SEMESTER I 2023/24

MH1100 - Calculus I

Midterm Test

QUESTION 1. (6 marks)

Find the limits if exist.

(a) $\displaystyle\lim_{x\to 5}\frac{3}{\sqrt{3x+1}+1}$ (b) $\displaystyle\lim_{x\to 1}\frac{x^{-1}-1}{x-1}$ (c) $\displaystyle\lim_{x\to -3}\frac{x^2-9}{\sqrt{x^2+16}-5}$

(d) $\displaystyle\lim_{x\to 1^-}\frac{\sqrt{2x}(x-1)}{|x-1|}$ (e) $\displaystyle\lim_{t\to 0}\frac{2t}{\tan t}$ (f) $\displaystyle\lim_{x\to -1^+}\left(\frac{1}{1+x}-\frac{3}{1+x^3}\right)$

Solution.

(a) By direct substitution,

$$\lim_{x\to 5}\frac{3}{\sqrt{3x+1}+1}=\frac{3}{\sqrt{3(5)+1}+1}=\frac{3}{5}.$$

(b)

$$\lim_{x\to 1}\frac{x^{-1}-1}{x-1}=\lim_{x\to 1}\frac{1-x}{x(x-1)}=\lim_{x\to 1}\frac{-(x-1)}{x(x-1)}=\lim_{x\to 1}\frac{-1}{x}=-1.$$

(c)

$$\begin{aligned}\lim_{x\to -3}\frac{x^2-9}{\sqrt{x^2+16}-5}&=\lim_{x\to -3}\frac{x^2-9}{\sqrt{x^2+16}-5}\times\frac{\sqrt{x^2+16}+5}{\sqrt{x^2+16}+5}\\&=\lim_{x\to -3}\frac{(x^2-9)(\sqrt{x^2+16}+5)}{x^2-9}\\&=\lim_{x\to -3}(\sqrt{x^2+16}+5)\\&=\sqrt{(-3)^2+16}+5\\&=10.\end{aligned}$$

(d) Consider $x < 1$, then $|x-1| = -(x-1)$. Therefore,

$$\lim_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|} = \lim_{x\to 1^-} \frac{\sqrt{2x}(x-1)}{-(x-1)} = \lim_{x\to 1^-} (-\sqrt{2x}) = -\sqrt{2}.$$

(e)

$$\lim_{t\to 0} \frac{2t}{\tan t} = \lim_{t\to 0} \frac{2t}{\frac{\sin t}{\cos t}} = \lim_{t\to 0} \left(2\cos t \cdot \frac{t}{\sin t}\right).$$

Using the fact that $\lim_{t\to 0} \frac{\sin t}{t} = 1$,

$$\lim_{t\to 0} \frac{t}{\sin t} = \lim_{t\to 0} \frac{1}{\frac{\sin t}{t}} = \frac{1}{\lim_{t\to 0} \frac{\sin t}{t}} = 1.$$

Also, $\lim_{t\to 0}(2\cos t) = 2\cos(0) = 2$. Therefore,

$$\lim_{t\to 0} \frac{2t}{\tan t} = \lim_{t\to 0} \left(2\cos t \cdot \frac{t}{\sin t}\right) = \lim_{t\to 0} (2\cos t) \cdot \lim_{t\to 0} \frac{t}{\sin t} = 2.$$

(f) Note that $1 + x^3 = (1+x)(1-x+x^2)$.

$$\begin{aligned}
\lim_{x\to -1^+} \left(\frac{1}{1+x} - \frac{3}{1+x^3}\right) &= \lim_{x\to -1^+} \left(\frac{1-x+x^2}{(1+x)(1-x+x^2)} - \frac{3}{(1+x)(1-x+x^2)}\right) \\
&= \lim_{x\to -1^+} \frac{x^2-x-2}{(1+x)(1-x+x^2)} \\
&= \lim_{x\to -1^+} \frac{(1+x)(x-2)}{(1+x)(1-x+x^2)} \\
&= \lim_{x\to -1^+} \frac{x-2}{1-x+x^2} \\
&= \frac{-1-2}{1-(-1)+(-1)^2} \\
&= -1.
\end{aligned}$$

$\square$

QUESTION 2. **(2 marks)**

Use the precise definition to prove the limit,

$$\lim_{x \to -2}(x^2 + 2x + 3) = 3.$$

Solution.

Let ϵ be a given positive number. To prove the limit, we only need to find a number $\delta > 0$ such that

$$\text{if } 0 < |x+2| < \delta \text{ then } |(x^2 + 2x + 3) - 3| = |x^2 + 2x| < \epsilon.$$

But

$$|x^2 + 2x| = |x(x+2)| = |x| \cdot |x+2|.$$

If $|x+2| < 1$, then $-3 < x < -1$ and $|x| < 3$. Thus,

$$|x^2 + 2x| = |x| \cdot |x+2| < 3|x+2|.$$

If $|x+2| < \frac{1}{3}\epsilon$, then

$$|x^2 + 2x| < 3 \cdot |x+2| < \epsilon.$$

This suggests that we should choose $\delta = \min\left\{1, \dfrac{1}{3}\epsilon\right\}$. $\square$

QUESTION 3. **(4 marks)**

Show that there is at least one root of the equation

$$x + 3\cos x = 0.$$

Solution.

Consider the function $f(x) = x + 3\cos x$. We apply the I.V.T. to this function on the interval $[-\pi, 0]$ with $N = 0$. The first thing we have to check is that $f(x)$ is continuous on $[-\pi, 0]$. This is true because

(i) $f(x)$ is continuous on the closed interval $[-\pi, 0]$.

(ii) $f(-\pi) = -\pi + 3\cos\pi = -\pi - 3 < 0$.

(iii) $f(0) = 0 + 3\cos 0 = 3 > 0$.

So because $f(0) > 0 > f(-\pi)$ we deduce from the I.V.T. that there exist a c in $(-\pi, 0)$ where $f(c) = 0$. This c will solve the given equation. $\square$

QUESTION 4. **(5 marks)**

Consider

$$f(x) = \begin{cases} -3, & x \le -1 \\ cx - d, & -1 < x < 1 \\ 3, & x \ge 1 \end{cases}$$

(a) For what values of c and d is $f(x)$ continuous at every x?

(b) With the values of c and d found in (a), find $f'(x)$ and its domain.

Solution.

(a) $f(x)$ is clearly continuous on $x < -1, -1 < x < 1$ and $x > 1$. It remains to check the points $x = -1$ and $x = 1$. By the definition of continuity,

$$\lim_{x\to -1} f(x) = f(-1) \text{ and } \lim_{x\to 1} f(x) = f(1).$$

For $\lim_{x\to -1} f(x)$ and $\lim_{x\to -1} f(x)$ to exist, we require

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) \implies -3 = c(-1) - d \implies c + d = 3.$$

and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) \implies c - d = 3.$$

Solving the equations, we obtain $c = 3$ and $d = 0$.

(b) We see that for $x < -1$ and $x > 1$, $f'(x) = 0$. For $-1 < x < 1$, $f'(x) = \frac{d}{dx}(3x) = 3$. It remains to check the differentiability at $x = -1$ and $x = 1$.

We see that $f'(-1)$ is not defined as $\lim_{x\to -1^-} f'(x) = 0$ and $\lim_{x\to -1^+} f'(x) = 3$ which implies $\lim_{x\to -1} f'(x)$ does not exist and f is not differentiable at $x = -1$.
Similarly, $f'(1)$ is not defined as $\lim_{x\to 1^-} f'(x) = 3$ and $\lim_{x\to 1^+} f'(x) = 0$ which implies f is not differentiable at $x = 1$. Therefore,

$$f'(x) = \begin{cases} 0, & x < -1 \text{ or } x > 1 \\ 3, & -1 < x < 1 \end{cases}$$

and its domain is $\mathbb{R}\backslash\{-1, 1\}$.

$\square$

QUESTION 5. **(3 marks)**

Evaluate the following derivatives.

(a) $\dfrac{d}{dx}\left(\dfrac{(x+1)(x^2-2x)}{x^4}\right)$ (b) $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}+\dfrac{x}{\sin x}\right)$ (c) $\dfrac{d^{110}}{dx^{110}}(\sin x+4\cos x)$

Solution.

(a) By expansion on the numerator, the function $\dfrac{(x+1)(x^2-2x)}{x^4}$ can be simplified to

$$\frac{x^3-x^2-2x}{x^4}$$

which can further simplfied to

$$\frac{1}{x}-\frac{1}{x^2}-\frac{2}{x^3}.$$

Therefore,

$$\frac{d}{dx}\left(\frac{(x+1)(x^2-2x)}{x^4}\right)=\frac{d}{dx}\left(\frac{1}{x}-\frac{1}{x^2}-\frac{2}{x^3}\right)=-\frac{1}{x^2}+\frac{2}{x^3}+\frac{6}{x^4}=\frac{-x^2+2x+6}{x^4}.$$

(b) We apply quotient rule to $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right)$ and we obtain:

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right)=\frac{x\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}(x)}{x^2}=\frac{x\cos x-\sin x}{x^2}.$$

Similarly, for $\dfrac{d}{dx}\left(\dfrac{x}{\sin x}\right)$ and we obtain:

$$\frac{d}{dx}\left(\frac{x}{\sin x}\right)=\frac{\sin x\frac{d}{dx}(x)-x\frac{d}{dx}(\sin x)}{\sin^2 x}=\frac{\sin x-x\cos x}{\sin^2 x}.$$

Therefore,

$$\frac{d}{dx}\left(\frac{\sin x}{x}+\frac{x}{\sin x}\right)=\frac{x\cos x-\sin x}{x^2}+\frac{\sin x-x\cos x}{\sin^2 x}.$$

(c) One can observe the fact that:

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x & \frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d^2}{dx^2}(\sin x) = \frac{d}{dx}(\cos x) &= -\sin x & \frac{d^2}{dx^2}(\cos x) = \frac{d}{dx}(-\sin x) &= -\cos x \\
\frac{d^3}{dx^3}(\sin x) = \frac{d}{dx}(-\sin x) &= -\cos x & \frac{d^3}{dx^3}(\cos x) = \frac{d}{dx}(-\cos x) &= \sin x \\
\frac{d^4}{dx^4}(\sin x) = \frac{d}{dx}(-\cos x) &= \sin x & \frac{d^4}{dx^4}(\cos x) = \frac{d}{dx}(\sin x) &= \cos x.
\end{aligned}$$

Note that the differentiation cycle is in a cycle of 4. Since 110 gives a reaminder of 2 when divided by 4, therefore,

$$\frac{d^{110}}{dx^{110}}(\sin x + 4\cos x) = \frac{d^2}{dx^2}(\sin x + 4\cos x) = -\sin x - 4\cos x.$$

$\square$

QUESTION 6 (Optional). **(1 bonus mark)**

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$.

Solution.

Suppose not. We assume that $L \neq M$. Without loss of generality, suppose $L > M$.

Given $\lim_{x \to a} f(x) = L$, by the precise definiton of a limit:

For every $\epsilon_1 > 0$, there exist a $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then $|f(x) - L| < \epsilon_1$.

Similarly, given $\lim_{x \to a} f(x) = M$, by the precise definiton of a limit:

For every $\epsilon_2 > 0$, there exist a $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $|f(x) - M| < \epsilon_2$.

Pick ϵ_1 and ϵ_2 to be $\frac{L-M}{2}$ which is clearly positive. To fulfill both inequalities, we pick $\delta = \min\{\delta_1, \delta_2\}$.

From $|f(x) - L| < \epsilon_1$,

$$\begin{aligned} |f(x) - L| < \frac{L-M}{2} &\implies -\frac{L-M}{2} < f(x) - L < \frac{L-M}{2} \\ &\implies L - \frac{L-M}{2} < f(x) < L + \frac{L-M}{2} \\ &\implies \frac{L+M}{2} < f(x) < \frac{3L-M}{2}. \end{aligned}$$

From $|f(x) - M| < \epsilon_2$,

$$\begin{aligned} |f(x) - M| < \frac{L-M}{2} &\implies -\frac{L-M}{2} < f(x) - M < \frac{L-M}{2} \\ &\implies M - \frac{L-M}{2} < f(x) < M + \frac{L-M}{2} \\ &\implies \frac{3M-L}{2} < f(x) < \frac{M+L}{2}. \end{aligned}$$

Clearly we see a contradiction with $\frac{L+M}{2} < f(x) < \frac{M+L}{2}$, therefore we must have $L = M$.

A similar argument follows if we suppose $L < M$. $\square$