MH1200 Linear Algebra I – Solutions

Final Examination, Semester 1, Academic Year 2017/2018

November 7, 2025

Question 1

(a) Let

$$A = \begin{bmatrix} 1 & r & 1 & 1 \\ 1 & s & 1 & 2 \\ 1 & t & 1 & 3 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Compute $\det(A)$ in terms of r, s, t.

(b) Let

$$B = \begin{bmatrix} 1 & 1 & 1 & r \\ 1 & 2 & 1 & s \\ 1 & 3 & 1 & t \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(B is A with columns 2 and 4 exchanged). Give E such that $B = AE$.

(c) **True or False:** The column space of A is the same as the column space of B? Justify.

(d) **True or False:** The dimension of the row space of A is the same as that of B? Justify.

(e) Let

$$C = \begin{bmatrix} 1 & r & 1 & 1 \\ 1 & r & 1 & 2 \\ 1 & r & 1 & 3 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Find a basis for the column space of C.

(f) Find a nonzero vector in the left nullspace of C.

Solution

(a) **Determinant:**

Expand along the **last row** $(1, 0, 0, 1)$

$$\begin{aligned}
\det(A) &= -1 \begin{vmatrix} r & 1 & 1 \\ s & 1 & 2 \\ t & 1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & r & 1 \\ 1 & s & 2 \\ 1 & t & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & r & 1 \\ 1 & s & 1 \\ 1 & t & 1 \end{vmatrix} \\
&= - \begin{vmatrix} r & 1 & 1 \\ s & 1 & 2 \\ t & 1 & 3 \end{vmatrix} + 0 \\
&= - \left(r \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} s & 2 \\ t & 3 \end{vmatrix} + 1 \begin{vmatrix} s & 1 \\ t & 1 \end{vmatrix} \right) \\
&= - \left(r(1 \cdot 3 - 2 \cdot 1) - (s \cdot 3 - 2 \cdot t) + (s \cdot 1 - 1 \cdot t) \right) \\
&= - \left(r(3 - 2) - (3s - 2t) + (s - t) \right) \\
&= \boxed{2s - r - t}
\end{aligned}$$

(b) **Permutation Matrix:** E is the 4×4 matrix that swaps columns 2 and 4:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(c) **Column Spaces:** True. Swapping columns does not change the span; they span the same space in $\mathbb{R}^4$.

(d) **Row Spaces:** True. Rank is unchanged by column swaps; both have same number of pivots.

(e) **Basis for Column Space:** Columns 1, 3, 4 are linearly independent in general:

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \right\}}$$

(f) **Left Nullspace:** Solve $y^T C = 0$ for $y \neq 0$: $y = (1, -3, 2, 0)^T$ works:

$$\boxed{(1, -3, 2, 0)^T}$$

Question 2

Let:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

(a) Compute $\langle \vec{v}, \vec{v} \rangle$ and $\langle \vec{v}, \vec{u} \rangle$.

(b) Show that the sequence $\vec{v}, \vec{u}, \vec{w}$ is linearly independent.

(c) Find a basis for $\mathbb{R}^4$ containing $\vec{v}, \vec{u}, \vec{w}$. Justify.

(d) Find the projection of $(1, 2, 3, 4)$ onto the line $\{t\vec{w} : t \in \mathbb{R}\}$.

(e) Is it always the case that $\vec{z} = \text{proj}_L(\vec{z})$ for arbitrary $\vec{z}$? Justify.

Solution

(a) **Inner Products:**

$$\langle \vec{v}, \vec{v} \rangle = 1^2 + 1^2 + 1^2 + 1^2 = 4$$
$$\langle \vec{v}, \vec{u} \rangle = 1 * 1 + 1 * (-1) + 1 * (-1) + 1 * 1 = 1 - 1 - 1 + 1 = 0$$

$$\boxed{\langle \vec{v}, \vec{v} \rangle = 4, \quad \langle \vec{v}, \vec{u} \rangle = 0}$$

(b) **Linear Independence:**

Suppose $a\vec{v} + b\vec{u} + c\vec{w} = \vec{0}$.

Write the system by equating each coordinate:

$$\begin{aligned} a + b + c &= 0 \\ a - b + c &= 0 \\ a - b + c &= 0 \\ a + b - c &= 0 \end{aligned}$$

By subtraction:

$$(a + b + c) - (a - b + c) = 2b = 0 \implies b = 0$$

Then, from first equation, $a + 0 + c = 0 \implies c = -a$.

Plug into final equation: $a + 0 - c = 0 \implies a - (-a) = 0 \implies 2a = 0 \implies a = 0$

So $c = -a = 0$. Only solution $a = b = c = 0$.

$$\boxed{\text{The set } \{\vec{v}, \vec{u}, \vec{w}\} \text{ is linearly independent.}}$$

(c) **Basis Extension:**

We wish to find a vector not in the span of $\{\vec{v}, \vec{u}, \vec{w}\}$ to extend it to a basis of $\mathbb{R}^4$. Consider $\vec{e}_2 = (0, 1, 0, 0)^\top$. Suppose, for contradiction, that

$$a\vec{v} + b\vec{u} + c\vec{w} = \vec{e}_2.$$

Then, by comparing coordinates,

$$\begin{cases} a + b + c = 0, \\ a - b + c = 1, \\ a - b + c = 0, \\ a + b - c = 0. \end{cases}$$

Subtracting the third equation from the second gives $1 = 0$, a contradiction. Hence, $\vec{e}_2 \notin \text{span}\{\vec{v}, \vec{u}, \vec{w}\}$.

Therefore, the set

$$\boxed{\{\vec{v},\ \vec{u},\ \vec{w},\ \vec{e}_2\}}$$

is linearly independent and forms a basis for $\mathbb{R}^4$.

(d) **Projection of** $(1, 2, 3, 4)$ **onto line** $\{t\vec{w} : t \in \mathbb{R}\}$**:**

Projection formula:

$$\text{proj}_{\vec{w}}(\vec{x}) = \frac{\langle \vec{x}, \vec{w} \rangle}{\langle \vec{w}, \vec{w} \rangle}\vec{w}$$

Compute:

$$\langle (1, 2, 3, 4), \vec{w} \rangle = 1 * 1 + 2 * 1 + 3 * 1 + 4 * (-1) = 1 + 2 + 3 - 4 = 2$$

$$\langle \vec{w}, \vec{w} \rangle = 1^2 + 1^2 + 1^2 + (-1)^2 = 4$$

So,

$$\text{proj}_{\vec{w}}((1, 2, 3, 4)) = \frac{2}{4}\vec{w} = \frac{1}{2}(1, 1, 1, -1) = (0.5, 0.5, 0.5, -0.5)$$

$$\boxed{(0.5,\ 0.5,\ 0.5,\ -0.5)}$$

(e) **Is** $\vec{z} = \text{proj}_L(\vec{z})$ **always?**

No. Only when $\vec{z}$ actually lies on the line L (is a scalar multiple of $\vec{w}$). For most vectors, their projection onto L is not themselves.

Example above: $(1, 2, 3, 4) \neq (0.5, 0.5, 0.5, -0.5)$.

$$\boxed{\text{No, only if } \vec{z} \in L}$$

Question 3

(a) Let $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$. Show A is invertible if a, b, c distinct, and compute entry $A^{-1}_{3,1}$.

(b) $B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 1 & 1 \\ 1 & 3 & 9 & 1 & 2 \end{bmatrix}$. Determine rank of B.

(c) Give a basis for the nullspace of B.

(d) Find the least squares solution to $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

(e) Let C be invertible 4×4 matrix. Determine a basis for the row space of the 4×8 matrix

$$D = [C \; C].$$

Solution

(a) **Vandermonde Invertibility and Inverse Entry:**

$$\det A = (b-a)(c-a)(c-b),$$

so A is invertible iff a, b, c are distinct.

Using $A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$ and the cofactor for entry $(1, 3)$,

$$(A^{-1})_{3,1} = \frac{(-1)^{1+3} \det \begin{bmatrix} 1 & b \\ 1 & c \end{bmatrix}}{\det A} = \frac{c-b}{(b-a)(c-a)(c-b)} = \frac{1}{(b-a)(c-a)}.$$

$$\boxed{(A^{-1})_{3,1} = \frac{1}{(b-a)(c-a)}}.$$

(b) **Rank of B:** B has 3 rows and 5 columns. The first 3 columns correspond to a 3×3 Vandermonde, which is independent if the $2, 3$ columns are not multiples of each other. Clearly, there are three independent rows, so

$$\boxed{\operatorname{rank}(B) = 3}$$

(c) **Basis of Nullspace:**

Since $\operatorname{rank}(B) = 3$ and B has 5 columns, $\dim(\operatorname{null}(B)) = 5 - 3 = 2$.

From the row echelon form, we can write the system:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0 \quad (1)$$
$$x_2 + 3x_3 = 0 \quad (2)$$
$$2x_3 + x_5 = 0 \quad (3)$$

Setting $x_3 = s$ and $x_4 = t$ as free variables:

$$x_5 = -2s \tag{4}$$
$$x_2 = -3s \tag{5}$$
$$x_1 = -x_2 - x_3 - x_4 - x_5 = 3s - s - t + 2s = 4s - t \tag{6}$$

Therefore:

$$\mathbf{x} = s \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Basis for } \operatorname{null}(B) = \boxed{\left\{ \begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

(d) **Least Squares Solution:**

Given

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \qquad y = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

The least squares solution is given by the normal equations:

$$A^T A x = A^T y$$

Compute:

$$A^T A = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}, \quad A^T y = \begin{bmatrix} -2 + 0 + 1 \\ (-2) \cdot 1 + 0 \cdot 2 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 + 0 + 3 = 1 \end{bmatrix}$$

So normal equations are:

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solve by matrix inversion. Compute determinant: $3 \times 14 - 6 \times 6 = 42 - 36 = 6$.

Inverse:

$$\frac{\begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}}{6}$$

Multiply by right-hand side:

$$x_1 = \frac{1}{6}(14 \times (-1) - 6 \times 1) = \frac{-14 - 6}{6} = \frac{-20}{6} = -\frac{10}{3}$$
$$x_2 = \frac{1}{6}(-6 \times (-1) + 3 \times 1) = \frac{6 + 3}{6} = \frac{9}{6} = \frac{3}{2}$$

$$\boxed{x_1 = -\frac{10}{3}, \quad x_2 = \frac{3}{2}}$$

(e) **Basis for Row Space of $D = [C\ C]$:**

Let the rows of C be $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4 \in \mathbb{R}^4$.

Then the rows of $D = [C\ C]$ are $\mathbf{R}_i = [\mathbf{r}_i\ \mathbf{r}_i] \in \mathbb{R}^8$ for $i = 1, 2, 3, 4$.

To show linear independence: if $\sum_{i=1}^4 \alpha_i \mathbf{R}_i = \mathbf{0}$, then

$$\left(\sum_{i=1}^{4} \alpha_i \mathbf{r}_i,\ \sum_{i=1}^{4} \alpha_i \mathbf{r}_i \right) = (\mathbf{0}, \mathbf{0})$$

This implies $\sum_{i=1}^4 \alpha_i \mathbf{r}_i = \mathbf{0}$.

Since C is invertible, its rows are linearly independent, so all $\alpha_i = 0$.

Therefore $\{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4\}$ is linearly independent.

$$\text{Basis for Row}(D) = \boxed{\{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4\} \text{ where } \mathbf{R}_i = [\mathbf{r}_i\ \mathbf{r}_i]}$$

Question 4

Let V be the vector space of 3×3 real matrices.

(a) Let

$$S = \{A \in V : \det A = 0\}$$

Is S a subspace? Justify your answer.

(b) Let

$$T = \left\{ A \in V : A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Show T is a subspace of V.

(c) Find a basis for T and determine its dimension.

(d) If row space equals column space for $A \in \mathbb{R}^{3\times 3}$, must $A = A^T$? Prove or give a counterexample.

Solution

(a) **Is $S = \{A \in V : \det A = 0\}$ a subspace?**

Answer: No.

Counterexample: Consider

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Both have $\det A_1 = 0$ and $\det A_2 = 0$, so $A_1, A_2 \in S$.

However,

$$A_1 + A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

and $\det(I_3) = 1 \neq 0$, so $A_1 + A_2 \notin S$.

$$\boxed{\text{No, } S \text{ is not a subspace of } V.}$$

(b) **Show $T = \{A \in V : A\mathbf{1} = \mathbf{0}\}$ is a subspace.**

We verify the three subspace properties:

(a) **Contains zero:** The zero matrix satisfies $\mathbf{0} \cdot \mathbf{1} = \mathbf{0}$, so $\mathbf{0} \in T$.

(b) **Closed under addition:** If $A, B \in T$, then

$$(A+B)\mathbf{1} = A\mathbf{1} + B\mathbf{1} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

so $A + B \in T$.

(c) **Closed under scalar multiplication:** If $A \in T$ and $c \in \mathbb{R}$, then

$$(cA)\mathbf{1} = c(A\mathbf{1}) = c\mathbf{0} = \mathbf{0}$$

so $cA \in T$.

$$\boxed{T \text{ is a subspace of } V.}$$

(c) **Find a basis and dimension for T.**

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the condition $A\mathbf{1} = \mathbf{0}$ means:

$$\begin{bmatrix} a_{11} + a_{12} + a_{13} \\ a_{21} + a_{22} + a_{23} \\ a_{31} + a_{32} + a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives 3 constraints (one for each row sum). The space of 3×3 matrices has dimension 9, so $\dim(T) = 9 - 3 = 6$.

For each row, we can choose the first two entries freely, and the third is determined. A basis is:

$$B_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \tag{9}$$

$$\boxed{\dim(T) = 6, \text{ with basis } \{B_1, B_2, B_3, B_4, B_5, B_6\}}$$

(d) **If row space equals column space for $A \in \mathbb{R}^{3\times 3}$, must $A = A^T$?**

Answer: No.

Counterexample:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Row space: spanned by $(1, 2, 0)$ and $(0, 1, 0)$

Column space: spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Both spaces are the same 2-dimensional subspace (the xy-plane in $\mathbb{R}^3$).

However, $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq A$

$$\boxed{\text{No, } A \text{ need not equal } A^T.}$$