Name: _____   Tutorial group: ____

Matriculation number: [ ][ ][ ][ ][ ][ ][ ][ ][ ]

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I 2022/23

MH1100 – Calculus I

23 September 2022   Midterm Test   90 minutes

<u>INSTRUCTIONS</u>

1. Do not turn over the pages until you are told to do so.

2. Write down your name, tutorial group, and matriculation number.

3. This test paper contains **SIX (6)** questions and comprises **SEVEN (7)** printed pages. Question 6 is optional.

4. The marks for each question are indicated at the beginning of each question.

QUESTION 1. (4 marks)

Use the ϵ, δ definition of a limit to prove that $\lim\limits_{x \to 2} f(x) = 5$ if

$$f(x) = \begin{cases} 2x + 1, & x \neq 2, \\ 0, & x = 2. \end{cases}$$

[Answer:] Let ϵ be a given positive number. To prove the limit, we only need to find a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - 2| < \delta \quad \text{then} \quad |(2x + 1) - 5| < \epsilon.$$

But

$$|(2x + 1) - 5| = |2x - 4| = 2|x - 2|.$$

If $|x - 2|$ is less than $\frac{1}{2}\epsilon$, then

$$|(2x + 1) - 5| = 2|x - 2| < 2 \times \frac{1}{2}\epsilon = \epsilon.$$

This suggests that we could choose $\delta = \frac{1}{2}\epsilon$.

QUESTION 2. (4 marks)

Find the limits if exist.

(a) $\lim\limits_{x\to 2}\dfrac{x^2-4x+4}{x+\frac{1}{x}+\sin(\sqrt{x})}$,

(b) $\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{x-1}$,

(c) $\lim\limits_{x\to 2}\dfrac{\sin(x-2)}{x^2-4}$,

(d) $\lim\limits_{x\to -2^+}\left(\dfrac{x}{1+x}\right)\left(\dfrac{2x+5}{x^2+x}\right)$.

[Answer:]

(a) The denominator $f(x)=x+\frac{1}{x}+\sin(\sqrt{x})$ is a function constructed by composition and algebraic operations of continous function. Thus, it is continuous on its domain. We know $x=2$ is in the domain of $f(x)$. Meanwhile, the numerator is continuous at every real number. Thus, the limit of the quotient can be evaluated by directly substituting $x=2$ in the function. The numerator is 0 when $x=2$. The denominator is non-zero. So the limit is 0.

(b) We rationalize the numerator.

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{x-1}=\lim_{x\to 1}\frac{1-\sqrt{x}}{x-1}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}}=\lim_{x\to 1}\frac{1-x}{x-1}\cdot\frac{1}{1+\sqrt{x}}$$
$$=\lim_{x\to 1}(-1)\cdot\frac{1}{1+\sqrt{x}}=(-1)\cdot\lim_{x\to 1}\frac{1}{1+\sqrt{x}}=-\frac{1}{2}.$$

(c)

$$\lim_{x\to 2}\frac{\sin(x-2)}{x^2-4}=\lim_{x\to 2}\frac{\sin(x-2)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{\sin(x-2)}{x-2}\cdot\frac{1}{x+2}$$
$$=\lim_{x\to 2}\frac{\sin(x-2)}{x-2}\cdot\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}.$$

(d) The function $\left(\frac{x}{1+x}\right)\left(\frac{2x+5}{x^2+x}\right)$ is continuous at $x=-2$. Using the direct substition property, we have

$$\lim_{x\to -2^+}\left(\frac{x}{1+x}\right)\left(\frac{2x+5}{x^2+x}\right)=\left(\frac{-2}{1+(-2)}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right)=1.$$

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QUESTION 3.   (4 marks)

Find contants a and b so that the following limit is true.

$$\lim_{x \to 0} \frac{\sqrt{ax+b} - 1}{x} = 1.$$

[Answer:] From

$$\lim_{x \to 0} \frac{\sqrt{ax+b} - 1}{x} = 1$$

and

$$\lim_{x \to 0} x = 0,$$

we have

$$\lim_{x \to 0} \left(\sqrt{ax+b} - 1 \right) = \lim_{x \to 0} \left(\frac{\sqrt{ax+b} - 1}{x} \cdot x \right) = \lim_{x \to 0} \frac{\sqrt{ax+b} - 1}{x} \cdot \lim_{x \to 0} x = 0.$$

Then, by the sum law and power law we have

$$\lim_{x \to 0} (ax+b) = 1$$

or

$$a \lim_{x \to 0} (x) = 1 - b.$$

Thus, $b = 1$.

We continue to evaluate the limit

$$\lim_{x \to 0} \frac{\sqrt{ax+b} - 1}{x} = \lim_{x \to 0} \frac{\sqrt{ax+1} - 1}{x} \cdot \frac{\sqrt{ax+1} + 1}{\sqrt{ax+1} + 1} = \lim_{x \to 0} \frac{ax}{x} \cdot \frac{1}{\sqrt{ax+1} + 1}$$

$$= a \lim_{x \to 0} \frac{1}{\sqrt{ax+1} + 1} = \frac{a}{2}.$$

We further obtain $a = 2$.

QUESTION 4. (4 marks)

Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. What, if anything, does this tell you about

$$\lim_{x \to 0} \frac{1 - \cos x}{x \sin x}.$$

[Answer:] It is easy to verify that

$$\lim_{x \to 0} \left(\frac{1}{2} - \frac{x^2}{24} \right) = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2}.$$

By the squeeze theorem, we have

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

Because

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

we obtain

$$\lim_{x \to 0} \frac{1 - \cos x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{x^2} \frac{x}{\sin x} = \lim_{x \to 0} \frac{1 - \cos x}{x^2} \lim_{x \to 0} \frac{x}{\sin x} = \frac{1}{2}.$$

QUESTION 5. (4 marks)

Show that the function $f(x) = x|x|$ is differentiable in its domain.

[Answer:] The domain of $f(x)$ is R. We consider 3 cases.

- Case 1: $x > 0$. The derivative of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)|x+h| - x|x|}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{(2x+h)h}{h} = 2x.$$

- Case 2: $x < 0$. The derivative of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)|x+h| - x|x|}{h}$$
$$= \lim_{h \to 0} \frac{-(x+h)^2 + x^2}{h} = \lim_{h \to 0} \frac{-(2x+h)h}{h} = -2x.$$

- Case 3: $x = 0$. The derivative of $f(x)$ is

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h|h| - 0}{h}$$
$$= \lim_{h \to 0} |h| = 0.$$

Thus, we obtain

$$f'(x) = \begin{cases} -2x, & x < 0, \\ 0, & x = 0, \\ 2x, & x > 0. \end{cases}$$

This proves that $f(x) = x|x|$ is differentiable in its domain.

QUESTION 6 (Optional). **(1 bonus mark)**

The function $f(x)$ is a continuous functions on the interval $[a,\ b]$. n is a positive integer. $x_1,\ \cdots,\ x_n \in [a,\ b]$. Prove that there exists a $\xi \in [a,\ b]$ such that

$$f(\xi) = \frac{1}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right].$$

[Answer:] It is obvious for $n = 1$. For $n > 1$, we define a new continuous function

$$F(x) = f(x) - \frac{1}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right].$$

Without loss of generality, we assume

$$f(x_I) = \min\left\{f(x_1),\ f(x_2),\ \cdots,\ f(x_n)\right\}$$

and

$$f(x_J) = \max\left\{f(x_1),\ f(x_2),\ \cdots,\ f(x_n)\right\}$$

with $x_I,\ x_J \in \{x_1,\ x_2,\ \cdots,\ x_n\}$. Obviously, we have

$$F(x_I) \leq 0 \qquad \text{and} \qquad F(x_J) \geq 0.$$

When $F(x_I) = 0$, we find $\xi = x_I$; when $F(x_J) = 0$, we find $\xi = x_J$. When $F(x_I) < 0$ and $F(x_J) > 0$, by the Intermediate Value Theorem, there exists a ξ in between x_I and x_J such that $F(\xi) = 0$. $F(\xi) = 0$ is equivalent to

$$f(\xi) = \frac{1}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right].$$

This completes the proof.