

MH1300 AY20/21 Final Exams

Solutions

Q1(a)

Let a and b be integers.

Need to prove $(a+b)^2$ is odd iff

a, b are of opposite parity.

First direction: Assume $(a+b)^2$ is odd. And a, b are of the same parity. (We want to derive a contradiction)

Since a, b are of the same parity, either a, b are both even, or a, b are both odd.

Case 1: a, b are both odd. Then, there exist integers k, l such that $a = 2k+1$ and $b = 2l+1$.

$$\begin{aligned}\text{Then, } (a+b)^2 &= (2k+1 + 2l+1)^2 \\ &= (2k+2l+2)^2 \\ &= 2(2(k+l+1))^2\end{aligned}$$

Since $2(k+l+1) \in \mathbb{Z}$, this means that $(a+b)^2$ is even, contradicting our hypothesis that it is odd.

Case 2 : a, b are both even. Then, there exist integers k', ℓ' such that $a = 2k'$ and $b = 2\ell'$.

$$\begin{aligned}\text{So, } (a+b)^2 &= (2k' + 2\ell')^2 \\ &= 2(2(k' + \ell')^2)\end{aligned}$$

Since $2(k' + \ell')^2 \in \mathbb{Z}$, this means $(a+b)^2$ is even, contradicting our hypothesis that it is odd.

In both cases, we conclude that

$(a+b)^2$ is odd $\Rightarrow a, b$ of opposite parity is true.

Second direction : Assume a, b are of opposite parity.

Then, either a is even and b is odd, or a is odd and b is even.

Case 1: a even & b is odd : let $n, m \in \mathbb{Z}$ such that

$$a = 2n \text{ and } b = 2m+1.$$

$$\begin{aligned}\text{Then, } (a+b)^2 &= (2n+2m+1)^2 \\ &= 4(n+m)^2 + 4(n+m) + 1\end{aligned}$$

$$= 2(2(n+m)^2 + 2(n+m)) + 1$$

Since $2(n+m)^2 + 2(n+m) \in \mathbb{Z}$, this means that

$(a+b)^2$ is odd.

Case 2: a odd and b even! Similar to case 1.

In each case, we conclude $(a+b)^2$ is odd.

So, a, b of opposite parity $\Rightarrow (a+b)^2$ is odd
is true.

Remarks on Q1ca) : You can use the theorems
we discuss in class like $\text{odd} \times \text{odd} = \text{odd}$
 $\text{odd} \times \text{even} = \text{even}$
etc.

instead of working directly with the
definition of even / odd.

Q1(b) Let n be an odd integer.

By the Quotient Remainder Theorem to $d=4$,

$$n = 4K, 4K+1, 4K+2 \text{ or } 4K+3 \text{ for some } K \in \mathbb{Z}.$$

Since we've assumed that n is odd, we can only have $n = 4K+1$ or $4K+3$.

Case 1: $n = 4K+1$: Then, $n^2+3 = 16K^2 + 8K + 1 + 3$
 $= 4(4K^2 + 2K + 1)$

$$\begin{aligned} n^2+7 &= 16K^2 + 8K + 1 + 7 \\ &= 8(2K^2 + K + 1) \end{aligned}$$

$$\text{So, } (n^2+3)(n^2+7) = 32(4K^2+2K+1)(2K^2+K+1)$$

$$\text{Hence, } 32 \mid (n^2+3)(n^2+7).$$

Case 2: $n = 4K+3$: Then, $n^2+3 = 16K^2 + 24K + 9 + 3$

$$= 4(4K^2 + 6K + 3)$$

$$n^2 + 7 = 16k^2 + 24k + 9 + 7$$

$$= 8(2k^2 + 3k + 2)$$

Hence, $(n^2 + 3)(n^2 + 7) = 32(4k^2 + 6k + 3)(2k^2 + 3k + 2)$

So, $32 \mid (n^2 + 3)(n^2 + 7)$.

In both cases, we obtain $(n^2 + 3)(n^2 + 7)$ is divisible by 32.

Remarks: We need to apply QRT to $d=4$. If you

merely write $n=2k+1$ by applying the definition that n is odd, you will not be able to easily show that $(n^2 + 3)(n^2 + 7)$ is div. by 32.

Q1(c)

Using truth table:

p	q	$p \rightarrow q$	$(p \rightarrow q) \leftrightarrow q$	Output ↓ $((p \rightarrow q) \leftrightarrow q) \rightarrow p$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	F
F	F	T	F	T

Since the output column contains both 'T' and 'F' values, this statement form is neither a contradiction nor a tautology.

Q2(a) This statement is false. Suppose it is true,

let x be a non zero rational number and

y be a non zero irrational number and

$$\frac{1}{x} + \frac{x}{y} = 1. \quad \text{Then, } (xy) \left(\frac{1}{x} + \frac{x}{y} \right) = xy$$

$$\text{And so } y + x^2 = xy$$

$$\text{So } (1-x)y = -x^2$$

Note that $x \neq 1$, because otherwise, $\frac{1}{x} + \frac{x}{y} = 1$ means $\frac{1}{y} = 0$, which is impossible.

$$\text{So, } 1-x \neq 0, \text{ which means } y = \frac{-x^2}{1-x}.$$

However, if x is rational, and $1-x \neq 0$, we

Conclude $\frac{-x^2}{1-x}$ is rational. Hence, y is rational,

A contradiction.

Q2b This is false. For a counter example, we need to find three sets, A, B, C such that $(A-B) \cup (A-C) \neq A - (B \cup C)$.

For example, $A = \{0, 1\}$, $B = \{0\}$, $C = \{1\}$.

Then $(A-B) \cup (A-C) = \{1\} \cup \{0\} = \{0, 1\}$
and $A - \{B \cup C\} = \{0, 1\} - \{0, 1\} = \emptyset$. not equal.

Q2c This is true. Suppose not. Then, there are integers n and m such that $3 \mid n$ and $3 \nmid m$ and $3 \mid n+m$.

Since $3 \mid n$ and $3 \mid n+m$, by a lemma in Section 4.8 of the lecture notes, we conclude $3 \mid (n+m) - n$
i.e. $3 \mid m$, a contradiction to the hypothesis.

Q3a Let $P(n)$ be the statement

$$1 \cdot 4 + 2 \cdot 7 + \dots + n(3n+1) = n(n+1)^2, \quad n \geq 1.$$

Base case: $P(1)$ is the statement

$$1 \cdot 4 = 1 \cdot (1+1)^2$$

$$\text{But } 1 \cdot 4 = 4 \text{ and } 1 \cdot (1+1)^2 = 1 \cdot 2^2 = 4.$$

So $P(1)$ is true.

Inductive step: Assume $k \geq 1$ and $P(k)$ is true.

$$\text{i.e. Assume } 1 \cdot 4 + 2 \cdot 7 + \dots + k(3k+1) = k(k+1)^2.$$

We want to show $P(k+1)$ is true.

$$\text{i.e. WTS } 1 \cdot 4 + 2 \cdot 7 + \dots + (k+1)(3(k+1)+1) = (k+1)(k+2)^2.$$

$$\text{So, } 1 \cdot 4 + 2 \cdot 7 + \dots + (k+1)(3(k+1)+1)$$

$$= [1 \cdot 4 + 2 \cdot 7 + \dots + k(3k+1)] + (3(k+1)+1)(k+1)$$

$$= (\text{By IH}) \quad k(k+1)^2 + (3(k+1)+1)(k+1)$$

$$= (k+1)(k(k+1) + 3k+4)$$

$$= (k+1)(k^2 + 4k + 4)$$

$$= (k+1)(k+2)^2 \quad \text{So, } P(k+1) \text{ is true.}$$

Q3b Let $P(n)$ be the statement

$6 \cdot 7^n - 2 \cdot 3^n$ is divisible by 4, $n \geq 1$.

Base case: $6 \cdot 7 - 2 \cdot 3$ is div by 4.

Then $6 \cdot 7 - 2 \cdot 3 = 42 - 6 = 36$, which is div by 4.

So, $P(1)$ is true.

Inductive step: Let $k \geq 1$ and assume $P(k)$.

i.e. let M be such that M is an integer and

$$4M = 6 \cdot 7^k - 2 \cdot 3^k.$$

$$\begin{aligned} \text{Consider } 6 \cdot 7^{k+1} - 2 \cdot 3^{k+1} &= 42 \cdot 7^k - 6 \cdot 3^k \\ &= 6 \cdot 7^k + 36 \cdot 7^k - 2 \cdot 3^k - 4 \cdot 3^k \\ &= (6 \cdot 7^k - 2 \cdot 3^k) + 36 \cdot 7^k - 4 \cdot 3^k \\ &= (\text{By IH}) \quad 4M + 4(9 \cdot 7^k - 3^k) \\ &= 4(M + 9 \cdot 7^k - 3^k) \end{aligned}$$

Since $M + 9 \cdot 7^k - 3^k \in \mathbb{Z}$, this means

$6 \cdot 7^{k+1} - 2 \cdot 3^{k+1}$ is div by 4, hence,

$P(k+1)$ is true.

Q4(a)

let r be a real number.

Case 1: $r \geq 0$: Then $|r| = r$ (by definition of $|r|$).
and $-r \leq 0$. So, $|-r| = -(-r)$ (by definition of $|-r|$)

Since $r = -(-r)$, So,

$$|r| = |-r|.$$

Case 2: $r < 0$: Then, $|r| = -r$.

Since $-r > 0$, So, $|-r| = -r$.

Hence, $|r| = |-r| = -r$.

In both cases, $|r| = |-r|$.

Q4(b)

let $a, b, c, d \in \mathbb{Z}$ such that $d \neq 0$ and
 $d|a$ and $d|b$ and $d \nmid c$.

Assume that there are integers x and y such
that $ax + by = c$.

Since $d|a$ and $d|b$, by lemma in Section 4.8,
 $d|ax + by$. Hence, $d|c$.

This contradicts our assumptions.

So, there cannot be integers x and y

such that $ax + by = c$.

Q 4(c)

Let B, C be sets such that
 $B \cup C = B \cap C$.

Then $B = C$.

Let $x \in B$. Then, $x \in B \cup C$. Since $B \cup C = B \cap C$,

we have $x \in B \cap C$. So, $x \in B$ and $x \in C$.

Hence, $x \in C$. This shows $B \subseteq C$.

Now let $x \in C$. Then, $x \in C \cup B$. Since $C \cup B = B \cap C$,

we have $x \in B \cap C$. So, $x \in B$ and $x \in C$.

Hence, $x \in B$. This shows $C \subseteq B$.

We conclude, by the element method, that

$B \subseteq C$ and $C \subseteq B$, hence, $B = C$.

Q 5(a) Let $A = \{\emptyset, \{\emptyset\}\}$.

Then, $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, A\}$.

$$A \times \mathcal{P}(A) = \left\{ (\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\emptyset, \{\{\emptyset\}\}), (\emptyset, A), \right. \\ \left. (\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\emptyset\}, \{\{\emptyset\}\}), \right. \\ \left. (\{\emptyset\}, A) \right\}, \text{ altogether 8 elements.}$$

Q 5(b) Let $B = \{b_1, \dots, b_k\}$.

Any equivalence relation on B needs to be reflexive,
i.e. $(b_1, b_1), (b_2, b_2), \dots, (b_k, b_k) \in R$ for any
equivalence relation R on B .

Let $R = \{(b_1, b_1), \dots, (b_k, b_k)\}$.

Then R is clearly reflexive

To see that R is symmetric,

take $(b, b') \in R$

Then, $b = b'$, and so, $(b', b) = (b, b') \in R$.

To see that R is transitive,

take (b, b') and (b', b'') in R .

Then, $b = b'$ and $b' = b''$.

So, $(b, b'') = (b, b') \in R$.

So, R is an equivalence relation on B .

Furthermore, any equivalence relation on B must have R as a subset of it, and so must have size at least k . Hence, R is the smallest equivalence relation on B .

The largest equivalence relation on B is $B \times B$, which is clearly reflexive, symmetric and transitive.

Any (equivalence) ^{binary} relation on B is a subset of $B \times B$.

Q5ccj Suppose C is a set and R is a reflexive and transitive relation on C .

$R \subseteq R \circ R$: let $(x, y) \in R$. Since R is reflexive, $(x, x) \in R$.

So, $(x, y) \in R \circ R$.

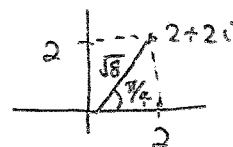
$R \circ R \subseteq R$: let $(x, y) \in R \circ R$. Then $(x, z) \in R$ and

$(z, y) \in R$ for some $z \in C$.

Since R is transitive, $(x, y) \in R$.

Q6(a)

$$z^3 - 2 - 2i = 0$$



$$z^3 = 2 + 2i$$

$$r = \sqrt{2^2 + 2^2} = 2\sqrt{2} \quad \theta = \tan^{-1} \frac{2}{2} = \frac{\pi}{4}$$

$$z^3 = 2\sqrt{2} e^{i\frac{\pi}{4}}$$

$$z = \sqrt[3]{2\sqrt{2}} e^{i\theta} \quad \text{where } \theta = \frac{\pi}{12}, \frac{9\pi}{12}, \frac{17\pi}{12}$$

$$\text{i.e. } \sqrt[3]{2\sqrt{2}} e^{i\frac{\pi}{12}}, \sqrt[3]{2\sqrt{2}} e^{i\frac{3\pi}{4}}, \sqrt[3]{2\sqrt{2}} e^{i\frac{17\pi}{12}}$$

Q6(b)

Let $f: A \rightarrow B$, $g: B \rightarrow C$.

Suppose f, g are surjective.

Let $z \in C$. Since g is surjective, let $y \in B$

such that $g(y) = z$. Since f is surjective,

let $x \in A$ such that $f(x) = y$.

Since $f(x) = y$ and $g(y) = z$, so

$$(g \circ f)(x) = g(f(x)) = g(y) = z.$$

So, $g \circ f$ is surjective.

Q6(c)

Suppose $h: D \rightarrow E$.

First direction: Suppose h is injective.

WTS: For any $X \subseteq D$, $h^{-1}(h(X)) = X$. So, we fix such
an $X \subseteq D$. $\left[\begin{array}{l} \text{Recall: } h(X) = \{ h(b) \mid b \in X \} \\ h^{-1}(Y) = \{ x \in D \mid h(x) \in Y \} \end{array} \right]$

$h^{-1}(h(X)) \subseteq X$: let $x \in h^{-1}(h(X))$. By definition

of h^{-1} , there is some $y \in h(X)$ such that

$h(x) = y$. Since $y \in h(X)$, there is some $a \in X$

such that $h(a) = y$. Since h is injective,

and $h(a) = y = h(x)$, we have $a = x$.

So, $x = a \in X$.

$X \subseteq h^{-1}(h(X))$: Let $x \in X$. Now by definition

of $h(X)$, we know $h(x) \in h(X)$.

Since $h(x) \in h(X)$, we conclude

$x \in h^{-1}(h(X))$.

We conclude that $h^{-1}(h(X)) = X$.

Second direction: Suppose that for any $X \subseteq D$,
$$h^{-1}(h(X)) = X.$$

WTS: h is injective. Fix $p, q \in D$ such that
$$h(p) = h(q).$$

Let $X = \{p\}$. Then, $X \subseteq D$ (since $p \in D$)

By assumption, $h^{-1}(h(X)) = X$.

However, note that since $h(q) = h(p)$ and
 $h(p) \in h(X)$, so, $h(q) \in h(X)$, which
means that $q \in h^{-1}(h(X))$.

Since $h^{-1}(h(X)) = X$, so

$q \in X$. Since $X = \{p\}$ this means $p = q$.

Q7(ai) Let E, F be equivalence relations on A .

$E \cap F$ is reflexive: let $a \in A$. Then $(a, a) \in E$ and $(a, a) \in F$ since E, F are reflexive. So, $(a, a) \in E \cap F$.

$E \cap F$ is symmetric: let $(a, b) \in E \cap F$. Then $(a, b) \in E$ and $(a, b) \in F$. Since E & F are symmetric, so $(b, a) \in E$ and $(b, a) \in F$. So, $(b, a) \in E \cap F$.

$E \cap F$ is transitive: let $(a, b) \in E \cap F$ and $(b, c) \in E \cap F$.

Then $(a, b) \in E$, $(b, c) \in E$, $(a, b) \in F$ and $(b, c) \in F$.

Since E and F are transitive, so $(a, c) \in E$ and $(a, c) \in F$. Hence, $(a, c) \in E \cap F$.

So, $E \cap F$ is an equivalence relation on A .

Q7a(ii)

Each equivalence class of ENF is of the form $X \cap Y$ where X is an equivalence class of E and Y is an equivalence class of F , and $X \cap Y \neq \emptyset$, and vice versa.

$$\bullet A/_{ENF} = \{ X \cap Y \mid X \in A/E \text{ and } Y \in A/F \text{ and } X \cap Y \neq \emptyset \}$$

$$\bullet \text{ For each } a \in A, [a]_{ENF} = [a]_E \cap [a]_F$$

You can describe the equivalence classes of ENF in any of the ways above.

To prove it,

we let $a \in A$ and WTS: $[a]_{ENF} = [a]_E \cap [a]_F$.

let $b \in [a]_{ENF}$. Then $(a,b) \in ENF$, and

$(a,b) \in E$ and $(a,b) \in F$, so, $b \in [a]_E \cap [a]_F$.

let $b \in [a]_E \cap [a]_F$. Then $(a,b) \in E$ and $(a,b) \in F$.

So, $(a,b) \in ENF$ and so $b \in [a]_{ENF}$.

Q7a(iii) EUF is not necessarily transitive

(although it has to be both reflexive and symmetric)

Take $A = \{0, 1, 2\}$ and E to be $\{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$

and F to be $\{(0, 0), (1, 1), (2, 2), (1, 2), (2, 1)\}$.

Then, E and F are clearly equivalence relations

on A . But $E \cup F = \{(0, 0), (1, 1), (2, 2), (1, 2), (2, 1), (0, 1), (1, 0)\}$

isn't transitive since $(0, 1) \in E \cup F$ and

$(1, 2) \in E \cup F$ and

$(0, 2) \notin E \cup F$

7b $198 = 168 \times 1 + 30$

$$168 = 30 \times 5 + 18$$

$$30 = 18 \times 1 + 12$$

$$18 = 12 \times 1 + \boxed{6}$$

$$12 = 6 \times 2$$



$$\gcd(168, 198) = 6.$$