NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2015-2016

MH1300– Foundations of Mathematics

December 2015 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (10 marks)

Prove or disprove each of the following.

(a) $\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ \exists z \in \mathbb{R},\ \ z^2 > x - y.$

(b) $\forall x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ \ x < y \Rightarrow (\exists z \in \mathbb{Q},\ x < z < y).$

QUESTION 2. (15 marks)

(a) Prove that if $a \in \mathbb{Z}$, then $a(a^2 + 2)$ is divisible by 3.

(b) Suppose that $q \in \mathbb{Z}$ and $q > 1$, and that for any integers a, b,

q divides ab implies that q divides a or q divides b.

Prove that q is prime.

QUESTION 3. (10 marks)

Prove by mathematical induction that for every positive integer n,

$$\sum_{k=1}^{n} \frac{k^2}{(2k-1)(2k+1)} = \frac{n(n+1)}{4n+2}.$$

QUESTION 4. (10 marks)

(a) Use the Well-ordering Principle to prove that for any real number $a > 0$ and any positive integer n,
$$a^n > 0.$$

(b) Prove that if n is a positive integer then one of the numbers
$$n,\ n+3,\ n+6,\ n+9$$
is a multiple of 4.

QUESTION 5. (20 marks)

(a) Find non-empty sets A, B, C and functions $f_0 : A \to B$, $g_0 : B \to C$, $f_1 : A \to B$ and $g_1 : B \to C$ such that

(i) f_0 is onto but $g_0 \circ f_0$ is not onto.

(ii) $g_1 \circ f_1$ is 1-1 but g_1 is not 1-1.

Justify your answer.

(b) Let $h : (1, \infty) \to (1, \infty)$ be defined by
$$h(x) = \frac{x}{x-1}.$$
Is h 1-1? Is h onto? Justify your answer.

(c) Let $F : A \to B$. Prove that if $X \subseteq A$ and F is 1-1 then
$$F(A - X) = F(A) - F(X).$$

QUESTION 6. (15 marks)

(a) Suppose that R and S are equivalence relations on a non-empty set A. Let $A/R = \{[a]_R \mid a \in A\}$ be the set of equivalence classes of R. Similarly $A/S = \{[a]_S \mid a \in A\}$ is the set of equivalence classes of S. Prove that if $A/R \subseteq A/S$, then $R = S$.

(b) Let T be a relation on the set of positive integers defined by

$$(a, b) \in T \text{ if and only if } \frac{a}{b} = 2^m \text{ for some } m \in \mathbb{Z}.$$

Prove that T is an equivalence relation.

QUESTION 7. (20 marks)

(a) Prove that for any sets A and B,

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

(b) For any set A let $\mathscr{P}(A) = \{X \mid X \subseteq A\}$ be the powerset of A. Are there sets A and B such that

$$\mathscr{P}(A - B) = \mathscr{P}(A) - \mathscr{P}(B)?$$

Justify your answer.

(c) Are the following true for any non-empty sets A, B, C and D? In each case, prove or give a counter-example.

(i) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

(ii) $(C \times C) - (A \times B) = (C - A) \times (C - B)$.

END OF PAPER