NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2018-2019

Suggested Solutions (Chunfei, Linus)

MH1100 – Calculus I

December 2018 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises **THREE (3)** printed pages.
2. Answer all questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT an OPEN BOOK** exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.
6. All questions are the property of Nanyang Technological University.

QUESTION 1.

Evaluate the limits

(a)

$$\lim_{x\to\infty} \frac{1}{x} \sin\left(\frac{1}{x}\right).$$

(b)

$$\lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)}\right).$$

(Hint: $\lim\limits_{n\to\infty} \frac{1}{n} = 0$)

(13 marks)

Solution:

(a) For this limit we can let $u = \frac{1}{x}$, so as $x \to \infty$, $\frac{1}{x} \to 0^+$. Therefore,

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x} \sin\left(\frac{1}{x}\right) &= \lim_{u\to 0^+} u \sin u \\ &= \lim_{u\to 0^+} u \cdot \lim_{u\to 0^+} \sin u \\ &= 0\end{aligned}$$

(b) We first think of a partial fraction decomposition for $\frac{1}{n\cdot(n+1)}$. We can obtain the following:

$$\frac{1}{n\cdot(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Therefore we can say that the limit

$$\lim_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)}\right).$$

is now equivalent to

$$
\begin{aligned}
&\lim_{n\to\infty}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n=1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]\\
&=\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]\\
&=1-\lim_{n\to\infty}\frac{1}{n+1}\\
&=1-0\\
&=1
\end{aligned}
$$

$\square$

QUESTION 2.

Find the derivatives of the following functions. **You do not need to simplify.**

(a)

$$h(t) = \sin(t^2 + \cos(t));$$

(b)

$$g(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}.$$

(13 marks)

Solution:

(a)

$$\begin{aligned} h(t) &= \sin(t^2 + \cos t) \\ h'(t) &= [\cos(t^2 + \cos t)] \cdot (t^2 + \cos t)' \\ &= [\cos(t^2 + \cos t)] \cdot (2t - \sin t) \end{aligned}$$

(b)

$$\begin{aligned} g(x) &= \sqrt{x + \sqrt{x + \sqrt{x}}} \\ (g(x))^2 &= x + \sqrt{x + \sqrt{x}} \end{aligned}$$

Differentiating w.r.t x implicitly on both sides, we have

$$\begin{aligned} 2(g(x)) \cdot g'(x) &= 1 + \frac{1}{2}(x + \sqrt{x})^{-\frac{1}{2}} \cdot (x + \sqrt{x})' \\ 2(g(x)) \cdot g'(x) &= 1 + \frac{1}{2\sqrt{x + \sqrt{x}}} \left(1 + \cdot \frac{1}{2\sqrt{x}}\right) \\ g'(x) &= \frac{1 + \frac{1}{2\sqrt{x+\sqrt{x}}} \left(1 + \cdot \frac{1}{2\sqrt{x}}\right)}{2g(x)} \\ g'(x) &= \frac{1 + \frac{1}{2\sqrt{x+\sqrt{x}}} \left(1 + \cdot \frac{1}{2\sqrt{x}}\right)}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \end{aligned}$$

Comment: This would be sufficient as your final answer. If you would want to simplify further,

$$g'(x) = \frac{4\sqrt{x+\sqrt{x}}\sqrt{x} + 2\sqrt{x} + 1}{8\sqrt{x+\sqrt{x+\sqrt{x}}}\sqrt{x+\sqrt{x}}\sqrt{x}}$$

Alternatively, you may use a direct chain rule to solve this question.

$\square$

QUESTION 3.

Prove that the equation

$$\sqrt{x+5} = \frac{1}{x+3}$$

has at least one root. **(13 marks)**

Solution: Define $f(x) = \sqrt{x+5} - \frac{1}{x+3}$. The domain of $f(x)$ is $[-5, \infty)\backslash\{-3\}$. We proceed by applying IVT on the closed interval $[-2.5, -1]$. We first show that f is continuous on $[-2.5, -1]$. Take $a \in (-2.5, -1)$, then

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(\sqrt{x+5} - \frac{1}{x+3}\right)\\ &= \sqrt{a+5} - \frac{1}{a+3}\\ &= f(a)\end{aligned}$$

Therefore, by the definition of continuity, $f(x)$ is continuous for the open interval $(-2.5, -1)$. Also,

$$\lim_{x\to -2.5^+} f(x) = \lim_{x\to -2.5^+}\left(\sqrt{x+5} - \frac{1}{x+3}\right) = f(-2.5)$$

and

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}\left(\sqrt{x+5} - \frac{1}{x+3}\right) = f(-1).$$

Thus, f is also continuous from the right of -2.5 and continuous from the left of -1. This shows that f is continuous on $[-2.5, -1]$. Finally, by trial and error, we note that, $f(-2.5) = \sqrt{-2.5+5} - \frac{1}{-2.5+3} = -0.419 < 0$ and $f(-1) = \sqrt{-1+5} - \frac{1}{-1+3} = \frac{3}{2} > 0$. By the Intermediate Value Theorem, there exist a $c \in (-2.5, -1)$ such that $f(c) = 0$. Therefore, we can conclude that $f(x) = 0$ has at least one root. $\square$

QUESTION 4.

Show that the tangent line to the ellipse

$$\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{c^2} + \frac{y_0 y}{d^2} = 1$$

(Note that number c and d are constant value.)

(13 marks)

Solution: Differentiating implicitly yields:

$$\frac{2x}{c^2} + \frac{2y}{d^2} \cdot \frac{dy}{dx} = 0$$

which simplifies to

$$\frac{dy}{dx} = -\frac{xd^2}{yc^2}$$

At the point (x_0, y_0), $\frac{dy}{dx} = -\frac{x_0 d^2}{y_0 c^2}$. Therefore the equation of the tangent line at (x_0, y_0) is:

$$\begin{aligned}
y - y_0 &= -\frac{x_0 d^2}{y_0 c^2}(x - x_0) \\
y &= y_0 - \frac{x_0 x d^2}{y_0 c^2} + \frac{{x_0}^2 d^2}{y_0 c^2} \\
\frac{y}{d^2} &= \frac{y_0}{d^2} - \frac{x_0 x}{y_0 c^2} + \frac{{x_0}^2}{y_0 c^2} \\
\frac{y_0 y}{d^2} &= \frac{{y_0}^2}{d^2} - \frac{x_0 x}{c^2} + \frac{{x_0}^2}{c^2} \\
\frac{y_0 y}{d^2} + \frac{x_0 x}{c^2} &= \frac{{y_0}^2}{d^2} + \frac{{x_0}^2}{c^2}
\end{aligned}$$

Since (x_0, y_0) is a point that lie on the ellipse,

$$\frac{{x_0}^2}{c^2} + \frac{{y_0}^2}{d^2} = 1$$

Therefore, the tangent line at the point (x_0, y_0) on the ellipse is

$$\frac{x_0 x}{c^2} + \frac{y_0 y}{d^2} = 1$$

$\square$

QUESTION 5.

Hot air is leaking out from a spherical balloon. The volume of the spherical balloon decreases at a rate of $10m^3/s$. How fast is the radius of the balloon decreasing when the diameter is $5m$?

(13 marks)

Solution: Let V be the volume of the spherical volume with radius r metres. Recall that the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3.$$

Differentiating both sides with respect to t, we have

$$\begin{aligned} \frac{dV}{dt} &= \frac{4}{3}(3)(\pi)r^2 \cdot \frac{dr}{dt} \\ \frac{dV}{dt} &= 4\pi r^2 \cdot \frac{dr}{dt} \end{aligned}$$

Given that $\frac{dV}{dt} = -10m^3/s$ and $r = \frac{5m}{2} = 2.5m$, we have

$$\begin{aligned} -10 &= 4\pi(2.5)^2 \cdot \frac{dr}{dt} \\ \frac{dr}{dt} &= -\frac{2}{5\pi}m^3/s \end{aligned}$$

Therefore, the radius of the balloon is decreasing at a rate of $\frac{2}{5\pi}m^3/s$ when the diameter is $5m$. $\square$

QUESTION 6.

For a curve defined by $y = x^2 + x + 1$, we find three points with x coordinates $x_1 = 0, x_2 = -1$ and $x_3 = -\frac{1}{2}$ respectively. prove that the normal lines at these three points intersect with each other at one point.

(13 marks)

Solution: We first find the derivative of the function, $y = x^2 + x + 1$,

$$\frac{dy}{dx} = (x^2 + x + 1)' = 2x + 1$$

When $x_1 = 0$, $y = 1$.
When $x_2 = -1$, $y = 1$.
When $x_3 = -\frac{1}{2}$, $y = \frac{3}{4}$.
Therefore, the equation of normal line at $x_1 = 0$ is

$$\begin{aligned} y - 1 &= -\frac{1}{2(0) + 1}(x - 0) \\ y &= -x + 1. \end{aligned} \tag{1}$$

The equation of normal line at $x_2 = -1$ is

$$\begin{aligned} y - 1 &= -\frac{1}{2(-1) + 1}(x - (-1)) \\ y &= x + 2. \end{aligned} \tag{2}$$

The equation of normal line at $x_3 = -\frac{1}{2}$ is

$$x = -\frac{1}{2}$$

as the gradient of normal line at the point $x_3 = -\frac{1}{2}$ is undefined, which means the normal line is parallel to the $y-$axis.

For the normal lines to intersect with each other at one point, the normal lines must pass through the point $\left(-\frac{1}{2}, \frac{3}{4}\right)$. Solving the system of linear equations defined in (1) and (2), there is a unique solution and it is indeed $\left(-\frac{1}{2}, \frac{3}{4}\right)$.

$\square$

QUESTION 7.

Use the $\epsilon - \delta$ definition to prove if we have $\lim_{x\to a} f(x) = f(a) \neq 0$, then we have $\lim_{x\to a} \dfrac{1}{f(x)} = \dfrac{1}{f(a)}$.

(11 marks)

Solution: Since $\lim_{x\to a} f(x) = f(a) \neq 0$, for any $\epsilon_1 > 0$, there exist a $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then $|f(x) - f(a)| < \epsilon_1$

Given $\epsilon > 0$, we want to find $\delta > 0$ such that if $0 < |x-a| < \delta$, then $\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| < \epsilon$.
We have,

$$\begin{aligned}
\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| &= \left|\frac{1}{f(x)f(a)}\,(f(a) - f(x))\right| \\
&= \left|\frac{1}{f(x)f(a)}\right| |f(a) - f(x)| \\
&= \frac{1}{|f(a)|\,|f(x)|}\,|f(x) - f(a)|
\end{aligned}$$

We find a lower bound for $|f(x)|$ since we want to estimate the term $\frac{1}{|f(x)|}$.
By the Triangle Inequality,

$$|f(a)| = |f(a) - f(x) + f(x)|$$

$$\leq |f(a) - f(x)| + |f(x)|$$

$$\Rightarrow |f(a)| - |f(x) - f(a)| \leq |f(x)|$$

$$\Rightarrow |f(a)| - \epsilon_1 \leq |f(x)|$$

Since ϵ_1 can be any positive number, we find a $\delta_1 > 0$ such that
if $0 < |x-a| < \delta_1$ then $|f(x) - f(a)| < \epsilon_1 = \frac{1}{2}|f(a)|$.

Therefore,

$$\begin{aligned}
|f(x)| &\geq |f(a)| - \epsilon_1 \\
&= |f(a)| - \frac{|f(a)|}{2} \\
&= \frac{|f(a)|}{2} \\
\frac{1}{|f(x)|} &\leq \frac{2}{|f(a)|}
\end{aligned}$$

Now for any $\epsilon > 0$, we let $\epsilon_2 = \frac{|f(a)|^2\epsilon}{2}$. We find a $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then $|f(x) - f(a)| \leq \frac{|f(a)|^2\epsilon}{2}$. Therefore, we have

$$\left|\frac{1}{f(x)} - \frac{1}{f(a)}\right| < \frac{\frac{|f(a)|^2\epsilon}{2}}{|f(a)|} \cdot \frac{2}{|f(a)|} = \epsilon$$

We choose $\delta = \min\{\delta_1, \delta_2\}$ to meet the restrictions of the inequalities. $\square$

QUESTION 8.

Prove that if $f(x)$ and $g(x)$ are continuous on $[a, b]$, and differentiable on (a, b) then there exists a $\xi \in (a, b)$, that satisfies

$$\begin{vmatrix} f(b) - f(a) & g(b) - g(a) \\ f'(\xi) & g'(\xi) \end{vmatrix} = 0$$

(11 marks)

Solution: Since $f(x)$ and $g(x)$ are continuous on $[a, b]$, and differentiable on (a, b), by MVT, there exists a $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi), \frac{g(b) - g(a)}{b - a} = g'(\xi)$$

Case 1: Suppose there exists a $\xi \in (a, b)$ such that both $f'(\xi)$ and $g'(\xi)$ are non-zero. Then,

$$\Rightarrow \frac{b - a}{f(b) - f(a)} = \frac{1}{f'(\xi)}, \frac{b - a}{g(b) - g(a)} = \frac{1}{g'(\xi)}$$

$$\Rightarrow \frac{f(b) - f(a)}{f'(\xi)} = \frac{g(b) - g(a)}{g'(\xi)}$$

$$\Rightarrow [f(b) - f(a)] \cdot g'(\xi) = [g(b) - g(a)] \cdot f'(\xi)$$

$$\Rightarrow [f(b) - f(a)] \cdot g'(\xi) - [g(b) - g(a)] \cdot f'(\xi) = 0$$

$$\Rightarrow \begin{vmatrix} f(b) - f(a) & g(b) - g(a) \\ f'(\xi) & g'(\xi) \end{vmatrix} = 0.$$

Case 2: Suppose for all $\xi \in (a, b)$, $f'(\xi) = 0$. From lecture, we note that f must be a constant function on [a,b]. Therefore, we have $f(a) = f(b)$ which implies $f(b) - f(a) = 0$. The determinant expression is now reduced to

$$\begin{vmatrix} 0 & g(b) - g(a) \\ 0 & g'(\xi) \end{vmatrix}$$

which is 0.

The case where for all $\xi \in (a,b)$, $g'(\xi) = 0$ follows a similar argument.

Therefore we conclude that there exist a $\xi \in (a,b)$, that satisfies

$$\begin{vmatrix} f(b) - f(a) & g(b) - g(a) \\ f'(\xi) & g'(\xi) \end{vmatrix} = 0$$

$\square$

END OF PAPER