SPMS / Division of Mathematical Sciences

MH1300 Foundations of Mathematics
2017/2018 Semester 1

MID-TERM EXAM SOLUTIONS

QUESTION 1. **(10 marks)**

Derive the following logical equivalence **without** using truth tables:

$$(p \wedge q) \leftrightarrow p \;\equiv\; p \to q$$

You should use the list of logical equivalences in Theorem 2.1.1 of the lecture notes. You do not need to state the name of the logical equivalence at each step.

SOLUTION . We start from the LHS:

$$\begin{aligned}
(p \wedge q) \leftrightarrow p &\equiv ((p \wedge q) \to p) \;\wedge\; (p \to (p \wedge q)) && \text{[Definition of } \leftrightarrow\text{]}\\
&\equiv (\neg(p \wedge q) \vee p) \;\wedge\; (\neg p \vee (p \wedge q)) && \text{[2.2.13a of Tutorial 3]}\\
&\equiv (\neg(p \wedge q) \vee p) \;\wedge\; ((\neg p \vee p) \wedge (\neg p \vee q)) && \text{[Distributive law]}\\
&\equiv (\neg(p \wedge q) \vee p) \;\wedge\; (\mathbf{T} \wedge (\neg p \vee q)) && \text{[Negation law]}\\
&\equiv (\neg(p \wedge q) \vee p) \;\wedge\; (\neg p \vee q) && \text{[Identity law]}\\
&\equiv ((\neg p \vee \neg q) \vee p) \;\wedge\; (\neg p \vee q) && \text{[De Morgan's law]}\\
&\equiv ((\neg q \vee \neg p) \vee p) \;\wedge\; (\neg p \vee q) && \text{[Commutative law]}\\
&\equiv (\neg q \vee (\neg p \vee p)) \;\wedge\; (\neg p \vee q) && \text{[Associative law]}\\
&\equiv (\neg q \vee \mathbf{T}) \;\wedge\; (\neg p \vee q) && \text{[Negation law]}\\
&\equiv \mathbf{T} \;\wedge\; (\neg p \vee q) && \text{[Universal Bound law]}\\
&\equiv \neg p \vee q && \text{[Identity law]}\\
&\equiv p \to q && \text{[2.2.13a of Tutorial 3]}
\end{aligned}$$

□

QUESTION 2 **(10 marks)**

Let $A = \{4, 8\}, B = \{2, 4\}$ and $C = \{1, 2, 4\}$. Determine if each of the following is true or false. Justify your answer.

(a) $\forall x \in A,\ \forall y \in B,\ \exists z \in C$ such that $x = yz$.

(b) $\exists x \in A$ such that $\forall y \in B,\ \forall z \in C,\ x = yz \to x = y + z$.

SOLUTION . (a) $\forall x \in A,\ \forall y \in B,\ \exists z \in C$ such that $x = yz$.

This is true. We have to go through every pair of elements $x \in A$ and $y \in B$, and for each pair, find some $z \in C$ which works. There are four pairs of elements x, y:

- $x = 4$, $y = 2$: Take $z = 2$. Then $x = 4 = 2 \cdot 2 = yz$ is true.
- $x = 4$, $y = 4$: Take $z = 1$. Then $x = 4 = 4 \cdot 1 = yz$ is true.
- $x = 8$, $y = 2$: Take $z = 4$. Then $x = 8 = 2 \cdot 4 = yz$ is true.
- $x = 8$, $y = 4$: Take $z = 2$. Then $x = 8 = 4 \cdot 2 = yz$ is true.

(b) $\exists x \in A$ such that $\forall y \in B,\ \forall z \in C,\ x = yz \to x = y + z$.

This is false. We need to prove the negation, which is the statement

$$\forall x \in A, \exists y \in B,\ \exists z \in C \text{ such that } x = yz \text{ and } x \neq y + z.$$

To do this, we go through every element $x \in A$, and for each such x, we must find a pair of $y \in B$ and $z \in C$ which works.

- $x = 4$: Take $y = 4$ and $z = 1$. Then $x = 4 = 4 \cdot 1 = yz$ is true and $x = 4 \neq 4 + 1 = y + z$ is true.
- $x = 8$: Take $y = 4$ and $z = 2$. Then $x = 8 = 4 \cdot 2 = yz$ is true and $x = 8 \neq 4 + 2 = y + z$ is true. (It is also possible to take $y = 2$ and $z = 4$).

□

QUESTION 3. **(15 marks)**

Prove the following statements:

(a) Let x be an integer. If 3 does not divide x^2+2, then 3 divides x.

(b) Let a, b, c be integers. If a^2 does not divide bc, then either a does not divide b or a does not divide c.

SOLUTION . (a) Let x be an integer. If 3 does not divide x^2+2, then 3 divides x.

Method 1: Proof using the Division Algorithm: Using the Division Algorithm on $d = 3$ (why choose $d = 3$? obviously because the question is asking about whether or not x is divisible by 3), the number x is of the form $3q, 3q+1$ or $3q+2$ for some integer q. Therefore, there are three cases:

(Case 1) $x = 3q$: In this case, x is divisible by 3, and so the statement "If 3 does not divide x^2+2, then 3 divides x" is true.

(Case 2) $x = 3q+1$: In this case, $x^2+2 = (3q+1)^2+2 = (9q^2+6q+1)+2 = 3(3q^2+2q+1)$. Since $3q^2+2q+1 \in \mathbb{Z}$, this means that 3 divides x^2+2. Hence the statement "If 3 does not divide x^2+2, then 3 divides x" is true.

(Case 3) $x = 3q+2$: In this case, $x^2+2 = (3q+2)^2+2 = (9q^2+12q+4)+2 = 3(3q^2+4q+2)$. Since $3q^2+4q+2 \in \mathbb{Z}$, this means that 3 divides x^2+2. Hence the statement "If 3 does not divide x^2+2, then 3 divides x" is true.

In all three cases, the statement to be proved "If 3 does not divide x^2+2, then 3 divides x" is true.

Method 2: Prove the contrapositive: To prove this, we fix an integer x, and prove the contrapositive. Suppose that 3 does not divide x. We want to get to the conclusion that $3|(x^2+2)$. By the Quotient Remainder Theorem with $d = 3$, we can write $x = 3q$ or $3q+1$ or $3q+2$ for some q. Since 3 does not divide x, the first case is not possible. Thus, we are left with two cases:

(Case 1) $x = 3q+1$: To get to the conclusion we want (that $3|(x^2+2)$), we have to compute the quantity x^2+2. We have $x^2+2 = (3q+1)^2+2 = (9q^2+6q+1)+2 = 3(3q^2+2q+1)$. Since $3q^2+2q+1$ is an integer, we conclude that 3 divides x^2+2.

(Case 2) $x = 3q+2$: We have $x^2+2 = (3q+2)^2+2 = (9q^2+12q+4)+2 = 3(3q^2+4q+2)$. Since $3q^2+2q+2$ is an integer, we conclude that 3 divides x^2+2.

In either case, we conclude that 3 divides x^2+2.

(b) Let a, b, c be integers. If a^2 does not divide bc, then either a does not divide b or a does not divide c.

Method 1: Proof using cases: We split into three cases:

(Case 1) a does not divide b: In this case, the statement "either $a \nmid b$ or $a \nmid c$" is true, and so the statement we want to prove "If $a^2 \nmid bc$, then either $a \nmid b$ or $a \nmid c$" is true, since it has a true conclusion.

(Case 2) a does not divide c: Similar to Case 1.

(Case 3) a divides b and a divides c: Then there exists integers k and l such that $ak = b$ and $al = c$. Multiplying these two quantities together, we get $(ak)(al) = bc$, which means that $a^2(kl) = bc$. Since kl is an integer, we conclude that $a^2|bc$. So the statement we want to prove "If $a^2 \nmid bc$, then either $a \nmid b$ or $a \nmid c$" is true, since it has a false premise.

Note: It should be clear that Case 3 covers the situation where Case 1 and Case 2 are both false, and so we can say that these three cases together cover all possible cases.

Method 2: Prove the contrapositive: Let a, b, c be integers. We have to show $a^2 \nmid bc \to (a \nmid b \ \vee \ a \nmid c)$. It is much easier to prove the contrapositive of this, which is $(a|b \ \wedge \ a|c) \to a^2|bc$. Assume that $a|b$ and $a|c$. Then there exists integers k and l such that $ak = b$ and $al = c$. Multiplying these two quantities together, we get $(ak)(al) = bc$, which means that $a^2(kl) = bc$. Since kl is an integer, we conclude that $a^2|bc$.

$\square$

QUESTION 4. **(15 marks)**

(a) Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$. For integers x and y, let $P(x)$ be the predicate "$7x + 4$ is odd" and let $Q(y)$ be the predicate "$5y + 9$ is odd". Let

$$S = \{(x, y) : x, y \in \mathbb{Z} \text{ and } \neg\,(P(x) \to Q(y))\}\,.$$

Write down the elements of $S \cap (A \times B)$ and $S \cap (B \times A)$.
(*Recall that $S \cap (A \times B)$ is the set of all elements belonging to both S and $A \times B$.*)

(b) Prove or disprove:

For all integers m and n, if $2m + 5n$ is odd then m and n are both odd.

SOLUTION . (a) If $(x, y) \in S$ then $P(x)$ is true and $Q(y)$ is false. In other words, $7x + 4$ is odd and $5y + 9$ is even. This is equivalent to saying that $7x$ is odd and $5y$ is odd. This in turn is equivalent to saying that x and y are both odd.

Therefore $S \cap (A \times B)$ contains exactly those pairs (x, y) such that $x \in A$ and $y \in B$ and x, y are both odd. This is the set

$$S\cap(A\times B) = \{(1,1),(1,3),(1,5),(1,7),(3,1),(3,3),(3,5),(3,7),(5,1),(5,3),(5,5),(5,7)\}\,.$$

Similarly,

$$S\cap(B\times A) = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(7,1),(7,3),(7,5)\}\,.$$

(b) Counterexample: Take $m = 0$ and $n = 1$. Then $2m + 5n = 5$ is odd, but $m = 0$ and $n = 1$ are not both odd, since 0 is even.

$\square$