Name: ________________________________ Tutorial group: ____

Matriculation number: | | | | | | | | |

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2021/22

MH1100 – Calculus I

24 September 2021 Midterm Test 90 minutes

<u>INSTRUCTIONS</u>

1. Do not turn over the pages until you are told to do so.
2. Write down your name, tutorial group, and matriculation number.
3. This test paper contains **SIX (6)** questions and comprises **SEVEN (7)** printed pages.
4. The marks for each question are indicated at the beginning of each question.

For graders only	Question	1	2	3	4	5	6	Total
	Marks							

QUESTION 1. **(2 marks)**

Let ϵ be a given positive number. Verify that a possible choice of δ for showing that

$$\lim_{x \to 4} x^2 = 16$$

is

$$\delta = \min\{2, \frac{\epsilon}{10}\}.$$

SOLUTION Given $\epsilon > 0$, choose $\delta = \min\{2, \epsilon/10\}$. We have $\delta \leq 2$ and $\delta \leq \epsilon/10$. On the one hand, if $0 < |x - 4| < \delta \leq 2$, then $-2 < x - 4 < 2$ or $6 < x + 4 < 10$. On the other hand, if $0 < |x - 4| < \delta \leq \frac{\epsilon}{10}$, then $|x - 4| < \frac{\epsilon}{10}$. So, we can say that

$$|x^2 - 16| = |(x + 4)(x - 4))| = |(x + 4)| \cdot |(x - 4))| < 10 \times \frac{\epsilon}{10} = \epsilon.$$

Thus,

$$\delta = \min\{2, \frac{\epsilon}{10}\}$$

is a possible choice.

QUESTION 2. **(4 marks)**

Find the limits if exist.

(a) $\lim_{t\to 2}\left(\frac{t^2+6t-16}{t^2-5t+7}\right)^{100}$, (b) $\lim_{x\to 0}\frac{\sqrt{9+x}-3}{3x}$,

(c) $\lim_{x\to 2}\frac{x^2+ax+a+8}{(x-2)^2}$, where a is a real number, (d) $\lim_{x\to 0}\frac{|x|}{\sin x}$.

SOLUTION

(a) The function

$$g(t)=\frac{t^2+6t-16}{t^2-5t+7}$$

is a rational function and it is continuous on its domain. We know $t=2$ is in the domain of $g(t)$. So,

$$\lim_{t\to 2}g(t)=\frac{2^2+6\times 2-16}{2^2-5\times 2+7}=0.$$

Given that $f(t)=t^{100}$ is a continuous function and the original function can be written as $f(g(t))$, we have

$$\lim_{t\to 2}f(g(t))=f\left(\lim_{t\to 2}g(t)\right)=f(0)=0.$$

(b) We rationalize the numerator.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{9+x}-3}{3x}&=\lim_{x\to 0}\frac{\sqrt{9+x}-3}{3x}\cdot\frac{\sqrt{9+x}+3}{\sqrt{9+x}+3}=\lim_{x\to 0}\frac{9+x-9}{3x}\cdot\frac{1}{\sqrt{9+x}+3}\\&=\lim_{x\to 0}\frac{x}{3x}\cdot\frac{1}{\sqrt{9+x}+3}=\lim_{x\to 0}\frac{1}{3}\cdot\lim_{x\to 0}\frac{1}{\sqrt{9+x}+3}=\frac{1}{3}\times\frac{1}{6}=\frac{1}{18}.\end{aligned}$$

(c) We assume the limit exists and the limit value is L. It is obvious that

$$\lim_{x\to 2}(x-2)^2=0.$$

By the product law, we know that the limit as x approaches 2 of the numerator exists,

$$\lim_{x\to 2}\left(x^2+ax+a+8\right)=\lim_{x\to 2}\frac{x^2+ax+a+8}{(x-2)^2}\times\lim_{x\to 2}(x-2)^2=L\times 0=0.$$

But

$$\lim_{x\to 2}\left(x^2+ax+a+8\right)=4+2a+a+8=3a+12.$$

So, we have $3a+12=0$ or $a=-4$. Now we go to evaluate the limit value L.

$$L=\lim_{x\to 2}\frac{x^2+ax+a+8}{(x-2)^2}=\lim_{x\to 2}\frac{x^2-4x+4}{(x-2)^2}=1.$$

When $a\neq -4$, the limit does not exist. As 2 approaches 0, the denominator approaches 0 but the numerator approaches a non-zero value. Thus, the limit does not exist.

(d) We consider the two one-sided limits:

$$\lim_{x\to 0^+} \frac{|x|}{\sin x} = \lim_{x\to 0^+} \frac{x}{\sin x} = 1, \qquad \lim_{x\to 0^-} \frac{|x|}{\sin x} = \lim_{x\to 0^-} \frac{-x}{\sin x} = -1.$$

The two one-sided limits are not equal. The limit does not exist.

QUESTION 3. **(4 marks)**

(a) If $\lim_{x\to 0} \frac{f(x)}{x^2} = 10$, find $\lim_{x\to 0} \frac{f(x)}{x}$.

(b) Suppose that $|f(x)| \leq x^2$ for all x. Find $\lim_{x\to 0} f(x)$.

SOLUTION

(a) We know that

$$\frac{f(x)}{x} = \frac{f(x)}{x^2} \cdot x$$

and $\lim_{x\to 0} x = 0$. Following the product law, we have

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x)}{x^2} \times \lim_{x\to 0} x = 10 \times 0 = 0.$$

(b) For any given real number a, we have $-|a| \leq a \leq |a|$. Equivalently,

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

We further have $|f(x)| \leq x^2$, which also indicates that $-|f(x)| \geq -x^2$. Combining all the inequalities together, we can obtain that

$$-x^2 \leq f(x) \leq x^2.$$

Obviously, $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$. The squeeze theorem yields that $\lim_{x\to 0} f(x) = 0$.

QUESTION 4. **(3 marks)**

Show that there is at least one root of the equation

$$\ln x + \sqrt{4 - x^2} = 1.$$

SOLUTION We consider the function $f(x) = \ln x + \sqrt{4 - x^2} - 1$. Its domain is $(0,\, 2]$. The function $f(x)$ is continuous in its domain. We can apply the I.V.T. to this function on a closed interval to prove that there exists a number c such that $f(c) = 0$. By trial and error, we find that $[1,\, 2]$ is a possible choice. Obviously, $f(x)$ is continuous on $[1,\, 2]$. $f(1) = \sqrt{3} - 1 > 0$ and $f(2) = \ln 2 - 1 < 0$. We deduce from the I.V.T. that there exists a $c \in [1,\, 2]$ where $f(c) = 0$. This c will solve the given equation. We also find another closed interval $[e^{-2},\, 1]$ which contains at least one root of that equation.

QUESTION 5. **(4 marks)**

Let

$$f(x) = \begin{cases} x^2, & x \leq 4; \\ ax + b, & x > 4. \end{cases}$$

(a) Find the values of a and b that make $f(x)$ continuous everywhere.

(b) Find the values of a and b that make $f(x)$ differentiable everywhere.

SOLUTION

(a) Give $c < 4$ or $c > 4$, the function $f(x)$ is continuous at c as $\lim_{x\to c} f(x) = f(c)$. Now we consider $c = 4$. The two one-sided limits are computed

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} x^2 = 16, \qquad \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (ax + b) = 4a + b.$$

To make $f(x)$ continuous at $x = 4$, we must have $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$. Here $f(4) = 16$. Thus, $4a + b = 16$. To conclude, the condition $4a + b = 16$ must be satisfied to make $f(x)$ be continuous everywhere.

(b) Again, given $c < 4$ or $c > 4$, the function $f(x)$ is differentiable at c. At $c = 4$, if $f(x)$ is differentiable, then $f(x)$ is continuous. So, we have $4a + b = 16$ from Part (a). By definition, the derivative of $f(x)$ at $x = 4$ is

$$f'(4) = \lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x\to 4} \frac{f(x) - 16}{x - 4}.$$

$f(x)$ has different expressions on the two sides of $x = 4$. We turn to evaluate the two one-sided limits.

$$\lim_{x\to 4^+} \frac{f(x) - 16}{x - 4} = \lim_{x\to 4^+} \frac{ax + b - 16}{x - 4} = \lim_{x\to 4^+} \frac{ax + (16 - 4a) - 16}{x - 4} = a,$$

and

$$\lim_{x\to 4^-} \frac{f(x) - 16}{x - 4} = \lim_{x\to 4^-} \frac{x^2 - 16}{x - 4} = \lim_{x\to 4^-} (x + 4) = 8.$$

So, $f'(4)$ is defined only when $a = 8$ and $b = -16$. In summary, $a = 8$ and $b = -16$ make $f(x)$ differentiable everywhere.

QUESTION 6. **(3 marks)**

Find equations of the tangent lines to the curve

$$y = \frac{x-2}{x+2}$$

that are parallel to the line $x - 4y = 4$.

SOLUTION The slope of the curve at x is $y'(x)$. By the quotient rule, we have

$$y'(x) = \frac{(x+2)-(x-2)}{(x+2)^2} = \frac{4}{(x+2)^2}.$$

The slope of the line $x - 4y = 4$ is $1/4$. We can find that the curve has a slope of $1/4$ at $x_1 = 2$ and $x_2 = -6$, where $4/(x+2)^2 = 1/4$. $(2, 0)$ and $(-6, 2)$ are the two points on the curve. We can find the equations of the two tangent lines as

$$y = \frac{1}{4}(x-2), \qquad\qquad y - 2 = \frac{1}{4}(x+6)$$

or

$$x - 4y = 2, \qquad\qquad x - 4y = -14.$$