Solutions to MH1300 Final Exam AY 15/16

Q1 (a) $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \exists z \in \mathbb{R}\ (z^2 > x \cdot y)$

True. Take $x=0$. Now for any arbitrary $y \in \mathbb{R}$,

we chose $z = \sqrt{|y|} + 1$

Then $z > \sqrt{|y|}$

and so $z^2 > |y|$ since both are non negative.

This means $z^2 > |y| \geq -y = 0 - y = x - y$.

(b) $\forall x \in \mathbb{Q}\ \forall y \in \mathbb{Q}\ \ x<y \Rightarrow \exists z \in \mathbb{Q}\ \ x<z<y$

Fix arbitrary $x, y \in \mathbb{Q}$ and assume that $x<y$.

We pick $z = \dfrac{x+y}{2}$. Then,

$$x = \tfrac{1}{2}x + \tfrac{1}{2}x < \tfrac{1}{2}x + \tfrac{1}{2}y = z$$
$$< \tfrac{1}{2}y + \tfrac{1}{2}y = y$$

So, $x<z<y$.

$\boxed{Q2}$ (a) Fix $a \in \mathbb{Z}$.

By the Quotient remainder theorem, a is of the form $3k$, $3k+1$ or $3k+2$ for some $k \in \mathbb{Z}$.

Case 1 $a = 3k$. Then $\underbrace{a(a^2+2)}_{(3k)(9k^2+2)}$ is divisible by 3

Case 2 $a = 3k+1$ Then $a^2+2 = (3k+1)^2 + 2$

$= (9k^2+6k+1)+2$

$= 3(3k^2+2k+1)$

is divisible by 3.

Case 3 $a = 3k+2$. Then $a^2+2 = (3k+2)^2+2$

$= (9k^2+6k+4)+2$

$= 3(3k^2+2k+2)$

is divisible by 3.

In all three cases, $a(a^2+2)$ is divisible by 3.

(b) let $q \in \mathbb{Z}$ and $q > 1$. Suppose q is not prime. let $q = a.b$ where $1 < a < q$
$1 < b < q$.

Now we show $\neg (\forall\ a, b \in \mathbb{Z}\ \ q | ab \Rightarrow q | a \text{ or } q | b)$

Take a, b as above. Then $q | ab$ is true since $q = a.b$.

but $(q | a \text{ or } q | b)$ is false since $|a| < |q|$
and $|b| < |q|$

Q3 Let $P(n)$ be the statement

$$\sum_{k=1}^{n} \frac{k^2}{(2k-1)(2k+1)} = \frac{n(n+1)}{4n+2}, \quad n \geq 1.$$

Base case: $n=1$. $P(n)$ is the statement

$$\sum_{k=1}^{1} \frac{k^2}{(2k-1)(2k+1)} = \frac{1(1+1)}{4+2}$$

$$LHS = \frac{1}{(1)(3)} = \frac{1}{3}$$

$$RHS = \frac{2}{6} = \frac{1}{3}.$$

So $P(1)$ holds.

Inductive step: Assume $P(n)$ holds. We now prove $P(n+1)$.

$$\text{LHS of } P(n+1) = \sum_{k=1}^{n+1} \frac{k^2}{(2k-1)(2k+1)}$$

$$= \sum_{k=1}^{n} \frac{k^2}{(2k-1)(2k+1)} + \frac{(n+1)^2}{(2(n+1)-1)(2(n+1)+1)}$$

Apply $P(n)$,

$$= \frac{n(n+1)}{4n+2} + \frac{(n+1)^2}{(2n+1)(2n+3)}$$

$$= \frac{n(n+1)(2n+3) + 2(n+1)^2}{2(2n+1)(2n+3)}$$

$$= \frac{(n+1)\left(n(2n+3) + 2(n+1) \right)}{2(2n+1)(2n+3)}$$

$$= \frac{(n+1)\left(2n^2 + 3n + 2n + 2 \right)}{2(2n+1)(2n+3)}$$

$$= \frac{(n+1)(2n^2+5n+2)}{2(2n+1)(2n+3)}$$

$$= \frac{(n+1)(2n+1)(n+2)}{2(2n+1)(2n+3)}$$

$$= \frac{(n+1)(n+2)}{4n+6}$$

$=$ RHS of $P(n+1)$

Since LHS = RHS, we conclude $P(n+1)$ holds.

So $P(n)$ holds for all $n \geq 1$.

$\boxed{Q4}$ (a) Suppose not. Then there exists a real number $a > 0$ and some positive integer n_0 such that $a^n \leq 0$.

Let $S = \{ n \in \mathbb{Z} \mid n > 0 \text{ and } a^n \leq 0 \}$.

Then $S \neq \phi$ since $n_0 \in S$.

By the WOP, S has a least element, say $m \in S$.

Clearly $1 \notin S$ because $a^1 = a > 0$ by assumption.

So $m > 1$ (since $m \in S$).

Since $a^m \leq 0$ and $a > 0$

this means $\dfrac{a^m}{a} \leq 0$

So $a^{m-1} \leq 0$.

as $m > 1$ we have $m - 1 > 0$

So $m-1 \in S$, contradicting m is least in S.

(b) Fix a positive integer $n \geq 4$.

By the Quotient Remainder Theorem, $d = 4$,

n is of the form $4k$, $4k+1$, $4k+2$ or $4k+3$

for some k.

Case 1: $n = 4k$. Div by 4.

Case 2: $n = 4k+1$, $n+3 = 4k+4 = 4(k+1)$
is div by 4.

Case 3: $n = 4k+2$. $n+6 = 4k+8 = 4(k+2)$
is div by 4.

Case 4: $n = 4k+3$. $n+9 = 4k+12 = 4(k+3)$
is div by 4.

In all cases, one of $n, n+3, n+6$ or $n+9$ is

div by 4.

Q5 (a) Let $A = B = C = [0,1]$

(i) Take $f_0(x) = x$

$g_0(x) = \frac{1}{2}x.$

Then $(g_0 \circ f_0)(x) = g_0(f_0(x))$

$= g_0(x)$

$= \frac{1}{2}x.$

Then $g_0 \circ f_0$ is not onto, for instance take $y = 1$. There is no $x \in [0,1]$ such that $\frac{1}{2}x = y = 1.$

But f_0 is onto.

(ii) Take $f_1(x) = \frac{1}{2}x$

$g_1(x) = |x - \frac{1}{2}|$

Then $(g_1 \circ f_1)(x) = g_1(\frac{1}{2}x)$

$= |\frac{1}{2}x - \frac{1}{2}|$

$= \frac{1}{2}|x-1|.$

Since $x \in [0,1]$, So $x - 1 \leq 0$

$(g_1 \circ f_1)(x) = -\frac{1}{2}(x-1)$

$= \frac{1}{2}(1-x)$ is of course 1-1. (linear function).

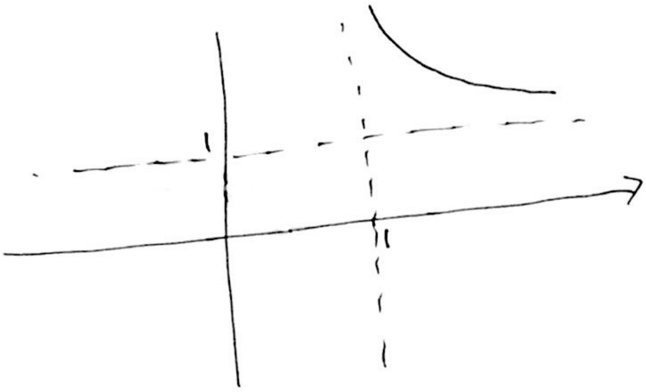
But g_1 is not 1-1 because

$g_1(0) = \frac{1}{2} = g_1(1)$

(b) $h(x) = \dfrac{x}{x-1}$

$= 1 + \dfrac{1}{x-1}$

Sketch:



From the sketch, $h(x)$ is 1-1 and onto.

let's prove it.

$h(x)$ is 1-1: Suppose $h(x) = h(y)$.

Then $1 + \dfrac{1}{x-1} = 1 + \dfrac{1}{y-1}$

$\dfrac{1}{x-1} = \dfrac{1}{y-1}$

Cross multiply,

$x - 1 = y - 1$

$x = y$

<u>$h(x)$ is onto</u>: Take $y \in (1, \infty)$. Then $y > 1$.

Solve for some $x \in (1, \infty)$ such that

$$h(x) = y$$

$$1 + \frac{1}{x-1} = y$$

$$\frac{1}{x-1} = y-1$$

$$x-1 = \frac{1}{y-1}$$

$$x = 1 + \frac{1}{y-1}$$

Since $y > 1$, so $y-1 > 0$

and $\frac{1}{y-1}$ is defined, and > 0.

$$1 + \frac{1}{y-1} > 1.$$

So $x \in (1, \infty)$.

(c) let $F: A \mapsto B$ and assume $X \subseteq A$

and F is 1-1.

$F(A-X) \subseteq F(A) - F(X)$:

Start with $F(x) \in \overbrace{F(A-X)}^{LHS}$, where $x \in A - X$.

Then $x \in A$ and $x \notin X$.

Since $x \in A$ so $F(x) \in F(A)$.

But now we see that $F(x) \notin F(X)$,

otherwise $F(x) = F(y)$ for some $y \in X$.

Since $x \notin X$ so $x \neq y$ which contradicts

F is 1-1.

So $F(x) \notin F(X)$.

This means $F(x) \in F(A) - F(X)$

$= RHS$.

$F(A) - F(X) \subseteq F(A-X)$:

let $y \in F(A) - F(X)$.

Since $y \in F(A)$ so $y = F(x)$ some $x \in A$.

If $x \in X$ then $y = F(x) \in F(X)$ contradicts first line above. So $x \notin X$. Thus, $x \in A - X$.

This means $y = F(x) \in F(A-X)$

$\boxed{\text{Q6}}$ (a) Suppose $A/R \subseteq A/S$.

Recall that A/R and A/S are both partitions of the set A.

$\underline{R \subseteq S:}$ Suppose $(a,b) \in R$.

Then $b \in [a]_R$. In fact, $a, b \in X$, for some $X \in A/R$.

Since $A/R \subseteq A/S$,

So, $X \in A/S$.

Since $a, b \in X \in A/S$,

hence $(a,b) \in S$.

$\underline{S \subseteq R:}$ Suppose $(a,b) \in S$.

Then there is some $X \in A/S$ such that $a, b \in X$.

Now $[a]_R \in A/R$. Since $[a]_R \in A/S$ and A/S is a partition of A,

and $a \in X \cap [a]_R \neq \phi$,

this means $X = [a]_R$.

So $b \in X = [a]_R$.

So $(a,b) \in R$

(b) <u>Reflexive:</u> $\frac{a}{a} = 1 = 2^0$ and $0 \in \mathbb{Z}$

So $(a,a) \in T$.

<u>Symmetric:</u> Suppose $(a,b) \in T$.

Then $\frac{a}{b} = 2^m$ for some $m \in \mathbb{Z}$.

Then $\frac{b}{a} = 2^{-m}$, and $-m \in \mathbb{Z}$

So $(b,a) \in T$.

<u>Transitive:</u> Suppose (a,b), $(b,c) \in T$.

Then there exists $m, n \in \mathbb{Z}$

where $\frac{a}{b} = 2^m$, $\frac{b}{c} = 2^n$.

Then $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = 2^n \times 2^m$

$= 2^{n+m}$.

Since $n+m \in \mathbb{Z}$, so

$(a,c) \in T$.

$\boxed{Q7}$ (a) Show $(A-B) \cup (B-A)$

$= (A \cup B) - (A \cap B)$.

$x \in LHS = (A-B) \cup (B-A)$

$\Leftrightarrow$ $x \in A-B$ or $x \in B-A$

$\Leftrightarrow$ $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$.

(Distributive law)

$\Leftrightarrow$ $(x \in A$ or $x \in B)$ and $(x \notin B$ or $x \in B)$ and

$(x \in A$ or $x \notin A)$ and $(x \notin B$ or $x \notin A)$.

$\Leftrightarrow$ $(x \in A$ or $x \in B)$ and T and T and

$(x \notin B$ or $x \notin A)$

$\Leftrightarrow$ $(x \in A$ or $x \in B)$ and $(x \notin A$ or $x \notin B)$

(De Morgan's Law)

$\Leftrightarrow$ $(x \in A \cup B)$ and $\neg(x \in A$ and $x \in B)$

$\Leftrightarrow$ $x \in A \cup B$ and $\neg(x \in A \cap B)$

$\Leftrightarrow$ $x \in A \cup B - (A \cap B)$

(b) Suppose there are A, B s.t.

$$\mathcal{P}(A-B) = \mathcal{P}(A) - \mathcal{P}(B).$$

Note that $\emptyset \in \mathcal{P}(X)$ for any set X.

So $\emptyset \in \mathcal{P}(A-B)$

but $\emptyset \notin \mathcal{P}(A) - \mathcal{P}(B)$.

So contradiction.

(c) (i) False. Take $A = \{0\} = B$

$C = \{1\}$, $D = \{2\}$

$(A \times B) \cup (C \times D)$

$= \{(0,0)\} \cup \{(1,2)\}$

$= \{(0,0), (1,2)\}$.

$(A \cup C) \times (B \cup D) = \{0,1\} \times \{0,2\}$

$= \{(0,0), (0,2), (1,0), (1,2)\}$

Obviously not equal.

(ii) False. Take $C = \{0\} = B$

$A = \{1\}$.

$(C \times C) - (A \times B) = \{(0,0)\} - \{(1,0)\}$

$= \{(0,0)\}$

$(C - A) \times (C - B) = \{0\} \times \phi$

$= \phi$

Not equal.