Name: ________________________________ Tutorial group: ____

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2019/20

MH1100 & SM2MH1100 – Calculus I

20 September 2019 Midterm Test 90 minutes

<u>INSTRUCTIONS</u>

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **SIX (6)** questions and comprises **SEVEN (7)** printed pages. Question 6 is optional.
4. The marks for each question are indicated at the beginning of each question.

For graders only	Question	1	2	3	4	5	6	Total
	Marks							

QUESTION 1. **(3 marks)**

Use the ϵ, δ definition of a limit to prove the following statement

$$\lim_{x\to 3}\left(\frac{1}{x}+\frac{1}{3}\right)=\frac{2}{3}.$$

[Answer:] Let ϵ be a given positive number. To prove the limit, we only need to find a number $\delta > 0$ such that

$$\text{if} \quad 0<|x-3|<\delta \quad \text{then} \quad |\left(\frac{1}{x}+\frac{1}{3}\right)-\frac{2}{3}|=|\frac{1}{x}-\frac{1}{3}|<\epsilon.$$

But

$$|\frac{1}{x}-\frac{1}{3}|=|\frac{3-x}{3x}|=\frac{1}{3}|x-3|\cdot|\frac{1}{x}|.$$

If $|x-3|<1$, then $2<x<4$ and $|\frac{1}{x}|<\frac{1}{2}$. Thus,

$$|\frac{1}{x}-\frac{1}{3}|=\frac{1}{3}|x-3|\cdot|\frac{1}{x}|<\frac{1}{6}|x-2|.$$

If $|x-2|$ is less than 6ϵ, then

$$|\frac{1}{x}-\frac{1}{3}|<\frac{1}{6}\cdot|x-2|<\epsilon.$$

This suggests that we should choose $\delta=\min\{1,6\epsilon\}$.

QUESTION 2. **(5 marks)**

Find the limits if exist.

(a) $\lim_{x\to 1} \frac{x^4+\sqrt{x}-2}{x^2+\cos x+e^x}$ (b) $\lim_{x\to 0} \frac{2-\sqrt{4-x^2}}{2x}$ (c) $\lim_{x\to 2} \frac{x^3+x^2+1}{(x-2)^2}$

(d) $\lim_{h\to 0} \left[\frac{(x+2h)^2-(x-3h)^2}{5h}\right]$ (e) $\lim_{x\to 1^+} \left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$.

[Answer:]

(a) $f(x)=\frac{x^4+\sqrt{x}-2}{x^2+\cos x+e^x}$ is an algebraic function and it is continuous on its domain. We know $x=1$ is in the domain of $f(x)$. The limit can be evaluated by directly substituting $x=1$ in the function. The numerator is 0 when $x=1$. So $\lim_{x\to 1} f(x) = f(1)=0$.

(b) We rationalize the numerator.

$$\begin{aligned}\lim_{x\to 0}\frac{2-\sqrt{4-x^2}}{2x} &= \lim_{x\to 0}\frac{2-\sqrt{4-x^2}}{2x}\cdot\frac{2+\sqrt{4-x^2}}{2+\sqrt{4-x^2}} = \lim_{x\to 0}\frac{4-4+x^2}{2x}\cdot\frac{1}{2+\sqrt{4-x^2}}\\ &= \lim_{x\to 0}\frac{x}{2}\cdot\frac{1}{2+\sqrt{4-x^2}} = \lim_{x\to 0}\frac{x}{2}\cdot\lim_{x\to 0}\frac{1}{2+\sqrt{4-x^2}} = 0\times\frac{1}{4}=0.\end{aligned}$$

(c) As x approaches 0, the denominator approaches 0 but the numerator approaches 13. Thus, the limit does not exist. Or the function has an infinite limit at $x=2$. One can use the definition of an infinite limit to prove that the limit does not exist.

(d)

$$\begin{aligned}\lim_{h\to 0}\left[\frac{(x+2h)^2-(x-3h)^2}{5h}\right] &= \lim_{h\to 0}\left[\frac{(x+2h+x-3h)(x+2h-x+3h)}{5h}\right]\\ &= \lim_{h\to 0}\left[\frac{(2x-3h)(5h)}{5h}\right] = \lim_{h\to 0}(2x-3h)=2x.\end{aligned}$$

(e)

$$\begin{aligned}\lim_{x\to 1^+}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right) &= \lim_{x\to 1^+}\left(\frac{1+x+x^2}{1-x^3}-\frac{3}{1-x^3}\right) = \lim_{x\to 1^+}\frac{1+x+x^2-3}{1-x^3}\\ &= \lim_{x\to 1^+}\frac{-2+x+x^2}{1-x^3} = \lim_{x\to 1^+}\frac{(x-1)(x+2)}{(1-x)(1+x+x^2)}\\ &= \lim_{x\to 1^+}\frac{-(x+2)}{1+x+x^2} = -1.\end{aligned}$$

QUESTION 3. **(4 marks)**

Show that there is at least one root of the equation

$$\sin x = x + \frac{1}{2}$$

between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

[Answer:] Consider the function $f(x) = \sin x - x - \frac{1}{2}$. We apply the I.V.T. to this function on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $N = 0$. The first thing we have to check is that $f(x)$ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This is true because

(i) $f(x)$ is continuous on the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(ii) $f\left(-\frac{\pi}{2}\right) = -1 + \frac{\pi}{2} - \frac{1}{2} > 0$.

(iii) $f\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{2} - \frac{1}{2} < 0$.

So because $f(-\frac{\pi}{2}) > 0 > f(\frac{\pi}{2})$ we deduce from the I.V.T. that there exists a c where $f(c) = 0$. This c will solve the given equation.

QUESTION 4. **(4 marks)**

Find the value of a that makes the following function continuous for all x-values.

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0; \\ a + x, & x \leq 0. \end{cases}$$

[Answer:] That $f(x)$ is differentiable for all x-values indicates that $f(x)$ is continuous

at every number including $x = 0$. Thus,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0).$$

We have $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (a + x) = a$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x \sin \frac{1}{x}$. We know $-|x| \leq x \sin \frac{1}{x} \leq |x|$ and $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} (-|x|) = 0$. So, $\lim_{x \to 0^+} x \sin \frac{1}{x} = 0$. This gives that $a = 0$. If $x_0 < 0$, $\lim_{x \to x_0} f(x) = a + x_0 = f(x_0)$. If $x > 0$, $\lim_{x \to x_0} f(x) = x_0 \sin \frac{1}{x_0} = f(x_0)$. Thus, $a = 0$ can make the function continuous everywhere on its domain.

QUESTION 5. **(4 marks)**

Consider the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

(a) Show that $f(x)$ is continuous in its domain.

(b) Find the derivative of $f(x)$ at $x = 0$ if exists.

[Answer]

(a) When $x \neq 0$, $f(x)$ is obviously continuous. At $x = 0$, using the squeeze theorem by selecting $g(x) = -x^2$ and $h(x) = x^2$ with $g(x) \leq f(x) \leq h(x)$, we can find that

$$0 = \lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = \lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

As $f(0) = 0$, the function $f(x)$ is continuous at $x = 0$. So, $f(x)$ is continuous in its domain $\mathbb{R}$.

(b) At the point $x = 0$

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} \\ &= \lim_{h\to 0} \left[h \sin\left(\frac{1}{h}\right)\right] \\ &= 0. \end{aligned}$$

So $f'(0) = 0$.

QUESTION 6 (Optional). **(1 bonus mark)**

Suppose $f(x)$ and $g(x)$ are continuous functions on the interval I. Let

$$F(x) = \max\{f(x), g(x)\} \quad \text{and} \quad G(x) = \min\{f(x), g(x)\}.$$

Show that both $F(x)$ and $G(x)$ are continuous on I.

[Answer:] We can rewrite $F(x)$ and $G(x)$ as

$$F(x) = \frac{1}{2}\left(f(x) + g(x) + |f(x) - g(x)|\right)$$

and

$$G(x) = \frac{1}{2}\left(f(x) + g(x) - |f(x) - g(x)|\right).$$

Given that $f(x)$ and $g(x)$ are continuous functions, $f(x) - g(x)$ is continuous. We know that the absolute value function $h(x) = |x|$ is continuous. Since a continuous function of a continuous function is continuous, $|f(x) - g(x)|$ is continuous. Both $F(x)$ and $G(x)$ can be obtained from the continuous functions $f(x)$, $g(x)$ and $|f(x) - g(x)|$. Thus, *both* $F(x)$ and $G(x)$ are continuous.