Exercises on the geometry of linear equations

Problem 1.1: (1.3 #4. *Introduction to Linear Algebra:* Strang) Find a combination $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3$ that gives the zero vector:

我得到

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{w}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \mathbf{w}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

$$\begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = 1 \end{cases}$$

Those vectors are (independent)(dependent).

The three vectors lie in a <u>plane</u>. The matrix W with those columns is *not invertible.*

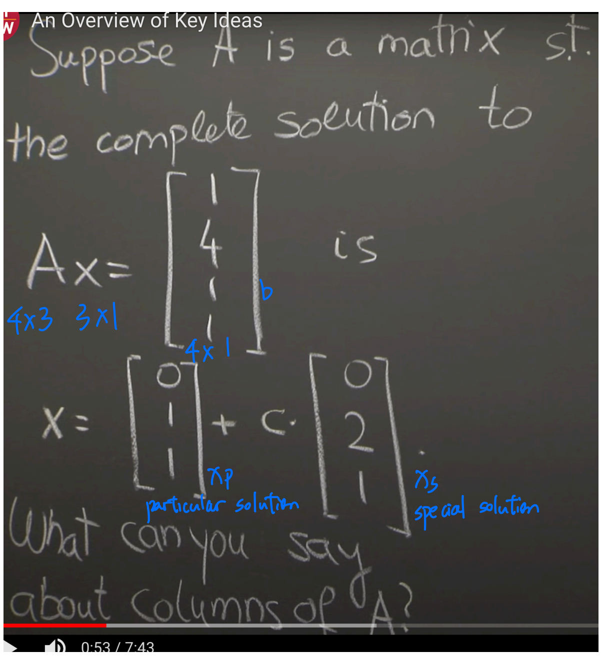
不 可逆的

Problem 1.2: Multiply: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$

$$\begin{bmatrix} -1 \\ 9 \\ 11 \end{bmatrix}$$

Problem 1.3: True or false: A 3 by 2 matrix A times a 2 by 3 matrix B equals a 3 by 3 matrix AB. If this is false, write a similar sentence which is correct.

True



For any c. $A(x_p + c\cdot x_s) = b$

$c=0 \quad A\cdot x_p = b$

$c=1 \quad A\cdot x_p + A\cdot x_s = b$

$A\cdot x_s = 0$

设 $\underset{4\times 3}{A} = [c_1, c_2, c_3]$. $A\cdot x_p = b \Rightarrow c_2 + c_3 = b$

$A\cdot x_s = 0 \Rightarrow 2c_2 + c_3 = 0$

$\begin{cases} c_2 = -b \\ c_3 = 2b \end{cases}$

$A\cdot x_s = 0$ 只有一个 special solution. $\dim N(A) = 1$

$r(A) = 3 - 1 = 2$

因此 c_1 not a multiple of b.

否则 $r \neq 2$.

Exercises on elimination with matrices

Problem 2.1: In the two-by-two system of linear equations below, what multiple of the first equation should be subtracted from the second equation when using the method of elimination? Convert this system of equations to matrix form, apply elimination (what are the pivots?), and use back substitution to find a solution. Try to check your work before looking up the answer.

$$\begin{bmatrix} 2 & 3 & 5 \\ 6 & 15 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{2} & 3 & 5 \\ 0 & \boxed{6} & -3 \end{bmatrix}$$

$$2x + 3y = 5$$
$$6x + 15y = 12$$

$$\begin{cases} y = -\frac{1}{2} \\ x = \frac{13}{4} \end{cases}$$

Problem 2.2: (2.3 #29. *Introduction to Linear Algebra:* Strang) Find the triangular matrix E that reduces "*Pascal's matrix*" to a smaller Pascal:

$$E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Which matrix M (multiplying several E's) reduces Pascal all the way to I?

(1) ① $\mathrm{I} \times (-1) + \mathrm{II}$, $+\mathrm{III}$, $+\mathrm{IV}$ $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$ ② $\mathrm{II} \times (-1) + \mathrm{III}$, $\mathrm{II} \times (-2) + \mathrm{IV}$

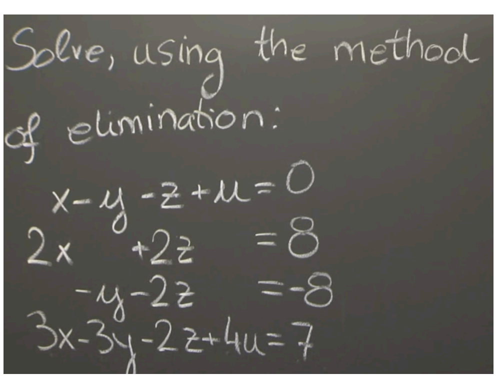
$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

(2) ① $\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 3 & 1 \end{bmatrix}$ $\mathrm{II} \times (-2) + \mathrm{III}$, $\mathrm{II} \times (-3) + \mathrm{IV}$ $\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

$\mathrm{III} \times (-3) + \mathrm{IV} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$



$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{array}\right] \xrightarrow[\mathrm{I} \times (-3) + \mathrm{IV}]{\mathrm{I} \times (-2) + \mathrm{II}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & 2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right] \xrightarrow{\mathrm{II} \cdot \frac{1}{2} + \mathrm{III}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right]$$

$$\xrightarrow{\mathrm{III} \Leftrightarrow \mathrm{IV}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & -1 & -4 \end{array}\right] \qquad \begin{cases} u = 4 \\ z = 3 \\ y = 2 \\ x = 1 \end{cases}$$

Exercises on multiplication and inverse matrices

Problem 3.1: Add AB to AC and compare with $A(B+C)$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 10 & 12 \\ 20 & 24 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 11 & 12 \\ 23 & 24 \end{bmatrix} = A(B+C)$$

Problem 3.2: (2.5 #24. *Introduction to Linear Algebra:* Strang) Use Gauss-Jordan elimination on $[U \;\; I]$ to find the upper triangular U^{-1} :

$$UU^{-1} = I \quad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & \\ x_1 & x_2 & x_3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{II} \times (-a) + \text{I}} \left[\begin{array}{ccc|ccc} 1 & 0 & b-ca & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{III} \times (ca-b) + \text{I}]{\text{III} \times (-c) + \text{II}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ca-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Exercises on factorization into $A = LU$

Problem 4.1: What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$\mathrm{I} \times (-2) + \mathrm{II}$
$\mathrm{I} \times (-2) + \mathrm{III}$ $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & 1 \end{bmatrix}$

$\mathrm{II} \times (-3) + \mathrm{III}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$ $L = E^{-1}$ (Gauss-Jordan) $= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

Problem 4.2: (2.6 #13. *Introduction to Linear Algebra:* Strang) Compute L and U for the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

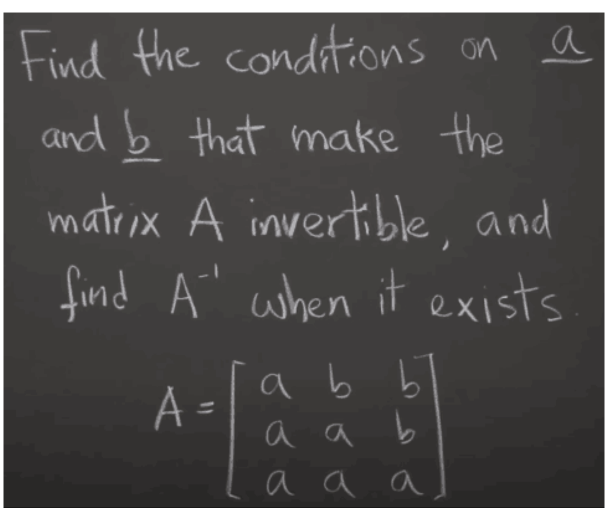
条件 → 算一步得一个条件
而非由最终diagnal得

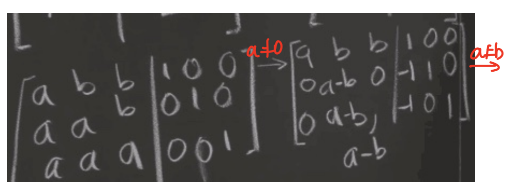
Find four conditions on a, b, c, d to get $A = LU$ with four pivots. $\neq 0 \Rightarrow \begin{cases} a \neq 0 \\ a \neq b \\ b \neq c \\ c \neq d \end{cases}$

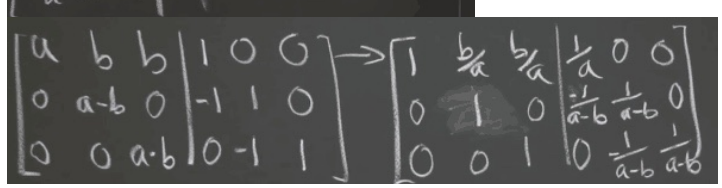
$A \Rightarrow \overset{a \neq 0}{\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}} \xrightarrow{a \neq b} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-a \end{bmatrix} \xrightarrow{b \neq c} \overset{c \neq d}{\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}} = U$

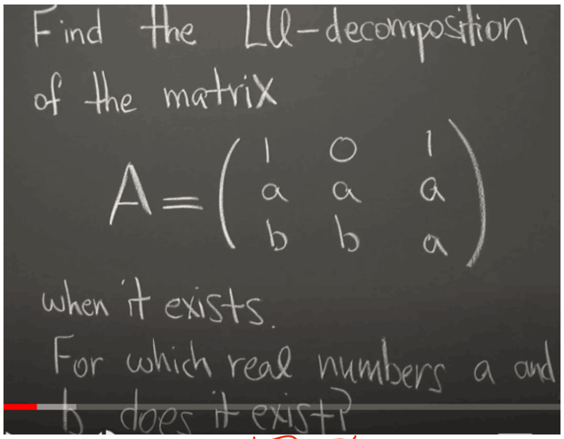
直接把E^{-1}对应位置元素写入

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$



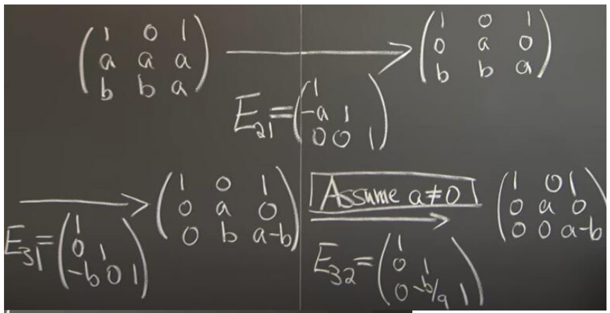


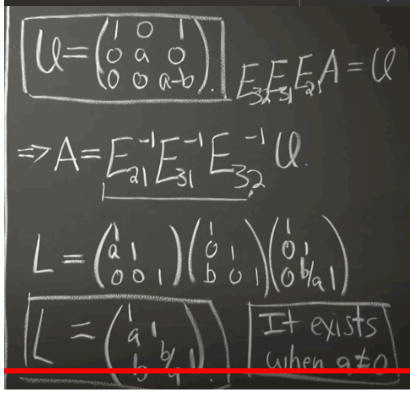




只需 $a \neq 0$。 无需 $a \neq b$

Singular matrix can have LU decomposition





Exercises on transposes, permutations, spaces

Problem 5.1: (2.7 #13. *Introduction to Linear Algebra:* Strang)

a) Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$). $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

b) Find a 4 by 4 permutation $\widehat{P}$ with $\widehat{P}^4 \neq I$. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

变量

Problem 5.2: Suppose A is a four by four matrix. How many entries of A can be chosen independently if:

a) A is symmetric? 10

b) A is *skew-symmetric*? ($A^T = -A$) 6 $\begin{array}{|c|c|c|c|} \hline 0 & -1 & & \\ \hline 1 & 0 & & \\ \hline & & 0 & \\ \hline & & & 0 \\ \hline \end{array}$

Problem 5.3: (3.1 #18.) True or false (check addition or give a counterexample):

对线性运算封闭

a) The symmetric matrices in M (with $A^T = A$) form a subspace.

b) The skew-symmetric matrices in M (with $A^T = -A$) form a subspace.

c) The unsymmetric matrices in M (with $A^T \neq A$) form a subspace.

(a) $A^T = A$ $B^T = B$

$(A+B)^T = A^T + B^T = A+B$

$(c \cdot A)^T = c \cdot A^T = cA$.

True

(b) $A^T = -A$ $B^T = -B$

$(A+B)^T = -A-B$

$(c \cdot A)^T = -cA$.

True

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Exercises on column space and nullspace

Problem 6.1: (3.1 #30. *Introduction to Linear Algebra:* Strang) Suppose **S** and **T** are two subspaces of a vector space **V**.

a) **Definition:** The sum $\mathbf{S}+\mathbf{T}$ contains all sums $\mathbf{s}+\mathbf{t}$ of a vector $\mathbf{s}$ in **S** and a vector $\mathbf{t}$ in **T**. Show that $\mathbf{S}+\mathbf{T}$ satisfies the requirements (addition and scalar multiplication) for a vector space. $(s+t)+(s'+t') = (s+s')+(t+t')$

$c(s+t) = cs+ct$

b) If **S** and **T** are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S}+\mathbf{T}$ and $\mathbf{S}\cup\mathbf{T}$? That union contains all vectors from **S** and **T** or both. Explain this statement: *The span of* $\mathbf{S}\cup\mathbf{T}$ *is* $\mathbf{S}+\mathbf{T}$. S+T 是plane SUT是2 lines

The span of SUT 是 plane

Problem 6.2: (3.2 #18.) The plane $x-3y-z=12$ is parallel to the plane $x-3y-x=0$. One particular point on this plane is $(12,0,0)$. All points on the plane have the form (fill in the first components)

$x = 12+3y+z$

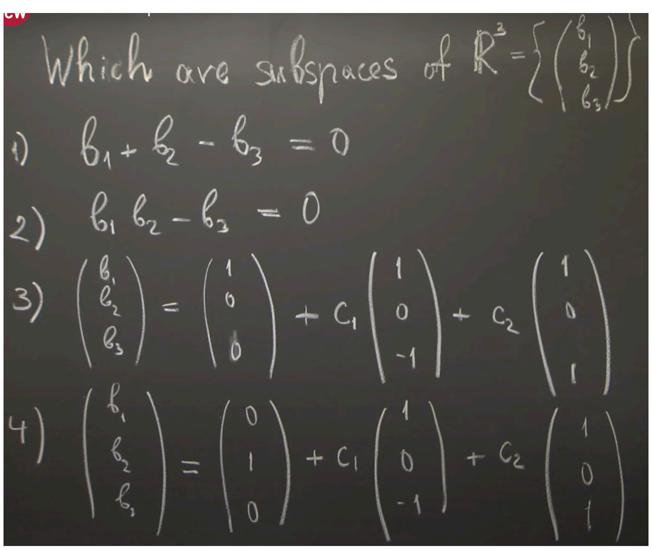
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 6.3: (3.2 #36.) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

$Cx=0$ $\begin{bmatrix} Ax \\ Bx \end{bmatrix} = 0$

$Ax=0$ 且 $Bx=0$.

$N(C) = N(A) \cap N(B)$



1) $(1,1,-1)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0$ $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ span 到 $(1,1,-1)$的 nullspace 是 subspace

2) $b_1b_2-b_3=0$. ⇒ $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$在其中. $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$不在. 因此不是subspace

3) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$在 $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$和$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ span 的平面上. 是

4) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$不在$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$和$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ span 的平面上. $b_2=1$. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$不在space中. 不是

Exercises on solving $A\mathbf{x} = \mathbf{0}$: pivot variables, special solutions

Problem 7.1:

a) Find the row reduced form of:

$$A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 4 & 1 & 7 \\ 2 & -2 & 11 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{23}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{7}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

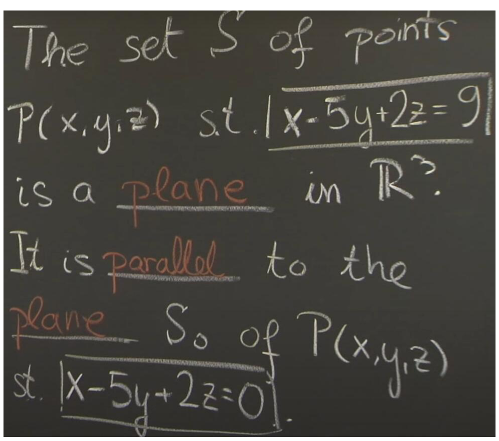
b) What is the rank of this matrix? 2

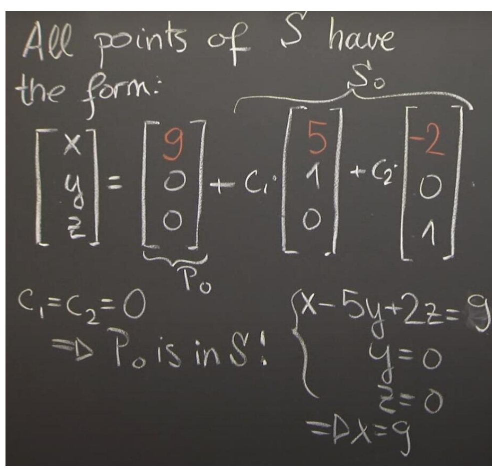
c) Find any special solutions to the equation $A\mathbf{x} = \mathbf{0}$.

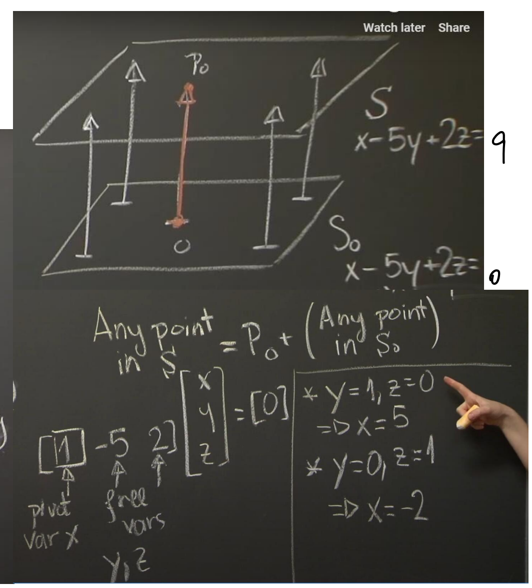
$$\begin{bmatrix} -\frac{23}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -\frac{1}{4} \\ -\frac{7}{4} \\ 0 \\ 1 \end{bmatrix}$$

Problem 7.2: (3.3 #17.b *Introduction to Linear Algebra:* Strang) Find A_1 and A_2 so that rank(A_1B) = 1 and rank(A_2B) = 0 for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$A_1 = I$ $\quad A_2 = 0$







Exercises on solving $A\mathbf{x} = \mathbf{b}$ and row reduced form R

Problem 8.1: (3.4 #13.(a,b,d) *Introduction to Linear Algebra:* Strang) Explain why these are all false:

错，加的系数为1

a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$. 仅当有解

b) The system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution. 可无解

c) If A is invertible there is no solution $\mathbf{x}_n$ in the nullspace. 有0

Problem 8.2: (3.4 #28.) Let

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{cases} \quad x = c\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

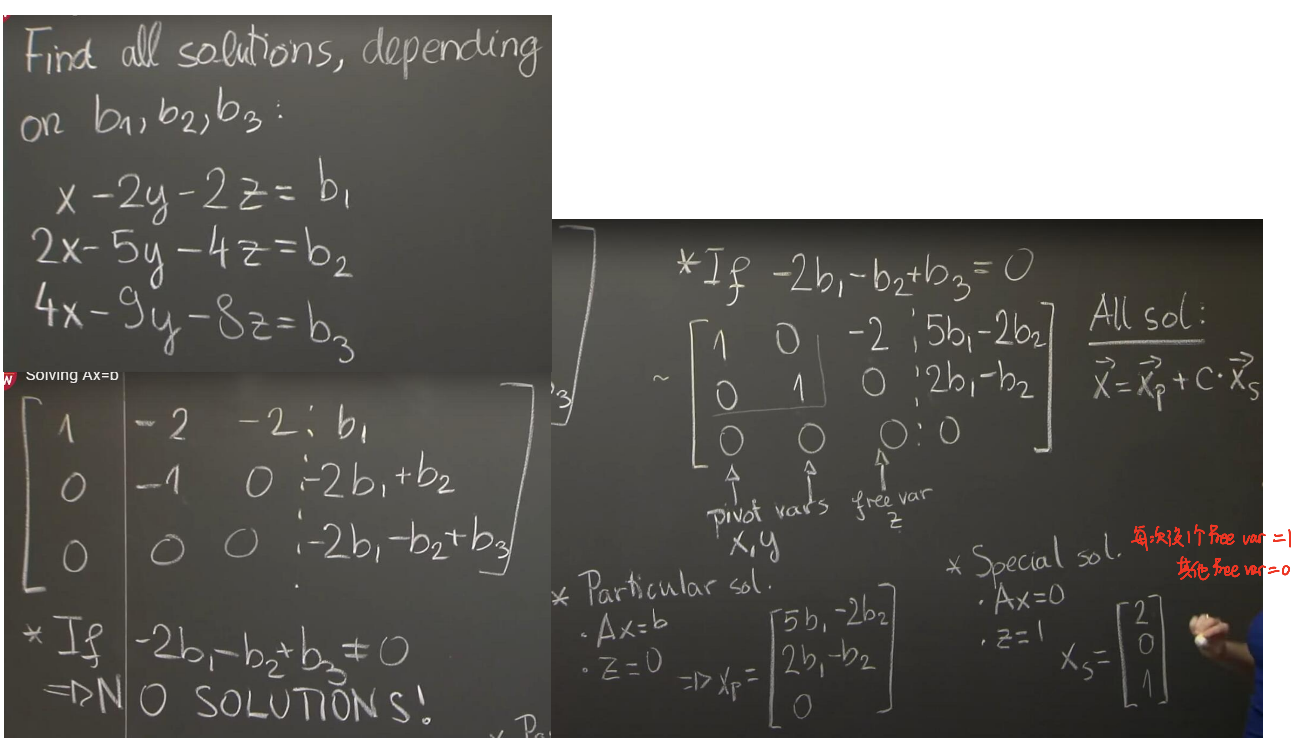
$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{cases} x_1 + 2x_2 = -1 \\ x_3 = 2 \end{cases} \quad x = c\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Use Gauss-Jordan elimination to reduce the matrices $[U \ \ 0]$ and $[U \ \ \mathbf{c}]$ to $[R \ \ 0]$ and $[R \ \ \mathbf{d}]$. Solve $R\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{d}$.

Check your work by plugging your values into the equations $U\mathbf{x} = \mathbf{0}$ and $U\mathbf{x} = \mathbf{c}$.

Problem 8.3: (3.4 #36.) Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$? Yes



Exercises on independence, basis, and dimension

Problem 9.1: (3.5 #2. *Introduction to Linear Algebra:* Strang) Find the largest possible number of independent vectors among:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

$v_4 = v_2 - v_1$
$v_5 = v_3 - v_1$
$v_6 = v_3 - v_2$

3.

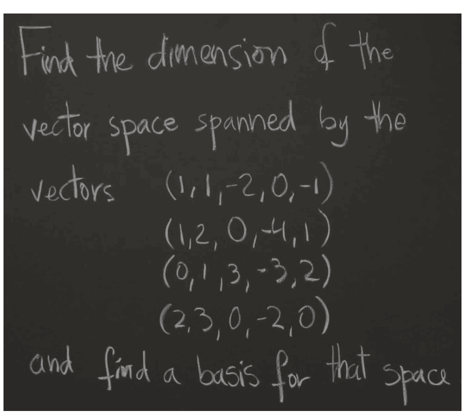
Problem 9.2: (3.5 #20.) Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. Then find a basis for the intersection (交叉) of that plane with the xy plane. Then find a basis for all vectors perpendicular (垂直) to the plane.

(1) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ 和 $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad Ax = 0.$

$N(A)$ 的 basis. 即第一问

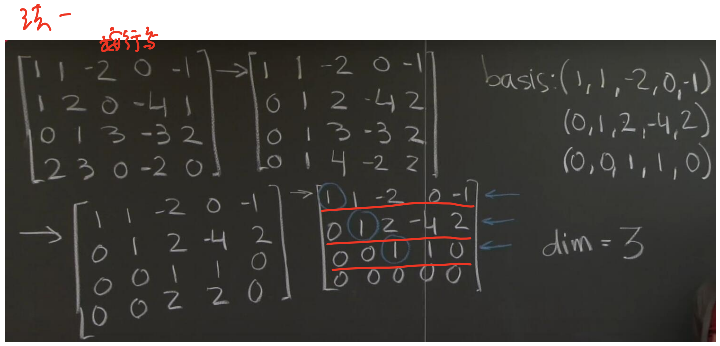
(2) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(3) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ 叉积



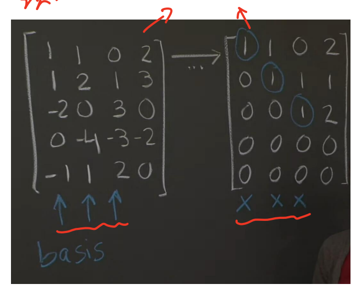
法一

按行写



法二

二者的 C(A) 不同.



Exercises on the four fundamental subspaces

Problem 10.1: (3.6 #11. *Introduction to Linear Algebra:* Strang) A is an m by n matrix of rank r. Suppose there are right sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has *no solution*.

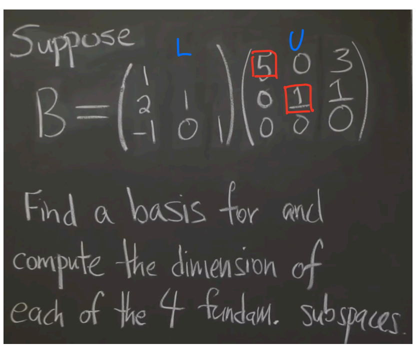
a) What are all the inequalities ($<$ or $\leq$) that must be true between m, n, and r?

$r = n < m$ / $r < m$. $r < n$.

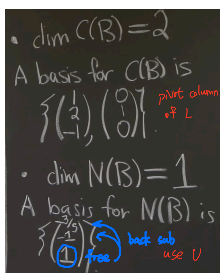
b) How do you know that $\underset{n\times m}{A^T}\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

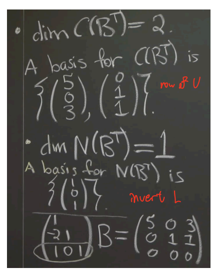
$r<m$. have free var.

Problem 10.2: (3.6 #24.) $A^T\mathbf{y} = \mathbf{d}$ is solvable when $\mathbf{d}$ is in which of the four subspaces? row space The solution $\mathbf{y}$ is unique when the left null contains only the zero vector.



2 pivots.





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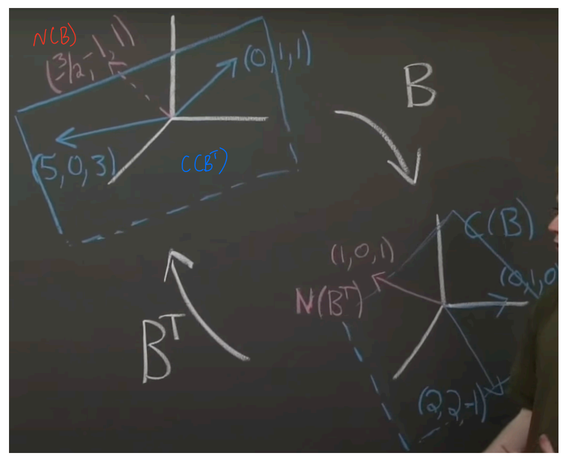


B kill N(B) to zero.

B take (any thing else including row space) into (column space)

B^T kill $N(B^T)$ to zero

B^T take (any thing else including column space) into (row space)

Solution: The other five permutation matrices are:

$$P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix},$$

$$P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \text{ and } P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

Since $P_{21} + P_{31} + P_{32}$ is the all ones matrix and $P_{32}P_{21} + P_{21}P_{32}$ is the matrix with zeros on the diagonal and ones elsewhere,

$$I = P_{21} + P_{31} + P_{32} - P_{32}P_{21} - P_{21}P_{32}.$$

For the second part, setting $c_1P_1 + \cdots + c_5P_5$ equal to zero gives:

$$\begin{bmatrix} c_3 & c_1 + c_4 & c_2 + c_5 \\ c_1 + c_5 & c_2 & c_3 + c_4 \\ c_2 + c_4 & c_3 + c_5 & c_1 \end{bmatrix} = 0.$$

So $c_1 = c_2 = c_3 = 0$ along the diagonal, and $c_4 = c_5 = 0$ from the off-diagonal entries.

Exercises on matrix spaces; rank 1; small world graphs

Problem 11.1: [Optional] (3.5 #41. *Introduction to Linear Algebra:* Strang) Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. (Assume a combination gives $c_1P_1 + \cdots + c_5P_5 = 0$ and check
元素 entries to prove c_i is zero.) The five permutation matrices are a basis for the subspace of three by three matrices with row and column sums all equal.

Problem 11.2: (3.6 #31.) **M** is the space of three by three matrices. Multiply each matrix X in **M** by:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Notice that $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

a) Which matrices X lead to $AX = 0$? $\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$

b) Which matrices have the form AX for some matrix X? $AX = B$. $B = \begin{bmatrix} a & b & c \\ d & e & f \\ -a-d & -b-e & -c-f \end{bmatrix}$

c) Part (a) finds the "nullspace" (dim = 3) of the operation AX and part (b) finds the "column space." (dim = 6) What are the dimensions of those two subspaces of **M**? Why do the dimensions add to $(n - r) + r = 9$?
dim of M

Show that the set of 2x3 matrices whose nullspace contains $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a vector subspace, and find a basis for it.
What about the set of those whose column space contains $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

(1)

$A\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $B\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(A+B)\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = A\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + B\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

$A\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, c a scalar

$(cA)\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = c\left(A\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

Each row of A
- must be $\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$ $(2a+b+c=0)$
- must be $\begin{bmatrix} a & b & -2a-b \end{bmatrix} = \begin{bmatrix} a & 0 & -2a \end{bmatrix} + \begin{bmatrix} 0 & b & -b \end{bmatrix}$
- must be a lin. comb. of $\begin{bmatrix} 1 & 0 & -2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$

basis: $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

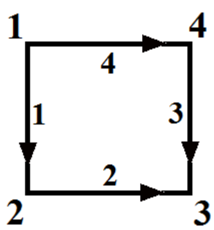
(2) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 不在其中. 不是

Exercises on graphs, networks, and incidence matrices

Problem 12.1: (8.2 #1. *Introduction to Linear Algebra:* Strang) Write down the four by four incidence matrix A for the square graph, shown below. (Hint: the first row has -1 in column 1 and +1 in column 2.) What vectors (x_1, x_2, x_3, x_4) are in the nullspace of A? How do you know that (1,0,0,0) is not in the row space of A?

(1, 0, 0, 0) 与 null space non-orthogonal

因此 不在 row space



$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} x_2 - x_1 = 0 \\ x_3 - x_2 = 0 \\ x_3 - x_4 = 0 \\ x_4 - x_1 = 0 \end{cases}$$

$$\begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix}$$

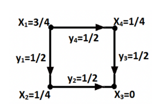
Problem 12.2: (8.2 #7.) Continuing with the network from problem one, suppose the conductance matrix is

电导

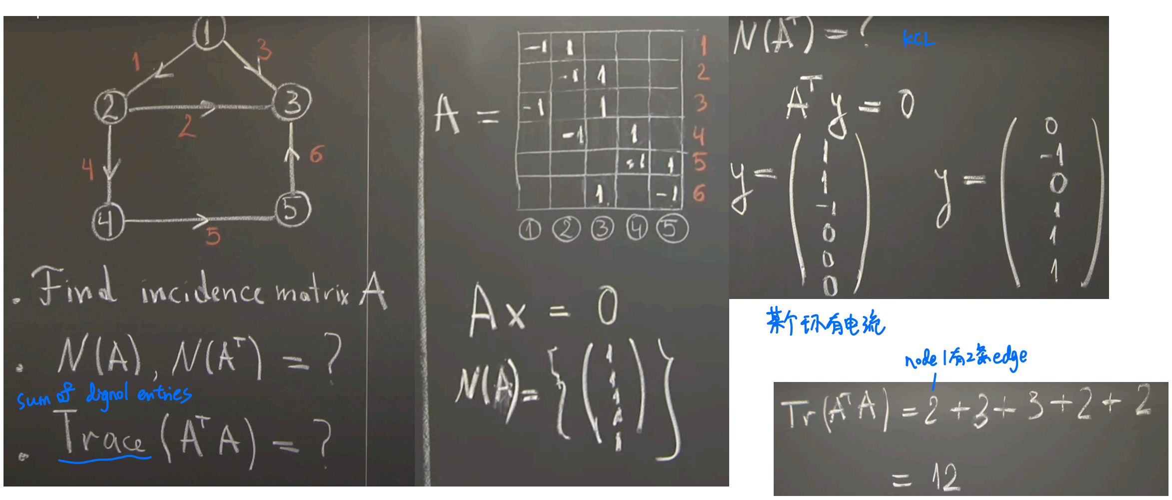
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & -1 & 1 \\ -1 & 3 & -2 & 0 & 0 \\ 0 & -2 & 4 & -2 & -1 \\ -1 & 0 & -2 & 3 & 0 \end{array}\right]$$

$\Rightarrow x$ 取 $\begin{bmatrix} 3/4 \\ 1/4 \\ 0 \\ 1/4 \end{bmatrix}$ $\quad y = -CAx = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$



Multiply matrices to find $A^T CA$. For $\mathbf{f} = (1, 0, -1, 0)$, find a solution to $A^T CA\mathbf{x} = \mathbf{f}$. Write the potentials $\mathbf{x}$ and currents $\mathbf{y} = -CA\mathbf{x}$ on the square graph (see above) for this current source $\mathbf{f}$ going into node 1 and out from node 3.



5x6 6x5

$A^T \cdot A$ 的对角线元素是

A 的 column 元素的平方之和

即某个 node 连了几条 edge

Exercises on orthogonal vectors and subspaces

Problem 16.1: (4.1 #7. *Introduction to Linear Algebra:* Strang) For every system of m equations with no solution, there are numbers $y_1, ..., y_m$ that multiply the equations so they add up to $0 = 1$. This is called *Fredholm's Alternative*:

> Exactly one of these problems has a solution:
> $A\mathbf{x} = \mathbf{b}$ OR $A^T\mathbf{y} = \mathbf{0}$ with $\mathbf{y}^T\mathbf{b} = 1$.

If $\mathbf{b}$ is not in the column space of A it is not orthogonal to the nullspace of A^T. Multiply the equations $x_1 - x_2 = 1$, $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2 and y_3 chosen so that the equations add up to $\underbrace{0}_{y^Tb} = 1$. $\frac{1}{2}$ $\frac{1}{2}$ $-\frac{1}{2}$

$y^Tb = 1$

Problem 16.2: (4.1#32.) Suppose I give you four nonzero vectors $\mathbf{r}$, $\mathbf{n}$, $\mathbf{c}$ and $\mathbf{l}$ in $\mathbb{R}^2$.

a) What are the conditions for those to be bases for the four fundamental subspaces $C(A^T)$, $N(A)$, $C(A)$, and $N(A^T)$ of a 2 by 2 matrix?

b) What is one possible matrix A?

a) In order for $\mathbf{r}$ and $\mathbf{n}$ to be bases for $N(A)$ and $C(A^T)$, we must have

$$\mathbf{r} \cdot \mathbf{n} = 0,$$

as the row space and null space must be orthogonal. Similarly, in order for $\mathbf{c}$ and $\mathbf{l}$ to form bases for $C(A)$ and $N(A^T)$ we need

$$\mathbf{c} \cdot \mathbf{l} = 0,$$

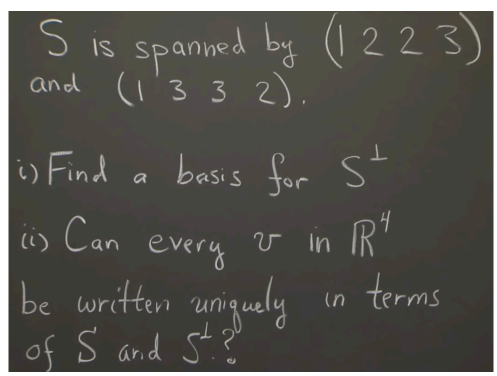
as the column space and the left nullspace are orthogonal. In addition, we need:

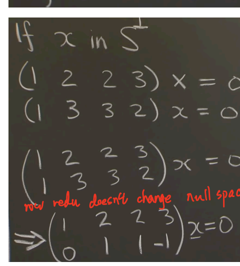
$$\dim N(A) + \dim C(A^T) = n \quad \text{and} \quad \dim N(A^T) + \dim C(A) = m;$$

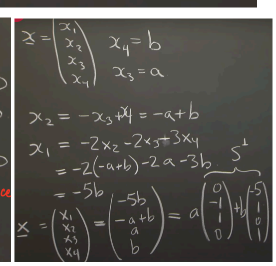
however, in this case $n = m = 1$, and as the four vectors we are given are nonzero both of these equations reduce to $1 + 1 = 2$, which is automatically satisfied.

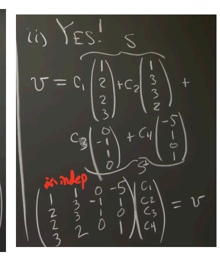
b) One possible such matrix is $A = \mathbf{c}\mathbf{r}^T$.

Note that each column of A will be a multiple of $\mathbf{c}$, so it will have the desired column space. On the other hand, each row of A will be a multiple of $\mathbf{r}$, so A will have the desired row space. The nullspaces don't need to be checked, as any matrix with the correct row and column space will have the desired nullspaces (as the nullspaces are just the orthogonal complements of the row and column spaces).









$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$Ax = b$

$A^TA\hat{x} = A^Tb$

$\hat{x} = (A^TA)^{-1}A^Tb$

$A\hat{x} = p$ $\quad p = A(A^TA)^{-1}A^Tb$

$P = A(A^TA)^{-1}A^T$

Solution: P will be four by four since we are projecting a 4-dimensional vector to another 4-dimensional vector. We will have:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This can be seen by observing that the column space of A is the wxy–space, so we just need to subtract the z coordinate from the 4-dimensional vector (w, x, y, z) we're projecting. The projection of $\mathbf{b}$ is therefore:

$$\mathbf{p} = P\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

Exercises on projections onto subspaces

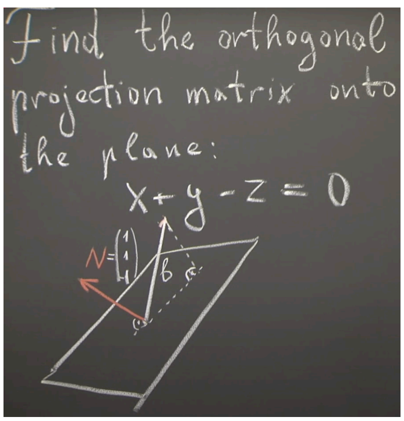
Problem 15.1: (4.2 #13. *Introduction to Linear Algebra:* Strang) Suppose A is the four by four identity matrix with its last column removed; A is four by three. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Problem 15.2: (4.2 #17.) If $P^2 = P$, show that $(I - P)^2 = I - P$. For the matrices A and P from the previous question, P projects onto the column space of A and $I - P$ projects onto the left null.

$$(I-P)^2 = I^2 - IP - PI + P^2 = I - 2P + P^2 = I - 2P + P = I - P.$$

Using the matrices A and P from the previous question,

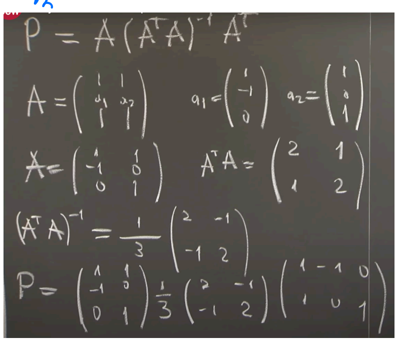
$$I - P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



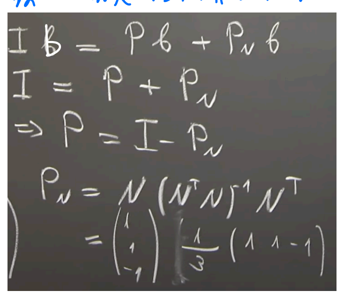
A's column: basis of the subspace to be projected onto

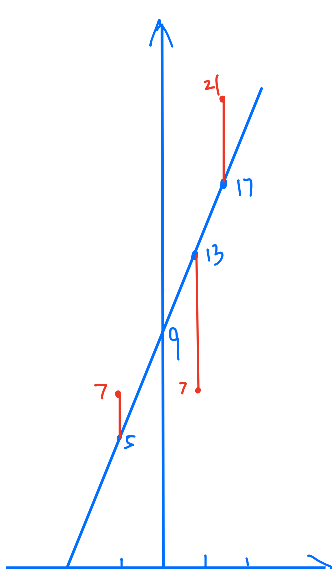
总算 ; Pn == subspace 的 $\vec{n}$

法一



法二：P_N是一维，计算方便。再 $P = I - P_N$





16.1 $\quad A \quad x \quad b$

$$\begin{bmatrix}1&-1\\1&1\\1&2\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix}=\begin{bmatrix}7\\7\\21\end{bmatrix}$$

$$A^TA\hat{x}=A^Tb$$

$$\begin{bmatrix}1&1&1\\-1&1&2\end{bmatrix}\left[\begin{array}{cc|c}1&-1&7\\1&1&7\\1&2&21\end{array}\right]\Rightarrow\left[\begin{array}{cc|c}3&2&35\\2&6&42\end{array}\right]\Rightarrow \hat{x}=\begin{bmatrix}9=C\\4=D\end{bmatrix}$$

$b=9+4t$

16.2 $p=(5,13,17)$

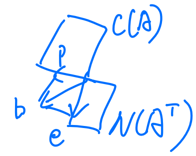
$b=(7,7,21)$

$e=(2,-6,4)$

$Pe=P(b-p)=Pb-Pp=p-p=0$

16.3 $\quad A \quad x \quad e$

$$\begin{bmatrix}1&-1\\1&1\\1&2\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix}=\begin{bmatrix}2\\-6\\4\end{bmatrix}$$



Exercises on projection matrices and least squares

Problem 16.1: (4.3 #17. *Introduction to Linear Algebra:* Strang) Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\hat{\mathbf{x}} = (C, D)$ and draw the closest line.

Problem 16.2: (4.3 #18.) Find the projection $\mathbf{p} = A\hat{\mathbf{x}}$ in the previous problem. This gives the three heights of the closest line. Show that the error vector is $\mathbf{e} = (2, -6, 4)$. Why is $P\mathbf{e} = \mathbf{0}$?

Problem 16.3: (4.3 #19.) Suppose the measurements at $t = -1, 1, 2$ are the errors 2, -6, 4 in the previous problem. Compute $\hat{\mathbf{x}}$ and the closest line to these new measurements. Explain the answer: $\mathbf{b} = (2, -6, 4)$ is perpendicular to __C(A)__ so the projection is $\mathbf{p} = \mathbf{0}$.

Problem 16.4: (4.3 #20.) Suppose the measurements at $t = -1, 1, 2$ are $\mathbf{b} = (5, 13, 17)$. Compute $\hat{\mathbf{x}}$ and the closest line and $\mathbf{e}$. The error is $\mathbf{e} = \mathbf{0}$ because this $\mathbf{b}$ is __in C(A)__.

Problem 16.5: (4.3 #21.) Which of the four subspaces contains the error vector $\mathbf{e}$? Which contains $\mathbf{p}$? Which contains $\hat{\mathbf{x}}$? What is the nullspace of A?

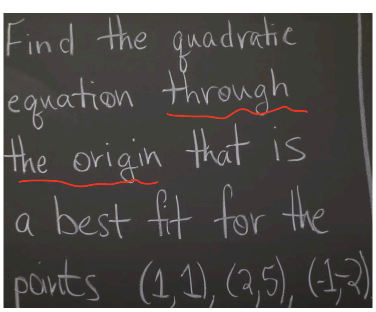
$N(A^T)$ $\quad C(A)$ $\quad C(A^T)$ $\quad \{0\}$

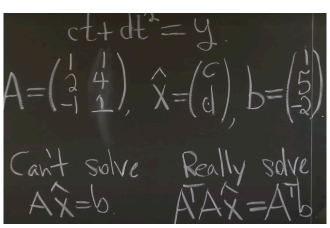
Problem 16.6: (4.3 #22.) Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

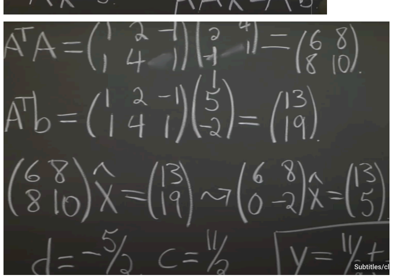
Symmetric $t \Rightarrow$ diagonal A^TA

$$\begin{bmatrix}1&-2\\1&-1\\1&0\\1&1\\1&2\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix}=\begin{bmatrix}4\\2\\-1\\0\\0\end{bmatrix} \qquad \begin{bmatrix}5&0\\0&10\end{bmatrix}\begin{bmatrix}C\\D\end{bmatrix}=\begin{bmatrix}5\\-10\end{bmatrix}$$

$C=1 \quad D=-1$







Solution: By definition, Q is a matrix whose columns are orthonormal, and so we know that $Q^TQ = I$ (where Q may be rectangular). Then:

$$Q\mathbf{x} = \mathbf{0} \Longrightarrow Q^TQ\mathbf{x} = Q^T\mathbf{0} \Longrightarrow I\mathbf{x} = \mathbf{0} \Longrightarrow \mathbf{x} = \mathbf{0}.$$

Thus the nullspace of Q is the zero vector, and so the columns of Q are linearly independent. There are no non-zero linear combinations of the columns that equal the zero vector. Thus, orthonormal vectors are automatically linearly independent.

Exercises on orthogonal matrices and Gram-Schmidt

Problem 17.1: (4.4 #10.b *Introduction to Linear Algebra:* Strang)
Orthonormal vectors are automatically linearly independent.
Matrix Proof: Show that $Q\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$. Since Q may be rectangular, you can use Q^T but not Q^{-1}.

Problem 17.2: (4.4 #18) Given the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ listed below, use the Gram-Schmidt process to find orthogonal vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ that span the same space.

$$\mathbf{a} = (1, -1, 0, 0), \mathbf{b} = (0, 1, -1, 0), \mathbf{c} = (0, 0, 1, -1).$$

Show that $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are bases for the space of vectors perpendicular to $\mathbf{d} = (1, 1, 1, 1)$.

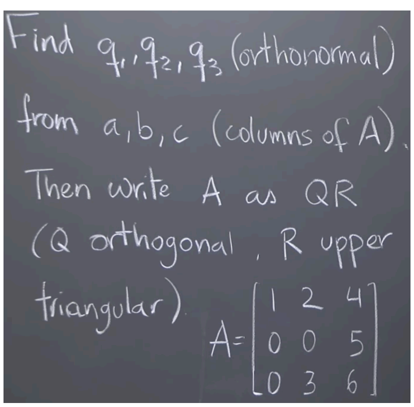
$a' = (1,-1,0,0)$

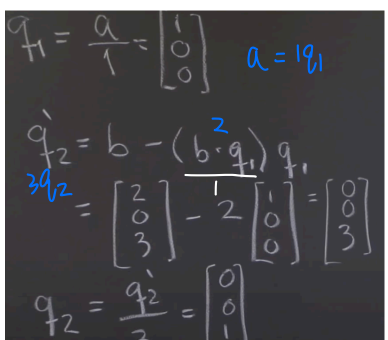
$b' = b - \frac{(a\cdot b)}{a^2}\cdot a = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\-1\\0\end{pmatrix}$

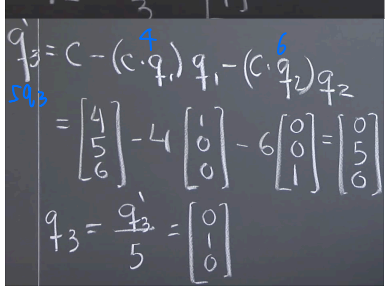
$c' = c - \frac{(a\cdot c)}{a^2}\cdot a - \frac{(b'\cdot c)}{b'^2}\cdot b' = c - \frac{-1}{\frac{3}{2}}b' = c + \frac{2}{3}b' = \begin{pmatrix}1/3\\1/3\\1/3\\-1\end{pmatrix}$

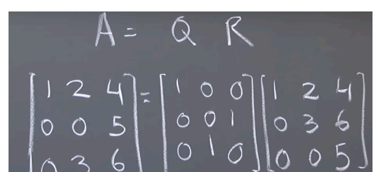
We know from the first problem that the elements of the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ are linearly independent, and each vector is orthogonal to (1,1,1,1). The space of vectors perpendicular to $\mathbf{d}$ is three dimensional (since the row space of $(1,1,1,1)$ is one-dimensional, and the number of dimensions of the row space added to the number of dimensions of the nullspace add to 4). Therefore $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ forms a basis for the space of vectors perpendicular to $\mathbf{d}$.

Similarly, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis for the space of vectors perpendicular to $\mathbf{d}$ because the vectors are linearly independent, orthogonal to (1,1,1,1), and because there are three of them.









Permutation