NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2019-2020

MH1300– Foundations of Mathematics

December 2019 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS** a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **BOTH SIDES OF ONE A4-SIZED HELPSHEET.**
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (15 marks)

(a) Suppose that x, y, z are real numbers. Let x be irrational and $y \neq 0$ be rational. Prove that $xy + z$ or $xy - z$ is irrational.

(b) Let n be a positive composite integer. Show that there is some positive integer m such that $m \mid n$ and $1 < m \leq \sqrt{n}$.

(c) Show that the following argument form is valid. State all rules of inferences used.

$$\begin{array}{ll} & \neg p \to (q \to \neg r) \\ & r \to \neg p \\ & (\neg s \vee p) \to \neg\neg r \\ & \neg s \\ \therefore & \neg q. \end{array}$$

QUESTION 2. (12 marks)

Determine if each of the following is true or false. Justify your answer.

(a) There is a non-empty set A and a set B such that $A \in B$ and $A \subseteq B$.

(b) There is an integer n such that $n^4 + n^3 + n^2 + n$ is odd.

(c) There exist odd integers a and b such that $4 \mid (3a^2 + 7b^2)$.

QUESTION 3. (15 marks)

(a) Use mathematical induction to prove that

$$1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n$$

for every positive integer n.

(b) Prove that for every positive integer n,

$$3^{4n+1} - 5^{2n-1} \text{ is divisible by 7.}$$

QUESTION 4. (16 marks)

(a) If n is an integer, prove that $(n-1)^3 + n^3 + (n+1)^3$ is divisible by 9.

(b) Prove that an integer m is divisible by 4 if the last two digits of m form a number that is divisible by 4.

(c) Suppose that $p > 1$ is an integer with the property:

for any two integers a and b, $p \mid ab$ implies that $p \mid a$ or $p \mid b$.

Prove that $\sqrt{p}$ is irrational.

QUESTION 5. (15 marks)

(a) Let A, B and C be sets. Define the symmetric difference of A and B to be $A \Delta B = (A - B) \cup (B - A)$.

(i) Prove that $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

(ii) Write down a statement in terms of $A \Delta B$ that is both sufficient and necessary for A and B to be equal. Prove that your statement is equivalent to $A = B$.

(b) Let D be a finite set with k many elements. Calculate the size of the set $\mathcal{P}(D \times D) \times \mathcal{P}(D \times D)$ and the size of the set $\mathcal{P}(\mathcal{P}(D))$ in terms of k. Here, $\mathcal{P}(D)$ is the power set of D. Show your working.

(c) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined by $f(n) = 5n + 2$. Determine if f is one-to-one, and if f is onto. Justify your answer.

QUESTION 6. (12 marks)

(a) Suppose that $z = a + bi$ and $w = c + di$ are complex numbers. Determine when $|z + w| = |z| + |w|$ holds.

(b) Express the cube roots of 1 both in the form $re^{i\theta}$ and in the form $x + yi$.

(c) Suppose that R_1 and R_2 are transitive relations on a set A. Must $R_1 \cup R_2$ be transitive? Justify your answer.

QUESTION 7. (15 marks)

(a) Let R be the relation defined on $\mathbb{N}$ by $a\ R\ b$ iff $a^2 + b^2$ is even.

(i) Prove that R is an equivalence relation.

(ii) Describe the distinct equivalence classes of R.

(b) Use the Euclidean algorithm to find the greatest common divisor of the pair

$$126 \qquad \text{and} \qquad 224.$$

END OF PAPER