Name: ______________________________ Tutorial group: ____

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2025/26

MH1100 – Calculus I

19 September 2025 Midterm Test 90 minutes

<u>INSTRUCTIONS</u>

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **SIX (6)** questions and comprises **SEVEN (7)** printed pages. Question 6 is optional.
4. The marks for each question are indicated at the beginning of each question.

For graders only	Question	1	2	3	4	5	6	Total
	Marks							

QUESTION 1. **(3 marks)**

Prove that the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$.

[Answer:] To prove continuity at $x = 0$, we check whether

$$\lim_{x \to 0} f(x) = f(0).$$

We have

$$f(0) = 0.$$

For $x \neq 0$, since $|\sin(1/x)| \leq 1$,

$$|f(x)| = \left| x^2 \sin\left(\frac{1}{x}\right) \right| \leq x^2.$$

Therefore, by the Squeeze Theorem, as $x \to 0$,

$$-x^2 \leq f(x) \leq x^2 \quad \Longrightarrow \quad \lim_{x \to 0} f(x) = 0.$$

Since

$$\lim_{x \to 0} f(x) = f(0) = 0,$$

the function f is continuous at $x = 0$.

QUESTION 2. **(6 marks)**

(a) Evaluate the limits:

$$\lim_{x\to 3}\frac{x^2-9}{x-3}, \qquad \lim_{x\to 0} x^2\, e^{-\frac{1}{x^2}}.$$

(b) Compute the one-sided limits:

$$\lim_{x\to 2^-}\frac{|x-2|}{x-2}, \qquad \lim_{x\to 2^+}\frac{|x-2|}{x-2}.$$

What can you conclude about $\lim\limits_{x\to 2}\frac{|x-2|}{x-2}$?

(c) Find the vertical asymptote(s) of:

$$f(x)=\frac{2x}{x^2-1}.$$

[Answer:]

(a)

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6.$$

For $\lim_{x\to 0} x^2\, e^{-\frac{1}{x^2}}$, since $0\le e^{-1/x^2}\le 1$ for all $x\ne 0$, we have

$$0\le x^2e^{-1/x^2}\le x^2.$$

By the Squeeze Theorem, $\lim_{x\to 0} x^2e^{-1/x^2}=0$.

(b) - For $x\to 2^-$, $x-2<0 \implies |x-2|=-(x-2)$, so

$$\frac{|x-2|}{x-2}=\frac{-(x-2)}{x-2}=-1 \implies \lim_{x\to 2^-}\frac{|x-2|}{x-2}=-1.$$

- For $x\to 2^+$, $x-2>0 \implies |x-2|=x-2$, so

$$\frac{|x-2|}{x-2}=\frac{x-2}{x-2}=1 \implies \lim_{x\to 2^+}\frac{|x-2|}{x-2}=1.$$

Since the left-hand and right-hand limits are not equal, the two-sided limit does not exist:

$$\lim_{x\to 2}\frac{|x-2|}{x-2} \text{ does not exist.}$$

(c) Vertical asymptotes occur where the denominator is zero but the numerator is nonzero:

$$x^2-1=0 \implies x=\pm 1.$$

Check the numerator: $2x\ne 0$ at $x=\pm 1$. Thus, the vertical asymptotes are

$$x=1 \quad \text{and} \quad x=-1.$$

QUESTION 3. **(4 marks)**

(a) Show that $x^3 + x - 1 = 0$ has at least one real root in the interval (0, 1).

(b) Show that the equation $e^x = 3x$ has a solution in the interval [0, 2].

[Answer:] **(a)** Define $f(x) = x^3 + x - 1$. The function $f(x)$ is a polynomial, and therefore continuous on $[0, 1]$. Evaluate f at the endpoints:

$$f(0) = 0^3 + 0 - 1 = -1,$$

$$f(1) = 1^3 + 1 - 1 = 1.$$

Since $f(0) = -1 < 0$ and $f(1) = 1 > 0$, and f is continuous on $[0, 1]$, the Intermediate Value Theorem guarantees that there exists some $c \in (0, 1)$ such that

$$f(c) = 0.$$

Therefore, the equation $x^3 + x - 1 = 0$ has at least one real root in the interval $(0, 1)$.
(b) Define

$$f(x) = e^x - 3x.$$

The function $f(x)$ is continuous on $[0, 2]$ because both e^x and $3x$ are continuous functions. Evaluate f at the endpoints:

$$f(0) = e^0 - 3 \cdot 0 = 1 - 0 = 1 > 0,$$

$$f(2) = e^2 - 3 \cdot 2 \approx 7.389 - 6 = 1.389 > 0.$$

Since $f(0)$ and $f(2)$ are both positive, we need to check inside the interval for a sign change. Evaluate at $x = 1$:

$$f(1) = e^1 - 3 \cdot 1 \approx 2.718 - 3 = -0.282 < 0.$$

Observe that $f(0) > 0$ and $f(1) < 0$. By the Intermediate Value Theorem, because f is continuous on $[0, 1]$, there exists some $c \in (0, 1)$ such that

$$f(c) = 0 \quad \Longrightarrow \quad e^c = 3c.$$

Therefore, the equation $e^x = 3x$ has at least one solution in the interval $[0, 2]$. If the closed interval [1, 2] is considered, you can find another solution in the interval $[1, 2]$ for the equation.

QUESTION 4. **(4 marks)**

Determine whether the piecewise function

$$f(x) = \begin{cases} x+1, & x < 1, \\ 3-x, & x \geq 1 \end{cases}$$

is continuous at $x = 1$.

[Answer:] Check the left-hand limit:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+1) = 1 + 1 = 2.$$

Check the right-hand limit:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3-x) = 3 - 1 = 2.$$

Compute the function value:

$$f(1) = 3 - 1 = 2.$$

Since

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 2,$$

the function is continuous at $x = 1$.

QUESTION 5. **(3 marks)**

(a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.

(b) Prove that if f is a continuous function on an interval, then so is $|f|$.

(c) Is the converse true? That is, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.

[Answer:]

(a) The absolute value function is defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Both x and $-x$ are continuous functions, and at $x = 0$:

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0, \quad \lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0, \quad |0| = 0.$$

Hence, $|x|$ is continuous at $x = 0$, and being continuous elsewhere, it is continuous everywhere.

(b) Since both the absolute value function and f are continuous functions, $|f(x)|$ is a continuous function of a continuous function. So, it is continuous.

Alternatively, let f be continuous at a point x_0. Then

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Since the absolute value function is continuous everywhere (from part (a)), we have

$$\lim_{x \to x_0} |f(x)| = |\lim_{x \to x_0} f(x)| = |f(x_0)|.$$

Hence, $|f|$ is continuous at x_0. Since x_0 is arbitrary, $|f|$ is continuous on the interval.

(c) The converse is **not necessarily true**. **Counterexample:** Define

$$f(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

Then $|f(x)| = 1$ for all x, which is continuous. However, $f(x)$ has a jump discontinuity at $x = 0$.

Hence, continuity of $|f|$ does not guarantee continuity of f.

QUESTION 6 (Optional). **(1 bonus mark)**

Let $\lim_{x\to a} f(x) = L$ and $L \neq 0$. Use the ϵ-δ definition to prove that

$$\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{L}.$$

[Answer:] Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that

$$0 < |x-a| < \delta \implies \left|\frac{1}{f(x)} - \frac{1}{L}\right| < \varepsilon.$$

Observe that

$$\left|\frac{1}{f(x)} - \frac{1}{L}\right| = \left|\frac{L - f(x)}{f(x)L}\right| = \frac{|f(x) - L|}{|f(x)|\,|L|}.$$

Since $L \neq 0$, choose $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \implies |f(x) - L| < \frac{|L|}{2}.$$

Then

$$|f(x)| = |f(x) - L + L| \geq |L| - |f(x) - L| > |L| - \frac{|L|}{2} = \frac{|L|}{2}.$$

Therefore,

$$\left|\frac{1}{f(x)} - \frac{1}{L}\right| < \frac{2}{|L|^2}|f(x) - L|.$$

To ensure $\left|\frac{1}{f(x)} - \frac{1}{L}\right| < \varepsilon$, it suffices to have

$$|f(x) - L| < \frac{|L|^2}{2}\varepsilon.$$

Since $\lim_{x\to a} f(x) = L$, there exists $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \implies |f(x) - L| < \frac{|L|^2}{2}\varepsilon.$$

Finally, let

$$\delta = \min\{\delta_1, \delta_2\}.$$

Then for $0 < |x-a| < \delta$, we have

$$\left|\frac{1}{f(x)} - \frac{1}{L}\right| < \varepsilon.$$

This completes the proof.