NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2024-2025

MH1100 – Calculus I

December 2024 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **CLOSED** book exam.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1 **(16 marks)**

Evaluate the limits

(a)

$$\lim_{x\to 1^+}[\ln(x^7-1)-\ln(x^5-1)].$$

(b)

$$\lim_{x\to\infty}\left(1+\frac{2024}{x}\right)^{2x}.$$

[Solution:]

(a) Note that

$$\begin{aligned}\lim_{x\to 1^+}[\ln(x^7-1)-\ln(x^5-1)] &= \lim_{x\to 1^+}[\ln\frac{x^7-1)}{x^5-1}] \\ &= \lim_{x\to 1^+}[\ln\frac{(x-1)(x^6+...+1)}{(x-1)(x^4+...+1)}] \\ &= \ln\left(\frac{7}{5}\right).\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{2024}{x}\right)^{2x} &= \lim_{x\to\infty}\left(1+\frac{1}{x/2024}\right)^{(x/2024)*4048} \\ &= \lim_{x\to\infty}\left(1+\frac{1}{x/2024}\right)^{(x/2024)*4048} \\ &= e^{4048}.\end{aligned}$$

QUESTION 2 **(11 marks)**

Use the ϵ-δ definition to prove if we have $\lim_{x\to a}f(x)=f(a)$ and $\lim_{x\to a}g(x)=g(a)$, then we have $\lim_{x\to a}(f(x)-g(x))=f(a)-g(a)$.

[Solution:] For $\forall\varepsilon>0$, since we have $\lim_{x\to a}f(x)=f(a)$ and $\lim_{x\to a}g(x)=g(a)$, there exists δ_1, such that when $|x-a|<\delta_1$, we have

$$|f(x)-f(a)|\leq\varepsilon/2,$$

there exists δ_2, such that when $|x-a|<\delta_2$, we have

$$|g(x)-g(a)| \le \varepsilon/2.$$

We can let $\delta = \min\{\delta_1, \delta_2\}$, when $|x-a|<\delta$, we have

$$|(f(x)-g(x))-(f(a)-g(a))| = |(f(x)-f(a))-(g(x)-g(a))| \le |(f(x)-f(a))|+|(g(x)-g(a))| \le \varepsilon$$

More specifically, that is to say, $\forall \varepsilon > 0$, there exists δ. When $|x-a|<\delta$, we have

$$|(f(x)-g(x))-(f(a)-g(a))| \le \varepsilon$$

Thus $\lim\limits_{x\to a}(f(x)-g(x)) = f(a)-g(a)$.

QUESTION 3 **(16 marks)**

Suppose that $f(x)$ and $g(x)$ are continuous functions on $[a,b]$ and differentiable on (a,b). Suppose also that $f(a)=g(a)$ and $f'(x)<g'(x)$ for $a<x<b$. Prove that $f(b)<g(b)$.

[Solution:] Let $F(x)=f(x)-g(x)$. Since $f(x)$ and $g(x)$ are continuous functions on $[a,b]$ and differentiable on (a,b), the function $F(x)$ is continuous on $[a,b]$ and differentiable on (a,b).

From the mean value theorem, there exists $\xi \in (a,b)$ such that

$$\frac{F(b)-F(a)}{b-a} = F'(\xi)$$

Since $f'(x)<g'(x)$ for $a<x<b$, we have $F'(x)<0$ for $a<x<b$.
From $\frac{F(b)-F(a)}{b-a} = F'(\xi) < 0$, we have

$$F(b)-F(a)<0$$

$$(f(b)-g(b))-(f(a)-g(a))<0$$

Since $f(a)=g(a)$, we have $f(b)<g(b)$.

QUESTION 4 **(16 marks)**

If $xy + y^3 = 1$, find the value of y'' at the point where $x = 0$.

[Solution:] If $x = 0$ in $xy + y^3 = 1$, then we have $y = 1$ and the point where $x = 0$ is (0,1). From the given equation, we have

$$xy' + y + 3y^2y' = 0$$

At the point (0,1), we have $y' = -\frac{1}{3}$. We differentiate the above equation again

$$y' + xy'' + y' + 6y(y')^2 + 3y^2y'' = 0$$

At point (0,1) with $y' = -\frac{1}{3}$, we have $y'' = 0$.

QUESTION 5 **(12 marks)**

Show that the sum of the $x-$ and $y-$intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c. (Note that number c is a positive constant value.)

[Solution:] From $\sqrt{x} + \sqrt{y} = \sqrt{c}$, we have

$$\frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0$$

thus $y' = -\frac{\sqrt{y}}{\sqrt{x}}$. The equation of tangent line at (x_0, y_0) can be expressed as

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

Let $x = 0$, we have the y-intercept

$$y = y_0 + \frac{\sqrt{y_0}}{\sqrt{x_0}}x_0 = y_0 + \sqrt{y_0x_0}$$

Let $y = 0$, we have the x-intercept

$$y_0 = \frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

$$x = x_0 + \sqrt{y_0x_0}$$

The sum of the intercepts is

$$y_0 + \sqrt{y_0 x_0} + x_0 + \sqrt{y_0 x_0} = (\sqrt{x_0} + \sqrt{y_0})^2 = c$$

QUESTION 6 **(12 marks)**

Suppose that a function $f(x)$ is differentiable on $[a, b]$. Prove that there exists a number ξ such that

$$2\xi[f(b) - f(a)] = (b^2 - a^2)f'(\xi).$$

[Solution:]

Let $F(x) = (b^2 - a^2)f(x) - x^2(f(b) - f(a))$, since $f(x)$ is differentiable on $[a, b]$, $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

We have

$$F(a) = (b^2 - a^2)f(a) - a^2(f(b) - f(a)) = b^2 f(a) - a^2 f(b)$$

$$F(b) = (b^2 - a^2)f(b) - b^2(f(b) - f(a)) = b^2 f(a) - a^2 f(b)$$

From the Rolle's theorem, there exists a number $\xi \in (a, b)$ such that $F'(\xi) = 0$, that is

$$(b^2 - a^2)f'(\xi) - 2\xi(f(b) - f(a)) = 0$$

$$2\xi[f(b) - f(a)] = (b^2 - a^2)f'(\xi)$$

QUESTION 7 **(12 marks)**

Suppose that a function f is continuous on $[0, 1]$ and $f(0) = f(1)$. Prove that for any positive integer number n, there exists ξ such that

$$f(\xi + \frac{1}{n}) = f(\xi).$$

[Solution:] Let $F(x) = f(x + \frac{1}{n}) - f(x)$. Since f is continuous on $[0, 1]$, function $F(x)$ is continuous in the region of $[0, (n-1)/n]$.

Since $f(0) = f(1)$, we have

$$F(0) + F(1/n) + \cdots + F((n-1)/n) = 0$$

There are two possibilities. First, we have $F(x) = 0$ in the entire region, then we can choose ξ to be any value in the region of $[0, (n-1)/n]$. Here $F(\xi) = 0$ means $f(\xi + \frac{1}{n}) - f(\xi) = 0$, thus we have $f(\xi + \frac{1}{n}) = f(\xi)$.

Second, we have values i/n and j/n such that $F(i/n) < 0$ and $F(j/n) > 0$. Here $i \neq j$ and $i \in \{0, 1, \cdots, n-1\}, j \in \{0, 1, \cdots, n-1\}$. Since $F(x)$ is continuous in the region of $[0, (n-1)/n]$, there exists ξ in $(i/n, j/n)$ or $(j/n, i/n)$, such that $F(\xi) = 0$, which means $f(\xi + \frac{1}{n}) - f(\xi) = 0$. Thus we have $f(\xi + \frac{1}{n}) = f(\xi)$.

END OF PAPER