Name: ______________________________ Tutorial group: ____

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2024/25

MH1100 – Calculus I

27 September 2024 Midterm Test 90 minutes

INSTRUCTIONS

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **SIX (6)** questions and comprises **SEVEN (7)** printed pages. Question 6 is optional.
4. The marks for each question are indicated at the beginning of each question.

For graders only	Question	1	2	3	4	5	6	Total
	Marks							

QUESTION 1. **(2 marks)**

Use the ϵ, δ definition of a limit to prove

$$\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}.$$

[Answer:] To prove that

$$\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}$$

using the ϵ-δ definition of a limit, we need to show that for every $\varepsilon>0$, there exists a $\delta>0$ such that if $0<|x-1|<\delta$, then

$$\left|\frac{1}{x+1}-\frac{1}{2}\right|<\varepsilon.$$

Start with the limit expression:

$$\left|\frac{1}{x+1}-\frac{1}{2}\right|=\left|\frac{2-(x+1)}{2(x+1)}\right|=\left|\frac{1-x}{2(x+1)}\right|.$$

We want to bound this expression by ε:

$$\left|\frac{1-x}{2(x+1)}\right|<\varepsilon.$$

So, we try to find a suitable bound for $|x+1|$. Since we are taking the limit as x approaches 1, we can restrict x to be in a small neighborhood around 1. If we let $\delta<1$, then x will be within the interval $(0,2)$:

$$1-\delta<x<1+\delta \quad \text{implies} \quad 1<x+1<2.$$

Thus, we have:

$$|x+1|\geq 1.$$

Therefore, we can express our inequality as:

$$\left|\frac{1-x}{2(x+1)}\right|\leq\frac{|1-x|}{2} \quad (\text{since } |x+1|\geq 1).$$

We want:

$$\frac{|1-x|}{2}<\varepsilon \implies |1-x|<2\varepsilon.$$

Therefore, we can choose δ as:

$$\delta=\min(1,2\varepsilon).$$

This ensures that if $0<|x-1|<\delta$, then:

$$|1-x|<2\varepsilon \implies \left|\frac{1}{x+1}-\frac{1}{2}\right|<\varepsilon.$$

By the ϵ-δ definition of a limit, we have shown that:

$$\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}.$$

QUESTION 2. **(4 marks)**

Find the limits, if they exist:

(a) $\lim\limits_{x\to 0}\left(\dfrac{1}{1+2x}-\dfrac{1}{1-3x}\right)$, (b) $\lim\limits_{h\to 0}\dfrac{(x-3h)^2-(x+2h)^2}{\sin h}$,

(c) $\lim\limits_{x\to 0^+}\left(\dfrac{x^2}{2}-\dfrac{1}{2x}\right)$, (d) $\lim\limits_{x\to 0}\sqrt[3]{x}\sin\left(\dfrac{1}{x^2}\right)$.

[Answer:]

(a) We compute the limit:

$$\lim_{x\to 0}\left(\frac{1}{1+2x}-\frac{1}{1-3x}\right).$$

Finding a common denominator:

$$=\lim_{x\to 0}\left(\frac{(1-3x)-(1+2x)}{(1+2x)(1-3x)}\right)=\lim_{x\to 0}\left(\frac{-5x}{(1+2x)(1-3x)}\right).$$

Substituting $x=0$:

$$=\frac{-5(0)}{(1+0)(1-0)}=0.$$

(b) We compute the limit:

$$\lim_{h\to 0}\frac{(x-3h)^2-(x+2h)^2}{\sin h}.$$

Calculating the numerator:

$$(x-3h)^2-(x+2h)^2=x^2-6xh+9h^2-(x^2+4xh+4h^2)=-10xh+5h^2.$$

Thus,

$$=\lim_{h\to 0}\frac{-10xh+5h^2}{\sin h}=\lim_{h\to 0}(-10x+5h)\frac{h}{\sin h}.$$

Using $\lim_{h\to 0}\frac{h}{\sin h}=1$:

$$=(-10x+5\cdot 0)(1)=-10x.$$

(c) We compute the limit:

$$\lim_{x\to 0^+}\left(\frac{x^2}{2}-\frac{1}{2x}\right).$$

As x approaches 0^+, $-\frac{1}{2x}\to-\infty$, hence:

$$\lim_{x\to 0^+}\left(\frac{x^2}{2}-\frac{1}{2x}\right)=-\infty.$$

(d) We compute the limit:

$$\lim_{x \to 0} \sqrt[3]{x} \sin\left(\frac{1}{x^2}\right).$$

Since $\sin\left(\frac{1}{x^2}\right)$ oscillates between -1 and 1, we have:

$$-|\sqrt[3]{x}| \leq \sqrt[3]{x} \sin\left(\frac{1}{x^2}\right) \leq |\sqrt[3]{x}|.$$

Taking the limit as $x \to 0$:

$$\lim_{x \to 0} \sqrt[3]{x} = 0.$$

Thus,

$$\lim_{x \to 0} |\sqrt[3]{x}| = 0.$$

By the Squeeze Theorem:

$$\lim_{x \to 0} \sqrt[3]{x} \sin\left(\frac{1}{x^2}\right) = 0.$$

QUESTION 3. **(4 marks)**

(a) Let $f(x) = \cos x - x$. Show that there is a solution to the equation $\cos x = x$ between 0 and $\frac{\pi}{3}$.

(b) Suppose the temperature at 6 AM is 25°C and at 2 PM is 34°C. Assuming the temperature is a continuous function of time, show that at some point during the day, the temperature was exactly 30°C.

[Answer:] **(a)** We are asked to show that the equation $\cos x = x$ has a solution in the interval $[0, \frac{\pi}{3}]$. This is equivalent to finding a root of the function $f(x) = \cos x - x$ in the same interval.

To apply the Intermediate Value Theorem (IVT), we first evaluate $f(x)$ at the endpoints of the interval:

$$f(0) = \cos(0) - 0 = 1 - 0 = 1$$

$$f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) - \frac{\pi}{3} = \frac{1}{2} - \frac{\pi}{3} \approx \frac{1}{2} - 1.047 \approx -0.547$$

Since $f(0) = 1$ and $f\left(\frac{\pi}{3}\right) \approx -0.547$, we observe that $f(0) > 0$ and $f\left(\frac{\pi}{3}\right) < 0$.

By the Intermediate Value Theorem, because $f(x)$ is continuous on $[0, \frac{\pi}{3}]$ and changes sign over this interval, there must be some $c \in (0, \frac{\pi}{3})$ such that $f(c) = 0$, or equivalently, $\cos c = c$. Therefore, there is a solution to the equation $\cos x = x$ in the interval $[0, \frac{\pi}{3}]$.

(b) Let $T(t)$ represent the temperature at time t, where t is the time of day between 6 AM and 2 PM. We are given that:

$$T(6) = 25°\text{C}, \quad T(14) = 34°\text{C}$$

We need to show that there exists some time t between 6 AM and 2 PM (i.e., $t \in [6, 14]$) such that $T(t) = 30°\text{C}$. Since $T(6) = 25$ and $T(14) = 34$, and $T(t)$ is continuous, by the Intermediate Value Theorem, there must exist some $t \in (6, 14)$ such that:

$$T(t) = 30°\text{C}$$

Therefore, at some point between 6 AM and 2 PM, the temperature must have been exactly 30°C.

QUESTION 4. **(4 marks)**

Compute the derivatives of the functions in (a)-(c):

(a) $f(x) = (3x^2+5)(2x^3-4)$, (b) $g(x) = \dfrac{x^4-2x+1}{x^2+3}$, (c) $h(x) = \sqrt{x+7}$,

(d) Let $k(x) = x(x+1)(x+2)\cdots(x+2024)$, compute $k'(0)$.

[Answer:]

(a) We apply the product rule:

$$(fg)' = f'g + fg'$$

Let $u(x) = 3x^2+5$ and $v(x) = 2x^3-4$. First, compute their derivatives:

$$u'(x) = 6x, \quad v'(x) = 6x^2$$

Now apply the product rule:

$$f'(x) = u'(x)v(x) + u(x)v'(x)$$

$$f'(x) = (6x)(2x^3-4) + (3x^2+5)(6x^2)$$

Simplify:

$$f'(x) = 12x^4 - 24x + 18x^4 + 30x^2$$

$$f'(x) = 30x^4 + 30x^2 - 24x$$

(b) We apply the quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Let $f(x) = x^4 - 2x + 1$ and $g(x) = x^2 + 3$. First, compute their derivatives:

$$f'(x) = 4x^3 - 2, \quad g'(x) = 2x$$

Now apply the quotient rule:

$$g'(x) = \frac{(4x^3-2)(x^2+3) - (x^4-2x+1)(2x)}{(x^2+3)^2}$$

Simplify the numerator:

$$(4x^3-2)(x^2+3) = 4x^5 + 12x^3 - 2x^2 - 6$$

$$(x^4-2x+1)(2x) = 2x^5 - 4x^2 + 2x$$

Now subtract the two:

$$(4x^5 + 12x^3 - 2x^2 - 6) - (2x^5 - 4x^2 + 2x) = 2x^5 + 12x^3 + 2x^2 - 2x - 6$$

Thus, the derivative is:

$$g'(x) = \frac{2x^5 + 12x^3 + 2x^2 - 2x - 6}{(x^2+3)^2}$$

(c) We use the definition of a derivative.

$$\begin{aligned}
h'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h+7}-\sqrt{x+7}}{h} \\
&= \lim_{h\to 0}\left(\frac{\sqrt{x+h+7}-\sqrt{x+7}}{h}\cdot\frac{\sqrt{x+h+7}+\sqrt{x+7}}{\sqrt{x+h+7}+\sqrt{x+7}}\right) \qquad \text{Rationalize the numerator} \\
&= \lim_{h\to 0}\frac{(x+h+7)-(x+7)}{h\left(\sqrt{x+h+7}+\sqrt{x+7}\right)} = \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h+7}+\sqrt{x+7}\right)} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h+7}+\sqrt{x+7}} = \frac{1}{\sqrt{x+7}+\sqrt{x+7}} = \frac{1}{2\sqrt{x+7}}.
\end{aligned}$$

(d) Compute $k'(0)$ for $k(x) = x(x+1)(x+2)\cdots(x+2024)$.

We use the definition of a derivative.

$$\begin{aligned}
k'(0) &= \lim_{h\to 0}\frac{k(0+h)-k(0)}{h} = \lim_{h\to 0}\frac{h(h+1)(h+2)\cdots(h+2024)-0}{h} \\
&= \lim_{h\to 0}(h+1)(h+2)\cdots(h+2024) \\
&= 2024!.
\end{aligned}$$

Thus:

$$k'(0) = 2024!$$

QUESTION 5. **(6 marks)**

(a) Determine if the function is differentiable at $x = 1$:

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ 2x + 1, & \text{if } x > 1. \end{cases}$$

(b) Determine if the function is differentiable at $x = 0$:

$$g(x) = \begin{cases} 2x - x^3 - 1, & \text{if } x \leq 0 \\ x - \frac{1}{x+1}, & \text{if } x > 0. \end{cases}$$

[Answer:]

(a) To determine if $f(x)$ is differentiable at $x = 1$, we must check two conditions:

- Continuity at $x = 1$
- The existence of the derivative at $x = 1$

For $f(x)$ to be differentiable at $x = 1$, it must first be continuous at $x = 1$. Therefore, we need to check if:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

- For $x \to 1^-$ (from the left), $f(x) = x^2$. So,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1$$

- For $x \to 1^+$ (from the right), $f(x) = 2x + 1$. So,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x + 1) = 2(1) + 1 = 3$$

- At $x = 1$, $f(1) = 1^2 = 1$.

Since the left-hand limit and right-hand limit are not equal, the function is **not continuous** at $x = 1$. Because $f(x)$ is not continuous at $x = 1$, it cannot be differentiable at $x = 1$. Therefore, the function is **not differentiable** at $x = 1$.

(b) We need to determine if the function $g(x)$ is differentiable at $x = 0$. We will first confirm that $g(x)$ is continuous at this point. The function is continuous at $x = 0$ if:

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = g(0)$$

We calculate $g(0)$:

$$g(0) = 2(0) - (0)^3 - 1 = -1,$$

the left-hand limit $\lim_{x \to 0^-} g(x)$:

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (2x - x^3 - 1) = 2(0) - (0)^3 - 1 = -1,$$

and the right-hand limit $\lim_{x\to 0^+} g(x)$:

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \left(x - \frac{1}{x+1}\right) = 0 - \frac{1}{0+1} = -1.$$

Since both limits equal $g(0)$:

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^+} g(x) = g(0) = -1$$

Thus, $g(x)$ is continuous at $x = 0$.

Next, we check differentiability by computing the left-hand and right-hand derivatives at $x = 0$:

$$g'(0^-) = \lim_{h\to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^-} \frac{(2h - h^3 - 1) - (-1)}{h} = \lim_{h\to 0^-} \frac{2h - h^3}{h} = \lim_{h\to 0^-} (2-h^2) = 2$$

$$\begin{aligned} g'(0^+) &= \lim_{h\to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^+} \frac{\left(h - \frac{1}{h+1}\right) - (-1)}{h} = \lim_{h\to 0^+} \frac{h + 1 - \frac{1}{h+1}}{h} \\ &= \lim_{h\to 0^+} \frac{(h+1)^2 - 1}{h(h+1)} = \lim_{h\to 0^+} \frac{h^2 + 2h}{h(h+1)} = \lim_{h\to 0^+} \frac{h+2}{h+1} = 2 \end{aligned}$$

Since the left-hand derivative $g'(0^-) = 2$ and the right-hand derivative $g'(0^+) = 2$ are equal, $g(x)$ is differentiable at $x = 0$.

Thus, we conclude that:

$$\text{The function } g(x) \text{ is continuous at } x = 0 \text{ differentiable at } x = 0.$$

QUESTION 6 (Optional). **(1 bonus mark)**

Find the limit

$$\lim_{h\to 0}\frac{h}{f(a-2h)-f(a+3h)},$$

given that $f'(a)=-1$.

[Answer:]

$$\begin{aligned}
L &= \lim_{h\to 0}\frac{h}{f(a-2h)-f(a+3h)} = \lim_{h\to 0}\frac{1}{\frac{f(a-2h)-f(a+3h)}{h}}\\
&= \lim_{h\to 0}\frac{1}{\frac{f(a-2h)-f(a)-[f(a+3h)-f(a)]}{h}} = \lim_{h\to 0}\frac{1}{\frac{f(a-2h)-f(a)}{h}-\frac{f(a+3h)-f(a)}{h}}\\
&= \lim_{h\to 0}\frac{1}{\frac{f(a-2h)-f(a)}{-2h}\cdot(-2)-\frac{f(a+3h)-f(a)}{3h}\cdot(3)}\\
&= \frac{1}{\lim\limits_{h\to 0}\frac{f(a-2h)-f(a)}{-2h}\cdot(-2)-\lim\limits_{h\to 0}\frac{f(a+3h)-f(a)}{3h}\cdot(3)}\\
&= \frac{1}{f'(a)\cdot(-2)-f'(a)\cdot(3)} = \frac{1}{-5f'(a)}\\
&= \frac{1}{5}.
\end{aligned}$$