Name: ______________________________ Tutorial group: ____

Matriculation number: | | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2018/19

MH1100 & SM2MH1100 – Calculus I

28 September 2018 Midterm Test 90 minutes

INSTRUCTIONS

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **FIVE (5)** questions and comprises **SIX (6)** printed pages.
4. Answer **all** questions. The marks for each question are indicated at the beginning of each question.

For graders only	Question	1	2	3	4	5	Total
	Marks						

QUESTION 1. **(5 marks)**

Find the limits if exist.

(a) $\lim_{x\to 0}(x^2+1)(2+\cos x)$ (b) $\lim_{x\to -2}\frac{\sqrt{x^2+5}-3}{x+2}$ (c) $\lim_{x\to 2}\frac{1}{|x-2|}$

(d) $\lim_{x\to 0^+}\left[x\sin\left(\frac{1}{x^2}\right)\right]$ (e) If $\lim_{x\to 0^+}[8g(x)]^{\frac{1}{3}}=3$, find $\lim_{x\to 0^+}g(x)$.

[Answer:]

(a) The function $f(x)=(x^2+1)(2+\cos x)$ is continuous at $x=0$. So $\lim_{x\to 0}f(x)=f(0)$. Plugging in $x=0$, we get

$$\lim_{x\to 0}f(x)=f(0)=3.$$

(b) We rationalize the numerator.

$$\begin{aligned}\lim_{x\to -2}\frac{\sqrt{x^2+5}-3}{x+2}&=\lim_{x\to -2}\frac{\sqrt{x^2+5}-3}{x+2}\cdot\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}\\&=\lim_{x\to -2}\frac{(x^2+5)-9}{x+2}\cdot\frac{1}{\sqrt{x^2+5}+3}\\&=\lim_{x\to -2}\frac{x^2-4}{x+2}\cdot\frac{1}{\sqrt{x^2+5}+3}=\lim_{x\to -2}\frac{x-2}{\sqrt{x^2+5}+3}\\&=-\frac{2}{3}.\end{aligned}$$

(c) We check the two one-sided limits.

$$\lim_{x\to 2^-}\frac{1}{|x-2|}=\lim_{x\to 2^-}\frac{1}{2-x}=+\infty.$$

$$\lim_{x\to 2^+}\frac{1}{|x-2|}=\lim_{x\to 2^+}\frac{1}{x-2}=+\infty.$$

Thus, the limit does not exist. Or the function has an infinite limit at $x=2$.

(d) Let $f(x)=-x$, $g(x)=x\sin\left(\frac{1}{x^2}\right)$, and $h(x)=x$. It is easy to find out that $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}h(x)=0$. When $x>0$, we also have $f(x)\le g(x)\le h(x)$. Using the squeeze theorem, $\lim_{x\to 0^+}g(x)=\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}h(x)=0$. Thus, $\lim_{x\to 0^+}\left[x\sin\left(\frac{1}{x^2}\right)\right]=0$.

(e) Given that $\lim_{x\to 0^+}[8g(x)]^{\frac{1}{3}}=3$ and $\lim_{x\to 0^+}\left[\frac{1}{8}\right]^{\frac{1}{3}}=\frac{1}{2}$, we have $\lim_{x\to 0^+}[8g(x)]^{\frac{1}{3}}\cdot\left[\frac{1}{8}\right]^{\frac{1}{3}}=\lim_{x\to 0^+}[g(x)]^{\frac{1}{3}}=\frac{3}{2}$. Using the Power Law, we obtain that $\lim_{x\to 0^+}g(x)=\frac{27}{8}$.

QUESTION 2. **(3 marks)**

Find the derivatives of the functions.

(a) $y = \dfrac{x^2}{x+1}$ (b) $w = (z-2)(z+2)(z^2+4)$ (c) $s = \cos\left(\sqrt{2t+1}\right)$

[Answer:]

(a) Using the Quotient Rule,

$$y' = \frac{(x+1)(x^2)' - x^2(x+1)'}{(x+1)^2} = \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2}.$$

(b) We rewrite the function $w = w(z)$ as

$$w = (z-2)(z+2)(z^2+4) = (z^2-4)(z^2+4) = z^4 - 16.$$

Thus, the derivative of w is $w' = 4z^3$.

(c) Using the chain rule, we can get

$$\begin{aligned} s' &= -\sin\left(\sqrt{2t+1}\right) \cdot \frac{1}{2}(2t+1)^{\frac{1}{2}-1}(2t+1)' \\ &= -\sin\left(\sqrt{2t+1}\right)(2t+1)^{-\frac{1}{2}} \\ &= \frac{-\sin\left(\sqrt{2t+1}\right)}{\sqrt{2t+1}}. \end{aligned}$$

QUESTION 3. **(4 marks)**

(a) Use the ϵ-δ definition to prove that $\lim_{x\to 2}(3x-4) = 2$.

(b) Show that the function $f(x) = 3x - 4$ is continuous in its domain.

[Answer:]

(a) (1) Preliminary analysis of the problem (guessing a value for δ).

Let ϵ be a given positive number. We want to find a number δ such that

$$\text{if} \quad 0 < |x-2| < \delta \quad \text{then} \quad |(3x-4)-2| < \epsilon.$$

Now we need to work out an inequality for $|x-2|$ from the second inequality $|(3x-4)-2| < \epsilon$.

But

$$|(3x-4)-2| = |3x-6| = 3|x-2|$$

Therefore we want δ such that

$$\text{if} \quad 0 < |x-2| < \delta \quad \text{then} \quad 3|x-2| < \epsilon$$

that is,

$$\text{if} \quad 0 < |x-2| < \delta \quad \text{then} \quad |x-2| < \frac{1}{3}\epsilon$$

This suggests that we should choose $\delta = \frac{1}{3}\epsilon$.

(2) Proof (showing that this δ works).

Given $\epsilon > 0$, choose $\delta = \frac{1}{3}\epsilon$. If $0 < |x-2| < \delta = \frac{1}{3}\epsilon$, then

$$|(3x-4)-2| = |3x-6| = 3|x-2| < 3\delta = 3(\frac{1}{3}\epsilon) = \epsilon$$

Thus

$$\text{if} \quad 0 < |x-2| < \delta \quad \text{then} \quad |(3x-4)-2| < \epsilon$$

Therefore, by the definition of a limit,

$$\lim_{x\to 4}(3x-4) = 2.$$

(b) The domain of $f(x)$ is $\mathbb{R}$. Let a be any given real number. We have $f(a) = 3a - 4$. Meanwhile,

$$\lim_{x\to a}(3x-4) = 3a - 4.$$

Thus, $\lim_{x\to a} f(x) = f(a)$, indicating that $f(x) = 3x - 4$ is continuous in its domain.

QUESTION 4. **(4 marks)**

Find the values of a and b that make the following function differentiable for all x-values.

$$g(x) = \begin{cases} ax+b, & x<1; \\ bx^2-3, & x \geq 1. \end{cases}$$

[Answer:]

(a) That $f(x)$ is differentiable for all x-values indicates that $f(x)$ is continuous at every number including $x = 1$. Thus,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1).$$

We have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(ax+b) = a+b$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(bx^2-3) = b-3$. So, $a+b = b-3$ gives that $a = -3$.

(b) If $x < 1$, $f'(x) = a$. If $x > 1$, $f'(x) = 2bx$. We need to consider the derivative of $f(x)$ at $x = 1$.

(c)

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}.$$

The limit $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$ exists if and only if $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$ and $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$ both exist and are equal. We have

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{bx^2-3-(b-3)}{x-1} = \lim_{x\to 1^+} b(x+1) = 2b.$$

Given that $a = -3$,

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{ax+b-(b-3)}{x-1} = \lim_{x\to 1^-} \frac{-3x+3}{x-1} = -3.$$

So $2b = -3$ gives $b = -\frac{3}{2}$. In summary, if $a = -3$ and $b = -3/2$, $f(x)$ is differentiable for all x-values.

QUESTION 5. **(4 marks)**

Let $f(x) = x + 2\sin\left(\frac{1}{x}\right)$.

(a) Find the domain of $f(x)$.

(b) Show that $f(x)$ is an odd function in its domain.

(c) Prove that the equation $f(x) = 0$ has a root in its domain.

[Answer]

(a) The domain of $f(x)$ is $\mathbb{R}\backslash\{0\}$.

(b) Since

$$f(-x) = -x + 2\sin\left(\frac{1}{-x}\right) = -x - 2\sin\left(\frac{1}{x}\right) = -\left[x + 2\sin\left(\frac{1}{x}\right)\right] = -f(x),$$

$f(x)$ is an odd function.

(c) We know $f(x)$ is continuous on its domain. But $x = 0$ is a hole of this function. We know $-2 \leq 2\sin\left(\frac{1}{x}\right) \leq 2$. By trial and error, we find the function values $f(10) = 10 + 2\sin(1/10) \geq 10 - 2 = 8 > 0$ and $f(\frac{2}{19\pi}) = \frac{2}{19\pi} + 2\sin\left(\frac{19\pi}{2}\right) = \frac{2}{19\pi} - 2 < 0$. Note that $f(x)$ is continuous on the closed interval $[\frac{2}{19\pi}, 10]$. Since the value $y = 0$ is between the two numbers $f(\frac{2}{19\pi}) < 0$ and $f(10) > 0$, by the Intermediate Value Theorem there is a number $c \in [\frac{2}{19\pi}, 10]$ such that $f(c) = 0$. That is, the number c solves the equation $f(x) = 0$.