Solutions to final exam
MH1300 2016/2017

$\boxed{\text{Q1(a)}}$ $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}\ \forall z \in \mathbb{R}\quad xy \leq z^2.$

True. Fix arbitrary $y \in \mathbb{R}$. Take $x = -y \in \mathbb{R}$.

Then we shall show " $\forall z \in \mathbb{R}\ \ xy \leq z^2$ ".

let z be arbitrary. Then $z^2 \geq 0$.

But $xy = (-y)(y) = -y^2 \leq 0$.

So, $xy \leq 0 \leq z^2$.

$\boxed{\text{Q1(b)}}$ Suppose x is not odd.

Then x is even. let $x = 2k$ for some $k \in \mathbb{Z}$.

Then $x^2 = 4k^2$. Since $k^2 \in \mathbb{Z}$, hence

x^2 is divisible by 4.

$\boxed{Q1(c)}$ Suppose statement is false.

Then, $x+y$ is even, y is odd and x is even,

let $x = 2k$, $y = 2l+1$ for some $k, l \in \mathbb{Z}$.

Then $x+y = 2(k+l)+1$.

So, $x+y$ is odd. Contradicting our assumption that $x+y$ is even.

$\boxed{Q1(d)}$ Let $M = 51$. Then M is an odd integer.

We now show "$\forall r \in \mathbb{R},\ r > m \Rightarrow \frac{1}{2r} < 0.01$"

Fix $r > m$. Then $r > 51$.

So $2r > 102$.

$$\frac{1}{2r} < \frac{1}{102} < 0.01$$

Q2(a) Let n be a positive integer.

By the quotient remainder theorem,

$n = 5K, \quad 5K+1, \quad 5K+2, \quad 5K+3$ or $5K+4$.

$$n(n^4-1) = n(n^2+1)(n^2-1)$$
$$= n(n-1)(n+1)(n^2+1).$$

If $n = 5K$, then n div by 5.

If $n = 5K+1$, then $n-1$ div by 5.

If $n = 5K+4$, then $n+1$ div by 5.

We only left with $5K+2$ & $5K+3$.

Case 1: $n = 5K+2$.

$$n^2+1 = 25k^2 + 20K + 4 + 1$$
$$= 5(5k^2 + 4k + 1)$$ is div by 5.

Case 2: $n = 5K+3$.

$$n^2+1 = 25K^2 + 30K + 9 + 1$$
$$= 5(5K^2 + 6K + 2)$$ is div by 5.

In any case, $n(n^4-1)$ is div by 5.

$\boxed{2b}$ Since $x+y=n$, so $x=n-y$.

We need to show $\lceil x \rceil = n - \lfloor y \rfloor$

So $\lceil n-y \rceil = n - \lfloor y \rfloor$.

Since $\lfloor y \rfloor \leq y$, so

$n - \lfloor y \rfloor \geq n-y$.

But $y < \lfloor y \rfloor + 1$, so

$n-y > n - \lfloor y \rfloor - 1$.

So,
$$n - \lfloor y \rfloor - 1 < n-y \leq n - \lfloor y \rfloor$$

So, $\lceil n-y \rceil = n - \lfloor y \rfloor$.

$\boxed{2c}$ Take $x=2$, $y=\frac{1}{2}$.

$\lceil xy \rceil = \lceil 1 \rceil = 1$

But, $\lceil x \rceil \lceil y \rceil = \lceil 2 \rceil \lceil \frac{1}{2} \rceil = 2$.

Q3 let $P(n)$:

$$3^n > n^2.$$

$P(1)$: $3^1 > 1^2$

$3 > 1$ which is true.

$P(2)$: $3^2 > 2^2$

$9 > 4$

which is true.

Assume $P(n)$ holds, i.e. assume $3^n > n^2$; $n \geq 2$.

Since $n \geq 2$, so $n^2 > 2n$ and $n^2 > 1$

So, $n^2 + n^2 > 2n+1$

So, $2n^2 > 2n+1$

So, $3n^2 > n^2 + 2n+1 = (n+1)^2$

Now, $3^{n+1} = 3 . 3^n > 3 . n^2$ (by Inductive hyp)

$> (n+1)^2$ (by above)

So $P(n+1)$ holds.

Q4a First we show $\subseteq$.

Let $x \in LHS$. Then $x = 4n$ for some $n \in \mathbb{Z}$.

So $x = 2(2n)$. So $x \in RHS$, as $2n \in \mathbb{Z}$.

Now Take $2 \in RHS$, Since $2 = 2.1$

But if $2 = 4n$ for some n, then $n = \frac{1}{2} \notin \mathbb{Z}$.

So $2 \notin LHS$.

Q4(b) let n be a positive integer.

Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$.

Then, $7n + 4 = 7(2k) + 4 = 2(7k+2)$

is even.

Now suppose n is odd. Then $n = 2m+1$ some $m \in \mathbb{Z}$

So, $7m + 4 = 7(2m+1) + 4 = 14m + 11$

$=$

$= 2(7m+5) + 1$ is odd.

$\boxed{\text{Q4(c)}}$ Assume n and $n+2$ are both perfect squares. let k and $l \in \mathbb{Z}$ s.t.

$k^2 = n$, $l^2 = n+2$. Assume $l, k > 0$.

Then, $l^2 - k^2 = 2$.

$(l+k)(l-k) = 2$.

Since 2 is prime, and $l+k > 0$, this means in particular that $l-k > 0$, and

Since $l+k > l-k$, we certainly must have $l-k = 1$ and $l+k = 2$.

So, substituting, $(1+k) + k = 2$

$\Rightarrow k = \frac{1}{2}$, Contradiction.

$\boxed{5(a)}$ f is not 1-1, let $(1,1) \neq (2,2)$

but $f(1,1) = 0 = f(2,2)$.

f is onto. Take any $y \in \mathbb{Z}$. Then $f(y,0) = y$.

$\boxed{5(b)}$ (i) let $x \in f^{-1}(B-E)$.

Then by definition, $f(x) \in B-E$.

Since $f(x) \in B$, so $x \in A$.

Now $f(x) \notin E$, so by definition of the inverse image, $x \notin f^{-1}(E)$.

So $x \in A - f^{-1}(E)$

(ii) Suppose f is 1-1 and onto, and assume $f(D) = E$.

let $x \in f^{-1}(E)$.

Then, $f(x) \in E$.

Since $E = f(D)$, so $f(x) \in f(D)$.

So, $\exists y \in D$ s.t. $f(x) = f(y)$.

Since f is 1-1, so $x = y$.

So, $x \in D$.

Hence, $f^{-1}(E) \subseteq D$.

Now take $x \in D$. Since $f(D) = E$, so $f(x) \in f(D) = E$.

So, $x \in f^{-1}(E)$.

So $D \subseteq f^{-1}(E)$. Hence, $D = f^{-1}(E)$.

Q6a Suppose R is transitive.

let $(x,y) \in R^{-1}$ and $(y,z) \in R^{-1}$.

Then $(y,x) \in R$ and $(z,y) \in R$.

Since R is transitive, so, $(z,x) \in R$.

Hence, $(x,y) \in R^{-1}$.

Hence R^{-1} is transitive.

Q6b Reflexive: let $n \geq 1$. Then, the smallest prime number dividing n is clearly equal to the smallest prime dividing n. So, $n S n$.

Suppose $n S m$. Then the smallest prime dividing n is the smallest prime dividing m.

Clearly $m S n$.

Suppose $n S m$ and $m S k$.

Then the smallest prime dividing n equals to the smallest prime dividing m, and this is equal to the smallest prime factor of k.

So, $n S k$.

The distinct classes of S are

$[p]_S$ where p is a prime number.

Then $[p]_S$ (or $p/_S$) $= \{ n \in \mathbb{Z} \mid n > 1$ and

p is the smallest prime factor of $n \}$.

$\boxed{Q7(a)}$ Suppose $A = B$. Then $A - B = A - A$
$= B - A$.

Now suppose $A - B = B - A$.

But if $A - B \neq \phi$, then let $x \in A - B$.

Then $x \in A$ and $x \notin B$. But since $x \in B - A$

hence $x \in B$ and $x \notin A$. Contradiction.

So $\underbrace{A - B = \phi}_{\text{means that } A \subseteq B} = \underbrace{B - A}_{\text{means } B \subseteq A}$.

Alternatively, Assume $A - B = B - A$ but $A \neq B$.

let $x \in A$ but $x \notin B$, without loss of generality.

Then $x \in A - B$. Since $A - B = B - A$, so $x \in B - A$.

So $x \in B$ & $x \notin A$. Contradiction.

$\boxed{Q7(b)}$

$$14038 = 1529 \cdot 9 + 277$$
$$1529 = 277 \cdot 5 + 144$$
$$277 = 144 \cdot 1 + 133$$
$$144 = 133 \cdot 1 + 11$$
$$133 = 11 \cdot 12 + 1$$

gcd.

$\boxed{\text{Q7 (c)}}$ Suppose $A \subseteq C$ and $B \subseteq D$.

let $(a,b) \in A \times B$. Then $a \in A$ and $b \in B$.

Since $A \subseteq C$ and $B \subseteq D$, so

$a \in C$ and $b \in D$.

So $(a,b) \in C \times D$.

Now suppose $A \times B \subseteq C \times D$.

let $a \in A$. Since $B \neq \phi$, fix $b_0 \in B$.

So $(a, b_0) \in A \times B$. So, $(a, b_0) \in C \times D$.

Here $a \in C$. So, $A \subseteq C$.

To show $B \subseteq D$, similar, use $A \neq \phi$:

let $b \in B$. Since $A \neq \phi$, fix $a_0 \in A$.

Then, $(a_0, b) \in A \times B$. So, $(a_0, b) \in C \times D$.

So, $b \in D$. Here, $B \subseteq D$.