MH1200 Linear Algebra I – Revision Notes

Quantitative Research Society @NTU

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Course Overview & Topic Map

Topic Area	Lectures	Key Concepts	ILO
Linear Equations & Systems	1-5	Systems, REF, RREF, Solutions	1, 2
Matrix Theory & Operations	6-8	Addition, Multiplication, Inverses	2, 3
Determinants	9-10	Cofactor expansion, Properties	3
Vector Spaces & Subspaces	11-15	Span, Independence, Basis	4

This revision guide covers all 15 lectures organized by mathematical topics rather than chronological order, ensuring comprehensive coverage while highlighting conceptual connections.

Contents

Complete Formula & Facts Reference

Linear Systems & Solutions

Basic Forms:

- $a_1x_1 + \cdots + a_nx_n = b$
- $\mathbf{Ax} = \mathbf{b}$ (system)
- $[\mathbf{A}|\mathbf{b}]$ (augmented)
- Homogeneous: $\mathbf{Ax} = \mathbf{0}$

Solution Types:

- Unique: $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}|\mathbf{b}]) = n$
- None: $\text{rank}(\mathbf{A}) < \text{rank}([\mathbf{A}|\mathbf{b}])$
- Infinite: $\text{rank}(\mathbf{A}) = \text{rank}([\mathbf{A}|\mathbf{b}]) < n$
- Free vars: $n - \text{rank}(\mathbf{A})$

ERO:

- $R_i \leftrightarrow R_j$ (swap)
- kR_i $(k \neq 0)$ (scale)
- $R_i + kR_j$ (add)

Matrix Operations

Basic:

- $(A+B)_{ij} = a_{ij} + b_{ij}$, $(kA)_{ij} = ka_{ij}$
- $(AB)_{ij} = \sum_k a_{ik}b_{kj}$
- $(A^T)_{ij} = a_{ji}$

Properties:

- $(AB)C = A(BC)$, $A(B+C) = AB + AC$
- $(AB)^T = B^TA^T$, $(AB)^{-1} = B^{-1}A^{-1}$
- $AB \neq BA$, $\text{tr}(A) = a_{11} + \ldots + a_{nn}$

Special:

- $I_n : (I_n)_{ij} = \delta_{ij}$
- O : all 0; Diagonal: $a_{ij} = 0 (i \neq j)$
- Symmetric: $A = A^T$; Triangular: $a_{ij} = 0$ $(i > j \text{ or } i < j)$

Matrix Inverses

Existence:

- A^{-1} iff $\det A \neq 0$
- $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = I$

Methods:

- Gauss-Jordan: $[\mathbf{A}|\mathbf{I}] \to [\mathbf{I}|\mathbf{A}^{-1}]$
- Adjoint: $A^{-1} = (1/\det A)\,\text{adj}(A)$
- Elementary: $\mathbf{A} = E_k \cdots E_1$

Determinants

Definition:

- Cofactor: $\det \mathbf{A} = \sum_j a_{ij}C_{ij}$; $C_{ij} = (-1)^{i+j}$ minor

Properties:

- $\det AB = \det \mathbf{A} \det \mathbf{B}$
- $\det \mathbf{A}^T = \det \mathbf{A}$
- $\det \mathbf{A}^{-1} = 1/\det \mathbf{A}$
- $\det k\mathbf{A} = k^n \det \mathbf{A}$
- Triangular: $\det \mathbf{A} = a_{11}a_{22}\cdots a_{nn}$

ERO Effects:

- Swap: $\det \to -\det$
- Scale k: $\det \to k\det$
- Add row: det unchanged

Applications:

- Invertibility: $\det \mathbf{A} \neq 0 \Leftrightarrow$ invertible
- Cramer: $x_i = \det \mathbf{A}_i / \det \mathbf{A}$
- Area/vol: $|\det \mathbf{A}|$ scaling

Vector Spaces & Subspaces

Subspace: (i) $0 \in V$, (ii) $\mathbf{u}, \mathbf{v} \in V \Rightarrow \mathbf{u} + \mathbf{v} \in V$, (iii) $c\mathbf{u} \in V$ **Subspaces:**

- $\text{span}\{\mathbf{v}_i\}$
- $\text{nullspace}(\mathbf{A})$
- Row/column space

Independence:

- $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ independent $\Leftrightarrow c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k = 0 \implies c_i = 0$
- Test: Form $[\mathbf{v}_i]$, RREF; all columns pivot $\iff$ independent

Basis:

- Linearly independent, spanning set
- All basis have same size = dimension
- Std: $\{\mathbf{e}_i\}$ $(\dim \mathbb{R}^n = n)$

Rank-Nullity:

- $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = \#\text{cols}$
- rank = dim(row/col space)
- nullity = dim(nullspace)

Key Algorithms (Summary)

- **Gaussian Elimination:** $[\mathbf{A}|\mathbf{b}]$ →REF→back-sub
- **Gauss-Jordan:** $[\mathbf{A}|\mathbf{b}]$ →RREF→solutions
- **Matrix Inverse:** $[\mathbf{A}|\mathbf{I}] \to [\mathbf{I}|\mathbf{A}^{-1}]$
- **Basis:** Row RREF→nonzero rows; Columns→pivot cols; Null→solve $\mathbf{Ax} = 0$

1 Topic 1: Linear Equations and Systems

1.1 Linear Equations and Solution Sets

Definition

Linear Equation: An equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where $a_i, b \in \mathbb{R}$ and not all $a_i = 0$.

Definition

Solution Set: The collection of all solutions to a linear equation, forming a subset $S \subseteq \mathbb{R}^n$.

Key Properties:

- Solution set of $ax + by = c$ in $\mathbb{R}^2$ is a line (unless $a = b = 0$)
- Solution set of $ax + by + cz = d$ in $\mathbb{R}^3$ is a plane (unless $a = b = c = 0$)
- Solution sets can be parameterized using free variables

Vector Operations:

- Addition: $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$
- Scalar multiplication: $k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$

Example

Parameterization Example: The equation $2x_1 + 3x_2 = 6$ has general solution $(3 - \frac{3t}{2}, t)$ where $t \in \mathbb{R}$.
Three-Variable Example: The equation $x + 2y + z = 5$ has general solution $(5 - 2s - t, s, t)$ where $s, t \in \mathbb{R}$.

1.2 Systems of Linear Equations

Definition

System of Linear Equations: A collection of linear equations involving the same variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Definition

Augmented Matrix: The matrix $[\mathbf{A}|\mathbf{b}]$ where $\mathbf{A}$ contains the coefficients and $\mathbf{b}$ the constants.

Solution Classification:

- Consistent: Has at least one solution

- Inconsistent: Has no solutions
- A system has exactly 0, 1, or infinitely many solutions

Theorem

Fundamental Theorem of Linear Systems: Elementary row operations preserve the solution set of a linear system.

Each ERO corresponds to a valid algebraic manipulation: swapping equations, multiplying by non-zero constant, or adding equations.

1.3 Elementary Row Operations and Gaussian Elimination

Definition

Elementary Row Operations (EROs):

1. $R_i \leftrightarrow R_j$: Swap rows i and j
2. kR_i (where $k \neq 0$): Multiply row i by non-zero scalar k
3. $R_i + kR_j$: Add k times row j to row i

Gaussian Elimination Algorithm:

1. Form augmented matrix $[\mathbf{A}|\mathbf{b}]$
2. Use EROs to create zeros below each leading entry
3. Continue until matrix is in row echelon form
4. Apply back-substitution to find solutions

Definition

Row Echelon Form (REF):

1. All zero rows are at the bottom
2. The leading entry of each non-zero row is to the right of the leading entry of the row above
3. All entries below a leading entry are zero

Example

REF Example: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Back-substitution: From bottom up: $x_3 = 2$, $x_2 = -1-2(2) = -5$, $x_1 = 3-2(-5)+2 = 15$

1.4 Reduced Row Echelon Form and Complete Solutions

Definition

Reduced Row Echelon Form (RREF): REF with additional requirements:

1. The leading entry in each non-zero row is 1
2. Each column containing a leading 1 has zeros elsewhere

RREF Uniqueness: Every matrix has a unique reduced row echelon form.
Gauss-Jordan Elimination:

1. Apply Gaussian elimination to reach REF
2. Make all leading entries equal to 1
3. Use EROs to create zeros above each leading entry
4. Result is RREF; solutions can be read directly

Key Properties:

- Solution set of $ax + by = c$ in $\mathbb{R}^2$ is a line (unless $a = b = 0$)
- Solution set of $ax + by + cz = d$ in $\mathbb{R}^3$ is a plane (unless $a = b = c = 0$)
- Solution sets can be parameterized using free variables

Vector Operations:

- Addition: $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$
- Scalar multiplication: $k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$

RREF with Parameters:

Consider the system depending on parameter k:

$$\begin{cases} x + 2y = 3 \\ 2x + ky = 6 \end{cases}$$

Analysis:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & k-4 & 0 \end{bmatrix}$$

- If $k \neq 4$: Unique solution $(3, 0)$
- If $k = 4$: Infinitely many solutions $(3 - 2t, t)$

Example

RREF with Parameters: Consider the system depending on parameter k:

$$\begin{cases} x + 2y = 3 \\ 2x + ky = 6 \end{cases}$$

Analysis: $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & k & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & k-4 & 0 \end{array}\right]$

-
- If $k \neq 4$: Unique solution $(3, 0)$
- If $k = 4$: Infinitely many solutions $(3 - 2t, t)$

1.5 Homogeneous Systems

Definition

Homogeneous System: A system of the form $\mathbf{Ax} = \mathbf{0}$ where all constant terms are zero.

Theorem

Trivial Solution Existence: Every homogeneous system has the trivial solution $\mathbf{x} = \mathbf{0}$.

Direct verification: $\mathbf{A} \cdot \mathbf{0} = \mathbf{0}$ by properties of matrix multiplication.

Theorem

Solution Structure: A homogeneous system has either:

1. Only the trivial solution (if $\text{rank}(\mathbf{A}) = n$)
2. Infinitely many solutions including non-trivial ones (if $\text{rank}(\mathbf{A}) < n$)

Geometric Interpretation:

- In $\mathbb{R}^2$: Lines through origin
- In $\mathbb{R}^3$: Planes through origin
- Solution space always contains the origin

Example

Non-trivial Solutions: Consider $\begin{cases} x + y + z = 0 \\ 2x + 2y + 2z = 0 \end{cases}$

RREF: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

General solution: $(-s - t, s, t)$ where $s, t \in \mathbb{R}$. Infinitely many non-trivial solutions exist.

Key Insight: If homogeneous system has more variables than equations ($n > m$), then non-trivial solutions always exist.

2 Topic 2: Matrix Theory and Operations

2.1 Matrix Fundamentals

Definition

Matrix: A rectangular array of numbers arranged in rows and columns. An $m \times n$ matrix has m rows and n columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Matrix Equality: Two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal if they have the same dimensions and $a_{ij} = b_{ij}$ for all i, j.

Special Matrices:

- **Zero Matrix O:** All entries are 0
- **Identity Matrix $\mathbf{I}_n$:** $(I_n)_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
- **Diagonal Matrix:** $a_{ij} = 0$ for $i \neq j$
- **Upper Triangular:** $a_{ij} = 0$ for $i > j$
- **Lower Triangular:** $a_{ij} = 0$ for $i < j$
- **Symmetric:** $\mathbf{A} = \mathbf{A}^T$ (i.e., $a_{ij} = a_{ji}$)

2.2 Matrix Operations

Definition

- **Matrix Addition:** For matrices of the same size: $(\mathbf{A} + \mathbf{B})_{ij} = a_{ij} + b_{ij}$
- **Scalar Multiplication:** $(k\mathbf{A})_{ij} = ka_{ij}$ for scalar k
- **Matrix Transpose:** $(\mathbf{A}^T)_{ij} = a_{ji}$ (rows become columns)
- **Matrix Trace:** $\text{tr}(\mathbf{A}) = a_{11} + a_{22} + \cdots + a_{nn}$ (sum of diagonal entries, only for square matrices)

Properties of Matrix Operations:

- $(A + B) + C = A + (B + C)$ (associative)
- $A + B = B + A$ (commutative)
- $A + O = O + A = A$ (additive identity)
- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(kA)^T = kA^T$

- $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$
- $\text{tr}(kA) = k\,\text{tr}(A)$

Example

Basic Operations: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$, $2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $\text{tr}(A) = 5$

2.3 Matrix Multiplication

Definition

Matrix Multiplication: For $\mathbf{A}_{m\times p}$ and $\mathbf{B}_{p\times n}$:

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

The number of columns in $\mathbf{A}$ must equal the number of rows in $\mathbf{B}$.

Dimensional Analysis: $A_{m\times p} \cdot B_{p\times n} = C_{m\times n}$
Properties of Matrix Multiplication:

- $(AB)C = A(BC)$ (associative)
- $A(B+C) = AB + AC$ (left distributive)
- $(A+B)C = AC + BC$ (right distributive)
- $\mathbf{I}_n\mathbf{A} = \mathbf{A}\mathbf{I}_m = \mathbf{A}$ (identity property)
- $AB \neq BA$ in general (not commutative)
- $(AB)^T = B^T A^T$
- $A \cdot O = O \cdot A = O$

Example

Multiplication Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1(5)+2(7) & 1(6)+2(8) \\ 3(5)+4(7) & 3(6)+4(8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

Non-commutativity: Note that $BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix} \neq AB$

Theorem

Linear Combination Property: If $\mathbf{v}_1$ and $\mathbf{v}_2$ are solutions to the homogeneous system $\mathbf{Ax} = \mathbf{0}$, then any linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is also a solution.

$\mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1(\mathbf{A}\mathbf{v}_1) + c_2(\mathbf{A}\mathbf{v}_2) = c_1\mathbf{0} + c_2\mathbf{0} = \mathbf{0}$

2.4 Elementary Matrices

Definition

Elementary Matrix: A square matrix obtained from an identity matrix by performing a single elementary row operation.

Types of Elementary Matrices:

- **Type I**: Row swap $R_i \leftrightarrow R_j$
- **Type II**: Row scaling kR_i (where $k \neq 0$)
- **Type III**: Row addition $R_i + kR_j$

Theorem

Fundamental Fact about Elementary Matrices: If E is an elementary matrix, then $E\mathbf{A}$ is the matrix obtained by performing the corresponding elementary row operation on $\mathbf{A}$.

Direct verification by examining how matrix multiplication implements each row operation.

Theorem

Elementary Matrices are Invertible: Every elementary matrix has an inverse that is also elementary.

Elementary Matrix Examples:

- **Row swap:**

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_1^{-1} = E_1$$

- **Row scaling:**

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Row addition:**

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.5 Matrix Invertibility

Definition

Invertible Matrix: A square matrix $\mathbf{A}$ is invertible if there exists a matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Characterizations of Invertibility

For an $n \times n$ matrix $\mathbf{A}$, the following are equivalent:

1. $\mathbf{A}$ is invertible
2. $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution
3. $\text{rref}\mathbf{A} = \mathbf{I}$
4. $\mathbf{A}$ can be written as a product of elementary matrices

Gauss-Jordan Method:

1. Form the augmented matrix $[\mathbf{A}|\mathbf{I}]$
2. Use elementary row operations to transform it to $[\mathbf{I}|\mathbf{B}]$
3. If successful, then $\mathbf{B} = \mathbf{A}^{-1}$
4. If $\mathbf{A}$ cannot be reduced to $\mathbf{I}$, then $\mathbf{A}$ is not invertible

Properties of Matrix Inverses:

- If $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is unique
- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$
- $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1} \quad (\text{for } k \neq 0)$

Example

Computing Matrix Inverse: Find $\mathbf{A}^{-1}$ where $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right]$$

Therefore, $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Verification: $\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

3 Topic 3: Determinants

3.1 Mathematical Induction (Prerequisite)

Definition

Mathematical Induction: A proof technique for statements involving natural numbers:

1. **Base case:** Prove $P(1)$ is true
2. **Induction hypothesis:** Assume $P(k)$ is true for some $k \geq 1$
3. **Induction step:** Prove $P(k+1)$ is true
4. **Conclusion:** $P(n)$ is true for all $n \geq 1$

Example

Classic Example: Prove $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
Base case: $n = 1$: $1 = \frac{1(2)}{2} = 1$
Induction step: Assume true for k: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$
For $k+1$: $1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

3.2 Definition and Basic Properties

Definition

Minor: For an $n \times n$ matrix $\mathbf{A}$, the (i, j)-minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j.

Definition

Cofactor: The (i, j)-cofactor is $C_{ij} = (-1)^{i+j} M_{ij}$.

Definition

Determinant (Cofactor Expansion): For an $n \times n$ matrix $\mathbf{A}$:

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{ij} C_{ij} \text{ (expansion along row } i\text{)}$$

$$\det \mathbf{A} = \sum_{i=1}^{n} a_{ij} C_{ij} \text{ (expansion along column } j\text{)}$$

Base Cases:

- 1×1: $\det [a] = a$
- 2×2: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

Theorem

Cofactor Expansion Invariance: The determinant can be computed using cofactor expansion along any row or column, yielding the same result.

Example

3×3 Determinant: $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Expanding along row 1: $= 1\det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2\det\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3\det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} = 1(45-48) - 2(36-42) + 3(32-35) = -3 + 12 - 9 = 0$

3.3 Properties of Determinants

Six Elementary Properties of Determinants

Let $\mathbf{A}$ be an $n \times n$ matrix:

1. $\det \mathbf{A}^T = \det \mathbf{A}$ (transpose invariance)
2. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by swapping two rows, then $\det \mathbf{B} = -\det \mathbf{A}$
3. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by multiplying a row by scalar k, then $\det \mathbf{B} = k \det \mathbf{A}$
4. If $\mathbf{A}$ has two identical rows, then $\det \mathbf{A} = 0$
5. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by adding a multiple of one row to another, then $\det \mathbf{B} = \det \mathbf{A}$
6. Determinant is linear in each row (multilinear property)

Theorem

Triangular Matrix Determinant: If $\mathbf{A}$ is upper or lower triangular, then $\det \mathbf{A} = a_{11}a_{22}\cdots a_{nn}$ (product of diagonal entries).

Example

Effects of Row Operations: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, so $\det \mathbf{A} = -2$

- Row swap: $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ has determinant $-(-2) = 2$
- Scale row: $\begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}$ has determinant $2(-2) = -4$
- Add rows: $\begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$ has determinant -2 (unchanged)

3.4 Alternative Definitions

Definition

Permutation Definition: For an $n \times n$ matrix $\mathbf{A}$:

$$\det \mathbf{A} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where S_n is the set of all permutations of $\{1, 2, \dots, n\}$ and $\operatorname{sgn}(\sigma)$ is the sign of permutation σ.

Computational Complexity:

- Cofactor expansion: $O(n!)$ operations
- Using row operations: $O(n^3)$ operations
- Direct formula impractical for $n \geq 4$

Example

Permutation Formula for 3×3: $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - afh - bdi$

This corresponds to the 6 permutations: $(1, 2, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1), (1, 3, 2), (2, 1, 3)$ with signs $+, +, +, -, -, -$ respectively.

3.5 Major Theorems and Applications

Theorem 1: Determinant Detects Invertibility

An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$.

($\Rightarrow$) If $\mathbf{A}$ invertible, then $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, so $\det \mathbf{A} \det \mathbf{A}^{-1} = \det \mathbf{I} = 1$, thus $\det \mathbf{A} \neq 0$. ($\Leftarrow$) If $\det \mathbf{A} \neq 0$, then $\operatorname{rref}\mathbf{A} = \mathbf{I}$ (since triangular form would have non-zero diagonal product), so $\mathbf{A}$ is invertible.

Theorem 2: Multiplicativity of Determinants

For $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$: $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$

Using elementary matrix factorization: if $\mathbf{A} = E_k \cdots E_1$, then $\det \mathbf{AB} = \det E_k \cdots \det E_1 \det \mathbf{B} = \det \mathbf{A} \det \mathbf{B}$.

Important Corollaries:

- $\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}$ (when $\mathbf{A}$ is invertible)
- $\det k\mathbf{A} = k^n \det \mathbf{A}$ for $n \times n$ matrix
- $\det \mathbf{A}^k = (\det \mathbf{A})^k$

3.6 Computational Methods

Efficient Determinant Calculation

1. Use row operations to transform matrix to triangular form
2. Keep track of sign changes from row swaps
3. Keep track of scaling factors from row multiplications
4. Compute product of diagonal entries
5. Apply accumulated sign and scale factors

Strategic Tips:

- Expand along row/column with most zeros
- Use row operations to create more zeros before expanding
- For larger matrices, reduce to triangular form

Example

Efficient Calculation: $\det \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$

Step 1: $R_3 - 2R_1$ gives $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -5 \end{bmatrix}$ (determinant unchanged)

Step 2: Product of diagonal $= 2(-1)(-5) = 10$

3.7 Applications

Definition

Adjoint Matrix: The adjoint (or adjugate) of an $n \times n$ matrix $\mathbf{A}$ is $\text{adj}(\mathbf{A}) = [C_{ji}]$ where C_{ij} are the cofactors.

Adjoint Formula for Inverse: If $\det \mathbf{A} \neq 0$, then $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj}(\mathbf{A})$
Fundamental Identity: $\mathbf{A} \cdot \text{adj}(\mathbf{A}) = \text{adj}(\mathbf{A}) \cdot \mathbf{A} = \det \mathbf{A} \cdot \mathbf{I}$

Definition

Cramer's Rule: For the system $\mathbf{Ax} = \mathbf{b}$ where $\det \mathbf{A} \neq 0$:

$$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}}$$

where $\mathbf{A}_i$ is the matrix obtained by replacing column i of $\mathbf{A}$ with $\mathbf{b}$.

Example

Cramer's Rule Example: Solve $\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$

$\det \mathbf{A} = \det \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = -3$

$x = \frac{\det \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}}{\det \mathbf{A}} = \frac{-6}{-3} = 2$

$y = \frac{\det \begin{bmatrix} 2 & 5 \\ 1 & 1 \end{bmatrix}}{\det \mathbf{A}} = \frac{-3}{-3} = 1$

Geometric Applications:

- Area of parallelogram $= |\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}|$
- Volume of parallelepiped $= |\det \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}|$
- Determinant measures how linear transformation scales areas/volumes

4 Topic 4: Vector Spaces and Subspaces

4.1 Introduction to Subspaces

Definition

Subspace: A subset $V \subseteq \mathbb{R}^n$ is a subspace if it satisfies three axioms:

1. **Non-empty:** $\mathbf{0} \in V$
2. **Closed under addition:** If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$
3. **Closed under scalar multiplication:** If $\mathbf{v} \in V$ and $c \in \mathbb{R}$, then $c\mathbf{v} \in V$

Theorem

Zero Vector Theorem: Every subspace contains the zero vector.

For any $\mathbf{v} \in V$ and scalar $c = 0$: $0\mathbf{v} = \mathbf{0} \in V$ by closure under scalar multiplication.

Theorem

Intersection Theorem: If V_1 and V_2 are subspaces of $\mathbb{R}^n$, then $V_1 \cap V_2$ is also a subspace.

Verify three axioms: (i) $\mathbf{0} \in V_1$ and $\mathbf{0} \in V_2$, so $\mathbf{0} \in V_1 \cap V_2$ (ii) If $\mathbf{u}, \mathbf{v} \in V_1 \cap V_2$, then $\mathbf{u} + \mathbf{v} \in V_1$ and $\mathbf{u} + \mathbf{v} \in V_2$ (iii) Similar argument for scalar multiplication.

Geometric Interpretation:

- In $\mathbb{R}^2$: Lines through origin
- In $\mathbb{R}^3$: Lines and planes through origin
- Always contains origin (distinguishes from general lines/planes)

Example

Subspace Verification: Show $V = \{(x, y, z) : x + 2y - z = 0\}$ is a subspace of $\mathbb{R}^3$
(i) $\mathbf{0} = (0, 0, 0)$: $0 + 2(0) - 0 = 0$ (ii) Let $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in V$: $(u_1 + v_1) + 2(u_2 + v_2) - (u_3 + v_3) = (u_1 + 2u_2 - u_3) + (v_1 + 2v_2 - v_3) = 0 + 0 = 0$ (iii) For $c\mathbf{u} = (cu_1, cu_2, cu_3)$: $cu_1 + 2(cu_2) - cu_3 = c(u_1 + 2u_2 - u_3) = c(0) = 0$
Counterexample: $W = \{(x, y) : x + y = 1\}$ is NOT a subspace because $\mathbf{0} = (0, 0) \notin W$ since $0 + 0 = 0 \neq 1$.

Theorem

Homogeneous Solution Spaces: The solution set of any homogeneous linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

Let $S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{0}\}$ (i) $\mathbf{A}\mathbf{0} = \mathbf{0}$, so $\mathbf{0} \in S$ (ii) If $\mathbf{u}, \mathbf{v} \in S$, then $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ (iii) If $\mathbf{v} \in S$ and $c \in \mathbb{R}$, then $\mathbf{A}(c\mathbf{v}) = c(\mathbf{A}\mathbf{v}) = c\mathbf{0} = \mathbf{0}$

4.2 Linear Combinations and Span

Linear Combination: A linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is an expression of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ where $c_i \in \mathbb{R}$.

Span: The span of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is the set of all their linear combinations:

$$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_i \in \mathbb{R}\}$$

Theorem

Span is a Subspace: For any finite set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, their span is a subspace.

Let $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$:

- $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_k \in V$
- If $\mathbf{u} = \sum a_i\mathbf{v}_i$ and $\mathbf{w} = \sum b_i\mathbf{v}_i$ are in V, then $\mathbf{u} + \mathbf{w} = \sum(a_i + b_i)\mathbf{v}_i \in V$
- If $\mathbf{u} = \sum a_i\mathbf{v}_i \in V$ and $c \in \mathbb{R}$, then $c\mathbf{u} = \sum(ca_i)\mathbf{v}_i \in V$

Definition

Spanning Set: A set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans a subspace V if $V = \text{span}(S)$, i.e., every vector in V can be written as a linear combination of vectors in S.

Theorem

Spanning Set Comparison: If $S \subseteq \text{span}(T)$, then $\text{span}(S) \subseteq \text{span}(T)$.

Testing Span Membership:

1. Form the augmented matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \cdots \;\; \mathbf{v}_k \mid \mathbf{w}]$
2. Apply row operations to reach REF
3. $\mathbf{w} \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ if and only if the system is consistent

Example

Span Membership: Is $(2, -1, 3) \in \text{span}\{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$?
Set up the equation:

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

The system is consistent (no row $[0 \;\; 0 \;\; 0 \;\; | \;\; c]$ for $c \neq 0$), so $(2, -1, 3) \in \text{span}\{(1, 0, 1), (0, 1, 1), (1, 1, 0)\}$.
Solution: $c_1 = 0$, $c_2 = 1$, $c_3 = 2$:

$$(2, -1, 3) = 0(1, 0, 1) + 1(0, 1, 1) + 2(1, 1, 0)$$

Connection to Matrix Subspaces:

- span(columns of $\mathbf{A}$) is the column space of $\mathbf{A}$
- span(rows of $\mathbf{A}$) is the row space of $\mathbf{A}$

4.3 Linear Independence

Definition

Linear Dependence: Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent if there exist scalars $c_1, c_2, \dots, c_k$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$.
Equivalently, at least one vector can be written as a linear combination of the others.

Definition

Linear Independence: Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent if they are not linearly dependent, i.e., $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ implies $c_1 = c_2 = \cdots = c_k = 0$.

Theorem

Equivalence of Definitions: The two definitions of linear dependence are equivalent.

Characterization of Linear Dependence and Independence:

($\Rightarrow$) If $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ for some $c_i \neq 0$, then $\mathbf{v}_i$ can be written as a linear combination of the other vectors:

$$\mathbf{v}_i = -\frac{1}{c_i}(c_1\mathbf{v}_1 + \cdots + c_{i-1}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_{i+1} + \cdots + c_k\mathbf{v}_k)$$

($\Leftarrow$) If $\mathbf{v}_j = \sum_{i \neq j} d_i\mathbf{v}_i$, then rearranging gives a nontrivial dependence relation:

$$\sum_{i \neq j} d_i\mathbf{v}_i - \mathbf{v}_j = \mathbf{0}$$

Testing Linear Independence:

1. Form the matrix $\mathbf{A} = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \cdots \ \ \mathbf{v}_k]$
2. Solve the homogeneous system $\mathbf{Ac} = \mathbf{0}$
3. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent if and only if the only solution is $\mathbf{c} = \mathbf{0}$ (i.e., all $c_i = 0$), which is equivalent to every column of $\mathbf{A}$ being a pivot column in its RREF.

Example

Independence Test: Are $(1, 2, 1), (2, 1, 0), (1, -1, -1)$ linearly independent in $\mathbb{R}^3$?

Form matrix and find RREF: $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Only 2 pivot columns, so the vectors are linearly dependent. Dependence relation: $c_1(1, 2, 1) + c_2(2, 1, -1) + c_3(1, -1, -1) = \mathbf{0}$ From RREF: $c_1 - c_3 = 0$ and $c_2 + c_3 = 0$, so $c_1 = c_3, c_2 = -c_3$ Choose $c_3 = 1$: $(1, 2, 1) - (2, 1, -1) + (1, -1, -1) = (0, 0, 0)$

Important Facts:

- Any set containing the zero vector is linearly dependent
- More than n vectors in $\mathbb{R}^n$ must be linearly dependent
- Standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ is linearly independent

4.4 Basis and Dimension

Definition

Basis: A set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a basis for subspace V if:

1. $\text{span}\{\mathcal{B}\} = V$ (spanning property)
2. $\mathcal{B}$ is linearly independent (independence property)

A basis is a "minimal" spanning set or "maximal" independent set.

Definition

Standard Basis: The standard basis of $\mathbb{R}^n$ is $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ where $\mathbf{e}_i$ has 1 in position i and 0 elsewhere.

Theorem

Basis Size Invariance: All bases of a finite-dimensional subspace have the same number of vectors.

Definition

Dimension: The dimension of a subspace V, denoted $\dim V$, is the number of vectors in any basis of V.

Standard Dimensions:

- $\dim \{\mathbf{0}\} = 0$ (zero subspace)
- $\dim \mathbb{R}^n = n$
- $\dim \text{line through origin} = 1$
- $\dim \text{plane through origin in } \mathbb{R}^3 = 2$

Finding a Basis from a Spanning Set

Given spanning set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for subspace V:

1. Form matrix $\mathbf{A} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_k]$
2. Find RREF of $\mathbf{A}$
3. The columns of $\mathbf{A}$ corresponding to pivot columns form a basis for V

Example

Basis from Spanning Set: Find basis for $V = \text{span}\{(1,2,1),(2,4,2),(1,1,0),(0,1,1)\}$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns are 1 and 3, so basis is $\{(1,2,1),(1,1,0)\}$ and $\dim V = 2$.

Basis for Solution Spaces: For homogeneous system $\mathbf{Ax} = \mathbf{0}$: 1. Find RREF of $\mathbf{A}$ 2. Identify free variables 3. Express basic variables in terms of free variables 4. Write general solution in parametric form 5. Basis vectors correspond to individual parameters

Example

Null Space Basis: Find basis for null space of $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 \end{bmatrix}$

RREF: $\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Free variables: x_2, x_4 General solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-2t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Basis for null space: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$, $\dim \text{null}(\mathbf{A}) = 2$

4.5 Matrix Subspaces

Definition

Row Space: $\text{row}(\mathbf{A}) = \text{span}\{\text{rows of } \mathbf{A}\} \quad \mathbb{R}^n$ (for mn matrix)
Column Space: $\text{col}(\mathbf{A}) = \text{span}\{\text{columns of } \mathbf{A}\} \quad \mathbb{R}^m$ (for mn matrix)
Null Space: $\text{null}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{0}\} \quad \mathbb{R}^n$

Row Equivalence Preserves Row Space: If matrices $\mathbf{A}$ and $\mathbf{A}'$ are row equivalent, then $\text{row}(\mathbf{A}) = \text{row}(\mathbf{A}')$.

Elementary row operations create linear combinations of existing rows, so new rows are in original row space. Process is reversible, so spaces are equal.

Finding Bases for Matrix Subspaces

For mn matrix $\mathbf{A}$:
Row Space Basis:

1. Find RREF of $\mathbf{A}$
2. Non-zero rows of RREF form basis for row space

Column Space Basis:

1. Find RREF of $\mathbf{A}$
2. Identify pivot columns in RREF
3. Corresponding columns of original matrix $\mathbf{A}$ form basis for column space

Null Space Basis: Solve $\mathbf{A}\mathbf{x} = \mathbf{0}$ and parameterize solutions

Definition

Rank: $\text{rank}(\mathbf{A}) = \dim \text{row}(\mathbf{A}) = \dim \text{col}(\mathbf{A}) =$ number of pivots in RREF
Nullity: $\text{nullity}(\mathbf{A}) = \dim \text{null}(\mathbf{A}) =$ number of free variables

Rank-Nullity Theorem

For any mn matrix $\mathbf{A}$: $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$

Number of pivot columns + number of free variables = total number of columns.

Example

Complete Subspace Analysis:

For $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$,

RREF: $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row Space: $\text{span}\{(1,0,2,-1),\ (0,1,-1,2)\} \subseteq \mathbb{R}^4$, $\dim = 2$

Column Space: $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$, $\dim = 2$

Null Space: $\text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$, $\dim = 2$

Verification: $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 2 + 2 = 4 = n$

Geometric Relationships:

- colspace($\mathbf{A}$) represents all possible outputs of the linear transformation $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$
- nullspace($\mathbf{A}$) represents all inputs that map to zero
- rowspace($\mathbf{A}$) is orthogonal to nullspace($\mathbf{A}$) (advanced topic)

5 Complete Notation Reference

Symbol	Meaning	Context
$\mathbb{R}^n$	Set of all n-tuples of real numbers	Vector spaces
$\mathbf{0}$	Zero vector	All topics
$[A\|b]$	Augmented matrix for system $Ax = b$	Linear systems
rref(A)	Reduced row echelon form of A	Systems, inverses
$R_i \leftrightarrow R_j$	Swap rows i and j	Elementary operations
kR_i	Multiply row i by scalar k	Elementary operations
$R_i + kR_j$	Add k times row j to row i	Elementary operations
A^T	Transpose of matrix A	Matrix theory
tr$(()A)$	Trace of matrix A (sum of diagonal)	Matrix theory
A^{-1}	Inverse of matrix A	Matrix theory
det(A)	Determinant of matrix A	Determinants
M_{ij}	(i, j)-minor of matrix	Determinants
C_{ij}	(i, j)-cofactor: $(-1)^{i+j}M_{ij}$	Determinants
adj(A)	Adjoint (adjugate) matrix	Determinants
span$\{v_1, \ldots, v_k\}$	Span of vectors	Vector spaces
$V \subseteq W$	V is subspace of W	Vector spaces
dim(V)	Dimension of subspace V	Basis and dimension
$\{e_1, \ldots, e_n\}$	Standard basis of $\mathbb{R}^n$	Basis and dimension
null(A)	Null space: $\{x : Ax = 0\}$	Matrix subspaces
row(A)	Row space: span of rows	Matrix subspaces
col(A)	Column space: span of columns	Matrix subspaces
rank(A)	Rank: dimension of row/column space	Matrix subspaces
nullity(A)	Nullity: dimension of null space	Matrix subspaces

Key Algorithms Summary:

- **Gaussian Elimination:** $[A|b] \to$ REF $\to$ back-substitution
- **Gauss-Jordan:** $[A|b] \to$ RREF $\to$ read solutions directly
- **Matrix Inverse:** $[A|I] \to [I|A^{-1}]$ (if possible)
- **Determinant:** Cofactor expansion or row reduction to triangular
- **Basis Finding:** RREF to identify pivot columns/rows
- **Independence Test:** Form matrix, find RREF, check for pivots in all columns