MH1300 Final Exam Solutions

AY 23/24 Sem 1

$\boxed{Q1(a)}$ Suppose there are positive integers a, b such that $a^2 + a + 1 = b^2$.

Then $b^2 = a^2 + a + 1 > a^2$

$\Rightarrow b > a$. (as both a, b are positive)

Mtd 1: $b^2 = a^2 + a + 1$ (completing the square)

$= (a+1)^2 - a$

So, $a = (a+1)^2 - b^2$

$= (a+1+b)(a+1-b)$

Since $b > a$, so $a+1-b \leq 0$, and $a+1+b \geq 0$

which means the product above ≤ 0

But $a \leq 0$ is a contradiction.

Mtd 2: $b^2 = a^2 + a + 1$

$\Rightarrow b^2 - a^2 = a+1$

$\Rightarrow (b+a)(b-a) = a+1$

Since $b+a > a+a = 2a$, and $b-a > 0$

So $\underbrace{(b+a)(b-a)}_{a+1} > 2a \Rightarrow a+1 > 2a$

$\Rightarrow 1 > a$

Contradiction.

$\boxed{Q1(b)}$ Let c be an integer.

Suppose c is divisible by 3.

Let $k \in \mathbb{Z}$ be such that $c = 3k$.

then $c^2 = 3ck = 3(ck)$.

Since $ck \in \mathbb{Z}$, we conclude that $3 \mid c^2$.

Suppose c^2 is divisible by 3. By QRT, there are 3 cases: $c = 3k,\ 3k+1,\ 3k+2$ for some $k \in \mathbb{Z}$.

We suppose that c isn't divisible by 3.

then, $c = 3k+1$ or $c = 3k+2$ for some $k \in \mathbb{Z}$.

(Goal: to obtain a contradiction in each case).

Case 1: $c = 3k+1$: $c^2 = (3k+1)^2 = 9k^2 + 6k + 1$

$= 3(3k^2 + 2k) + 1.$

So c^2 is not divisible by 3, contradiction.

Case 2: $c = 3k+2$: $c^2 = (3k+2)^2 = 9k^2 + 12k + 4$

$= 3(3k^2 + 4k + 1) + 1$

So, c^2 is not divisible by 3, Contradiction.

Therefore we conclude that c must be divisible by 3.

Q1(c) We show they are logically equivalent:

$$p \to (q \vee r) \equiv \neg p \vee (q \vee r) \quad \text{(logical equivalence for conditional)}$$

$$\equiv (\neg p \vee q) \vee r \quad \text{(associative law)}$$

$$\equiv (q \vee \neg p) \vee r \quad \text{(Commutative law)}$$

$$\equiv q \vee (\neg p \vee r) \quad \text{(associative law)}$$

$$\equiv (\neg\neg q) \vee (\neg p \vee r) \quad \text{(double negation)}$$

$$\equiv \neg q \to (\neg p \vee r) \quad \text{(logical law for Conditional)}$$

Truth table Solution is also fine

Q2(a) False. We want to show that there are no positive real numbers x, y such that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

Suppose there are such $x, y > 0$.

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}$$

$$(\sqrt{x+y})^2 = (\sqrt{x} + \sqrt{y})^2$$

$$x + y = x + y + 2\sqrt{x}\sqrt{y}$$

$$2\sqrt{xy} = 0$$

$$\sqrt{xy} = 0$$

$$xy = 0 \Rightarrow x = 0 \text{ or } y = 0 \text{ (by zero product property)}.$$

Contradiction

$\boxed{\text{Q2(b)}}$ This is true. Fix a rational number $p > 0$.

Take $z = \frac{p}{\sqrt{2}}$. Why do we choose $z = \frac{p}{\sqrt{2}}$?

Recall that $\sqrt{2} \approx 1.4$

So $\frac{1}{\sqrt{2}}$ is between 0 and 1

$\Rightarrow \quad 0 < z < p$

Now furthermore, z is irrational, because let's suppose it is rational. Let $p = \frac{a}{b}$ and $z = \frac{c}{d}$ for some integers a, b, c, d and $b \neq 0$, $d \neq 0$. We also know $c \neq 0$ since $z > 0$.

Then $\frac{p}{\sqrt{2}} = z = \frac{c}{d} \Rightarrow \sqrt{2} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

Since $bc \neq 0$ (by zero product property)

$\sqrt{2}$ is rational, a contradiction.

$\boxed{\text{Q2(c)}}$ This is false, students need to write down sets A, B, C and check the equality fails.

Eg. $A = \{0, 2\}$, $B = \{0, 1\}$, $C = \{2, 3\}$

Then LHS $= (A - B) \cap (A - C) = \{2\} \cap \{0\} = \emptyset$

RHS $= A - (B \cap C) = A - \{\} = \{0, 2\}$.

LHS $\neq$ RHS.

Alternatively, take $A = \mathbb{Z}$, B = set of even integers, C = set of odd integers.

Check $\underbrace{\text{LHS}}_{\emptyset} \neq \overbrace{\text{RHS}}^{\mathbb{Z}}$.

$\boxed{Q3(a)}$ Let $P(n)$ be the property

"there are non negative integers c, d

such that $n = 7c + 3d$".

Use Strong MI, $a = 12$, $b = 14$.

Base case Verify $P(12)$. We need to find $c, d \geq 0$

s.t. $12 = 7c + 3d$.

Take $c = 0$, $d = 4$, then $7c + 3d = 0 + 12 = \underset{n}{12}$.

So $P(12)$ is true.

Verify $P(13)$: Take $c = 1$, $d = 2$. then

$7c + 3d = 7 + 6 = \underset{n}{13}$.

So, $P(13)$ is true.

Verify $P(14)$: Take $c = 2$, $d = 0$. Then,

$7c + 3d = 14 + 0 = \underset{n}{14}$

So, $P(14)$ is true.

Inductive step: Now fix $k \geq b = 14$ and assume

$P(i)$ true for all $\underset{12}{a} \leq i \leq k$

WTS: $P(k+1)$ is true. Take $i = k+1-3 = k-2 \geq 14-2 = 12$.

Since $P(i)$ is true (by IH), there are integers $c, d \geq 0$

such that $i = 7c + 3d$.

$\therefore k+1 = i + 3 = (7c + 3d) + 3 = 7c + 3(d+1)$

So, $P(k+1)$ is true, and hence $P(n)$ true for all $n \geq 12$.

Q3(b) Let $P(n)$:

$$1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1) = \frac{(2n+2)!}{(n+1)! \cdot 2^{n+1}}$$

<u>Base case $P(0)$:</u>

LHS $= 1$

RHS $= \dfrac{(0+2)!}{1! \cdot 2} = \dfrac{2!}{2} = 1$ $\quad \therefore$ $P(0)$ is true.

<u>Inductive Step:</u> Let $k \geq 0$ and assume $P(k)$ is true.

$$\boxed{\text{Inductive Hyp: } 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1) = \frac{(2k+2)!}{(k+1)!\, 2^{k+1}}}$$

we assume this.

$$\boxed{\text{Need to show } P(k+1)\text{: } 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1)\,(2(k+1)+1) = \frac{[2(k+1)+2]!}{(k+2)!\, 2^{k+2}}}$$

we need to prove this

start from LHS of $P(k+1)$ $= \underbrace{1 \cdot 3 \cdot 5 \cdots (2k+1)}\,(2k+3)$

Apply IH above $= \dfrac{(2k+2)!}{(k+1)!\, 2^{k+1}} \cdot (2k+3)$

consider RHS of $P(k+1)$ $= \dfrac{(2k+4)!}{(k+2)!\, 2^{k+2}}$

$$= \frac{(2k+2)!\,(2k+3)\cancel{(2k+4)}}{(k+1)!\,\cancel{(k+2)} \cdot 2^{k+1}\,\cancel{2}}$$

$$= \frac{(2k+2)!\,(2k+3)}{(k+1)! \cdot 2^{k+1}}$$

EQUAL

So, $P(k+1)$ is true, and $P(n)$ true for all $n \geq 0$.

$\boxed{\text{Q4(a)}}$ Suppose $A \times C = B \times C$, and $C \neq \phi$.

Since $C \neq \phi$, let $x \in C$.

$A \subseteq B$: let $a \in A$. Then $(a, x) \in A \times C$.

Since $A \times C = B \times C$, $(a, x) \in B \times C$.

This means $a \in B$.

$B \subseteq A$: let $b \in B$. Then $(b, x) \in B \times C = A \times C$.

So, $b \in A$.

If $C = \phi$ then $A \times C = \phi$ and $B \times C = \phi$ for any sets A and B. So the property is false.

For example, $A = \mathbb{Z}$ and $B = \mathbb{R}$, $C = \phi$

then $A \times C = B \times C$ but $A \neq B$.

$\boxed{\text{Q4(b)}}$ Let $D = \{0, 1\}$.

First write down $\mathcal{P}(D) = \{\phi, \{0\}, \{1\}, D\}$

So, $D \times \mathcal{P}(D) = \{ (0, \phi), (0, \{0\}), (0, \{1\}), (0, D), \underbrace{(1, \phi), (1, \{0\}), (1, \{1\}), (1, D)}_{\text{8 elements.}} \}$

$\boxed{Q4(c)}$ This is similar to a tutorial problem where you showed $\sqrt{2}+\sqrt{3}$ is irrational.

Suppose $\sqrt{2}+\sqrt{7}$ is rational. Let a, b be integers such that

$$\sqrt{2}+\sqrt{7} = \frac{a}{b}, \quad b \neq 0.$$

$a = b(\sqrt{2}+\sqrt{7})$, so $a \neq 0$ as well.

$$\sqrt{7} = \frac{a}{b} - \sqrt{2}$$

$$7 = \left(\frac{a}{b} - \sqrt{2}\right)^2 = \frac{a^2}{b^2} - 2\frac{a}{b}\sqrt{2} + 2$$

$$\frac{2a}{b}\sqrt{2} = \frac{a^2}{b^2} - 5$$

$$\sqrt{2} = \frac{b}{2a}\left(\frac{a^2-5b^2}{b^2}\right) = \frac{a^2-5b^2}{2ab}$$

Since $a \neq 0$, $b \neq 0$, so $2ab \neq 0$ (Zero product property)

so $\sqrt{2}$ is rational, Contradiction.

Q5(a) (i) A function $f: A \to B$ is surjective if for every $b \in B$ there is some $a \in A$ such that $f(a) = b$.

(ii) $g: C \to D$ is one-to-one if

$\forall a, b \in C$ if $g(a) = g(b) \Rightarrow a = b$

alternatively, $\forall a, b \in C$ if $a \neq b \Rightarrow g(a) \neq g(b)$

Q5(b) (ii) Suppose S is symmetric. We show $\bar{S}$ is symmetric: let $(x,y) \in \bar{S}$. Then $(x,y) \notin S$. If $(y,x) \in S$ then $(x,y) \in S$ (by the fact that S is symmetric)

So, $(y,x) \notin S$.

So, $(y,x) \in \bar{S}$.

(iii) This is false. For example, let $B = \mathbb{Z}$, S be the "divides" relation, i.e. $nSm \Leftrightarrow n|m$.

Then S is transitive (shown in lecture).

But $\bar{S}$ is not: for example,

$\underbrace{2 \nmid 5}_{(2,5) \in \bar{S}}$ and $\underbrace{5 \nmid 8}_{(5,8) \in \bar{S}}$ and $\underbrace{2 | 8}_{(2,8) \notin \bar{S}}$

(i) False. let $B = \mathbb{Z}$, and S be $=$ relation,

i.e. nSm iff $n = m$.

Then S is reflexive as $n = n$ holds for all $n \in \mathbb{Z}$.

$\bar{S}$ isn't reflexive as $0 \neq 0$ is false, $0 \in \mathbb{Z}$.

$\boxed{\text{Q5(c)}}$ gcd(12345, 67890)

$67890 = 12345 \times 5 + 6165$

$12345 = 6165 \times 2 + \underline{\underline{15}}$

$6165 = 15 \times 411 + 0.$

So, gcd(12345, 67890) = 15.

$\boxed{Q6(a)}$ $z^5 + 32 = 0$

$z^5 = -32 = 32 \cdot e^{i\pi}$

$z = 2 e^{i(\pi + \frac{2k\pi}{5})}$, $k = 0,1,2,3,4$

$= 2e^{i\pi}$, $2e^{i(\pi + \frac{2\pi}{5})}$, $2e^{i(\pi + \frac{4\pi}{5})}$, $2e^{i(\pi + \frac{6\pi}{5})}$, $2e^{i(\pi + \frac{8\pi}{5})}$

$= 2e^{i\pi}$, $2e^{i\frac{7\pi}{5}}$, $2e^{i\frac{9\pi}{5}}$, $2e^{i\frac{11\pi}{5}}$, $2e^{i\frac{13\pi}{5}}$

$\boxed{Q6(b)}$ There are many options:

(i) $f_0(n) = 2n$ is one to one:

$f_0(n) = f_0(m)$

$\Rightarrow 2n = 2m$

$\Rightarrow n = m$

not onto: There is no n s.t. $f_0(n) = 2n = 1$ since $n = \frac{1}{2} \notin \mathbb{Z}$.

(ii) $f_1(n) = \lfloor \frac{1}{2}n \rfloor$.

$f_1(n)$ not one to one! $f_1(0) = \lfloor 0 \rfloor = 0$
$f_1(1) = \lfloor \frac{1}{2} \rfloor = 0$

$f_1(0) = f_1(1)$ but $0 \neq 1$.

$f_1(n)$ is onto: Given any $m \in \mathbb{Z}$.
$f_1(2m) = \lfloor \frac{1}{2} \cdot 2m \rfloor = \lfloor m \rfloor = m$.

(iii) $f_2(n) = |n|$. alternatively $f_2(n) = n^2$

$f_2(n)$ not one-to-one: $f_2(-1) = |-1| = 1 = f_2(1)$

$f_2(n)$ is not onto: there is no n such that $f_2(n) = |n| = -1$ as $|n| \geq 0$.

Q6(c) Suppose $g: A \to B$, $C, D \subseteq B$.

$g^{-1}(C \cup D) \subseteq g^{-1}(C) \cup g^{-1}(D)$:

let $a \in g^{-1}(C \cup D)$. Then $g(a) \in C \cup D$.

- If $g(a) \in C$, then $a \in g^{-1}(C) \Rightarrow a \in g^{-1}(C) \cup g^{-1}(D)$
- If $g(a) \in D$, then $a \in g^{-1}(D) \Rightarrow a \in g^{-1}(C) \cup g^{-1}(D)$.

$g^{-1}(C) \cup g^{-1}(D) \subseteq g^{-1}(C \cup D)$:

let $a \in g^{-1}(C) \cup g^{-1}(D)$.

- If $a \in g^{-1}(C)$ then $g(a) \in C$. So $g(a) \in C \cup D$ and hence $a \in g^{-1}(C \cup D)$.
- If $a \in g^{-1}(D)$ then $g(a) \in D$. So $g(a) \in C \cup D$ and hence $a \in g^{-1}(C \cup D)$.

Q7 (a) (i) R is reflexive: given any $a \in \mathbb{Z}$,

$a - a = 0 = 8 \cdot 0$ so $a - a \in K$.

Hence aRa.

(ii) R is symmetric. Suppose $(a,b) \in R$.

Then $a - b \in K$ and so $a - b = 8m$ for some $m \in \mathbb{Z}$

$b - a = -8m = 8(-m)$. So $b - a \in K$

and $(b,a) \in R$.

(iii) R transitive. Suppose $(a,b) \in R$ and $(b,c) \in R$.

Then $a - b \in K$ and $b - c \in K$.

So $a - b = 8m$ and $b - c = 8p$ for some $m, p \in \mathbb{Z}$.

$$a - c = (a-b) + (b-c) = 8m + 8p$$
$$= 8(m+p).$$

So, $a - c \in K$, and $(a,c) \in R$.

Note aRb iff $a - b \in K$

iff $8 \mid a - b$

iff $a \equiv b \pmod 8$

There are 8 equivalence classes,

$[0], [1], \ldots, [7]$

where $[i] = \{ 8k + i \mid k \in \mathbb{Z} \}$, $i = 0, 1, 2, \cdots, 7$

$\boxed{Q7(b)}$ Suppose S is an equivalence relation.

Then S is obviously reflexive.

To show S is round, let $x, y, z \in A$ and

assume $x S y$ and $y S z$ hold.

Since S is transitive, $x S z$ holds.

Since S is symmetric, $z S x$ holds.

Now assume that S is reflexive and round.

To show S is an equivalence relation, we need

to show S is symmetric and transitive.

Symmetric:

Suppose $x, y \in A$ and assume $x S y$.

Since S is reflexive, $y S y$ holds.

Since S is round, $y S x$ holds.

Transitive: Suppose $x, y, z \in A$ and assume $x S y$ and $y S z$ hold.

Since S is round, $z S x$ holds.

Since S is symmetric, $x S z$ holds

(from above)

So, S is an equivalence relation.