NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2020-2021

MH1300– Foundations of Mathematics

November 2020 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS** a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **BOTH SIDES OF ONE A4-SIZED HANDWRITTEN HELPSHEET.**
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (15 marks)

(a) Let a and b be integers. Prove that $(a+b)^2$ is odd if and only if a and b are of opposite parity.

(b) Let n be an odd integer. Prove that $(n^2+3)(n^2+7)$ is divisible by 32.

(c) Is the following statement form a tautology, a contradiction, or neither?

$$((p \to q) \leftrightarrow q) \to p$$

Justify your answer.

QUESTION 2. (12 marks)

Determine if each of the following is true or false. Justify your answers.

(a) There is a rational number $x \neq 0$ and an irrational number $y \neq 0$ such that $\frac{1}{x} + \frac{x}{y} = 1$.

(b) If A, B and C are sets then $(A - B) \cup (A - C) = A - (B \cup C)$.

(c) For any integers n and m, if $3 \mid n$ and $3 \nmid m$ then $3 \nmid (n+m)$.

QUESTION 3. (15 marks)

(a) Use mathematical induction to prove that for every positive integer n,

$$1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n(3n+1) = n(n+1)^2.$$

(b) Prove that for every positive integer n,

$$6 \cdot 7^n - 2 \cdot 3^n \text{ is divisible by } 4.$$

QUESTION 4. (12 marks)

(a) Prove by the definition of the absolute value function that

$$|r| = |-r|$$

holds for every real number r.

(b) Suppose that a, b, c and d are integers such that $d \mid a$ and $d \mid b$ but $d \nmid c$, where $d \neq 0$. Prove that there are no integers x and y such that $ax + by = c$.

(c) Let B and C be sets where $B \cup C = B \cap C$. Write down a different relationship between B and C, and prove it.

QUESTION 5. (14 marks)

(a) Let A be the set $\{\emptyset, \{\emptyset\}\}$. Write down all the elements of $A \times \mathcal{P}(A)$.

(b) Let $B = \{b_1, b_2, \cdots, b_k\}$ be a set with k elements, for some positive integer k. What are the smallest and the largest equivalence relation on B, in terms of size? Justify your answer.

(c) Prove that if R is a reflexive and transitive relation on a non-empty set C, then $R \circ R = R$.

QUESTION 6. (14 marks)

(a) Find all complex numbers z satisfying the equation $z^3 - 2 - 2i = 0$.

(b) Suppose that $f : A \to B$ and $g : B \to C$ for sets A, B and C. Prove that if f and g are surjective, then $g \circ f$ is surjective.

(c) Suppose that D and E are sets and $h : D \to E$. Prove that h is injective if and only if $h^{-1}(h(X)) = X$ for every set $X \subseteq D$.

QUESTION 7. (18 marks)

(a) Let E and F be equivalence relations on a non-empty set A.

(i) Show that $E \cap F$ is an equivalence relation on A.

(ii) Describe the equivalence classes of $E \cap F$ in terms of the equivalence classes of E and the equivalence classes of F. Justify your answer.

(iii) Is $E \cup F$ an equivalence relation on A? Prove it, or give a counter-example.

(b) Use the Euclidean algorithm to find the greatest common divisor of the pair

$$168 \qquad \text{and} \qquad 198.$$

END OF PAPER