**SPMS / Division of Mathematical Sciences**

**MH1300 Foundations of Mathematics**
**2019/2020 Semester 1**

**MID-TERM EXAM SOLUTIONS**

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**QUESTION 1.** **(14 marks)**

Solve the following **without** using truth tables. You will **need to state** the logical equivalence used at each step.

(a) Determine if the following is a tautology, a contradiction, or neither: (5m)

$$((p \to \neg q) \wedge p) \wedge q.$$

(b) Determine which of the following is logically equivalent to $(p \vee q) \to r$: (9m)

(i) $(p \to r) \wedge (q \to r)$.

(ii) $(p \to r) \vee (q \to r)$.

(iii) $p \to (q \to r)$.

For each part (i), (ii) and (iii), if it is logically equivalent to $(p \vee q) \to r$, prove it without using truth tables. If it is not logically equivalent to $(p \vee q) \to r$, explain why not.

SOLUTION . (a) This is a contradiction.

$$\begin{aligned}
((p \to \neg q) \wedge p) \wedge q &\equiv (p \to \neg q) \wedge (p \wedge q) && [\text{Associative law}] \\
&\equiv (\neg p \vee \neg q) \wedge (p \wedge q) && [\text{Using } a \to b \equiv \neg a \vee b] \\
&\equiv \neg(p \wedge q) \wedge (p \wedge q) && [\text{De Morgan's Law}] \\
&\equiv (p \wedge q) \wedge \neg(p \wedge q) && [\text{Commutative Law}] \\
&\equiv \mathbf{F} && [\text{Negation Law}]
\end{aligned}$$

(b)(i) This is logically equivalent.

$$\begin{aligned}
(p \to r) \wedge (q \to r) &\equiv (\neg p \vee r) \wedge (\neg q \vee r) && [\text{Using } a \to b \equiv \neg a \vee b] \\
&\equiv (r \vee \neg p) \wedge (r \vee \neg q) && [\text{Commutative Law } \times 2] \\
&\equiv r \vee (\neg p \wedge \neg q) && [\text{Distributive Law}] \\
&\equiv (\neg p \wedge \neg q) \vee r && [\text{Commutative Law}] \\
&\equiv \neg(p \vee q) \vee r && [\text{De Morgan's Law}] \\
&\equiv (p \vee q) \to r && [\text{Using } a \to b \equiv \neg a \vee b]
\end{aligned}$$

(b)(ii) They are not logically equivalent. We wish to show that $(p \vee q) \to r \not\equiv (p \to r) \vee (q \to r)$. For example, take $p$ to be true, $q$ and $r$ to be false. Then $(p \vee q) \to r$ is $(T \vee F) \to F \equiv T \to F \equiv F$, while $(q \to r)$ is true since $q$ is false. Hence, $(p \to r) \vee (q \to r)$ is true, and so $(p \vee q) \to r \not\equiv (p \to r) \vee (q \to r)$.

(b)(iii) They are not logically equivalent. Take the same truth values as in (b)(ii), i.e. take $p$ to be true, $q$ and $r$ to be false. Then $(p \vee q) \to r$ is false, while $p \to (q \to r)$ is $T \to (F \to F) \equiv T \to T \equiv T$. Hence, $(p \vee q) \to r \not\equiv p \to (q \to r)$. $\square$

**QUESTION 2** **(12 marks)**

Determine if each of the following is true or false. Justify your answer.

(a) For each positive integer $a$ there is a positive integer $b$ such that $\dfrac{1}{2b^2 + b} < \dfrac{1}{ab^2}$.

(b) For each pair of integers $x$ and $y$, there is an integer $z$ such that $z^2 + 2xz - y^2 = 0$.

(c) There is some positive integer $p$ such that $p^2 - 2$ is divisible by $3$.

SOLUTION . (a) This is false. We need to show the negation of the statement, i.e. we need to show that there exists some positive integer $a$ such that for every positive integer $b$, we have $\dfrac{1}{2b^2 + b} \geq \dfrac{1}{ab^2}$.

Take $a = 3$. We need to show that this choice of $a$ works. Now fix a positive integer $b$; we want to show that $\dfrac{1}{2b^2 + b} \geq \dfrac{1}{3b^2}$. Since $b \geq 1$, we know that $b^2 \geq b$. Hence, $3b^2 \geq 2b^2 + b$. Since these quantities are all strictly positive, this means that $\dfrac{1}{3b^2} \leq \dfrac{1}{2b^2 + b}$, which is what we want.

(b) This is false. Take $x = 1$ and $y = 1$. Then we need to show that for every integer $z$, $z^2 + 2z - 1 \neq 0$. Suppose for a contradiction that $z^2 + 2z - 1 = 0$ for some integer $z$. Then completing the square gives $(z + 1)^2 - 2 = 0$ and so $(z + 1)^2 = 2$. Hence $z + 1 = \sqrt{2}$ or $-(z + 1) = \sqrt{2}$. Since $z + 1$ and $-(z + 1)$ are both integers, this means that $\sqrt{2}$ is an integer, a contradiction to the irrationality of $\sqrt{2}$. Thus, for every integer $z$, $z^2 + 2z - 1 \neq 0$.

Alternatively, to show this without using contradiction, you can (either by completing the square or applying the quadratic formula) conclude that the roots of the equation $z^2 + 2z - 1 = 0$ are $-1 \pm \sqrt{2}$, and both roots are not an integer because $\sqrt{2}$ is irrational and hence not an integer. Therefore, there is no integer solution to $z^2 + 2z - 1 = 0$.

A note on this part; you must pick $x$ and $y$ to be both non-zero integers. This is because the discriminant of the quadratic equation $z^2 + 2z - 1 = 0$ is $4(x^2 + y^2)$, and we do not want this to be a perfect square. So any choice of $x, y$ such that $x^2 + y^2$ is not a perfect square will be okay.

(c) This statement is false. We need to prove the negation. Given a positive integer $p$, we want to show that $p^2 - 2$ is not divisible by $3$. We apply the Quotient Remainder Theorem with $d = 3$; hence $p$ is of the form $3k$, $3k + 1$ or $3k + 2$ for some integer $k$. There are three cases:

$p = 3k$ **for some integer** $k$**:** Then $p^2 - 2 = (3k)^2 - 2 = 9k^2 - 2 = 3(3k^2 - 1) + 1$. By the uniqueness of the quotient and the remainder, $p^2 - 2$ cannot be divisible by $3$, since $(p^2 - 2) \mod 3 = 1$.

$p = 3k + 1$ **for some integer** $k$**:** Then $p^2 - 2 = (3k + 1)^2 - 2 = (9k^2 + 6k + 1) - 2 = 3(3k^2 + 2k - 1) + 2$. By the uniqueness of the quotient and the remainder, $p^2 - 2$ cannot be divisible by $3$, since $(p^2 - 2) \mod 3 = 2$.

$p = 3k + 2$ **for some integer** $k$**:** Then $p^2 - 2 = (3k + 2)^2 - 2 = (9k^2 + 12k + 4) - 2 = 3(3k^2 + 4k) + 2$. By the uniqueness of the quotient and the remainder, $p^2 - 2$ cannot be divisible by $3$, since $(p^2 - 2) \mod 3 = 2$.

$\square$

**QUESTION 3.** **(14 marks)**

(a) Let $p, q$ be non-zero integers. If $p \mid q$ and $q \mid p$, show that $p = q$ or $p = -q$. (6m)

(b) Let $x, y$ be real numbers. Prove from the definition of the absolute value function that $|xy| = |x||y|$. (8m)

SOLUTION . (a) Let $p, q$ be non-zero integers such that $p \mid q$ and $q \mid p$. Thus there are integers $k$ and $l$ such that $pk = q$ and $ql = p$. We have $(ql)k = q$ and hence $q(lk) = q$. Since $q \neq 0$ we can divide both sides by $q$ and obtain $lk = 1$. Since $l, k$ are both integers, they are both divisors of $1$. By Theorem 4.3.2 on Slide 38 of the lecture notes, $l$ is either $1$ or $-1$. If $l = 1$ then $p = q$ and if $l = -1$ then $p = -q$. Hence, we conclude that $p = q$ or $p = -q$.

(b) Let $x, y$ be real numbers. Recall that the definition of the absolute value function is

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

We wish to show that $|xy| = |x||y|$. First of all, if $xy = 0$ then by the Zero Product Property, $x = 0$ or $y = 0$. Hence $|xy| = 0$, and $|x||y| = 0$, and so they are equal. Therefore, we will assume that $xy \neq 0$ and hence $x$ and $y$ are *both* not zero.

We split into two cases:

**Case 1:** $xy > 0$**.** Then $|xy| = xy$ according to the definition of $|xy|$. Since $xy > 0$, either $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$ (otherwise the product $xy$ will be strictly negative). If $x > 0$ and $y > 0$, then by definition of $|x|$ and $|y|$, we have $|x| = x$ and $|y| = y$ and so $|x||y| = xy = |xy|$. If $x < 0$ and $y < 0$ then by definition of $|x|$ and $|y|$, we have $|x| = -x$ and $|y| = -y$ and so $|x||y| = (-x)(-y) = xy = |xy|$.

**Case 2:** $xy < 0$**.** Then $|xy| = -xy$ according to the definition of $|xy|$. Since $xy < 0$, either $x > 0$ and $y < 0$, or $x < 0$ and $y > 0$. If $x > 0$ and $y < 0$, then by definition of $|x|$ and $|y|$, we have $|x| = x$ and $|y| = -y$ and so $|x||y| = x(-y) = -xy = |xy|$. If $x < 0$ and $y > 0$ then by definition of $|x|$ and $|y|$, we have $|x| = -x$ and $|y| = y$ and so $|x||y| = (-x)y = -xy = |xy|$.

In any case, we conclude that $|xy| = |x||y|$.

Note: You can also divide into four cases, according to whether $x \geq 0$ or $x < 0$, and whether $y \geq 0$ or $y < 0$.

$\square$

**QUESTION 4.** **(10 marks)**

Prove that for any integer $n \geq 1$, $n^5 - n$ is divisible by $5$.
Hint: *You may wish to first factorize* $n^5 - n$ *completely.*

SOLUTION . Let $n$ be an integer such that $n \geq 1$. We follow the hint and factorize completely $n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = n(n-1)(n+1)(n^2+1)$. (Without this step the rest of the proof will be very tedious). Applying the Quotient Remainder Theorem to $d = 5$, we have the following five cases:

$n = 5k$ **for some integer** $k$**:** Then $5$ divides $n$, which in turn divides $n^5 - n$.

$n = 5k + 1$ **for some integer** $k$**:** Then $n - 1 = (5k+1) - 1 = 5k$, and so $5$ divides $n - 1$, which in turn divides $n^5 - n$.

$n = 5k + 2$ **for some integer** $k$**:** Then $n^2 + 1 = (5k+2)^2 + 1 = (25k^2 + 20k + 4) + 1 = 5(5k^2 + 4k + 1)$. So $5$ divides $n^2 + 1$, which in turn divides $n^5 - n$.

$n = 5k + 3$ **for some integer** $k$**:** Then $n^2 + 1 = (5k+3)^2 + 1 = (25k^2 + 30k + 9) + 1 = 5(5k^2 + 6k + 2)$. So $5$ divides $n^2 + 1$, which in turn divides $n^5 - n$.

$n = 5k + 4$ **for some integer** $k$**:** Then $n + 1 = (5k+4) + 1 = 5(k+1)$, and so $5$ divides $n + 1$, which in turn divides $n^5 - n$.

In all cases, we conclude that $5 \mid (n^5 - n)$. $\square$