

Problem 1 Answer

The future value FV of two bonds at horizon:

$$\begin{aligned} FV &= FV_1 + FV_2 \\ &= P_1(1+y)^D + P_2(1+y)^D \end{aligned} \quad (1)$$

where P_1 is the price of the first bond; P_2 is the price of the second bond. D is the horizon. y is the interest rate.

For each bond, its price is

$$P = \sum_{i=1}^n \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n} \quad (2)$$

where n is the term to maturity. C is the coupon payment. F is par value of the bond.

Use P in (2) to substitute P_1 and P_2 in (1):

$$\begin{aligned} FV &= \left(\sum_{i=1}^{n_1} \frac{C_1}{(1+y)^i} + \frac{F_1}{(1+y)^{n_1}} \right) (1+y)^D + \left(\sum_{i=1}^{n_2} \frac{C_2}{(1+y)^i} + \frac{F_2}{(1+y)^{n_2}} \right) (1+y)^D \\ &= \sum_{i=1}^{n_1} C_1(1+y)^{D-i} + F_1(1+y)^{D-n_1} + \sum_{i=1}^{n_2} C_2(1+y)^{D-i} + F_2(1+y)^{D-n_2} \end{aligned}$$

At horizon,

$$\frac{\partial FV}{\partial y} = \frac{\partial FV_1}{\partial y} = \frac{\partial FV_2}{\partial y} = 0 \quad (3)$$

The partial derivative FV over y :

$$\begin{aligned} \frac{\partial FV}{\partial y} &= \sum_{i=1}^{n_1} (D-i)C_1(1+y)^{D-i-1} + (D-n_1)F_1(1+y)^{D-n_1-1} \\ &\quad + \sum_{i=1}^{n_2} (D-i)C_2(1+y)^{D-i-1} + (D-n_2)F_2(1+y)^{D-n_2-1} \end{aligned} \quad (4)$$

According to (3) and (4):

$$\begin{aligned}
& \sum_{i=1}^{n_1} (D-i)C_1(1+y)^{D-i-1} + (D-n_1)F_1(1+y)^{D-n_1-1} \\
& + \sum_{i=1}^{n_2} (D-i)C_2(1+y)^{D-i-1} + (D-n_2)F_2(1+y)^{D-n_2-1} = 0 \\
(1+y)^{D-1} & \left[\sum_{i=1}^{n_1} \frac{(D-i)C_1}{(1+y)^i} + \frac{(D-n_1)F_1}{(1+y)^{n_1}} + \sum_{i=1}^{n_2} \frac{(D-i)C_2}{(1+y)^i} + \frac{(D-n_2)F_2}{(1+y)^{n_2}} \right] = 0 \\
& \sum_{i=1}^{n_1} \frac{(D-i)C_1}{(1+y)^i} + \frac{(D-n_1)F_1}{(1+y)^{n_1}} + \sum_{i=1}^{n_2} \frac{(D-i)C_2}{(1+y)^i} + \frac{(D-n_2)F_2}{(1+y)^{n_2}} = 0 \\
& \sum_{i=1}^{n_1} \left[\frac{C_1 D}{(1+y)^i} - \frac{C_1 i}{(1+y)^i} \right] + \frac{F_1 D}{(1+y)^{n_1}} - \frac{F_1 n_2}{(1+y)^{n_1}} \\
& + \sum_{i=1}^{n_2} \left[\frac{C_2 D}{(1+y)^i} - \frac{C_2 i}{(1+y)^i} \right] + \frac{F_2 D}{(1+y)^{n_2}} - \frac{F_2 n_2}{(1+y)^{n_2}} = 0 \\
& - \left[\sum_{i=1}^{n_1} \frac{C_1 i}{(1+y)^i} + \frac{F_1 n_1}{(1+y)^{n_1}} \right] + D \left[\sum_{i=1}^{n_1} \frac{C_1}{(1+y)^i} + \frac{F_1}{(1+y)^{n_1}} \right] \\
& - \left[\sum_{i=1}^{n_2} \frac{C_2 i}{(1+y)^i} + \frac{F_2 n_2}{(1+y)^{n_2}} \right] + D \left[\sum_{i=1}^{n_2} \frac{C_2}{(1+y)^i} + \frac{F_2}{(1+y)^{n_2}} \right] = 0
\end{aligned} \tag{5}$$

Macauley duration MD of a coupon bond is:

$$MD = \frac{1}{P} \sum_{i=1}^n \left[\frac{Ci}{(1+y)^i} + \frac{Fn}{(1+y)^n} \right] \tag{6}$$

substitute (6) and (2) to (5):

$$\begin{aligned}
& -P_1 D_1 + D P_1 - P_2 D_2 + D P_2 = 0 \\
& D(P_1 + P_2) = P_1 D_1 + P_2 D_2 \\
& \frac{P_1}{P_1 + P_2} D_1 + \frac{P_2}{P_1 + P_2} D_2 = D
\end{aligned}$$

Write $\frac{P_1}{P_1+P_2}$ as ω_1 and $\frac{P_2}{P_1+P_2}$ as ω_2 :

$$\omega_1 + \omega_2 = 1$$

$$\omega_1 D_1 + \omega_2 D_2 = D$$