NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2023-2024

MH1300– Foundations of Mathematics

December 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS** a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN ON THE A4 PAPER WITHOUT ANY ATTACHMENTS (e.g. sticky notes, post-it notes, gluing or stapling of additional papers)**.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (14 marks)

(a) Prove that there do not exist positive integers a, b such that $a^2 + a + 1 = b^2$.

(b) Let c be an integer. Prove that c is divisible by 3 if and only if c^2 is divisible by 3.

(c) Are the following pair of statements logically equivalent?

$$p \to (q \vee r) \qquad \text{and} \qquad \neg q \to (\neg p \vee r).$$

Justify your answer.

QUESTION 2. (12 marks)

Determine if each of the following is true or false. Justify your answers.

(a) There are positive real numbers x, y such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

(b) For every rational number $p > 0$ there is an irrational number z such that $p > z > 0$.

(c) If A, B and C are sets then $(A - B) \cap (A - C) = A - (B \cap C)$.

QUESTION 3. (16 marks)

(a) Use mathematical induction or strong mathematical induction to prove that for every integer $n \geq 12$, there are non-negative integers c and d such that

$$n = 7c + 3d.$$

(b) Prove that for every non-negative integer n,

$$1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1) = \frac{(2n+2)!}{(n+1)! \cdot 2^{n+1}}.$$

QUESTION 4. (12 marks)

(a) Let A, B, C be sets. If $A \times C = B \times C$ and $C \neq \emptyset$, prove that $A = B$. Explain what happens if $C = \emptyset$.

(b) Let D be the set $\{0, 1\}$. Write down all the elements of $D \times \mathcal{P}(D)$. Recall that $\mathcal{P}(D)$ is the power set of D.

(c) Prove that $\sqrt{2} + \sqrt{7}$ is irrational.

QUESTION 5. (15 marks)

(a) State the definition of each of the following:

(i) A surjective function.

(ii) A one-to-one function.

(b) Suppose that S is a relation on a set B. Define $\bar{S} = \{(x, y) \in B \times B \mid (x, y) \notin S\}$.

(i) If S is symmetric, must $\bar{S}$ be symmetric?

(ii) If S is reflexive, must $\bar{S}$ be reflexive?

(iii) If S is transitive, must $\bar{S}$ be transitive?

Justify your answers.

(c) Use the Euclidean algorithm to find the greatest common divisor of the pair

$$12345 \qquad \text{and} \qquad 67890.$$

QUESTION 6. (15 marks)

(a) Find all complex numbers z satisfying the equation $z^5 + 32 = 0$.

(b) Write down three functions $f_0 : \mathbb{Z} \to \mathbb{Z}$, $f_1 : \mathbb{Z} \to \mathbb{Z}$ and $f_2 : \mathbb{Z} \to \mathbb{Z}$ such that:

(i) f_0 is one-to-one but not onto.

(ii) f_1 is onto but not one-to-one.

(iii) f_2 is neither one-to-one nor onto.

Justify your answers.

(c) Suppose that $g : A \to B$ is a function. Prove that if $C \subseteq B$ and $D \subseteq B$, then

$$g^{-1}(C \cup D) = g^{-1}(C) \cup g^{-1}(D).$$

QUESTION 7. (16 marks)

(a) Let K be the set $\{8k \mid k \in \mathbb{Z}\}$. Define a relation R on $\mathbb{Z}$ by $a\ R\ b$ if and only if $a - b \in K$, for every $a, b \in \mathbb{Z}$.

(i) Show that R is an equivalence relation on $\mathbb{Z}$.

(ii) Describe the equivalence classes of R.

(b) Let S be a relation on a non-empty set A. We say that S is *round* if for every $x, y, z \in A$, if $x\ S\ y$ and $y\ S\ z$ then $z\ S\ x$. Prove that S is an equivalence relation if and only if S is reflexive and round.

END OF PAPER