NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2021-2022

MH1300– Foundations of Mathematics

November 2021 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS** a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **BOTH SIDES OF ONE A4-SIZED HANDWRITTEN HELPSHEET.**
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (18 marks)

(a) Let a and b be integers. Prove that $a - b$ and $a^2 + b^2$ have the same parity.

(b) Is the following statement form a tautology, a contradiction, or neither?

$$(p \to (q \wedge \neg r)) \to (\neg q \to \neg p)$$

Justify your answer.

(c) Write down a tautology using only the statement variables s and t, and the connective $\to$. All three symbols must be used at least once in your statement form, and you cannot use any other variables or connectives. Prove that your statement form is a tautology.

QUESTION 2. (14 marks)

Determine if each of the following is true or false. Justify your answers.

(a) If a and b are composite numbers, then $a + b$ is composite.

(b) For all positive integers c, d, e, if $c \mid e$ and $d \mid e$, then either $c = e$, $d = e$ or $cd \mid e$.

(c) Let A be a subset of a set B. Then $A \times A \subseteq B \times B$.

(d) If C, D and E are sets, then $(C \cup D) \cap E = C \cup (D \cap E)$.

QUESTION 3. (18 marks)

(a) Let $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function satisfying: $F(a, 0) = a$, and $F(a, b+1) = F(a, b) + 1$, for each $a, b \in \mathbb{N}$. Using the definition of F, prove that for every $a, b, c \in \mathbb{N}$, $F(F(a, b), c) = F(a, F(b, c))$.

(b) Prove that for every positive integer m,

$$4^{m+1} + 5^{2m-1} \text{ is divisible by } 21.$$

QUESTION 4. (10 marks)

(a) Let a, b, c, d be integers such that $d \mid a$, $d \mid b$ and $d \mid c$. Prove that $d^2 \mid ab + ac + bc$.

(b) Prove that every non-zero rational number is the product of two irrational numbers. You may use the fact that the product of a non-zero rational number with an irrational number is irrational.

QUESTION 5. (12 marks)

(a) Let A and B be sets and $f : A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$. Prove that

$$f\left(f^{-1}(Y)\right) \subseteq Y \text{ and } X \subseteq f^{-1}\left(f(X)\right).$$

(b) Let $g : \mathbb{Z} \to \mathbb{Z}$ be the function $g(n) = 2n \bmod 3$.

(i) Is g injective?

(ii) Is g surjective?

(iii) What is the range of g?

Justify your answers.

QUESTION 6. (12 marks)

(a) Find all sixth roots of unity. That is, find all complex numbers z satisfying $z^6 - 1 = 0$. Leave your answer in terms of $re^{i\theta}$.

(b) Determine if the following argument is valid. State all rules of inference used.

$$\begin{array}{ll} & \neg p \vee \neg q \\ & \neg r \to (p \wedge q) \\ & \neg r \vee s \\ & s \to (t \wedge u) \\ \therefore & u \end{array}$$

QUESTION 7. (16 marks)

(a) For the relation R below defined on the set $\mathbb{R}^2$, determine if it is reflexive, if it is symmetric and if it is transitive. Justify your answer.

$$(x_1, x_2)\ R\ (x_3, x_4) \leftrightarrow x_i = x_j \text{ for some } i \neq j, \text{ and } i, j = 1, 2, 3 \text{ or } 4.$$

(b) Supose that T is a reflexive relation on a set A such that for every $x, y, z \in A$, if $x\ T\ y$ and $x\ T\ z$ then $y = z$. Show that T is an equivalence relation, and describe the equivalence classes of T.

END OF PAPER