MH1300 Foundations of Mathematics

AY 2017 / 2018 Final Exams Solutions

$\boxed{\text{Q1ai}}$ disprove.

For any A, B, $A - B = B - A$.

False, take $A = \{0\}$ and $B = \phi$.

Then $A - B = \{0\} - \phi = \{0\}$

But $B - A = \phi - \{0\} = \phi$

$\boxed{\text{Q1aii}}$ For any A, B, C, if $A \cup B = A \cup C$ then B and C are disjoint.

False, take $A = B = C = \{0\}$

can take any non-empty set instead

Then $A \cup B = A \cup C = A$

and $B \cap C = \{0\} \neq \phi$, so B, C not disjoint.

$\boxed{Q1b}$ For any A, B, C,

$$[A \cap (B \cup C)] \cup [B \cap (A \cup C)] \subseteq (A \cup B) \cap (A \cup C)$$

This is true. To prove it, use either the element method, or use identities.

Element method: Let $x \in LHS$. Then, $x \in A \cap (B \cup C)$ or $x \in B \cap (A \cup C)$.

Case 1: $x \in A \cap (B \cup C)$. Then, $x \in A$.

So, $x \in A \cup B$ and $x \in A \cup C$.

Therefore, $x \in (A \cup B) \cap (A \cup C) = RHS$.

Case 2: $x \in B \cap (A \cup C)$. So, $x \in B$ and $x \in A \cup C$.

This means $x \in A \cup B$ and $x \in A \cup C$.

So, $x \in (A \cup B) \cap (A \cup C) = RHS$.

Hence, $LHS \subseteq RHS$.

∵ We may also use identities:

$$LHS = [A \cap (B \cup C)] \cup [B \cap (A \cup C)]$$

$$= [(A \cap B) \cup (A \cap C)] \cup [B \cap (A \cup C)]$$

(Distributive law)

$$\subseteq [(A \cap A) \cup (A \cap C)] \cup [B \cap (A \cup C)]$$

(Because $A \cap B \subseteq A \cap A$)

$$= [A \cap (A \cup C)] \cup [B \cap (A \cup C)]$$

(Distributive law)

$$= (A \cup B) \cap (A \cup C)$$

(Distributive law)

$$= RHS.$$

$\boxed{\text{Q 2a}}$ $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad |xy| < 1 \rightarrow x+y > 2$

$\equiv \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad |xy| \geqslant 1$ or $x+y > 2$.

This is true. Fix $x \in \mathbb{R}$. take $y = 3 - x \in \mathbb{R}$.

Then $y = 3 - x > 2 - x$

So, $x + y > 2$.

So, $|xy| \geqslant 1$ or $x+y > 2$ is true.

$\boxed{\text{Q2b}}$ $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \quad x^2 < y^2 \rightarrow x < y$.

This is false. We need counterexamples for x, y.

Take $\underbrace{x = 0}_{\text{in } \mathbb{Z}}$, $\underbrace{y = -1}_{\text{in } \mathbb{Z}}$. Then $x^2 = 0 \quad y^2 = 1$

and so $x^2 < y^2$ true.

and $y < x$.

$\boxed{\text{Q 2c}}$ $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, y^2 - x < 100.$

This is false. We need to show the negation, which is

$\exists x \in \mathbb{Z}, \quad \forall y \in \mathbb{Z}, \quad y^2 - 100 \geq x$

Obviously, we take $x = -100$. Then, we need to show that our choice of x works, i.e. we need to show $\forall y \in \mathbb{Z}, \quad y^2 - 100 \geq \underbrace{-100}_{\substack{x \\ \text{we choose}}}$

Fix $y \in \mathbb{Z}$. Then $y^2 \geq 0$. So, $y^2 - 100 \geq -100$

So, $y^2 - 100 \geq x$ is true.

$\boxed{Q3}$ Prove that for every integer $n > 0$,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$$

Let $P(n)$ be the above.

Base case $P(1)$ is $\underbrace{1 \cdot 2}_{=2} = \underbrace{\frac{1}{3} \cdot 1 \cdot 2 \cdot 3}_{=2}$

So $P(1)$ is true.

Assume $P(n)$ holds, i.e. assume that

$$1 \cdot 2 + \cdots + n(n+1) = \frac{1}{3} n(n+1)(n+2).$$

We need to show $P(n+1)$ holds. Since $P(n+1)$ is an equality, we begin with

$$\begin{aligned}
LHS &= 1 \cdot 2 + \cdots + n(n+1) + (n+1)(n+2) \\
&\quad \text{|| apply IH} \\
&= \frac{1}{3} n(n+1)(n+2) + (n+1)(n+2) \\
&= (n+1)(n+2)\left[\frac{1}{3} n + 1\right] \\
&= (n+1)(n+2)\left[\frac{1}{3}(n+3)\right] \\
&= \frac{1}{3}(n+1)(n+2)(n+3)
\end{aligned}$$

So $P(n+1)$ is true. By MI, $P(n)$ is true for all $n > 0$.

Q4a Let n be an integer. Suppose that $3 \mid 2n$.

WTS : $3 \mid n$

By hypothesis, there exists $k \in \mathbb{Z}$ such that

$3k = 2n$.

Since $2n$ is even, thus, $3k$ is even.

Therefore, k is even, (otherwise if k is odd then $3k$ = odd × odd = odd, which is impossible since $3k$ is even)

Since k is even, let $k = 2l$ for some integer l.

So, $2n = 3k = 3(2l)$

So, $n = 3l$. Hence, $3 \mid n$

Alternative proof : Suppose $3k = 2n$. By the unique factorization theorem, since 3 is prime, 3 must appear in a factor of $2n$. Since 2 is also prime, and $2 \neq 3$, 3 must appear as one of the prime factors of n. Hence, $3 \mid n$.

$\boxed{Q4b}$ Let n and m be integers.

Suppose that n is even and m is odd.

Let K, ℓ be integers such that

$$n = 2K \quad \text{and} \quad m = 2\ell + 1.$$

Now we evaluate the expression

$$\begin{aligned} n^2 + 2m^2 &= (2K)^2 + 2(2\ell+1)^2 \\ &= 4K^2 + 2(4\ell^2 + 4\ell + 1) \\ &= 4(K^2 + 2\ell^2 + 2\ell) + 2 \end{aligned}$$

By the quotient remainder theorem, $n^2 + 2m^2 \bmod 4 = 2$

Hence, $n^2 + 2m^2$ is not divisible by 4.
(Because otherwise $n^2 + 2m^2 \bmod 4 = 0$)

<u>Alternative method</u>: We proceed by contradiction.

Suppose not. Let n be even, m is odd, and $4 \mid (n^2 + 2m^2)$. Let K, ℓ, j be integers such that $4j = n^2 + 2m^2$, $n = 2K$, $m = 2\ell + 1$.

Then, $4j = n^2 + 2m^2$

$= 4(k^2 + 2l^2 + 2l) + 2.$

So, $4(j - k^2 - 2l^2 - 2l) = 2$

$2(j - k^2 - 2l^2 - 2l) = 1.$

So 1 is odd, a contradiction.

Hence the statement

n even & m odd $\rightarrow$ $4 \nmid (n^2 + 2m^2)$ is true.

$\boxed{\text{Q4c}}$ Suppose not. Then there exist real numbers a, b, c, d, e such that all five numbers are smaller than their average.

let $m = \frac{1}{5}(a+b+c+d+e)$.

Since $a < m$, $b < m$, $\cdots$, $e < m$,

So $a+b+c+d+e < m+m+m+m+m$

$= 5m$

$= 5 \cdot \frac{1}{5}(a+b+c+d+e)$

$= a+b+c+d+e.$

This is a contradiction because no real number can be smaller than itself.

$\boxed{\text{Q5a}}$ Let $f: A \to A$. Suppose that $f \circ f$ is injective.

Let $f(x) = f(y)$. Since f is a function,

hence $f(f(x)) = f(f(y))$

So $(f \circ f)(x) = (f \circ f)(y)$

Since $f \circ f$ is injective, so $x = y$.

Hence we have shown $f(x) = f(y) \to x = y$.

So f is injective.

Alternative Proof

Suppose not. Then let $f \circ f$ be injective but f is not injective.

Since f is not injective, there exist $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$. Since f is a function,

$f(f(x)) = f(f(y))$.

So $(f \circ f)(x) = (f \circ f)(y)$. But this means that, as $x \neq y$, $f \circ f$ is not injective.

Contradiction.

$\boxed{Q5b}$ Let $f(n+m) = f(n) + f(m)$, and $a = f(1)$.

A formula for $f(1)$ can be deduced by writing down the values of $f(2), f(3) \ldots$

$$f(2) = f(1+1) = f(1) + f(1) = 2f(1)$$
$$f(3) = f(2+1) = f(2) + f(1) = 2f(1) + f(1)$$
$$= 3f(1)$$
$$f(4) = f(3+1) = f(3) + f(1) = 4f(1)$$

So we guess a formula is

$$f(n) = n.f(1).$$

To prove it, we apply induction.

<u>$n=0$</u>: $f(0) = f(0+0) = f(0) + f(0). = 2f(0)$

Hence $f(0) = 0. = 0.f(1)$

assume $f(n) = n.f(1)$.

$$f(n+1) = f(n) + f(1) = n.f(1) + f(1)$$
$$= (n+1).f(1).$$

So by MI, $f(n) = n.f(1)$ for all $n \in \mathbb{N}$.

$\boxed{5c}$ $h(x) = \dfrac{3x-1}{x}$, $x \neq 0$.

If $x \neq 0$, $h(x) = 3 - \frac{1}{x}$.

To show h is 1-1, Let $h(x) = h(y)$,

where $x, y \neq 0$.

Then $3 - \frac{1}{x} = 3 - \frac{1}{y}$.

So, $\frac{1}{x} = \frac{1}{y}$.

and $x = y$.

Hence, h is injective.

$\boxed{Q6a}$ Find all fourth roots of $4-4i$

First express $4-4i$ in the form $re^{i\theta}$

$$r = \sqrt{4^2+(-4)^2} = \sqrt{32} = 4\sqrt{2}.$$

$$\theta = \frac{7\pi}{4}.$$

So the 4th roots of $re^{i\theta}$ is given by

$$r^{\frac{1}{4}} e^{i\frac{\theta+2k\pi}{4}} \qquad k=0,1,2,3$$

$$= 32^{\frac{1}{8}} e^{i\left(\frac{\frac{7\pi}{4}+2k\pi}{4}\right)}$$

$$= \underbrace{32^{\frac{1}{8}} e^{i\frac{7\pi}{16}},\ 32^{\frac{1}{8}} e^{i\frac{15\pi}{16}},\ 32^{\frac{1}{8}} e^{i\frac{23\pi}{16}},\ 32^{\frac{1}{8}} e^{i\frac{31\pi}{16}}}_{\text{four roots.}}$$

$\boxed{Q6bi}$ $P(x)$: $\frac{1}{2} < x < \frac{5}{2}$

$Q(x)$: $x \in \mathbb{Z}$

$R(x)$: $x^2 = 1$

$S(x)$: $x = 2$

$\forall x \in \mathbb{R}$, $P(x) \to R(x)$. This is obviously false.

Take $x = 2$. Then $P(2)$ holds since $\frac{1}{2} < 2 < \frac{5}{2}$.

$R(2)$ is false since $2^2 \neq 1$.

$\boxed{Q6bii}$ $\forall x \in \mathbb{R}$, $Q(x) \to R(x)$

This is false. Again take $x = 2$.

Then $Q(2)$ true and $R(2)$ false.

$\boxed{Q6biii}$ $\forall x \in \mathbb{R}$ $(P(x) \wedge Q(x)) \to (R(x) \vee S(x))$

This is true. Fix $x \in \mathbb{R}$ and assume $P(x) \wedge Q(x)$ holds. Then $\frac{1}{2} < x < \frac{5}{2}$ and $x \in \mathbb{Z}$.

This means $x = 1$ or $x = 2$.

If $x = 1$, then $x^2 = 1$ and so $R(x)$ true.

Hence, $R(x) \vee S(x)$ true.

If $x = 2$, then $S(x)$ true. So, $R(x) \vee S(x)$ true.

In either case, $R(x) \vee S(x)$ true.

$\boxed{\text{Q6biv}}$ $\exists x \in \mathbb{R} \quad S(x) \to R(x)$

$\equiv \exists x \in \mathbb{R} \quad \neg S(x) \vee R(x)$.

Can take $x=1$. Then $x \neq 2$ so $\neg S(x)$ true. So, $\neg S(x) \vee R(x)$ true.

$\boxed{Q7a}$ Let XRY iff $X \cap B = Y \cap B$.

R is reflexive: Take $X \in \mathcal{P}(A)$. Then $\underbrace{X \cap B = X \cap B}_{\text{obviously}}$.

So, XRX holds.

R is symmetric: Take $X, Y \in \mathcal{P}(A)$ and assume XRY.

So, $X \cap B = Y \cap B$. So, $Y \cap B = X \cap B$.

Hence YRX holds.

R is transitive: Take $X, Y, Z \in \mathcal{P}(A)$ and assume XRY and YRZ.

So, $\underbrace{X \cap B = Y \cap B}_{\because XRY \text{ true}}$ and $\underbrace{Y \cap B = Z \cap B}_{\because YRZ \text{ holds}}$.

So, $X \cap B = Y \cap B = Z \cap B$.

So, $X \cap B = Z \cap B$.

So, XRZ true.

Hence R is reflexive, symmetric and transitive

Let $A = \{1,2,3,4,5\}$ and $B = \{3,4,5\}$.

Determine all $Y \in \mathcal{P}(A)$ such that $Y \mathrel{R} \{2,3,4\}$ true. We need to find all $Y \subseteq A$ such that

$$Y \cap \{3,4,5\} = \{2,3,4\} \cap \{3,4,5\}$$
$$= \{3,4\}.$$

So Y must contain 3 and 4 but not 5.

Choices for Y are $\left.\begin{array}{l}\{3,4\},\ \{2,3,4\},\\ \{1,3,4\},\ \{1,2,3,4\}.\end{array}\right\}$ Eq class of X

Now determine all $Y \in \mathcal{P}(A)$ such that $Y \mathrel{R} B$.

We need to find all $Y \subseteq A$ such that

$$Y \cap \{3,4,5\} = \{3,4,5\}.$$

Choices for Y are $\left.\begin{array}{l}\{3,4,5\},\ \{2,3,4,5\},\\ \{1,3,4,5\},\ \{1,2,3,4,5\}.\end{array}\right\}$ Eq class of B

$\boxed{Q7b}$

$$1188 = 385 \times 3 + 33$$

$$385 = 33 \times 11 + 22$$

$$33 = 22 \times 1 + 11$$

$$22 = 11 \times 2 + 0$$

last non zero remainde $= 11.$

So $\gcd(1188, 385) = 11$

To check, factorize $1188 = 11 \times 3^3 \times 2^2$

and $385 = 11 \times 5 \times 7$

So obviously the $\gcd = 11.$