NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2019-2020

MH1100 – Calculus I

December 2019 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED** book exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1 **(13 marks)**

Evaluate the limits (You can use l'Hospital's Rule.)

(a)

$$\lim_{x\to 0}\frac{|x|}{x^3};$$

(b)

$$\lim_{x\to \pi}\frac{e^{\sin x}-1}{x-\pi}.$$

[Solution:]

(a)

$$\lim_{x\to 0^+}\frac{|x|}{x^3}=\lim_{x\to 0^+}\frac{1}{x^2}=\infty$$

$$\lim_{x\to 0^-}\frac{|x|}{x^3}=\lim_{x\to 0^-}\frac{-1}{x^2}=-\infty.$$

thus

$$\lim_{x\to 0^+}\frac{|x|}{x^3}\ does\ not\ exist.$$

(b) Use L'Hospital's Rule

$$\lim_{x\to \pi}\frac{e^{\sin x}-1}{x-\pi}=\lim_{x\to \pi}\frac{\cos x e^{\sin x}}{1}=-1.$$

QUESTION 2 **(13 marks)**

Find the derivatives of the following functions. **You do not need to simplify.**

(a)

$$h(t)=\frac{1}{1+\frac{1}{1+\frac{1}{t}}};$$

(b)

$$g(x)=\sin(x^2+\sqrt{\tan x}).$$

[Solution:]

(a)

$$h(t) = h(t) = \frac{1}{1 + \frac{1}{1+\frac{1}{t}}} = \frac{1}{1 + \frac{t}{1+t}} = \frac{t+1}{2t+1}$$

$$h'(t) = \frac{(2t+1) - (t+1)2}{(2t+1)^2} = \frac{-1}{(2t+1)^2}$$

(b)

$$g'(x) = \cos(x^2 + \sqrt{\tan x})(2x + \frac{\sec^2 x}{2\sqrt{\tan x}})$$

QUESTION 3 **(13 marks)**

Suppose $f(x)$ and $g(x)$ are differentiable functions such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that

$$g'(x) = \frac{1}{1+x^2}.$$

[Solution:] From $f(g(x)) = x$, we have $f'(g(x))g'(x) = 1$, thus

$$g'(x) = \frac{1}{f'(g(x))}$$

From $f'(x) = 1 + [f(x)]^2$, we replace x by function $g(x)$, and have $f'(g(x)) = 1 + [f(g(x))]^2$. Since $f(g(x)) = x$, we have $f'(g(x)) = 1 + x^2$. In this way, we have

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1+x^2}.$$

QUESTION 4 **(13 marks)**

Suppose that $f(x)$ is differentiable in the whole domain $\mathbf{R}$ and have N distinct roots (Note that $N > 1$ and N is an integer). Show that $f'(x)$ has at least $N-1$ distinct roots.

[Solution:] If we denote the N roots of $f(x)$ as x_1, x_2,..., x_N. We can take any two adjacent roots x_i and x_{i+1} with $i = 1, 2, ..., N-1$. We have $f(x_i) = 0$ and $f(x_{i+1}) = 0$. Since $f(x)$ is differential in the whole domain R, we have
(1) $f(x)$ is continuous in the region $[x_i, x_{i+1}]$;
(2) $f(x)$ is differential in the region (x_i, x_{i+1}).
From the Mean Value Theorem, there exists a number c in (x_i, x_{i+1}) such that

$$f'(c) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = 0$$

Therefore $f'(x)$ has a root in (x_i, x_{i+1}) for $i = 1, 2, ..., N-1$. Totally, there are $N-1$ different roots for $f'(x)$. In this way, $f'(x)$ has at least $N-1$ different roots.

QUESTION 5 **(13 marks)**

Find the equations of both tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point (12, 3).

[Solution:] We can calculate the slope of the tangent line as follows,

$$2x + 8yy' = 0$$

$$y' = -\frac{x}{4y}$$

If we assume (x_0, y_0) is the point on the ellipse that its tangent lines pass through the point (12, 3), we have

$$y'|_{(x_0,y_0)} = -\frac{x_0}{4y_0} = \frac{y_0 - 3}{x_0 - 12}$$

$$12x_0 - x_0^2 = 4y_0^2 - 12y_0$$

$$12x_0 + 12y_0 = 4y_0^2 + x_0^2$$

Since (x_0, y_0) is on the ellipse $x^2 + 4y^2 = 36$, thus $x_0^2 + 4y_0^2 = 36$. So we have

$$12x_0 + 12y_0 = 4y_0^2 + x_0^2 = 36$$

$$x_0 = 3 - y_0$$

We can take it to the equation $x_0^2 + 4y_0^2 = 36$ and have

$$(3 - y_0)^2 + 4y_0^2 = 36$$

$$(y_0 - 3)(5y_0 + 9) = 0$$

When $y_0 = 3$, we have $x_0 = 0$; when $y_0 = -9/5$, $x_0 = 24/5$.
For point $(0, 3)$, the tangent line is $y = 3$;
For point $(24/5, -9/5)$, the tangent line is

$$y - 3 = -\frac{24/5}{4 * (-9/5)}(x - 12)$$

$$y = 2/3x - 5.$$

QUESTION 6 **(13 marks)**

A stone was dropped off a cliff and hit the ground with a speed of 98 m/s (meter/second). What is the height of the cliff? How long does it take for the stone to hit the ground? (Hint: the acceleration is -9.8 m/s^2)

[Solution:] As the stone was dropped off the cliff, the initial velocity is $v(t = 0) = 0$ m/s. The velocity can be expressed as

$$v(t) = -9.8t$$

We assume the height of cliff as h_0, the height function can be expressed as

$$h(t) = h_0 - 4.9t^2$$

The time that the stone touches the ground is

$$-9.8t = -98$$

we have $t = 10$ second. From the height function we have

$$0 = h_0 - 4.9 * 10^2$$

Thus $h_0 = 490$ meter.

QUESTION 7 **(11 marks)**

Use the ϵ-δ definition to prove if we have $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$, then we have $\lim_{x\to a}(f(x) + g(x)) = f(a) + g(a)$.

[Solution:] For $\forall\varepsilon > 0$, since we have $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$, there exists δ_1, such that when $|x - a| < \delta_1$, we have

$$|f(x) - f(a)| \le \varepsilon/2,$$

there exists δ_2, such that when $|x - a| < \delta_2$, we have

$$|g(x) - g(a)| \le \varepsilon/2.$$

We can let $\delta = \min\{\delta_1, \delta_2\}$, when $|x - a| < \delta$, we have

$$|(f(x)+g(x))-(f(a)+g(a))| = |(f(x)-f(a))+(g(x)-g(a))| \le |(f(x)-f(a))|+|(g(x)-g(a))| \le \varepsilon$$

More specifically, that is to say, $\forall\varepsilon > 0$, there exists δ. When $|x - a| < \delta$, we have

$$|(f(x) + g(x)) - (f(a) + g(a))| \le \varepsilon$$

Thus $\lim_{x\to a}(f(x) + g(x)) = f(a) + g(a)$.

QUESTION 8 **(11 marks)**

We have an infinite sequence 1, $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$, $\sqrt[5]{5}$, ..., $\sqrt[N]{N}$,.... Find out and prove which term has the largest value in this sequence.

[Solution:] Consider the function $f(x) = x^{\frac{1}{x}}$ and $x > 0$. We have $\ln f(x) = \frac{1}{x}\ln x$

$$\frac{f'(x)}{f(x)} = -\frac{1}{x^2}\ln x + \frac{1}{x^2} = \frac{1 - \ln x}{x^2}$$

thus we have

$$f'(x) = x^{1/x}\frac{1 - \ln x}{x^2}$$

Further, we have that
(1) At $(0, e)$, $f'(x) > 0$, thus $f(x)$ systematically increases;
(2) At (e, ∞), $f'(x) < 0$, thus $f(x)$ systematically decreases;
(3) At $x = e$, $f'(x) = 0$, thus $(x, f(x))$ is a critical point.
In this way, $f(x = e)$ is the maximal value for $f(x)$. Further, since $\sqrt{2} < \sqrt[3]{3}$, therefore $\sqrt[3]{3}$ is the largest value.

END OF PAPER