NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2022-2023

MH1100 – Calculus I

December 2022 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED** book exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1 **(16 marks)**

(a) Find the limit or show it does not exist

$$\lim_{x\to\infty} x^{1/3}\sin(\frac{1}{\sqrt{x}}).$$

(b) Find the limit (you can use l'Hospital's Rule)

$$\lim_{x\to 0}\frac{\ln(1+\ln(1+3x^2))}{x^2}.$$

[Solution:]

(a)

$$\lim_{x\to\infty} x^{1/3}\sin(\frac{1}{\sqrt{x}}) = \lim_{x\to\infty}\frac{\sin(\frac{1}{\sqrt{x}})}{\frac{1}{\sqrt{x}}}x^{-1/6} = \lim_{x\to\infty}\frac{\sin(\frac{1}{\sqrt{x}})}{\frac{1}{\sqrt{x}}}\lim_{x\to\infty}x^{-1/6} = 0.$$

(b)

$$\lim_{x\to 0}\frac{\ln(1+\ln(1+3x^2))}{x^2} = \lim_{x\to 0}\frac{\frac{\frac{6x}{1+3x^2}}{1+\ln(1+3x^2)}}{2x} = 3.$$

QUESTION 2 **(16 marks)**

Let a be a fixed real number. Use the ϵ-δ definition to prove the following: if we have $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$, then we have $\lim_{x\to a}(f(x) - g(x)) = f(a) - g(a)$.

[Solution:] For $\forall\varepsilon > 0$, since we have $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$, there exists δ_1, such that when $|x-a| < \delta_1$, we have

$$|f(x) - f(a)| \le \varepsilon/2,$$

there exists δ_2, such that when $|x-a| < \delta_2$, we have

$$|g(x) - g(a)| \le \varepsilon/2.$$

We can let $\delta = \min\{\delta_1, \delta_2\}$, when $|x - a| < \delta$, we have

$|(f(x)-g(x))-(f(a)-g(a))| = |(f(x)-f(a))-(g(x)-g(a))| \leq |f(x)-f(a)|+|g(x)-g(a)| \leq \varepsilon$

More specifically, that is to say, $\forall \varepsilon > 0$, there exists δ. When $|x - a| < \delta$, we have

$$|(f(x) - g(x)) - (f(a) - g(a))| \leq \varepsilon$$

Thus $\lim_{x \to a}(f(x) - g(x)) = f(a) - g(a)$.

QUESTION 3 **(16 marks)**

Find the explicit expression for $f(x)$ such that

$$f'''(x) = \sin x, \quad f''(0) = 1, \; f'(0) = 2, \; f(0) = 3.$$

[Solution:] We have

$$f'''(x) = \sin x \Rightarrow f''(x) = -\cos x + C \Rightarrow f''(0) = -1 + C$$

Since $f''(0) = 1$, we have $f''(x) = -\cos x + 2$. Further

$$f''(x) = -\cos x + 2 \Rightarrow f'(x) = -\sin x + 2x + D \Rightarrow f'(0) = D$$

Since $f'(0) = 2$, we have $f'(x) = -\sin x + 2x + 2$. Further

$$f'(x) = -\sin x + 2x + 2 \Rightarrow f(x) = \cos x + x^2 + 2x + E \Rightarrow f(0) = 1 + E$$

Since $f(0) = 3$, we have

$$f(x) = \cos x + x^2 + 2x + 2.$$

QUESTION 4 **(16 marks)**

Suppose that y is an implicit function of x satisfying that

$$y \ln(x^{2022}) - x \ln(y^{2022}) = 0,$$

find y'.

[Solution:] We have

$$2022y\ln(x) - 2022x\ln(y) = 0 \Rightarrow y\ln(x) = x\ln(y).$$

Take the derivative with respect to x,

$$y'\ln x + \frac{y}{x} = \ln y + \frac{xy'}{y} \Rightarrow y'(\ln x - \frac{x}{y}) = \ln y - \frac{y}{x} \Rightarrow y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}.$$

QUESTION 5 **(12 marks)**

Suppose that $n > 0$ is an integer and $a_0, ..., a_n$ are real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n = 0.$$

Prove that the function

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

has at least one root in $(0, 1)$.

[Solution:] Let

$$F(x) = \frac{a_0}{n+1}x^{n+1} + \frac{a_1}{n}x^n + \cdots + \frac{a_{n-1}}{2}x^2 + a_nx.$$

We have $F(x)$ is continuous and differentiable on $[0, 1]$. Further,

$$F(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n = 0 \quad \text{and} \quad F(0) = 0.$$

From Rolle's theorem, there is a number c in $(0, 1)$, such that $F'(c) = f(c) = 0$. That is the function has a root in the region.

QUESTION 6 **(12 marks)**

Suppose $f(x)$ has second derivative in $[a,b]$, and $f(a)=f(b)=0$. Prove that for every $x\in(a,b)$, there exists a $\xi\in(a,b)$, such that

$$f(x)=\frac{f''(\xi)}{2}(x-a)(x-b).$$

[Solution:] For any $x\in(a,b)$, we define a function $F(t)$ as

$$F(t)=f(t)-\frac{f(x)(t-a)(t-b)}{(x-a)(x-b)}.$$

Note that $F(t)$ is a continues and differentiable function on $[a,b]$. Since $f(a)=f(b)=0$, we have $F(a)=F(b)=0$. Further, we have $F(x)=0$. We consider region $[a,x]$, from Rolle's theorem, there is a $\xi_1\in(a,x)$, such that $F'(\xi_1)=0$. Similarly, at region $[x,b]$, there is a $\xi_2\in(x,b)$, such that $F'(\xi_2)=0$. Since $F'(t)$ is continues and differentiable function on $[\xi_1,\xi_2]$, from Rolle's theorem, there is $\xi\in(\xi_1,\xi_2)$, such that $F''(\xi)=0$. Note that

$$F''(t)=f''(t)-\frac{2f(x)}{(x-a)(x-b)}$$

so we have

$$F''(\xi)=f''(\xi)-2\frac{f(x)}{(x-a)(x-b)}=0.$$

Therefore we have

$$f(x)=\frac{f''(\xi)}{2}(x-a)(x-b).$$

QUESTION 7 **(12 marks)**

Suppose we have

$$f(x)=\frac{2x}{1-x^2},$$

find $f^{(n)}(x)$.
(Note that $f^{(n)}(x)$ is the n-th order derivative of $f(x)$.)

[Solution:] we have

$$f(x)=\frac{1}{1-x}-\frac{1}{1+x}=(1-x)^{-1}-(1+x)^{-1}.$$

so

$$f'(x) = (-1)(-1)(1-x)^{-2} - (-1)(1+x)^{-2} = (-1)((-1)(1-x)^{-2} - (1+x)^{-2}).$$

$$f''(x) = (-1)(-2)((-1)^2(1-x)^{-3} - (1+x)^{-3}).$$
$$f^{(3)}(x) = (-1)(-2)(-3)((-1)^3(1-x)^{-4} - (1+x)^{-4}).$$

thus we can find that

$$f(n)(x) = (-1)^n n!((-1)^n(1-x)^{-(n+1)} - (1+x)^{-(n+1)}).$$

It is equal to

$$f(n)(x) = n!(\frac{1}{(1-x)^{n+1}} - \frac{(-1)^n}{(1+x)^{n+1}}).$$

END OF PAPER