NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2017-2018

MH1300– Foundations of Mathematics

December 2017 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (15 marks)

(a) Disprove the following:

(i) For any sets A and B, $A - B = B - A$.

(ii) For any sets A, B and C, if $A \cup B = A \cup C$ then B and C are disjoint.

(b) Prove that for any sets A, B and C,

$$(A \cap (B \cup C)) \cup (B \cap (A \cup C)) \subseteq (A \cup B) \cap (A \cup C).$$

QUESTION 2. (15 marks)

Determine if the following are true or false. Justify your answer.

(a) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ \ |xy| < 1 \rightarrow x + y > 2.$

(b) $\forall x \in \mathbb{Z},\ \forall y \in \mathbb{Z},\ \ x^2 < y^2 \rightarrow x < y.$

(c) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z},\ \ y^2 - x < 100.$

QUESTION 3. (10 marks)

Prove that for every integer $n > 0$,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

QUESTION 4. (15 marks)

(a) Let n be an integer. Prove that if 3 divides $2n$ then 3 divides n.

(b) Let n and m be integers. Prove that if n is even and m is odd, then 4 does not divide $n^2 + 2m^2$.

(c) Let a, b, c, d, e be real numbers. The average of these five numbers is $\frac{a+b+c+d+e}{5}$. Prove that one of the five numbers is at least as large as their average.

QUESTION 5. (18 marks)

(a) Let A be a set and $f : A \to A$. Prove that if $f \circ f$ is injective, then f is injective.

(b) Let $g : \mathbb{N} \to \mathbb{N}$ be a function such that $g(n+m) = g(n) + g(m)$ for all $n, m \in \mathbb{N}$. Let $a = g(1)$. Write down a formula for $g(n)$ in terms of n and a and prove that it holds.

(c) The function h is defined on a set of real numbers. Determine whether or not h is injective and justify your answer.

$$h(x) = \frac{3x-1}{x}, \text{ for all real numbers } x \neq 0.$$

QUESTION 6. (12 marks)

(a) Find all the fourth complex roots of $4 - 4i$.

(b) For a real number x, define the predicates $P(x)$ by "$\frac{1}{2} < x < \frac{5}{2}$", $Q(x)$ by "x is an integer", $R(x)$ by "$x^2 = 1$" and $S(x)$ be "$x = 2$". Which of the following are true? Justify your answer.

(i) $\forall x \in \mathbb{R},\ P(x) \to R(x)$.

(ii) $\forall x \in \mathbb{R},\ Q(x) \to R(x)$.

(iii) $\forall x \in \mathbb{R},\ (P(x) \wedge Q(x)) \to (R(x) \vee S(x))$.

(iv) $\exists x \in \mathbb{R},\ S(x) \to R(x)$.

QUESTION 7. (15 marks)

(a) Let A be a non-empty set and B be a fixed subset of A. Let $\mathscr{P}(A)$ be the power set of A. Define the relation R on $\mathscr{P}(A)$ by $X\ R\,Y$ if and only if $X \cap B = Y \cap B$.

(i) Prove that R is an equivalence relation.

(ii) Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5\}$. Determine the equivalence class of $X = \{2, 3, 4\}$ and the equivalence class of B.

(b) Use the Euclidean algorithm to find the greatest common divisor of the pair

$$1188 \qquad \text{and} \qquad 385.$$

END OF PAPER