# MH1300 2018-19 Final Solutions

$\boxed{\text{Q1ai}}$ Disprove. Let $a = 2$, $b = \frac{1}{2}$. Then $a$ and $b$ are rational real numbers, but $a^b = a^{\frac{1}{2}} = \sqrt{2}$ is irrational.

$\boxed{\text{Q1aii}}$ Disprove. Take $x = 2$. Then $\lceil x \rceil = 2$

and $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{2} \rfloor = 1$

and $\sqrt{\lceil x \rceil} = \sqrt{2}$

$\boxed{\text{Q1b}}$ Assume that $(A-C) \cup (C-A) = (B-C) \cup (C-B)$

First show $\underline{A \subseteq B}$: Let $x \in A$. There are two cases:

<u>Case 1</u>: $x \notin C$. Then $x \in A$ and $x \notin C$. So $x \in A - C$. Hence $x \in LHS$. Therefore, $x \in (B-C) \cup (C-B)$. Since $x \notin C$ this means $x \notin C - B$. Hence $x \in B - C$. In particular, $x \in B$.

<u>Case 2</u>: $x \in C$. So, $x \in A \cap C$. We wish to conclude that $x \in B$. Suppose not. Then $x \notin B$ and $x \in C$. So, $x \in C - B$. Hence $x \in RHS$.

This means $x \in (A-C) \cup (C-A)$. Since $x \in C$, So $x \notin A-C$. So, $x \in C-A$. This is a Contradiction to our assumption that $x \in A$. So, we conclude that $x \in B$.

In either case, we conclude that $x \in B$. Hence, $A \subseteq B$.

To show $B \subseteq A$, we apply same argument.

$\boxed{Q1c}$ $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

$\equiv \neg((p \vee q) \wedge (\neg p \vee r)) \vee (q \vee r)$ [Equivalence $a \rightarrow b \equiv \neg a \vee b$]

$\equiv \neg(p \vee q) \vee \neg(\neg p \vee r) \vee (q \vee r)$ [De Morgan's Law]

$\equiv (\neg p \wedge \neg q) \vee (\neg\neg p \wedge \neg r) \vee (q \vee r)$ [De Morgan's Law]

$\equiv (\neg p \wedge \neg q) \vee (p \wedge \neg r) \vee (q \vee r)$ [Double Negation Law]

$\equiv (q \vee (\neg p \wedge \neg q)) \vee (r \vee (p \wedge \neg r))$ [Commutative and associative laws]

$\equiv ((q \vee \neg p) \wedge (q \vee \neg q)) \vee ((r \vee p) \wedge (r \vee \neg r))$ [Distributive Law]

$\equiv ((q \vee \neg p) \wedge T) \vee ((r \vee p) \wedge T)$

$\equiv (q \vee \neg p) \vee (r \vee p)$

$\equiv (p \vee \neg p) \vee (q \vee r)$ [Commutative and associative laws]

$\equiv T \vee (q \vee r)$

$\equiv T$

An answer or solution by truth tables is also acceptable.

$\boxed{Q2a}$ $\exists n \in \mathbb{Z}, \exists m \in \mathbb{Z}, \ n^2 + m^3 = 15.$

This is true. Take $n = 4$, $m = -1$.

Then $n^2 + m^3 = 4^2 + (-1)^3 = 16 - 1 = 15.$

$\boxed{Q2b}$ $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \ xy > x.$

This is false. Take $x = 0 \in \mathbb{Z}$. We wish to show

$\forall y \in \mathbb{Z} \quad xy \leq x$. For any given integer $y$,

$x \cdot y = 0 \cdot y = 0$ which is $\leq x = 0$.

So, $\forall y \in \mathbb{Z}, \ xy \leq x$ is true.

$\boxed{Q2c}$ $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, \ xy \geq x.$

This is true. Fix any $y \in \mathbb{Z}$. We pick $x = 0 \in \mathbb{Z}$.

Then $xy = 0 \geq 0 = x$. So $xy \geq x$ is true.

$\boxed{\text{Q3a}}$ Let $P(n)$ be the statement

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}.$$

$P(2)$: $\displaystyle\sum_{k=1}^{2} \frac{1}{k^2} < 2 - \frac{1}{2}$

LHS $= \frac{1}{1^2} + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{5}{4}$

RHS $= 2 - \frac{1}{2} = \frac{3}{2}$.

Obviously, $\frac{5}{4} < \frac{3}{2}$.

Assume $P(n)$ is true, $n \geq 2$. Now we show $P(n+1)$.

LHS of $P(n+1) = \displaystyle\sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2}$

$\left[\text{Apply } P(n)\right] < 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$

Note that $n.(n+1) < (n+1)^2$

and so $\frac{1}{n(n+1)} > \frac{1}{(n+1)^2}$

Since $\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$, this tells us that

$$\frac{1}{n} - \frac{1}{n+1} > \frac{1}{(n+1)^2}$$

So, $$-\frac{1}{n+1} > \frac{1}{(n+1)^2} - \frac{1}{n}$$

and $$2 - \frac{1}{n} + \frac{1}{(n+1)^2} < 2 - \frac{1}{n+1}.$$

This means $$\sum_{k=1}^{n+1} \frac{1}{k^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2} < \underbrace{2 - \frac{1}{n+1}}_{\text{RHS of } P(n+1).}$$

So, $P(n+1)$ is true.

By Math Induction, $P(n)$ true for all $n \geq 2$.

**Q3b** Let $P(n)$ be the statement

$4^{n+1} + 5^{2n-1}$ is divisible by 21.

$P(1)$: $4^{1+1} + 5^{2-1} = 4^2 + 5 = 16+5 = 21$

is divisible by 21. So the base case, $P(1)$ is true.

Assume $P(n)$ is true, $n \geq 1$. Let $m \in \mathbb{Z}$ such that

$$21m = 4^{n+1} + 5^{2n-1}.$$

We wish to prove $P(n+1)$. So, we examine the expression

$$5^{2(n+1)-1} + 4^{(n+1)+1} = 4^{n+2} + 5^{2n+1}$$

$$= 4 . 4^{n+1} + 5^2 . 5^{2n-1}$$

$$= 4 . 4^{n+1} + 21 . 5^{2n-1} + 4 . 5^{2n-1}$$

$$= 4(4^{n+1} + 5^{2n-1}) + 21 . 5^{2n-1}$$

[Apply $P(n)$] $= 4(21m) + 21 . 5^{2n-1}$

$$= 21(4m + 5^{2n-1}).$$

Since $n \geq 1$, so $2n-1 \geq 1$ so $5^{2n-1} \in \mathbb{Z}$.

Hence, $4m + 5^{2n-1} \in \mathbb{Z}$. So, $5^{2(n+1)-1} + 4^{(n+1)+1}$ is divisible by 21. So, $P(n+1)$ is true.

By math induction, $P(n)$ true for all $n \geq 1$.

$\boxed{\text{Q4a}}$ If $y$ is any real number, then $y \geq 0$.

So let $x$ be a nonzero real number.

Then, $(\underbrace{x - \tfrac{1}{x}}_{\in \mathbb{R}.})^2 \geq 0$

So, $0 \leq (x - \frac{1}{x})^2 = x^2 - 2(x)(\frac{1}{x}) + \frac{1}{x^2}$

$= x^2 - 2 + \frac{1}{x^2}$

So, $x^2 + \frac{1}{x^2} \geq 2$.

$\boxed{\text{Q4b}}$ Fix arbitrary integers $a, b$. Suppose $a, b$ have the same parity.

Case 1: $a, b$ are both even. Let $a = 2k$ and $b = 2l$ for some integers $k, l$. Let $c = k + l$.

Then $|a - c| = |2k - (k+l)| = |k - l|$

and $|b - c| = |2l - (k+l)| = |l - k|$

So, $|a - c| = |b - c|$.

Case 2: $a, b$ are both odd. Let $a = 2m+1$ and $b = 2n+1$ for some integers $m, n$. Let $c = m+n+1$

Then $|a - c| = |2m+1 - (m+n+1)| = |m - n|$

and $|b - c| = |2n+1 - (m+n+1)| = |n - m|$

So $|a - c| = |b - c|$

In either case, we conclude $|a-c| = |b-c|$ for some integer $c$.

Now assume that there is an integer $c$ such that $|a-c| = |b-c|$. Then $a-c = b-c$ or $a-c = -(b-c)$.

In the first case, $a-c = b-c$ and so $a = b$. So, $a$ and $b$ have the same parity.

In the second case, $a-c = -(b-c)$ and so $a+b = 2c$. This means $a+b$ is even. If $a$ and $b$ have different parity then $a+b =$ even + odd = odd, which cannot be. So $a$ and $b$ have the same parity.

**Q4c** By the Quotient remainder theorem,

$m = 4k,\ 4k+1,\ 4k+2$ or $4k+3$ for some $k$.

Since $m$ is odd, there are only two cases:

<u>Case 1</u> $m = 4k+1$: Then $m^2 - 1 = (4k+1)^2 - 1$

$= (16k^2 + 8k + 1) - 1 = 8(2k^2 + k)$

So, 8 divides $m^2 - 1$.

<u>Case 2</u>: $m = 4k+3$: Then $m^2 - 1 = (4k+3)^2 - 1$

$= (16k^2 + 24k + 9) - 1$

$= 16k^2 + 24k + 8$

$= 8(2k^2 + 3k + 1)$

So, 8 divides $m^2 - 1$.

In either case, $8 \mid (m^2 - 1)$ and so $m^2 \equiv 1 \pmod 8$.

$\boxed{Q5a}$ Let $f(n,m) = |n| - |m|$.

f <u>is not one-one</u>:

$$f(1,1) = |1| - |1| = 0$$
$$f(-1,-1) = |-1| - |-1| = 0$$

but $(1,1) \neq (-1,-1)$.

f <u>is onto</u>: Let $y \in \mathbb{Z}$ be given. If $y \geq 0$,

then $y = |y|$ and so $f(y,0) = |y| - |0|$
$= |y| = y$

If $y < 0$ then $y = -|y|$ and $f(0,y) = |0| - |y|$
$= y$.

$\boxed{Q5b}$ This is false. We let $f(x) = 0$ for all $x \in \mathbb{R}$,
$C_0 = \{0\}$ and $C_1 = \{1\}$.

Then $C_0 \cap C_1 = \phi$

and $f(C_0 \cap C_1) = f(\phi) = \{y \in \mathbb{R} : y = f(x) \text{ some } x \in \phi\}$
$= \phi$

and $f(C_0) = f(\{0\}) = \{0\}$

and $f(C_1) = f(\{1\}) = \{0\}$

So $f(C_0) \cap f(C_1) = \{0\}$.

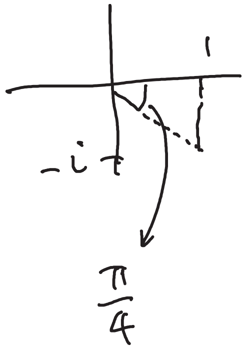
$\boxed{\text{Q5C}}$ Write down the power set of

$$\{1, \{1,2\}, \{1,2,3\}\}$$

This set has 3 elements, so the power set contains $2^3 = 8$ elements.

$$\{\phi, \{1\}, \{\{1,2\}\}, \{\{1,2,3\}\},$$
$$\{1, \{1,2\}\}, \{1, \{1,2,3\}\}, \{\{1,2\}, \{1,2,3\}\},$$
$$\{1, \{1,2\}, \{1,2,3\}\}\}$$

**Q6a**




$$\theta = \frac{7\pi}{4}$$

$$1 - i = re^{i\theta} = \sqrt{2}\, e^{i\frac{7\pi}{4}}$$

The fifth roots are $2^{\frac{1}{10}} e^{i\frac{\frac{7\pi}{4} + 2k\pi}{5}}$

It's more convenient to use $-\frac{\pi}{4}$ rather than $\frac{7\pi}{4}$

So, we can also write

$$2^{\frac{1}{10}} e^{i\left(\frac{-\frac{\pi}{4} + 2k\pi}{5}\right)} = 2^{\frac{1}{10}} e^{i\frac{-\pi + 8k\pi}{20}}, \quad k = 1, 2, 3, 4, 5$$

$\boxed{Q6b}$ $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

Let $(x,y) \in (A \cap B) \times (C \cap D)$. Then, $x \in A \cap B$ and $y \in C \cap D$. So, $x \in A$ and $y \in C$. This means $(x,y) \in A \times C$. Similarly, $x \in B$ and $y \in D$. So $(x,y) \in B \times D$. So, $(x,y) \in (A \times C) \cap (B \times D)$.

Let $(x,y) \in (A \times C) \cap (B \times D)$.

Then $(x,y) \in A \times C$ and $(x,y) \in B \times D$.

So $x \in A$ and $y \in C$, and $x \in B$ and $y \in D$

This means $x \in A$ and $x \in B$ and $y \in C$ and $y \in D$.

So, $x \in A \cap B$ and $y \in C \cap D$.

So, $(x,y) \in (A \cap B) \times (C \cap D)$

$\boxed{\text{Q6ci}}$ This is false. Let $A_0 = \phi$

$B_0 = \{0\}$

$C_0 = \{0\}$

$D_0 = \phi$.

Then $(A_0 \cup B_0) \times (C_0 \cup D_0)$

$= \{0\} \times \{0\} = \{(0,0)\}$

and $(A_0 \times C_0) \cup (B_0 \times D_0) = (\phi \times \{0\}) \cup (\{0\} \times \phi)$

$= \phi \cup \phi = \phi$, so not equal.

Alternatively, if you don't like empty set, can try another counterexample. Let $A_0$ = even integers

$B_0$ = odd integers

$C_0$ = odd integers

$D_0$ = even integers.

Then, $A_0 \cup B_0 = C_0 \cup D_0$ = set of all integers,

and $(A_0 \cup B_0) \times (C_0 \cup D_0) = \mathbb{Z} \times \mathbb{Z}$.

However, $(A_0 \times C_0) \cup (B_0 \times D_0) \neq \mathbb{Z} \times \mathbb{Z}$

because $(2,2) \in \mathbb{Z} \times \mathbb{Z}$ but

$(2,2) \notin A_0 \times C_0$ and $(2,2) \notin B_0 \times D_0$.

$\boxed{\text{Q6cii}}$ Let $R = \mathbb{Z}$. (or any set with at least two elements).

let $R = \{(0,0)\}$.

Then $R$ is symmetric and transitive.

$R \neq \phi$. However $R$ is not reflexive

since $(1,1) \notin R$.

$\boxed{Q7a}$ T is reflexive: let $x \in \mathbb{R}$. Then

$x^2 - x^2 = 0 \in \mathbb{Z}$, so $xTx$ is true.

T is symmetric: let $x, y \in \mathbb{R}$ and assume $xTy$. So, $x^2 - y^2$ is an integer.

But $y^2 - x^2 = -(x^2 - y^2)$ is also an integer.

So $yTx$ is true.

T is transitive: let $x, y, z \in \mathbb{R}$ and assume $xTy$ and $yTz$.

So $x^2 - y^2$ and $y^2 - z^2$ are both integers.

This means their sum is also an integer.

$(x^2 - y^2) + (y^2 - z^2) = x^2 - z^2$ is an integer.

So $xTz$ is true.

If $x$ and $y$ are both integers, then $x^2 - y^2$ is obviously an integer, so $xTy$ holds.

So any two integers are in the same equivalence class. So, $\mathbb{Z} \subseteq [0]_T$. Since distinct equivalence classes are disjoint, only $[0]_T$ contains an integer.

**Q7b**

$$662 = 414 \times 1 + 248$$

$$414 = 248 \times 1 + 166$$

$$248 = 166 \times 1 + 82$$

$$166 = 82 \times 2 + \boxed{2}$$

$$82 = 2 \times 41$$

So, $\gcd(662, 414) = 2$

last nonzero remainder