SPMS / Division of Mathematical Sciences

MH1300 Foundations of Mathematics
2021/2022 Semester 1

MID-TERM EXAM SOLUTIONS

QUESTION 1. **(20 marks)**

Show the following. Justify all of your answers.

(a) Using logical equivalences, deduce whether $p \to (q \leftrightarrow (p \wedge q))$ is a tautology, contradiction, or neither. You may use the fact that $a \leftrightarrow b \equiv (a \to b) \wedge (b \to a)$ for every statement forms a, b.

(b) Let $Q(m)$ be the predicate "m is even". Write down predicates $P(m)$ and $R(m)$ such that:

(i) For every $m \in \mathbb{Z}$, $P(m)$ is sufficient for $Q(m)$, but $P(m)$ is not necessary for $Q(m)$ for some $m \in \mathbb{Z}$.

(ii) For every $m \in \mathbb{Z}$, $R(m)$ is necessary for $Q(m)$, but $R(m)$ is not sufficient for $Q(m)$ for some $m \in \mathbb{Z}$.

You need to explain your answers.

SOLUTION . (a) The question specifically asks for a solution using logical equivalences:

$$
\begin{aligned}
& p \to (q \leftrightarrow (p \wedge q)) && [\text{Using } a \to b \equiv \neg a \vee b] \\
\equiv\ & \neg p \vee (q \leftrightarrow (p \wedge q)) && [\text{Using } a \leftrightarrow b \equiv (a \to b) \wedge (b \to a)] \\
\equiv\ & \neg p \vee ([q \to (p \wedge q)] \wedge [(p \wedge q) \to q]) && [\text{Using } a \to b \equiv \neg a \vee b] \\
\equiv\ & \neg p \vee ([\neg q \vee (p \wedge q)] \wedge [\neg(p \wedge q) \vee q]) && [\text{De Morgan's law}] \\
\equiv\ & \neg p \vee ([\neg q \vee (p \wedge q)] \wedge [(\neg p \vee \neg q) \vee q]) && [\text{Associative law}] \\
\equiv\ & \neg p \vee ([\neg q \vee (p \wedge q)] \wedge [\neg p \vee (\neg q \vee q)]) && [\text{Negation law}] \\
\equiv\ & \neg p \vee ([\neg q \vee (p \wedge q)] \wedge [\neg p \vee \mathbf{T}]) && [\text{Universal bound law}] \\
\equiv\ & \neg p \vee ([\neg q \vee (p \wedge q)] \wedge \mathbf{T}) && [\text{Identity law}] \\
\equiv\ & \neg p \vee (\neg q \vee (p \wedge q)) && [\text{Associative law}] \\
\equiv\ & (\neg p \vee \neg q) \vee (p \wedge q) && [\text{De Morgan's law}] \\
\equiv\ & \neg(p \wedge q) \vee (p \wedge q) && [\text{Negation law}] \\
\equiv\ & \mathbf{T}
\end{aligned}
$$

It is a tautology.

(b)(i) Let $P(m)$ be the predicate "m is divisible by 4" and $Q(m)$ be "m is even". Then $P(m)$ is sufficient for $Q(m)$ for every m, because if $P(m)$ holds then $m = 4k$ for some $k \in \mathbb{Z}$, which means that $m = 2(2k)$ and hence m is even. On the other hand, $P(m)$ is not necessary for $Q(m)$ for $m = 2$, because $Q(2)$ is true (as 2 is even) but $P(2)$ is not (as 2 is not divisible by 4).

You can also take $P(m)$ to be "m is divisible by 6" or "m is even and positive", and many others.

(b)(ii) Let $R(m)$ be the predicate "n is an integer". Then $R(m)$ is necessary for $Q(m)$ for every $m \in \mathbb{Z}$, because if $R(m)$ is always true for any $m \in \mathbb{Z}$. On the other hand, $R(m)$ is not sufficient for $Q(m)$ for $m = 1$, because $R(1)$ is true (as 1 is an integer but $Q(1)$ is not (as 1 is not even).

You can also take $R(m)$ to be "m is even or $m > 0$" □

QUESTION 2 **(10 marks)**

Determine if the following is true or false. Justify your answer.

There are positive integers n and m such that $2m^2 + 3n^2 = 31$.

SOLUTION . This statement is false. We need to prove that for any two positive integers n, m, $2m^2 + 3n^2 \neq 30$. Let n, m be positive integers.

Case 1: $m > 3$: Then $2m^2 \geq 2 \cdot 4^2 = 32$ which means that $2m^2 + 3n^2 \geq 2m^2 \geq 32$ and so $2m^2 + 3n^2 \neq 31$.

Case 2: $n > 3$: Then $3n^2 \geq 3 \cdot 4^2 = 48$ which means that $2m^2 + 3n^2 \geq 3n^2 \geq 48$ and so $2m^2 + 3n^2 \neq 31$.

Case 3: $m \leq 3$ and $n \leq 3$: Then the only possibilities are $m = 1, 2, 3$ and $n = 1, 2, 3$. We just check exhaustively every pair of values in this case, none of them gives the answer 31.

$m = 1, n = 1$: $2m^2 + 3n^2 = 5$
$m = 1, n = 2$: $2m^2 + 3n^2 = 14$
$m = 1, n = 3$: $2m^2 + 3n^2 = 29$
$m = 2, n = 1$: $2m^2 + 3n^2 = 11$
$m = 2, n = 2$: $2m^2 + 3n^2 = 20$
$m = 2, n = 3$: $2m^2 + 3n^2 = 35$
$m = 3, n = 1$: $2m^2 + 3n^2 = 21$
$m = 3, n = 2$: $2m^2 + 3n^2 = 30$
$m = 3, n = 3$: $2m^2 + 3n^2 = 45$. □

QUESTION 3. **(12 marks)**

Using the definition of $|x|$, prove that for all real numbers x and all positive real numbers d,

$$|x| < d \text{ if and only if } -d < x < d.$$

SOLUTION . Let $x, d \in \mathbb{R}$ and $d > 0$. There are two directions. First we prove "$|x| < d \to -d < x < d$". Assume that $|x| < d$. Recall the definition of

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Case 1: $x \geq 0$. Then $|x| = x$ by the definition of $|x|$. By assumption, $|x| < d$ which means that $x < d$. Since $x \geq 0$ and $d > 0$ we have $x \geq 0 > -d$. Therefore, $-d < x$ and $x < d$. So we obtain the conclusion $-d < x < d$.

Case 2: $x < 0$. Then $|x| = -x$ by the definition of $|x|$. By assumption, $|x| < d$ which means that $-x < d$ and so $x > -d$. Since $x < 0$ and $d > 0$ we have $d > 0 > x$. Therefore, $-d < x$ and $x < d$. So we obtain the conclusion $-d < x < d$.

In both cases we obtain the conclusion $-d < x < d$.

Now we prove the other direction "$-d < x < d \to |x| < d$". Assume that $-d < x$ and $x < d$. Again there are two cases.

Case 1: $x \geq 0$. Then $|x| = x$ by the definition of $|x|$. But then since $x < d$ and $|x| = x$ we have $|x| < d$.

Case 2: $x < 0$. Then $|x| = -x$ by the definition of $|x|$. But since $x > -d$ we get $-x < d$ and since $|x| = -x$ we have $|x| < d$.

In both cases we obtain the conclusion $|x| < d$. □

QUESTION 4. **(8 marks)**

Show that the following argument is valid. If you've used any rule of inference, state them.

$$\begin{array}{ll} & \neg p \to (q \to \neg r) \\ & r \to \neg p \\ & (\neg s \vee p) \to \neg\neg r \\ & \neg s \\ \therefore & \neg q \end{array}$$

SOLUTION . To show that the argument is valid, you can use truth tables. In this solution we present using rules of inference.

$\neg s$	[Premise #4]
$\neg s \vee p$	[Generalization]
$(\neg s \vee p) \to \neg\neg r$	[Premise #3]
$\neg\neg r$	[Modus Ponens]
r	[Rule of inference: Double negation]
$r \to \neg p$	[Premise #2]
$\neg p$	[Modus Ponens]
$\neg p \to (q \to \neg r)$	[Premise #1]
$q \to \neg r$	[Modus Ponens]
$\neg\neg r$	[Previously obtained]
$\neg q$	[Modus Tollens]

$\square$