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Tutorial group: \_\_\_\_\_

Matriculation number:

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**NANYANG TECHNOLOGICAL UNIVERSITY**

SEMESTER I 2017/18

**MH1100 & SM2MH1100 – Calculus I**

13 October 2017

Midterm Test

90 minutes

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**INSTRUCTIONS**

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages.
4. Answer **all** questions. The marks for each question are indicated at the beginning of each question.

For graders only	Question	1	2	3	4	Total
	Marks					

## QUESTION 1. (7 marks)

Find the limits if exist.

$$(a) \lim_{x \rightarrow 2} \frac{(2x+4)(x+2)}{x^2+5x+6} \quad (b) \lim_{x \rightarrow 0} \frac{\frac{1}{x-1} + \frac{1}{x+1}}{2x} \quad (c) \lim_{x \rightarrow 4} \cos\left(\frac{x-4}{\sqrt{x}-2}\pi\right)$$

$$(d) \text{ If } \lim_{x \rightarrow 1} \frac{f(x)-5}{x-2} = 1, \text{ find } \lim_{x \rightarrow 1^+} f(x).$$

[Answer:]

(a) The function  $f(x) = \frac{(2x+4)(x+2)}{x^2+5x+6}$  is continuous at  $x = 2$ . So  $\lim_{x \rightarrow 2} f(x) = f(2)$ . Plugging in  $x = 2$ , we get

$$\lim_{x \rightarrow 2} f(x) = f(2) = \frac{8}{5}.$$

(b) This is a limit of  $\frac{0}{0}$ . We need to cancel zeros.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\frac{1}{x-1} + \frac{1}{x+1}}{2x} &= \lim_{x \rightarrow 0} \frac{\frac{1}{x-1} + \frac{1}{x+1}}{2x} \cdot \frac{(x-1)(x+1)}{(x-1)(x+1)} \\ &= \lim_{x \rightarrow 0} \frac{(x+1) + (x-1)}{2x} \cdot \frac{1}{(x-1)(x+1)} \\ &= \lim_{x \rightarrow 0} \frac{2x}{2x} \cdot \frac{1}{(x-1)(x+1)} = \lim_{x \rightarrow 0} \frac{1}{(x-1)(x+1)} \\ &= -1. \end{aligned}$$

(c) Let  $f(x) = \cos x$  and  $g(x) = \frac{x-4}{\sqrt{x}-2}\pi$ .  $f(x)$  is continuous on the real line. We have

$$\begin{aligned} \lim_{x \rightarrow 4} g(x) &= \lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2}\pi = \pi \lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2} = \pi \lim_{x \rightarrow 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} \\ &= \pi \lim_{x \rightarrow 4} (\sqrt{x}+2) = 4\pi. \end{aligned}$$

Thus, according to  $\lim_{x \rightarrow 4} f(g(x)) = f(\lim_{x \rightarrow 4} g(x))$ , we get the limit equal to  $\cos 4\pi = 1$ .

(d) When  $x$  is close to 1, we can express  $f(x)$  as

$$f(x) = \frac{f(x)-5}{x-2} \cdot (x-2) + 5.$$

Given  $\lim_{x \rightarrow 1} \frac{f(x)-5}{x-2} = 1$ ,  $\lim_{x \rightarrow 1} (x-2) = -1$ , and  $\lim_{x \rightarrow 1} 5 = 5$ , we have

$$\begin{aligned} \lim_{x \rightarrow 4} f(x) &= \lim_{x \rightarrow 4} \left[ \frac{f(x)-5}{x-2} \cdot (x-2) + 5 \right] \\ &= \lim_{x \rightarrow 4} \frac{f(x)-5}{x-2} \cdot \lim_{x \rightarrow 4} (x-2) + \lim_{x \rightarrow 4} 5 \\ &= 1 \cdot (-1) + 5 = 4. \end{aligned}$$

**QUESTION 2.**

(3 marks)

- (a) Let  $a$  and  $L$  be real numbers. State the  $\epsilon$ - $\delta$  definition of the equation  $\lim_{x \rightarrow a} f(x) = L$ .

- (b) Prove that

$$\lim_{x \rightarrow 0} \left( -\frac{1}{x^2} \right) = -\infty.$$

[Answer:]

- (a) Let  $f(x)$  be a function defined on some open interval that contains  $a$ , except possibly at  $a$  itself. Then we say that **the limit of  $f(x)$  as  $x$  approaches  $a$  is  $L$** , and we write

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number  $\epsilon > 0$  there is a number  $\delta > 0$  such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

- (b) Let  $N$  be a given negative number. We want to find a number  $\delta > 0$  such that

$$\text{if } 0 < |x - 0| < \delta \text{ then } -\frac{1}{x^2} < N.$$

But

$$-\frac{1}{x^2} < N \iff x^2 < -\frac{1}{N} \iff \sqrt{x^2} < \sqrt{-\frac{1}{N}} \iff |x| < \frac{1}{\sqrt{-N}}$$

So if we choose  $\delta = 1/\sqrt{-N}$  or a smaller number and  $0 < |x| < \delta \leq 1/\sqrt{-N}$ , then  $-1/x^2 < N$ . This shows that

$$\lim_{x \rightarrow 0} \left( -\frac{1}{x^2} \right) = -\infty.$$

**QUESTION 3.**

**(5 marks)**

Let  $L$  be a real number. The function  $f(x)$  is defined on the real line as

$$f(x) = \begin{cases} x^2 \cos \frac{1}{x}, & x < 0; \\ L, & x = 0; \\ \sqrt{x}, & x > 0. \end{cases}$$

- (a) Use the squeeze theorem to prove that  $\lim_{x \rightarrow 0^-} f(x) = 0$ .
- (b) Find  $\lim_{x \rightarrow 0^+} f(x)$ .
- (c) Based on your conclusions in parts (a) and (b), can you say anything about the limit  $\lim_{x \rightarrow 0} f(x)$ ?
- (d) Can you say anything about the continuity of  $f(x)$  at  $x = 0$ .

[Answer:]

- (a) Let  $f_1(x) = -x^2$  and  $f_2(x) = x^2$ . We know that  $f_1(x) \leq f(x) \leq f_2(x)$ . The one-sided limits of  $f_1(x)$  and  $f_2(x)$  as  $x$  approaches 0 from the left are 0. Using the squeeze theorem, we have

$$\lim_{x \rightarrow 0^-} f(x) = 0.$$

- (b) Based on the Root Law,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sqrt{x} = \sqrt{\lim_{x \rightarrow 0^+} x} = \sqrt{0} = 0.$$

- (c) Both the left-sided and right-sided limits exist and equal 0. So, the limit  $\lim_{x \rightarrow 0} f(x)$  exists and equals 0.
- (d) If  $L \neq 0$ , then  $\lim_{x \rightarrow 0} f(x) \neq f(0)$ . That means  $f(x)$  is not continuous at  $x = 0$ . If  $L = 0$ , then  $\lim_{x \rightarrow 0} f(x) = f(0) = 0$ . In this case,  $f(x)$  is continuous at  $x = 0$ .

**QUESTION 4.**

**(5 marks)**

Let  $f(x) = \sqrt{2x+5} + x^2 - 4$ .

- (a) Find the domain of  $f(x)$ .
- (b) Use the definition of continuity to show that  $f(x)$  is continuous on its domain.
- (c) Use the definition to find the derivative function  $f'(x)$ .
- (d) Prove that the equation  $f(x) = 0$  has a root in its domain. (Hint: Use the Intermediate Value Theorem.)

[Answer]

(a) The domain of  $f(x)$  is  $[-\frac{5}{2}, \infty)$ .

(b) Let  $a$  be a real number. When  $a \in (-\frac{5}{2}, \infty)$ ,

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (\sqrt{2x+5} + x^2 - 4) = \lim_{x \rightarrow a} \sqrt{2x+5} + \lim_{x \rightarrow a} (x^2 - 4) = \sqrt{2a+5} + a^2 - 4 = f(a).$$

When  $a = -\frac{5}{2}$ ,

$$\lim_{x \rightarrow (-\frac{5}{2})^+} f(x) = \lim_{x \rightarrow (-\frac{5}{2})^+} (\sqrt{2x+5} + x^2 - 4) = \frac{9}{4} = f(-\frac{5}{2}).$$

Thus,  $f(x)$  is continuous on its domain.

- (c) We use the definition of derivative, which requires us to calculate  $f(x+h)$  and then subtract  $f(x)$  to obtain the numerator in the difference quotient. We have

$$f(x) = \sqrt{2x+5} + x^2 - 4 \quad \text{and} \quad f(x+h) = \sqrt{2(x+h)+5} + (x+h)^2 - 4,$$

so

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{2(x+h)+5} + (x+h)^2 - 4) - (\sqrt{2x+5} + x^2 - 4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{2(x+h)+5} - \sqrt{2x+5} + (x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{2(x+h)+5} - \sqrt{2x+5}) \frac{\sqrt{2(x+h)+5} + \sqrt{2x+5}}{\sqrt{2(x+h)+5} + \sqrt{2x+5}} + (x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{[2(x+h)+5] - (2x+5)}{\sqrt{2(x+h)+5} + \sqrt{2x+5}} + 2xh + h^2}{h} = \lim_{h \rightarrow 0} \frac{\frac{2h}{\sqrt{2(x+h)+5} + \sqrt{2x+5}} + 2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} \left[ \frac{2}{\sqrt{2(x+h)+5} + \sqrt{2x+5}} + 2x + h \right] \\ &= \frac{1}{\sqrt{2x+5}} + 2x. \end{aligned}$$

- (d) We know  $f(x)$  is continuous on its domain. By trial and error, we find the function values  $f(0) = \sqrt{5} - 4 < 0$  and  $f(2) = 3$ . Note that  $f(x)$  is continuous on the closed interval  $[0, 2]$ . Since the value  $y = 0$  is between the two numbers  $\sqrt{5} - 4$  and 3, by the Intermediate Value Theorem there is a number  $c \in [0, 2]$  such that  $f(c) = 0$ . That is, the number  $c$  solves the equation  $f(x) = 0$ .