SPMS / Division of Mathematical Sciences

MH1300 Foundations of Mathematics
2016/2017 Semester 1

MID-TERM EXAM SOLUTIONS

QUESTION 1. **(15 marks)**

Determine if the following pairs of statement forms are logically equivalent. If a pair is logically equivalent, prove it. If a pair is not logically equivalent, find truth values for the symbols p, q and r so that the statements forms have different truth values.

(a) Is $p \to (q \vee r) \ \equiv \ (p \wedge \neg r) \to q$?

(b) Is $p \wedge (\neg q \vee r) \ \equiv \ p \vee (q \wedge \neg r)$?

(c) Is $(p \to q) \to r \ \equiv \ p \to (q \to r)$?

SOLUTION . (a) Yes this pair is logically equivalent. You can prove this by truth tables, or by logical equivalences. We proceed by logical equivalences. It is slightly easier to start from the RHS.

$$\begin{aligned}
(p \wedge \neg r) \to q &\equiv \neg (p \wedge \neg r) \vee q && \textit{(Slide 37, Lecture 2.2)} \\
&\equiv (\neg p \vee \neg\neg r) \vee q && \textit{(De Morgan's Law)} \\
&\equiv (\neg p \vee r) \vee q && \textit{(Double Negation)} \\
&\equiv \neg p \vee (r \vee q) && \textit{(Associative Law)} \\
&\equiv \neg p \vee (q \vee r) && \textit{(Commutative Law)} \\
&\equiv p \to (q \vee r) && \textit{(Slide 37, Lecture 2.2)}
\end{aligned}$$

(b) This pair is not logically equivalent. We take p to be T, q to be T and r to be F. Then $p \wedge (\neg q \vee r)$ has truth value $T \wedge (\neg T \vee F) = T \wedge (F \vee F) = F$, while $p \vee (q \wedge \neg r)$ has truth value $T \vee (T \wedge \neg F) = T$.

(c) This pair is not logically equivalent. We take p to be F, q to be T and r to be F. Then $(p \to q) \to r$ has truth value $(F \to T) \to F = T \to F = F$. On the other hand, $p \to (q \to r)$ has truth value $F \to (T \to F) = F \to F = T$.

$\square$

QUESTION 2 **(10 marks)**

Let $P(x, y)$ be the predicate

$$x \geq y \to x^2 > y^2,$$

and the domain $D = \{-2, -1, 0, 1, 2\}$. Determine if each of the following is true. Justify your answer.

(a) $\forall x \in D\ \exists y \in D\ P(x, y)$

(b) $\forall y \in D\ \exists x \in D\ P(x, y)$

SOLUTION . (a) $\forall x \in D\ \exists y \in D\ P(x, y)$. To show that it is true, we have to go through every $x = -2, -1, 0, 1, 2$, and for each x, pick some y which works for it. We summarize this in a table:

Given x	Pick y to be	$x \geq y$	$x^2 > y^2$	$x \geq y \to x^2 > y^2$
$x = -2$	$y = 2$	F	F	T
$x = -1$	$y = 2$	F	F	T
$x = 0$	$y = 2$	F	F	T
$x = 1$	$y = 2$	F	F	T
$x = 2$	$y = 1$	T	T	T

So the statement $\forall x \in D\ \exists y \in D\ P(x, y)$ is true.

(b) $\forall y \in D\ \exists x \in D\ P(x, y)$. This is false. In other words, we have to show its negation

$$\exists y \in D,\ \forall x \in D,\quad x \geq y \text{ and } x^2 \leq y^2$$

is true. We need to pick some example $y \in D$. We pick $y = -2$. Now we need to check that our example $y = -2$ works. In other words, we need to show that

$$\forall x \in D,\quad x \geq -2 \text{ and } x^2 \leq (-2)^2.$$

But it is easy to check that for $x = -2, -1, 0, 1$ and 2, we have $x \geq -2$ and $x^2 \leq 4$.

□

QUESTION 3. **(10 marks)**

Determine if the following are true or false. Justify your answer.

(a) $\{\emptyset\} \subsetneq \{\emptyset, \{\emptyset\}\}$

(b) $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$

(c) $\{1, 2\} \subseteq \{x \in \mathbb{R} \mid x^3 - 6x^2 + 11x = 6\}$.

SOLUTION . (a) This is true, because the set on LHS contains only one element, $\emptyset$, while the set on the RHS contains two elements, $\emptyset$ and $\{\emptyset\}$. So LHS is a subset of RHS. On the other hand, LHS $\neq$ RHS since they have different number of elements.

(b) This is true, because the RHS set contains two elements, $\emptyset$ and $\{\emptyset\}$. So, $\{\emptyset\} \in$ RHS.

(c) This is true, because $1^3 - 6 \cdot 1^2 + 11 = 6$ so $1 \in \{x \in \mathbb{R} \mid x^3 - 6x^2 + 11x = 6\}$. Also $2^3 - 6 \cdot 2^2 + 11 \cdot 2 = 6$ so we also have $2 \in \{x \in \mathbb{R} \mid x^3 - 6x^2 + 11x = 6\}$. Therefore, $\{1, 2\} \subseteq \{x \in \mathbb{R} \mid x^3 - 6x^2 + 11x = 6\}$.

$\square$

QUESTION 4. **(15 marks)**

(a) Let $Q(x)$ be the predicate

$$x < \frac{1}{x}.$$

Find the truth set of $Q(x)$ for the domain $\mathbb{R}$ and the truth set of $Q(x)$ for the domain $\mathbb{Z}$. Explain your answer.

(b) Write the negation of the statement form

$$(p \to r) \leftrightarrow (q \to r)$$

without using the symbols $\to$ and $\leftrightarrow$.

SOLUTION . (a) First of all, if $x > 0$ then we can multiply x on both sides of the inequality to get $x^2 < 1$, which means $0 < x < 1$. Now if $x < 0$ then we multiply $-x$ on both sides of the inequality to get $x^2 > 1$, which means that $x < -1$. So the truth set of $Q(x)$ for domain $\mathbb{R}$ is

$$\{x \in \mathbb{R} \mid x < -1 \text{ or } 0 < x < 1\}.$$

The truth set for domain $\mathbb{Z}$ is

$$\{\cdots, -4, -3, -2\}.$$

(b) We rewrite:

$$\begin{aligned}(p \to r) \leftrightarrow (q \to r) &\equiv \Big[(p \to r) \to (q \to r)\Big] \wedge \Big[(q \to r) \to (p \to r)\Big] \\ &\equiv \Big[\neg(p \to r) \vee (q \to r)\Big] \wedge \Big[\neg(q \to r) \vee (p \to r)\Big]\end{aligned}$$

The negation of this is:

$$\begin{aligned}\neg\Big[\neg(p \to r) \vee (q \to r)\Big] \vee \neg\Big[\neg(q \to r) \vee (p \to r)\Big] &\equiv \Big[(p \to r) \wedge \neg(q \to r)\Big] \vee \Big[(q \to r) \wedge \neg(p \to r)\Big] \\ &\equiv \Big[(\neg p \vee r) \wedge (q \wedge \neg r)\Big] \vee \Big[(\neg q \vee r) \wedge (p \wedge \neg r)\Big]\end{aligned}$$

$\square$