NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024-2025

MH1300– Foundations of Mathematics

December 2024 TIME ALLOWED: 2 HOURS

SEAT NUMBER:

MATRICULATION NUMBER:

INSTRUCTIONS TO CANDIDATES

1. This question cum answer booklet contains **SEVEN (7)** questions and comprises **TWENTY-THREE (23)** printed pages, including three spill over pages at the end.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Write your answers in the **space provided** after each question.
4. Write your matriculation number on the **cover page**, and at the **bottom right-hand corner of every odd numbered page**.
5. The last three pages are spill over pages. Only use them if you run out of space for your answers. If you use them, please indicate clearly which question(s) you are answering.
6. This question cum answer booklet **IS NOT** to be removed from the examination hall.
7. This **IS** a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **ONE DOUBLE-SIDED A4-SIZE REFERENCE SHEET WITH TEXTS HANDWRITTEN ON THE A4 PAPER WITHOUT ANY ATTACHMENTS (e.g. sticky notes, post-it notes, gluing or stapling of additional papers)**.
8. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1.

(a) Prove that for every integer n, if $n^4 - 1$ is not divisible by 5 then n is divisible by 5.

(6 marks)

Question 1 continues on the next page

QUESTION 1.

(b) Let a, b and d be integers where $d > 1$. Prove that if $a \equiv b \pmod{d}$, then $a^2 \equiv b^2$ ($\mathrm{mod}\ d$).

(5 marks)

Question 1 continues on the next page

MATRIC NO.: ____________________

QUESTION 1.

(c) Are the following pair of statements logically equivalent?

$$(p \to q) \to (p \wedge r) \qquad \text{and} \qquad p \wedge (q \to r).$$

Justify your answer.

(5 marks)

End of Question 1

QUESTION 2.

(a) Determine if the following is true or false. Justify your answer.

There are distinct positive integers n and m such that $\frac{1}{m} + \frac{1}{n}$ is an integer.

(6 marks)

Question 2 continues on the next page MATRIC NO.: ____________________

QUESTION 2.

(b) Determine if the following is true or false. Justify your answer.

Let $a > 1$ be an integer. If a is a perfect square, then $\sqrt[3]{a}$ is irrational.

(3 marks)

Question 2 continues on the next page

QUESTION 2.

(c) Determine if the following is true or false. Justify your answer.

If D and E are finite sets such that E has at least one more element than D, then $\mathcal{P}(E)$ has at least two more elements than $\mathcal{P}(D)$. Here, $\mathcal{P}(X)$ is the power set of X.

(3 marks)

End of Question 2 MATRIC NO.: ____________________

QUESTION 3.

(a) Use mathematical induction to prove that for every integer $n \geq 1$,

$$\sum_{j=1}^{3n} j(j-1) = n(9n^2-1).$$

(8 marks)

Question 3 continues on the next page

QUESTION 3.

(b) Use mathematical induction to prove that for every integer $n \geq 1$, and every sequence of non-negative real numbers $x_1, x_2, \cdots, x_n$,

$$\text{if } x_1 + x_2 + \cdots + x_n = 0, \text{ then } x_1 = x_2 = \cdots = x_n = 0.$$

(9 marks)

End of Question 3

MATRIC NO.: ____________________

QUESTION 4.

(a) If X and Y are sets, prove that

$$\mathcal{P}(X-Y)-\{\emptyset\} \subseteq \mathcal{P}(X)-\mathcal{P}(Y).$$

Give a counterexample to show that $\mathcal{P}(X-Y)-\{\emptyset\} = \mathcal{P}(X)-\mathcal{P}(Y)$ is false for some X and Y.

(5 marks)

Question 4 continues on the next page

QUESTION 4.

(b) Let A, B and C be sets. Prove that

$$(A \cap (A - B)) \cup (A^c \cup B)^c = A - B.$$

(4 marks)

Question 4 continues on the next page

MATRIC NO.: ____________________

QUESTION 4.

(c) Prove or disprove the following statements:

(i) For every real number x, $\lfloor -x \rfloor = -\lceil x \rceil$.

(ii) For every real number x, $\lfloor -x \rfloor = -\lfloor x \rfloor$.

(5 marks)

End of Question 4

QUESTION 5.

(a) Let x and y be two real numbers such that $0 < x < y$. Prove that there are integers n and m such that $nx \leq m \leq ny$.

(6 marks)

Question 5 continues on the next page MATRIC NO.: ____________________

QUESTION 5.

(b) Prove that if a is an odd integer then $a^3 - a$ is a multiple of 8.

(5 marks)

Question 5 continues on the next page

QUESTION 5.

(c) Use the Euclidean algorithm to find the greatest common divisor of the pair

$$630 \qquad \text{and} \qquad 96.$$

(5 marks)

End of Question 5

MATRIC NO.: ____________________

QUESTION 6.

(a) Find all complex numbers z satisfying the equation $z^3 = 3 + 3i$.

(4 marks)

Question 6 continues on the next page

QUESTION 6.

(b) Let $g : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R}^2)$ be defined by $g(A) = A \times A$. Determine if g is one-to-one and if g is onto. Justify your answers.

(6 marks)

End of Question 6 MATRIC NO.: ____________________

QUESTION 7.

(a) State the definition of each of the following:

(i) A symmetric binary relation R on a set A.

(ii) A transitive binary relation R on a set A.

(3 marks)

Question 7 continues on the next page

QUESTION 7.

(b) The relation R on $\mathbb{R}^2$ is defined by $(a, b)R(x, y)$ if and only if $a < x$ or ($a = x$ and $b < y$).

(i) Is R reflexive?

(ii) Is R symmetric?

(iii) Is R transitive?

Justify your answers.

(6 marks)

Question 7 continues on the next page MATRIC NO.: ____________________

QUESTION 7.

(c) Let $X = \mathbb{R}^2 - \{(0,0)\}$, and define the relation T on the set X by $(a,b)T(x,y)$ if and only if there is some real number $c \neq 0$ such that $ca = x$ and $cb = y$.

(i) Show that T is an equivalence relation on X.

(ii) Describe the equivalence class of $(1,2)$.

(6 marks)

END OF PAPER

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You may use this page if you run out of space for your answers. If you do so, indicate clearly which question you are answering on this page.

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