

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2021-2022

MH1100 – Calculus I

December 2021

TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED** book exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1**(16 marks)**

(a) Evaluate the limit

$$\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x^2 - 1}.$$

(b) Use L'Hospital's Rule to evaluate the limit

$$\lim_{x \rightarrow 0^+} (1 + \sin 3x)^{\cot x}.$$

[Solution:]

(a)

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x^2 - 1} &= \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x^2 - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \lim_{x \rightarrow 1} \frac{x - 1}{(x + 1)(x - 1)} \cdot \frac{1}{\sqrt{x} + 1} \\ &= \lim_{x \rightarrow 1} \frac{(x - 1)}{(x + 1)(x - 1)} \cdot \frac{1}{\sqrt{x} + 1} = \lim_{x \rightarrow 1} \frac{1}{(x + 1)} \cdot \lim_{x \rightarrow 1} \frac{1}{\sqrt{x} + 1} \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{1} + 1} = \frac{1}{4}. \end{aligned}$$

(b) As $x \rightarrow 0^+$, we have $1 + \sin 3x \rightarrow 1$ and $\cot x \rightarrow \infty$, so the given limit is indeterminate (type 1^∞). Let $y = (1 + \sin 3x)^{\cot x}$, we have

$$\ln y = \cot x \ln(1 + \sin 3x) = \frac{\ln(1 + \sin 3x)}{\tan x}.$$

Use L'Hospital's Rule,

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin 3x)}{\tan x} = \lim_{x \rightarrow 0^+} \frac{\frac{3 \cos 3x}{1 + \sin 3x}}{\sec^2 x} = 3.$$

To find limit of y , we use the fact that $y = e^{\ln y}$:

$$\lim_{x \rightarrow 0^+} (1 + \sin 3x)^{\cot x} = \lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} e^{\ln y} = e^3$$

QUESTION 2**(16 marks)**

Use the ϵ - δ definition to prove the limit

$$\lim_{x \rightarrow 1} x^2 - 1 = 0.$$

[Solution:]

1. Guessing a value for δ .

Let ϵ be a given positive number. We want to find a number $\delta > 0$ such that

$$\text{if } 0 < |x - 1| < \delta \text{ then } |x^2 - 1| < \epsilon.$$

We write

$$|x^2 - 1| = |(x + 1)(x - 1)| = |x + 1| \cdot |x - 1|$$

Then we want

$$\text{if } 0 < |x - 1| < \delta \text{ then } |x + 1| \cdot |x - 1| < \epsilon.$$

In fact, since we are interested only in values of x that are close to 1, it is reasonable to assume that x is within a distance of $\frac{1}{2}$ from 1, that is $|x - 1| < \frac{1}{2}$. Then we have $\frac{1}{2} < x < \frac{3}{2}$, so $\frac{3}{2} \leq |x + 1| < \frac{5}{2}$. Thus, when $\frac{1}{2} < x < \frac{3}{2}$, it follows that

$$|x + 1| \cdot |x - 1| < \frac{5}{2}|x - 1|.$$

If $|x - 1| < \frac{2}{5}\epsilon$ is also satisfied when $|x - 1| < \frac{1}{2}$, we have

$$|x^2 - 1| < \frac{5}{2}|x - 1| < \epsilon.$$

Therefore, we can take δ to be the smaller of the two numbers $\frac{1}{2}$ and ϵ . The notation for this is $\delta = \min\{\frac{1}{2}, \frac{2}{5}\epsilon\}$.

2. Showing that this δ works.

Given $\epsilon > 0$, let $\delta = \min\{1/2, \frac{2}{5}\epsilon\}$. If $0 < |x - 1| < \delta$, then

$$|x - 1| < 1/2 \Rightarrow 1/2 < x < 3/2 \Rightarrow |1 + x| < \frac{5}{2}.$$

We also have $|x - 1| < \frac{2}{5}\epsilon$, so

$$|x^2 - 1| = |1 + x| \cdot |x - 1| < \frac{5}{2}|x - 1| < \epsilon.$$

Therefore,

$$\lim_{x \rightarrow 1} (x^2 - 1) = 0.$$

QUESTION 3 (16 marks)

Find the derivatives of the following functions. (**You do not need to simplify the answers**)

(a)

$$g(x) = \cos(x^5 + \sqrt{\tan x}).$$

(b)

$$f(x) = (\sin x)^{\ln x}.$$

[Solution:]

(a)

$$g'(x) = -\sin(x^5 + \sqrt{\tan x}) \left(5x^4 + \frac{\sec^2 x}{2\sqrt{\tan x}} \right)$$

(b) we have $\ln f(x) = \ln x \ln(\sin x)$, then take the derivative on the both sides with respect to x ,

$$\frac{f'}{f} = \frac{\ln(\sin x)}{x} + \frac{\ln x \cos x}{\sin x} \Rightarrow f' = (\sin x)^{\ln x} \left(\frac{\ln(\sin x)}{x} + \frac{\ln x \cos x}{\sin x} \right)$$

QUESTION 4 (12 marks)

Prove that the function

$$f(x) = x^{201} + x^{101} + x + 1,$$

has neither a local maximum nor a local minimum.

[Solution:] Note that $f(x)$ is continuous and differentiable in the whole domain. Therefore, if $f(x)$ has a local extreme value at, say $x = c$, then $f'(c) = 0$. However, we have

$$f'(x) = 201x^{200} + 101x^{100} + 1 = 201(x^{100})^2 + 101(x^{50})^2 + 1 > 0.$$

So $f'(x)$ is always positive (nonzero) in the whole domain. In this way, there is no local extreme values.

QUESTION 5 (12 marks)

Suppose $f(x)$ has third order derivative on $[0, 1]$ (that is $f'''(x)$ exists on $[0, 1]$), and $f(0) = f(1) = 0$. Show that for the function $F(x) = x^2 f(x)$, there exists a number $c \in (0, 1)$, such that $F'''(c) = 0$.

[Solution:] Since $f(x)$ is third order derivative on $[0, 1]$, functions $f(x)$, $f'(x)$, and $f''(x)$ are all continuous and differential on $[0, 1]$. For $F(x) = x^2 f(x)$, we have

$$F'(x) = 2xf(x) + x^2 f'(x)$$

$$F''(x) = 2f(x) + 4xf'(x) + x^2 f''(x)$$

$$F'''(x) = 6f'(x) + 6xf''(x) + x^2 f'''(x)$$

so that $F(x)$, $F'(x)$, and $F''(x)$ are all continuous and differential at $[0, 1]$, and $F'''(x)$ exists at $[0, 1]$. Further,

$$F(0) = F(1) = F'(0) = F''(0) = 0.$$

From $F(0) = F(1) = 0$ and $F'(x)$ exists at $[0, 1]$, there exist a number $c_1 \in (0, 1)$, such that $F'(c_1) = 0$.

From $F'(0) = F'(c_1) = 0$ and $F''(x)$ exists at $[0, 1]$, there exist a number $c_2 \in (0, c_1)$, such that $F''(c_2) = 0$.

From $F''(0) = F''(c_2) = 0$ and $F'''(x)$ exists at $[0, 1]$, there exist a number $c \in (0, c_2)$, such that $F'''(c) = 0$.

QUESTION 6**(12 marks)**

Show that the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1.$$

[Solution:]

Differentiate both sides of the equation with respect to x

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

We now solve for y'

$$y' = -\frac{b^2 x}{a^2 y}$$

The tangent line at (x_0, y_0) is

$$y - y_0 = y'(x_0, y_0)(x - x_0)$$

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

$$\frac{yy_0 - y_0^2}{b^2} = -\frac{xx_0 - x_0^2}{a^2}$$

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$$

Note that (x_0, y_0) is a point on the ellipse, so that

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$$

Therefore

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$$

QUESTION 7**(16 marks)**

Suppose we have

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that

$$f''(0) = 0.$$

[Solution:] we have

$$f'(0) = \lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}} - 0}{x - 0} = \lim_{y \rightarrow \infty} y e^{-y^2} = 0$$

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$$f''(0) = \lim_{x \rightarrow 0} \frac{\frac{2}{x^3} e^{-\frac{1}{x^2}} - 0}{x - 0} = \lim_{y \rightarrow \infty} 2y^2 e^{-y} = 0$$

END OF PAPER