SPMS / Division of Mathematical Sciences

MH1300 Foundations of Mathematics
2023/2024 Semester 1

MID-TERM EXAM SOLUTIONS

QUESTION 1. **(15 marks)**

Prove each of the following statements.

(a) Let n be an integer. Show that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

(b) By using the definition of the absolute value function $|x|$, show that for any two real numbers x, y, $|xy| = |x||y|$.

SOLUTION . (a) Suppose that $2 \mid (n^4 - 3)$. Then there is some integer k such that $2k = n^4 - 3$, which means that $n^4 = 2k + 3$. Hence n^4 is odd, and therefore n is odd. Let $n = 2l + 1$ for some integer l. Therefore $n^2 + 3 = (2l+1)^2 + 3 = 4l^2 + 4l + 1 + 3 = 4(l^2 + l + 1)$. Since $l^2 + l + 1 \in \mathbb{Z}$, therefore, $4 \mid (n^2 + 3)$.

Now suppose that $4 \mid (n^2 + 3)$. Let k' be an integer such that $4k' = n^2 + 3$. Therefore $n^2 = 4k' - 3 = 2(2k' - 2) + 1$ which is odd. Since n^2 is odd, n is also odd. But this means that n^4 is also odd, which means that $n^4 - 3$ is even. Therefore $2 \mid (n^4 - 3)$.

(b) Since we must use the definition of the absolute function, we must consider cases:

Case 1: $x, y \geq 0$ Then $xy \geq 0$ and so $|xy| = xy$ and $|x| = x$ and $|y| = y$ by the definition. So $|xy| = xy = |x||y|$.

Case 2: $x \geq 0$ **and** $y < 0$ Then $xy \leq 0$ and so $|xy| = -xy$ and $|x| = x$ and $|y| = -y$ by the definition. So $|xy| = -xy = x(-y) = |x||y|$.

Case 3: $x < 0$ **and** $y \geq 0$ Then $xy \leq 0$ and so $|xy| = -xy$ and $|x| = -x$ and $|y| = y$ by the definition. So $|xy| - xy = (-x)y = |x||y|$. (*This case may be omitted if you mention that it is similar to case 2*).

Case 4: $x < 0$ **and** $y < 0$ Then $xy > 0$ and so $|xy| = xy$ and $|x| = -x$ and $|y| = -y$ by the definition. So $|xy| = xy = (-x)(-y) = |x||y|$.

Common Mistakes: Some students applied the definition only to $|x|$ and forgot about $|y|$. (There should be a total of 4 cases). Some students considered the two cases $xy > 0$ and $xy \leq 0$, but then proceeded to declare that in the first case we have $x > 0$ and $y > 0$ (and missed out the case $x < 0$ and $y < 0$). Some students applied the definition wrongly: They said that if $x < 0$ and $y < 0$ then $|xy| = |(-x)(-y)| = |-x||-y| = |x||y|$. This string of equations is of course correct, but it's not how the definition should be applied.

$\square$

QUESTION 2 **(15 marks)**

Determine if each of the following is true or false. Justify your answer.

(a) For every integer a, $5a - 9$ is odd if and only if $3a + 4$ is even.

(b) Let n be a positive integer. If $n^2 + \dfrac{1}{n^2} > 4$ then $n + \dfrac{2}{n} \geq 3$.

(c) There is an integer m such that $m^6 + 2m^4 + m^2 - 5 = 0$.

SOLUTION . (a) This is true. Suppose $5a - 9$ is odd for a given integer a. Then $5a = (5a - 9) + 9$ is even (since odd + odd = even). Since $5a$ is even, a must be even (otherwise 5 times odd will give odd). This means that $3a$ is even, and so $3a + 4$ is even.

Now suppose that $3a + 4$ is even. Then $3a = (3a + 4) - 4$ is even (since even $-$ even $=$ even). This means that a is even, and so $5a$ is even. Therefore $5a - 9$ is odd (since even $-$ odd $=$ odd).

Common Mistakes: This part is easy, and has appeared before in past years in a similar form. The only main problem is that some students proved only one direction, where the question is stated as "if and only if".

(b) Suppose n is a positive integer. If $n = 1$ then $n + \dfrac{2}{n} = 1 + 2 \geq 3$. If $n = 2$ then $n + \dfrac{2}{n} = 2 + 1 \geq 3$. If $n \geq 3$ then $n + \dfrac{2}{n} \geq n \geq 3$. Therefore, given any positive integer n, $n + \dfrac{2}{n} \geq 3$ is always true. Therefore the statement

$$\text{If } n^2 + \frac{1}{n^2} > 4 \text{ then } n + \frac{2}{n} \geq 3$$

is true for any positive integer n.

Common Mistakes: Most people did not realise the contrapositive statement "$n + \frac{2}{n} < 3 \rightarrow n^2 + \frac{1}{n^2} \leq 4$" is vacuously true if $n \geq 2$, so there is no need to try and do a complicated argument. Some students missed out the case $n = 1$.

(c) This is false. To disprove it, we need to show its negation, ie we need to show that for any integer m, $m^6 + 2m^4 + m^2 - 5 \neq 0$. Let m be an integer. If $m = 0$ then $m^6 + 2m^4 + m^2 - 5 = -5$. If $m = \pm 1$ then $m^6 + 2m^4 + m^2 - 5 = -1$. If $m \geq 2$ or if $m \leq -2$ then $m^6 + 2m^4 + m^2 - 5 \geq 2^6 + 2 \cdot 2^4 + 2^2 - 5 = 95$. In all cases, $m^6 + 2m^4 + m^2 - 5 \neq 0$.

Common Mistakes: Again, this part is very similar to questions that have appeared in past years. You have to argue using cases, by considering large (positive and negative) values of m and then argue exhaustively for small values of m. Some students missed out negative values of m.

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QUESTION 3. **(12 marks)**

Show each of the following statements. Justify your answers.

(a) Write down a biconditional predicate whose negation is logically equivalent to the following predicate: n^3 and $3n + 4$ are both odd or n^3 and $3n + 4$ are both even.

(b) Find all values of the integers a and b (if any) such that $n^2 + an + b$ is odd for every integer n. You need to justify why these are <u>all</u> the values.

SOLUTION . (a) The negation of the biconditional statement $p \leftrightarrow q$ is $p \wedge \neg q$ or $q \wedge \neg p$. So comparing with the given predicate, we can take p to be "n^3 is odd" and q to be "$3n+4$ is even". So one possible answer is "n^3 is odd $\leftrightarrow 3n+4$ is even". Alternatively you can also take "n^3 is even $\leftrightarrow 3n + 4$ is odd".

Common Mistakes: The following are some common wrong answers:

- *"n^3 and $3n + 4$ are both odd if and only if n^3 and $3n + 4$ are both even",*
- *"n^3 is even and $3n + 4$ is odd if and only if n^3 is odd and $3n + 4$ is even",*
- *"n^3 and $3n + 4$ are both even or n^3 and $3n + 4$ are both odd",*
- *"n^3 is odd and $3n + 4$ are is even or n^3 is even and $3n + 4$ is odd".*

Your statement has to be bi-conditional. The first two are actually contradictions (never true for any value of n), while the given statement is a tautology (true for any value of n), so they would be okay if properly justified.

(b) The answer is: a and b are both odd numbers. We need to justify two things: First, we need to check that if a and b are both odd numbers, then $n^2 + an + b$ is odd for every integer n. Suppose that a and b are both odd numbers. Now n is either even or odd. If n is even, then $n^2 + an + b =$ even$^2+$ odd$\cdot$even $+$odd $=$ even $+$ even $+$ odd $=$ odd. On the other hand, if n is odd, then $n^2 + an + b =$ odd$^2+$ odd$\cdot$odd $+$odd $=$ odd $+$ odd $+$ odd $=$ odd.

Now to get full credit, you will also need to explain why you cannot choose a to be even or b to be even. First of all, suppose that b is even. Then we check that the property "$n^2 + an + b$ is odd for every integer n" is false. Take $n = 0$, then $n^2 + an + b = b$ is not odd. So b even does not work. Now suppose that b is odd and a is even. Then taking $n = 1$ we have $n^2 + an + b = 1 + a + b$ which is even, so the property "$n^2 + an + b$ is odd for every integer n" is also false in this case.

Common Mistakes: Many students considered different cases for n. You are asked to write down all values of a and b that works for all n, so your final answer cannot depend on n.
Many students did the following argument: Since $n^2 + an + b$ is odd for every odd n, we conclude that b is odd. Now we know that b is odd, we consider $n = 1$, which means that $1 + a + b$ is odd, which means that a is odd. While this reasoning is correct, it only explains that the condition "a, b odd" is a necessary condition. You will still need to explain why the condition is sufficient, i.e. if a, b both odd, then $n^2 + an + b$ is odd for every integer n.

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QUESTION 4. **(8 marks)**

Find two predicates $P(n)$ and $Q(n)$ with domain $E =$ the set of positive even numbers, such that $P(n) \wedge \neg Q(n)$ holds for infinitely many values of $n \in E$, and $\neg P(n) \wedge Q(n)$ holds for infinitely many values of $n \in E$. You should find a single pair of predicates $P(n)$ and $Q(n)$ satisfying both the conditions above. Justify your answer.

SOLUTION . There are many different choices one can take, but the important thing is to realise that you should only write down a single choice for $P(n)$ and a single choice of $Q(n)$ such that $P(n) \wedge \neg Q(n)$ holds for infinitely many $n \in E$ and $\neg P(n) \wedge Q(n)$ holds for infinitely many (other) values of $n \in E$.

One possible choice: $P(n)$ is "n is divisible by 3" and "$Q(n)$ is n is divisible by 5". Then for any integer of the form $3 \cdot 2^k$ for a positive integer k is even and is divisible by 3 but not divisible by 5, because $2, 3$ and 5 are prime numbers and by the Unique Factorization Theorem, 5 is not a factor of the expression $3 \cdot 2^k$. Therefore $P(3 \cdot 2^k) \wedge \neg Q(3 \cdot 2^k)$ holds. Similarly $\neg P(5 \cdot 2^k) \wedge Q(5 \cdot 2^k)$ holds for any positive integer k.

Another justification could be something like: Any integer of the form $30k + 6$ is even (since $30k + 6 = 2(15k + 3)$) and divisible by 3 (since $30k + 6 = 3(10k + 2)$). However, it isn't divisible by 5, because $30k + 6 = 5(6k + 1) + 1$ and it has a remainder of 1 when divided by 5.

Some unacceptable choices are: $P(n)$ is "$n \geq 0$" and $Q(n)$ is "$n \leq 0$", since $Q(n)$ never holds for any $n \in E$. Another unacceptable choice is $P(n)$ is "n is divisible by 4" and "$Q(n)$ is n is divisible by 8", since $Q(n) \wedge \neg P(n)$ is never true for any $n \in E$. Another unacceptable choice is $P(n)$ is "$n > 100$" and "$Q(n)$ is $n < 1000$", since $Q(n)$ is true for only finitely many values of $n \in E$.

Common Mistakes: Many students came up with something similar to the unacceptable choices listed above; the problem is that you need the properties $P(n) \wedge \neg Q(n)$ and $\neg P(n) \wedge Q(n)$ to hold for infinitely many values n in E, so n cannot be negative. Many students got the correct predicates $P(n)$, $Q(n)$, but did not fully explain why they hold for infinitely many values of n. For example, saying something like $P(n) \wedge \neg Q(n)$ holds for $n = 3, 6, 9, 12, 18, 21, 24, 27, 33, \cdots$ isn't enough. Some also had insufficient justification, e.g. how do you know that $3 \cdot 2^k$ is not divisible by 5? You should mention the Unique Factorization Theorem . $\square$