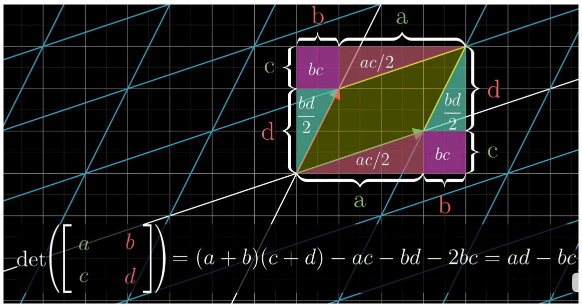
det = 0 ⟺ columns are linear dependent. 可能压缩到面/线/点

det < 0　i' j' k' 不再满足右手螺旋

$\det\left(\begin{bmatrix}a&0\\0&d\end{bmatrix}\right)=ad.$

$\det\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = ad - b\cdot 0 = ad$ 

b.c 表示平行四边形在对角方向上伸缩程度



$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\cdot\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\cdot\det + c\cdot\det = V$平行六面体

$\det(M_1\cdot M_2) = \det(M_1)\cdot\det(M_2)$. 缩放比例

A^{-1}. 逆变换 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$A^{-1}\cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 恒等变换

方程组 $\underset{A}{\begin{bmatrix} & & \\ & 3\times 3 & \\ & & \end{bmatrix}} \cdot \underset{\vec{x}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{\vec{v}}{\begin{bmatrix} 3\times 1 \end{bmatrix}}$

$A\cdot\vec{x} = \vec{v}$. $\vec{x}$ 经 linear tran A 得 $\vec{v}$

当 $\det A \neq 0$ 有唯一解. $\vec{x} = A^{-1}\cdot\vec{v}$

当 $\det A = 0$ 无 A^{-1}. 降到二维 A秩 rank = 2

linear tran 不可升维 降到一维 rank = 1

秩: linear tran 后的空间维数

A的 column space $\Rightarrow$ A 的 span. (column $\Rightarrow$ i, j 张成的空间)

秩: dimension in column space.

$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$. rank = 1

rank $\leq$ column. 列数.

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ - 定在 column space 中 (原点不变)

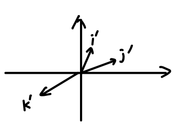
full rank 满秩. 变换后仅原点位置不变

not full rank. 变换后落在原点的向量 $\Rightarrow$ 矩阵的 null space / kernel
零空间 核

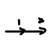
当 $A\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ null space 就是所有可能解

非方阵 $\begin{array}{cc} i' & j' \end{array}$ $\begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 5 & 9 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$
满秩

把 xy 平面投射到三维空间中的一个二维平面.

$\begin{array}{ccc} i' & j' & k' \end{array}$
$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}$ 

三维 → 二维

$\begin{bmatrix} 1 & 2 \end{bmatrix}$ 

二维 → 一维

点积. $[3.4]\cdot[7.2] \overset{\text{duality}}{\Longleftrightarrow} \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} = 29$ $7i + 2j$

$\begin{array}{cc} i & j \end{array}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

二维转一维 linear trans $i \to i' = 3$

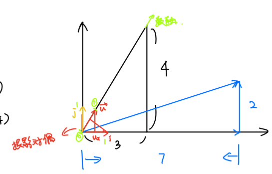
$j \to j' = 4$

$\begin{array}{cc} i' & j' \end{array}$ $(u_x . u_y) = \left(\frac{3}{5}, \frac{4}{5}\right)$

投影 $|\ |\cdot|\ |\cdot\cos$

再伸缩. $\times 5 \Rightarrow (3.4)$

把 $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ 变到一维数轴上 (方向. (3.4))



叉积. cross product

$$\begin{cases} 大小 \begin{cases} \pm S_{平行四边形} / \pm |\,|\cdot|\,|\cdot \sin \\ \begin{bmatrix}3\\1\end{bmatrix} \times \begin{bmatrix}2\\-1\end{bmatrix} = \det\left(\begin{bmatrix}3 & 2\\1 & -1\end{bmatrix}\right) \end{cases} \\ 方向.\ 右手螺旋 \end{cases}$$

(转置det值不变)

$v \times w > 0$　　$i \times j > 0$

$w \times v < 0$　　$j \times i < 0$

$$\Rightarrow \det \begin{bmatrix} \vec{i} & v_1 & w_1 \\ \vec{j} & v_2 & w_2 \\ \vec{k} & v_3 & w_3 \end{bmatrix}$$

证: $\det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} = \pm V_{平行六面体}$

$$f\begin{bmatrix}x\\y\\z\end{bmatrix} = \det \begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}$$

线性变换

$$\Rightarrow \begin{array}{c} p_1\ p_2\ p_3 \\ [?\ ?\ ?] \end{array} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = \det \begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}$$

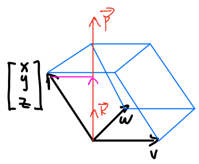
三维→一维

$$\begin{cases} P_1 = v_2 w_3 - v_3 w_2 & \vec{i} \\ P_2 = v_3 w_1 - v_1 w_3 & \vec{j} \\ P_3 = v_1 w_2 - v_2 w_1 & \vec{k} \end{cases}$$

$[p_1\ p_2\ p_3]$ linear tran. 把 $\begin{bmatrix}x\\y\\z\end{bmatrix}$ 投影到垂直于 $\vec{v}$ 和 $\vec{w}$ 方向上

$$\det \begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix} = S_{平行四边形} \cdot \underset{(n \cdot \vec{k})}{高} = \vec{P} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix}$$

$= 点积: \begin{bmatrix}x\\y\\z\end{bmatrix}$ 投影到 $\vec{P}$ 方向上. 长度相乘 ($\begin{bmatrix}x\\y\\z\end{bmatrix}$ 投影为高. $\vec{P}$ 长度为 $S_{平行四边形}$)



$\therefore \vec{P} \begin{cases} 大小为 S_{平行四边形} \\ 方向为 \vec{k}. \end{cases}$

$$\vec{P} = \begin{cases} v_2 w_3 - v_3 w_2 \cdot \vec{i} \\ v_3 w_1 - v_1 w_3 \cdot \vec{j} \\ v_1 w_2 - v_2 w_1 \cdot \vec{k} \end{cases}$$

$$\Rightarrow \therefore \vec{P} = \vec{w} \times \vec{v} = \begin{cases} 大小.\ S_{平行四边形} \\ 方向.\ \vec{k} \end{cases} = \det \begin{bmatrix} \vec{i} & v_1 & w_1 \\ \vec{j} & v_2 & w_2 \\ \vec{k} & v_3 & w_3 \end{bmatrix}$$

基变换

basis $\vec{i}\,\vec{j}$　　basis 2 $\begin{cases} i' = (2.1) \\ j' = (-1.1) \end{cases}$

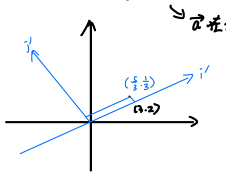
$\vec{a_1} = A \cdot \vec{a_2}$

$A^{-1} \cdot \vec{a_1} = \vec{a_2}$

$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

几何上. basis 1 网格 → basis 2 → 坐标系仍是 $\vec{i}.\vec{j}$. $\vec{a} \to \vec{a'}$

数值上. basis 2 语言 → basis 1 → $\vec{a}$ 在 $\vec{i}\vec{j}$ 坐标系下的表示形式



$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$?

basis 2 中 (-1.2) 逆时针 90° 的 linear tran

$$\underset{\text{转回 basis 2}}{A^{-1}} \cdot \underset{\text{向 90°}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \cdot \underset{\text{转到 basis 1 } \vec{i}\,\vec{j}}{A} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A^{-1} \cdot \underbrace{M}_{变换} \cdot A \cdot \vec{v}$$

特征值 & 特征向量 eigenvector

$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ 后 $\begin{bmatrix} x \\ y \end{bmatrix}$ 未偏离原本 span / 方向

则 $\begin{bmatrix} x \\ y \end{bmatrix}$ 为特征向量. 伸缩的值为特征值 (负为反向)

(三维. 不偏离 span ⇒ 旋转轴. 特征值为 1)

(1.0)	3
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	2

$$A \cdot \vec{v} = \lambda \cdot \vec{v} \Rightarrow A \cdot \vec{v} = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix} \vec{v} = \lambda \cdot \underset{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}{I} \cdot \vec{v} \Rightarrow (A - \lambda I)\vec{v} = \vec{0}$$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 1 & 5-\lambda & 9 \\ 2 & 6 & 5-\lambda \end{bmatrix}$$

↓

将空间降维，才可 $A\vec{v}=\vec{0}$

$\det(A-\lambda I)=0$

eg: $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ $\quad \det\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = 0$

$\lambda = 3 / 2$

↓

解 $\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix}$ 落在 span 为含 $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ 的对角线上

二维 linear tran 不一定有特征向量 / 仅有1个特征向量 / 仅有1个特征值，有无数特征向量

逆时针 90°. $\quad$ shear $\quad$ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ 全部 ×2

diagonal matrix
对角阵

$$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \cdots \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{100次}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3^{100}x \\ 2^{100}y \end{bmatrix}$$

Eigenbasis . 必然是对角阵
特征基

仅伸缩

相似对角化

? 计算 $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}^{100}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

↓

转到特征基空间

向量类比到函数，具有线性，求导看作 linear tran. $1, x, x^2, x^3 \cdots$ 看作基，空间: all polynomials

$$\frac{d}{dx}(1x^3+5x^2+4x+5) = 3x^2+10x+4$$

$$\begin{bmatrix} 0 & 1 & & \cdots \\ 0 & & 2 & \\ 0 & & & 3 \\ \vdots & & & \cdots \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 3 \\ \vdots \end{bmatrix}$$

↓

求导 linear tran. 来自 $\begin{bmatrix} (1)' & (x)' & & \cdots \\ & & (x^2)' & \\ & & & (x^3)' \\ \cdots & & & \cdots \end{bmatrix}$

线性代数中的概念	应用于函数时的别名
Linear transformations 线性变换	Linear operators 线性算子
Dot products 点积	Inner products 内积
Eigenvectors 特征向量	Eigenfunctions 特征函数

线代可应用到其他主体，此学科中主体为向量

线代是一套 Axioms 公理，是抽象的而非具体.

Rules for vectors addition and scaling

向量加法和数乘的规则

1. $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
2. $\vec{v}+\vec{w}=\vec{w}+\vec{v}$
3. There is a vector $\mathbf{0}$ such that $\mathbf{0}+\vec{v}=\vec{v}$ for all $\vec{v}$
4. For every vector $\vec{v}$ there is a vector $-\vec{v}$ so that $\vec{v}+(-\vec{v})=\mathbf{0}$
5. $a(b\vec{v})=(ab)\vec{v}$
6. $1\vec{v}=\vec{v}$
7. $a(\vec{v}+\vec{w})=a\vec{v}+a\vec{w}$
8. $(a+b)\vec{v}=a\vec{v}+b\vec{v}$

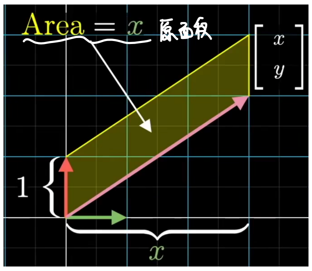
“公理”

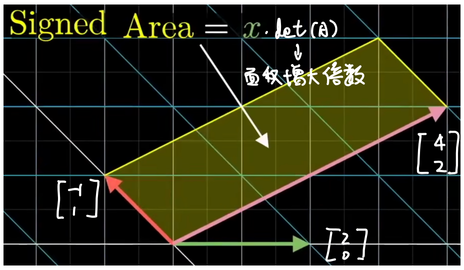
“Axioms”

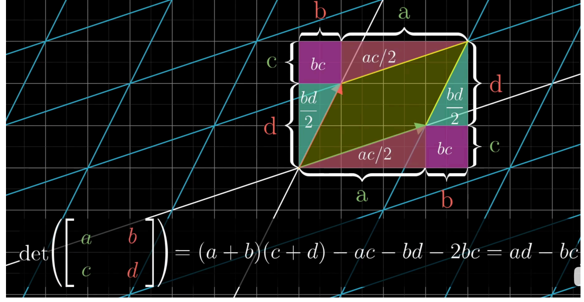
Cramer's rule
克莱姆法则（高斯消元法更快） 仅考虑 $\det \neq 0$ 情况

（基向量的点积在线性变化前后通常会改变
Orthonormal tran 正交变换. 基在变换前后长度不变，仍相互垂直（如 旋转 90°））

$$\begin{cases} 2x - y = 4 \\ 0x + y = 2 \end{cases} \qquad \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$







$$x = \frac{\text{Area}}{\det A} = \frac{\det\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}}{\det\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}}$$

扩展到三维

$$\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 2 \\ -4 & 6 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \\ 3 \end{bmatrix}$$

$$x = \frac{\det\left(\begin{bmatrix} 7 & 2 & 3 \\ -8 & 0 & 2 \\ 3 & 6 & -9 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 2 \\ -4 & 6 & -9 \end{bmatrix}\right)} \quad y = \frac{\det\left(\begin{bmatrix} -4 & 7 & 3 \\ -1 & -8 & 2 \\ -4 & 3 & -9 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 2 \\ -4 & 6 & -9 \end{bmatrix}\right)} \quad z = \frac{\det\left(\begin{bmatrix} -4 & 2 & 7 \\ -1 & 0 & -8 \\ -4 & 6 & 3 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 2 \\ -4 & 6 & -9 \end{bmatrix}\right)}$$

MIT 18.06

LEC1 row picture / column picture

$Ax=b$.

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

row picture 直线相交

☆ column picture 列向量的线性组合 $x\cdot\begin{bmatrix} 2 \\ -1 \end{bmatrix}+y\begin{bmatrix} -1 \\ 2 \end{bmatrix}=\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

高维时列图像更清洁

LEC1.5 乘法

另一种角度 linear comb of column vector

$$\begin{array}{ccc} u & v & w \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\underset{u}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} + x_2\underset{v}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} + x_3\underset{w}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} x_1 \\ x_2-x_1 \\ x_3-x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

difference. matrix $Ax=b$.

其逆运算

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1+b_2 \\ b_1+b_2+b_3 \end{bmatrix}$$

sum matrix $A^{-1}\cdot b = x$

Overview

$A=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$: $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} x_1 \\ x_2-x_1 \\ x_3-x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $Ax=0$ $\Rightarrow$ 解 $x_1=x_2=x_3=0$

可逆

⇓

$C=\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$: $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} x_1-x_3 \\ x_2-x_1 \\ x_3-x_2 \end{bmatrix}=\begin{bmatrix} b_1=0 \\ b_2=0 \\ b_3=0 \end{bmatrix}$ $Cx=0$ $\Rightarrow$ 解 $x_1=x_2=x_3=c$

不可逆

$\begin{cases} \exists \vec{x}\neq\vec{0}.\ 使 Ax=0. \Rightarrow A 不可逆 \\ A 不可逆时. Ax=0 解不唯一 \end{cases}$

$C=\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ 的 vector space 等价于 Cx. 是平面 $b_1+b_2+b_3=0$

是 R^3 的 subspace

等价: ① independent.

② 5-dim basis

③ $\underbrace{[\,0\,0\,0\,0\,0\,]}_{5\text{ vector}}$ is invertible 可逆的

subspace $\begin{cases} 0\text{-dim} \Rightarrow \text{a point. 仅含 } \vec{0} \\ 1\text{-dim} \Rightarrow \text{a line} \\ 2\text{-dim} \Rightarrow \text{a plane} \\ 3\text{-dim} \Rightarrow \text{space 本身} \end{cases}$

7×3 rectangular matrix A. 不可逆

$\underset{3\times7}{A^T}\cdot\underset{7\times3}{A} \Rightarrow 3\times3$. symmetric. (A. KVL. A^T. KCL)

LEC2 gauss elimination

$Ax=b$

first pivot second pivot det = 10

A消元 (行变换) $\Rightarrow$ 上三角阵 $U=\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$ (diagonal 上有0 $\Rightarrow$ 矩阵不可逆. 消元后无解)

软件中解方程. $[A\ b]$ augmented matrix (增广) $\Rightarrow$ ① Gauss Elimination 消元 得 $[U.c]$ ② Back-substitution 回带求解

pivot ≠0. (换行处理. temp failure ; 全0列 complete failure)

消元矩阵 E

行变换. 左乘. 消元阵 × A

单位阵 $I\times A=A$.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\times A$. 2行互换

$A\times\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 2列互换

$PA\neq AP$ not commutative

消元矩阵 由I变化得到.

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

第一行×-3. 加到第二行 → 将2行1列置. 记作E_{21}.

若 $E_{32}E_{21}A=U$. 记 $E=E_{32}\cdot E_{21}$ 为A的消元阵. associative

$E^{-1}\cdot E = I$ $\quad Ax=b.$

$E\cdot Ax = E\cdot b$

$U\cdot x = E\cdot b$

LEC3 矩阵乘法

$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\cdot\begin{bmatrix}3\\4\\5\end{bmatrix} = 3\cdot\begin{bmatrix}1\\4\\7\end{bmatrix}+4\cdot\begin{bmatrix}2\\5\\8\end{bmatrix}+5\begin{bmatrix}3\\6\\9\end{bmatrix}$

$[1\ 2\ 7]\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} = 1\cdot[1\,2\,3]+2\cdot[4\,5\,6]+7\cdot[7\,8\,9]$

$\underset{4\times4}{A}\cdot\begin{bmatrix}a&b&c&d\\a&b&c&d\\a&b&c&d\\a&b&c&d\end{bmatrix} = \left[A\cdot\begin{bmatrix}a\\a\\a\\a\end{bmatrix}\ A\cdot\begin{bmatrix}b\\b\\b\\b\end{bmatrix}\ A\cdot\begin{bmatrix}c\\c\\c\\c\end{bmatrix}\ A\cdot\begin{bmatrix}d\\d\\d\\d\end{bmatrix}\right]$

$\begin{bmatrix}a&a&a&a\\b&b&b&b\\c&c&c&c\end{bmatrix}\cdot\underset{4\times4}{A} = \begin{bmatrix}[a\ a\ a\ a]\cdot A\\ [b\ b\ b\ b]\cdot A\\ [c\ c\ c\ c]\cdot A\end{bmatrix}$

$\begin{bmatrix}2&7\\3&8\\4&9\end{bmatrix}\begin{bmatrix}1&6\\0&0\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}[1\quad 6]+\begin{bmatrix}7\\8\\9\end{bmatrix}[0\quad 0]$

$\begin{bmatrix}2\\3\\4\end{bmatrix}[1\ 6] = \begin{bmatrix}2&12\\3&18\\4&24\end{bmatrix}$ (2,12) (3,18) (4,24) on same line

(2,3,4) (12,18,24) on same line

$\Rightarrow$ row/column space is a single line

逆矩阵
Inverses

方阵. 唯一. $\quad A\cdot A^{-1}=A^{-1}\cdot A = I \Rightarrow$ invertible / nonsingular 可逆

$\exists\vec{x}\neq\vec{0}$. 使 $Ax=0$. $\Rightarrow$ A. singular 不可逆

$\begin{bmatrix}1&3\\2&6\end{bmatrix}$ 不可逆 det=0 $\quad\begin{bmatrix}1&3\\2&6\end{bmatrix}\underset{\vec{x}}{\begin{bmatrix}3\\-1\end{bmatrix}}=\begin{bmatrix}0\\0\end{bmatrix}$

$(AB)^{-1}=B^{-1}A^{-1} \Leftarrow AB\cdot B^{-1}\cdot A^{-1}=I$

$(A^{-1})^T=(A^T)^{-1}$ transpose 和 inverse 可交换 $\Leftarrow (AA^{-1})^T=(A^{-1})^T\cdot A^T = I$

$A^T\cdot(A^T)^{-1} = I$

求逆矩阵
高斯-若当.
Gauss-Jordan

$\underset{A}{\begin{bmatrix}1&3\\2&7\end{bmatrix}}\underset{A^{-1}}{\begin{bmatrix}a&b\\c&d\end{bmatrix}} = I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\left[\begin{array}{cc|cc}1&3&1&0\\2&7&0&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&3&1&0\\0&1&-2&1\end{array}\right]\to\left[\begin{array}{cc|cc}1&0&7&-3\\0&1&-2&1\end{array}\right]$

$E\cdot A = I \quad E\cdot I = A^{-1}$

LEC4 LU分解
Factorization

$A = LU$ (Pivot $\neq 0$), $L = E^{-1}$ $\qquad U = E\cdot A \qquad A = E^{-1}\cdot U$

$A=LU$ { L is lower triangular with 1 on diagonal; U is upper triangular with pivots on diagonal }

$\overset{A}{\begin{bmatrix}2&1\\8&7\end{bmatrix}} = \overset{L}{\begin{bmatrix}1&0\\4&1\end{bmatrix}}\overset{U}{\begin{bmatrix}2&1\\0&3\end{bmatrix}}.$

$\overset{A}{\begin{bmatrix}2&1\\8&7\end{bmatrix}} = \overset{L}{\begin{bmatrix}1&0\\4&1\end{bmatrix}}\overset{D}{\begin{bmatrix}2&0\\0&3\end{bmatrix}}\overset{U'}{\begin{bmatrix}1&1/2\\0&1\end{bmatrix}}.$

三维: $E_{32}\cdot E_{31}\cdot E_{21}\cdot A = U$ (only 高斯消元. no row exchange, pivot $\neq 0$)

顺序 $E_{21}^{-1}\cdot E_{31}^{-1}\cdot E_{32}^{-1}\cdot U = A$

$E_{21}^{-1}\cdot E_{31}^{-1}\cdot E_{32}^{-1}\cdot$ 比 $E_{32}\cdot E_{31}\cdot E_{21}$ 更好 $\Rightarrow E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}$ $E_{31}=I$ $E_{21}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$ $\quad E=\begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}$

$E_{21}^{-1}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}$ $E_{31}^{-1}=I$ $E_{32}^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}$ $\quad L=\begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix}$

if no row exchanges. multipliers go directly into L

求逆复杂度

计算机 求矩阵逆.

将 A (100×100) 经行变换得 U. 要计算几次?

第一列 $\vdots$. 运算 100次. 第一行100个元素. $\therefore 100^2$次

第二列 99^2次

计算量 $\sum_1^n k^2 \approx \int_1^n k^2 dk$ 次 $=\frac{1}{3}n^3$

置换矩阵 P. Permutation　　用于 row changes (pivot=0)

n 阶方阵有 $n!$ 个置换矩阵

$P \cdot P^{-1} = I$　　$P^{-1} = P^T$

求矩阵 $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ 的所有置换矩阵，并判断其性质。

一共有 6 个置换矩阵：

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

LEC5

symmetric　　对称阵 $A = A^T$.　　主对角线

$(AB)^T = B^T A^T$

$\Downarrow$

$(B \cdot B^T)$ 是对称阵 (B 不一定是方阵) $\Leftarrow$ $(B \cdot B^T)^T = {B^T}^T B^T = B B^T$

vector space　　向量空间/子空间. 对线性运算封闭. 必须包含原点. 即 $\vec{0}$

推广一下，R^3 的子空间就是如下三个：

（1） 穿过原点的平面

（2） 穿过原点的直线

（3） Z，原点。$\vec{0}$

2个子空间 $\cap$ 仍是子空间

$\cup$ 不是子空间

LEC6

column space

$A = \begin{bmatrix}1&3\\2&3\\4&1\end{bmatrix}$　C(A) 列空间 $= A \cdot x = $ 由 $\begin{bmatrix}1\\2\\4\end{bmatrix}\begin{bmatrix}3\\3\\1\end{bmatrix}$ span 的 R^2. 是 R^3 的子空间 (列向量 3 维)

$x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

$x_1 \cdot \begin{bmatrix}1\\2\\4\end{bmatrix} + x_2 \begin{bmatrix}3\\3\\1\end{bmatrix}$

$A = \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}$　$\underbrace{\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + \begin{bmatrix}1\\1\\1\\1\end{bmatrix}}^{\text{主列}} = \begin{bmatrix}2\\3\\4\\5\end{bmatrix}$　C(A) span 到 R^2. 是 R^4 的子空间

$Ax = b$　① 当 b 落在 C(A) 中. 有解

② 当 $b = \vec{0}$. 必有解. 因为任何子空间都含 $\vec{0}$

null space　　$Ax = 0$ 的 x span 的空间 $\Rightarrow$ null space. N(A)

eg: 求 $A = \begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}$ 的零空间　　$1+1=2$　　$Ax = 0$.　$x = \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix} \cdots$

$c \cdot x$

$N(A) = c \cdot \begin{bmatrix}1\\1\\-1\end{bmatrix}$　N(A) span 到 R^1. 是 R^3 的子空间

证 null space 是 space　　$Av = 0$, $Aw = 0 \Rightarrow A(v+w) = 0$

$A \cdot v = 0 \Rightarrow A \cdot (cv) = 0$

$Ax = b$.　③ $b = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$　$x = \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\-1\\1\end{bmatrix} \cdots$　是一条不过原点的直线

非 space

LEC7

compute null space

$Ax = 0$

homo linear sys

$Ax = 0$　以增广的形式右侧 $= 0$. 因此省略. 仅处理 A

$Ax = 0$ 与 $Ux = 0$ 解出同样 null space

$$A = \begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix} \xrightarrow{} \begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix} \rightarrow \begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix} = U$$

$\times 2 =$

echelon 阶梯形式

pivot column

free column

rank = num of pivots $= 2$

special solutions

back substitution $\Rightarrow$ $N = x = c_1 \cdot \boxed{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}} + c_2 \boxed{\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}}$

row = m column = n rank = r

free variable : n − r

rref
reduced row
echelon form
$U \to R$

还是行变换. $Rx=0$ 与 $Ax=0$ 解出同样 null space

$$\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix} \to \begin{bmatrix}1&2&0&-2\\0&0&2&4\\0&0&0&0\end{bmatrix} \to \begin{bmatrix}\boxed{1}&2&0&-2\\0&0&\boxed{1}&2\\0&0&0&0\end{bmatrix} = R$$

matlab: rref(A)

pivots matrix $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ free matrix $\begin{bmatrix}2&-2\\0&2\end{bmatrix}$ 取负

$$N = c_1 \cdot \begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + c_2 \cdot \begin{bmatrix}+2\\0\\-2\\1\end{bmatrix}$$

F can be partly mixed into I

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \leftarrow r\text{行}. \qquad RN=0 \qquad N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

r列 pivot n−r列 free

2 pivot column
1 free column

$$A = \begin{bmatrix}1&2&3\\2&4&6\\2&6&8\\2&8&10\end{bmatrix} \longrightarrow U = \begin{bmatrix}1&2&3\\0&2&2\\0&0&0\\0&0&0\end{bmatrix} \quad N = c\begin{bmatrix}-1\\-1\\1\end{bmatrix} \longrightarrow R = \begin{array}{c} \begin{matrix} I & & F \end{matrix} \\ \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix} \end{array} \quad N = c \cdot \begin{bmatrix} -F \\ I \end{bmatrix}$$

$1+2=3$

rank = 2 free column

LEC8

compute $Ax=b$
non-homo linear sys

$$Ax=b \to \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \Rightarrow U = \begin{bmatrix}1&2&2&2&b_1\\0&0&2&4&b_2-2b_1\\0&0&0&0&b_3-b_2-b_1\end{bmatrix}$$

$$\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}$$

free column

1° 当 $Ax=b$ 有解. $b \in C(A)$

以 $\vec{b} = \begin{bmatrix}1\\5\\6\end{bmatrix}$ 为例

algorithm : ① 求 $x_{particular}$. set free var = 0. $x_2=0$. $x_4=0$

back subs $\Rightarrow x_p = \begin{bmatrix}-2\\0\\\frac{2}{3}\\0\end{bmatrix}$

② 求 x_{null}

③ $x = x_p + x_n$

$A \cdot x_p = b$
$A \cdot x_n = 0$
$A \cdot (x_n + x_p) = b$

$$x_{complete} = \underbrace{\begin{bmatrix}-2\\0\\\frac{2}{3}\\0\end{bmatrix}}_{\text{平移}} + \underbrace{c_1 \cdot \begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}2\\0\\-2\\1\end{bmatrix}}_{\text{subspace}}$$

解是 ⇒ 经过 x_p, 不过原点 的平面

solvability

$r \le m \quad r \le n$

① $r=n<m$ no free var. $N = \{\text{zero vector}\}$
full column rank 若有解. $x = x_p$. unique
$\boxed{0}$ or 1 solution

$\begin{bmatrix} I & F & \\ 0 & 0 & b\neq 0 \end{bmatrix}$

eg: $A = \begin{bmatrix}1&3\\2&1\\6&1\\5&1\end{bmatrix} \to R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

当 $b=\begin{bmatrix}4\\3\\7\\6\end{bmatrix}$. 有1解. 大多数时候 0解. 如 $0x=b$

② $r=m<n$ 对 $\forall b$. 都有解
full row rank free var = n − r
∞ solution

eg: $\begin{bmatrix}1&2&6&5\\3&1&1&1\end{bmatrix} \to R = \begin{bmatrix} I & F \end{bmatrix}$

③ $r=m=n$ invertible matrix. 方阵
full rank $N = \{\text{zero vector}\}$
对 $\forall b$. 都有解 + unique solution

eg: $A = \begin{bmatrix}1&2\\3&1\end{bmatrix} \to R = I$

④ $r<m$ $\quad$ 0 or ∞ solution $\quad$ eg: $R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

$r<n$

LEC9 $\quad$ independent

① 对 $x_1, x_2 \cdots x_n$. 除了 $\vec{c}=0$ 时, $\vec{c}\cdot\vec{x}\neq 0$. $x_1 x_2 \cdots x_n$ independent

∃ $\vec{c}\neq 0$ 使 $\vec{c}\cdot\vec{x}=0$. 则 dependent

因此 $\vec{x_1}$ 和 $\vec{0}$ dependent

② $N(A)=\{\text{zero vector}\}$. 则 A 的列向量 independent $\quad$ $r=n$ $\quad$ no free var $\Rightarrow N(A)=\{0\}$

($Ax=0$)

∃ $x\neq 0$ 使 $Ax=0$. 则 dependent $\quad$ $r<n$ $\quad$ have free var $\Rightarrow N(A)=c_1[\ ]+c_2[\ ]$

span. basis. dimension

R^6 的 basis 是 6 个 6 维向量

rank of A = dim of C(A) = num of pivot

dim of N(A) = num of free var = n−r

standard basis $\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$

LEC10 $\quad$ 4 subspace:

A : m×n

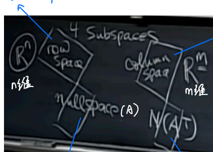
column space. C(A) . R^m

null space . N(A) . R^n

row space. $C(A^T)$. R^n

left null space $N(A^T)$. R^m

dim = r . pivot



dim = r pivot

dim = n−r free var

dim = m−r free var

dim of column space = dim of row space = r

行变换 preserve row space, change column space

$A=\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix} \rightarrow R=\begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix}$ (I, F)

$C(A)\neq C(R)$

basis for row space of A / R is first r rows of R

A. 不一定对

$N(A^T) \rightarrow A^T\cdot y=0$ $\quad$ ($\rightarrow y^T\cdot A=0$. "left")

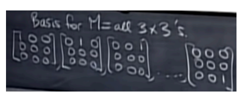
Gauss−Jordan. $[A_{m\times n} \quad I_{m\times m}] \rightarrow [R_{m\times n} \quad E_{m\times m}]$ $\quad$ (I $\quad A^{-1}$)

$E\cdot A=R$

$E=\begin{bmatrix}-1&2&0\\1&-1&0\\-1&0&1\end{bmatrix}$ basis for $N(A^T)$. $[-1\ 0\ 1]\cdot A=[0\ 0\ 0]$

$E\cdot A=R \quad \begin{bmatrix}-1&2&0\\1&-1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}=\begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix}$

space with no vector



all 3x3 matrix → span → matrix space M → dim = 9

subspace of M → all U / symmetric matrix / diagnol matrix (前两者取交集, D = S∩U)

↳ 仅考虑相加和数乘封闭 $\quad$ dim = 6 $\quad$ (smaller)

不考虑矩阵乘法 $\quad$ dim = 3

$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&3&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&7\end{bmatrix}$

diagnol matrix space 的 3 个 basis

dim = 6 $\quad$ 6 $\quad$ 9

S∪U 不是 subspace. 但 S+U = all 3x3 matrix (sum)

(comb of S and U)

S∩U (intersection) dim = 3

6+6 = 3+9

LEC11 $\quad$ 微分方程空间 $\quad \frac{d^2y}{dx^2}+y=0$ $\quad$ $y=\cos x / \sin x \Rightarrow$ solution / null space

$y=c_1\underbrace{\cos x}+c_2\underbrace{\sin x}_{\text{basis}} \Rightarrow$ complete solution dim = 2

$e^{ix}\cdot e^{-ix} \Rightarrow$ another basis

rank 1 matrix

$\underset{2\times3}{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$. $r=1$. basis for row space : $[1\ 4\ 5]$

$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} [1\ 4\ 5] = \overset{2\times1}{u} \cdot \overset{1\times3}{v^T}$

rank one matrix is building block

5x17 . rank 4 的矩阵 可拆成 4个 rank 1 矩阵的组合 ?

M = all 5x17 rank 4 matrix . 不是subspace 对加法不封闭

$\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$

subspace example

In $\mathbb{R}^4$. $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$. S = all v in $\mathbb{R}^4$ with $v_1+v_2+v_3+v_4=0 \Rightarrow$ 是 subspace

$A\cdot v = 0 \qquad S = N(A)$

$\begin{array}{cccc} \text{pivot} & & & \\ \end{array}$ $[1\ 1\ 1\ 1]$. (free)

$r(A)=1 \quad \dim N(A) = n - r = 3$ $\quad$ dim=3

basis for $N(A)$: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$C(A) = \mathbb{R}^1$ (dim=1) $\quad N(A^T) = \{0\}$ (dim=0) $\qquad 3+1=4=n$

all comb of 4×1 $\qquad\qquad 1+0=1=m$

$C(A^T)$ dim = 1

LEC12

Graph

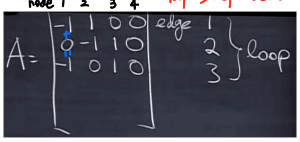
Insidence matrix

Graph . nodes (n) + edges (m)

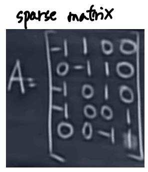
全国友谊 Graph . 6人 ⇒ small world

以 potential & current (电压电流) 为例

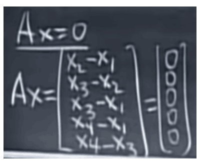
node 1 2 3 4 $\qquad$ loop ⇒ dependent



sparse matrix



$e = Ax$



$x \to$ potential

$Ax \to$ potential difference

basis for $N(A) \Rightarrow x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

all in $N(A) \Rightarrow x = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ 恒电势、无电流

$\dim N(A) = 1$

$\text{rank}(A) = 3$

令 $x_4=0$ 接地. 不再是 node . 剩下 3 column independent

$y = Ce$ 令 current on edges $y_1 . y_2 . y_3 . y_4 . y_5$

KCL ☆ $\underset{\substack{4\times5 \\ n\times m}}{A^T y = 0} \Rightarrow \dim N(A^T) = m - r = 2$

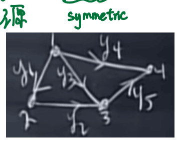
\# loops $= \#\text{edge} - (\#\text{node} - 1)$

$r = n-1$. (one dependency)

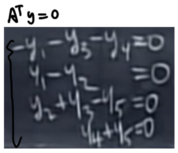
$\underset{\text{dim0}}{\#\text{node}} - \underset{\text{dim1}}{\#\text{edges}} + \underset{\text{dim2}}{\#\text{loop}} = 1$

Euler's formula

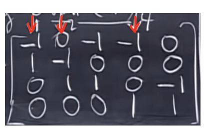
$A^T y = f$ (电流源) $\Rightarrow \underbrace{A^T C A}_{\text{symmetric}} x = f$. balance equation



$A^T y = 0$



basis for $N(A^T)$: 仅某一圈上有电流 $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$



$C(A^T) = r = 3$. basis for $C(A^T)$. 图中不形成 loop 的三条 edge

4 node, 3 edge ⇒ tree (graph with no loop)

LEC14 orthognal

vector/subspace

vector perpendicular

$x^T \cdot y = 0$ $\quad x^T \cdot x + y^T \cdot y = \underbrace{(x+y)^T (x+y)}_{x^Tx + y^Ty + x^Ty + y^Tx}$

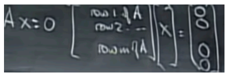
$\vec{0} \perp \vec{a}$

subspace orthognal：2个space中所有vector orthognal

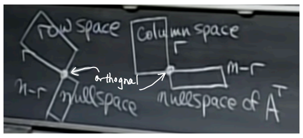
2 space meet. then non-orthognal. unless only meet at $\vec{0}$

row space is orthognal to nullspace

lin comb of rows



orthognal complement



$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$ $\quad n=3 \quad r=1 \quad \dim N(A)=2$

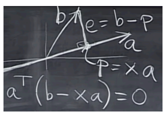
row space is a line $(1,2,5)$, $(0,0,0)$

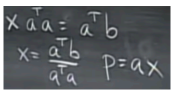
row space and nullspace are orthognal complements in R^n

$r + n - r = n$

nullspace contains all vectors $\perp$ row space

LEC15 projection 投影





$p = \left(a \cdot \frac{a^T b}{a^T a}\right)$

$p = Pb$

projection matrix $P = \frac{aa^T}{a^Ta}$

$C(P)$ = line through a

$r(P) = 1$

$P^T = P$ symmetric

$P^2 = P$ 投影第二次无效果

$Ax=b$ with no solution

$Ax=b \quad m>n.$ 1000次测量. 1000个方程. 解6个参数

A^TA. square symmetric. not always invertible $\quad A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$

$r=1 \quad\quad r=1$

$A^TA\hat{x} = A^Tb$ $\quad \hat{x} \neq x$. $\hat{x}$是一个best solution

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$ $\quad r=2 \quad A^TA = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$

$N(A^TA) = N(A)$

$\text{rank}(A^TA) = \text{rank}(A)$

A^TA is invertible when $N(A) = \{0\}$

A has independent column

$Ax=b$. $\Rightarrow$ $A\hat{x} = p$. $\quad p$ is projection of b onto col space

no solution

$b - A\hat{x} = e$. $e \perp C(A)$.

$C(A)$ 的 basis a_1, a_2. $\begin{cases} a_1^T(b - A\hat{x}) = 0 \\ a_2^T(b - A\hat{x}) = 0 \end{cases}$

$A^T \underbrace{(b - A\hat{x})}_{} = 0$

e. in $N(A^T)$. $N(A^T)$ and $C(A)$ orthognal

$e \perp C(A)$

$A^TA\hat{x} = A^Tb \Rightarrow \hat{x} = (A^TA)^{-1}A^Tb$ $\quad$ 1D : $\frac{aa^T}{a^Ta}$

$p = A\hat{x} = A(A^TA)^{-1}A^Tb$

$P = A(A^TA)^{-1}A^T$

(b保持原位置不变)

若A是 invertible square. $A \cdot A^{-1} \cdot (A^T)^{-1} \cdot A^T = I$. 把b投影到整个空间. 并非subspace

A是一个 $m>>n$ 的不可逆矩阵. 因此不可拆分. 把b投影到subspace

$P^T = P$. $P^2 = P$

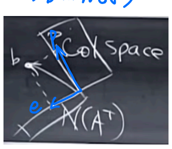
LEC16

b in $C(A)$. $Pb = b$ $\Rightarrow p = Pb = A(A^TA)^{-1}A^T \cdot \underbrace{b}_{Ax} \Rightarrow \overset{b}{Pb} = A \cdot \overbrace{(A^TA)^{-1} \cdot A^TA}^{I} \cdot x \Rightarrow Ax = b$

Ax (things in $C(A)$ are Ax)

$b \perp C(A)$. $Pb = 0$

↳ b in $N(A^T)$



$\begin{matrix} p & + & e & = b \\ \downarrow & & \searrow & \\ Pb & & (I-P)b & \end{matrix}$

project to $\perp$ space $\quad I-P$ also symmetric. $(I-P)^2 = I-P$

application: least square fit

fit (1,1) (2,2) (3,2) 误差min的 line : $\hat{C}+\hat{D}x$ outlier 异常值. least squares overcompensate for outlier because of squaring

$\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix}=\begin{bmatrix}1\\2\\2\end{bmatrix} \Rightarrow A^TA\hat{x}=A^Tb$. $A^TA=\begin{bmatrix}3&6\\6&14\end{bmatrix}$

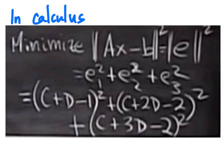
$A \quad x = b$

symmetric
invertible
positive definite

$\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\left[\begin{array}{cc|c}1&1&1\\1&2&2\\1&3&2\end{array}\right]=\left[\begin{array}{cc|c}3&6&5\\6&14&11\end{array}\right]$

$\begin{cases}3\hat{C}+6\hat{D}=5\\6\hat{C}+14\hat{D}=11\end{cases}$

In calculus

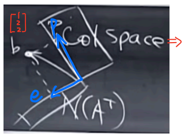


求导

$\Rightarrow \overset{\hat{C}}{\frac{2}{3}}x+\overset{\hat{D}}{\frac{1}{2}}$. $\hat{x}=\begin{bmatrix}\hat{C}\\\hat{D}\end{bmatrix}$ $p=A\cdot\hat{x}$

给定点b. fit line 点p. 误差e

$\vec{e}=(-\frac{1}{6},\frac{2}{6},-\frac{1}{6})$. $\vec{p}=(\frac{7}{6},\frac{5}{3},\frac{13}{6})$. $\vec{e}+\vec{p}=\vec{b}$.



⇒ spanned by $\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}$

$\vec{e}\cdot\vec{p}=0$. $\vec{e}\perp C(A)$

C and D are comb of 2 columns that gives p

prove . A has indep col $\Rightarrow A^TA$ invertible

即 $A^TAx=0 \Rightarrow x=0$

$x^TA^TAx=0$

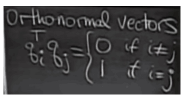
$\underbrace{(Ax)^T(Ax)}_{\text{square}}=0 \Rightarrow Ax=0$. ∵ A has indep col

∴ $x=0$

LEC17 orthognal basis/matrix

col definitely indep if they're perp. unit vector
orthonormal vector

eg: $\left[\begin{array}{c|c|c}0&1&0\\0&0&0\\1&0&1\end{array}\right]$ $\left[\begin{array}{c|c}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{array}\right]$



$Q=\begin{bmatrix}q_1&\cdots&q_n\end{bmatrix}$ (square)

$Q^TQ=I$
$Q^T=Q^{-1}$

eg: $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ $\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$

Adhemar

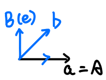
$\frac{1}{3}\begin{bmatrix}1&-2&2\\2&-1&-2\\2&2&1\end{bmatrix}$

project onto Q's column space . $P=Q(\underbrace{Q^TQ}_{I})^{-1}Q^T=\underbrace{QQ^T}_{\text{不一定}I}=I$ (if Q is square)

$\underbrace{Q^TQ}_{I}\hat{x}=Q^Tb \Rightarrow \hat{x}=Q^Tb$

$\hat{x}_i=q_i^Tb$

Graham-Schmidt
$A\to Q$

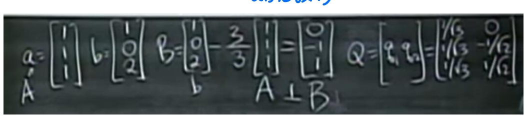


a,b (in R^n) $\Rightarrow B=b-\underbrace{\frac{A^Tb}{A^TA}A}_{\text{a方向上投影}}$ $\Rightarrow$ orthognol $A\perp B$ $\Rightarrow$ orthonormal $q_1=\frac{A}{|A|}$ $q_2=\frac{B}{|B|}$

A^TB
$=A^T(b-\frac{A^Tb}{A^TA}A)$
$=0$

a,b,c . ⇒ A.B 同上 . $C=c-\underbrace{\frac{A^Tc}{A^TA}A}_{\text{a方向投影}}-\frac{B^Tc}{B^TB}B$

eg:



$C(\underbrace{\begin{bmatrix}1&1\\1&0\\1&2\end{bmatrix}}_{A})=C(Q)$

$A=QR$. $R=\begin{bmatrix} & \\ 0 & \end{bmatrix}$. upper triangular
$[a_1\ a_2]$ $[q_1 q_2]$

$a_1^Tq_2$