MH5100 Advanced Investigations into Calculus I

Revision Notes

Quantitative Research Society @NTU

Academic Year 2025–2026
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About These Notes

These comprehensive revision notes are designed specifically for the MH5100 Final Test (Week 13, 50 marks). They prioritize Chapters 01 and 02 and thoroughly cover Chapters 03–06 (recommended) from Rudin's *Principles of Mathematical Analysis*. The notes build upon the midterm summary and incorporate numerous worked examples from all problem sheets and the midterm examination, presented in brown boxes for easy reference.

Contents

Exam Strategy and Structure

Test Format

- **Total Marks:** 50 marks
- **Duration:** Typically 2–2.5 hours
- **Question Types:** Mix of proofs (theory-based), computational problems, and applications
- **Coverage:** Chapters 01–06 of Rudin, with emphasis on 01–02

Time Allocation Strategy

- **Read-through (5 minutes):** Quickly scan all questions; identify easier and harder problems.
- **Question allocation:**
  - Theory/Proof questions (15–20 marks): 35–45 minutes
  - Computational/Limit problems (10–15 marks): 20–30 minutes
  - Sequence/Series problems (10–15 marks): 20–30 minutes
  - Continuity/Differentiation/Integration (5–10 marks): 15–20 minutes
- **Review (10 minutes):** Check calculations, re-read problem statements, ensure proof logic is clear.

Common Problem Formats

1. **Epsilon–delta proofs:** Definition of limit, continuity, uniform continuity.
2. **Sequence convergence:** Monotone convergence theorem, Cauchy criterion, limit computations.
3. **Series convergence tests:** Comparison, ratio, root, integral tests; absolute vs. conditional convergence.
4. **Topology in $\mathbb{R}^k$:** Open/closed sets, limit points, compactness, connectedness.
5. **Continuity theorems:** Intermediate Value Theorem (IVT), Extreme Value Theorem (EVT), uniform continuity on compact sets.
6. **Differentiation:** Mean Value Theorem (MVT), L'Hôpital's rule, Taylor's theorem, derivative of inverse functions.
7. **Riemann–Stieltjes integration:** Definition, properties, Fundamental Theorem of Calculus (FTC).

Mark Distribution (Typical)

- Chapter 01 (Real/Complex Numbers, Ordered Sets): 5–8 marks
- Chapter 02 (Topology): 10–12 marks
- Chapter 03 (Sequences/Series): 10–12 marks
- Chapter 04 (Continuity): 8–10 marks

- Chapter 05 (Differentiation): 6–8 marks
- Chapter 06 (Riemann–Stieltjes Integral): 4–6 marks

1 Chapter 01: The Real and Complex Number Systems

1.1 Ordered Sets and Bounds

Definition 1.1 (Ordered Set). *A set S with an order relation $<$ is called an* ordered set *if for any $x, y \in S$, exactly one of the following holds: $x < y$, $x = y$, or $y < x$. The order is transitive: if $x < y$ and $y < z$, then $x < z$.*

Definition 1.2 (Upper Bound, Supremum). *Let $E \subset S$ where S is ordered.*

- *$\beta \in S$ is an* upper bound *of E if $x \le \beta$ for all $x \in E$.*
- *$\alpha = \sup E$ (the* supremum *or* least upper bound*) if:*
  1. *α is an upper bound of E, and*
  2. *if $\gamma < \alpha$, then γ is not an upper bound of E (i.e., $\exists x \in E$ such that $x > \gamma$).*

Similarly define lower bound *and* infimum $\inf E$.

Theorem 1.3 (Least-Upper-Bound Property). *An ordered set S has the* least-upper-bound property *if every nonempty subset $E \subset S$ that is bounded above has a supremum in S.*

Remark 1.4. *The real numbers $\mathbb{R}$ satisfy the least-upper-bound property; the rationals $\mathbb{Q}$ do not.*

Supremum in $\mathbb{Q}$ (Problem Sheet Week 2, Q4)

Let $E = \{x \in \mathbb{Q} : x^2 < 2\}$.
Analysis. The set E is bounded above in $\mathbb{Q}$ (for example, 2 is an upper bound since if $x \in E$ then $x^2 < 2 < 4$, so $|x| < 2$). However, $\sup E = \sqrt{2}$ which is irrational. Therefore E has no supremum in $\mathbb{Q}$, demonstrating that $\mathbb{Q}$ does not have the least-upper-bound property.
This example illustrates the fundamental difference between $\mathbb{Q}$ and $\mathbb{R}$: $\mathbb{R}$ is *complete* (has the LUB property) while $\mathbb{Q}$ is not.

Proving Irrationality (Problem Sheet Week 2, Q4)

Prove that $\sqrt{2} + \sqrt{3}$ is irrational.
Proof. Suppose for contradiction that $\sqrt{2} + \sqrt{3} = r$ for some $r \in \mathbb{Q}$. Then

$$\sqrt{3} = r - \sqrt{2}.$$

Squaring both sides:

$$3 = r^2 - 2r\sqrt{2} + 2 \implies \sqrt{2} = \frac{r^2 - 1}{2r}.$$

Since $r \in \mathbb{Q}$ and $r \neq 0$ (as $\sqrt{2} + \sqrt{3} > 0$), the right side is rational. But $\sqrt{2}$ is irrational, a contradiction. Therefore $\sqrt{2} + \sqrt{3}$ is irrational. $\square$

1.2 Fields and the Real Numbers

Definition 1.5 (Field). *A* field *is a set F with two operations (addition $+$ and multiplication $\cdot$) satisfying:*

- *(A) Axioms of addition: closure, commutativity, associativity, identity (0), inverses.*
- *(M) Axioms of multiplication: closure, commutativity, associativity, identity (1), inverses (for nonzero elements).*

- *(D) Distributive law:* $a \cdot (b + c) = a \cdot b + a \cdot c$.

Theorem 1.6 (Archimedean Property)**.** *For any* $x, y \in \mathbb{R}$ *with* $x > 0$*, there exists a positive integer* n *such that* $nx > y$*.*

Theorem 1.7 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *Between any two distinct real numbers there exists a rational number. That is, if* $x, y \in \mathbb{R}$ *and* $x < y$*, then there exists* $r \in \mathbb{Q}$ *with* $x < r < y$*.*

Application of Archimedean Property (Problem Sheet Week 2)

Show that for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.
Proof. Let $\epsilon > 0$. Apply the Archimedean property with $x = \epsilon$ and $y = 1$: there exists $n \in \mathbb{N}$ such that $n\epsilon > 1$. Thus $\frac{1}{n} < \epsilon$. □
This result is fundamental for epsilon–delta proofs involving sequences.

1.3 The Extended Real Number System

Definition 1.8 (Extended Reals)**.** $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ *with conventions:*

- $-\infty < x < +\infty$ *for all* $x \in \mathbb{R}$*.*
- $x + \infty = +\infty$*,* $x - \infty = -\infty$ *for* $x \in \mathbb{R}$*.*
- $\frac{x}{\pm\infty} = 0$ *for* $x \in \mathbb{R}$*.*
- *Indeterminate forms:* $\infty - \infty$*,* $0 \cdot \infty$*,* $\frac{\infty}{\infty}$*,* $\frac{0}{0}$ *are* not defined.

Every nonempty subset of $\overline{\mathbb{R}}$ *has a supremum and infimum in* $\overline{\mathbb{R}}$*.*

1.4 The Complex Field

Definition 1.9 (Complex Numbers)**.** $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ *where* $i^2 = -1$*. For* $z = a + bi$*:*

- $Re(z) = a$*,* $Im(z) = b$*.*
- $\overline{z} = a - bi$ *(complex conjugate).*
- $|z| = \sqrt{a^2 + b^2}$ *(modulus).*

Theorem 1.10 (Schwarz Inequality (Cauchy–Schwarz))**.** *For* $a_j, b_j \in \mathbb{C}$ *(*$j = 1, \dots, n$*),*

$$\left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

Theorem 1.11 (Triangle Inequality)**.** $|z_1 + z_2| \leq |z_1| + |z_2|$ *for all* $z_1, z_2 \in \mathbb{C}$*.*

Modulus Calculations (Problem Sheet Week 2)

Let $z_1 = 3 + 4i$ and $z_2 = -1 + 2i$. Verify the triangle inequality.
Solution. First compute:

$$|z_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5, \quad |z_2| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}.$$

Now $z_1 + z_2 = (3 - 1) + (4 + 2)i = 2 + 6i$, so

$$|z_1 + z_2| = \sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10} \approx 6.32.$$

We verify: $|z_1 + z_2| = 2\sqrt{10} < 5 + \sqrt{5} \approx 7.24 = |z_1| + |z_2|$. □

Pitfalls

Common Mistakes:

- Confusing $\sup E$ (least upper bound) with $\max E$ (maximum element). $\sup E$ may not be in E.
- Forgetting that $\mathbb{C}$ is *not* an ordered field: there is no order relation $<$ compatible with field operations.
- Misapplying Archimedean property: it requires $x > 0$.

2 Chapter 02: Basic Topology

2.1 Metric Spaces

Definition 2.1 (Metric Space)**.** *A* metric space *is a set X together with a function (metric) $d : X \times X \to \mathbb{R}$ satisfying for all $x, y, z \in X$:*

1. $d(x, y) \geq 0$*;* $d(x, y) = 0 \iff x = y$*.*
2. $d(x, y) = d(y, x)$ *(symmetry).*
3. $d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality).*

Example

In $\mathbb{R}^k$, the usual metric is the Euclidean distance:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}.$$

Definition 2.2 (Open Ball, Neighborhood)**.** *For $x \in X$ and $r > 0$, the* open ball *centered at x with radius r is*

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

A neighborhood *of x is any set containing an open ball around x.*

2.2 Open and Closed Sets

Definition 2.3 (Open Set)**.** *A set $E \subset X$ is* open *if for every $x \in E$, there exists $r > 0$ such that $B_r(x) \subset E$.*

Definition 2.4 (Closed Set)**.** *A set $E \subset X$ is* closed *if its complement $E^c = X \setminus E$ is open.*

Theorem 2.5 (Properties of Open/Closed Sets)**.**

1. *The union of any collection of open sets is open.*
2. *The intersection of finitely many open sets is open.*
3. *The intersection of any collection of closed sets is closed.*
4. *The union of finitely many closed sets is closed.*
5. *$\emptyset$ and X are both open and closed.*

Open and Closed Sets in $\mathbb{R}$ (Problem Sheet Week 3, Q1)

Determine whether the following sets are open, closed, both, or neither:

1. $E_1 = (0, 1)$
2. $E_2 = [0, 1]$
3. $E_3 = [0, 1)$
4. $E_4 = \mathbb{R}$

Solution.

1. $E_1 = (0, 1)$ is **open** (every point has a neighborhood in E_1) but not closed (does not contain limit points $0, 1$).
2. $E_2 = [0, 1]$ is **closed** (complement is $(-\infty, 0) \cup (1, \infty)$, which is open) but not open (endpoint 0 has no neighborhood entirely in E_2).
3. $E_3 = [0, 1)$ is **neither** open (0 is not interior) nor closed (does not contain limit point 1).
4. $E_4 = \mathbb{R}$ is **both** open and closed (by convention, $\mathbb{R}$ and $\emptyset$ are clopen).

Definition 2.6 (Limit Point, Closure, Interior)**.** *Let $E \subset X$.*

- *x is a* limit point *of E if every neighborhood of x contains a point $y \in E$ with $y \neq x$.*
- *The* closure *$\overline{E}$ is E together with all its limit points.*
- *The* interior *E° is the set of all interior points of E (points $x \in E$ such that some neighborhood of x is contained in E).*
- *E is closed $\iff$ $E = \overline{E}$ $\iff$ E contains all its limit points.*

Limit Points and Closure (Problem Sheet Week 3, Q2)

Let $E = (0, 1) \cup \{2\}$ in $\mathbb{R}$. Find the limit points, closure, and interior of E.
Solution.

- **Limit points:** For any $x \in [0, 1]$, every neighborhood of x contains points of $(0, 1)$. The point 2 is isolated (has a neighborhood $B_{0.5}(2) = (1.5, 2.5)$ containing only 2 from E), so 2 is not a limit point. Thus the set of limit points is $[0, 1]$.
- **Closure:** $\overline{E} = E \cup \{\text{limit points}\} = (0, 1) \cup \{2\} \cup [0, 1] = [0, 1] \cup \{2\}$.
- **Interior:** $E^\circ = (0, 1)$ (the point 2 is isolated, so not interior).

2.3 Compact Sets

Definition 2.7 (Open Cover, Compact Set)**.** *A collection $\{G_\alpha\}$ of open sets is an* open cover *of E if $E \subset \bigcup_\alpha G_\alpha$. A set K is* compact *if every open cover of K has a finite subcover.*

Theorem 2.8 (Heine–Borel Theorem)**.** *A subset K of $\mathbb{R}^k$ is compact if and only if K is closed and bounded.*

Theorem 2.9 (Properties of Compact Sets)**.** *1. Closed subsets of compact sets are compact.*

2. *Compact subsets of metric spaces are closed.*

3. *If $\{K_\alpha\}$ is a collection of compact sets with the finite intersection property, then $\bigcap_\alpha K_\alpha \neq \emptyset$.*

Testing Compactness (Problem Sheet Week 3, Q3)

Determine whether the following sets in $\mathbb{R}$ are compact:

1. $A = [0, 1]$
2. $B = (0, 1)$
3. $C = [0, \infty)$

Solution. Use Heine–Borel: compact $\iff$ closed and bounded.

1. $A = [0, 1]$ is closed and bounded, hence **compact**.
2. $B = (0, 1)$ is bounded but not closed, hence **not compact**.
3. $C = [0, \infty)$ is closed but not bounded, hence **not compact**.

Finite Intersection Property (Problem Sheet Week 3)

Let $K_n = [n, \infty)$ for $n \in \mathbb{N}$. Show that $\bigcap_{n=1}^{\infty} K_n = \emptyset$ and explain why this does not contradict the finite intersection property for compact sets.
Solution. For any $x \in \mathbb{R}$, choose $n > x$. Then $x \notin K_n$, so $x \notin \bigcap_{n=1}^{\infty} K_n$. Thus $\bigcap_{n=1}^{\infty} K_n = \emptyset$.
This does not contradict the compact case because each K_n is *not compact* (unbounded). The finite intersection property holds only for compact sets.

2.4 Perfect Sets and Connected Sets

Definition 2.10 (Perfect Set)**.** *E is* perfect *if E is closed and every point of E is a limit point of E.*

Example

$[a, b]$ is perfect; (a, b) is not (not closed); $\{1, 2, 3\}$ is not (isolated points).

Definition 2.11 (Connected Set)**.** *$E \subset X$ is* connected *if E is not the union of two nonempty separated sets. (Sets A, B are* separated *if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.)*

Theorem 2.12. *A subset E of $\mathbb{R}$ is connected if and only if E is an interval.*

Connected Sets (Problem Sheet Week 3)

Show that $E = [0, 1] \cup [2, 3]$ is not connected.
Proof. Let $A = [0, 1]$ and $B = [2, 3]$. Then $E = A \cup B$. We have:

$$\overline{A} = [0, 1], \quad \overline{B} = [2, 3].$$

Thus $\overline{A} \cap B = [0, 1] \cap [2, 3] = \emptyset$ and $A \cap \overline{B} = [0, 1] \cap [2, 3] = \emptyset$. So A and B are separated nonempty sets with $E = A \cup B$, proving E is not connected. □

Techniques & Tips

Proving compactness:

- In $\mathbb{R}^k$: show closed and bounded (Heine–Borel).
- In general metric spaces: use sequential compactness (every sequence has a convergent subsequence).
- Use the definition: given an arbitrary open cover, extract a finite subcover.

Pitfalls

Common Mistakes:

- Confusing open and closed: a set can be both (e.g., X itself), neither (e.g., $[0, 1)$), or one but not the other.
- Assuming bounded $\implies$ compact: must also be closed.
- Forgetting that a sequence can have many limit points; a set's limit points are different from sequence limits.

3 Chapter 03: Numerical Sequences and Series

3.1 Convergent Sequences

Definition 3.1 (Convergence). *A sequence $\{p_n\}$ in a metric space X* converges *to $p \in X$ if for every $\epsilon > 0$, there exists N such that $n \geq N \implies d(p_n, p) < \epsilon$. Write $\lim_{n\to\infty} p_n = p$ or $p_n \to p$.*

Theorem 3.2 (Uniqueness of Limits). *If $p_n \to p$ and $p_n \to q$, then $p = q$.*

Theorem 3.3 (Convergent Sequences are Bounded). *If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.*

Theorem 3.4 (Subsequences). *If $\{p_n\}$ converges to p, then every subsequence $\{p_{n_k}\}$ converges to p.*

Epsilon–N Proof of Convergence (Problem Sheet Week 4, Q1)

Prove using the epsilon–N definition that $\lim_{n\to\infty} \frac{2n+3}{n+1} = 2$.
Proof. Let $\epsilon > 0$ be given. We need to find N such that $n \geq N$ implies

$$\left|\frac{2n+3}{n+1} - 2\right| < \epsilon.$$

Compute:

$$\left|\frac{2n+3}{n+1} - 2\right| = \left|\frac{2n+3-2(n+1)}{n+1}\right| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1}.$$

For $n \geq N$, we have $\frac{1}{n+1} \leq \frac{1}{N+1}$. Choose N such that $\frac{1}{N+1} < \epsilon$, i.e., $N > \frac{1}{\epsilon} - 1$. Then for $n \geq N$,

$$\left|\frac{2n+3}{n+1} - 2\right| = \frac{1}{n+1} \leq \frac{1}{N+1} < \epsilon.$$

Therefore $\lim_{n\to\infty} \frac{2n+3}{n+1} = 2$. □

Definition 3.5 (Cauchy Sequence). *$\{p_n\}$ is a* Cauchy sequence *if for every $\epsilon > 0$, there exists N such that $m, n \geq N \implies d(p_m, p_n) < \epsilon$.*

Theorem 3.6 (Cauchy Criterion). *In $\mathbb{R}^k$ (or any complete metric space), a sequence converges if and only if it is a Cauchy sequence.*

Theorem 3.7 (Monotone Convergence Theorem). *A monotone sequence in $\mathbb{R}$ converges if and only if it is bounded.*

- *If $\{s_n\}$ is increasing and bounded above, then $\lim_{n\to\infty} s_n = \sup_n s_n$.*
- *If $\{s_n\}$ is decreasing and bounded below, then $\lim_{n\to\infty} s_n = \inf_n s_n$.*

Monotone Convergence (Midterm 2025, Q5; Problem Sheet Week 4)

Prove that the sequence $u_n = \left(1+\frac{1}{n}\right)^n$ converges.
Proof. We show $\{u_n\}$ is monotone increasing and bounded above.
Step 1: Monotonicity. Using the binomial theorem,

$$u_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=0}^{k-1} \left(1 - \frac{j}{n}\right).$$

Each term increases as n increases, and u_{n+1} has one more term than u_n. Thus $u_n < u_{n+1}$.
Step 2: Boundedness. Note that $\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right) \le 1$, so

$$u_n \le \sum_{k=0}^{n} \frac{1}{k!} < \sum_{k=0}^{\infty} \frac{1}{k!} = e < 3.$$

By the Monotone Convergence Theorem, $\{u_n\}$ converges (in fact, to e). □

3.2 Series

Definition 3.8 (Series, Partial Sum). *Given a sequence $\{a_n\}$, the* series $\sum_{n=1}^{\infty} a_n$ *is defined by its sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$. The series* converges *if* $\lim_{n\to\infty} s_n$ *exists.*

Theorem 3.9 (Cauchy Criterion for Series). *$\sum a_n$ converges $\iff$ for every $\epsilon > 0$, there exists N such that $m \ge n \ge N \implies |\sum_{k=n}^{m} a_k| < \epsilon$.*

Corollary 3.10 (Necessary Condition). *If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. (Converse is false: $\sum \frac{1}{n}$ diverges.)*

Theorem 3.11 (Convergence Tests).

1. ***Comparison Test:*** *If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n \ge b_n \ge 0$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.*
2. ***Ratio Test:*** *Suppose $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.*
   - *If $L < 1$, then $\sum a_n$ converges.*
   - *If $L > 1$ (or $L = \infty$), then $\sum a_n$ diverges.*
   - *If $L = 1$, the test is inconclusive.*
3. ***Root Test:*** *Suppose $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \alpha$.*
   - *If $\alpha < 1$, then $\sum a_n$ converges absolutely.*
   - *If $\alpha > 1$, then $\sum a_n$ diverges.*
   - *If $\alpha = 1$, the test is inconclusive.*
4. ***Integral Test:*** *If $f(x)$ is positive, continuous, and decreasing on $[1,\infty)$, then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.*

p-Series Test (Problem Sheet Week 4, Q2)

Determine for which values of p the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.
Solution. Use the integral test. Consider $f(x) = \frac{1}{x^p}$ for $x \geq 1$. Then

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \left[\frac{x^{1-p}}{1-p}\right]_1^{\infty} & \text{if } p \neq 1 \\ [\ln x]_1^{\infty} & \text{if } p = 1 \end{cases}.$$

- If $p > 1$: $\int_1^{\infty} x^{-p} dx = \frac{1}{p-1} < \infty$, so the series **converges**.
- If $p = 1$: $\int_1^{\infty} \frac{1}{x} dx = \infty$, so the harmonic series $\sum \frac{1}{n}$ **diverges**.
- If $p < 1$: $\int_1^{\infty} x^{-p} dx = \infty$, so the series **diverges**.

Conclusion: The series converges if and only if $p > 1$.

Ratio Test Application (Problem Sheet Week 4, Q3)

Test the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ for convergence.
Solution. Let $a_n = \frac{n!}{n^n}$. Apply the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{n+1}{(n+1)^{n+1}} \cdot n^n = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+1/n}\right)^n.$$

Taking the limit:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^n = \frac{1}{e} < 1.$$

By the ratio test, the series **converges**.

Theorem 3.12 (Absolute Convergence)**.** *If $\sum |a_n|$ converges, then $\sum a_n$ converges. (The converse is false: $\sum \frac{(-1)^{n+1}}{n}$ converges conditionally.)*

Theorem 3.13 (Alternating Series Test (Leibniz))**.** *If $a_n \geq 0$, a_n decreasing, and $\lim_{n\to\infty} a_n = 0$, then $\sum (-1)^{n+1} a_n$ converges.*

Conditional Convergence (Problem Sheet Week 4, Q4)

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, but does not converge absolutely.
Solution.

- **Convergence:** Apply the alternating series test with $a_n = \frac{1}{n}$. We have $a_n > 0$, a_n is decreasing, and $\lim a_n = 0$. Thus the series converges.
- **Not absolutely convergent:** $\sum \left|\frac{(-1)^{n+1}}{n}\right| = \sum \frac{1}{n}$ is the harmonic series, which diverges.

Therefore the series converges **conditionally** but not absolutely.

3.3 Limsup and Liminf

Definition 3.14 (Limsup, Liminf)**.** *Let $\{s_n\}$ be a sequence in $\overline{\mathbb{R}}$.*

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty} \sup_{k\geq n} s_k, \quad \liminf_{n\to\infty} s_n = \lim_{n\to\infty} \inf_{k\geq n} s_k.$$

Theorem 3.15. $\lim_{n\to\infty} s_n$ *exists* $\iff$ $\limsup s_n = \liminf s_n$*, and then* $\lim s_n$ *equals this common value.*

Techniques & Tips

Techniques for sequences and series:

- **Monotone sequences:** Check boundedness and monotonicity, apply monotone convergence theorem.
- **Squeeze theorem:** Bound the sequence between two convergent sequences with the same limit.
- **Series:** First check if $a_n \to 0$. If not, series diverges. Then apply ratio, root, or comparison test.
- **Absolute vs. conditional:** Check $\sum |a_n|$ first. If it converges, original series converges absolutely.

Pitfalls

Common Mistakes:

- Assuming $a_n \to 0 \implies \sum a_n$ converges (false: harmonic series).
- Misapplying ratio/root test when limit equals 1 (test is inconclusive).
- Confusing convergence of $\{s_n\}$ with convergence of $\sum a_n$ (they are related but distinct concepts).

4 Chapter 04: Continuity

4.1 Limits of Functions

Definition 4.1 (Limit of a Function). *Let $f : X \to Y$ be a function between metric spaces, and let p be a limit point of the domain of f. We say $\lim_{x\to p} f(x) = q$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$0 < d_X(x,p) < \delta \implies d_Y(f(x),q) < \epsilon.$$

Theorem 4.2 (Sequential Criterion for Limits). *$\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\}$ in the domain of f with $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$.*

Limit Computation (Midterm 2025, Q1a)

Compute $\lim_{x\to 0} \cos\left(\frac{\frac{1}{x-1}+\frac{1}{x+1}}{2x}\pi\right)$.

Solution. Simplify the argument of cosine:

$$\frac{\frac{1}{x-1}+\frac{1}{x+1}}{2x} = \frac{(x+1)+(x-1)}{2x(x-1)(x+1)} \cdot \frac{(x-1)(x+1)}{1} = \frac{2x}{2x(x^2-1)} = \frac{1}{x^2-1}.$$

Thus

$$\lim_{x\to 0} \cos\left(\frac{\pi}{x^2-1}\right) = \cos\left(\frac{\pi}{-1}\right) = \cos(-\pi) = -1. \quad \square$$

Limit at Infinity (Midterm 2025, Q1b)

Compute $\lim_{x\to -\infty} \frac{e^x + e^{-x}}{e^x - 3e^{-x}}$.

Solution. Divide numerator and denominator by e^{-x}:

$$\frac{e^x + e^{-x}}{e^x - 3e^{-x}} = \frac{e^{2x}+1}{e^{2x}-3}.$$

As $x \to -\infty$, $e^{2x} \to 0$. Thus

$$\lim_{x\to -\infty} \frac{e^{2x}+1}{e^{2x}-3} = \frac{0+1}{0-3} = -\frac{1}{3}. \quad \square$$

Squeeze Theorem (Problem Sheet Week 5, Q1)

Let n be a positive integer. Show that $\lim_{x\to 0} x^n \sin\frac{1}{x} = 0$.

Proof. Note that $|\sin\frac{1}{x}| \leq 1$ for all $x \neq 0$. Thus

$$|x^n \sin\frac{1}{x}| \leq |x^n| \cdot 1 = |x|^n.$$

Since $\lim_{x\to 0} |x|^n = 0$ and $\lim_{x\to 0}(-|x|^n) = 0$, by the squeeze theorem,

$$\lim_{x\to 0} x^n \sin\frac{1}{x} = 0. \quad \square$$

4.2 Continuous Functions

Definition 4.3 (Continuity at a Point). *f is* continuous at p *if $\lim_{x\to p} f(x) = f(p)$. Equivalently, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x,p) < \delta \implies d(f(x),f(p)) < \epsilon$.*

Definition 4.4 (Continuity on a Set)**.** *f is* continuous *on E if f is continuous at every point of E.*

Theorem 4.5 (Composition of Continuous Functions)**.** *If $f : X \to Y$ is continuous at p and $g : Y \to Z$ is continuous at $f(p)$, then $g \circ f : X \to Z$ is continuous at p.*

Continuity vs. Absolute Value (Problem Sheet Week 6, Q1)

Explain why if $f(x)$ is continuous, then so is $|f(x)|$. Give an example where $|f(x)|$ is continuous but $f(x)$ is not.

Solution.

- If f is continuous at p, then for any $\epsilon > 0$, there exists $\delta > 0$ such that $|x-p| < \delta \implies |f(x)-f(p)| < \epsilon$. By the reverse triangle inequality, $||f(x)|-|f(p)|| \leq |f(x)-f(p)| < \epsilon$, so $|f|$ is continuous at p.
- **Counterexample:** Let
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}.$$
Then f is discontinuous everywhere (any neighborhood of any point contains both rationals and irrationals). However, $|f(x)| = 1$ for all x, which is continuous.

Piecewise Continuity (Midterm 2025, Q2; Problem Sheet Week 3)

Given
$$f(x) = \begin{cases} x^2 + a & x \geq 0 \\ -bx + 1 & x < 0 \end{cases},$$
find values of a and b such that f is continuous and differentiable at $x = 0$.

Solution.

- **Continuity:** We need $\lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x)$.
$$\lim_{x\to 0^+}(x^2 + a) = a, \quad f(0) = 0^2 + a = a, \quad \lim_{x\to 0^-}(-bx + 1) = 1.$$
Thus $a = 1$.
- **Differentiability:** We need $f'(0^+) = f'(0^-)$.
$$f'(0^+) = \lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^2 + 1 - 1}{h} = \lim_{h\to 0^+} h = 0.$$
$$f'(0^-) = \lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-bh + 1 - 1}{h} = \lim_{h\to 0^-}(-b) = -b.$$
Thus $-b = 0$, so $b = 0$.

Answer: $a = 1$, $b = 0$. □

4.3 Continuity and Compactness

Theorem 4.6 (Continuous Image of Compact Set is Compact)**.** *If $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is compact.*

Theorem 4.7 (Extreme Value Theorem (EVT))**.** *If $f : K \to \mathbb{R}$ is continuous and K is compact,* *then f attains its maximum and minimum on K. That is, there exist $p, q \in K$ such that*

$$f(p) \le f(x) \le f(q) \quad \text{for all } x \in K.$$

Theorem 4.8 (Intermediate Value Theorem (IVT)). *If $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) < \gamma < f(b)$ (or $f(b) < \gamma < f(a)$), then there exists $c \in (a, b)$ such that $f(c) = \gamma$.*

IVT Application (Problem Sheet Week 6, Q3)

Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(a) < a$ and $f(b) > b$. Prove there exists $\xi \in (a, b)$ such that $f(\xi) = \xi$.

Proof. Define $g(x) = f(x) - x$. Then g is continuous on $[a, b]$ (as the difference of continuous functions), and

$$g(a) = f(a) - a < 0, \quad g(b) = f(b) - b > 0.$$

Since $g(a) < 0 < g(b)$ and g is continuous, by IVT there exists $\xi \in (a, b)$ such that $g(\xi) = 0$, i.e., $f(\xi) = \xi$. □

This is a **fixed-point theorem**: continuous functions satisfying certain conditions must have fixed points.

Fixed Point Application (Problem Sheet Week 9, Q5)

Prove that the equation $e^x = 2 - x$ has at least one solution in $(0, 1)$.

Proof. Let $f(x) = e^x + x - 2$. Then f is continuous on $[0, 1]$. Evaluate:

$$f(0) = e^0 + 0 - 2 = 1 - 2 = -1 < 0,$$

$$f(1) = e^1 + 1 - 2 = e - 1 > 0.$$

By IVT, there exists $c \in (0, 1)$ such that $f(c) = 0$, i.e., $e^c = 2 - c$. □

4.4 Uniform Continuity

Definition 4.9 (Uniform Continuity). *$f : X \to Y$ is* uniformly continuous *on X if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$,*

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

Theorem 4.10 (Uniform Continuity on Compact Sets). *If f is continuous on a compact set K, then f is uniformly continuous on K.*

Remark 4.11. *Uniform continuity differs from pointwise continuity: the δ depends only on ϵ, not on the point.*

Non-uniform Continuity (Problem Sheet Week 6, Q5)

Show that $f(x) = x^2$ is continuous on $\mathbb{R}$ but not uniformly continuous.
Proof.

- **Continuity:** For any $a \in \mathbb{R}$ and $\epsilon > 0$, choose $\delta = \min\{1, \frac{\epsilon}{2|a|+1}\}$. Then if $|x - a| < \delta$,

$$|f(x) - f(a)| = |x^2 - a^2| = |x + a| \cdot |x - a| < (2|a| + 1)\delta \leq \epsilon.$$

  Thus f is continuous at a.

- **Not uniformly continuous:** Let $\epsilon = 1$. For any $\delta > 0$, choose n large enough so that $\frac{1}{n} < \delta$ but $2n + 1 > \frac{1}{\delta}$. Let $x = n$ and $y = n + \frac{1}{2n}$. Then

$$|x - y| = \frac{1}{2n} < \delta, \quad \text{but} \quad |f(x) - f(y)| = |n^2 - (n + \frac{1}{2n})^2| \approx 1 + \frac{1}{4n^2} > 1 = \epsilon.$$

  Thus no single δ works for all x, y.

Techniques & Tips

Proving continuity:

- **Epsilon–delta:** Given $\epsilon > 0$, find $\delta > 0$ such that $|x-p| < \delta \implies |f(x)-f(p)| < \epsilon$.
- **Sequential criterion:** Show $x_n \to p \implies f(x_n) \to f(p)$.
- **Composition:** Use that composition of continuous functions is continuous.
- **Piecewise:** Check continuity at boundary points by verifying left and right limits match function value.

Pitfalls

Common Mistakes:

- Forgetting to check continuity at boundary points of piecewise functions.
- Assuming IVT gives uniqueness of c (it does not).
- Confusing uniform continuity with continuity: δ in uniform continuity must work for all points simultaneously.

5 Chapter 05: Differentiation

5.1 The Derivative

Definition 5.1 (Derivative). *Let $f : [a, b] \to \mathbb{R}$. The* derivative *of f at $x \in [a, b]$ is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. If $f'(x)$ exists, we say f is differentiable at x.

Theorem 5.2 (Differentiability Implies Continuity). *If f is differentiable at x, then f is continuous at x. (The converse is false: $f(x) = |x|$ is continuous at 0 but not differentiable there.)*

Theorem 5.3 (Differentiation Rules). *If f and g are differentiable at x, then:*

- $(f + g)'(x) = f'(x) + g'(x)$.
- $(cf)'(x) = cf'(x)$ *for constant c.*
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ *(product rule).*
- $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ *if $g(x) \neq 0$ (quotient rule).*
- $(f \circ g)'(x) = f'(g(x))g'(x)$ *(chain rule).*

Chain Rule Application (Midterm 2025, Q3)

Compute the derivative of $f(x) = e^{\sin(\sqrt[3]{x})}$.
Solution. Apply the chain rule repeatedly:

$$f'(x) = e^{\sin(\sqrt[3]{x})} \cdot \frac{d}{dx}\left[\sin(\sqrt[3]{x})\right] = e^{\sin(\sqrt[3]{x})} \cdot \cos(\sqrt[3]{x}) \cdot \frac{d}{dx}\left[x^{1/3}\right].$$

Now $\frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3}$. Thus

$$f'(x) = e^{\sin(\sqrt[3]{x})} \cdot \cos(\sqrt[3]{x}) \cdot \frac{1}{3}x^{-2/3} = \frac{e^{\sin(\sqrt[3]{x})}\cos(\sqrt[3]{x})}{3x^{2/3}}. \quad \square$$

Simplify Before Differentiating (Problem Sheet Week 10, Q3)

Simplify and differentiate $f(x) = \frac{\sin^2 x}{1+\cot x} + \frac{\cos^2 x}{1+\tan x}$.
Solution. First simplify. Note that

$$\frac{\sin^2 x}{1+\cot x} = \frac{\sin^2 x}{1+\frac{\cos x}{\sin x}} = \frac{\sin^3 x}{\sin x + \cos x},$$

$$\frac{\cos^2 x}{1+\tan x} = \frac{\cos^2 x}{1+\frac{\sin x}{\cos x}} = \frac{\cos^3 x}{\cos x + \sin x}.$$

Thus

$$f(x) = \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x}.$$

Using the factorization $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$:

$$\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x) = (\sin x + \cos x)(1 - \sin x \cos x).$$

Therefore

$$f(x) = 1 - \sin x \cos x = 1 - \frac{1}{2}\sin 2x.$$

Differentiate:

$$f'(x) = -\frac{1}{2} \cdot 2\cos 2x = -\cos 2x. \quad \square$$

5.2 Mean Value Theorems

Theorem 5.4 (Rolle's Theorem)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Theorem 5.5 (Mean Value Theorem (MVT))**.** *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 5.6. *If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on (a,b).*

Functional Equation (Midterm 2025, Q4; Problem Sheet Week 8, Q4)

Suppose $f(x)f(y) = f(x+y)$ for all $x, y \in \mathbb{R}$, f is differentiable, and $f \not\equiv 0$. Prove $f(x) = e^{cx}$ for some constant c.
Proof.

1. Set $x = y = 0$: $f(0)^2 = f(0)$, so $f(0) = 0$ or $f(0) = 1$. Since $f \not\equiv 0$, we have $f(0) = 1$.
2. Compute $f'(0)$: let $f'(0) = c$. Then
$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - 1}{h} = c.$$
3. Differentiate $f(x+y) = f(x)f(y)$ with respect to x:
$$f'(x+y) = f'(x)f(y).$$
Set $y = 0$: $f'(x) = f'(x)f(0) = f'(x) \cdot 1$, which is consistent. Set $x = 0$:
$$f'(y) = f'(0)f(y) = cf(y).$$
4. Thus f satisfies the differential equation $f'(x) = cf(x)$ with $f(0) = 1$. The unique solution is $f(x) = e^{cx}$. □

Polynomial Derivative Identity (Problem Sheet Week 8, Q1)

Let $f(x) = (x-a)(x-b)(x-c)(x-d)(x-e)$ where a, b, c, d, e are distinct reals. If $f'(k) = (k-a)(k-b)(k-c)(k-d)$ for some k, show that $k = e$.
Proof. By the product rule,

$$f'(x) = (x-b)(x-c)(x-d)(x-e) + (x-a)(x-c)(x-d)(x-e) + \cdots + (x-a)(x-b)(x-c)(x-d).$$

At $x = e$:

$$f'(e) = (e-b)(e-c)(e-d)\cdot 0 + \cdots + (e-a)(e-b)(e-c)(e-d) = (e-a)(e-b)(e-c)(e-d).$$

Given that $f'(k) = (k-a)(k-b)(k-c)(k-d)$, and since the polynomial $f'(x) - (x-a)(x-b)(x-c)(x-d)$ has degree 4 and vanishes at $x = e$, we conclude $k = e$ (uniqueness of roots). □

5.3 Derivatives and Limits

Theorem 5.7 (L'Hôpital's Rule)**.** *Suppose f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, and either:*

- $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$, *or*
- $\lim_{x\to a^+} g(x) = \pm\infty$.

If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ (finite or infinite), then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

(Similar statement for $x \to b^-$, $x \to \infty$, etc.)

Symmetric Difference Quotient (Problem Sheet Week 8, Q2)

Let f be differentiable at a. Evaluate $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{h}$.
Solution. Rewrite:

$$\frac{f(a+h)-f(a-h)}{h} = \frac{f(a+h)-f(a)}{h} + \frac{f(a)-f(a-h)}{h}.$$

As $h \to 0$:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a),$$

$$\lim_{h\to 0} \frac{f(a)-f(a-h)}{h} = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = f'(a).$$

Therefore

$$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{h} = f'(a)+f'(a) = 2f'(a). \quad \square$$

Implicit Differentiation (Problem Sheet Week 9, Q1)

Let f be differentiable with $f(f(x)) = x$ and $f'(x) = 1+[f(x)]^2$. Show that $f'(f(x)) = \frac{1}{1+x^2}$.
Solution. Differentiate $f(f(x)) = x$ using the chain rule:

$$f'(f(x)) \cdot f'(x) = 1.$$

Thus $f'(f(x)) = \frac{1}{f'(x)}$. Substitute $f'(x) = 1+[f(x)]^2$:

$$f'(f(x)) = \frac{1}{1+[f(x)]^2}.$$

Now let $y = f(x)$, so $x = f(y)$ (since $f(f(x)) = x$). Then

$$f'(y) = \frac{1}{1+[f(y)]^2} = \frac{1}{1+x^2}.$$

Replacing y with $f(x)$:

$$f'(f(x)) = \frac{1}{1+x^2}. \quad \square$$

5.4 Higher Derivatives and Taylor's Theorem

Theorem 5.8 (Taylor's Theorem with Remainder)**.** *Suppose $f : [a,b] \to \mathbb{R}$ has n continuous derivatives on $[a,b]$ and $f^{(n+1)}$ exists on (a,b). Then for any $x, x_0 \in [a,b]$, there exists c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Techniques & Tips

Differentiation techniques:

- **Chain rule:** Carefully identify inner and outer functions.
- **Implicit differentiation:** Differentiate both sides with respect to x, treating y as a function of x.
- **Logarithmic differentiation:** Useful for products, quotients, and powers involving x: take ln of both sides first.
- **MVT applications:** Use MVT to prove inequalities or estimates involving derivatives.

Pitfalls

Common Mistakes:

- Assuming differentiable $\implies$ f' is continuous (false: consider $f(x) = x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$).
- Misapplying L'Hôpital's rule when hypotheses are not satisfied (e.g., limit of ratio of derivatives doesn't exist).
- Forgetting to check that $g'(x) \neq 0$ in L'Hôpital's rule.

6 Chapter 06: The Riemann–Stieltjes Integral

6.1 Definition and Existence

Definition 6.1 (Partition and Riemann–Stieltjes Sum). *A* partition P *of* $[a,b]$ *is a finite set* $\{x_0, x_1, \dots, x_n\}$ *with* $a = x_0 < x_1 < \cdots < x_n = b$. *The* mesh *of* P *is* $\|P\| = \max_i (x_i - x_{i-1})$.

For functions $f, \alpha : [a,b] \to \mathbb{R}$, *the* Riemann–Stieltjes sum *is*

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i, \quad \text{where } t_i \in [x_{i-1}, x_i],\ \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

Definition 6.2 (Riemann–Stieltjes Integral). *We say* f *is* Riemann–Stieltjes integrable with respect to α *on* $[a,b]$ *(write* $f \in \mathcal{R}(\alpha)$*) if there exists* $A \in \mathbb{R}$ *such that for every* $\epsilon > 0$, *there exists* $\delta > 0$ *with*

$$\|P\| < \delta \implies |S(P, f, \alpha) - A| < \epsilon.$$

We write $A = \int_a^b f\, d\alpha$.

Theorem 6.3 (Existence of Integral).

1. *If* f *is continuous on* $[a,b]$ *and* α *is of bounded variation on* $[a,b]$, *then* $f \in \mathcal{R}(\alpha)$.
2. *If* f *is bounded and has only finitely many discontinuities, and* α *is continuous at each discontinuity of* f, *then* $f \in \mathcal{R}(\alpha)$.

6.2 Properties of the Integral

Theorem 6.4 (Linearity). *If* $f, g \in \mathcal{R}(\alpha)$ *on* $[a,b]$ *and* $c_1, c_2 \in \mathbb{R}$, *then*

$$\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f\, d\alpha + c_2 \int_a^b g\, d\alpha.$$

Theorem 6.5 (Integration by Parts). *If* $f \in \mathcal{R}(\alpha)$ *on* $[a,b]$, *then* $\alpha \in \mathcal{R}(f)$ *and*

$$\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a).$$

Theorem 6.6 (Change of Variable). *Suppose* $f \in \mathcal{R}(\alpha)$ *on* $[a,b]$, φ *is a strictly increasing continuous function mapping* $[A,B]$ *onto* $[a,b]$, *and* $g(s) = f(\varphi(s))$. *Then*

$$\int_A^B g(s)\, d\alpha(\varphi(s)) = \int_a^b f(x)\, d\alpha(x).$$

6.3 Fundamental Theorem of Calculus

Theorem 6.7 (Fundamental Theorem of Calculus, Part I). *If* $f \in \mathcal{R}(\alpha)$ *on* $[a,b]$ *and* $\alpha(x) = x$, *define*

$$F(x) = \int_a^x f(t)\, dt.$$

Then F *is continuous on* $[a,b]$. *If* f *is continuous at* $x_0 \in [a,b]$, *then* F *is differentiable at* x_0 *and* $F'(x_0) = f(x_0)$.

Theorem 6.8 (Fundamental Theorem of Calculus, Part II). *If* f *has a continuous derivative* f' *on* $[a,b]$, *then*

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Basic Integration (Problem Sheet Week 10)

Compute $\int_0^1 x^2\,dx$.

Solution. The antiderivative of x^2 is $F(x) = \frac{x^3}{3}$ (since $F'(x) = x^2$). By FTC Part II,

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}. \quad \square$$

Integration by Substitution (Problem Sheet Week 10)

Evaluate $\int_0^{\pi/2} \sin x \cos x\,dx$.

Solution. Use substitution $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \pi/2$, $u = 1$. Thus

$$\int_0^{\pi/2} \sin x \cos x\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}. \quad \square$$

Integration by Parts (Problem Sheet Week 10)

Evaluate $\int_0^1 xe^x\,dx$.

Solution. Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. By integration by parts,

$$\int_0^1 xe^x\,dx = [xe^x]_0^1 - \int_0^1 e^x\,dx = (1 \cdot e - 0) - [e^x]_0^1 = e - (e - 1) = 1. \quad \square$$

Techniques & Tips

Integration techniques:

- **Direct application of FTC:** Find antiderivative F with $F' = f$, then $\int_a^b f = F(b) - F(a)$.
- **Integration by parts:** Use $\int u\,dv = uv - \int v\,du$.
- **Substitution:** Change variables to simplify integrand.
- **Riemann sums:** For existence proofs or approximations, use definition with partitions.

Pitfalls

Common Mistakes:

- Forgetting the role of α in Riemann–Stieltjes integrals: if $\alpha(x) = x$, recover usual Riemann integral.
- Misapplying FTC when f is not continuous or f' does not exist.
- Incorrectly applying integration by parts: remember the product rule connection and the signs.

7 Problem-Solving Checklist for the Exam

General Approach to Any Problem

1. **Read carefully:** Identify what is given, what is to be proved/computed, and any special conditions.
2. **Identify the relevant chapter/topic:** Is this about limits, sequences, series, continuity, differentiation, or integration?
3. **Recall key definitions and theorems:** Write down relevant definitions, inequalities, or theorem statements.
4. **Plan your strategy:**
   - For proofs: identify the proof technique (direct, contradiction, epsilon–delta, induction, etc.).
   - For computations: identify applicable rules (limit laws, differentiation rules, FTC, etc.).
5. **Execute:** Work through the problem step-by-step, justifying each step with a theorem or definition.
6. **Check:** Does your answer make sense? Are all hypotheses of theorems satisfied? Are there edge cases?

Checklist by Problem Type

Epsilon–Delta Proofs

- Write: "Let $\epsilon > 0$ be given."
- Work backwards from $|f(x) - L| < \epsilon$ to find a suitable $\delta > 0$.
- State the choice of δ clearly.
- Verify the implication $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$ by forward reasoning.

Sequence Convergence

- Check if sequence is monotone and bounded (use monotone convergence theorem).
- If not monotone, try to show it is Cauchy.
- Use squeeze theorem if you can bound the sequence between two convergent sequences.
- For existence of limit, show that limsup = liminf.

Series Convergence

- First: Does $a_n \to 0$? If not, series diverges.
- Choose a test: comparison, ratio, root, integral, alternating series test.
- For absolute convergence, test $\sum |a_n|$ first.
- Remember: ratio and root tests are inconclusive when limit equals 1.

Continuity Proofs

- For piecewise functions: check continuity at boundary points by computing left and right limits.
- Use sequential criterion: $x_n \to p \implies f(x_n) \to f(p)$.
- Apply composition theorem for continuous functions.
- For uniform continuity on compact sets, invoke the theorem directly.

Application of IVT/EVT/MVT

- **IVT:** Check f is continuous on $[a, b]$ and there is a value between $f(a)$ and $f(b)$.
- **EVT:** Check f is continuous on a compact set (closed and bounded in $\mathbb{R}^k$).
- **MVT:** Check f is continuous on $[a, b]$, differentiable on (a, b); conclude there exists c with $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Differentiation

- Apply chain rule carefully: identify inner and outer functions.
- For implicit differentiation, differentiate both sides with respect to x.
- For product/quotient, use product/quotient rules.
- Check if f' is continuous (not automatic even if f is differentiable).

Integration

- For definite integrals, use FTC if possible: find antiderivative, evaluate at endpoints.
- For existence, check if f is continuous or has finitely many discontinuities.
- For Riemann–Stieltjes, remember the role of α.
- Integration by parts: $\int u\,dv = uv - \int v\,du$.

8 Practice Problems with Solutions

This section contains 8 representative practice questions covering the full exam scope, with fully worked solutions and time recommendations.

Practice Problem 1 (Epsilon–Delta Proof — Chapter 02, 04)

Time: 6–8 minutes **Marks:** 6

Problem: Let $f(x) = 3x+2$. Use the epsilon–delta definition to prove that $\lim_{x\to 1} f(x) = 5$.

Solution:

Proof. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that

$$0 < |x-1| < \delta \implies |f(x) - 5| < \epsilon.$$

We have

$$|f(x) - 5| = |(3x+2) - 5| = |3x - 3| = 3|x-1|.$$

To ensure $3|x-1| < \epsilon$, it suffices to choose $|x-1| < \frac{\epsilon}{3}$.

Thus, choose $\delta = \frac{\epsilon}{3}$. Then

$$0 < |x-1| < \delta \implies |f(x) - 5| = 3|x-1| < 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\lim_{x\to 1} f(x) = 5$. □ □

Marking Scheme:

- Stating "Let $\epsilon > 0$ be given": 1 mark
- Computing $|f(x) - 5| = 3|x-1|$: 2 marks
- Choosing $\delta = \epsilon/3$: 1 mark
- Verification of implication: 2 marks

Practice Problem 2 (Sequence Convergence — Chapter 03)

Time: 8–10 minutes **Marks:** 8

Problem: Let $a_n = \frac{2n+1}{3n-2}$. Prove that $\{a_n\}$ converges and find $\lim_{n\to\infty} a_n$.

Solution:

Proof. We claim that $\lim_{n\to\infty} a_n = \frac{2}{3}$. Let $\epsilon > 0$ be given. We have

$$\left|a_n - \frac{2}{3}\right| = \left|\frac{2n+1}{3n-2} - \frac{2}{3}\right| = \left|\frac{3(2n+1) - 2(3n-2)}{3(3n-2)}\right| = \left|\frac{6n+3-6n+4}{3(3n-2)}\right| = \frac{7}{3(3n-2)}.$$

For $n \geq 2$, we have $3n - 2 \geq 3 \cdot 2 - 2 = 4$, so

$$\frac{7}{3(3n-2)} \leq \frac{7}{12}.$$

More precisely, for large n, $3n - 2 \approx 3n$, so $\frac{7}{3(3n-2)} \approx \frac{7}{9n}$. To ensure $\frac{7}{9n} < \epsilon$, we need $n > \frac{7}{9\epsilon}$.

Choose $N = \lceil \frac{7}{9\epsilon} \rceil + 1$. Then for $n \geq N$,

$$\left|a_n - \frac{2}{3}\right| = \frac{7}{3(3n-2)} < \frac{7}{9n} < \frac{7}{9 \cdot \frac{7}{9\epsilon}} = \epsilon.$$

Therefore, $\lim_{n\to\infty} a_n = \frac{2}{3}$. □ □

Marking Scheme:

- Stating the claim $\lim a_n = 2/3$: 1 mark
- Computing $|a_n - 2/3|$ correctly: 3 marks
- Choosing appropriate N in terms of ϵ: 2 marks
- Verification: 2 marks

Practice Problem 3 (Series Convergence — Chapter 03)

Time: 6–8 minutes **Marks:** 6

Problem: Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges. Justify your answer.

Solution:

Proof. We apply the ratio test. Let $a_n = \frac{n^2}{2^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{(n+1)^2}{n^2} \cdot \frac{1}{2} = \frac{(n+1)^2}{2n^2}.$$

Taking the limit,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2n^2} = \lim_{n\to\infty} \frac{n^2(1+1/n)^2}{2n^2} = \frac{1}{2} < 1.$$

By the ratio test, the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges. □ □

Marking Scheme:

- Identifying ratio test as appropriate: 1 mark
- Computing a_{n+1}/a_n correctly: 2 marks
- Taking limit correctly: 2 marks
- Conclusion from ratio test: 1 mark

Practice Problem 4 (Topology — Chapter 02)

Time: 7–9 minutes **Marks:** 7

Problem: Let $E = \{1/n : n \in \mathbb{N}\}$. Find the set of limit points of E, the closure $\overline{E}$, and determine whether E is open, closed, compact, or connected.

Solution:

Proof. **Limit points:** Let $x \in \mathbb{R}$. If $x > 0$ and $x \neq 1/n$ for any n, then there exists $\epsilon > 0$ such that $B_\epsilon(x)$ contains at most one point of E (namely, the $1/n$ closest to x), so x is not a limit point. If $x \leq 0$ or $x > 1$, then some neighborhood of x contains no points of E, so x is not a limit point. However, for any $\epsilon > 0$, the interval $(0, \epsilon)$ contains points of E (take n large enough so that $1/n < \epsilon$). Thus 0 is a limit point of E.

Since the elements of E are isolated (each $1/n$ has a neighborhood containing no other points of E), the only limit point is 0.

Closure: $\overline{E} = E \cup \{0\} = \{0\} \cup \{1/n : n \in \mathbb{N}\}$.

Open: E is not open because $1 \in E$ and every neighborhood of 1 contains points not in E (e.g., 0.9).

Closed: E is not closed because it does not contain its limit point 0.

Compact: E is not compact. It is bounded (contained in $[0, 1]$) but not closed. Alternatively, the open cover $\{(1/(n + 1) - 0.1, 1/(n - 1) + 0.1) : n \geq 2\} \cup \{(0.9, 1.1)\}$ has no finite subcover.

Connected: E is not connected. For example, $E = A \cup B$ where $A = \{1\}$ and $B = \{1/n : n \geq 2\}$. Then $\overline{A} = \{1\}$ and $\overline{B} = \{0\} \cup B$, so $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. Thus A and B are separated. □ □

Marking Scheme:

- Identifying limit points (just $\{0\}$): 2 marks
- Closure $\overline{E}$: 1 mark
- Open/closed: 1 mark each (total 2 marks)
- Compact: 1 mark
- Connected: 1 mark

Practice Problem 5 (Continuity and IVT — Chapter 04)

Time: 7–9 minutes **Marks:** 7

Problem: Prove that the equation $x^3 + x - 1 = 0$ has at least one solution in the interval $(0, 1)$.

Solution:

Proof. Let $f(x) = x^3 + x - 1$. Note that f is a polynomial, hence continuous on $\mathbb{R}$.

Evaluate at the endpoints:

$$f(0) = 0^3 + 0 - 1 = -1 < 0,$$

$$f(1) = 1^3 + 1 - 1 = 1 > 0.$$

Since $f(0) < 0 < f(1)$ and f is continuous on $[0, 1]$, by the Intermediate Value Theorem, there exists $c \in (0, 1)$ such that $f(c) = 0$. Therefore, the equation $x^3 + x - 1 = 0$ has at least one solution in $(0, 1)$. □ □

Marking Scheme:

- Defining $f(x)$ and noting continuity: 1 mark
- Computing $f(0)$ and $f(1)$: 2 marks
- Stating IVT correctly: 2 marks
- Conclusion: 2 marks

Practice Problem 6 (Differentiation — Chapter 05)

Time: 6–8 minutes **Marks:** 6

Problem: Let $f(x) = \sqrt{1+x^2}$. Find $f'(x)$ and verify that $f'(0) = 0$.

Solution:

Proof. Write $f(x) = (1+x^2)^{1/2}$. By the chain rule,

$$f'(x) = \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{1+x^2}}.$$

Evaluating at $x = 0$,

$$f'(0) = \frac{0}{\sqrt{1+0^2}} = 0.$$

Thus $f'(0) = 0$ as required. □ □

Marking Scheme:

- Applying chain rule correctly: 3 marks
- Simplifying $f'(x)$: 2 marks
- Verifying $f'(0) = 0$: 1 mark

Practice Problem 7 (Mean Value Theorem — Chapter 05)

Time: 7–9 minutes **Marks:** 7

Problem: Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose $|f'(x)| \leq M$ for all $x \in (a,b)$. Prove that $|f(b) - f(a)| \leq M(b-a)$.

Solution:

Proof. Since f is continuous on $[a,b]$ and differentiable on (a,b), by the Mean Value Theorem, there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Thus

$$f(b) - f(a) = f'(c)(b-a).$$

Taking absolute values and using $|f'(c)| \leq M$,

$$|f(b) - f(a)| = |f'(c)| \cdot |b-a| \leq M(b-a).$$

Therefore, $|f(b) - f(a)| \leq M(b-a)$. □ □

Marking Scheme:

- Stating MVT and hypotheses: 2 marks
- Applying MVT to get $f(b) - f(a) = f'(c)(b-a)$: 2 marks
- Taking absolute values and using $|f'(c)| \leq M$: 2 marks
- Conclusion: 1 mark

Practice Problem 8 (Riemann–Stieltjes Integral — Chapter 06)

Time: 6–8 minutes **Marks:** 6

Problem: Compute $\int_0^2 x^2\,dx$ using the Fundamental Theorem of Calculus.

Solution:

Proof. The function $f(x) = x^2$ is continuous on $[0, 2]$. An antiderivative of $f(x)$ is $F(x) = \frac{x^3}{3}$ (since $F'(x) = x^2$). By the Fundamental Theorem of Calculus (Part II),

$$\int_0^2 x^2\,dx = F(2) - F(0) = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3} - 0 = \frac{8}{3}.$$

Therefore, $\int_0^2 x^2\,dx = \frac{8}{3}$. □ □

Marking Scheme:

- Identifying antiderivative $F(x) = x^3/3$: 2 marks
- Applying FTC: 2 marks
- Computing $F(2) - F(0)$: 1 mark
- Correct final answer: 1 mark

9 Quick Reference / Cheat Sheet

This one-page summary contains essential definitions, theorems, and frequently used inequalities for quick recall during the exam.

Chapter 01: Real and Complex Numbers

- **Supremum:** $\alpha = \sup E$ if α is an upper bound and no smaller number is an upper bound.
- **Archimedean Property:** For $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$ for any $y \in \mathbb{R}$.
- **Density of $\mathbb{Q}$:** Between any two distinct reals, there exists a rational.
- **Triangle Inequality:** $|z_1 + z_2| \leq |z_1| + |z_2|$; $|z_1 - z_2| \geq ||z_1| - |z_2||$.
- **Cauchy–Schwarz:** $\left|\sum a_i \overline{b_i}\right|^2 \leq \sum |a_i|^2 \sum |b_i|^2$.

Chapter 02: Topology

- **Metric space:** Set X with metric d satisfying $d(x, y) \geq 0$; $d(x, y) = 0 \iff x = y$; $d(x, y) = d(y, x)$; $d(x, z) \leq d(x, y) + d(y, z)$.
- **Open set:** Every point has a neighborhood entirely in the set.
- **Closed set:** Complement is open; equivalently, contains all its limit points.
- **Limit point:** Every neighborhood of x contains a point of E distinct from x.
- **Compact in $\mathbb{R}^k$:** Closed and bounded (Heine–Borel).
- **Connected in $\mathbb{R}$:** E is an interval.

Chapter 03: Sequences and Series

- **Convergence:** $p_n \to p$ if for every $\epsilon > 0$, there exists N such that $n \geq N \implies d(p_n, p) < \epsilon$.
- **Cauchy:** For every $\epsilon > 0$, there exists N such that $m, n \geq N \implies d(p_m, p_n) < \epsilon$.
- **Monotone Convergence:** Increasing and bounded above $\implies$ converges to sup.
- **Series:** $\sum a_n$ converges if partial sums $s_n = \sum_{k=1}^{n} a_k$ converge.
- **Necessary condition:** If $\sum a_n$ converges, then $a_n \to 0$.
- **Ratio test:** $L = \lim \frac{a_{n+1}}{a_n}$: $L < 1 \implies$ converges; $L > 1 \implies$ diverges; $L = 1 \implies$ inconclusive.
- **Root test:** $\alpha = \limsup \sqrt[n]{|a_n|}$: $\alpha < 1 \implies$ converges; $\alpha > 1 \implies$ diverges.
- **Alternating series:** If $a_n \geq 0$ decreasing and $a_n \to 0$, then $\sum (-1)^{n+1} a_n$ converges.

Chapter 04: Continuity

- **Continuity at p:** $\lim_{x\to p} f(x) = f(p)$. Equivalently, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x,p) < \delta \implies d(f(x), f(p)) < \epsilon$.
- **Sequential criterion:** f continuous at p iff $x_n \to p \implies f(x_n) \to f(p)$.
- **IVT:** $f : [a,b] \to \mathbb{R}$ continuous, $f(a) < \gamma < f(b) \implies$ there exists $c \in (a,b)$ with $f(c) = \gamma$.
- **EVT:** f continuous on compact $K \implies f$ attains max and min on K.
- **Uniform continuity:** δ depends only on ϵ, not on the point. Continuous on compact $\implies$ uniformly continuous.

Chapter 05: Differentiation

- **Derivative:** $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.
- **Differentiable $\implies$ continuous.** (Converse false.)
- **Chain rule:** $(f \circ g)'(x) = f'(g(x))g'(x)$.
- **MVT:** f continuous on $[a,b]$, differentiable on $(a,b) \implies$ there exists $c \in (a,b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- **Rolle's Theorem:** $f(a) = f(b)$ and f differentiable on $(a,b) \implies$ there exists $c \in (a,b)$ with $f'(c) = 0$.
- **L'Hôpital:** If $\lim f(x) = \lim g(x) = 0$ or $\pm\infty$ and $\lim \frac{f'(x)}{g'(x)} = L$, then $\lim \frac{f(x)}{g(x)} = L$.

Chapter 06: Riemann–Stieltjes Integral

- **Definition:** $\int_a^b f\, d\alpha = \lim_{\|P\|\to 0} \sum f(t_i)\Delta\alpha_i$.
- **Existence:** f continuous and α of bounded variation $\implies f \in \mathcal{R}(\alpha)$.
- **Linearity:** $\int (c_1 f + c_2 g)\, d\alpha = c_1 \int f\, d\alpha + c_2 \int g\, d\alpha$.
- **FTC Part I:** If $F(x) = \int_a^x f(t)\, dt$ and f continuous at x_0, then $F'(x_0) = f(x_0)$.
- **FTC Part II:** If f' continuous on $[a,b]$, then $\int_a^b f'(x)\, dx = f(b) - f(a)$.
- **Integration by parts:** $\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a)$.

Frequently Used Inequalities

- **Bernoulli:** $(1+x)^n > 1 + nx$ for $x > -1$, $x \neq 0$, $n \geq 2$.
- **AM–GM:** $\frac{a_1+\cdots+a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$ for $a_i \geq 0$.
- **Cauchy–Schwarz:** $|\sum a_i b_i| \leq \sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$.
- **Triangle inequality:** $|a+b| \leq |a| + |b|$; $|a-b| \geq ||a| - |b||$.

10 Further Reading and Revision Plan

Recommended Reading

- **Primary textbook:** Walter Rudin, *Principles of Mathematical Analysis*, 3rd edition, Chapters 01–06.
- **Supplementary:** Terence Tao, *Analysis I* and *Analysis II*, for additional intuition and examples.
- **Problem practice:** Review all problem sheets (Weeks 2–10) and the midterm exam.

7-Day Revision Plan

Day 7 (One week before exam):

- Review Chapters 01–02 (Real numbers, topology).
- Re-read definitions: supremum, limit point, compact set, connected set.
- Work through Practice Problems 1, 4 in these notes.
- Review Problem Sheets Week 2–3.

Day 6:

- Focus on Chapter 03 (Sequences and series).
- Memorize convergence tests (ratio, root, comparison, alternating series).
- Work through Practice Problems 2, 3.
- Review Problem Sheets Week 4–5.

Day 5:

- Study Chapter 04 (Continuity).
- Practice epsilon–delta proofs and IVT/EVT applications.
- Work through Practice Problem 5.
- Review Problem Sheets Week 5–6.

Day 4:

- Focus on Chapter 05 (Differentiation).
- Review chain rule, product rule, MVT, L'Hôpital's rule.
- Work through Practice Problems 6, 7.
- Review Problem Sheets Week 8–10.

Day 3:

- Study Chapter 06 (Riemann–Stieltjes integral).
- Practice FTC applications and integration by parts.
- Work through Practice Problem 8.

- Review any remaining problem sheets.

Day 2:

- Complete a full-length mock exam (use midterm as template, extend to full scope).
- Time yourself strictly (2 hours).
- Review mistakes and identify weak areas.

Day 1 (Night before exam):

- Review the Quick Reference / Cheat Sheet.
- Skim through worked examples in brown boxes throughout these notes.
- Get a good night's sleep—rest is crucial for exam performance.

Final Tips

- **Understand, don't memorize:** Focus on understanding the logic and structure of proofs, not rote memorization.
- **Practice writing:** Real analysis exams require clear, rigorous writing. Practice writing out full proofs.
- **Use theorem names:** In your exam answers, explicitly cite theorems by name (e.g., "By the Intermediate Value Theorem...").
- **Check hypotheses:** Before applying a theorem, verify that all hypotheses are satisfied.
- **Time management:** Don't spend too long on one problem. If stuck, move on and return later.
- **Partial credit:** Even if you can't complete a proof, write down relevant definitions and start the argument—you may earn partial marks.

Good luck with your MH5100 final test!

Remember: rigorous reasoning, clear writing, and systematic problem-solving are your keys to success. The brown-boxed examples throughout these notes illustrate exactly the kind of step-by-step reasoning expected on the exam.