**NANYANG TECHNOLOGICAL UNIVERSITY**

**SEMESTER I EXAMINATION 2017-2018**

**SM2(MH1100) – Calculus I**

December 2017 TIME ALLOWED: 2 HOURS

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INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED** book exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

**QUESTION 1** **(12 marks)**

Evaluate the limits

(a)

$$\lim_{x\to 0}\frac{x^2-3x}{\tan(2x)};$$

(b)

$$\lim_{x\to 1}\frac{\sqrt{3x+6}-3}{x^2-1}.$$

[Solution:]

(a)

$$\lim_{x\to 0}\frac{x^2-3x}{\tan(2x)}=\lim_{x\to 0}\frac{2x}{\sin(2x)}\frac{\cos(2x)}{2}\frac{x-3}{1}=1\cdot\frac{1}{2}\cdot\frac{-3}{1}=-\frac{3}{2}$$

(b)

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{3x+6}-3}{x^2-1}&=\lim_{x\to 1}\frac{\sqrt{3x+6}-3}{x^2-1}\cdot\frac{\sqrt{3x+6}+3}{\sqrt{3x+6}+3}=\lim_{x\to 1}\frac{3x+6-9}{(x+1)(x-1)}\cdot\frac{1}{\sqrt{3x+6}+3}\\&=\lim_{x\to 1}\frac{3(x-1)}{(x+1)(x-1)}\cdot\frac{1}{\sqrt{3x+6}+3}=\lim_{x\to 1}\frac{3}{(x+1)}\cdot\lim_{x\to 1}\frac{1}{\sqrt{3x+6}+3}\\&=\frac{3}{2}\cdot\frac{1}{\sqrt{9}+3}=\frac{1}{4}.\end{aligned}$$

**QUESTION 2** **(12 marks)**

Use the $\epsilon$-$\delta$ definition to prove the limit $\lim_{x\to 2}(x^2+2x-7)=1$.

[Solution:]

1. Guessing a value for $\delta$.
Let $\epsilon$ be a given positive number. We want to find a number $\delta>0$ such that

$$\text{if}\quad 0<|x-2|<\delta\quad\text{then}\quad|(x^2+2x-7)-1|<\epsilon.$$

To connect $|(x^2+2x-7)-1|$ with $|x-2|$, we write

$$|(x^2+2x-7)-1| = |x^2+2x-8| = |(x-2)(x+4)| = |x-2|\cdot|x+4|.$$

Then we want

$$\text{if} \quad 0<|x-2|<\delta \quad \text{then} \quad |x-2|\cdot|x+4|<\epsilon.$$

However, $|x+4|$ has no upper bound in its domain $\mathbb{R}$. We can only find such a number $C$ if we restrict $x$ to lie in some interval centered at 2. In fact, since we are interested only in values of $x$ that are close to 2, it is reasonable to assume that $x$ is within a distance 1 from 2, that is $|x-2|<1$. Then we have $1<x<3$, so $5<x+4<7$. Thus $|x+4|<7$ when $|x-2|<1$. Now there are two restrictions on $|x-3|$, namely

$$|x-2|<1 \quad \text{and} \quad |x-2|<\frac{\epsilon}{|x+4|}=\frac{\epsilon}{7}.$$

To make sure that both of these inequalities are satisfied, we take $\delta$ to be the smaller of the two numbers 1 and $\epsilon/7$. The notation for this is $\delta=\min\{1,\epsilon/7\}$.

2. Showing that this $\delta$ works.
Given $\epsilon>0$, let $\delta=\min\{1,\epsilon/7\}$. If $0<|x-2|<\delta$, then

$$|x-2|<1 \Rightarrow 1<x<3 \Rightarrow |x+4|<7.$$

We also have $|x-2|<\epsilon/7$, so

$$|(x^2+2x-7)-1| = |x^2+2x-8| = |(x-2)(x+4)| = |x-2|\cdot|x+4| < \frac{\epsilon}{7}\cdot 7=\epsilon.$$

Therefore,

$$\lim_{x\to 2}(x^2+2x-7)=1.$$

**QUESTION 3** **(12 marks)**

Find the derivatives of the following functions. **You do not need to simplify.**

(a)

$$g(x)=(\frac{1}{x^2}+3x)(\frac{1}{\sqrt{x}}+1);$$

(b)

$$h(t) = 7\sec(t)\tan(\frac{3}{t}).$$

[Solution:]

(a)

$$g'(x) = [-2x^{-3} + 3][\frac{1}{\sqrt{x}+1}] + [\frac{1}{x^2} + 3x][-\frac{1}{2}x^{-3/2}]$$

(b)

$$h'(t) = 7[\sec(t)\tan(t)]\tan(\frac{3}{t}) + 7\sec(t)[\sec^2(\frac{3}{t})(-3t^{-2})]$$

**QUESTION 4** **(12 marks)**

(a) State the Intermediate Value Theorem (IVT).

(b) Use the IVT to show that the equation $\sin(x) = x^2 - 1$ has at least one root. (Remember to state why you can apply the IVT)

[Solution:]

(a) If (i) $f$ is continuous on a closed interval $[a, b]$ and (ii) $N$ is an intermediate value between $f(a)$ and $f(b)$, then there exists a $c \in (a, b)$, such that $f(c) = N$.

(b) Take $f(x) = \sin(x) + 1 - x^2$ then $f(x)$ is continuous everywhere. Now $f(0) = 1$ and $f(2) = \sin(2) - 3 < 0$ (since $\sin(2) < 1$) so by the IVT there is a $c \in (0, 2)$ where $f(c) = \sin(c) = c^2 - 1 = 0$ so therefore $\sin(c) = c^2 - 1$ as desired.

**QUESTION 5** **(12 marks)**

Find the equation of the tangent line to the curve $y^2 + 13x = x^2y + 13$ at the point $P(4, 3)$.

[Solution:]

$$y^2 + 13x = x^2y + 13$$
$$2yy' + 13 = 2xy + x^2y'$$

At point $P(4, 3)$, we have $x = 4, y = 3$, so

$$2(3)y' + 13 = 2(4)(3) + (4)^2y' \iff 6y' + 13 = 24 + 16y' \iff 10y' = -11 \iff y' = -\frac{11}{10}.$$

The equation of the tangent line at $(4, 3)$ is

$$y - 3 = -\frac{11}{10}(x - 4) \iff 11x + 10y - 74 = 0.$$

**QUESTION 6** **(12 marks)**

Suppose you throw a stone upward from the top edge of a 100 meter tall building at time $t = 0$ and the stone's height is equal to $h(t) = 5t - 5t^2 + 100$ meters at $t$ seconds.

(a) When will the stone reach the highest point?

(b) With what velocity will the stone hit the ground?

[Solution:]

(a) Velocity is given by $v(t) = 5 - 10t$. At the highest point, the velocity is zero,

$$5 - 10t = 0$$

so $t = 0.5$ sec.

(b) The time at which the stone hits the ground is given by:

$$0 = 5t - 5t^2 + 100 \iff 0 = t - t^2 + 20 \iff 0 = (t + 4)(5 - t)$$

so $t = -4, t = 5$ but $t = -4$ makes no sense in the context of this problem so $t = 5$. Velocity is given by $v(t) = 5 - 10t$ so the velocity when the stone hits the ground is:

$$v(5) = 5 - 10(5) = -45 \text{ m/sec.}$$

**QUESTION 7** **(12 marks)**

Find the horizontal asymptote for the function

$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}.$$

[Solution:]

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{\frac{3x^2-x-2}{x^2}}{\frac{5x^2+4x+1}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \frac{\lim\limits_{x\to\infty}(3 - \frac{1}{x} - \frac{2}{x^2})}{\lim\limits_{x\to\infty}(5 + \frac{4}{x} + \frac{1}{x^2})}$$

$$= \frac{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty}\frac{1}{x} - \lim\limits_{x\to\infty}\frac{2}{x^2}}{\lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty}\frac{4}{x} + \lim\limits_{x\to\infty}\frac{1}{x^2}} = \frac{3-0-0}{5+0+0} = \frac{3}{5}$$

The horizontal asymptote is

$$y = \frac{3}{5}.$$

**QUESTION 8** **(16 marks)**

Find the values of $a$, $b$, $c$, and $d$ that make $f$ differentiable everywhere.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x < 2; \\ ax^2 + bx + 6, & \text{if } 2 \le x < c; \\ 2x + d, & \text{if } x \ge c. \end{cases}$$

[Solution:] $f(x)$ is differentiable everywhere. So $f(x)$ is continuous everywhere. We have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x).$$

The one-sided limits of $f(x)$ at $x = 2$ can be evaluated as

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2^-}(x+2) = 4$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(ax^2 + bx + 6) = 4a + 2b + 6.$$

Thus, $f(2) = 4$ and $4 = 4a + 2b + 6$.
The derivative function of $f(x)$ in the open intervals $(-\infty, 2)$, $(2, c)$, and $(c, \infty)$ can be expressed as

$$f'(x) = \begin{cases} 1, & \text{if } x < 2; \\ 2ax + b, & \text{if } 2 < x < c; \\ 2, & \text{if } x > c. \end{cases}$$

$f(x)$ is differentiable everywhere. So it is differentiable at $x = 2$. Based on the definition, when $x = 2$,

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{f(2+h) - 4}{h} \\ &= \begin{cases} \lim_{h\to 0^-} \frac{(2+h+2)-4}{h} = 1, \\ \lim_{h\to 0^+} \frac{[a(2+h)^2+b(2+h)+6]-(4a+2b+6)}{h} = 4a + b. \end{cases} \end{aligned}$$

That the above two one-sided limits are equal indicates that $1 = 4a + b$. Previously we have $4a + 2b + 6 = 4$. So we can get $a = 1$ and $b = -3$. Now we rewrite $f(x)$ as

$$f(x) = \begin{cases} x + 2, & \text{if } x < 2; \\ x^2 - 3x + 6, & \text{if } 2 \le x < c; \\ 2x + d, & \text{if } x \ge c. \end{cases}$$

$f(x)$ is differentiable at $x = c$. So $f(x)$ is continuous at $c$. This implies that

$$c^2 - 3c + 6 = \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = 2c + d.$$

That is $c^2 - 3c + 6 = 2c + d$. According to that $f(x)$ is differentiable at $x = 2$, from the definition we have

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h}.$$

In details,

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} &= \lim_{h\to 0^-} \frac{[(c+h)^2 - 3(c+h) + 6] - (2c + d)}{h} \\ &= \lim_{h\to 0^-} \frac{[(c+h)^2 - 3(c+h) + 6] - (c^2 - 3c + 6)}{h} \\ &= 2c - 3. \end{aligned}$$

and

$$\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0^+}\frac{[2(c+h)+d]-(2c+d)}{h}=2.$$

So $2c-3=2$ and then $c=\frac{5}{2}$. We can solve $d$ from our previously obtained equation $c^2-3c+6=2c+d$ and have $d=-\frac{1}{4}$. In summary, we get the values of $a$, $b$, $c$, and $d$ as

$$a=1, b=-3, c=\frac{5}{2}, d=-\frac{1}{4}.$$

The function $f(x)$ is finally expressed as

$$f(x)=\begin{cases} x+2, & \text{if } x<2; \\ x^2-3x+6, & \text{if } 2\le x<\frac{5}{2}; \\ 2x-\frac{1}{4}, & \text{if } x\ge\frac{5}{2}. \end{cases}$$

**END OF PAPER**