

# Stochastic Methods + Lab

## Assignment Sheet 10

Due on November 25, 2019

### Problem 1 [6 points]

A system of linear equations of the form

$$\begin{pmatrix} b_1 & c_1 & & \cdots & 0 \\ a_1 & b_2 & c_2 & & \\ & a_2 & b_3 & \ddots & \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & & a_{n-1} & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix},$$

where the  $n \times n$  matrix on the left-hand side is tridiagonal (i.e., the matrix entries are zero except for the main diagonal and the diagonal above and below), can be solved in  $O(n)$  steps.

- (a) Write out the expressions which arise when performing Gaussian elimination on this system.
- (b) Write a tridiagonal solver as a Python function.
- (c) Scipy has a build-in **banded matrix** solver:

```
from scipy.linalg import solve_banded
```

Look up the documentation and use it to compare the result and the computing time with your tridiagonal solver from b) for the case of  $a_i = c_i = 1$  for  $i = 1, \dots, n-1$ ,  $b_i = -2$  for  $i = 1, \dots, n$ ,  $n$  large, and the right-hand side some random vector.

### Problem 2 [2 points]

Consider the Black-Scholes partial differential equation for the price  $C(S, t)$  of a European call option as a function of the current stock price  $S$  and time  $t$ ,

$$\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C = 0,$$

where  $\sigma$  is the volatility of the underlying stock and  $r$  the risk-free interest rate. Use the chain rule to show that under the change of variable  $S = \exp(X)$  and  $V(X, t) = C(S, t)$ , the Black-Scholes equation turns into the constant coefficient drift-diffusion equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial X^2} + \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial V}{\partial X} - r V = 0.$$

**Problem 3 [12 points]**

- (a) Using the conventions from Problem 2, the explicit form of the finite difference approximation to the Black-Scholes equation reads

$$\frac{V_n^m - V_n^{m-1}}{\Delta t} + \frac{\sigma^2}{2} \frac{V_{n-1}^m - 2V_n^m + V_{n+1}^m}{\Delta X^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{V_{n+1}^m - V_{n-1}^m}{2\Delta X} - rV_n^m = 0.$$

Write a code which uses the explicit finite difference scheme to price a European call option. Make sure to include a discussion of the boundary conditions.

- (b) Show that the explicit code becomes unstable unless the time step  $\Delta t$  is much smaller than  $\Delta X$ .
- (c) Modify your code to use the implicit finite difference scheme

$$\frac{V_n^{m+1} - V_n^m}{\Delta t} + \frac{\sigma^2}{2} \frac{V_{n-1}^m - 2V_n^m + V_{n+1}^m}{\Delta X^2} + \left(r - \frac{\sigma^2}{2}\right) \frac{V_{n+1}^m - V_{n-1}^m}{2\Delta X} - rV_n^m = 0.$$

Show that it is stable even when the time step  $\Delta t$  is large.

- (d) Demonstrate the order of convergence of the implicit finite difference method with the same number of meshpoints in the  $t$  and in the  $X$  direction.