NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2023-2024

MH1100 – Calculus I

December 2023 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED** book exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1 **(16 marks)**

Evaluate the limits

(a)

$$\lim_{x\to 0}\frac{\sin 2023x \cdot \sin 2x}{x^2}.$$

(b)

$$\lim_{x\to\infty}\frac{2022x^2-1}{x^2+x-2023}.$$

[Solution:]

(a) Note that

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2023x \cdot \sin 2x}{x^2} &= \lim_{x\to 0}\frac{\sin 2023x}{2023x}\cdot\frac{\sin 2x}{2x}\cdot\frac{4046x^2}{x^2}\\ &= \lim_{x\to 0}\frac{\sin 2023x}{2023x}\cdot\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\lim_{x\to 0}\frac{4046x^2}{x^2}\\ &= 4046\end{aligned}$$

Therefore we have

$$\lim_{x\to 0}\frac{\sin 2023x \cdot \sin 2x}{x^2} = 4046$$

(b)

$$\begin{aligned}\lim_{x\to\infty}\frac{2022x^2-1}{x^2+x-2023} &= \lim_{x\to\infty}\frac{2022-1/x^2}{1+1/x-2023/x^2}\\ &= \frac{\lim_{x\to\infty}2022-\lim_{x\to\infty}1/x^2}{\lim_{x\to\infty}1+\lim_{x\to\infty}1/x-\lim_{x\to\infty}2023/x^2}\\ &= 2022\end{aligned}$$

QUESTION 2 **(16 marks)**

Use the ϵ-δ definition to prove the following limit

$$\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}=\frac{2}{3}.$$

[Solution:] When $x\neq 1$, we have

$$\left|\frac{x^2-1}{2x^2-x-1}-\frac{2}{3}\right|=\left|\frac{x+1}{2x+1}-\frac{2}{3}\right|=\frac{|x-1|}{3|2x+1|}$$

If we let $|x-1|<1$, then $x>0$ and $|2x+1|>1$. For $\forall\varepsilon>0$, we let $\delta=\min\{3\varepsilon,1\}$, when $|x-1|<\delta$,

$$\left|\frac{x^2-1}{2x^2-x-1}-\frac{2}{3}\right|=\frac{|x-1|}{3|2x+1|}<\frac{|x-1|}{3}<\varepsilon$$

Thus $\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}=\frac{2}{3}$.

QUESTION 3 **(16 marks)**

Use Newton's method to approximate the root of the following equation

$$x^3+3x+1=0.$$

Please start with $x_0=0$, and find the second approximation x_2.

[Solution:] We apply Newton's method with

$$f(x)=x^3+3x+1 \text{ and } f'(x)=3x^2+3$$

The Newton's formula is

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^3+3x_n+1}{3x_n^2+3}$$

When $n=0$, we have

$$x_1=x_0-\frac{x_0^3+3x_0+1}{3x_0^2+3}=-\frac{1}{3}$$

When $n = 1$, we have

$$x_2 = x_1 - \frac{x_1^3 + 3x_1 + 1}{3x_1^2 + 3} = -\frac{29}{90} \approx -0.32222.$$

QUESTION 4 **(16 marks)**

Suppose that y is an implicit function of x satisfying that

$$x^y = y^x,$$

find y'.

[Solution:] Take the logrithm on both sides

$$x^y = y^x \Rightarrow y \ln x = x \ln y$$

Take the derivative with respect to x,

$$y' \ln x + \frac{y}{x} = \ln y + \frac{xy'}{y} \Rightarrow y'(\ln x - \frac{x}{y}) = \ln y - \frac{y}{x} \Rightarrow y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

QUESTION 5 **(12 marks)**

Evaluate the limit

$$\lim_{n\to\infty} (1 + \frac{1}{n} - \frac{1}{n^2})^n.$$

[Solution:] Note that

$$(1 + \frac{1}{n} - \frac{1}{n^2})^n < (1 + \frac{1}{n})^n$$

and

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e.$$

Further, we have

$$(1+\frac{1}{n}-\frac{1}{n^2})^n = (1+\frac{n-1}{n^2})^{\frac{n^2}{n-1}-\frac{n}{n-1}} > (1+\frac{n-1}{n^2})^{\frac{n^2}{n-1}-2} = (1+\frac{1}{\frac{n^2}{n-1}})^{\frac{n^2}{n-1}-2}$$

since

$$\lim_{n\to\infty}(1+\frac{1}{\frac{n^2}{n-1}})^{\frac{n^2}{n-1}} = e$$

and

$$\lim_{n\to\infty}(1+\frac{1}{\frac{n^2}{n-1}})^{-2} = 1$$

From the squeeze theorem, we have

$$\lim_{n\to\infty}(1+\frac{1}{n}-\frac{1}{n^2})^n = e.$$

QUESTION 6 **(12 marks)**

Show that the function $y = Ae^{-x} + Bxe^{-x}$ (with A and B two arbitrary constants) satisfies the differential equation

$$y'' + 2y' + y = 0.$$

[Solution:] We have

$$y' = -Ae^{-x} + Be^{-x} - Bxe^{-x}$$

and

$$y'' = Ae^{-x} - 2Be^{-x} + Bxe^{-x}.$$

We can put them into the equation,

$$y''+2y'+y = Ae^{-x}-2Be^{-x}+Bxe^{-x}+2(-Ae^{-x}+Be^{-x}-Bxe^{-x})+Ae^{-x}+Bxe^{-x} = 0.$$

QUESTION 7 **(12 marks)**

A number a is called a fixed point of a function $f(x)$ if $f(a) = a$. Show that if $f'(x) \neq 1$ for all real numbers, then f has at most one fixed point.

[Solution:]

Since $f'(x) \neq 1$ for all real numbers, function f is differential and continuous.

Assume that f has two fixed points a and b, i.e., $f(a) = a$ and $f(b) = b$. Suppose that $b > a$.

From the mean value theorem, when f is continuous on $[a, b]$, and differential on (a, b), then there exists a number $c \in (a, b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 1,$$

which contradicts with the condition that $f'(x) \neq 1$ for all real numbers. This shows that our assumption is wrong, thus f has at most one fixed point.

END OF PAPER