MH1300 Final Exam Solutions

AY 21/22

$\boxed{\text{Q1(a)}}$ Let a, b be arbitrary integers.

Mtd 1: We divide into cases.

Case 1: $a-b$ is odd. Then $a+b = a-b+2b$ = odd + even is odd. Therefore, $(a+b)(a-b)$ = odd × odd = odd.

Therefore, $a^2-b^2 = (a+b)(a-b)$ is odd.

So, $a^2+b^2 = a^2-b^2+2b^2$ = odd + even = odd.

So, $a-b$ and a^2+b^2 have same parity.

Case 2: $a-b$ is even. Then $a+b = a-b+2b$ = even + even = even. Thus, $a^2-b^2 = (a+b)(a-b)$ = even × even = even.

So, $a^2+b^2 = a^2-b^2+2b^2$ = even + even = even.

So, $a-b$ and a^2+b^2 have same parity.

Q 1(a)

Mtd 2 Divide into four cases:

Case 1: a even, b even

Case 2: a even, b odd

Case 3: a odd, b even

Case 4: a odd, b odd

In each case, determine the parity of $a-b$ and a^2+b^2, and conclude they have the same parity.

Mtd 3: Since $\text{even}^2 = \text{even}$ and $\text{odd}^2 = \text{odd}$,

See that a and a^2 have same parity,

and $-b$ and b^2 have same parity.

Hence, $a-b$ and a^2+b^2 have the same parity.

Q1(b)

Mtd 1 : Using truth tables

p	q	r	$q \wedge \neg r$	$p \to (q \wedge \neg r)$	$\neg q \to \neg p$	$[p \to (q \wedge \neg r)] \to [\neg q \to \neg p]$
T	T	T	F	F	T	T
T	T	F	T	T	T	T
T	F	T	F	F	F	T
T	F	F	F	F	F	T
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

all values of the out put column is true, so it is a tautology.

Mtd 2 : Using logical equivalences

$[p \to (q \wedge \neg r)] \to (\neg q \to \neg p)$ [Using $a \to b \equiv \neg a \vee b$]

$\equiv [\neg p \vee (q \wedge \neg r)] \to [\neg\neg q \vee \neg p]$ [using $a \to b \equiv \neg a \vee b$]

$\equiv \neg [\neg p \vee (q \wedge \neg r)] \vee [\neg\neg q \vee \neg p]$ [distributive law]

$\boxed{\text{Q1(b)}}$

$\equiv \neg[(\neg p \vee q) \wedge (\neg p \vee \neg r)] \vee [\neg\neg q \vee \neg p]$ [De Morgan's Law]

$\equiv [\neg(\neg p \vee q) \vee \neg(\neg p \vee \neg r)] \vee [\neg\neg q \vee \neg p]$ [Associative law]

$\equiv \neg(\neg p \vee q) \vee [\neg(\neg p \vee \neg r) \vee (\neg\neg q \vee \neg p)]$ [Commutative law]

$\equiv \neg(\neg p \vee q) \vee [(\neg\neg q \vee \neg p) \vee \neg(\neg p \vee \neg r)]$ [associative law]

$\equiv [\neg(\neg p \vee q) \vee (\neg\neg q \vee \neg p)] \vee \neg(\neg p \vee \neg r)$ [Double Negation]

$\equiv [\neg(\neg p \vee q) \vee (q \vee \neg p)] \vee \neg(\neg p \vee \neg r)$ [Commutative law]

$\equiv [\neg(\neg p \vee q) \vee (\neg p \vee q)] \vee \neg(\neg p \vee \neg r)$ [Negation law]

$\equiv T \vee \neg(\neg p \vee \neg r)$ [Universal Bound law]

$\equiv T$

So, $[p \rightarrow (q \wedge \neg r)] \rightarrow [\neg q \rightarrow \neg p]$ is a tautology.

$\boxed{\text{Q1(c)}}$ There are many possible answers:

* $s \to (t \to t) \equiv \neg s \vee (t \to t)$ [using $a \to b \equiv \neg a \vee b$]

$\equiv \neg s \vee (\neg t \vee t)$ [using $a \to b \equiv \neg a \vee b$]

$\equiv \neg s \vee T$ [Negation law]

$\equiv T$ [Universal Bound]

* $s \to (t \to s) \equiv \neg s \vee (\neg t \vee s)$ [$a \to b \equiv \neg a \vee b$]

$\equiv \neg s \vee (s \vee \neg t)$ [Commutative law]

$\equiv (\neg s \vee s) \vee \neg t$ [Associative law]

$\equiv T \vee \neg t$ [Negation law]

$\equiv T$ [Universal Bound]

$$\left.\begin{array}{l} * \ (s \to s) \to (t \to t) \\ * \ (s \to t) \to (s \to t) \end{array}\right\} \text{can also show tautology.}$$

$\boxed{Q2(a)}$ False. Take $a=4$, $b=9$. Then a, b are composite, since $a, b > 1$ and $a = 2 \times 2$, $b = 3 \times 3$

but $a+b = 13$ is prime.

$\boxed{Q2(b)}$ False. Take $c = d = 4$ and $e = 8$

Then c, d, e are positive integers and

$c | e$ is true (since $4 \cdot 2 = 8$)

$d | e$ is true (since $4 \cdot 2 = 8$)

and $c \neq e$ and $d \neq e$ and $c \cdot d = 16$ does not divide $8 = e$.

$\boxed{Q2(c)}$ True. let $A \subseteq B$.

let $(a,b) \in A \times A$. Then, $a \in A$ and $b \in A$.

Since $A \subseteq B$, so, $a \in B$ and $b \in B$.

So, $(a,b) \in B \times B$.

Hence, $A \times A \subseteq B \times B$.

$\boxed{Q2(d)}$ Take $C = \{0\}$, $D = E = \phi$.

Then $(C \cup D) \cap E = (\{0\} \cup \phi) \cap \phi = \phi$

$C \cup (D \cap E) = C \cup \phi = C = \{0\}$.

So, $(C \cup D) \cap E \neq C \cup (D \cap E)$

Q3(a) Notice that $F(a,b)$ is defined recursively in the second input b. So, we prove by induction

$$\forall c \in \mathbb{N}, \text{ if } c \geq 0 \text{ then } \underbrace{F(F(a,b),c) = F(a, F(b,c))}_{P(c)}$$

where $a, b \in \mathbb{N}$ are fixed.

$P(0)$: $F(F(a,b),0) = F(a,b) = F(a, F(b,0))$

Fix $k \geq 0$ and assume $P(k)$, i.e. $F(F(a,b),c) = F(a, F(b,c))$

$$\begin{aligned} F(F(a,b), c+1) &= F(F(a,b),c)+1 && (\text{def of } F) \\ &= F(a, F(b,c))+1 && (\text{by IH}) \\ &= F(a, F(b,c)+1) && (\text{def of } F) \\ &= F(a, F(b,c+1)) && (\text{def of } F) \end{aligned}$$

By math induction, $F(F(a,b),c) = F(a, F(b,c))$ for all $a, b, c \in \mathbb{N}$.

Q3(b) Let $P(n)$: $4^{n+1} + 5^{2n-1}$ is divisible by 21. 8

$a = 1$.

Basis step: when $n=1$, $4^{1+1} + 5^{2\times1-1} = 4^2 + 5^1$

$= 16+5 = 21$, which is divisible by 21.

So, $P(1)$ is true.

Inductive step: Fix $k \geq 1$ and assume $P(k)$ is true,

i.e. $4^{k+1} + 5^{2k-1} = 21x$ for some $x \in \mathbb{Z}$.

Now, $4^{(k+1)+1} + 5^{2(k+1)-1} = 4 . 4^{k+1} + 5^{2k-1+2}$

$= 4 . 4^{k+1} + 25 . 5^{2k-1}$

$= 4 \cdot 4^{k+1} + 4 . 5^{2k-1} + 21 . 5^{2k-1}$

$= 4 (4^{k+1} + 5^{2k-1}) + 21 . 5^{2k-1}$

(by IH) $= 4 . 21x + 21 . 5^{2k-1}$

$= 21 (4x + 5^{2k-1})$.

Since $k \geq 1$, $5^{2k-1} \in \mathbb{Z}$ and so $4x + 5^{2k-1} \in \mathbb{Z}$.
So, 21 divides $4^{(k+1)+1} + 5^{2(k+1)-1}$, and $P(k+1)$ true

By MI, $P(n)$ is true for all $n > 0$

$\boxed{\text{Q4(a)}}$ Let a, b, c, d be integers such that

$d \mid a$ and $d \mid b$ and $d \mid c$.

By the definition of divisibility, let $x, y, z \in \mathbb{Z}$

such that $dx = a$, $dy = b$ and $dz = c$.

Then $ab + ac + bc = (dx)(dy) + (dx)(dz) + (dy)(dz)$

$= d^2xy + d^2xz + d^2yz$

$= d^2(xy + xz + yz)$.

Since $xy + xz + yz \in \mathbb{Z}$, so, we conclude that

$d^2 \mid ab + ac + bc$.

$\boxed{\text{Q4(b)}}$ Let $r \in \mathbb{Q}$ such that $r \neq 0$.

let $x = r\sqrt{2}$ and $y = \frac{1}{\sqrt{2}}$.

Then x is irrational as it is the product of a $\overbrace{\text{nonzero rational number}}^{r}$ with an $\overbrace{\text{irrational}}^{\sqrt{2}}$ number.

Also, $y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2} \cdot \sqrt{2}$ is also irrational as it is also the product of a $\underbrace{\text{non zero rational number}}_{\frac{1}{2}}$ with an $\underbrace{\text{irrational}}_{\sqrt{2}}$.

we can express $r = xy$.

Q5 (a) Let A, B be sets and $f: A \to B$.

let $X \subseteq A$, $Y \subseteq B$.

$f(f^{-1}(Y)) \subseteq Y$: Let $y \in f(f^{-1}(Y))$.

Then $y = f(x)$ for some $x \in f^{-1}(Y)$.

Since $x \in f^{-1}(Y)$, we know that $f(x) \in Y$.

So, $y = f(x) \in Y$.

$X \subseteq f^{-1}(f(X))$: Let $x \in X$. Therefore, $f(x) \in f(X)$

Since $f(x) \in f(X)$, this means that

$x \in f^{-1}(f(X))$.

Note: The relevant definitions used here are:

$$f(X) = \{ f(a) \in B \mid a \in X \}$$

$$f^{-1}(Y) = \{ a \in A \mid f(a) \in Y \}$$

Q 5(b) Let $g: \mathbb{Z} \to \mathbb{Z}$ be $g(n) = 2n \bmod 3$

Then for each $n \in \mathbb{Z}$, $2n = 3(2n \operatorname{div} 3) + (2n \bmod 3)$

where $0 \leq 2n \bmod 3 < 3$.

Hence $g(n) = 0, 1$ or 2. So g is not surjective,

for example, $g(n) \neq 3$ for all $n \in \mathbb{Z}$.

So we see that $\text{range}(g) \subseteq \{0,1,2\}$.

Now we show $\{0,1,2\} \subseteq \text{range}(g)$.

$$\left.\begin{aligned} g(0) &= 0 \bmod 3 = 0 \\ g(1) &= 2 \bmod 3 = 2 \\ g(2) &= 4 \bmod 3 = 1 \end{aligned}\right\} \text{ So, } 0,1,2 \in \text{range}(g).$$

$\therefore \text{range}(g) = \{0,1,2\}$.

g is not injective since $0 \neq 3$ but $g(0) = 0 \bmod 3 = 0$

and $g(3) = 3 \bmod 3 = 0$.

So $g(0) = g(3)$.

Q6(a)

Solve $z^6 = 1 = e^{i0} \rightarrow r=1,\ \theta=0.$

$$z = r^{\frac{1}{n}} e^{i\frac{\theta+2k\pi}{n}}, \quad n=6,\ \theta=0,\ k=0,1,2,\dots,5$$
$$r=1$$

↑
general formula
for n^{th} root

$$z = e^{i0},\ e^{i\frac{2\pi}{6}},\ e^{i\frac{4\pi}{6}},\ e^{i\frac{6\pi}{6}},\ e^{i\frac{8\pi}{6}},\ e^{i\frac{10\pi}{6}}$$
$$= 1,\ e^{i\frac{\pi}{3}},\ e^{i\frac{2\pi}{3}},\ e^{i\pi},\ e^{i\frac{4\pi}{3}},\ e^{i\frac{5\pi}{3}}.$$

Q6(b)

$\neg p \vee \neg q$
$\neg r \rightarrow (p \wedge q)$
$\neg r \vee s$
$s \rightarrow (t \wedge u)$
$\therefore\ u$

Argument:

$\neg p \vee \neg q$	(Premise #1)
$\neg (p \wedge q)$	(De Morgan's Law)
$\neg r \rightarrow (p \wedge q)$	(Premise #2)
$\neg\neg r$	(Modus Tollens)
$\neg r \vee s$	(Premise #3)
s	(Elimination)
$s \rightarrow (t \wedge u)$	(Premise #4)
$t \wedge u$	(Modus Ponens)
u	(Specialisation)

Q7(a) $(x_1, x_2)\ R\ (x_3\ x_4)$ iff $x_i = x_j$ for some $i \neq j$. 13

<u>reflexive</u>: Let $(x, y) \in \mathbb{R}^2$. Then as $x = x$,

$(x, y)\ R\ (x, y)$ holds.

<u>Symmetric</u>: Let $(x_1, x_2)\ R\ (x_3, x_4)$. Then $x_i = x_j$ for some $i \neq j$,

$i, j = 1, 2, 3$ or 4.

Then $x_i = x_j$ for some $i \neq j$, $i, j = 3, 4, 1$ or 2.

So, $(x_3, x_4)\ R\ (x_1, x_2)$.

<u>Transitive</u>: Not transitive.

Example, $(1, 2)\ R\ (1, 3)$ since $1 = 1$

and, $(1, 3)\ R\ (4, 3)$ since $3 = 3$

but $(1, 2)\ \not R\ (4, 3)$ since $1, 2, 4, 3$ are distinct.

Another example, $(0, 0)\ R\ (x, y)$ holds for every $(x, y) \in \mathbb{R}^2$.

But we can certainly pick two pairs unrelated, so for instance, $(1, 2)\ R\ (0, 0)$

$(0, 0)\ R\ (3, 4)$

but $(1, 2)\ \not R\ (3, 4)$.

$\boxed{Q7(b)}$ Suppose T is reflexive on A

s.t. $\forall x, y, z \in A$, if xTy & $xTz \Rightarrow y = z$.

To show T is an equiv. relation, we need to show

T is symmetric and T is transitive.

T symmetric: let $x, y \in A$ s.t. xTy.

Since T is reflexive, xTx. Hence, $x = y$.

So, yTx is true.

T transitive: let $x, y, z \in A$ s.t. xTy & yTz.

Since T is reflexive, xTx and yTy.

So, $x = y$ and $y = z$. Therefore, xTz.

If $x \in A$ then $[x] = \{x\}$, because if $y \in [x]$

then xTy. But as T is reflexive, xTx and

hence $x = y$. So the equivalence classes of

T are $\{x\}$ for each $x \in A$.