NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024-2025

MH1200 – LINEAR ALGEBRA I

November 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the start of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. **(20 marks)**

Consider a 4×4 matrix $\mathbf{A} = [a_{ij}]_{4\times4}$ determined by the formula

$$a_{ij} = \max(6 - i - j, 0),$$

where the notation $\max(a, b)$ denotes the function which returns the larger of the two numbers a and b.

(a) Write down the matrix $\mathbf{A}$.

(b) What is the reduced row echelon form of $\mathbf{A}$? Give working or justification to obtain this matrix without using a calculator, although you may use a calculator to check your answer if you wish.

(c) How can you determine the determinant of $\mathbf{A}$ from your answer to Part (b)?

QUESTION 2. **(10 marks)**

Consider a homogeneous system of linear equations which has more variables than equations. What are the possibilities for the number of solutions? Briefly justify.

QUESTION 3. **(15 marks)**

Let m and n be positive integers. Let $\mathbf{A}$ and $\mathbf{X}$ be matrices $\mathbf{A}, \mathbf{X} \in \mathbb{M}_{m\times n}(\mathbb{R})$.

(a) Write down a formula for $\mathrm{tr}(\mathbf{A}\mathbf{X}^T)$ in terms of the entries of $\mathbf{A}$ and $\mathbf{X}$.

(b) Now assume that $\mathbf{X} \in \mathbb{M}_{\mathbf{m}\times\mathbf{n}}(\mathbb{R})$ is a matrix with the property that

$$\mathrm{tr}(\mathbf{A}\mathbf{X}^T) = 0$$

for every possible matrix $\mathbf{A} \in \mathbb{M}_{\mathbf{m}\times\mathbf{n}}(\mathbb{R})$. Show that $\mathbf{X}$ equals the zero matrix.

QUESTION 4. **(20 marks)**

Consider the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 9 & 1 \\ 0 & 2 & 10 & 2 \\ 5 & 1 & 1 & 1 \end{bmatrix}$$

(a) Calculate the rank of $\mathbf{A}$. Show working to obtain this answer without using a calculator, although you may check your answer with a calculator if you wish.

(b) Determine a subset of rows of $\mathbf{A}$ which forms a basis for the row space of $\mathbf{A}$. Justify your answer.

QUESTION 5. **(20 marks)**

Consider some integer $n > 0$ and n vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^n$, where the i-th vector is given by constants $a_{i,j} \in \mathbb{R}$ as follows:

$$\vec{v}_i = (a_{i,1}, \ldots, a_{i,n}).$$

Assume that this list of vectors $\vec{v}_1, \ldots, \vec{v}_n$ forms a basis of $\mathbb{R}^n$.

(a) Is it true or false that for every vector $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ there exist unique real numbers $c_1, \ldots, c_n$ such that

$$\vec{x} = \sum_{i=1}^{n} c_i \vec{v}_i \quad ?$$

Justify starting from the definition of the term basis discussed in the course.

(b) Apply Cramer's rule to write down a formula expressing these constants c_i as a function of the numbers $\{a_{i,j}\}$ and $\{x_i\}$.

QUESTION 6. **(15 marks)**

Consider an $m \times n$ matrix $\mathbf{X}$ and let k be an integer such that $k \leq \min(m, n)$. There is a theorem that the rank of $\mathbf{X}$ is less than k if and only if all the k-minors of $\mathbf{X}$ are zero.

(a) Illustrate this theorem using an example of a 3×3 matrix of rank 1.

(b) Prove this theorem. You may use any theorems stated in the lectures in your proof.

Recall that as defined in lectures, a k-minor of $\mathbf{X}$ is the determinant of a $k \times k$ matrix obtained from $\mathbf{X}$ by possibly deleting some rows and columns.

END OF PAPER

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Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.