NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2016-2017

MH1300– Foundations of Mathematics

November 2016 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. (15 marks)

Prove or disprove each of the following.

(a) $\forall y \in \mathbb{R},\ \exists x \in \mathbb{R},\ \forall z \in \mathbb{R},\ \ xy \leq z^2.$

(b) Let x be an integer. If x^2 is not divisible by 4, then x is odd.

(c) Let x and y be integers. If $x + y$ is even and y is odd, then x is odd.

(d) There exists an odd integer M such that for all real numbers $r > M$, we have $\dfrac{1}{2r} < 0.01$.

QUESTION 2. (15 marks)

(a) Let n be a positive integer. Prove that $n(n^4 - 1)$ is divisible by 5.

(b) Let x and y be any real numbers such that $x + y = n$ where n is an integer. Prove that

$$\lceil x \rceil + \lfloor y \rfloor = n.$$

Here, $\lceil x \rceil$ is the ceiling function and $\lfloor y \rfloor$ is the floor function.

(c) Prove the following or give a counter-example:

$$\forall x, y \in \mathbb{R},\ \ \lceil xy \rceil = \lceil x \rceil \cdot \lceil y \rceil.$$

QUESTION 3. (10 marks)

Prove by mathematical induction that for every integer $n \geq 2$,

$$3^n > n^2.$$

QUESTION 4. (15 marks)

(a) Prove that
$$\{4n \mid n \in \mathbb{Z}\} \subsetneq \{2n \mid n \in \mathbb{Z}\}.$$

(b) Let n be a positive integer. Prove that n is even if and only if $7n + 4$ is even.

(c) Recall that a number n is a perfect square if there is an integer k such that $k^2 = n$. Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

QUESTION 5. (15 marks)

(a) Let $f : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$ be a function defined by $f(m, n) = m - n$.

  (i) Is f one-one? Justify your answer.

  (ii) Is f onto? Justify your answer.

(b) Let f be a function from A to B and let $D \subseteq A$ and $E \subseteq B$. Prove each of the following.

  (i) $f^{-1}(B - E) \subseteq A - f^{-1}(E)$.

  (ii) If f is a bijection and $f(D) = E$, prove that $f^{-1}(E) = D$.

QUESTION 6. (12 marks)

(a) Let R be a relation on a set A, and define $R^{-1} = \{(a, b) \mid (b, a) \in R\}$. Prove that if R is transitive, then R^{-1} is transitive.

(b) Let S be a relation on the set of integers larger than 1 defined by: $n \; S \; m$ if and only if the smallest prime number dividing n equals the smallest prime number dividing m. Prove that S is an equivalence relation, and describe the distinct equivalence classes of S.

QUESTION 7. (18 marks)

(a) Prove that for any sets A and B, $A = B$ if and only if $A - B = B - A$.

(b) Use the Euclidean algorithm to find the greatest common divisor of the pair

$$1529 \qquad \text{and} \qquad 14038.$$

(c) Prove that for any non-empty sets A and B, and any sets C and D,

$$A \times B \subseteq C \times D \quad \text{if and only if} \quad A \subseteq C \text{ and } B \subseteq D.$$

END OF PAPER