Problem 1 Answer

It was enough to show for zero-coupon bond. Given points:4

The future value FV of two bonds at horizon:

$$\begin{aligned} FV &= FV_1 + FV_2 \\ &= P_1(1+y)^D + P_2(1+y)^D \end{aligned} \tag{1}$$

where P_1 is the price of the first bond; P_2 is the price of the second bond. D is the horizon. y is the interest rate.
For each bond, its price is

$$P = \sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n} \tag{2}$$

where n is the term to maturity. C is the coupon payment. F is par value of the bond.
Use P in (2) to substitute P_1 and P_2 in (1):

$$\begin{aligned} FV &= \left(\sum_{i=1}^{n_1} \frac{C_1}{(1+y)^i} + \frac{F_1}{(1+y)^{n_1}}\right)(1+y)^D + \left(\sum_{i=1}^{n_2} \frac{C_2}{(1+y)^i} + \frac{F_2}{(1+y)^{n_2}}\right)(1+y)^D \\ &= \sum_{i=1}^{n_1} C_1(1+y)^{D-i} + F_1(1+y)^{D-n_1} + \sum_{i=1}^{n_2} C_2(1+y)^{D-i} + F_2(1+y)^{D-n_2} \end{aligned}$$

At horizon,

$$\frac{\partial FV}{\partial y} = \frac{\partial FV_1}{\partial y} \overset{\boxed{+}}{=} \frac{\partial FV_2}{\partial y} = 0 \tag{3}$$

The partial derivative FV over y:

$$\begin{aligned} \frac{\partial FV}{\partial y} &= \sum_{i=1}^{n_1}(D-i)C_1(1+y)^{D-i-1} + (D-n_1)F_1(1+y)^{D-n_1-1} \\ &+ \sum_{i=1}^{n_2}(D-i)C_2(1+y)^{D-i-1} + (D-n_2)F_2(1+y)^{D-n_2-1} \end{aligned} \tag{4}$$

According to (3) and (4):

$$
\begin{aligned}
\sum_{i=1}^{n_1}(D-i)C_1(1+y)^{D-i-1}+(D-n_1)F_1(1+y)^{D-n_1-1}& \\
+\sum_{i=1}^{n_2}(D-i)C_2(1+y)^{D-i-1}+(D-n_2)F_2(1+y)^{D-n_2-1}&=0 \\
(1+y)^{D-1}\left[\sum_{i=1}^{n_1}\frac{(D-i)C_1}{(1+y)^i}+\frac{(D-n_1)F_1}{(1+y)^{n_1}}+\sum_{i=1}^{n_2}\frac{(D-i)C_2}{(1+y)^i}+\frac{(D-n_2)F_2}{(1+y)^{n_2}}\right]&=0 \\
\sum_{i=1}^{n_1}\frac{(D-i)C_1}{(1+y)^i}+\frac{(D-n_1)F_1}{(1+y)^{n_1}}+\sum_{i=1}^{n_2}\frac{(D-i)C_2}{(1+y)^i}+\frac{(D-n_2)F_2}{(1+y)^{n_2}}&=0 \\
\sum_{i=1}^{n_1}\left[\frac{C_1D}{(1+y)^i}-\frac{C_1i}{(1+y)^i}\right]+\frac{F_1D}{(1+y)^{n_1}}-\frac{F_1n_2}{(1+y)^{n_1}}& \\
+\sum_{i=1}^{n_2}\left[\frac{C_2D}{(1+y)^i}-\frac{C_2i}{(1+y)^i}\right]+\frac{F_2D}{(1+y)^{n_2}}-\frac{F_2n_2}{(1+y)^{n_2}}&=0 \\
-\left[\sum_{i=1}^{n_1}\frac{C_1i}{(1+y)^i}+\frac{F_1n_1}{(1+y)^{n_1}}\right]+D\left[\sum_{i=1}^{n_1}\frac{C_1}{(1+y)^i}+\frac{F_1}{(1+y)^{n_1}}\right]& \\
-\left[\sum_{i=1}^{n_2}\frac{C_2i}{(1+y)^i}+\frac{F_2n_2}{(1+y)^{n_2}}\right]+D\left[\sum_{i=1}^{n_2}\frac{C_2}{(1+y)^i}+\frac{F_2}{(1+y)^{n_2}}\right]&=0
\end{aligned}
\tag{5}
$$

Macaulay duration MD of a coupon bond is:

$$MD=\frac{1}{P}\sum_{i=1}^{n}\left[\frac{Ci}{(1+y)^i}+\frac{Fn}{(1+y)^n}\right] \tag{6}$$

substitute (6) and (2) to (5):

$$
\begin{aligned}
-P_1D_1+DP_1-P_2D_2+DP_2&=0 \\
D(P_1+P_2)&=P_1D_1+P_2D_2 \\
\frac{P_1}{P_1+P_2}D_1+\frac{P_2}{P_1+P_2}D_2&=D
\end{aligned}
$$

Write $\frac{P_1}{P_1+P_2}$ as ω_1 and $\frac{P_2}{P_1+P_2}$ as ω_2:

$$\omega_1+\omega_2=1$$

$$\omega_1D_1+\omega_2D_2=D$$