

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH1300– Foundations of Mathematics

November 2022

TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam. You are only allowed to bring in **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HAND-WRITTEN OR TYPED ON THE A4 PAPER** (no sticky notes/post-it notes on the reference sheet).
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1.

(18 marks)

(a) Show that the following argument form is valid. State all rules of inferences used.

$$\begin{array}{l}
 p \vee q \\
 (\neg q) \vee s \\
 r \rightarrow (\neg s) \\
 \neg p \\
 \therefore \neg r.
 \end{array}$$

(b) Is the following statement form a tautology, a contradiction, or neither?

$$(p \wedge \neg q) \vee ((\neg p \wedge q) \vee (\neg p \vee q))$$

Justify your answer.

(c) Let $P(x, y)$ be the predicate $x^2 + y < 0$, where the domain for x and y is the set of real numbers. Determine if each of the following sentences is true or false and justify your answers:

(i) $(\forall y)(\exists x) P(x, y)$.

(ii) $(\exists y)(\forall x) P(x, y)$.

QUESTION 2.

(12 marks)

Determine if each of the following is true or false. Justify your answers.

(a) For any integer n , 4 does not divide $n^2 + 1$.

(b) For every sets A, B and C ,

$$A \times (B \cap C) = (A \times B) \cap C.$$

(c) If S and R are relations on a set X , then $S \circ R = R \circ S$.

QUESTION 3.

(18 marks)

Use Mathematical Induction to prove the following.

- (a) For every positive integer n ,

$$1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \cdots + n! \cdot n = (n+1)! - 1.$$

- (b) Define the sequence $\{a_n\}_{n=0}^{\infty}$ by the following: $a_0 = 1, a_1 = 2, a_2 = 3$ and $a_n = a_{n-1} + 3a_{n-3} + 1$ for all $n \geq 3$. Prove that $a_n \leq 2^n$ for all $n \geq 0$.

QUESTION 4.

(14 marks)

- (a) Given any real number x , what are the possible value(s) of $\lfloor x \rfloor + \lfloor -x \rfloor$? Justify your answer.
- (b) A set $S \subseteq \mathbb{R}$ is said to have the *Archimedean property* if for every $a, b \in S$ there is a positive integer n such that $na > b$. Find a condition which is both sufficient and necessary for a set S to have the Archimedean property. Justify your answer.

QUESTION 5.

(12 marks)

- (a) Prove that for any finite sets A and B , the average of the size of A and the size of B cannot exceed the size of $A \cup B$.
- (b) Let $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ be defined by

$$f(x) = x^2 \operatorname{div} 3.$$

Is f injective? Is f surjective? Justify your answers. Here, \mathbb{Z}^+ is the set of all positive integers.

QUESTION 6.

(12 marks)

- (a) Let a, b and c be integers. Prove that if a does not divide b^6 and a divides c then a does not divide $b^2 + c^2$.
- (b) Find all complex numbers z satisfying $z^5 + i = 0$. Leave your answer in terms of $re^{i\theta}$.
- (c) Use the Euclidean Algorithm to find the greatest common divisor of 1989 and 6435.

QUESTION 7.

(14 marks)

- (a) Let the relation T be defined on the power set of \mathbb{Z} by

$$A T B \leftrightarrow A \cap B \neq \emptyset.$$

Determine whether T is reflexive, whether T is symmetric and if T is transitive. Justify your answers.

- (b) Define the relation R on \mathbb{R}^2 as follows:

$$(a, b) R (x, y) \leftrightarrow b = y.$$

Show that R is an equivalence relation and describe geometrically the distinct equivalence classes of R .

END OF PAPER

MH1300 FOUNDATIONS OF MATHEMATICS

Please read the following instructions carefully:

- 1. Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**
2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.
3. Please write your Matriculation Number on the front of the answer book.
4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.