MH1300 Foundations of Mathematics

Final Exam Solutions

AY 24/25

Q1(a) Let n be an integer. Suppose that n isn't divisible by 5. There is an integer q such that $n = 5q,\ 5q+1,\ 5q+2,\ 5q+3$ or $5q+4$

Since n isn't divisible by 5, the first case isn't possible.

We look at $n^4 - 1 = (n^2+1)(n^2-1) = (n^2+1)(n+1)(n-1)$

Case 1 $n = 5q+1$: Then
$$n^4 - 1 = (n^2+1)(n+1)(5q+1-1)$$
$$= 5q(n^2+1)(n+1)$$

Case 2 $n = 5q+2$: Then
$$n^4 - 1 = ((5q+2)^2+1)(n+1)(n-1)$$
$$= (25q^2 + 20q + 4 + 1)(n+1)(n-1)$$
$$= 5(5q^2 + 4q + 1)(n+1)(n-1)$$

Case 3 $n = 5q+3$: Then
$$n^4 - 1 = ((5q+3)^2+1)(n+1)(n-1)$$
$$= (25q^2 + 30q + 9 + 1)(n+1)(n-1)$$
$$= 5(5q^2 + 6q + 2)(n+1)(n-1)$$

Case 4 $n = 5q+4$: Then
$$n^4 - 1 = (n^2+1)(5q+4+1)(n-1)$$
$$= 5(q+1)(n^2+1)(n-1)$$

In all cases, $n^4 - 1$ is divisible by 5.

$\boxed{\text{Q1(b)}}$ Let $a, b, d > 1$ be integers.

Suppose that $a \equiv b \pmod d$. Then $d \mid (b-a)$

So there is some integer K such that $K \cdot d = b - a$

Then $Kd(b+a) = (b-a)(b+a) = b^2 - a^2$

Therefore, $d \mid (b^2 - a^2)$

Thus, $a^2 \equiv b^2 \pmod d$.

$\boxed{\text{Q1(c)}}$ They are logically equivalent.

$$
\begin{aligned}
(p \to q) \to (p \wedge r) &\equiv \neg(p \to q) \vee (p \wedge r) && \text{(Conditional rule)} \\
&\equiv \neg(\neg p \vee q) \vee (p \wedge r) && \text{(Conditional rule)} \\
&\equiv (\neg\neg p \wedge \neg q) \vee (p \wedge r) && \text{(De Morgan's Law)} \\
&\equiv (p \wedge \neg q) \vee (p \wedge r) && \text{(Double Negation Law)} \\
&\equiv p \wedge (\neg q \vee r) && \text{(Distributive Law)} \\
&\equiv p \wedge (q \to r) && \text{(Conditional rule)}
\end{aligned}
$$

You may also prove it using a truth table.

Q2(a) Let n, m be distinct positive integers.

Then $n \geq 1$ and $m \geq 1$.

Case 1: $n = 1$. Then since $n \neq m$, we have $m \geq 2$.

Thus $\frac{1}{n} + \frac{1}{m} = \frac{1}{1} + \frac{1}{m} \leq 1 + \frac{1}{2} = \frac{3}{2}$.

and since $\frac{1}{m} > 0$, we also have $\frac{1}{n} + \frac{1}{m} = 1 + \frac{1}{m} > 1$

Thus $1 < \frac{1}{n} + \frac{1}{m} \leq \frac{3}{2}$, hence $\frac{1}{n} + \frac{1}{m}$ is not an integer.

Case 2: $m = 1$: Then we argue exactly same as Case 1.

Case 3: $n = 2$: If $m = 1$ then we apply Case 2. Since $n \neq m$, we have $m \neq 2$. So we assume $m \geq 3$.

Then we have $\frac{1}{n} + \frac{1}{m} = \frac{1}{2} + \frac{1}{m} \leq \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

So $0 < \frac{1}{n} + \frac{1}{m} \leq \frac{5}{6}$, hence $\frac{1}{n} + \frac{1}{m}$ is not an integer.

Case 4: $m = 2$: We argue similar to Case 3.

Case 5: $n \neq 1$ & $n \neq 2$ & $m \neq 1$ & $m \neq 2$: Then $n \geq 3$ and $m \geq 3$. Thus $\frac{1}{n} + \frac{1}{m} \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

Since $0 < \frac{1}{n} + \frac{1}{m} \leq \frac{2}{3}$, hence $\frac{1}{n} + \frac{1}{m}$ is not an integer.

In all cases, $\frac{1}{n} + \frac{1}{m}$ is not an integer.

$\boxed{Q2(b)}$ This is false. Take any number which is both a perfect square and a perfect cube larger than 1. Eg, $a = 64$.

Then a is a perfect square ($a = 8^2$) but $\sqrt[3]{a} = 4$ is rational.

$\boxed{Q2(c)}$ This is false. Let $D = \phi$, and $E = \{0\}$ are both finite sets, and E has one more element than D.

However, $P(D) = \{\phi\}$ and
$P(E) = \{\phi, \{\phi\}\}$

and $P(E)$ has only one more element than $P(D)$

Q3(a) Let $P(n): \sum_{j=1}^{3n} j(j-1) = n(9n^2-1)$

Base case: $P(1)$ $\quad LHS = \sum_{j=1}^{3} j(j-1) = 1.0 + 2.1 + 3.2$

$$= 0 + 2 + 6$$
$$= 8$$

$$RHS = 1.(9.1-1) = 8$$

So $P(1)$ is true.

Now assume $P(k)$ is true, i.e. $\sum_{j=1}^{3k} j(j-1) = k(9k^2-1)$.

Check $P(k+1)$:

$$\sum_{j=1}^{3(k+1)} j(j-1) = \sum_{j=1}^{3k+3} j(j-1) = \sum_{j=1}^{3k} j(j-1) + (3k+1)(3k) + (3k+2)(3k+1) + (3k+3)(3k+2)$$

$$= k(9k^2-1) + (3k+1)(3k) + (3k+2)[3k+1+3k+3]$$

$$= k(3k+1)(3k-1) + (3k+1)(3k) + (3k+2)(6k+4)$$

$$= k(3k+1)[3k-1+3] + 2(3k+2)^2$$

$$= k(3k+1)(3k+2) + 2(3k+2)^2$$

$$= (3k+2)[k(3k+1) + 2(3k+2)]$$

$$= (3k+2)[3k^2 + k + 6k + 4]$$

$$= (3k+2)[3k^2 + 7k + 4]$$

$$= (3k+2)(k+1)(3k+4)$$

$$= (k+1)[(\underline{3k+3}+1)(\underline{3k+3}-1)]$$

$$= (k+1)[(3k+3)^2 - 1]$$

$$= (k+1)[9(k+1)^2 - 1] = RHS.$$

Q 3(b) Let $P(n)$: for every sequence $x_1, x_2, \cdots x_n$ of non negative real numbers, if $x_1 + x_2 + \cdots + x_n = 0$ then $x_1 = x_2 = \cdots = x_n = 0$.

Base case : $P(1)$. We need to check every non negative real number x_1:

If $x_1 = 0$ then $x_1 = 0$ is true.

Assume $P(k)$ is true, i.e. for every sequence $x_1, x_2, \cdots x_k$ of non negative real numbers, if $x_1 + x_2 + \cdots + x_k = 0$ then $x_1 = x_2 = \cdots = x_k = 0$.

Note that $P(k)$ is a conditional statement !

Now verify $P(k+1)$. Fix a sequence $x_1, x_2, \ldots, x_k, x_{k+1}$ of non negative real numbers, and assume that

$$x_1 + x_2 \cdots + x_k + x_{k+1} = 0.$$

Then $x_1 + x_2 + \cdots + x_k = -x_{k+1}$.

Since $x_{k+1} \geq 0$, hence, RHS ≤ 0.

Since $x_1 \geq 0$, $x_2 \geq 0$, $\ldots$ $x_k \geq 0$,

hence, LHS ≥ 0.

But LHS = RHS, therefore both sides must be 0.

So, $x_1 + x_2 \cdots + x_k = -x_{k+1} = 0$. ——————— $(*)$

From $P(k)$, we know/assumed $x_1 + x_2 \cdots + x_k = 0 \Rightarrow x_1 = x_2 \cdots = x_k = 0$.

We also know $x_1 + x_2 \cdots + x_k = 0$. from $(*)$

So therefore, we conclude $x_1 = x_2 = \cdots = x_k = 0$.

From $(*)$ we also know $-x_{k+1} = 0$, so $x_{k+1} = 0$.

Therefore, $x_1 = x_2 = \cdots = x_k = x_{k+1} = 0$.

We have shown that starting from $x_1 + x_2 \cdots + x_k + x_{k+1} = 0$

we derived $x_1 = x_2 = \cdots = x_k = x_{k+1} = 0$.

So, $P(k+1)$ is true.

$\boxed{Q4(a)}$ Let X and Y be sets.

Let $A \in P(X-Y) - \{\phi\}$.

Thus, $\underbrace{A \in P(X-Y)}$ & $\underbrace{A \notin \{\phi\}}$.

By definition of power set, this means $A \subseteq X-Y$.

A is not an element of set $\{\phi\}$ So $A \neq \phi$.

Since $A \subseteq X-Y$ and $X-Y \subseteq X$
Thus $A \subseteq X$, and so $A \in P(X)$.

Since $A \neq \phi$,
A has some element $a \in A$. Since $A \subseteq X-Y$, it means $a \in X$. & $a \notin Y$. so, $A \nsubseteq Y$, so $A \notin P(Y)$.
We conclude $A \in P(X) - P(Y) =$ RHS.

Counter example to $P(X-Y) - \{\phi\} = P(X) - P(Y)$

$Y = \{0\}$, $X = \{0,1\}$. Then $X-Y = \{1\}$.

LHS $= \{\phi, \{1\}\} - \{\phi\} = \{\{1\}\}$.

RHS $= \{\phi, \{0\}, \{1\}, \{0,1\}\} - \{\phi, \{0\}\}$

$= \{\{1\}, \{0,1\}\}$. not equal.

$\boxed{Q4(b)}$ Using set identities,

$(A \cap (A-B)) \cup (A^c \cup B)^c$ [Set difference Law]

$= (A \cap (A \cap B^c)) \cup (A^c \cup B)^c$ [De Morgan's Law]

$= (A \cap (A \cap B^c)) \cup ((A^c)^c \cap B^c)$ [Double complement Law]

$= (A \cap (A \cap B^c)) \cup (A \cap B^c)$ [Commutative Law]

$= (A \cap B^c) \cup ((A \cap B^c) \cap A)$ [Absorption Law]

$= A \cap B^c$ [Set difference Law]

$= A - B$

$\boxed{Q4(c)}$ By trying out a few values of x, it's easy to see that (i) is true, (ii) is false.

(i) By definition of $\lfloor -x \rfloor$,

$$\lfloor -x \rfloor \le -x < \lfloor -x \rfloor + 1$$

By definition of $\lceil x \rceil$,

$$\lceil x \rceil - 1 < x \le \lceil x \rceil$$

Adding the inequalities gives

$$\lfloor -x \rfloor + \lceil x \rceil - 1 < -x + x < \lfloor -x \rfloor + 1 + \lceil x \rceil$$

$$\underbrace{\lfloor -x \rfloor + \lceil x \rceil - 1 < 0} \underbrace{< \lfloor -x \rfloor + \lceil x \rceil + 1}$$

$\lfloor -x \rfloor + \lceil x \rceil < 1$

$\lfloor -x \rfloor + \lceil x \rceil \le 0$

Since it's an integer

$\lfloor -x \rfloor + \lceil x \rceil > -1$

$\lfloor -x \rfloor + \lceil x \rceil \ge 0$

Since it's an integer

We conclude $\lfloor -x \rfloor + \lceil x \rceil = 0$

$$\lfloor -x \rfloor = -\lceil x \rceil.$$

(ii) Take $x = \frac{1}{2}$,

$$\left.\begin{aligned} \lfloor -x \rfloor &= \lfloor -\tfrac{1}{2} \rfloor = -1 \\ -\lfloor x \rfloor &= -\lfloor \tfrac{1}{2} \rfloor = 0 \end{aligned}\right\} \text{not equal.}$$

Q5(a) Let $0 < x < y$. Let $n = \lceil \frac{1}{y-x} \rceil$ and $m = \lfloor ny \rfloor$.

Then n, m are integers.

By definition of $\lceil \frac{1}{y-x} \rceil$, we have $\frac{1}{y-x} \leq \lceil \frac{1}{y-x} \rceil$.

So, $1 \leq n(y-x)$ (inequality does not flip around as $y-x>0$).

So $nx+1 \leq ny$.

By definition of $\lfloor ny \rfloor$, $m \leq ny < m+1$.

Thus, $nx \leq ny-1 < m$ and so

$nx < m \leq ny$.

$nx \leq m \leq ny$.

Q5(b) Let a be an odd integer. Then $a = 2k+1$ for some integer k. Then $a^3 - a = a(a^2-1)$

$= (2k+1)((2k+1)^2-1) = (2k+1)(4k^2+4k+1-1)$

$= (2k+1)(4k^2+4k) = 8k^3 + 12k^2 + 4k$

$= 4k(2k^2+3k+1) = 4k(2k+1)(k+1)$

$= 4k(k+1)(2k+1)$

By a result in class, $k(k+1)$ is even. Let $k(k+1) = 2\ell$

So, $a^3 - a = 4(2\ell)(2k+1) = 8\ell(2k+1)$, is divisible by 8.

Alternatively, you can proceed by the following

let a be an odd integer.

Then $a = 4k+1$ or $a = 4k+3$ for some integer k.

Case 1: $a = 4k+1$. $a^3 - a = (4k+1)(16k^2 + 8k + 1 - 1)$

$$= (4k+1)(16k^2 + 8k)$$

$$= 8(4k+1)(2k^2 + k)$$

Case 2: $a = 4k+3$. $a^3 - a = (4k+3)(16k^2 + 24k + 9 - 1)$

$$= (4k+3)(16k^2 + 24k + 8)$$

$$= 8(4k+3)(2k^2 + 3k + 1)$$

In either case, $a^3 - a$ is divisible by 8.

Q5(c)

$630 = 96 \times 6 + 54$

$96 = 54 \times 1 + 42$

$54 = 42 \times 1 + 12$

$42 = 12 \times 3 + 6 \longrightarrow \gcd(630, 96) = 6.$

$12 = 6 \times 2 + 0$

$\boxed{Q6(a)}$ $z^3 = 3(1+i)$

$= 3e^{i\frac{\pi}{4}}$

$z = 18^{\frac{1}{6}} e^{i\frac{\frac{\pi}{4}+2k\pi}{3}}$ $k=0,1,2$

$= 18^{\frac{1}{6}} e^{i\frac{\pi}{12}}, 18^{\frac{1}{6}} e^{i\frac{9\pi}{12}}, 18^{\frac{1}{6}} e^{i\frac{17\pi}{12}}$

$\boxed{Q6(b)}$ Let $g : P(\mathbb{R}) \to P(\mathbb{R}^2)$,

$g(A) = A \times A$

g is one-to-one: Suppose $g(A) = g(B)$.

$A \times A = B \times B$

Show: $A = B$: for any x, $x \in A \iff (x,x) \in A \times A$

$\iff (x,x) \in B \times B$

$\iff x \in B$.

g is not onto:

Take $C = \{(0,1)\} \in P(\mathbb{R}^2\}$.

If $g(A) = C$

then $A \times A = \{(0,1)\}$ So, $(0,1) \in A \times A$.

So $0 \in A$. This means $(0,0) \in A \times A = g(A) = C$.

Contradiction.

$\boxed{Q7(a)}$ (i) A symmetric binary relation R is a relation on a set A such that for all $x, y \in A$, if $(x,y) \in R$ then $(y,x) \in R$.

(ii) A transitive binary relation R is a relation on a set A such that for every $x, y, z \in A$, if $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$.

$\boxed{Q7(b)}$ No it is not reflexive.

(i) $0 < 0$ is false, so $(0,0)\ R\ (0,0)$ is false.

(ii) No it is not symmetric.

Take $\overset{(a,b)}{(0,0)}$ and $\overset{(x,y)}{(1,0)}$.

Then $(0,0)\ R\ (1,0)$ is true as $a < x$

but $(1,0)\ R\ (0,0)$ is false because $1 < 0$ false and $1 = 0$ is false.

(iii) Yes it is transitive.

Fix (a,b), (x,y), (u,v) such that $(a,b)\ R\ (x,y)$ and $(x,y)\ R\ (u,v)$.

Case 1: $a < x$. Then since $(x,y)\ R\ (u,v)$ we have $x \le u$. This means $a < x \le u$ $\Rightarrow a < u$.

So $(a,b)\ R\ (u,v)$ is true.

Case 2: $x < u$. Then since $(a,b) \, R \, (x,y)$,

hence $a \leq x$. So, $a \leq x < u \Rightarrow a < u$.

So again, $(a,b) \, R \, (u,v)$ true.

Case 3: $a = x$ and $x = u$: Then since $(a,b) \, R (x,y)$,

we have $b < y$. Since $(x,y) \, R \, (u,v)$, we have $y < v$. So we have:

$a = x = u$ & $b < y < v$. So, $a = u$ & $b < v$.

Thus $(a,b) \, R \, (u,v)$ is true.

$\boxed{Q7(c)}$ T is reflexive: $1 \cdot a = a$ and $1 \cdot b = b$

So, $(a,b) \, R (a,b)$ true.

T is symmetric. Suppose $(a,b) \, T \, (x,y)$. let $c \neq 0$ s.t. $ca = x$ and $cb = y$. Then $a = \frac{1}{c} x$ and $b = \frac{1}{c} y$ where $\frac{1}{c} \neq 0$. So, $(x,y) \, T \, (a,b)$ true.

T is transitive. Suppose $(a,b) \, T \, (x,y)$ and $(x,y) \, T \, (u,v)$. let $c \neq 0$, $d \neq 0$ s.t. $ca = x$, $cb = y$, $dx = u$, $dy = v$.

Then $dc \, a = dx = u$ and $dcb = dy = v$. Since $dc \neq 0$, (by zero product property), so, $(a,b) \, T \, (u,v)$.

Equivalence class of $(1,2)$: $(a,b) \in [(1,2)]$

$\Leftrightarrow \exists c \neq 0$ s.t. $c \cdot 1 = a$ & $c \cdot 2 = b$

$\Leftrightarrow (a,b) = (c, 2c)$ some $c \neq 0$.

This is the line $y = 2x$ with $(0,0)$ removed.