SPMS / Division of Mathematical Sciences

MH1300 Foundations of Mathematics
2015/2016 Semester 1

MID-TERM EXAM SOLUTIONS

QUESTION 1(a). **(12 marks)**

Use a truth table to determine whether the pair of statement forms are logically equivalent. Include a few words of explanation.

$$(P \to Q) \to R, \text{ and } (P \land \neg Q) \lor R.$$

SOLUTION . Draw the truth table. There are a total of eight rows as there are three variables.

P	Q	R	$P \to Q$	$P \land \neg Q$	$(P \to Q) \to R$	$(P \land \neg Q) \lor R$
1	1	1	1	0	1	1
1	1	0	1	0	0	0
1	0	1	0	1	1	1
1	0	0	0	1	1	1
0	1	1	1	0	1	1
0	1	0	1	0	0	0
0	0	1	1	0	1	1
0	0	0	1	0	0	0

The last two columns are identical, and so we conclude that $(P \to Q) \to R,$ and $(P \land \neg Q) \lor R$.

1 mark for each row. 2 marks for stating the conclusion. 2 marks for putting it in a table and setting the table correctly. □

QUESTION 1(b). **(10 marks)**

Without using truth tables, show that the following logical equivalence holds

$$(P \land Q) \to R \equiv (P \land \neg R) \to \neg Q.$$

Supply a reason for each step.

SOLUTION .

$$
\begin{aligned}
(P \wedge Q) \to R &\equiv \neg(P \wedge Q) \vee R \quad \text{(Logical Equivalence)} \\
&\equiv (\neg P \vee \neg Q) \vee R \quad \text{(De Morgan's Law)} \\
&\equiv \neg P \vee (\neg Q \vee R) \quad \text{(Associative Law)} \\
&\equiv \neg P \vee (R \vee \neg Q) \quad \text{(Commutative Law)} \\
&\equiv (\neg P \vee R) \vee \neg Q \quad \text{(Associative Law)} \\
&\equiv (\neg P \vee \neg\neg R) \vee \neg Q \quad \text{(Double Negation Law)} \\
&\equiv \neg(P \wedge \neg R) \vee \neg Q \quad \text{(De Morgan's Law)} \\
&\equiv (P \wedge \neg R) \to \neg Q \quad \text{(Logical Equivalence)}
\end{aligned}
$$

2 marks each for Logical equivalence, De Morgan's Law, Double Negation Law □

QUESTION 2(a) **(8 marks)**

Let $T = \{3, 17\}, V = \{2, 3, 7, 26\}$. Which of the following quantified statements are true?

You DO NOT need to justify your answer. Circle the correct option.

(i)	T	F	$\exists x \in T,\ \ x$ is odd $\to x > 8$
(ii)	T	F	$\exists x \in V,\ \ x$ is odd $\to x > 8$
(iii)	T	F	$\forall x \in T,\ \ x$ is odd $\to x > 8$
(iv)	T	F	$\forall x \in V,\ \ x$ is even $\to x > 8$
(v)	T	F	$\forall x \in T,\ \exists y \in V,\ \ x$ is odd $\to y > 8$
(vi)	T	F	$\exists x \in T,\ \forall y \in V,\ \ x$ is odd $\to y > 8$
(vii)	T	F	$\forall x \in V,\ \exists y \in T,\ \ x$ is odd $\leftrightarrow xy < 24$

SOLUTION . (i) True. Take $x = 17 \in T$. Then "x is odd" is true and "$x > 8$" is also true. So the statement "x is odd $\to x > 8$" is true.

(ii) True. Take $x = 26 \in V$. Then "x is odd" is false. So the statement "x is odd $\to x > 8$" is true.

(iii) False. Take $x = 3 \in T$ to be a counter-example. Then "x is odd" is true and "$x > 8$" is false. So the statement "x is odd $\to x > 8$" is false when $x = 3$.

(iv) False. Take $x = 2 \in V$ to be a counter-example. Then "x is even" is true and "$x > 8$" is false. So the statement "x is even $\to x > 8$" is false.

(v) True. For $x = 3 \in T$ we can take $y = 26 \in V$. Then " x is odd $\to y > 8$ is true. For $x = 17 \in T$ we can also take $y = 26 \in V$. Then " x is odd $\to y > 8$" is true.

(vi) False. We check that no $x \in T$ can work for all $y \in V$. T only contains two elements, 3 and 17. If $x = 3 \in T$ then we can see that for $y = 2 \in V$, the statement " x is odd $\to y > 8$" is false. So $x = 3$ does not work for all $y \in V$. What about $x = 17$? Then we can also take $y = 2 \in V$, and so $x = 17$ also does not work for all y. Hence the statement " $\exists x \in T,\ \forall y \in V,\ \ x$ is odd $\to y > 8$" is false.

(vii) True. When $x = 2 \in V$ we take $y = 17 \in T$, then both sides of the biconditional statement are false. When $x = 3 \in V$ we take $y = 3 \in T$, then both sides of the biconditional statement are true. When $x = 7 \in V$ we take $y = 3 \in T$. Finally when $x = 26 \in V$ we can take $y = 3$ or $y = 17$ in T.

1 mark each part, 2 marks for last part. □

QUESTION 2(b). **(9 marks)**

What can be said about the truth value of Q in each of the following cases:

(i) P is true and $P \to Q$ is false,

(ii) P is false and $P \leftrightarrow Q$ is true,

(iii) P is true and $P \leftrightarrow Q$ is false,

(iv) P is true and $P \to (Q \to \neg P)$ is true,

(v) $(P \leftrightarrow Q) \to P$ is false,

(vi) R is false and $(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow R)$ is true.

Your answer should be either "Q is true", or "Q is false" in each case. Justify your answer in each case.

SOLUTION . (i) Q is false. If $P \to Q$ is false, then there is only one possibility, that is, when P is true and Q is false. *(1mark)*

(ii) Q is false. If $P \leftrightarrow Q$ is true, then P and Q have the same truth values. *(1mark)*

(iii) Q is false. If $P \leftrightarrow Q$ is false, then P and Q have the different truth values. *(1mark)*

(iv) Q is false. If P is true and $P \to (Q \to \neg P)$ is true, then by modus ponens, $Q \to \neg P$ is true. But since P is true, by modus tollens, Q is false. *(1mark)*

(v) Q is false. If $(P \leftrightarrow Q) \to P$ is false, then the only possibility is for $P \leftrightarrow Q$ to be true and P to be false. Since $P \leftrightarrow Q$ is true, that means P and Q have the same truth values. So Q is false. *(2marks)*

(vi) Q is true. How do we derive this? Note that we know that R is false, but we do not know the truth value of P. Let's first suppose that P is true. Then $\neg P \leftrightarrow R$ is true. Since $P \leftrightarrow Q$ and $\neg P \leftrightarrow R$ have the same truth values, we know that $P \leftrightarrow Q$ is true. Since P is true by assumption, we conclude that Q is true. So assuming the P is true leads us to the conclusion that Q is true.

Now what happens if P is false? Then $\neg P \leftrightarrow R$ is false, so we know that $P \leftrightarrow Q$ is also false. By assumption that P is false, we conclude that Q is true. So assuming that P is false leads us to the same conclusion that Q is true.

Hence Q must be true (regardless of the truth value of P). *(3marks)*

□

QUESTION 3(a). **(6 marks)**

Write down the converse, contrapositive and the negation of the following statement:

For all integers n, $n + 1 > 2$ is necessary for $n^2 > 5$.

SOLUTION . **Converse** $\forall n \in \mathbb{Z}, \quad n + 1 > 2 \rightarrow n^2 > 5$. Or, you can also write in informal form "For all integers n, $n + 1 > 2$ is sufficient for $n^2 > 5$".

Contrapositive $\forall n \in \mathbb{Z}, \quad n + 1 \leq 2 \rightarrow n^2 \leq 5$. Or, you can also write in informal form "For all integers n, $n + 1 \leq 2$ is sufficient for $n^2 \leq 5$".

Negation $\exists n \in \mathbb{Z}, \quad n^2 > 5 \ \wedge \ n + 1 \leq 2$. Or, you can also write in informal form "There is an integer n such that $n^2 > 5$ and $n + 1 \leq 2$".

2marks each part

□

QUESTION 3(b). **(5 marks)**

Give an example of a nonempty domain D, and predicates $P(x)$ and $Q(x)$ such that

$$\exists x \in D, \quad P(x) \vee Q(x) \ \text{ is true, and}$$

$$(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x)) \ \text{ is false.}$$

SOLUTION . This is exactly the same as Question 4 of last year's midterm. Take $D = \mathbb{Z}$ and $P(x) : x > 0$ and $Q(x) : x \leq 0$. Alternatively you can take $P(x)$ and $Q(x)$ to be both $x > 0$. Then the first statement is true by taking $x = 1$ (because $P(1)$ and $Q(1)$ are both true). The second statement is false, because $P(0)$ and $Q(0)$ are both false. So $\forall x \in D, P(x)$ and $\forall x \in D, Q(x)$ are both false.

Of course, there are many other choices you can take for D, $P(x)$ and $Q(x)$.

1mark for stating $D, P(x), Q(x)$ correctly. 2marks for justifying why the first statement is true, and 2marks for justifying why the second statement is false. $\square$