NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2012-2013

MH1200/MTH 114– Linear Algebra I

November 2012 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps of their solutions.

QUESTION 1. **(20 marks)**

1. Determine the number of solutions of the following linear system and write down the general solution of the system (unless the system is inconsistent).

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 3y + z &= 1 \\ 6x + 11y - 3z &= 13. \end{aligned}$$

Solution: The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 1 \\ 6 & 11 & -3 & 13 \end{array}\right),$$

and a row-echelon form of that matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The corresponding system is consistent, and the general solution has one free parameter, i.e., there are infinitely many solutions. The general solution of the system is $x = -7 - 5t, y = 5 + 3t, z = t$, where t is an arbitrary parameter.

2. For the following linear system, determine the values of the constants a, b for which the system has (i) no solution, (ii) exactly one solution, (iii) infinitely many solutions.

$$\begin{aligned} ax + y &= 1 \\ 2x + y &= b. \end{aligned}$$

Solution:

The augmented matrix is

$$\left(\begin{array}{cc|c} a & 1 & 1 \\ 2 & 1 & b \end{array}\right).$$

A reduced row echelon form is

$$\left(\begin{array}{cc|c} 1 & 1/2 & b/2 \\ 0 & 1-a/2 & 1-ab/2 \end{array}\right).$$

The system is inconsistent if $a = 2$ and $b \neq 1$, otherwise the system is consistent. The system has infinitely many solutions if $a = 2$ and $b = 1$, otherwise (i.e., $a \neq 2$) the system has exactly one solution.

Note that a solution that uses division by a needs to make a case analysis ($a = 0$ and $a \neq 0$).

3. For a quadratic curve with equation

$$y = a + bx + cx^2$$

that passes through the points $(-1, 6), (2, 0)$, and $(3, 2)$, find a, b, c.

Solution: Plugging the 3 points into the quadratic equation gives 3 linear equations:

$$\begin{aligned} a - b + c &= 6 \\ a + 2b + 4c &= 0 \\ a + 3b + 9c &= 2. \end{aligned}$$

Then the augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 9 & 2 \end{array}\right).$$

A row echelon form is

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

By back-substitution we can see that $c = 1$, $b = -3$, $a = 2$.

4. Solve the following linear system using Cramer's rule. Do not use any other method.

$$\begin{aligned} x + y + 2z &= 0 \\ 3x + y + z &= 0 \\ -x + 3y + 4z &= 1. \end{aligned}$$

Solution: The determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ -1 & 3 & 4 \end{vmatrix} = 8.$$

We need to compute the 3 determinants, where the vector of right hand sides replaces a column in the coefficient matrix.

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{vmatrix} = -1.$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 3 & 0 & 1 \\ -1 & 1 & 4 \end{vmatrix} = 5.$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 3 & 1 & 0 \\ -1 & 3 & 1 \end{vmatrix} = -2.$$

Hence the solution is $(-1/8, 5/8, -1/4)$.

QUESTION 2. **(25 marks)**

1. Compute the determinant of the following matrix and determine the values of the unknown constant a for which the matrix is invertible. Compute the inverse for those values.

$$A = \begin{pmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & a \end{pmatrix}.$$

Solution: With the rule of Sarrus it is easy to see that the determinant is $a-2$ and hence A is invertible whenever $a \neq 2$.

The least error-prone way to compute the inverse is by using the adjoint matrix. Entry i, j of the adjoint matrix is $(-1)^{i+j} M_{i,j}$, where $M_{i,j}$ is the matrix A with row i and column j deleted. Then we can find the inverse is

$$A^{-1} = \frac{1}{a-2} \begin{pmatrix} 4a+3 & 3-7a & -11 \\ -a-1 & 2a-1 & 3 \\ -1 & 1 & 1 \end{pmatrix}.$$

2. Express A as a product of elementary matrices for the case $a = 0$. **Solution:** Here the important part is to bring A into reduced row echelon form, and write A as the product of inverses of elementary matrices used in the process. There are many possible solutions.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Let
$$B = \begin{pmatrix} a & b & 0 & 0 \\ b & a & b & 0 \\ 0 & b & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix},$$
where a, b are real numbers. Compute the determinant of B.

Solution: Cofactor expansion (best to use the 4th row) and the rule of Sarrus show that the determinant is $a^4 - 2a^2b^2$.

4. For what values of a, b is the above matrix B invertible? (Justify your answer.)

Solution: The matrix is invertible if and only if the determinant is not 0, i.e., if and only if $a \neq 0$ and $b \neq \pm 1/\sqrt{2} \cdot a$.

5. Determine the rank and the nullity of B for all values of a, b.

Solution: Case 1: $a = b = 0$: rank is 0, nullity is 4-0=4.

Case 2: $a = 0$ and $b \neq 0$. Because the rank does not change when the matrix is scaled by a nonzero factor and the last row and column are 0, the rank of B is equal to the rank of
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
i.e., 2, and the nullity of B is 4-2=2.

Case 3: $a \neq 0$ and $b = 0$. In this case the matrix is diagonal with a on the diagonal, the rank is 4 and the nullity is 0.

Case 4: $a \neq 0, b \neq 0$. In this case we first observe it is enough to compute the rank of

$$\begin{pmatrix} a & b & 0 \\ b & a & b \\ 0 & b & a \end{pmatrix}$$

and add 1, because the last row is independent of the previous rows. We can bring this matrix into the following form with elementary row operations:

$$\begin{pmatrix} b & a & b \\ 0 & b & a \\ 0 & a^2/b - b & a \end{pmatrix}.$$

Then it is clear that the matrix has rank 3 (and hence B has rank 4) if $a^2/b - b \neq b$, and rank 2 (and hence B rank 3) otherwise.

So we have Case 4a: $a = \pm\sqrt{2}b$, rank 3 and nullity 1.

Case 4b: $a \neq \pm\sqrt{2}b$, rank 4 and nullity 0.

QUESTION 3. **(30 marks)**

Determine which of the following statements are true and which are false. If a statement is correct, explain the reasons (you can use any result from the lectures for this); if a statement is wrong, give a counterexample.

1. For two $n \times n$ matrices A and U the row space of UA is contained in the row space of A.
2. For two $n \times n$ matrices A and B the rank of AB is always at least as large as the rank of B.
3. If A, B, C are $n \times n$ matrices then $\operatorname{tr}(ABC) = \operatorname{tr}(ACB)$, where tr(M) denotes the trace of the matrix M, i.e., the sum of the diagonal entries of M.

Solution:

1) TRUE. The rows of UA are linear combinations of the rows of A, hence any row of UA is in the rowspace of A.

2) FALSE. Example: Let A be the 0-matrix, then the rank of AB is 0, and the rank of AB is smaller than the rank of B unless B is also the 0-matrix.

3) FALSE. While it is true that $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$, it is possible that $\operatorname{tr}(ABC) \neq \operatorname{tr}(ACB)$.

Example:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\operatorname{tr}(ABC) = 2$ and $\operatorname{tr}(ACB) = 1$.

4. The null space of a matrix A does not change when elementary column operations are applied to A (i.e., exchange columns, add a multiple of a column to another column, multiply columns by nonzero constants).

5. If U, V are vector spaces, then the set $\{(u, v) : u \in U, v \in V\}$ is also a vector space.

6. A basis $\{b_1, \ldots, b_k\}$ of any k-dimensional subspace S of an n-dimensional vector space V can be extended to a basis of V by including $n-k$ more vectors with the basis.

Solution:

4) FALSE. Example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The matrix

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

results from a column swap. The nullspace of A is the set $\{(0, t) : t \in R\}$, whereas the nullspace of B is the set $\{(t, 0) : t \in R\}$, so the nullspaces are not the same.

5) TRUE. Check by checking closure under scalar multiplication and addition.

6) TRUE. Induction over k. Certainly this is true for $k = n$. If $k < n$ then the span of k vectors is a proper subset of V, and any vector from $V - S$ can be added to a basis to find a $k + 1$ dimensional subspace. By induction this can be continued $n - k$ times to find a basis of V.

QUESTION 4. **(25 marks)**

A *circulant* matrix of order 4 takes the form

$$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix},$$

where a, b, c, d are real numbers.

1. Show that circulant matrices of order 4 form a subspace V of the vector space of 4×4 matrices (i.e., of $M(4,4)$).

   Solution: The set of circulant matrices of order 4 can be described as the solution set of a homogenous system of equations in the entries M_{ij} of the matrices. The equations that define circulant matrices are of the type $M_{11} = M_{22} = M_{33} = M_{44}$ etc. and can be rewritten as a system of homogenous equations by writing $M_{11} - M_{22} = 0$ and $M_{22} - M_{33} = 0$ and $M_{33} - M_{44} = 0$ etc. The solution set of a system of homogenous equations is always a subspace, hence the set of circulant matrices is a subspace.

2. Determine the dimension of this subspace and write down a basis for it.

   Solution: The following is a basis:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

   Clearly every circulant matrix can be written as a linear combination of these 4 matrices, and furthermore none of these 4 matrices can be written as a linear combination of the others (because each contains 1's where the others contain only 0's). So this is a basis, and the dimension of V is 4.

3. W denote the following subspace of the space of all 4×4 matrices:

$$W = \{M \in M(4,4) : \operatorname{tr}(M) = 0\}.$$

Determine the dimension of the span of the union of V and W and the dimension of the intersection of V and W.

Solution: The span of the union of V and W is the space of all 4×4 matrices: Given any matrix M with trace t write

$$M = M + \begin{pmatrix} -t/4 & 0 & 0 & 0 \\ 0 & -t/4 & 0 & 0 \\ 0 & 0 & -t/4 & 0 \\ 0 & 0 & 0 & -t/4 \end{pmatrix} + \begin{pmatrix} t/4 & 0 & 0 & 0 \\ 0 & t/4 & 0 & 0 \\ 0 & 0 & t/4 & 0 \\ 0 & 0 & 0 & t/4 \end{pmatrix}.$$

The sum of the first two matrices on the right hand side is in W, and the third matrix is in V, so M lies in the span of $W \cup V$ and that span is indeed the space of a 4×4 matrices. Hence the dimension of the span of the union of W and V is 16.

The intersection of W and V is the set of circulant matrices with $a = 0$, and has dimension 3, as can be seen from the fact that it is spanned by the last 3 matrices in the basis of V.

4. Show that V is closed under matrix multiplication, i.e., for all $A, B \in V$ we have $AB \in V$.

Solution:

Take any two circulant matrices

$$\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}$$

and

$$\begin{pmatrix} a' & b' & c' & d' \\ d' & a' & b' & c' \\ c' & d' & a' & b' \\ b' & c' & d' & a' \end{pmatrix}.$$

Their product is

$$\begin{pmatrix} aa' + bd' + cc' + db' & ab' + ba' + cd' + dc' & ac' + bb' + ca' + dd' & ad' + bc' + cb' + da' \\ da' + ad' + bc' + cb' & db' + aa' + bd' + cc' & dc' + ab' + ba' + cd' & dd' + ac' + bb' + ca' \\ ca' + dd' + ac' + bb' & cb' + da' + ad' + bc' & cc' + db' + aa' + bd' & cd' + dc' + ab' + ba' \\ ba' + cd' + dc' + ab' & bb' + ca' + dd' + ac' & bc' + cb' + da' + ad' & bd' + cc' + db' + aa' \end{pmatrix},$$

which is clearly circulant.

END OF PAPER