SPMS / Division of Mathematical Sciences

MH1300 Foundations of Mathematics
2024/2025 Semester 1

MID-TERM EXAM SOLUTIONS

QUESTION 1. **(16 marks)**

Prove that the cube of any integer is of the form $9k, 9k+1$ or $9k+8$ for some integer k.

SOLUTION . Let n be an integer. Applying the Quotient-Remainder Theorem to $d = 3$, we can assume that there is some k such that $n = 3k$ or $n = 3k+1$ or $n = 3k+2$.

$n = 3k$**:** Then $n^3 = (3k)^3 = 27k^3 = 9(3k^3)$.

$n = 3k+1$**:** Then $n^3 = (3k+1)^3 = 27k^3 + 27k^2 + 9k + 1 = 9(3k^3 + 3k^2 + k) + 1$.

$n = 3k+2$**:** Then $n^3 = (3k+2)^3 = 27k^3 + 54k^2 + 36k + 8 = 9(3k^3 + 6k^2 + 4k) + 8$.

$\square$

QUESTION 2. **(16 marks)**

Prove that for any two positive real numbers x and y,

$$\lfloor xy \rfloor \geq \lfloor x \rfloor \cdot \lfloor y \rfloor.$$

Is the above statement true for all real numbers x, y? Justify your answer.

SOLUTION . Let $x > 0$ and $y > 0$ be positive real numbers. Then note that $0 \leq \lfloor x \rfloor$ and $0 \leq \lfloor y \rfloor$ since the floor of a positive real number cannot be negative. By the definition of the floor function, we also have $\lfloor x \rfloor \leq x$ and $\lfloor y \rfloor \leq y$. Multiplying the inequalities together, we have

$$\lfloor x \rfloor \cdot \lfloor y \rfloor \leq xy,$$

and the inequality does not flip around since the terms are all non-negative.

Since $\lfloor x \rfloor \cdot \lfloor y \rfloor$ is an integer, and $\lfloor xy \rfloor$ is the largest integer less than or equal to xy, we conclude that $\lfloor x \rfloor \cdot \lfloor y \rfloor \leq \lfloor xy \rfloor$.

The statement is not true for all real numbers. For example, take $x = y = -\frac{1}{2}$. Then $\lfloor xy \rfloor = \lfloor \frac{1}{4} \rfloor = 0$, $\lfloor x \rfloor = \lfloor y \rfloor = -1$, and the inequality above is false as RHS=1 and LHS = 0. $\square$

QUESTION 3. **(16 marks)**

Determine if the following is true or false. Justify your answer.

For every three integers a, b and x, if $a \bmod 3 = b \bmod 3$ then $ax \bmod 3 = bx \bmod 3$.

Is the converse true for all integers a, b and x?

SOLUTION . The statement is true, so let us prove it. Let a, b and x be integers such that $a \bmod 3 = b \bmod 3$. This means that there are integers r, k, l such that $a = 3k + r$ and $b = 3l + r$ where $0 \leq r < 3$. Then $ax = 3kx + rx$ and $bx = 3lx + rx$. Since rx is an integer, we let $r' = rx \bmod 3$. In other words, we write $rx = 3m + r'$ for some integer m and $0 \leq r' < 3$. This means that $ax = 3kx + 3m + r' = 3(kx + m) + r'$ and $bx = 3(lx + m) + r'$. Since $0 \leq r' < 3$, we conclude that ax and bx have the same remainder when divided by 3.

The converse is false. For example, take $x = 3, a = 1, b = 2$. Then $ax = 3$ and $bx = 6$, so they both have the same remainder when divided by 3. However, $a \bmod 3 = 1$ and $b \bmod 3 = 2$. $\square$

QUESTION 4. **(18 marks)**

Prove that for any real number x, $x - \lfloor x \rfloor < \frac{1}{2}$ if and only if $\lfloor 2x \rfloor = 2\lfloor x \rfloor$.

SOLUTION . Let x be a real number such that $x - \lfloor x \rfloor < \frac{1}{2}$. By the definition of the floor of x, we have $\lfloor x \rfloor \leq x$. Therefore we have $2\lfloor x \rfloor \leq 2x$. However, from the assumption, we have $x - \lfloor x \rfloor < \frac{1}{2}$ which means $2x - 2\lfloor x \rfloor < 1$, and thus $2x < 2\lfloor x \rfloor + 1$. Together with the inequality above, we obtain $2\lfloor x \rfloor \leq 2x < 2\lfloor x \rfloor + 1$. Since $2\lfloor x \rfloor \in \mathbb{Z}$, this means that $\lfloor 2x \rfloor = 2\lfloor x \rfloor$.

Now conversely suppose that $\lfloor 2x \rfloor = 2\lfloor x \rfloor$. This means that $2\lfloor x \rfloor \leq 2x < 2\lfloor x \rfloor + 1$, by the definition of floor function. Dividing by 2, we obtain $x < \lfloor x \rfloor + \frac{1}{2}$ and so $x - \lfloor x \rfloor < \frac{1}{2}$. $\square$

QUESTION 5. **(18 marks)**

Is the following a tautology, contradiction, or neither? Justify your answer.

$$((p \vee q) \to r) \to ((p \wedge q) \to r).$$

SOLUTION . It is a tautology. You can do this with either a truth table (DIY), or with logical equivalences. We present the solution with logical equivalences.

$$\begin{aligned}
&((p \vee q) \to r) \to ((p \wedge q) \to r) && \text{(Implication rule)}\\
&\equiv (\neg(p \vee q) \vee r) \to (\neg(p \wedge q) \vee r) && \text{(Implication rule)}\\
&\equiv \neg\,(\neg(p \vee q) \vee r) \vee (\neg(p \wedge q) \vee r) && \text{(De Morgan's law)}\\
&\equiv ((p \vee q) \wedge \neg r) \vee (\neg(p \wedge q) \vee r) && \text{(De Morgan's law)}\\
&\equiv ((p \vee q) \wedge \neg r) \vee ((\neg p \vee \neg q) \vee r) && \text{(Commutative law)}\\
&\equiv ((\neg p \vee \neg q) \vee r) \vee ((p \vee q) \wedge \neg r) && \text{(Distributive law)}\\
&\equiv [((\neg p \vee \neg q) \vee r) \vee (p \vee q)] \wedge [((\neg p \vee \neg q) \vee r) \vee \neg r] && \text{(Associative and Commutative)}\\
&\equiv [(\neg p \vee p) \vee ((\neg q \vee q) \vee r)] \wedge [(\neg p \vee \neg q) \vee (r \vee \neg r)] && \text{(Negation laws)}\\
&\equiv [T \vee (T \vee r)] \wedge [(\neg p \vee \neg q) \vee T] && \text{(Universal Bound laws)}\\
&\equiv [T \vee T)] \wedge [(\neg p \vee \neg q) \vee T] && \text{(Universal Bound laws)}\\
&\equiv T \wedge T && \text{(Identity or Idempotent law)}\\
&\equiv T
\end{aligned}$$

□

QUESTION 6.

Let $\mathbb{Q}$ be the set of rational numbers and E be the set of even integers. Determine if each of the following is true or false, and justify your answers.

(a) $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ \exists z \in E,\ xyz = 3.$ **(8 marks)**

(b) $\exists x \in E,\ \exists y \in \mathbb{Q},\ \forall z \in E,\ x + yz = 4.$ **(8 marks)**

SOLUTION . (a) The property in the question states that for every rational number x, we can find some rational number y and some even integer z such that $xyz = 3$, in other words, $yz = \dfrac{3}{x}$. This is impossible if $x = 0$. So the property is false. To prove that the negation is true, we shall pick $x = 0$ (which is a rational number). Then no matter what y and z are given, $xyz = 0 \neq 3$.

(b) The question asks if you can pick an even number x and a rational number y, such that no matter what number z is given, we always have $x + yz = 4$ is true? On first glance, this may seem impossible because if we pick a value for x and y, say for example $x = 1, y = 2$, then the values of x and y are fixed numbers, and it seems impossible that $1 + 2z = 4$ can hold for all values of z.

However, this is actually possible if we choose the right value for y. If we choose $y = 0$, then the equation $x + yz = x + 0$ will become independent of z. If we also pick $x = 4$, then $x + yz = 4 + 0 = 4$ will hold for *any* value of z. Therefore, the statement is true. $\square$