NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2018-2019

MH1300– Foundations of Mathematics

December 2018 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS** a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **BOTH SIDES OF ONE A4-SIZED HELPSHEET.**
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (18 marks)

(a) Prove or disprove the following statements:

(i) If a and b are rational real numbers then a^b is rational.

(ii) For each positive real number x, $\left\lfloor \sqrt{\lceil x \rceil} \right\rfloor = \sqrt{\lceil x \rceil}$.

(b) Let A, B and C be sets such that $(A - C) \cup (C - A) = (B - C) \cup (C - B)$. Prove that $A = B$.

(c) Show that the following is a tautology:

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r).$$

QUESTION 2. (10 marks)

Determine if the following are true or false. Justify your answer.

(a) $\exists n \in \mathbb{Z},\ \exists m \in \mathbb{Z},\ n^2 + m^3 = 15.$

(b) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ xy > x.$

(c) $\forall y \in \mathbb{Z},\ \exists x \in \mathbb{Z},\ xy \geq x.$

QUESTION 3. (15 marks)

(a) Prove that for every integer $n \geq 2$,

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}.$$

(b) Prove that for every integer $n \geq 1$,

$$4^{n+1} + 5^{2n-1} \text{ is divisible by } 21.$$

QUESTION 4. (15 marks)

(a) By considering the term $\left(x - \frac{1}{x}\right)^2$, prove that if x is a nonzero real number, then $x^2 + \frac{1}{x^2} \geq 2$.

(b) Let a and b be integers. Prove that a and b have the same parity if and only if there is an integer c such that $|a - c| = |b - c|$. Recall that a and b have the same parity if either a and b are both even or a and b are both odd.

(c) Prove that if m is an odd positive integer, then $m^2 \equiv 1 \pmod 8$.

QUESTION 5. (12 marks)

(a) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n, m) = |n| - |m|$. Determine if f is one-one, and if f is onto. Justify your answer.

(b) Disprove the following using a counter-example: Let $f : \mathbb{R} \to \mathbb{R}$, $C_0 \subseteq \mathbb{R}$ and $C_1 \subseteq \mathbb{R}$. Then $f\,(C_0 \cap C_1) = f(C_0) \cap f(C_1)$.

(c) Write down the power set of the following set:

$$\{1, \{1, 2\}, \{1, 2, 3\}\}.$$

QUESTION 6. (15 marks)

(a) Find all the complex roots of the equation $z^5 = 1 - i$.

(b) Let A, B, C, D be four sets. Prove that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

(c) Disprove the following statements. You should provide a different counter-example in each part.

(i) Let A_0, B_0, C_0, D_0 be four sets. Then $(A_0 \cup B_0) \times (C_0 \cup D_0) = (A_0 \times C_0) \cup (B_0 \times D_0)$.

(ii) Let R be a binary relation on a set A. If R is symmetric, transitive and $R \neq \emptyset$, then R is reflexive.

QUESTION 7. (15 marks)

(a) Let T be a relation on the set $\mathbb{R}$ be defined by the following: $x\ T\ y$ if and only if $x^2 - y^2$ is an integer.

(i) Prove that T is an equivalence relation.

(ii) Exactly how many distinct equivalence classes of T contain an integer? Justify your answer.

(b) Use the Euclidean algorithm to find the greatest common divisor of the pair

$$414 \qquad \text{and} \qquad 662.$$

END OF PAPER