MH1300 AY 22/23 Final Exam

Solutions

$\boxed{\text{Q1(a)}}$

$$\left.\begin{array}{l} p \vee q \\ \neg q \vee s \\ r \rightarrow (\neg s) \end{array}\right\} \text{Premises}$$

$$\left.\begin{array}{l} \neg p \\ \therefore\ \neg r \end{array}\right\} \text{Conclusion}$$

$\neg p$ (premise)

$p \vee q$ (premise)

q (Elimination)

$\neg\neg q \equiv q$ (logical equivalence)

$\neg q \vee s$ (premise)

s (Elimination)

$\neg\neg s \equiv s$ (logical equivalence)

$r \rightarrow (\neg s)$ (premise)

$\therefore\ \neg r$ (Modus Tollens)

Q1(b) $(p \wedge \neg q) \vee ((\neg p \wedge q) \vee (\neg p \vee q))$

is a tautology.

Method 1: Use a truth table. This is easier than logical equivalences, as there are only 4 rows to do. Check that each of the 4 rows output T.

Method 2: Logical equivalences.

$$
\begin{aligned}
& (p \wedge \neg q) \vee ((\neg p \wedge q) \vee (\neg p \vee q)) & \\
\equiv\ & (p \wedge \neg q) \vee ((\neg p \wedge q) \vee (\neg p \vee \neg\neg q)) & \text{(Double Negation)} \\
\equiv\ & (p \wedge \neg q) \vee ((\neg p \wedge q) \vee \neg(p \wedge \neg q)) & \text{(De Morgan's Law)} \\
\equiv\ & (p \wedge \neg q) \vee (\neg(p \wedge \neg q) \vee (\neg p \wedge q)) & \text{(Commutative Law)} \\
\equiv\ & ((p \wedge \neg q) \vee \neg(p \wedge \neg q)) \vee (\neg p \wedge q) & \text{(Associative law)} \\
\equiv\ & T \vee (\neg p \wedge q) & \text{(Negation law)} \\
\equiv\ & T & \text{(Universal Bound)}
\end{aligned}
$$

$\boxed{\text{Q1(c)}}$ - $\forall y \exists x \; x^2+y<0$ is false

(need to show $\exists y \forall x \; x^2+y \geq 0$ is true)

Take $y=0$. Then for any given $x \in \mathbb{R}$,

$$x^2+y = x^2+0 = x^2 \geq 0.$$

- $\exists y \forall x \; x^2+y<0$ is false

(need to show $\forall y \exists x \; x^2+y \geq 0$ is true)

Given some $y \in \mathbb{R}$. Take $x = |y|+1$.

Then
$$\begin{aligned} x^2+y &= |y|^2 + 2|y| + 1 + y \\ &\geq 0 + |y| + 1 + y \\ &\geq -y+y \quad (\text{since } |y| \geq -y) \\ &= 0. \end{aligned}$$

Alternatively, after you are given y, you may consider cases:

<u>Case 1</u>: <u>$y \geq 0$</u> : Take $x=0$, then
$$x^2+y = y \geq 0.$$

<u>Case 2</u>: <u>$y<0$</u>: Take $x = -y+1$,

then $x>1$, so $x^2 > x = -y+1$

and $x^2+y > 1 > 0$

$\boxed{\text{Q2(a)}}$ The statement is true.

Need to show: For every integer n, $4 \nmid n^2+1$.

Let integer n be given, and suppose 4 divides n^2+1. Let $c \in \mathbb{Z}$ be such that $4c = n^2+1$.

Case 1 : n even. Then $n = 2k$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} \text{So,} \quad 4c &= n^2+1 \\ &= (2k)^2+1 \\ &= 4k^2+1 \end{aligned}$$

So, $4(c-k^2) = 1$. Since c, k are integers, $c-k^2$ is an integer.

$c-k^2 = \frac{1}{4}$

So, $c-k^2 \notin \mathbb{Z}$.

we obtain a contradiction in Case 1.

Case 2: n odd. Then $n = 2l+1$ for some $l \in \mathbb{Z}$.

$$\begin{aligned} \text{So,} \quad 4c &= n^2+1 \\ &= (2l+1)^2+1 \\ &= 4l^2+4l+2 \end{aligned}$$

$2(c-l^2-l) = 1$. Since c, l are integers, $c-l^2-l$ is an integer

$c-l^2-l = \frac{1}{2}$

So, $c-l^2-l \notin \mathbb{Z}$.

we also obtain a contradiction in Case 2.

Thus, we conclude 4 does not divide n^2+1.

$\boxed{\text{Q 2(b)}}$ Statement is false. We need to find sets A, B and C such that

$$A \times (B \cap C) \neq (A \times B) \cap C$$

There are many possible counterexamples.
For instance, you can take $A = \{0\}$
$B = \{0\}$
$C = A \times B = \{(0,0)\}$

Then $(A \times B) \cap C = \{(0,0)\}$

Since $B \cap C = \phi$, as $\underset{\substack{\downarrow \\ \text{number} \\ \text{in } \mathbb{R}}}{0} \neq \underset{\substack{\downarrow \\ \text{Vector in } \mathbb{R}^2}}{(0,0)}$

So, $A \times (B \cap C) = A \times \phi = \phi$.

NOT EQUAL.

Another example, $A = \{0\}$, $B = \{0\}$, $C = \{0\}$

So $A \times (B \cap C) = \{0\} \times \{0\}$
$= \{(0,0)\}$

and $(A \times B) \cap C = \{\underline{(0,0)}\} \cap \{\underline{0}\} = \phi$.

NOT EQUAL

not the same

Another example, $A = \mathbb{R}$, $B = \mathbb{R}$, $C = \mathbb{R}$

$A \times (B \cap C) = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$(A \times B) \cap C = \mathbb{R}^2 \cap \mathbb{R} = \phi$

NOT EQUAL

$\boxed{\text{Q2(c)}}$ Statement is false.

Need to find relations S, R on a set X.

and such that $S \circ R \neq R \circ S$.

For example, $X = \{0, 1\}$

$S = \{(0,1)\}$

$R = \{(1,0)\}$

Then $S \circ R = \{(1,1)\}$

and $R \circ S = \{(0,0)\}$.

$\boxed{Q\,3(a)}$ Let $P(n)$: $\sum_{i=1}^{n} i!\cdot i = (n+1)!-1$

Base case $P(1)$: LHS $= \sum_{i=1}^{1} i!\cdot i = (1!)(1) = 1$

RHS $= (1+1)! - 1 = 2! - 1 = 1$

$\therefore$ $P(1)$ is true.

Assume $P(k)$ is true, for $k \geq 1$.

$\hookrightarrow \sum_{i=1}^{k} i!\cdot i = (k+1)! - 1.$

$$\begin{aligned}
\text{LHS of } P(k+1) &= \sum_{i=1}^{k+1} i!\cdot i \\
&= \sum_{i=1}^{k} i!\cdot i + (k+1)!\cdot(k+1) \\
(\text{By IH}) \quad &= (k+1)! - 1 + (k+1)!\cdot(k+1) \\
&= (k+1)!\,(1 + (k+1)) - 1 \\
&= (k+1)!\,(k+2) - 1 \\
&= (k+2)! - 1 \\
&= \text{RHS of } P(k+1).
\end{aligned}$$

$\therefore$ By MI, $P(n)$ is true for all $n \geq 1$.

$\boxed{Q3(b)}$ $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1} + 3a_{n-3} + 1$ for $n \geq 3$.

Let $P(n)$: $a_n \leq 2^n$.

<u>Base cases</u> $P(0)$, $P(1)$ and $P(2)$.

$P(0)$: $a_0 = 1 \leq 2^0$

$P(1)$: $a_1 = 2 \leq 2^1$

$P(2)$: $a_2 = 3 \leq 4 = 2^2$

Now assume $P(0), P(1), \ldots, P(k-1)$ holds, where $k \geq 3$.

$$\begin{aligned}
\text{Then} \quad a_k &= a_{k-1} + 3a_{k-3} + 1 && (\text{since } k \geq 3). \\
&\leq 2^{k-1} + 3.2^{k-3} + 1 && (\text{By IH } P(k-1) \text{ and } P(k-3)) \\
&= 2^{k-1} + 2.2^{k-3} + 2^{k-3} + 1 \\
&= 2^{k-1} + 2^{k-2} + 2^{k-3} + 2^0 && \text{Since } k \geq 3, \text{ So } 2^{k-3} \geq 2^0 \\
&\leq 2^{k-1} + 2^{k-2} + 2^{k-3} + 2^{k-3} \\
&= 2^{k-3}(2^2 + 2 + 1 + 1) \\
&= 2^{k-3} . 8 \\
&= 2^k
\end{aligned}$$

By strong MI, $P(n)$ is true for all $n \geq 0$.

$\boxed{\text{Q4(a)}}$ By the definition of the floor function,

for any $x \in \mathbb{R}$, we know that

$$\left.\begin{aligned} \lfloor x \rfloor &\le x < \lfloor x \rfloor + 1 \\ \text{and } \lfloor -x \rfloor &\le -x < \lfloor -x \rfloor + 1 \end{aligned}\right\} \text{In Lecture Notes.}$$

Adding the inequalities together,

$$\lfloor x \rfloor + \lfloor -x \rfloor \le 0 < \lfloor x \rfloor + \lfloor -x \rfloor + 2$$

$$0 \le -(\lfloor x \rfloor + \lfloor -x \rfloor) < 2$$

Since $-(\lfloor x \rfloor + \lfloor -x \rfloor)$ is an integer,

we conclude that

$$-(\lfloor x \rfloor + \lfloor -x \rfloor) = 0 \text{ or } 1$$

i.e. $\lfloor x \rfloor + \lfloor -x \rfloor = 0$ or -1

$\boxed{\text{Q 4(b)}}$

Any of these answers is fine

- The condition is that $S \subsetneq (0, \infty)$,
- i.e. every element of S is positive,
- or S contains only positive real numbers,
- or for every $x \in S$, $x > 0$.

Let's verify that this condition is sufficient.

Suppose S contains only +ve numbers.

Let $a, b \in S$. Let $0 < a \leq b$.

Take $n = \lfloor \frac{b}{a} \rfloor$. Since $\frac{b}{a} \geq 1$, so $n \geq 1$.
↳ positive integer

We have $n \leq \frac{b}{a} < n+1$

$\Rightarrow b < a(n+1)$ (since $a > 0$)

So S has the Archimedean property.

Let's verify that this condition is necessary.

Suppose S contains an element $c \in S$, s.t. $c \leq 0$.

If S has the Archimedean property, then there is a positive integer n such that

$nc > c$.

But since $n \geq 1$ and $c \leq 0$, we also have $nc \leq c$.

$\therefore$ we have $c < nc \leq c \Rightarrow c < c$, a contradiction.

So S does not have the Archimedean property.

$\boxed{\text{Q 5(a)}}$ Let A, B be finite sets.

We want to show $\underbrace{\dfrac{|A|+|B|}{2}}_{\substack{\text{Average of} \\ |A|\ \&\ |B|}} \leq \underbrace{|A \cup B|}_{\text{size of } A \cup B}$

Now we know $A \subseteq A \cup B$

So, $|A| \leq |A \cup B|$.

We also know that $B \subseteq A \cup B$

So similarly, $|B| \leq |A \cup B|$.

Adding the two inequalities,

$$|A| + |B| \leq 2|A \cup B|.$$

$$\therefore \quad \frac{|A|+|B|}{2} \leq |A \cup B|.$$

$\boxed{\text{Q 5(b)}}$ f is not surjective: We verify that 2 is not in the range of f.

Note that
$$f(1) = 1 \text{ div } 3 = 0$$
$$f(2) = 4 \text{ div } 3 = 1$$

Alternatively, you can show that $-1 \notin$ range of f.

and for $n \geq 3$,
$$f(n) = n^2 \text{ div } 3$$
$$\geq 3^2 \text{ div } 3$$
$$= 9 \text{ div } 3$$
$$= 3$$

$\therefore f(n) \neq 2$ for any $n \in \mathbb{Z}^+$

Now we check that f is injective.

Let $n, m \in \mathbb{Z}^+$ such that $n \neq m$, and $f(n) = f(m) = q$ for some $q \in \mathbb{Z}$

Suppose $n < m$. (The other case $n > m$ is same).

$$n^2 = 3q + r \quad \text{and}$$
$$m^2 = 3q + r'$$

for some
$$0 \leq r < 3$$
$$0 \leq r' < 3.$$

Since $n < m$, hence, $n+1 \leq m$,

Furthermore,
$$m^2 - n^2 = (3q + r') - (3q + r)$$
$$= r' - r \leq r' \quad (\text{since } r \geq 0)$$

On the other hand,

since $m, n+1 > 0$
$$m^2 - n^2 \geq (n+1)^2 - n^2$$
$$= 2n + 1 \geq 3 \quad (\text{since } n \geq 1)$$

So, $r' \geq m^2 - n^2 \geq 3$, a contradiction.

So, f is injective.

$\boxed{\text{Q6(a)}}$ Given integers a, b, c. Suppose $a \nmid b^6$ and $a \mid c$ and $a \mid (b^2+c^2)$.

Let $k, l \in \mathbb{Z}$ such that

$$ak = c \quad \text{and} \quad al = b^2 + c^2$$

Then $al = b^2 + (ak)^2$

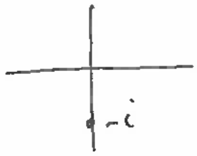
$$\Rightarrow \quad b^2 = al - a^2k^2$$

$$\begin{aligned} \Rightarrow \quad b^6 &= b^4 \cdot b^2 \\ &= b^4(al - a^2k^2) \\ &= a(b^4 l - a \cdot b^4 k^2) \end{aligned}$$

Since $a, b, k, l \in \mathbb{Z}$, $b^4 l - a b^4 k^2 \in \mathbb{Z}$ hence we conclude that $a \mid b^6$, a contradiction to our assumption.

$\boxed{Q6b}$ $z^5 = -i$

$= e^{i\frac{3\pi}{2}}$



Using formula, $\theta = \frac{3\pi}{2}$

$$z = e^{i\frac{\theta + 2k\pi}{5}}, \quad k = 0, 1, 2, 3, 4$$

$$= e^{i\frac{3\pi + 4k\pi}{10}}$$

$$= e^{i\frac{3\pi}{10}}, \quad e^{i\frac{7\pi}{10}}, \quad e^{i\frac{11\pi}{10}}, \quad e^{i\frac{15\pi}{10}}, \quad e^{i\frac{19\pi}{10}}$$

$\boxed{Q6c}$ To find $\gcd(1989, 6435)$

$$6435 = (1989) \cdot 3 + 468$$

$$1989 = (468) \cdot 4 + 117$$

$$468 = (117) \cdot 4 + 0$$

Last non-zero remainder $= 117$

$$\gcd(1989, 6435) = 117$$

$\boxed{Q7(a)}$ T is not reflexive: Need to find $A \in \mathcal{P}(\mathbb{Z})$ s.t. $A \not\!T A$.

we can let $A = \phi$.

Then $A \cap A = \phi$ and so $A \not\!T A$.

T is symmetric: let $A, B \in \mathcal{P}(\mathbb{Z})$ s.t.

ATB holds. So, $A \cap B \neq \phi$.

Since $B \cap A = A \cap B$, so $B \cap A \neq \phi$.

So, BTA holds.

T is not transitive: Need to find $A, B, C \in \mathcal{P}(\mathbb{Z})$

Such that ATB holds,
BTC holds,
$A \not\!T C$ holds.

we can take $A = \{0, 1\}$
$B = \{1, 2\}$
$C = \{2, 3\}$.

Then ATB holds as $A \cap B = \{1\} \neq \phi$
BTC holds since $B \cap C = \{2\} \neq \phi$
$A \not\!T C$ holds since $A \cap C = \phi$.

$\boxed{Q7(b)}$ R is defined on $\mathbb{R}^2$ by

$$(a,b)\ R\ (x,y) \iff b=y.$$

<u>R is reflexive</u>: let $(a,b)\in\mathbb{R}^2$.

Then $b=b$, so $(a,b)\ R\ (a,b)$ holds.

<u>R is symmetric</u>: let $(a,b),(x,y)\in\mathbb{R}^2$ s.t.

$(a,b)\ R\ (x,y)$ holds. So, $b=y$.

Hence $y=b$ and so

$(x,y)\ R\ (a,b)$ holds.

<u>R is transitive</u>: let $(a,b),(x,y),(u,v)\in\mathbb{R}^2$ s.t.

$(a,b)\ R\ (x,y)$ holds, and

$(x,y)\ R\ (u,v)$ holds.

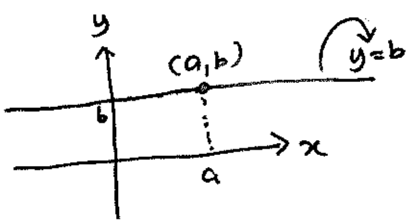
So, $b=y$ & $y=v$ holds.

So, $b=v$ holds.

Hence, $(a,b)\ R\ (u,v)$ holds.

If $(a,b)\in\mathbb{R}^2$, then $[(a,b)]=\{(x,y)\in\mathbb{R}^2 \mid y=b\}$.

Geometrically this is the line $y=b$.



The distinct equivalence classes of R are the lines $y=b$ for each $b\in\mathbb{R}$.