SPMS / Division of Mathematical Sciences

MH1300 Foundations of Mathematics
2022/2023 Semester 1

MID-TERM EXAM SOLUTIONS

QUESTION 1. **(18 marks)**

(a) Is the following pair of statement forms logically equivalent? Justify your answer.

$$p \to (q \to r) \text{ and } (p \wedge q) \to r.$$

(b) Let $P(x)$ and $Q(x)$ be predicates where the domain of x is the set of integers. Do the following always have the same truth values?

$$\forall x\,(\,P(x) \to Q(x)\,) \quad \text{and} \quad (\,\forall x P(x)\,) \to (\,\forall x Q(x)\,)$$

If yes, show it. If no, provide examples of predicates $P(x)$ and $Q(x)$ where the two statements above have different truth values. Justify your answer.

SOLUTION . (a) This pair is logically equivalent. You can either draw the truth tables for both statements and show that the output values are the same, or you can work it out by logical equivalences. We show it below by using logical equivalences.

$$\begin{aligned}
& p \to (q \to r) && [\text{Using } a \to b \equiv \neg a \vee b] \\
\equiv\ & \neg p \vee (q \to r) && [\text{Using } a \to b \equiv \neg a \vee b \text{ again}] \\
\equiv\ & \neg p \vee (\neg q \vee r) && [\text{Associative law}] \\
\equiv\ & (\neg p \vee \neg q) \vee r && [\text{De Morgan's law}] \\
\equiv\ & \neg\,(p \wedge q) \vee r && [\text{Using } a \to b \equiv \neg a \vee b] \\
\equiv\ & (p \wedge q) \to r
\end{aligned}$$

Common Mistakes: Generally okay, but some students dropped the associative law and did not put brackets around three terms, e.g. $\neg p \vee \neg q \vee r$ *instead of* $\neg p \vee (\neg q \vee r)$.

(b) They do not always have the same value. We need to provide examples of $P(x)$ and $Q(x)$ and show that for our examples chosen, the statements $\forall x\,(\,P(x) \to Q(x)\,)$

and ($\forall x P(x)$) $\to$ ($\forall x Q(x)$) have different truth values. There are many possible examples, and any correct answers are acceptable. However, you cannot choose $\neg P(x)$ to hold for every integer x, and you cannot choose $Q(x)$ to be true for every integer x, since in those cases both statements in the question would be true.

For instance, you can take $P(x)$ to be "x is even" and $Q(x)$ to be "x is odd". Then $\forall x$ ($P(x) \to Q(x)$) is false $P(0)$ is true but $Q(0)$ is false. On the other hand, ($\forall x P(x)$) $\to$ ($\forall x Q(x)$) is true since $\forall x P(x)$ is false as not every integer is even.

Common Mistakes: Some concluded with the correct answer but failed to provide a clear example. Your example should state clearly the properties for $P(x$ and $Q(x)$, and also explain clearly why one statement is true and why the second statement is false. Remember to show that $\forall x(P(x) \to Q(x))$ is false you need to show that $P(a) \to Q(a)$ is false for a specific integer a, such as $a = 0$ as above.

□

QUESTION 2 **(14 marks)**

Determine if each of the following is true or false. Justify your answers.

(a) There are positive integers n and m such that $n^2 = 10 + m^2$.

(b) For every integer n there is some integer k such that for every integer $m > k$, $(n - m)^2 > n^2$.

SOLUTION . (a) This statement is false. We need to prove that for any two positive integers n, m, $n^2 - m^2 \neq 10$. Let n, m be positive integers such that $n^2 - m^2 = (n + m)(n - m) = 10$. Since $n + m > 0$, it follows that $n - m$ must also be positive and that $n + m > n - m$. The positive factors of 10 are $1, 2, 5, 10$ and therefore there are only two possible cases:

Case 1: $n + m = 10$ and $n - m = 1$: Then $(n + m) + (n - m) = 10 + 1$, which means that $2n = 11$, which is not possible as 11 is odd.

Case 2: $n + m = 5$ and $n - m = 2$: Then $(n + m) + (n - m) = 5 + 2$, which means that $2n = 7$, which is not possible as 7 is odd.

Common Mistakes: Many students miss out on cases of n and m. Some students were also confused with the $\forall$ and $\exists$ symbol. Many students also attempted their proofs by stating a statement but did not further elaborate or explain why they think their statement is true.

(b) This statement is true. First let us examine why. Note that the expression $(n-m)^2 > n^2$ is equivalent to $n^2 - 2nm + m^2 > n^2$, which is equivalent to $m^2 > 2nm$. If m is positive then the expression is in fact equivalent to $m > 2n$. Hence, given any integer n, it is now clear that we should pick $k = 2n$. However this alone won't work, since we would like every $m > k$ to be positive (see the reason above). So we should also choose k to be ≥ 0.

Given the heuristics above, let's formally prove that the statement is true. Let n be a given integer. We pick $k = \max\{2n, 0\}$, which is also an integer. Now let $m > k$ be given. Then $m > k \geq 2n$. At the same time, $m > k \geq 0$, which means that if we multiply m to both sides of the inequality $m > 2n$, we get $m^2 > 2nm$. This means that $n^2 - 2nm + m^2 > n^2$ and therefore we get $(n - m)^2 > n^2$.

Common Mistakes: Most students did not understand what the question was asking for and how "k" matters in the question. Many students were also careless in the expansion of $(n - m)^2$. Students also get confused with the greater than or less than inequality sign. Likewise, some students also get confused with the $\forall$ and $\exists$ symbol. The order is: you are first given a value of n, then for that given value of n you must pick some value of k (depends on n), and then next assume that you are given some further $m > k$. This will allow you to relate the value of m to the value of n (indirectly through k) and thus allow you to show the required inequality, which involves n and m but not k. $\square$

QUESTION 3. (10 marks)

Let a, b be positive integers such that every divisor of a is a divisor of b.

(a) Is every multiple of a also a multiple of b?

(b) Is every multiple of b also a multiple of a?

If you answer "yes", prove it. If you answer "no", give a counter example for a, b and for the multiple.

SOLUTION . (a) This is false. Take $a = 2$ and $b = 4$. Then the divisors of a are ± 1 and ± 2, and all of these are divisors of $b = 4$. However, 2 is a multiple of a, but 2 is not a multiple of $b = 4$.

Common Mistakes: The counter example to part (a) should indicate why every divisor of a is a divisor of b. Many students just gave counter example and focussed on explaining why some multiple of a is not a multiple of b without also explaining the divisors.

(b) This is true. Let a, b be positive integers such that every divisor of a is a divisor of b. In particular, since a is a divisor of a, we can conclude from the assumption that a is a divisor of b, i.e. $a \mid b$. Now we want to show that every multiple of b is a multiple of a. Let x be a multiple of b. Then $b \mid x$. Since $a \mid b$, by the transitivity of divisibility (proved in the lecture), we have $a \mid x$. Hence, x is a multiple of a.

Common Mistakes: Some students (correctly) said that a divides b, without explaining why this is true. Some students also confused "multiple of" with "divisor of". A common wrong proof of this part is something like the following: Since every divisor of a is a divisor of b, let $k \mid a$. Then $kj = a$ for some integer j. Since k is a divisor of b as well, let $kl = b$ for some integer l. Now fix a multiple nb of b. We have $nb = n(kl) = \frac{nl}{j}a$ and so nb is a multiple of a. This is incorrect since $\frac{nl}{j}$ need not be an integer. Some students attempted this wrong proof with slightly more guile by taking $l = j$, which allowed them to show that $nb = na$ or something similar, but is equally fallacious and cannot fool the marker with three cups of coffee drank. $\square$

QUESTION 4. **(8 marks)**

Show that the following argument is valid. If you've used any rules of inference, state them.

$$\begin{array}{ll} & (\neg p \wedge q) \to r \\ & \neg p \\ & \neg(p \vee r) \\ \therefore & \neg q \end{array}$$

SOLUTION . To show that the argument is valid, you can use truth tables. In this solution we present using rules of inference.

$\neg(p \vee r)$	[Premise #3]
$\neg p \wedge \neg r$	[De Morgan's Law]
$\neg r$	[Specialisation]
$(\neg p \wedge q) \to r$	[Premise # 1]
$\neg(\neg p \wedge q)$	[Modus Tollens]
$\neg\neg p \vee \neg q$	[De Morgan's Law]
$p \vee \neg q$	[Double Negation Law]
$\neg p$	[Premise # 2]
$\neg q$	[Elimination]

Common Mistakes: Similar to Q1, many students did not cite the use of double negation and the associative laws, and some dropped the brackets around three terms. Many *students also did not show the correct steps in showing that an argument is valid, and said something like:* $\neg(p \vee r) \equiv T$ *and so* $p \vee r \equiv F$*. First of all, in the steps you cannot assign truth values to statement; the steps should contain statements themselves that follow from previous ones, and not truth values. Secondly, saying something like* $\neg(p \vee r) \equiv T$ *usually means that* $\neg(p \vee r)$ *is a tautology, which is not true.* $\square$