Exercises on the geometry of linear equations

Problem 1.1: (1.3 #4. *Introduction to Linear Algebra:* Strang) Find a combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3$ that gives the zero vector:

我得到

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \mathbf{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

$$\begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = 1 \end{cases}$$

Those vectors are (independent)(dependent) ✓.

The three vectors lie in a plane. The matrix W with those columns is *not invertible.*

不 可逆的

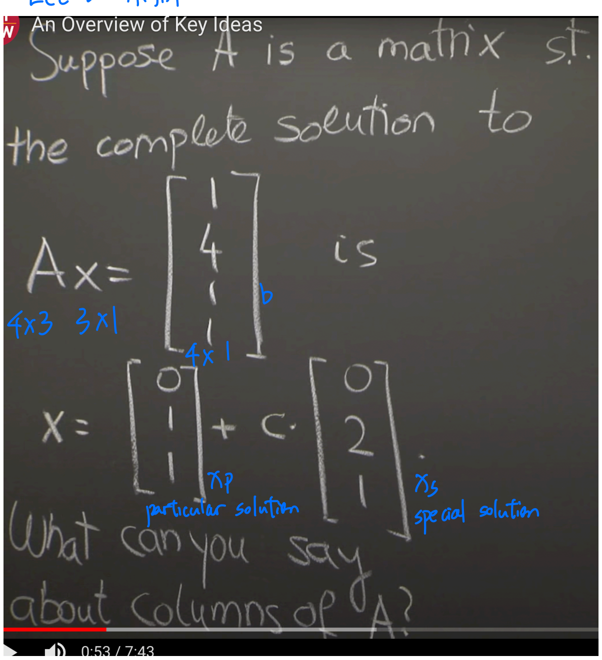
Problem 1.2: Multiply: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

Problem 1.3: True or false: A 3 by 2 matrix A times a 2 by 3 matrix B equals a 3 by 3 matrix AB. If this is false, write a similar sentence which is correct.

True

Lec 2 讲解



For any c. $A(x_p + c \cdot x_s) = b$

$c = 0 \quad A \cdot x_p = b$

$C = 1 \quad A \cdot x_p + A \cdot x_s = b$

$A \cdot x_s = 0$

设 $A = [c_1 . c_2 . c_3]$. $A \cdot x_p = b \Rightarrow c_2 + c_3 = b$

4x3 $\quad A \cdot x_s = 0 \Rightarrow 2c_2 + c_3 = 0$

$$\begin{cases} c_2 = -b \\ c_3 = 2b \end{cases}$$

7

Exercises on elimination with matrices

Problem 2.1: In the two-by-two system of linear equations below, what multiple of the first equation should be subtracted from the second equation when using the method of elimination? Convert this system of equations to matrix form, apply elimination (what are the pivots?), and use back substitution to find a solution. Try to check your work before looking up the answer.

$$\begin{bmatrix} 2 & 3 & 5 \\ 6 & 15 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{2} & 3 & 5 \\ 0 & \boxed{6} & -3 \end{bmatrix}$$

$$\begin{aligned} 2x + 3y &= 5 \\ 6x + 15y &= 12 \end{aligned}$$

$$\begin{cases} y = -\frac{1}{2} \\ x = \frac{13}{4} \end{cases}$$

Problem 2.2: (2.3 #29. *Introduction to Linear Algebra:* Strang) Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

$$E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Which matrix M (multiplying several E's) reduces Pascal all the way to I?

(1) ① I × (−1) + II, + III, + IV $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

② II × (−1) + III, II × (−2) + IV

$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$

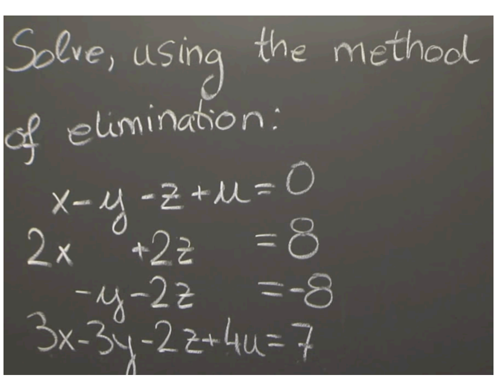
$\Rightarrow E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$

(2) ① $\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$

II × (−2) + III, II × (−3) + IV $\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

III × (−3) + IV $\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$



$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array}\right] \xrightarrow[\text{I}\times(-3)+\text{IV}]{\text{I}\times(-2)+\text{II}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & 2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right] \xrightarrow{\text{II}\cdot\frac{1}{2}+\text{III}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right]$$

$$\xrightarrow{\text{III}\Leftrightarrow\text{IV}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & -1 & -4 \end{array}\right]$$

$$\begin{cases} u = 4 \\ z = 3 \\ y = 2 \\ x = 1 \end{cases}$$

Exercises on multiplication and inverse matrices

Problem 3.1: Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 10 & 12 \\ 20 & 24 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 11 & 12 \\ 23 & 24 \end{bmatrix} = A(B+C)$$

Problem 3.2: (2.5 #24. *Introduction to Linear Algebra:* Strang) Use Gauss-Jordan elimination on $[U \;\; I]$ to find the upper triangular U^{-1} :

$$UU^{-1} = I \quad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\mathrm{II} \times (-a) + \mathrm{I}} \left[\begin{array}{ccc|ccc} 1 & 0 & b-ca & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\mathrm{III} \times (ca-b) + \mathrm{I}]{\mathrm{III} \times (-c) + \mathrm{II}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ca-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Exercises on factorization into $A = LU$

Problem 4.1: What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$\mathrm{I} \times (-2) + \mathrm{II}$
$\mathrm{I} \times (-2) + \mathrm{III}$ $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & 1 \end{bmatrix}$

$\mathrm{II} \times (-3) + \mathrm{III}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$ $L = E^{-1}$ (Gauss - Jordan) $= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

Problem 4.2: (2.6 #13. *Introduction to Linear Algebra:* Strang) Compute L and U for the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

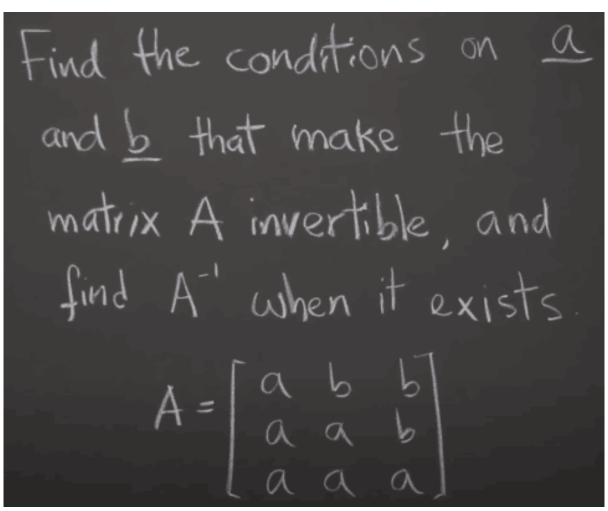
条件 → 算一步得一个条件 而非由最终diagonal得

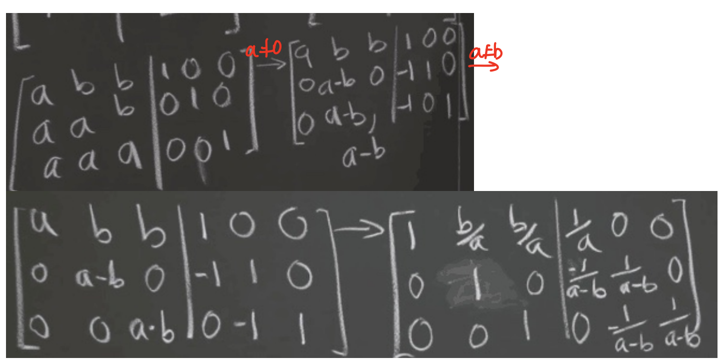
Find four conditions on a, b, c, d to get $A = LU$ with four pivots. $\neq 0 \Rightarrow \begin{cases} a \neq 0 \\ a \neq b \\ b \neq c \\ c \neq d \end{cases}$

$A \Rightarrow \overset{a\neq 0}{\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}} \xrightarrow{a \neq b} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-a \end{bmatrix} \xrightarrow{b \neq c} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix} \overset{c \neq d}{=} U$

直接把E^{-1}对应位置元素写入

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$





Find the LU-decomposition of the matrix

$$A=\begin{pmatrix}1&0&1\\a&a&a\\b&b&a\end{pmatrix}$$

when it exists.

For which real numbers a and b does it exist?

只需 $a\neq0$. 无需 $a\neq b$

Singular matrix can have LU decomposition

$$\begin{pmatrix}1&0&1\\a&a&a\\b&b&a\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&1\\0&a&0\\b&b&a\end{pmatrix}\qquad E_{21}=\begin{pmatrix}1&&\\-a&1&\\0&0&1\end{pmatrix}$$

$$\longrightarrow\begin{pmatrix}1&0&1\\0&a&0\\0&b&a-b\end{pmatrix}\qquad E_{31}=\begin{pmatrix}1&&\\0&1&\\-b&0&1\end{pmatrix}$$

$$\boxed{\text{Assume } a\neq0}\longrightarrow\begin{pmatrix}1&0&1\\0&a&0\\0&0&a-b\end{pmatrix}\qquad E_{32}=\begin{pmatrix}1&&\\0&1&\\0&-b/a&1\end{pmatrix}$$

$$\boxed{U=\begin{pmatrix}1&0&1\\0&a&0\\0&0&a-b\end{pmatrix}}.\qquad E_{32}E_{31}E_{21}A=U$$

$$\Rightarrow A=E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U.$$

$$L=\begin{pmatrix}1&&\\a&1&\\0&0&1\end{pmatrix}\begin{pmatrix}1&&\\0&1&\\b&0&1\end{pmatrix}\begin{pmatrix}1&&\\0&1&\\0&b/a&1\end{pmatrix}$$

$$\boxed{L=\begin{pmatrix}1&&\\a&1&\\b&b/a&1\end{pmatrix}}\qquad\boxed{\text{It exists when } a\neq0}$$

Exercises on transposes, permutations, spaces

Problem 5.1: (2.7 #13. *Introduction to Linear Algebra:* Strang)

a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$). $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Problem 5.2: Suppose A is a four by four matrix. How many entries (变量) of A can be chosen independently if:

a) A is symmetric? 10

b) A is *skew-symmetric*? ($A^T = -A$) 6 $\begin{bmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$

Problem 5.3: (3.1 #18.) True or false (check addition or give a counterexample):

a) The symmetric matrices in M (with $A^T = A$) form a subspace. 对线性运算封闭

b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

c) The unsymmetric matrices in M (with $A^T \neq A$) form a subspace.

(a) $A^T = A \quad B^T = B$

$(A+B)^T = A^T + B^T = A+B$

$(c \cdot A)^T = c \cdot A^T = cA$.

True

(b) $A^T = -A \quad B^T = -B$

$(A+B)^T = -A-B$

$(c \cdot A)^T = -cA$.

True

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Exercises on column space and nullspace

Problem 6.1: (3.1 #30. *Introduction to Linear Algebra:* Strang) Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$.

a) **Definition:** The sum $\mathbf{S}+\mathbf{T}$ contains all sums $\mathbf{s}+\mathbf{t}$ of a vector $\mathbf{s}$ in $\mathbf{S}$ and a vector $\mathbf{t}$ in $\mathbf{T}$. Show that $\mathbf{S}+\mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space. $(s+t)+(s'+t') = (s+s')+(t+t')$

$c(s+t) = cs+ct$

b) If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S}+\mathbf{T}$ and $\mathbf{S}\cup\mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and $\mathbf{T}$ or both. Explain this statement: *The span of* $\mathbf{S}\cup\mathbf{T}$ *is* $\mathbf{S}+\mathbf{T}$. S+T是plane SUT是2 lines

The span of SUT是plane

Problem 6.2: (3.2 #18.) The plane $x-3y-z=12$ is parallel to the plane $x-3y-x=0$. One particular point on this plane is $(12,0,0)$. All points on the plane have the form (fill in the first components)

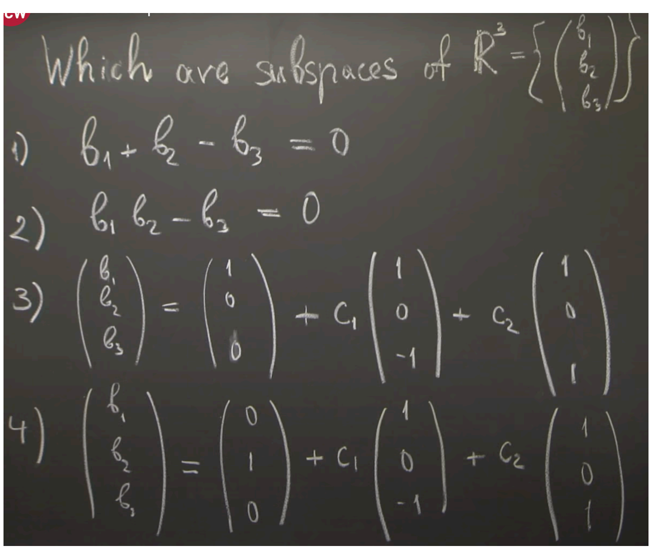
$x = 12+3y+z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 6.3: (3.2 #36.) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$? $Cx=0$ $\begin{bmatrix} Ax \\ Bx \end{bmatrix} = 0$

$Ax=0$ 且 $Bx=0$.

$N(C) = N(A) \cap N(B)$



1) $(1, 1, -1)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0$ $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ span 到 $(1,1,-1)$ 的 nullspace

是 subspace

2) $b_1 b_2 - b_3 = 0. \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ 在其中. $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ 不在. 因此不是 subspace

3) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ 在 $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ 和 $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ span 的平面上. 是

4) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ 不在 $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ 和 $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ span 的平面上. $b_2 = 1$. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ 不在 space 中. 不是