SPMS / Division of Mathematical Sciences

MH1300 Foundations of Mathematics
2018/2019 Semester 1

MID-TERM EXAM SOLUTIONS

QUESTION 1. **(15 marks)**

Solve the following **without** using truth tables. You should use the list of logical equivalences in the lecture notes. You do not need to state the name of the logical equivalence at each step.

(a) Show that the following is a tautology:

$$(p \wedge q) \to (p \to q).$$

(b) Show that the following logical equivalence holds:

$$\neg(p \leftrightarrow q) \equiv \neg p \leftrightarrow q.$$

SOLUTION . (a)

$$\begin{aligned}
(p \wedge q) \to (p \to q) &\equiv \neg(p \wedge q) \vee (p \to q) && \text{[Using } a \to b \equiv \neg a \vee b\text{]} \\
&\equiv \neg(p \wedge q) \vee (\neg p \vee q) && \text{[Using } a \to b \equiv \neg a \vee b\text{]} \\
&\equiv (\neg p \vee \neg q) \vee (\neg p \vee q) && \text{[De Morgan's Law]} \\
&\equiv (\neg p \vee \neg q) \vee (q \vee \neg p) && \text{[Commutative Law]} \\
&\equiv \neg p \vee (\neg q \vee (q \vee \neg p)) && \text{[Associative Law]} \\
&\equiv \neg p \vee ((\neg q \vee q) \vee \neg p) && \text{[Associative Law]} \\
&\equiv \neg p \vee (\mathbf{T} \vee \neg p) && \text{[Negation Law]} \\
&\equiv \neg p \vee \mathbf{T} && \text{[Universal Bound Law]} \\
&\equiv \mathbf{T} && \text{[Universal Bound Law]}
\end{aligned}$$

(b) We start from the LHS:

$$
\begin{aligned}
\neg(p \leftrightarrow q) &\equiv \neg((p \to q) \land (q \to p)) && [\text{Definition of } \leftrightarrow] \\
&\equiv \neg((\neg p \lor q) \land (\neg q \lor p)) && [\text{Using } a \to b \equiv \neg a \lor b] \\
&\equiv \neg(\neg p \lor q) \lor \neg(\neg q \lor p) && [\text{De Morgan's Law}] \\
&\equiv (\neg(\neg p) \land \neg q) \lor (\neg(\neg q) \land \neg p) && [\text{De Morgan's Law x2}] \\
&\equiv (p \land \neg q) \lor (q \land \neg p) && [\text{Double Negation Law x2}] \\
&\equiv ((p \land \neg q) \lor q) \land ((p \land \neg q) \lor \neg p) && [\text{Distributive Law}] \\
&\equiv (q \lor (p \land \neg q)) \land (\neg p \lor (p \land \neg q)) && [\text{Commutative Law x2}] \\
&\equiv ((q \lor p) \land (q \lor \neg q)) \land ((\neg p \lor p) \land (\neg p \lor \neg q)) && [\text{Distributive Law x2}] \\
&\equiv ((q \lor p) \land \mathbf{T}) \land (\mathbf{T} \land (\neg p \lor \neg q)) && [\text{Negation Law x2}] \\
&\equiv (q \lor p) \land (\neg p \lor \neg q) && [\text{Identity Law x2}] \\
&\equiv (p \lor q) \land (\neg q \lor \neg p) && [\text{Commutative Law x2}] \\
&\equiv (\neg(\neg p) \lor q) \land (\neg q \lor \neg p) && [\text{Double Negation Law}] \\
&\equiv (\neg p \to q) \land (q \to \neg p) && [\text{Using } a \to b \equiv \neg a \lor b \text{ x2}] \\
&\equiv \neg p \leftrightarrow q && [\text{Definition of } \leftrightarrow]
\end{aligned}
$$

$\square$

QUESTION 2 **(15 marks)**

Determine if each of the following is true or false. Justify your answer. The domain of x, y and z is the set of integers $\mathbb{Z}$.

(a) $\exists x\ \forall y\ x < y^2$.

(b) $\forall x\ \forall y\ \exists z$ such that $z = (x + y)/2$.

(c) $\exists x\ \exists y$ such that $x^2 + y^2 = 6$.

SOLUTION . (a) This is true. We take $x = -1$. Then we need to show $\forall y(-1 < y^2)$. This is true because for any $y \in \mathbb{Z}$, $y^2 \geq 0 > -1$. So, $y^2 > -1$.

(b) This is false. We need to show $\exists x\ \exists y\ \forall z,\ z \neq (x + y)/2$. For example, we can take $x = 1$ and $y = 0$. We need to show $\forall z,\ z \neq (1 + 0)/2$. Since the domain of z is $\mathbb{Z}$, we have $\forall z,\ z \neq \frac{1}{2}$ since $\frac{1}{2} \notin \mathbb{Z}$.

(c) This is false. We need to show $\forall x\ \forall y,\ x^2 + y^2 \neq 6$. Fix an arbitrary x and y. We have three cases.

Case 1: $|x| \geq 3$**.** Then $x^2 + y^2 \geq x^2 + 0 = (|x|)^2 \geq 9$ and so $x^2 + y^2 \neq 6$.

Case 2: $|y| \geq 3$**.** Then $x^2 + y^2 \geq y^2 + 0 = (|y|)^2 \geq 9$ and so $x^2 + y^2 \neq 6$.

Case 3: $|x| < 3$ **and** $|y| < 3$**.** Then we have the possibilities $x = -2, -1, 0, 1, 2$ and $y = -2, -1, 0, 1, 2$. Trying them all out, we prove that $x^2 + y^2 \neq 6$ by the method of exhaustion in this case 3.

	$x = -2$	$x = -1$	$x = 0$	$x = 1$	$x = 2$
$y = -2$	$x^2 + y^2 = 8$	5	4	5	8
$y = -1$	5	2	1	2	5
$y = 0$	4	1	0	1	4
$y = 1$	5	2	1	2	5
$y = 2$	8	5	4	5	8

The number 6 does not appear in the table, so we conclude that $x^2 + y^2 \neq 6$ in case 3.

$\square$

QUESTION 3. **(12 marks)**

Prove or disprove the following statements:

(a) Let x be a real number. If x^3 is irrational then x is also irrational.

(b) Let n be an integer. If $3n + 2$ is odd then $9n + 5$ is even.

(c) Let n and m be integers. If $n + m^2$ is divisible by 3 then n or m is divisible by 3.

SOLUTION . (a) This is true. We prove the contrapositive: "If x is rational, then x^3 is rational". Suppose x is an arbitrary rational number. Let a, b be integers with $b \neq 0$ such that $x = \frac{a}{b}$. Then $x^3 = \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$. Since a, b are integers, then a^3 and b^3 are integers. Since $b \neq 0$, hence $b^3 \neq 0$ by the zero product property. Hence x^3 is rational.

(b) This is true. Let n be an arbitrary integer. Assume that $3n + 2$ is odd. We want to show that n is odd. We proceed by contradiction. Suppose n is even. Then $3n + 2 =$ odd $\times$ even $+$ even $=$ even and contradicts the assumption that $3n + 2$ is odd. Therefore n is odd. This means that $9n + 5 =$ odd $\times$ odd $+$ odd $=$ odd $+$ odd $=$ even. Hence $9n + 5$ is even.

(c) This is false. We need to find counterexamples n and m. Take $n = 2$ and $m = 2$. Then $n + m^2 = 2 + 2^2 = 6$ which is divisible by 3. Furthermore neither n nor m is divisible by 3. Therefore, the statement "If $n + m^2$ is divisible by 3 then n or m is divisible by 3" is false for our choice of n and m.

$\square$

QUESTION 4. **(8 marks)**
Let a and b be real numbers. Show that the average of a and b is greater than a if and only if the average of a and b is less than b.

SOLUTION . Let a and b be arbitrary real numbers. We need to show two directions. First, we assume that the average of a and b is greater than a. This means that $\frac{a+b}{2} > a$. Hence, $a+b > 2a$ and so $a < b$. Now adding b to both sides, we get $a+b < 2b$ and so $\frac{a+b}{2} < b$. Thus the average of a and b is less than b.

Now we prove the reverse direction. Assume that $\frac{a+b}{2} < b$. We get $a+b < 2b$ and so $a < b$. Adding a to both sides, we obtain $2a < a+b$ and this $a < \frac{a+b}{2}$. This means that the average of a and b is larger than a.

An alternative solution is the following. Suppose $\frac{a+b}{2} > a$. Adding $\frac{b-a}{2}$ to both sides, we obtain $\frac{a+b}{2} + \frac{b-a}{2} > a + \frac{b-a}{2}$ so we obtain $b > \frac{a+b}{2}$. Now suppose that $b > \frac{a+b}{2}$. We add $\frac{a-b}{2}$ to both sides, and obtain $b + \frac{a-b}{2} > \frac{a+b}{2} + \frac{a-b}{2}$ and hence $\frac{a+b}{2} > a$. $\square$