MH1300 Final Exam Solutions

AY 19/20.

$\boxed{\text{Q1(a)}}$ Suppose x, y, z are real numbers, x is irrational, $y \neq 0$ and y is rational.

We proceed by contradiction and assume that the conclusion is false, i.e. both $xy + z$ and $xy - z$ are rational.

Then by Theorem 4.2.2. of the lecture notes, the sum of any two rational numbers is rational.

Since both $xy + z$ and $xy - z$ are rational, their sum $(xy+z) + (xy-z) = 2xy$ is rational.

By Ex. 4.2.16 in Tutorial 5, $\frac{2xy}{2y}$ is rational, because $2xy$ is rational and $2y \neq 0$ is rational.

Hence, x is rational, a contradiction to the hypothesis.

$\boxed{\text{Q1b}}$ Let n be a positive integer, and n composite.

Suppose for a contradiction that there is no divisor m of n such that $1 < m \leq \sqrt{n}$.

Since $n > 1$ is composite, let p, q be divisors of n such that $p \cdot q = n$, $1 < p < n$ and $1 < q < n$.

Since $p | n$, by our assumption, $1 < p \leq \sqrt{n}$ is false. So, $1 < p$ and $p \leq \sqrt{n}$ is false. Since $1 < p$, this means $p \leq \sqrt{n}$ is false. So, we conclude that $p > \sqrt{n}$. Similarly, $q > \sqrt{n}$, by a similar argument.

Therefore, $n = p \cdot q > \sqrt{n} \cdot \sqrt{n} = n$, and we obtain $n > n$, a contradiction.

Thus, there is some divisor $m | n$ such that $1 < m \leq \sqrt{n}$.

Q1 c

$\neg s$ (premise)

$\neg s \vee p$ (generalisation)

$(\neg s \vee p) \rightarrow \neg\neg r$ (premise)

$\neg\neg r$ (Modus Ponens)

r (Double Negation Law)

$r \rightarrow \neg p$ (Premise)

$\neg p$ (Modus Ponens)

$\neg p \rightarrow (q \rightarrow \neg r)$ (Premise)

$q \rightarrow \neg r$ (Modus Ponens)

$\neg q$ (Modus Tollens)

$\boxed{\text{Q2(a)}}$ This is true. Take $A = \{0\}$ and $B = \{0, \{0\}\}$. Then, $A \neq \phi$ since $0 \in A$ and $A \in B$ since $\{0\} \in B$. Furthermore, $A \subseteq B$ since $0 \in B$.

In fact, you can choose A to be any set such that $A = \{x\}$ and take $B = \{x, A\}$. However, you cannot choose $A = \phi$ and $B = \{\phi\}$ even though $A \in B$ and $A \subseteq B$ hold, since this case, A is not non-empty.

$\boxed{\text{Q2(b)}}$ This is false. We show the negation. Given any integer n, it is either even or odd.

If n is even, then n, n^2, n^3, n^4 are all even. Then, $n + n^2 + n^3 + n^4$ is the sum of 4 even numbers, which is even.

If n is odd, then n, n^2, n^3, n^4 are all odd. Then $n + n^2 + n^3 + n^4 = \text{odd} + \text{odd} + \text{odd} + \text{odd}$. The sum of two odd numbers is even, hence, $n + n^2$ and $n^3 + n^4$ are even. However the sum of two even numbers is even, so $(n + n^2) + (n^3 + n^4)$ is even.

Q2(c) This is false. We assume this statement is true, i.e. we assume that there exist a, b odd such that $4 \mid (3a^2 + 7b^2)$.

We want to derive a contradiction.

Since a, b are odd, let $a = 2k+1$ and $b = 2l+1$ for some integers k and l.

Since $4 \mid (3a^2 + 7b^2)$, let m be an integer such that $4m = 3a^2 + 7b^2$.

Thus,
$$
\begin{aligned}
4m &= 3(2k+1)^2 + 7(2l+1)^2 \\
&= 3(4k^2 + 4k + 1) + 7(4l^2 + 4l + 1) \\
&= 12k^2 + 12k + 3 + 28l^2 + 28l + 7
\end{aligned}
$$

$$4(m - 3k^2 - 3k - 7l^2 - 7l) = 10$$

Since $m - 3k^2 - 3k - 7l^2 - 7l \in \mathbb{Z}$,

this means that $\frac{10}{4} \in \mathbb{Z}$

So, $\frac{5}{2} \in \mathbb{Z}$, a contradiction.

(Alternatively, we get $2 \mid 5$ contradicting that 5 is prime)

Hence, our assumption "$\exists$ odd a, b s.t. $4 \mid (3a^2 + 7b^2)$" must be false.

Q3a Let $P(n)$ be the statement

$$1 + 5 + 9 + \cdots + (4n-3) = 2n^2 - n.$$

Base case: $P(1)$

LHS $= 1$

RHS $= 2(1)^2 - 1 = 2 - 1 = 1$

$\therefore$ LHS $=$ RHS.

Assume $P(k)$, i.e. assume $1 + 5 + 9 + \cdots (4k-3) = 2k^2 - k$.

Want to show $P(k+1)$.

LHS of $P(k+1) = 1 + 5 + 9 + \cdots + (4k-3) + (4k+1)$

(By IH) $= (2k^2 - k) + (4k+1)$

$= 2k^2 + 3k + 1$

RHS of $P(k+1) = 2(k+1)^2 - (k+1)$

$= 2(k^2 + 2k + 1) - k - 1$

$= 2k^2 + 4k + 2 - k - 1$

$= 2k^2 + 3k + 1$

$\therefore$ LHS $=$ RHS for $P(k+1)$

$\therefore$ $P(k+1)$ holds.

$\therefore$ By Mathematical Induction, $P(n)$ holds for all $n \geq 1$.

Q3(b)

Let $P(n)$ be the statement

" $3^{4n+1} - 5^{2n-1}$ is divisible by 7 "

Base case $P(1)$:

$$3^{4+1} - 5^{2-1} = 3^5 - 5 = 243 - 5$$
$$= 238 = 7(34)$$

$\therefore 3^{4+1} - 5^{2-1}$ is divisible by 7.

Assume $P(k)$ holds, i.e. $3^{4k+1} - 5^{2k-1}$ is div by 7.

let ℓ be such that $7\ell = 3^{4k+1} - 5^{2k-1}$.

Want to show $P(k+1)$.

$$3^{4(k+1)+1} - 5^{2(k+1)-1} = 3^{4k+5} - 5^{2k+1}$$
$$= 3^{4k+1} \cdot 3^4 - 5^{2k-1} \cdot 5^2$$
$$= 81 \cdot 3^{4k+1} - 25 \cdot 5^{2k-1}$$
$$= 77 \cdot 3^{4k+1} - 21 \cdot 5^{2k-1} + 4(3^{4k+1} - 5^{2k-1})$$
$$= 7(11 \cdot 3^{4k+1} - 3 \cdot 5^{2k-1}) + 4 \cdot 7\ell$$
$$= 7(11 \cdot 3^{4k+1} - 3 \cdot 5^{2k-1} + 4\ell)$$

Since $11 \cdot 3^{4k+1} - 3 \cdot 5^{2k-1} + 4\ell \in \mathbb{Z}$, this means that $3^{4(k+1)+1} - 5^{2(k+1)-1}$ is divisible by 7.

Hence $P(k+1)$ is true. By Mathematical Induction, $P(n)$ holds for all $n \geq 1$.

$\boxed{\text{Q 4(a)}}$ Let n be an integer.

$$(n-1)^3 + n^3 + (n+1)^3$$

$$= (n^3 - 3n^2 + 3n - 1) + n^3 + (n^3 + 3n^2 + 3n + 1)$$

$$= 3n^3 + 6n = 3n(n^2 + 2)$$

By the Quotient Remainder Theorem applied to $d = 3$, there are 3 cases:

<u>Case 1</u> $n = 3q$ for some q. Then $3n(n^2+2) = 9q(n^2+2)$

$$= 9(q(n^2+2)) \text{ is divisible by } 9.$$

<u>Case 2</u>: $n = 3q+1$ for some q. Then $3n(n^2+2) = 3n((3q+1)^2+2)$

$$= 3n(9q^2 + 6q + 1 + 2)$$

$$= 9n(3q^2 + 2q + 1) \text{ is divisible by } 9.$$

<u>Case 3</u>: $n = 3q+2$ for some q. Then $3n(n^2+2) = 3n((3q+2)^2+2)$

$$= 3n(9q^2 + 12q + 4 + 2)$$

$$= 9n(3q^2 + 4q + 2) \text{ is divisible by } 9.$$

In any case, $(n-1)^3 + n^3 + (n+1)^3$ is divisible by 9.

$\boxed{Q4(b)}$ Let m be an integer, with digits

$d_k d_{k-1} d_{k-2} \cdots d_3 d_2 d_1 d_0$.

Then, $m = 10^k d_k + 10^{k-1} d_{k-1} + \cdots + 10^2 d_2 + 10 d_1 + d_0$.

We assume $k \geq 2$, by choosing $d_k, \cdots d_1, d_0 = 0$ if necessary.

Suppose $d_1 d_0$ is a number divisible by 4.

Hence, $10d_1 + d_0$ is divisible by 4.

Let a be an integer such that $4a = 10d_1 + d_0$.

Then
$$
\begin{aligned}
m &= 10^k d_k + \cdots + 10^2 d_2 + (10d_1 + d_0) \\
&= 2^k 5^k d_k + \cdots + 2^2 5^2 d_2 + 4a \\
&= 2^2 (2^{k-2} 5^k d_k + \cdots + 5^2 d_2) + 4a \\
&= 4 (2^{k-2} 5^k d_k + \cdots + 5^2 d_2 + a)
\end{aligned}
$$

Since $2^{k-2} 5^k d_k + \cdots + 5^2 d_2 + a$ is an integer

as $k \geq 2$, this means m is divisible by 4.

$\boxed{Q4(c)}$ We proceed by contradiction.

Suppose p is a positive integer with the property

$$\forall a,b \in \mathbb{Z} \ (p|ab \rightarrow (p|a \text{ or } p|b)),$$

and that $\sqrt{p}$ is rational.

let $r, s \in \mathbb{Z}$, $s \neq 0$ such that $\sqrt{p} = \frac{r}{s}$, $\frac{r}{s}$ is in lowest form.

Then $p = \frac{r^2}{s^2}$ and $r^2 = s^2 p$.

Since $s^2 \in \mathbb{Z}$, this means $p | r^2$.

By the assumption on p, we have $p|r$ or $p|r$,

hence p must divide r.

let t be such that $p \cdot t = r$

So, $s^2 p = r^2 = (pt)^2 = p^2 t^2$

Since $p > 0$, we may divide p on both sides,

$$s^2 = pt^2.$$

Hence $p | s^2$. By assumption on p, we have $p|s$ or $p|s$.

Hence p must divide s.

So, $p|s$ and $p|r$. Since $p > 1$, this contradicts

our assumption that $\frac{r}{s}$ is in the lowest form.

$\boxed{Q\,5\,a\,(i)}$ WTS: $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$

$x \in A \cap (B \triangle C)$

$\Leftrightarrow$ $x \in A$ and $x \in B \triangle C$

$\Leftrightarrow$ $x \in A$ and ($x \in B - C$ or $x \in C - B$)

$\Leftrightarrow$ ($x \in A$ and $x \in B - C$) or ($x \in A$ and $x \in C - B$)

$\Leftrightarrow$ ($x \in A$ and $x \in B$ and $x \notin C$) or ($x \in A$ and $x \in C$ and $x \notin B$)

$\Leftrightarrow$ ($x \in A$ and $x \in B$ and ($x \notin A$ or $x \notin C$))

or ($x \in A$ and $x \in C$ and ($x \notin A$ or $x \notin B$))

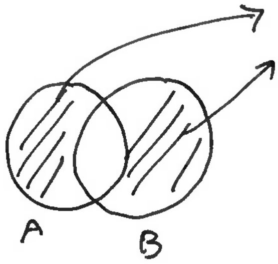
[Note: If $x \in A$ then $x \notin C$ is equivalent to $x \notin C$ or $x \notin A$.]

$\Leftrightarrow$ ($x \in A \cap B$ and $x \notin A \cap C$) or ($x \in A \cap C$ and $x \notin A \cap B$)

$\Leftrightarrow$ $x \in (A \cap B) - (A \cap C)$ or $x \in (A \cap C) - (A \cap B)$

$\Leftrightarrow$ $x \in (A \cap B) \triangle (A \cap C)$

$\boxed{\text{Q 5a(ii)}}$



the union of these two parts is $A \triangle B$.

If $A=B$ then this region is empty.

Condition is $A \triangle B = \phi$.

WTS: $A=B$ iff $A \triangle B = \phi$.

We prove contrapositive: $A \neq B \iff A \triangle B \neq \phi$.

$$
\begin{aligned}
A \neq B &\iff A \not\subseteq B \text{ or } B \not\subseteq A \\
&\iff \exists x \,(x \in A \ \& \ x \notin B) \text{ or } \exists y \,(y \in B \ \& \ y \notin A) \\
&\iff A - B \neq \phi \text{ or } B - A \neq \phi \\
&\iff (A-B) \cup (B-A) \neq \phi \\
&\iff A \triangle B \neq \phi.
\end{aligned}
$$

$\boxed{Q5b}$ In lecture, we said that

- number of elements in $A \times B$ = (number of elements in A).(number of elements in B)
- number of elements in $\mathcal{P}(A) = 2^{\text{number of elements in } A}$.

Using this facts, number of elements in $\mathcal{P}(D \times D) = 2^{k^2}$

and size of $\mathcal{P}(D \times D) \times \mathcal{P}(D \times D) = 2^{k^2} \cdot 2^{k^2}$

$= 2^{2k^2}$

Size of $\mathcal{P}(\mathcal{P}(D)) = 2^{2^k}$.

$\boxed{Q5c}$ Let $f: \mathbb{Z} \to \mathbb{Z}$ be $f(n) = 5n+2$.

f is one-one: let $5n+2 = 5m+2$

$\Rightarrow 5n = 5m$

$\Rightarrow n = m$.

f is not surjective/onto: let $y = 0 \in \mathbb{Z}$.

If it were surjective, then $f(n) = 0$ for some $n \in \mathbb{Z}$.

So, $5n+2 = 0$ for some $n \in \mathbb{Z}$

So, $n = -\frac{2}{5}$ for some $n \in \mathbb{Z}$, a contradiction.

So f is not surjective.

$\boxed{Q6(a)}$ $|z+w| = |(a+c)+(b+d)i| = \sqrt{(a+c)^2+(b+d)^2}$

$|z|+|w| = \sqrt{a^2+b^2} + \sqrt{c^2+d^2}$

$|z+w| = |z|+|w|$

$\Leftrightarrow (a+c)^2 + (b+d)^2 = \left(\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\right)^2$

$= (a^2+b^2) + (c^2+d^2) + 2\sqrt{(a^2+b^2)(c^2+d^2)}$

$\Leftrightarrow ac+bd = \sqrt{(a^2+b^2)(c^2+d^2)}$

This condition holds if and only if $ac+bd \geq 0$ and

$(ac+bd)^2 = (a^2+b^2)(c^2+d^2)$

$\Leftrightarrow a^2c^2+b^2d^2+2abcd = a^2c^2+b^2d^2+b^2c^2+b^2d^2$ and $ac+bd \geq 0$

$\Leftrightarrow 2abcd = a^2d^2+b^2c^2$ and $ac+bd \geq 0$

$\Leftrightarrow a^2d^2+b^2c^2-2abcd = 0$ and $ac+bd \geq 0$

$\Leftrightarrow (ad-bc)^2 = 0$ and $ac+bd \geq 0$

$\Leftrightarrow ad = bc$ and $ac+bd \geq 0$.

Note that under the condition $ad=bc$, whenever $ac \geq 0$ we must also have $bd \geq 0$, and vice versa. So, the acceptable answers are:

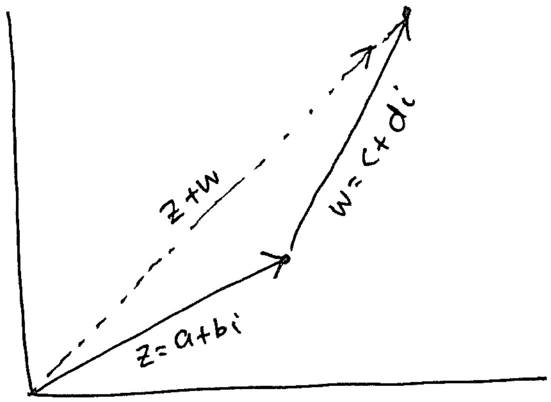
$$\left.\begin{array}{lll} ad = bc & \text{and} & ac+bd \geq 0 \\ ad = bc & \text{and} & ac \geq 0 \\ ad = bc & \text{and} & bd \geq 0 \end{array}\right\} \text{all acceptable answers.}$$

$\boxed{Q\,6(a)}$ Alternate solution.

$|z+w|$ = length of vector representing $z+w$

$|z|$ = length of vector representing z

$|w|$ = " " " " w



length of vector $z+w$ = length of z + length of w

exactly when z and w are parallel and pointing in the same direction.

exactly when $\frac{b}{a} = \frac{d}{c}$ ⇝ slopes are equal

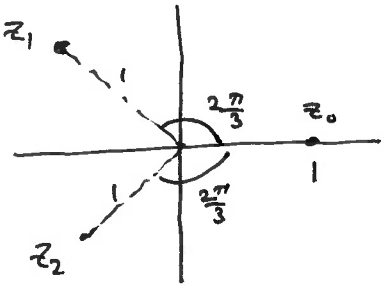
and $ac \geq 0$ ⇝ a & c are both +ve or both −ve

and $bd \geq 0$

$\boxed{Q6b}$ $z^3 = 1 = 1e^{i0}$, take $r=1$, $\theta = 0$.

$$z = r^{\frac{1}{3}} e^{i\frac{2\pi}{3}},\ r^{\frac{1}{3}} e^{i\frac{4\pi}{3}},\ r^{\frac{1}{3}} e^{i\frac{6\pi}{3}}$$

$$= e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}},\ e^{i2\pi}$$



$z_0 = 1$

$z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\boxed{Q6c}$ No. For example $R_1 = \{(0,1)\}$, $R_2 = \{(1,2)\}$ are transitive relations on $\mathbb{Z}$.

However, $R_1 \cup R_2 = \{(0,1), (1,2)\}$ is not transitive, since $(0,2) \notin R_1 \cup R_2$.

Q7a(i) Define relation R on $\mathbb{N}$ by

$(a,b) \in R$ iff a^2+b^2 is even.

R is reflexive : let $a \in \mathbb{N}$. Then $a^2+a^2 = 2a^2$ is even.

So, $(a,a) \in R$.

R is symmetric: let $a,b \in \mathbb{N}$ such that $(a,b) \in R$.

Then, a^2+b^2 is even. Since $b^2+a^2 = a^2+b^2$,

So b^2+a^2 is even also. So, $(b,a) \in R$.

R is transitive: let $a,b,c \in \mathbb{N}$ s.t. $(a,b) \in R$ and $(b,c) \in R$.

Then a^2+b^2 is even and b^2+c^2 is even.

let $k, l \in \mathbb{Z}$ s.t. $a^2+b^2 = 2k$, $b^2+c^2 = 2l$.

Then $(a^2+b^2)+(b^2+c^2) = 2(k+l)$.

$a^2+c^2 = 2(k+l) - 2b^2 = 2(k+l-b^2)$

So, a^2+c^2 is even. Hence, $(a,c) \in R$.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

$\boxed{Q7\ a(ii)}$

Notice that $(a,b) \in R$ iff $a^2 + b^2$ is even

iff a^2, b^2 are both even or a^2, b^2 both odd.

iff a, b both even or a, b both odd.

So if a is any even natural number,

$[a] = \{0, 2, 4, 6, \dots\}$

If a is any odd number,

$[a] = \{1, 3, 5, 7, \dots\}$.

There are only two distinct equivalence classes,

$[0]$ = set of even natural numbers,

$[1]$ = set of odd natural numbers.

$\boxed{Q7(b)}$

$$224 = 126 \times 1 + 98$$

$$126 = 98 \times 1 + 28$$

$$98 = 28 \times 3 + 14$$

$$28 = 14 \times 2 + 0$$

$\gcd(224, 126) = 14.$