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對於初始能譜長波段截止的研究

Ab initio investigation on the infrared cutoff of the primordial power spectrum

林裕翔
Yu-Hsiang Lin

指導教授：鄭士康 博士
陳丕燊 博士

Advisors：Shyh-Kang Jeng, Ph.D.
Pisin Chen, Ph.D.

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本論文係林裕翔君（R95942023）在國立臺灣大學電信工程學研究所完成之碩士學位論文，於民國一百年六月十四日承下列考試委員審查通過及口試及格，特此證明

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（指導教授）

吳俊輝

系主任、所長　陳光禎　（簽名）

誌謝

平常我的話很多，現在我盡量簡短一點。

人生常和我們想像中的不一樣，沒有按照我們幻想中的方式進行，也常常並不輕鬆順利。有時候每個選擇都很困難，我也曾經因為害怕痛苦而選擇逃避。但是就像某句話說：「沒有一個想法住在你的腦袋裡是不用付房租的。」我們的每一個選擇也都會要求我們付出代價。即使是你選擇當時看起來比較不辛苦，比較有安全感的道路，你付出的代價也不會比選擇看來比較辛苦，比較有風險的道路還少。「我們可以選擇痛苦活下去，也可以讓自己變堅強。花的力氣是一樣的。」這是我親身體會過之後，很喜歡的一句話。關鍵在哪裡呢？在於，我們必須盡一切的努力讓自己誠實面對自己。別人對你的肯定、否定、讚美、批評，幫你的所作所為所做的一切解釋，都沒辦法蓋過那個唯一重要的聲音：你心裡面真正的想法。你可能會無法肯定，到底自己這麼想對不對。我覺得，如果你覺得有需要，那麼多方徵詢別人的意見是好的，因為那可以給你靈感，讓你自己知道世界上還有其他的觀點，還有其他的可能性，也給你更多的資訊進行更完整的判斷。但是無論如何，最後你必須像法庭要做出一個裁決一樣，陪審團必須要關起門來，就所有有限的資訊，在有限的時間之內做出判斷。最後是你做選擇。你必須以最誠實的方式選擇一個方向，然後去試試看它是不是真的如你所想。這是唯一知道它對你來說是對還是錯的方式。其他人的說法都沒辦法代替你去走過這條路。而且每個人都是不同的，同樣一條路，不同的人來走都有不同的感受和結果。如果它對你真的很重要，你絕對不能把這個責任丟給別人。千萬不要只是因為別人的說法或壓力就讓別人為你做決定。因為絕對沒有任何人能夠為你負責。只有你自己能夠為你自己負責。

然後你也必須誠實看待結果，如果發現它糟透了，也不要害怕改變。「你的想法並不代表你這個人。」有句話這麼說，但是可能不是很好懂。我覺得簡單來說，就是你可以改變。改變是一件很自然的事情，你完全不需要忠於任何莫名其妙的「自己」，尤其當那樣的「自己」其實根本就一直為你帶來痛苦的時候。讓你的想法幫助你成為一個越來越快樂和滿足的人。

我想這就是我現在能夠分享給你的經驗。以後的事情就要等以後才會知道了。

謝謝到目前為止所有給我啟發和幫助的人，因為有你們使得我能夠更容易地成長。下面只是很有限的名字：鄭士康老師，陳丕燊老師，陳士元老師，吳俊輝老師，聯立老師，主任，榮宏，李忠霖，陳玉潔，涂楷旻，君朋學長，阿儒，士凱，老盧，蘇育正，林澍寰，王孝武，吳映嫻，何明潔，The Fourth的大家，b92的a cappella的你們，謝翰璋學長，Christine Gruber，Fabio，Florian，莊道茂，劉彥緯，董念恩，于虔，喬萱，莊忠成，陳人豪，林祐全，以及李師父和梅門的大家，還有我的爸爸媽媽。

中文摘要

有許多文獻探討宇宙背景輻射的非等向性溫度分布中，$l=2$ 模態的強度和理論預測相比異常偏低的問題。大部分的文獻都提出和標準暴漲模型不同的預測，認為宇宙初始能譜在大尺度的部分有截止的現象。我們研究一個在暴漲期之前具有一段物質主宰時期的早期宇宙模型，並且以第一原理進行計算，檢查這樣的截止現象是否存在。

Abstract

The problem of quadrupole anomaly in the cosmic microwave background temperature anisotropy spectrum is treated in many works. Most of them provide scenarios different from the one of standard inflation and point to a cutoff of power in the primordial power spectrum at large scales. We study the scenario of the early universe with a pre-inflation matter era, and make an *ab initio* calculation to check the existence of the infrared cutoff.

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Chapter 1

Introduction

1.1 The expanding universe

The most important discovery of modern cosmology is that the universe is expanding. The linear relation between redshifts and distances of the galaxies, found by E. P. Hubble in 1929 [1], is the first observational evidence of the expansion of the universe. This discovery lays the foundation for the research on the applications of the general relativity. The Einstein equation provides the theoretical framework for the model of the universe that contains matter and radiation, interacting with gravitation, or the geometrical structure of the universe. One of the most important consequences of the general relativity is the evolutionary model of the universe, introduced by A. Friedmann in 1920s [2]. In Friedmann's model, the universe may expand or contract with time, depending on the initial conditions and the composition of the universe. After Hubble's discovery, the universe is known to be expanding currently, so that, looking backward in time, the universe must have been in a state with extremely high temperature and energy density. This is the "Big Bang" model of the universe: the universe starts as a hot and dense fireball, and expands and cools down till now (figure 1.1).

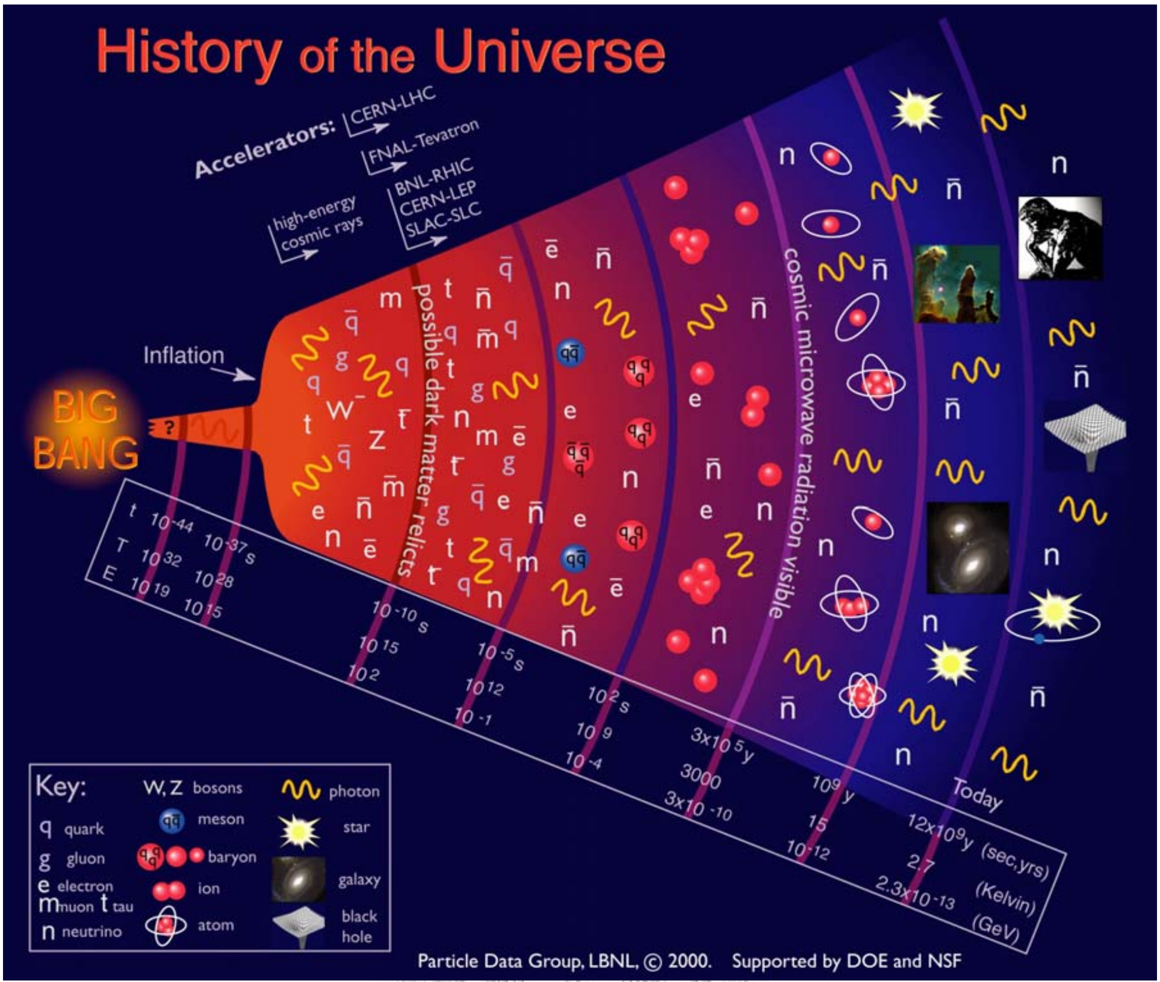


Fig 1.1 The “Big Bang” model of the universe. (Particle Data Group of Lawrence Berkeley National Laboratory, http://pdg.ge.infn.it/particleadventure/frameless/chart_print.html)

The history of the universe can be divided into two major eras: the matter-dominated era, in which most of the energy density in the universe is composed of the non-relativistic matters, such as galaxies and dark matter, and the radiation-dominated era, in which the universe is hot and the most important contributions to the energy density are from the highly relativistic particles, such as photons and neutrinos. At 1998, the observations of Type Ia supernovae made by two teams, the Supernova Cosmology Project [3] and the High-z Supernova Search Team [4], both confirm that the universe is currently accelerating. The acceleration may be caused (for reviews, see, for example, [5]) by some "vacuum energy" that does not change with position and time, behaving like the "cosmological constant," or by some dynamic field that mildly evolves, such as quintessence. It is also suggested that it may due to some modifications to the gravitation theory. Generally people call this new form of energy that causes the current acceleration of the universe as the "dark energy." The current concordance model of the composition of the universe, which is also known as the "ΛCDM" model, is: 70% dark energy, 25% dark matter, and 5% baryonic matter (see figure 1.3).

In the radiation-dominated era, the photons are tightly coupled to the baryons, and the Thomson scattering between the photons and the electrons keeps them in thermal equilibrium. (In cosmology, the term "baryons" refers to the nuclei and the electrons, which is a little bit misleading since the electrons are not baryons, but leptons. We will nevertheless follow the convention.) While the universe cools down and the temperature drops to about 3000K, the electrons and the protons combine into the neutral hydrogen atoms, a process known as "recombination," happening at about 380,000 years after the big bang. Ever since recombination, the photons decouple from the electrons and travel

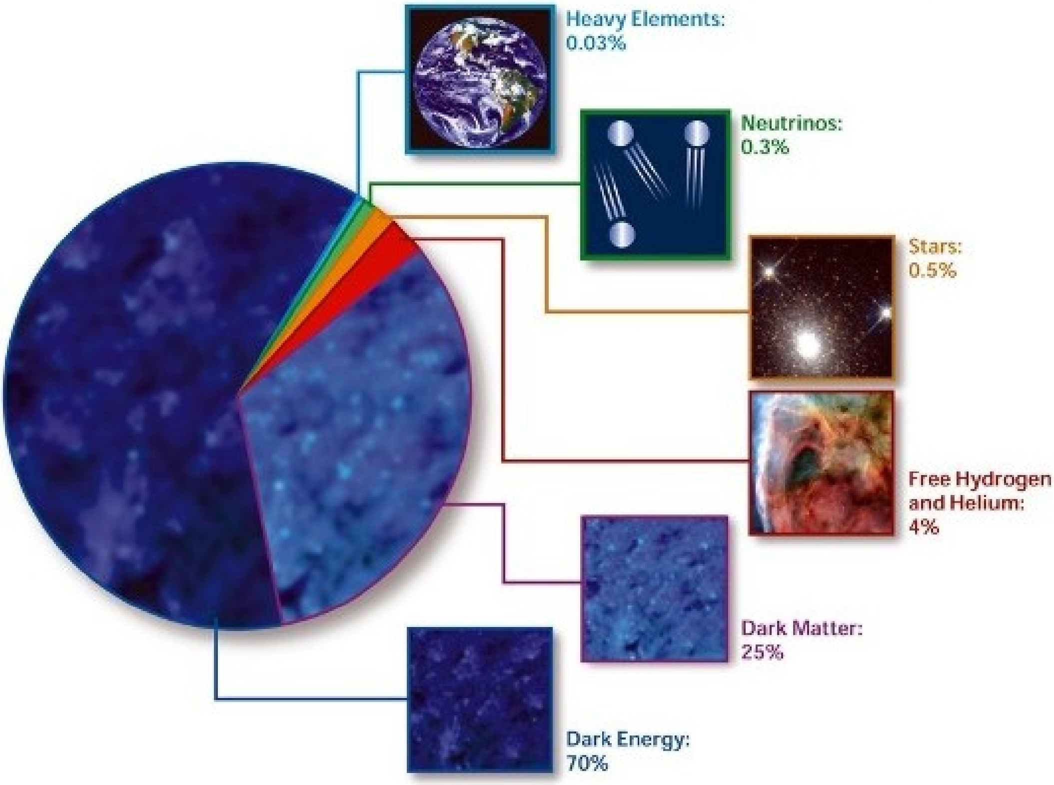


Fig 1.2 The composition of the universe. (Large Synoptic Survey Telescope, http://www.lsst.org/lsst/public/dark_energy)

freely in the transparent universe. These photons arriving at us today constitute the cosmic microwave background (CMB) we see from the antennas.

1.2 The CMB anisotropy and the quadrupole anomaly

Currently the CMB photons are observed to have an average temperature of 2.725K [6]. The fluctuations in temperature of the photons coming from different directions of the sky are at a scale of about tens of micro Kelvins, which is only about one part in 100,000 to the average temperature. This anisotropy of CMB has been accurately measured by Wilkinson Microwave Anisotropy Probe (WMAP) in the 2000s [6], shown in figure 1.3. C_l is the averaged squared coefficients of the different modes, defined as

$$\left\langle a_{lm}\, a^*_{l'm'} \right\rangle = \delta_{ll'}\delta_{mm'}C_l\,, \tag{1.1}$$

where a_{lm} is the coefficient of the temperature anisotropy expanded on the spherical harmonics,

$$\frac{\Delta T(\vec{x},\hat{p},t)}{\overline{T}(t)} = \sum_{l,m} a_{lm}(\vec{x},t) Y_l^m(\hat{p})\,, \tag{1.2}$$

with $\overline{T}$ and ΔT denoting the average temperature and the amount of temperature deviating from the average, respectively (see, for example, [7]). The observation of CMB and its anisotropies is the most important data of the modern cosmology, and also confirms the theoretical model we establish for the universe.

The physical mechanism responsible for the anisotropies of the CMB spectrum is currently best provided by the scenario of “inflation.” [8-10] Originally the notion of inflation is proposed to solve some issues of cosmology, such as the “horizon problem” and “flatness problem” [8], but it turns out to be a good candidate to explain many other phenomena, including the structure formation and CMB anisotropy. It is proposed that before the radiation-dominated era the universe goes through a period of “inflation,” in which the universe is dominated by some scalar field(s) and expands exponentially. The particles of the scalar field(s) are known as “inflatons.” They are created and annihilated constantly during the process, and leave their footprints in the gravitational field they produced. The perturbations of the gravitational field in turn arrange the distributions of the photons and other particles in a particular way, causing the anisotropy of the CMB.

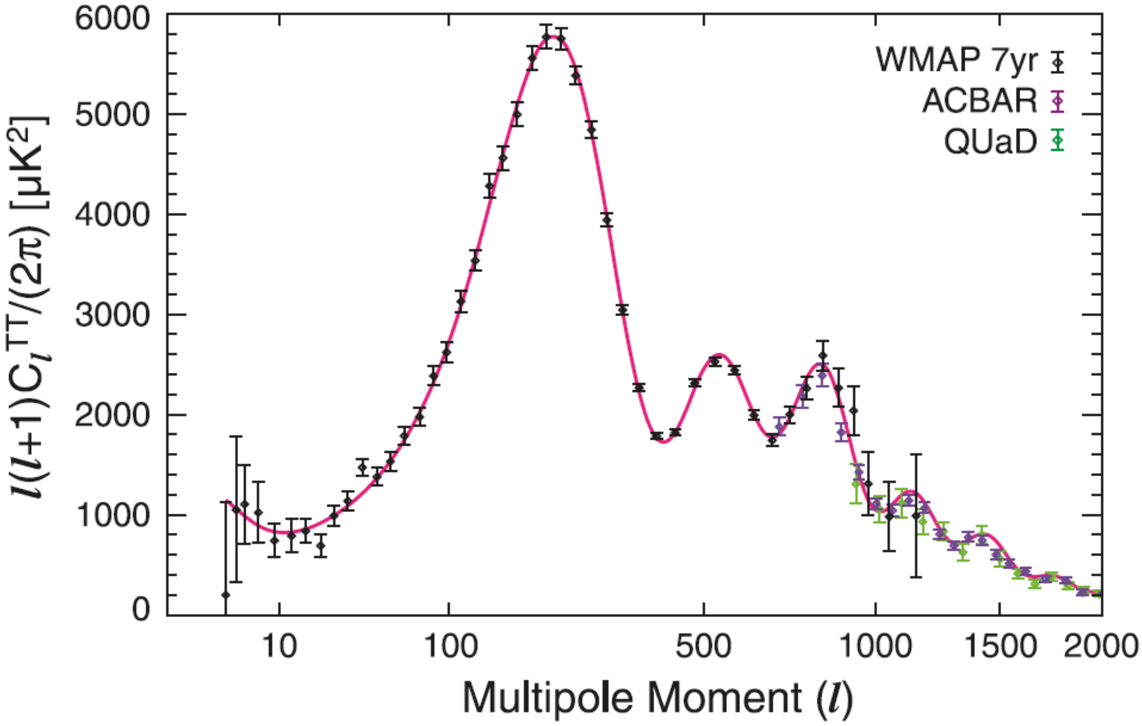


Fig 1.3 The CMB temperature anisotropy spectrum from WMAP 7-year observation [6], showing the averaged squared coefficients of the different modes of the spherical harmonics.

1.3 Proposed explanations to the quadrupole anomaly

Although the predictions made by inflation are very successful, there are still issures remain. One of them is the suppression of power to the quadrupole mode in the temperature anisotropy spectrum of CMB [11] (see figure 1.3). Many explanations have been proposed. Most of the works commonly point to a sharp drop in the primordial power spectrum for the modes whose wavelengths are at the scale of the Hubble radius today. We briefly review some of these ideas in the following paragraphs.

1.3.1 Different initial conditions

In the works of Contaldi *et al.* [12] and Donoghue *et al.* [13], they consider a period of strong kinetic domination at the beginning of the last 60-65 e-folds of inflation, with the chaotic inflationary potential. Since in chaotic inflation, there exists a wide range of initial conditions approaching to the same steady state, which is known as an "attractor" in the configuration space, it is reasonable to consider the possible effects caused by large initial velocity of the inflaton field. They show that a strong initial kinetic-dominated period can produce a cutoff at large scales of the primordial spectrum.

Powell *et al.* [14] and Wang *et al.* [15] consider a radiation-dominated period preceding the inflation epoch. They find that the altered vacuum state in the pre-inflation radiation era and the following phase transition into the inflation epoch significantly suppress the power of the primordial spectrum at large scales. Another scenario is provided in the works of Scardigli *et al.* [16] and Gruber [17], in which they introduce a pre-inflation matter era composed of the primordial micro black hole remnants, and demonstrate the resulting infrared cutoff of the primordial spectrum.

There are also works considering the general form of the initial conditions, as the ones of Sriramkumar *et al.* [18] and Boyanovsky *et al.* [19]. They study the effects caused by different forms of initial conditions, and also the inverse construction of such initial conditions from the given spectra. They point out that it is not sufficient to put any rigorous constraint on the trans-Planckian physics by CMB spectra only.

1.3.2 Different potentials

Some works resort to new forms of potential of the scalar field. In the work of Feng *et al.* [20], they consider a model of double inflation, which consists of two inflatons with different masses. They show that under some chosen parameters this model gives a smooth inflation, without interruptions, and also a power drop at the large scales of the primordial power spectrum.

Another example is the work of Jain *et al.* [21]. They show that the lack of power at large scales naturally arises in the scenario of "punctuated inflation," which is a kind of single field inflation that has a short period of rapid roll sandwiched by two stages of slow roll inflation. They also show that in this model, while the scalar power is lowered, the tensor power increases. This results in an increased tensor-to-scalar ratio at large scales of the primordial power spectra, which awaits the future confirmation or rejection from the CMB observations.

1.3.3 Reconstruction from CMB data

The feature of the infrared cutoff is also suggested by some model-independent reconstruction of the primordial power spectrum. In the work of Mukherjee *et al.* [22], they reconstruct the primordial power spectrum as a free function by two different methods, the wavelet band power method [23] and the top-hat binning method [24]. They analyze the results from the CMB data alone and the CMB data together with large-scale structure data. Either power spectrum indicates a lowered power at the large scales.

1.4 The pre-inflation matter era: issues and solutions proposed in the thesis

1.4.1 Issues in the calculation of the primordial spectrum

In the works by Scardigli *et al.* [16] and Gruber [17], a pre-inflation matter era is considered. The pre-inflation matter-dominated universe is constituted by the primordial micro black hole remnants, produced as the spontaneous formation of black holes out of the gravitational instabilities of space-time [25]. The black holes evaporate after the creation and are left as some remnants with their minimum mass restricted by the general uncertainty principle [26]. The total amount of the black hole remnants are then limited by the holographic principle [27], providing the estimation on the energy density of the pre-inflation matter era. The effect of the matter is more important to the modes of large scales, and may be irrelevant to the ones of small scales. This is because that whenever a mode exits the Hubble radius during the exponential expansion in the period of inflation, the evolution of the mode will freeze out until they re-enter the Hubble radius later after the end of inflation. Since the modes of larger scales leave the Hubble radius earlier, the effect of the pre-inflation matter era will be left on them more significantly. In the contrary, the smaller modes leave the radius at times well into the inflation, so the effect of the pre-inflation matter era has already been overwhelmed by inflation and leaves only little impact. The quadrupole moment of the CMB spectrum corresponds to modes that has just entered the Hubble radius today, and thus are the modes of larger scales that left the radius earlier during inflation. Therefore the lack of power to these modes may be explained by the effect of a pre-inflation matter era. Their conclusion points to a cutoff at large-scales in the primordial power spectrum, and also a suppression of the quadrupole moment of the CMB temperature anisotropy spectrum. However, some higher moments of the CMB spectrum are also disturbed, which is an unexpected effect.

Besides the higher-mode problem, the treatment in this work is not completely satisfactory either. Some comments are stated below:

(1) The scalar field of inflation is chosen to be a free field and massless, that is, with the potential equal to zero. A scalar field of this kind can not provide the exponential expansion as an inflation field. One therefore need to invoke another “cosmological constant” to make the universe “inflate,” which is a not so unified treatment.

(2) The perturbations of the metric tensor are not taken into account. This implants an intrinsic theoretical defect that the fluctuations of the scalar field have no way to leave their imprints to the following radiation era, and thus provide no physical mechanism for the CMB anisotropies.

(3) The primordial power spectrum under study is taken to be the one of the inflatons. It lacks physical meaning as the initial condition for the radiation era, since the power spectrum of the inflatons is significantly affected by the detailed physics at the end of inflation, which is not treated in the work.

(4) The comoving wave number k is shifted arbitrarily, and the physical wavelength of the corresponding mode is not handled properly. This can be seen in the various figures in the work, in which the values of k range from 0.1 to hundreds of Mpc^{-1}, while the k corresponds to the Hubble radius today is around 0.001 Mpc^{-1}. In fact, the modes smaller than

$k = 0.1 Mpc^{-1}$ re-enter the radius even before the time of recombination. Not surprisingly, a cutoff at such a large scale will induce the disturbance at the higher moments of the CMB spectrum.

1.4.2 *Ab initio* treatment proposed in the thesis

To understand the effects of the pre-inflation matter era from a more solid theoretical standing, we decide to find the primordial power spectrum in an *ab initio* way. First we adopt a massive scalar field responsible for the inflation, and discard the "cosmological constant" formulation. We solve the complete perturbed Einstein equations, with the perturbations to the scalar field and the metric both included, and obtain the initial conditions to the differential equations by quantizing the scalar field. The quantization is performed under some approximations; the conditions for these approximations to be valid are carefully analyzed. As for the primordial power spectrum, we use the conservative gauge-invariant variable and examine the time-evolutions of various quantities to make sure that the primordial power spectrum is not affected by the details at the end of inflation. Finally, to gain a full control to the evolution of the modes, so as to determine the suitable parameters of the scalar field and the value of k's we are going to study, we solve both the scale factors in ΛCDM model and inflation era numerically, and connect them to obtain the entire expansion history of the universe.

This enables us to have a better understanding on the behavior of the modes, when they are crossing out or in the Hubble radius, and the physical intuition about how the pre-inflation matter era affect the evolution of the modes. We then compare the results of the pure inflation case and the case with pre-inflation matter era at the end.

1.5 Chapter outline

The thesis is arranged as the following: in chapter 2 we find the scale factors in the ΛCDM model and the inflation era; in chapter 3 the cosmological perturbation theory is introduced; in chapter 4 we explain the details of the quantization of the scalar field; in chapter 5 the numerical results of the pure inflation case and the one with pre-inflation matter era are given; at the end we conclude the thesis in chapter 6.

Chapter 2

The zero-order universe

In this chapter we first introduce the scale factor in the Robertson-Walker metric and the Friedmann equations. Then we present how we calculate the scale factor in dark-energy, matter, and radiation-dominated eras (the ΛCDM model) through numerical integration of the Friedmann equations. In section 2.2 we give a brief introduction to the inflationary cosmology, and calculate the scale factor in inflation era. The calculation is performed in the Planck units, which is more suitable for the high-energetic environment in early universe.

2.1 The scale factor in ΛCDM model

2.1.1 The Robertson-Walker metric

After people found that the universe is more or less homogeneous and isotropic – at least at scales larger than 100 Mpc – physicists described the homogeneous universe by one kind of maximally symmetric metric tensor – the Robertson-Walker metric [28]. The line element of the Roberson-Walker metric is

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]. \tag{2.1}$$

We can see that this metric is not really maximally symmetric – the temporal and spatial variables are not generally on equal footing. When $a(t)$ is a constant – for simplicity we can take it to be unity – and k is zero, this metric reduces to a real maximally-symmetrized one – the Minkowsky space, in which the temporal and spatial variables are treated equally. If $a(t)$, which is generally called the "scale factor," is an incremental function of time while k kept to be zero, this metric describes a "flat Friedmann universe," in which all the galaxies sit still on the spatial coordinates – the "comoving coordinates" – and, along with the expansion of $a(t)$, get more and more away from each other. One should note that by assuming the form of $a(t)$ we have made an assumption that there exists a "synchronous hypersurface" at every moment of the history of the universe. When objects "co-move" from one moement to the next, the spatial coordinates of these objects do not change. This is why we use the term "comoving." It is the simplest model we give to describe the Hubble expansion. It is also a manifestation that although the space is symmetric, time has an absolute direction, from the past to the future – it is not symmetric in the dimension of time.

2.1.2 The Friedmann equations

The Einstein equation including the cosmological constant is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{2.2}$$

For Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

and the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = pg_{\mu\nu} + (p+\rho)u_\mu u_\nu \,, \tag{2.3}$$

where p and ρ are spatially homogeneous and are only functions of time, the Einstein equation reduces to two ordinary differential equations, known as Friedmann equations,

$$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{2.4}$$

$$\frac{1}{a}\frac{d^2 a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.5}$$

The conservation equation of energy-momentum tensor, which is not independent from Einstein equation and can be derived from it, is

$$T^{\mu}_{\nu;\mu} = 0, \tag{2.6}$$

where the summation over repeated indices is assumed throughout the text and the semicolon denotes the covariant derivative. Applying Robertson-Walker metric and the energy-momentum tensor of perfect fluid, the $\nu = 0$ component gives

$$\dot{\rho} = -3H(p+\rho) \tag{2.7}$$

while other components vanish automatically.

The constituents of the universe can be divided into two major categories, “matter” and “radiation.” “Matter” includes objects like galaxies and clusters that are composed of the baryons – the nuclei and electrons – and cold dark matter, who move at low speeds compared to the speed of light and contribute literally no pressure. “Radiation” means relativistic particles including photons and neutrinos, who move substantially in the speed of light. In this thesis we do not take into account the mass of neutrinos. The pressure of a system of relativistic particles in equilibrium is one third of its energy density,

$$p = \frac{1}{3}\rho . \qquad \text{(Relativistic)} \qquad (2.8)$$

From the conservation equation (2.7) we can deduce the evolution of the energy density ρ with the scale factor a. For matter ($p = 0$) we have

$$\rho_M = \frac{\rho_{M0} a_0^3}{a^3} \qquad \text{(Matter)} \qquad (2.9)$$

where the subscript 0 denotes the quantity at today. For radiation ($p = \frac{1}{3}\rho$) we have

$$\rho_R = \frac{\rho_{R0} a_0^4}{a^4} . \qquad \text{(Radiation)} \qquad (2.10)$$

We assume that the matter and radiation do not interact, so the energy conservation is satisfied by both constituents independently.

We note that if the universe is free of global curvature ($k = 0$) and cosmological constant, the expansion rate H and the energy density ρ is directly related,

$$H^2 = \frac{8\pi G}{3}\rho .$$

We denote the “critical density” today by

$$\rho_{cr0} = \frac{3H_0^2}{8\pi G}. \tag{2.11}$$

It is called "critical" because for energy density larger than ρ_{cr0} the universe would have positive curvature, which forms a "close" universe, and for energy density less than ρ_{cr0} the curvature would be negative, which is an "open" universe. Only when the energy density is just as large as ρ_{cr0}, the universe would be "flat."

It is customary to write the Friedmann equation in terms of the critical density *today* (be aware that the critical density ρ_{cr0} always refer to the value today),

$$H^2 = H_0{}^2\left[\frac{\Omega_{R0}a_0^4}{a^4} + \frac{\Omega_{M0}a_0^3}{a^3} + \frac{\Omega_{k0}a_0^2}{a^2} + \Omega_{\Lambda 0}\right], \tag{2.12}$$

where

$$\Omega_{R0} = \frac{\rho_{R0}}{\rho_{cr0}},\ \Omega_{M0} = \frac{\rho_{M0}}{\rho_{cr0}},\ \Omega_{k0} = -\frac{k}{H_0{}^2},\ \text{and}\ \Omega_{\Lambda 0} = \frac{\Lambda}{3H_0{}^2}. \tag{2.13}$$

Ω_{k0} and $\Omega_{\Lambda 0}$ can be viewed as the effective density parameters of curvature and the cosmological constant respectively. When calculating $a(t)$, we need only to solve the first-order differential equation (2.12). It is because that we have applied conservation equations to constrain the dependences of the energy densities on the scale factor. The price is that we need to provide two initial conditions: the scale factor today, which is customarily set to be unity, and the Hubble parameter today, which is based on the observation.

2.1.3 The numerical solution

The above equation (2.12) is then put into the numerical solver for differential equations. The basic idea is that given the few quantities observed today, one can then completely determine the whole expansion history of the universe. Therefore, what one really does is solving the differential equation backward in time. One can recognize from (2.12) that while the scale factor gets smaller, the energy densities of radiation and matter get larger, and in turn the expansion of the universe speeds up. There is a singular point at $t \to 0$, where $a \to 0$ and the energy densities as well as the expansion rate go to infinity. This singular point is what people called the "Big Bang." Since it is a mathematical singular point, one can only approach it more and more closely. In fact, by approaching the "Big Bang," from the time steps that are getting smaller, we are able to estimate the age of the universe.

Because our goal is to find the scale factor from present day, when the universe is tens of billions years old, down to about $t = 10^{-37}$ seconds after the Big Bang, when the temperature of the universe is about 10^{15} GeV, the energy scale of the grand unified theory (GUT), it is a challenge to keep accuracy for such a wide range of solution. We adopt the numerical solver provided by Mathematica[1], and develop a package[2] to solve the differential equation (2.12) down to arbitrary accuracy. The essential strategy is

1. Rewrite (2.12) in terms of $b(t) = \log\left(\frac{a(t)}{a_0}\right)$ and solve $b(t)$ instead. Note

that $a(t)$ is in the dimension of length.

2. In some unit of time, which we initially choose to be $\frac{1}{c}\frac{Mpc}{h}$, solve equation (2.12) for an trial range of time.
3. When the adaptive step size of t shrinks to essentially zero – that is, the value of it becomes smaller than the machine precision – while we proceed toward the singular point, stop the evaluation, rescale the unit of time, and solve the equation in the new unit for the following region with improved accuracy. In our case, we reduce the unit of time to 1000 times smaller when we detect a lose of precision.
4. Repeat the procedure until $a(t)$ is smaller than the goal set by the user.
5. Collect these solutions from different sections of time and normalize all the different time units into a single one, which in our case is second. Output a switch function that retrieve the value of $a(t)$ from the solution of the time section containing t quested by the user.

The reason we solve the logarithm of $a(t)$ instead of $a(t)$ itself is that it provides

---

[1]http://www.wolfram.com/

[2]A "package" is a set of program code that is organized and designed for public users. It is a structure supported by Mathematica corresponding to the "library" of some other programming languages as C/C++.

better accuracy when $a(t)$ getting 10^{-20} times smaller or further. The reason we do *not* transform t into $\log t$, regarding that t goes to extremely small as well when approaching the Big Bang at $t = 0$, is that we are actually solving the equation *backward* in time, so we do not really know when t is approaching "zero" *a priori*.

The scale factor obtained by the program is shown in figure 2.1-2.3. The parameters at the present time are: $H = 71.4\,km/s \cdot Mpc$, $\Omega_M = 0.262$, $\Omega_\Lambda = 0.738$, $T_\gamma = 2.725K$, adopted from the WMAP 7-year data [6].

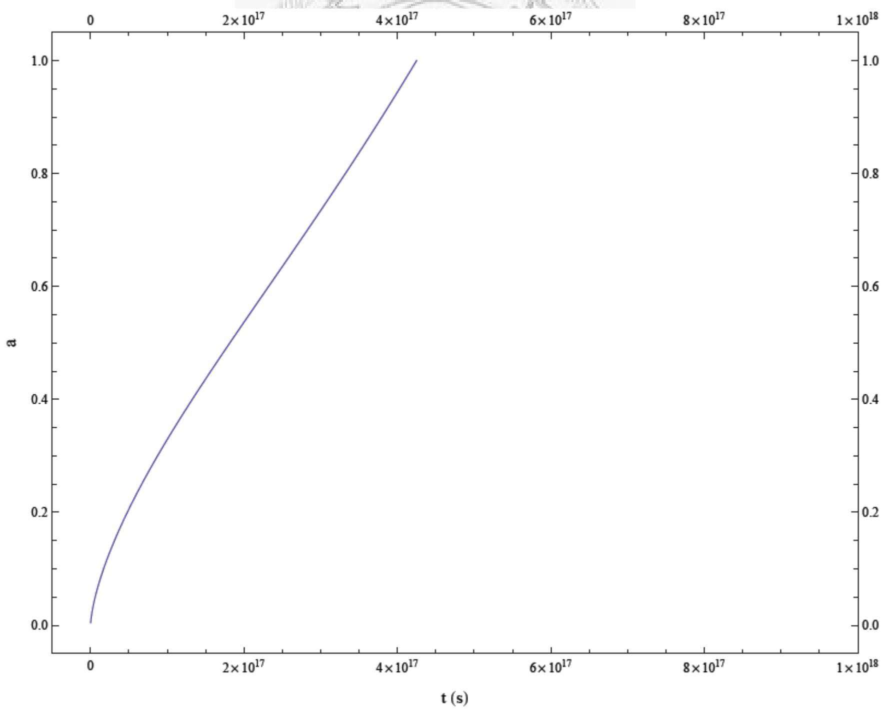


Fig 2.1 The scale factor of ΛCDM model.

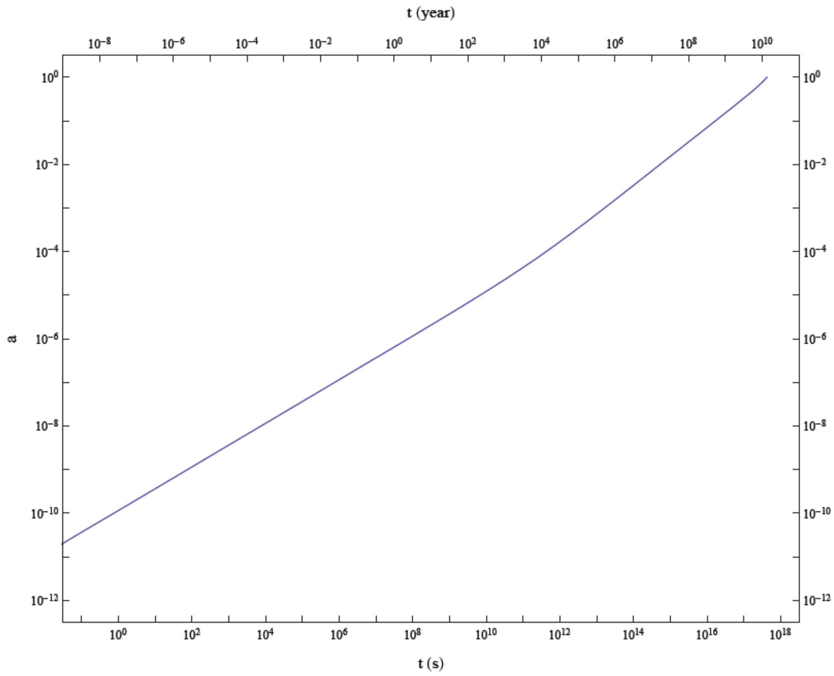


Fig 2.2 The scale factor of ΛCDM model, in log-log plot.

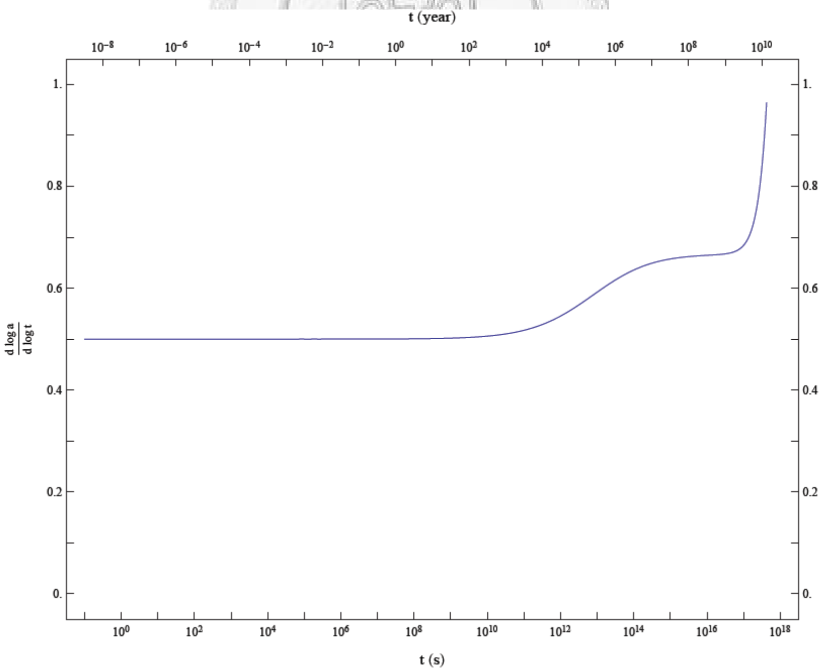


Fig 2.3 The power-law behavior of the scale factor of ΛCDM model. The transition from radiation ($a \propto t^{1/2}$) to matter ($a \propto t^{2/3}$) to dark energy ($a \propto e^{Ht}$) dominated era is clear.

2.2 The scale factor in inflation era

According to the simplest model of cosmology, the universe is created in the Big Bang, goes through a radiation dominant epoch in which all the particles in the universe are hot and relativistic, and then cools down to a matter dominant epoch till today. However, we run into some difficulties when we put this simple description together with the outcomes of the observations. It turns out that it takes some really specific conditions to make a universe as *this* one we live in, if this is the whole story. To make it more or less natural – that is, no matter how it was made initially – that the universe should be the way it is today, Alan Guth [8] in 1981 proposed a scenario of *inflationary cosmology*, which provided some plausible explanations to the questions of the cosmological evolution.

2.2.1 The horizon and flatness problems

The reason why people are in favor of the inflationary cosmology is most easily understood by considering the "horizon problem." (See, for example, [29].) When we look at the CMB temperature distribution, we find it almost perfectly homogeneous from all directions, with only tiny fluctuations in the order of 10^{-5} [6]. Is it natural that all CMB photons are so alike? We can estimate the size of causally connected region at the time when CMB photons decoupled, and see if the CMB photons we receive today were all causally connected then.

When the universe was 370,000 years old – the moment when temperature dropped below 3,000 K, so that electrons and protons combined into neutral hydrogen atoms, and CMB photons were no more scattered by free electrons – the physical particle horizon – the distance a photon can travel through without being disturbed from the beginning of the universe – was 0.258 Mpc (the number, and the numbers followed, are based on the ΛCDM model we introduced in the previous section). The CMB photons we receive today were then at a comoving distance of 14,100 Mpc from us. Knowing that the scale factor was 9.08×10^{-3} then, we get the physical distance between us and CMB photons at that time, which was 12.8 Mpc. Therefore, a distance of 0.258 Mpc spanned about 1.15° in the sky when the CMB photons "took off." This means that any CMB photons we receive today from two directions that are separated by more than 1.15° in the sky had no chance to interact with each other before they "took off," and since they didn't interact with each other on the way toward us either, they should naturally look "different" from each other. But the truth is: the temperature of CMB photons coming from *any* directions of the sky are all the same to a high accuracy. How could photons coming from portions of sky that were never in causal contact with each other be possibly all in the same temperature? This is the famous "horizon problem."

"Flatness problem" on the other hand has something to do with a more fundamental question about the universe. Recall that in the Roberson-Walker metric,

$$ds^2 = -dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{2.14}$$

we have a degree of freedom k describing the global curvature of the universe. For a flat universe, k is zero. For $k=1$ the metric describes a "close universe" with positive global curvature, and for $k=-1$ the metric describes an "open universe" with negative global curvature. It then raises a fundamental question: which kind of universe do we live in? If we write down the Einstein equation with Robertson-Walker metric, we arrive at the Friedmann equations

$$H^2 = \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{2.15}$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p). \tag{2.16}$$

It is easily seen that with different values of k we will get different solutions to $a(t)$, or different *expansion histories* of the universe. Therefore, by exploring the expansion rate of the universe at different times in history, we are able to tell the geometric structure of the universe.

Historically, various measurements and observations have been conducted to answer the question, including the observations on type Ia supernovae and CMB anisotropy [29]. All data converge to a conclusion: the universe is very close to a flat one. One might ask, what the problem is there? To see it, one needs to go back to the

Friedmann equation,

$$H^2 = \left(\frac{1}{a} \frac{da}{dt} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}, \tag{2.17}$$

and notices that it takes two quantities – the expansion rate H and the total energy density ρ – match exactly to make a flat universe. If we inspect more closely, we'll find that this match is unstable – as time passes by, any tiny discrepancy will be amplified significantly.

We can demonstrate this phenomenon by considering an open universe in a matter dominated era, in which the universe seems to be flat at some moment t_0. The Friedmann equation of an open universe is

$$H^2 = \frac{8\pi G}{3} \rho_m + \frac{1}{a^2}. \tag{2.18}$$

If at t_0 the contribution from curvature is overwhelmed by that from the matter density,

$$\frac{8\pi G}{3} \rho_{m0} >> \frac{1}{a_0^2}, \text{ or } \gamma \equiv \frac{3}{8\pi G \rho_{m0} a_0^2} << 1,$$

where subscript 0 denotes the value at t_0, the universe will look like a flat one as

$$H_0^2 = \frac{8\pi G}{3} \rho_{m0} (1+\gamma) \approx \frac{8\pi G}{3} \rho_{m0}.$$

However, since ρ_m evolves as $\frac{\rho_{m0} a_0^3}{a^3}$, which decreases much faster than $\frac{1}{a^2}$, the effect of curvature will soon become relevant. The Friedmann equation at later time reads

$$H^2 = \frac{8\pi G}{3}\rho_m + \frac{1}{a^2}$$

$$= \frac{8\pi G}{3}\rho_m\left(1 + \frac{3}{8\pi G\rho_m}\frac{1}{a^2}\right)$$

$$= \frac{8\pi G}{3}\rho_m\left(1 + \gamma\frac{a}{a_0}\right),$$

in which we can see that even γ is small, the expansion of the universe makes the ratio $\frac{a}{a_0}$ increase quickly, and the discrepancy between an open universe and a flat one becomes apparent. Therefore it is very unlikely that we are live in a special time in the history that the universe just looks like flat while it is indeed an open or close one. The more plausible conclusion should be that the universe is exactly a flat one. But since the amount of the energy density ρ and the expansion rate H are two independent quantities, how can they be so fine-tuned to match each other just right? This is the essence of the "flatness problem."

2.2.2 Inflation as a solution to the problems

Originally Guth [8] attempted to explain the horizon and flatness puzzles by the overcooling of the universe that experienced a phase transition from the "false vacuum" state into the "true vacuum" state. Guth argued that the energy density of space would stay constant in the period of overcooling. The persistent energy density, which mimics the "vacuum energy" that doesn't change with position and time, would produce the exponential expansion of the universe[3], a phenomenon named as “inflation.” Due to the radical expansion, the curvature and inhomogeneities of the universe were flattened and washed out, therefore providing the resolution to the flatness and horizon problems in one stroke.

---

[3]Here we would like to give a more intuitive explanation on why “persistent energy density” or “vacuum energy” would make the expansion of space in an “exponential form.” Let us concentrate at a volume of 1 cm^3 in a space filled with this “vacuum energy.” During the course of expansion, Assume that after 1 second the linear size of this volume doubles. Now if we take a look at any volume of 1 cm^3 inside of the original volume, it would look exactly the same as the original volume before expansion, since the energy density of the space stays constant. We can therefore conclude that the linear size of this subspace will also double in the following second. This can be understood either by the Friedmann equation or simply by symmetry. Because this argument applies to any interior region of the original volume, the linear size or the original volume will double again in the next second, making it four times large after overall 2 seconds. As the process goes, the size of the original volume will double and double again in every following seconds. This is exactly an exponential growth. The physical mechanism of this somehow miraculous exponential expansion is that the total energy of the system keeps increasing while it expands – it is the energy *density* that does not change with time. Until now there is no clear understanding on how it works.

Unfortunately, the “old-version” of inflation proposed by Guth does not work [30]. This is basically because that the overcooled “true vacuum” regions are trapped in some bubbles surrounded by the “false vacuum” background, while the background expands too fast to allow the bubbles to merge and form the present homogeneous universe. The “old inflation” was soon replaced by the “new inflation” proposed by Andrei Linde [9] and Andreas Albrecht and Paul Steinhardt [10]. The formulation was inspired by the symmetry-breaking of grand unified theory (GUT) in elementary particle physics [31], but the phenomenon of inflation could essentially be demonstrated by a single scalar field ϕ, known as the inflaton.

The basic idea is that in the very early time, when the temperature of the universe is higher than the energy scale of the GUT, there is one kind of dominant particles in the universe – the inflatons. In a stage of evolution of the inflatons, the energy density of them decreases so slowly that it stays almost constant. The persistent energy density, which mimics the “vacuum energy,” in turn causes the exponential expansion of the universe, the phenomenon known as “inflation.” At the end of inflation, these inflatons lose most of their energy and decay into the ordinary particles and radiation, a process known as “reheating.” The whole process is still a speculation, and further evidence will be revealed by the coming experiments on particle physics and astronomical observations.

2.2.3 The scalar field formulation

There have been many models of inflation investigated, including the chaotic inflation, power-law inflation, hybrid inflation, et cetera (for reviews, see, for example, [29]). In the thesis, we adopt the chaotic inflation model for its simplicity.

At the very early times, the universe was dominated by inflatons. Assuming that the universe was homogeneous and isotropic at the early times as well, the Friedmann equations were still applicable. The only extra information we need to put in is the energy-momentum tensor of the inflatons.

In the model of chaotic inflation, the inflatons are described by a single scalar field ϕ. The Lagrangian of the scalar field is

$$L = -\frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi - V(\phi), \tag{2.14}$$

where the potential we assume is that of a massive scalar field

$$V(\phi) = \frac{1}{2} m^2 \phi^2, \tag{2.15}$$

with m being the mass of the inflaton. The energy-momentum tensor is

$$\begin{aligned} T^\mu{}_\nu &= \partial_\nu \phi \frac{\partial L}{\partial(\partial_\mu \phi)} - \delta^\mu{}_\nu L \\ &= \partial^\mu \phi \partial_\nu \phi - \delta^\mu{}_\nu \left[-\frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi - V(\phi) \right]. \end{aligned} \tag{2.16}$$

Explicitly, with a homogeneous scalar field $\phi_0(t)$,

$$T^0{}_0 = -\frac{1}{2}\left(\frac{d\phi_0}{dt}\right)^2 - \frac{1}{2} m^2 \phi_0^2, \tag{2.17}$$

$$T^i{}_j = \delta^i{}_j\left[\frac{1}{2}\left(\frac{d\phi_0}{dt}\right)^2 - \frac{1}{2}m^2\phi_0^2\right]. \tag{2.18}$$

Throughout the text the Greek indices $\alpha, \beta, \gamma, \ldots$ run through 0, 1, 2, 3, and the Latin indices $i, j, k, \ldots$ run through only the spatial part 1, 2, 3, as the widely accepted convention. Compared to the general form of the energy-momentum tensor of a perfect fluid (2.3), here with mixed indices,

$$T^\mu{}_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{2.19}$$

the energy density and pressure of the scalar field are

$$\rho = \frac{1}{2}\left(\frac{d\phi_0}{dt}\right)^2 + \frac{1}{2}m^2\phi_0^2, \tag{2.20}$$

$$p = \frac{1}{2}\left(\frac{d\phi_0}{dt}\right)^2 - \frac{1}{2}m^2\phi_0^2. \tag{2.21}$$

Feeding (2.20) and (2.21) into the Friedmann equation (2.4), we have

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\left(\frac{d\phi_0}{dt}\right)^2 + \frac{1}{2}m^2\phi_0^2\right] - \frac{k}{a^2}, \tag{2.22}$$

where we have dropped the term of cosmological constant since it has negligible effect at early times. The conservation equation of energy-momentum tensor (2.6) for the inflatons gives

$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + m^2\phi_0 = 0. \tag{2.23}$$

Solving (2.22) and (2.23), we obtain the evolutions of the scale factor and the scalar field. Figure 2.4-2.7 show the results with $m = 2.04 \times 10^{-6} m_p$ and $k = 0$, and the initial conditions at $t = 10^3 t_p$ are $\phi_0 = 4.2 m_p$, $\frac{d\phi_0}{dt} = -0.001 \frac{m_p}{t_p}$, and $a = 1$. m_p and t_p are the Planck mass and the Planck time, respectively, which are defined in the following subsection.

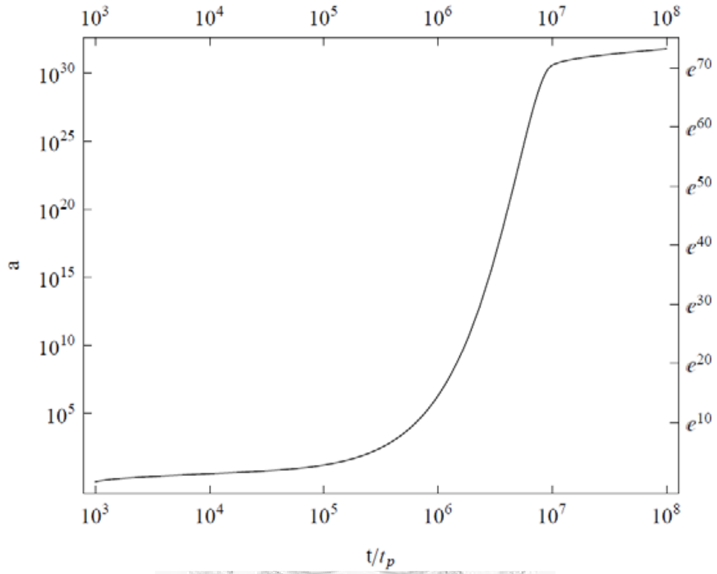


Fig 2.4 The scale factor in the inflation era.

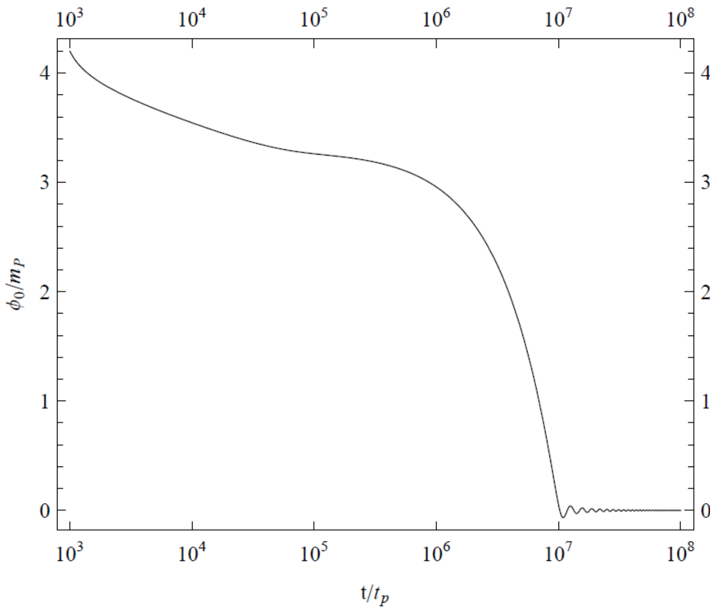


Fig 2.5 The scalar field in inflation era.

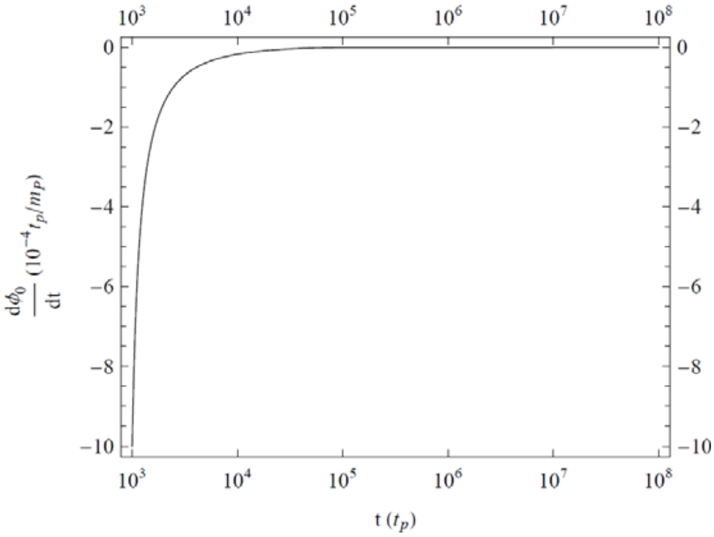


Fig 2.6 The time derivative of scalar field in inflation era.

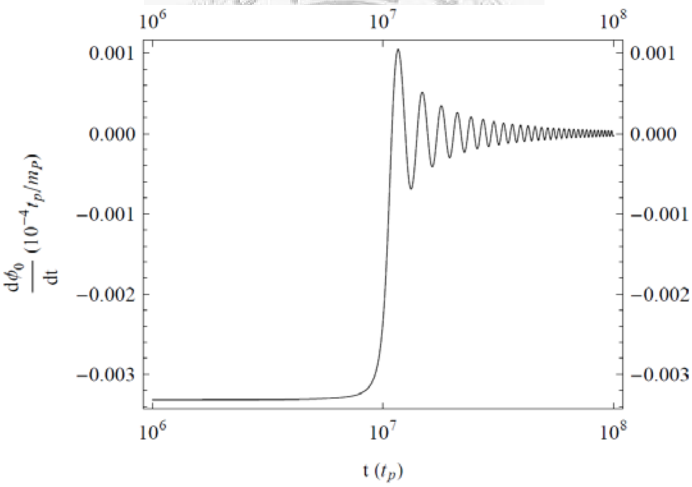


Fig 2.7 The time derivative of scalar field in inflation era, enlarged at the end of inflation.

2.2.4 The Planck units

While numerically solving for the scale factor $a(t)$ and the scalar field $\phi_0(t)$, we proceed with (b.22) and (b.23) in the Planck unit,

$$\text{unit of length } = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35}\ \text{m}, \tag{2.24}$$

$$\text{unit of time } = \sqrt{\frac{\hbar G}{c^5}} = 5.4 \times 10^{-44}\ \text{s}, \tag{2.25}$$

$$\text{unit of mass } = \sqrt{\frac{\hbar c}{G}} = 2.2 \times 10^{-8}\ \text{kg}, \tag{2.26}$$

$$\text{unit of temperature } = \frac{1}{k_B}\sqrt{\frac{\hbar c^5}{G}} = 1.4 \times 10^{32}\ \text{K}, \tag{2.27}$$

where the unit of mass is more commonly stated as the unit of energy,

$$\text{unit of energy } = \sqrt{\frac{\hbar c^5}{G}} = 10^{19}\ \text{GeV}. \tag{2.28}$$

The units are defined so that all the fundamental constants $c, \hbar, G$, and the Boltzmann constant k_B equal to 1 in this system. The units of different quantities are commonly called as “the Planck length” or “the Planck time,” et cetera.

Chapter 3

The cosmological perturbation theory in inflation era

One of the most significant success of the scenario of inflation is that it provides a physical mechanism to produce the seeds of the cosmological inhomogeneities we see today, especially the large-scale structures and the anisotropies of the cosmic microwave background. The inhomogeneities originated from the quantum fluctuations of the inflaton field, and along with the expansion of the universe, the tiny fluctuations were stretched to the cosmological scales we observed today. The equations governing the evolution of the fluctuations are the perturbed Einstein equations. In this chapter we introduce the perturbation theory of cosmology, which includes the perturbations of the metric tensor, the introduction of the concept of gauge, and the perturbations to the energy-momentum tensor, described in section 3.1 through 3.3. The overall perturbed Einstein equations are described in section 3.4.

3.1 The metric perturbations

The perfectly homogeneous flat Friedmann universe can be described by the

Robertson-Walker metric,

$$\bar{g}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix}, \tag{3.1}$$

where the over-bar denotes the unperturbed value. If there are some slight inhomogenieties in the gravitational sources, they will induce perturbations to the metric. The perturbations are structured in a way of "3+1 splitting" of spacetime [22], in which the metric tensor is viewed as consisting of 3 groups of elements: one time-time component, six space-time components, and nine space-space components. They look like a scalar, a vector – or one row vector and one column vector; their components are identical since the metric tensor is symmetric – and a tensor, respectively. The perturbations are classified as scalar, vector, and tensor perturbations as well, but *not* in the same way as the apparent structure. The time-time component is the simplest one:

$$\delta g_{00} = -E\ . \tag{3.2}$$

The space-time components, $\delta g_{0i} = \delta g_{i0}$, are further decomposed into a longitudinal (curless) and a transverse (divergenceless) part, satisfying

$$\delta g_{0i} = a\left[\frac{\partial F}{\partial x^i} + G_i\right], \tag{3.3}$$

in which

$$\frac{\partial G_i}{\partial x^i} = 0\ . \tag{3.4}$$

The curless and divergenceless vectors are just like the static electric and magnetic fields in electromagnetism, which can be derived by the scalar and vector potentials. The space-space components are also decomposed into longitudinal, solenoidal, and transverse parts in the way much alike:

$$\delta g_{ij} = a^2\left[A\delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} + \frac{\partial C_i}{\partial x^j} + \frac{\partial C_j}{\partial x^i} + D_{ij} \right], \tag{3.5}$$

where

$$\frac{\partial C_i}{\partial x^i} = 0\,,\ \ \frac{\partial D_{ij}}{\partial x^i} = \frac{\partial D_{ji}}{\partial x^i} = 0\,,\ \ D_{ii} = 0\,. \tag{3.6}$$

The perturbations A, B, E, and F are the *scalar* perturbations, C_i and G_i are the *vector* perturbations, and D_{ij} are the *tensor* perturbations.

The scalar perturbations are the most relevant ones because they are coupled with the energy density and pressure inhomogeneities that express themselves in the structure formation of the universe and the cosmic microwave background anisotropies. However, they are also mathematically the most complicated ones. In order to handle the mathematics in a more efficient and conceptually direct way, we have developed a symbolic computation system to facilitate the calculations involving tensors and perturbations.

The vector perturbations are related to the rotations of the cosmological fluid, and the tensor perturbations describe the gravitational waves. All these three classes of perturbations are decoupled with each other, therefore while we study any one of them we can neglect the effects from the others. In this thesis we will focus on the scalar perturbations.

3.2 The infinitesimal transformations and the gauges

3.2.1 The infinitesimal transformation of coordinate and the metric perturbation

The perturbations to the metric suffer from some “gauge problems” under coordinate transformations [33, 34]. To appreciate this, imagine that we are living in a perfectly homogeneous universe. In the unperturbed Robertson-Walker metric, we see homogeneous energy density throughout the space at any single moment. However, if we change the coordinates a little bit – for example, we adjust the clocks in the coordinate system so that at some places the time is a little bit faster than that of us and at some other places slower – we will find that the observers at different places do not see the same energy density “at the same time” anymore. In this coordinate system, we then detect the apparent “inhomogenieties.” The inhomogenieties are in fact just the side effects of the coordinate transformation, and are possible to be removed by another change of coordinates.

These “fictitious” perturbations need to be distinguished from the “real” or “physical” perturbations. One way to remove the degrees of freedom induced by the coordinate transformations is by choosing a “gauge.” Another way to extract the “physical” perturbations independent of the choices of coordinate systems is by adopting the “gauge-invariant variables.” [34] Here we are going to examine the coordinate transformations and the gauges of the scalar perturbations.

Consider the infinitesimal coordinate transformation

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \delta x^{\mu}(x^0, x^1, x^2, x^3) . \qquad (3.7)$$

Inversely, it reads

$$x^{\mu} = x'^{\mu} - \delta x^{\mu}(x^0, x^1, x^2, x^3) . \qquad (3.8)$$

We explicitly write out the dependences between two sets of coordinates to make the following analysis more clear.

The infinitesimal translation of the 4-coordinate can also be put into the “3+1 splitting” form:

$$\delta x^0(x^0, x^1, x^2, x^3) = -\delta x_0(x^0, x^1, x^2, x^3) , \qquad (3.9)$$

$$\delta x^i(x^0, x^1, x^2, x^3) = \frac{1}{a^2}\delta x_i(x^0, x^1, x^2, x^3) , \qquad (3.10)$$

where the indices are raised and lowered with the unperturbed Robertson-Walker metric, and the spatial components are decomposed into a curless and a divergenceless part:

$$\delta x_i = \partial_i \xi + \sigma_i , \qquad (3.11)$$

with

$$\frac{\partial \sigma_i}{\partial x^i} = 0 . \qquad (3.12)$$

Here again we consider only the scalar part:

$$\delta x_i = \partial_i \xi \, . \qquad (3.13)$$

Next we consider the perturbations to the metric tensor. We express the metric perturbations in the general way:

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x^0) + \delta g_{\mu\nu}(x^0, x^1, x^2, x^3) \, . \qquad (3.14)$$

The metric tensor transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \, . \qquad (3.15)$$

Using (3.8) and (3.14), (3.15) can be expressed as

$$\begin{aligned} & g'_{\mu\nu}(x') \\ &= \left[\delta^\alpha_\mu - \frac{\partial}{\partial x'^\mu} \delta x^\alpha (x^0, x^1, x^2, x^3) \right] \left[\delta^\beta_\nu - \frac{\partial}{\partial x'^\nu} \delta x^\beta (x^0, x^1, x^2, x^3) \right] \left[\bar{g}_{\alpha\beta}(x^0) + \delta g_{\alpha\beta}(x^0, x^1, x^2, x^3) \right] \\ &= \bar{g}_{\mu\nu}(x^0) + \delta g_{\mu\nu}(x^0, x^1, x^2, x^3) - \frac{\partial \delta x^\alpha (x^0, x^1, x^2, x^3)}{\partial x'^\mu} \bar{g}_{\alpha\nu}(x^0) - \frac{\partial \delta x^\beta (x^0, x^1, x^2, x^3)}{\partial x'^\nu} \bar{g}_{\mu\beta}(x^0). \end{aligned}$$

This can be further simplified by noticing that

$$\begin{aligned} \frac{\partial \delta x^\alpha (x^0, x^1, x^2, x^3)}{\partial x'^\mu} &= \frac{\partial \delta x^\alpha (x^0, x^1, x^2, x^3)}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\mu} \\ &= \frac{\partial \delta x^\alpha (x^0, x^1, x^2, x^3)}{\partial x^\sigma} \left[\delta^\sigma_\mu - \frac{\partial \delta x^\sigma (x^0, x^1, x^2, x^3)}{\partial x'^\mu} \right] \\ &= \frac{\partial \delta x^\alpha (x^0, x^1, x^2, x^3)}{\partial x^\mu} \, . \end{aligned}$$

At the second equality we have recursively expanded the derivative to the first order, and at the final result we keep only the first-order term. It is a useful general principle that when manipulating with a first-order quantity, one needs only to keep the zero-order parts for the rest of the expression, since any higher-order contributions from the rest of the expression will be at least second-order after multiplied by the first-order quantity. The transformed metric tensor can now be expressed totally in the unprimed coordinates:

$$g'_{\mu\nu}(x') = \bar{g}_{\mu\nu}(x^0) + \delta g_{\mu\nu}(x^0, x^1, x^2, x^3)$$
$$- \frac{\partial \delta x^\alpha (x^0, x^1, x^2, x^3)}{\partial x^\mu} \bar{g}_{\alpha\nu}(x^0) - \frac{\partial \delta x^\beta (x^0, x^1, x^2, x^3)}{\partial x^\nu} \bar{g}_{\mu\beta}(x^0). \quad (3.16)$$

On the other hand, we can also decompose the transformed metric tensor in the *primed* coordinate system:

$$g'_{\mu\nu}(x') = \bar{g}_{\mu\nu}(x'^0) + \delta g'_{\mu\nu}(x'^0, x'^1, x'^2, x'^3)\,. \quad (3.17)$$

The tricky part is that the unperturbed metric tensor is taken to be of the same functional form as that in the unprimed system; that is, instead of writing it as $\bar{g}'_{\mu\nu}(x'^0)$, we take it to be $\bar{g}_{\mu\nu}(x'^0)$ in much the same way in the unprimed system, with only x^0 replaced by x'^0. This make things a little bit more complicated at the beginning, but also leaves some advantages later, as we will see shortly.

We want to find the relation between the metric perturbations in the primed system, $\delta g'_{\mu\nu}$, and that in the unprimed system, $\delta g_{\mu\nu}$. To do so, we need to further expand $\bar{g}_{\mu\nu}(x'^0)$ in (3.17) into

$$\bar{g}_{\mu\nu}(x'^0) = \bar{g}_{\mu\nu}(x^0 + \delta x^0(x^0, x^1, x^2, x^3))$$
$$= \bar{g}_{\mu\nu}(x^0) + \frac{\partial \bar{g}_{\mu\nu}(x^0)}{\partial x^0} \delta x^0 (x^0, x^1, x^2, x^3)\,.$$

Equation (3.17) can then be put into

$$g'_{\mu\nu}(x') = \bar{g}_{\mu\nu}(x^0) + \delta g'_{\mu\nu}(x'^0, x'^1, x'^2, x'^3) + \frac{\partial \bar{g}_{\mu\nu}(x^0)}{\partial x^0} \delta x^0 (x^0, x^1, x^2, x^3)\,. \quad (3.18)$$

Comparing (3.16) and (3.18), we obtain the desired result,

$$\delta g'_{\mu\nu}(x'^0, x'^1, x'^2, x'^3) = \delta g_{\mu\nu}(x^0, x^1, x^2, x^3) - \frac{\partial \delta x^\alpha (x^0, x^1, x^2, x^3)}{\partial x^\mu} \bar{g}_{\alpha\nu}(x^0)$$
$$- \frac{\partial \delta x^\beta (x^0, x^1, x^2, x^3)}{\partial x^\nu} \bar{g}_{\mu\beta}(x^0) - \frac{\partial \bar{g}_{\mu\nu}(x^0)}{\partial x^0} \delta x^0 (x^0, x^1, x^2, x^3). \qquad (3.19)$$

It is remarkable that all the quantities and derivatives in the right hand side are in terms of the unprimed variables.

Let us consider the scalar perturbations. In the unprimed system, the perturbed metric tensor elements are

$$\delta g_{00} = -E, \qquad (3.20)$$

$$\delta g_{0i} = a \frac{\partial F}{\partial x^i}, \qquad (3.21)$$

$$\delta g_{ij} = a^2 A \delta_{ij} + a^2 \frac{\partial^2 B}{\partial x^i \partial x^j}, \qquad (3.22)$$

Putting the coordinate transformations (3.10), (3.13), and the scalar perturbations (3.20)-(3.22) into (3.19), we obtain the perturbations in the primed system:

$$\delta g'_{00} = -E + 2\frac{\partial \delta x^0}{\partial x^0}, \qquad (3.23)$$

$$\delta g'_{0i} = a \frac{\partial}{\partial x^i}\left[F + 2\frac{\dot{a}}{a^2}\xi - \frac{1}{a}\dot{\xi} + \delta x^0 \right], \qquad (3.24)$$

$$\delta g'_{ij} = a^2 \left[A - 2\frac{\dot{a}}{a}\delta x^0 \right] \delta_{ij} + a^2 \partial_i \partial_j \left[B - 2\frac{1}{a^2}\xi \right]. \qquad (3.25)$$

Here we have dropped all the explicit coordinate references in (3.19), but it should be clear that all the quantities at the right hand side are evaluated in the unprimed system, while the ones at the left hand side are evaluated in the primed system. We see that if we take

$$E' = E - 2\frac{\partial \delta x^0}{\partial x^0}, \tag{3.26}$$

$$F' = F + 2\frac{\dot{a}}{a^2}\xi - \frac{1}{a}\dot{\xi} + \delta x^0, \tag{3.27}$$

$$A' = A - 2\frac{\dot{a}}{a}\delta x^0, \tag{3.28}$$

$$B' = B - 2\frac{1}{a^2}\xi, \tag{3.29}$$

to be the corresponding scalar perturbations in the primed system, the form of perturbations will remain the same after transformation. More precisely, after such infinitesimal coordinate transformation, the scalar part of the metric perturbations and coordinate perturbations in the original system will induce *only* the scalar metric perturbations in the new system. Note that this is the manifestation of the decoupling of different classes of perturbations, stated in the previous section. Another point one should pay attention to is that E', F', A', B' are *not* the transformed E, F, A, B under the usual tensor transformation rules, but the corresponding perturbations in the new coordinate system. Since they are actually scalars, they remain unchanged under tensor transformations. In fact this is the general property to all the perturbed quantities in this gauge formulation.

3.2.2 Eliminate the gauge degrees of freedom by choosing a gauge

To specify an objective condition under which one performs the calculation, one chooses a gauge. For example, one of the most popular choice is working in a coordinate system in which the scalar perturbations B and F always vanish. This can be achieved by transforming into the coordinate system with a choice of ξ so that $B=0$ and a choice of δx^0 so that $F=0$. This particular coordinate system is known as the *Newtonian gauge*, with the metric tensor usually written as

$$g_{\mu\nu}=\begin{pmatrix} -1-2\Psi & 0 & 0 & 0 \\ 0 & a^2(1+2\Phi) & 0 & 0 \\ 0 & 0 & a^2(1+2\Phi) & 0 \\ 0 & 0 & 0 & a^2(1+2\Phi) \end{pmatrix}. \tag{3.30}$$

In this gauge, the coordinate system is fixed unambiguously. The metric perturbation Ψ is a generalization of the Newtonian potential, which explains the name of the gauge.

The *synchronous gauge* is widely used in the early literatures, especially the ground-breaking work on cosmological perturbation theory by Lifshitz in 1946 [35]. The name of the gauge comes from the property that there exists a set of comoving observers in this coordinate system who fall freely without changing their spatial coordinates [32]. However, it became not so popular after Bardeen pointed out that there are still residual gauge degrees of freedom in this gauge (for details, see [29]). In the synchronous gauge, by choosing δx^0 and ξ, one makes E and F vanish, respectively. The metric elements of this gauge are usually written as

$$g_{00}=-1,\quad g_{i0}=0,\quad g_{ij}=a^2\left[(1+A)\delta_{ij}+\frac{\partial^2 B}{\partial x^i \partial x^j}\right]. \tag{3.31}$$

Throughout the thesis, we will perform the calculations in the Newtonian gauge except otherwise stated.

3.3 The perturbation of energy-momentum tensor

The energy-momentum tensor of the inflatons is given by (2.16)

$$T^{\mu}{}_{\nu} = \partial^{\mu}\phi\partial_{\nu}\phi - \delta^{\mu}{}_{\nu}\left[-\frac{1}{2}\partial_{\alpha}\phi\partial^{\alpha}\phi - V(\phi)\right], \tag{3.32}$$

in which the potential is taken to be

$$V(\phi) = \frac{1}{2}m^2\phi^2 . \tag{3.33}$$

Inserting the perturbation to the scalar field

$$\phi(\bar{x},t) = \phi_0(t) + \delta\phi(\bar{x},t), \tag{3.34}$$

we can expand the energy momentum tensor as

$$\bar{T}^{0}{}_{0} = -\frac{1}{2}\dot{\phi}_0^2 - \frac{1}{2}m^2\phi_0^2, \tag{3.35}$$

$$\bar{T}^{0}{}_{i} = \bar{T}^{i}{}_{0} = 0, \tag{3.36}$$

$$\bar{T}^{i}{}_{j} = \delta^{i}{}_{j}\left[\frac{1}{2}\dot{\phi}_0^2 - \frac{1}{2}m^2\phi_0^2\right], \tag{3.37}$$

$$\delta T^{0}{}_{0} = \dot{\phi}_0^2\Psi - m^2\phi_0\delta\phi - \dot{\phi}_0\delta\dot{\phi}, \tag{3.38}$$

$$\delta T^{0}{}_{i} = -\dot{\phi}_0\frac{\partial\delta\phi}{\partial x^i}, \tag{3.39}$$

$$\delta T^{i}{}_{0} = \frac{1}{a^2}\dot{\phi}_0\frac{\partial\delta\phi}{\partial x^i}, \tag{3.40}$$

$$\delta T^{i}{}_{j} = \left[-\dot{\phi}_0^2\Psi - m^2\phi_0\delta\phi + \dot{\phi}_0\delta\dot{\phi}\right]\delta^{i}{}_{j}, \tag{3.41}$$

where Ψ is the scalar perturbation in the Newtonian gauge defined in (3.30). It is introduced when we raise the indices. One can notice again that the forms of perturbations of the energy-momentum tensor do depend on the gauge we choose.

3.4 The perturbed Einstein equation

We are now going to evaluate the Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{3.42}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{3.43}$$

is the Einstein tensor without cosmological constant. In the Newtonian gauge and inserting the energy-momentum tensor of the inflatons (3.35)-(3.41), we obtain the zero- and first-order parts of the Einstein equations:

$$\overline{G}^{0}{}_{0} : -3H^2 = 8\pi G\left(-\frac{1}{2}\dot{\phi}_0^2 - \frac{1}{2} m^2 \phi_0^2\right), \tag{3.44}$$

$$\overline{G}^{0}{}_{i}, \overline{G}^{i}{}_{0} : \ \text{vanish}, \tag{3.45}$$

$$\overline{G}^{i}{}_{j} : -\delta^{i}{}_{j}\left[H^2 + 2\frac{\ddot{a}}{a}\right] = 8\pi G \delta^{i}{}_{j}\left[\frac{1}{2}\dot{\phi}_0^2 - \frac{1}{2} m^2 \phi_0^2\right], \tag{3.46}$$

$$\delta G^{0}{}_{0} : 6H^2\Psi + 2\frac{1}{a^2}\nabla^2\Phi - 6H\dot{\Phi} = 8\pi G\left[\dot{\phi}_0^2\Psi - m^2\phi_0\delta\phi - \dot{\phi}_0\delta\dot{\phi}\right], \tag{3.47}$$

$$\delta G^{0}{}_{i} : 2\partial_i\left[\dot{\Phi} - H\Psi\right] = -8\pi G\dot{\phi}_0\frac{\partial\delta\phi}{\partial x^i}\,, \tag{3.48}$$

$$\delta G^{i}{}_{0} : -2\frac{1}{a^2}\partial_i\left[\dot{\Phi} - H\Psi\right] = 8\pi G\frac{1}{a^2}\dot{\phi}_0\frac{\partial\delta\phi}{\partial x^i}\,, \tag{3.49}$$

$$\begin{aligned}\delta G^{i}{}_{j} : \delta^{i}{}_{j}&\left[2H\left(H + 2\frac{\ddot{a}}{a}\right)\Psi + 2H\dot{\Psi} - 6H\dot{\Phi} - 2\ddot{\Phi} + \frac{1}{a^2}\nabla^2(\Phi + \Psi)\right] \\ &- \frac{1}{a^2}\partial_i\partial_j(\Phi + \Psi) = 8\pi G\delta^{i}{}_{j}\left[-\dot{\phi}_0^2\Psi - m^2\phi_0\delta\phi + \dot{\phi}_0\delta\dot{\phi}\right].\end{aligned} \tag{3.50}$$

The conservation equation of energy-momentum tensor is

$$T^{\mu}_{\nu;\mu} = 0 . \tag{3.51}$$

It can also be decomposed into zero- and first-order parts. The zero-order part of $\nu = 0$ component reads

$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + m^2\phi_0 = 0 . \tag{3.52}$$

The other zero-order components vanish. The first-order part of $\nu = 0$ component is

$$\begin{aligned} -\frac{d\phi_0}{dt}\frac{\partial^2\delta\phi}{\partial t^2} - \left(\frac{d^2\phi_0}{dt^2} + 6H\frac{d\phi_0}{dt} + m^2\phi_0\right)\frac{\partial\delta\phi}{\partial t} - \frac{d\phi_0}{dt}\left(m^2 - \frac{1}{a^2}\nabla^2\right)\delta\phi \\ + \frac{d\phi_0}{dt}\left(6H\frac{d\phi_0}{dt} + 2\frac{d^2\phi_0}{dt^2}\right)\Psi + \left(\frac{d\phi_0}{dt}\right)^2\left(\frac{\partial\Psi}{\partial t} - 3\frac{\partial\Phi}{\partial t}\right) = 0. \end{aligned} \tag{3.53}$$

Using (3.52) to eliminate the second and the fourth round brackets, dividing by $\frac{d\phi_0}{dt}$ and rearranging, (3.53) becomes

$$\frac{\partial^2\delta\phi}{\partial t^2} + 3H\frac{\partial\delta\phi}{\partial t} + \left(m^2 - \frac{\nabla^2}{a^2}\right)\delta\phi = -2m^2\phi_0\Psi + \frac{d\phi_0}{dt}\left(\frac{\partial\Psi}{\partial t} - 3\frac{\partial\Phi}{\partial t}\right). \tag{3.54}$$

The other first-order components trivially reduce to (3.52).

Chapter 4

Quantization of inflaton field under various scenarios

The quantum fluctuations of the inflaton field leave their footprints through the interactions with the metric perturbations. As inflation ends, these patterns in the metric perturbations in turn affect the distributions of the highly relativistic particles produced at the beginning of the following radiation era. The tiny inhomogeneities of the distributions of the particles will at the end evolve into the anisotropies of the comic microwave background and the large-scale structures of the universe. To quantitatively describe the imprints of the quantum fluctuations, in section 4.1 we first quantize the inflaton field in the pure inflation case to get the amplitudes of different modes. Since the inflaton field is coupled to the metric perturbations, we also find the amplitudes of the metric perturbations. The calculation is carried out in the Newtonian gauge. Some approximations are made to make the quantization possible, and the conditions under which this procedure is applicable are also examined carefully. In section 4.2, we propose another quantization of the inflaton field in the pre-inflation matter era. Similar analysis is carried out, but in a matter dominated universe. The approximations are also stated clearly. Finally, the expectation values of the amplitudes of the inflatons are given in section 4.3.

4.1 The pure inflation scenario

4.1.1 Quantization of the inflaton field at early times

We begin by considering the first-order Einstein equations. Equation (3.50) for the cases that $i \neq j$ reads

$$\partial_i \partial_j (\Psi + \Phi) = 0 . \tag{4.1}$$

Equation (3.48) and (3.49) both give

$$\partial_i \left(\frac{\partial \Phi}{\partial t} - H\Psi \right) = \partial_i \left(-4\pi G \frac{d\phi_0}{dt} \delta\phi \right). \tag{4.2}$$

The energy conservation equation (3.54) gives

$$\frac{\partial^2 \delta\phi}{\partial t^2} + 3H \frac{\partial \delta\phi}{\partial t} + \left(\frac{\partial^2 V(\phi_0)}{\partial {\phi_0}^2} - \frac{\nabla^2}{a^2} \right) \delta\phi = -2 \frac{\partial V(\phi_0)}{\partial \phi_0} \Psi + \frac{d\phi_0}{dt} \left(\frac{\partial \Psi}{\partial t} - 3 \frac{\partial \Phi}{\partial t} \right). \tag{4.3}$$

Here we keep the general potential of the scalar field. In Fourier space, (4.1) gives the useful relation

$$\Phi = -\Psi . \tag{4.4}$$

After applying relation (4.4), (4.2) and (4.3) reduce to

$$\frac{d\Psi}{dt} + H\Psi = 4\pi G \frac{d\phi_0}{dt} \delta\phi \tag{4.5}$$

and

$$\frac{d^2\delta\phi}{dt^2}+3H\frac{d\delta\phi}{dt}+\left(\frac{\partial^2 V(\phi_0)}{\partial{\phi_0}^2}+\frac{k^2}{a^2}\right)\delta\phi=-2\frac{\partial V(\phi_0)}{\partial\phi_0}\Psi+4\frac{d\phi_0}{dt}\frac{d\Psi}{dt}. \qquad (4.6)$$

To see how we might quantize the inflaton field $\delta\phi$, observe that there is an underlying structure

$$\frac{d^2\delta\phi}{dt^2}+\frac{k^2}{a^2}\delta\phi=0$$

in this complicated equation. To facilitate the quantization, however, some transformations and simplifications are needed. We first transform the derivatives against physical time t into that against conformal time η, and then substitute $\delta\phi$ and Ψ by $\frac{v}{a}$ and $\frac{u}{a}$, where v and u are functions of η. Note that for any function $f(t)$, the derivatives of t and η are related by

$$\frac{df}{dt}=\frac{1}{a}\frac{df}{d\eta},$$

$$\frac{d^2 f}{dt^2}=-\frac{H}{a}\frac{df}{d\eta}+\frac{1}{a^2}\frac{d^2 f}{d\eta^2},$$

recall that $\frac{d\eta}{dt}=\frac{1}{a}$. After applying these transformations to (4.6) and (4.5), we arrive at

$$\frac{d^2 v}{d\eta^2}+\left(\frac{\partial^2 V(\phi_0)}{\partial{\phi_0}^2}a^2+k^2-\frac{1}{a}\frac{d^2 a}{d\eta^2}\right)v$$

$$=-2a^2\frac{\partial V(\phi_0)}{\partial\phi_0}u-4a^2H\frac{d\phi_0}{dt}u+4a\frac{d\phi_0}{dt}\frac{du}{d\eta}, \qquad (4.7)$$

$$\frac{du}{d\eta}=4\pi Ga\frac{d\phi_0}{dt}v, \qquad (4.8)$$

where we have kept $\frac{d\phi_0}{dt}$ unchanged for later convenience.

To gain some ideas about the magnitudes of the various terms in the equations, note that for potential $V(\phi_0)=\frac{1}{2}m^2\phi_0^2$ during inflation era, as demonstrated in chapter

2, $\dfrac{\partial V(\phi_0)}{\partial \phi_0}$ and $\dfrac{\partial^2 V(\phi_0)}{\partial {\phi_0}^2}$ are of the order of m^2 since ϕ_0 is about unity, while $\dfrac{d\phi_0}{dt}$ is of the order of m. During inflation the term $\dfrac{1}{a}\dfrac{d^2 a}{d\eta^2}$ is about $\dfrac{2}{\eta^2}$, provided $\eta = 0$ at the end of inflation (so that η is negative during inflation). This can be obtained by considering that during inflation, the scale factor is

$$a(t) = a_i e^{H(t-t_i)}, \tag{4.9}$$

where t_i is the time when inflation begins and H stays literally constant. Integrating $\dfrac{d\eta}{dt} = \dfrac{1}{a}$ to solve for $\eta(t)$, we obtain

$$\eta(t_f) - \eta(t) = \frac{1}{a_i e^{-Ht_i}} \frac{1}{H} \left(\frac{1}{e^{Ht}} - \frac{1}{e^{Ht_f}} \right), \tag{4.10}$$

where t_f denotes the time of the end of inflation. As long as t is well inside the inflation, e^{Ht_f} is much greater than e^{Ht} and so can be neglected. Taking $\eta(t_f) = 0$, we then obtain

$$a = -\frac{1}{H\eta}. \quad \text{(Inflation)} \tag{4.11}$$

With this formula, the identity mentioned before can be easily proved.

Now if m, the mass of the inflaton, is very *small*, and $|\eta|$ (recall that η is negative during inflation), the magnitude of conformal time, is very *large*, (4.7) can be reduced to the equation of a simple harmonic oscillator,

$$\frac{d^2 v}{d\eta^2} + k^2 v = 0\,. \tag{4.12}$$

The rigorous criterions for being "large" and "small" will be given in the next section, but here we can first examine on the assertions to gain some faith. In most models of the inflation, the mass of the inflaton m is usually on the scale of 10^{-6} times smaller than ϕ_0; that is, it serves mathematically like a perturbation as $\delta\phi$ or Ψ, and at the very early times before the end of inflation, η is reasonable to be a large number. Therefore it is possible that the simplification is applicable.

The solutions to v in this limiting case are

$$v_{\pm} = f_{k,\pm} e^{\pm ik\eta}, \tag{4.13}$$

with $f_{k,\pm}$ are constants to be determined. u can also be obtained by (4.8) as

$$u_{\pm} = g_{k,\pm} e^{\pm ik\eta}, \tag{4.14}$$

where

$$g_{k,\pm} = \mp i4\pi G \frac{d\phi_0}{dt} \frac{1}{\left(\frac{k}{a}\right)} f_{k,\pm}. \tag{4.15}$$

While quantizing v, so as to determine the coefficient $f_{k,\pm}$, there are still difficulties arising from the complicated interaction between the inflaton field and the metric perturbation. To further simplify the problem, we take advantage of that at early times, when a is small, by (4.15) the magnitude of Ψ is negligible compared to that of $\delta\phi$. Under such approximation we can quantize the field in an *unperturbed* Robertson-Walker metric.

The action of the scalar field is

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi)\right)$$

$$= \int d^4x\, a^3\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2a^2}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^i} - V(\phi)\right], \tag{4.16}$$

from which we can obtain the canonical conjugate to ϕ

$$\pi = \frac{\partial L}{\partial\dot{\phi}}$$

$$= a^3\dot{\phi}, \tag{4.17}$$

with overdot denoting the derivative with respect to t and

$$L = a^3\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2a^2}\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^i} - V(\phi)\right]. \tag{4.18}$$

Promoting the classical field ϕ to an operator, we expand it in k space

$$\hat{\phi}(x,t) = \int d^3\vec{k}\left(\phi_k(t)\hat{\alpha}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + \phi_k{}^*(t)\hat{\alpha}_{\vec{k}}{}^+e^{-i\vec{k}\cdot\vec{x}}\right), \tag{4.19}$$

and also

$$\hat{\pi}(x,t) = a(t)^3\int d^3\vec{k}\left(\dot{\phi}_k(t)\hat{\alpha}_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + \dot{\phi}_k{}^*(t)\hat{\alpha}_{\vec{k}}{}^+e^{-i\vec{k}\cdot\vec{x}}\right), \tag{4.20}$$

where $\hat{\alpha}_{\vec{k}}{}^+$ and $\hat{\alpha}_{\vec{k}}$ are creation and annihilation operators respectively. The commutation relations are

$$\left[\hat{\phi}(\vec{x},t),\hat{\pi}(\vec{x}',t)\right] = i\delta^3(\vec{x}-\vec{x}'), \tag{4.21}$$

$$\left[\hat{\phi}(\vec{x},t),\hat{\phi}(\vec{x}',t)\right] = 0, \tag{4.22}$$

$$\left[\hat{\pi}(\vec{x},t),\hat{\pi}(\vec{x}',t)\right] = 0, \tag{4.23}$$

$$\left[\hat{\alpha}_{\vec{k}},\hat{\alpha}_{\vec{k}'}{}^+\right] = \delta^3\left(\vec{k}-\vec{k}'\right), \tag{4.24}$$

$$\left[\hat{\alpha}_{\vec{k}},\hat{\alpha}_{\vec{k}'}\right] = 0, \tag{4.25}$$

$$\left[\hat{\alpha}_{\vec{k}}^{\ +}, \hat{\alpha}_{\vec{k}'}^{\ +}\right] = 0\,. \tag{4.26}$$

The commutator (4.21) gives

$$\begin{aligned}\left[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{x}',t)\right] &= \hat{\phi}(\vec{x},t)\hat{\pi}(\vec{x}',t) - \hat{\pi}(\vec{x}',t)\hat{\phi}(\vec{x},t) \\ &= a^3 \int d^3\vec{k} \int d^3\vec{k}'\, e^{i(\vec{k}\cdot\vec{x}+\vec{k}'\cdot\vec{x}')} \left(\phi_k \dot{\phi}_{k'}^* \left[\hat{\alpha}_{\vec{k}}, \hat{\alpha}_{-\vec{k}'}^{\ +}\right] + \phi_k^* \dot{\phi}_{k'} \left[\hat{\alpha}_{-\vec{k}}^{\ +}, \hat{\alpha}_{\vec{k}'}\right]\right),\end{aligned}$$

where we have applied the vanishing commutators (4.25) and (4.26). Replacing the remaining commutators by (4.24) and integrate over $\vec{k}'$, we have

$$\left[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{x}',t)\right] = a^3 \int d^3\vec{k}\; e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(\phi_k \dot{\phi}_{k'}^* - \phi_k^* \dot{\phi}_{k'}\right). \tag{4.27}$$

Comparing this to (4.21), and recalling that

$$\delta^3(\vec{x}-\vec{x}') = \frac{1}{(2\pi)^3} \int d^3\vec{k}\; e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}, \tag{4.28}$$

we see that ϕ_k satisfies the quantization condition

$$\phi_k \dot{\phi}_k^* - \phi_k^* \dot{\phi}_k = \frac{i}{(2\pi)^3} \frac{1}{a^3}. \tag{4.29}$$

Because ϕ_k is a c-number, the commutators and the equation of quantization condition of the perturbation are the same as that of ϕ_k itself. Replacing $\delta\phi_k$ by the positive-frequency solution

$$\delta\phi_k = \frac{f_k}{a} e^{-ik\eta}, \tag{4.30}$$

the time derivative of $\delta\phi_k$ is

$$\delta\dot{\phi}_k = -\left(H + i\frac{k}{a}\right)\delta\phi_k\,. \tag{4.31}$$

Here we have added a subscript k to $\delta\phi$ for clarity. At very early times, when physical wavelength $a\dfrac{2\pi}{k}$ is much smaller than the Hubble radius $\dfrac{1}{H}$, we can drop the Hubble parameter in (4.31)

$$\delta\dot{\phi}_k = -i\frac{k}{a}\delta\phi_k \, ; \tag{4.32}$$

we then have

$$\delta\phi_k\delta\dot{\phi}_k^* - \delta\phi_k^*\delta\dot{\phi}_k = i\frac{k}{a^3}\left(f_k f_k^* + f_k^* f_k\right). \tag{4.33}$$

Comparing it with the quantization condition corresponding to (4.29)

$$\delta\phi_k\delta\dot{\phi}_k^* - \delta\phi_k^*\delta\dot{\phi}_k = \frac{i}{(2\pi)^3}\frac{1}{a^3}, \tag{4.34}$$

we see that by setting

$$f_k = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2k}} \tag{4.35}$$

the quantization condition (4.34) can then be satisfied. We finally arrive at the normalized solution

$$\delta\phi_k = \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2k}}\frac{1}{a}e^{-ik\eta}. \quad \text{(Early times)} \tag{4.36}$$

By (4.15) we also get the solution to Ψ_k

$$\Psi_k = \frac{1}{\left(\frac{k}{a}\right)}\cdot\frac{i4\pi G\dot{\phi}_0}{(2\pi)^{\frac{3}{2}}\sqrt{2k}}\frac{1}{a}e^{-ik\eta}, \quad \text{(Early times)} \tag{4.37}$$

where we have separate the factor $\frac{k}{a}$ to emphasize that Ψ_k is smaller than $\delta\phi_k$ by factors $\frac{k}{a}$ and $4\pi G\dot{\phi}_0$.

4.1.2 Validity conditions of the approximation

In this section we develop the quantitative conditions for the approximations made in the previous section to be satisfied. To approximate the initial $\delta\phi$ as an simple harmonic oscillator, we need to simplify (4.7), reproduced here,

$$\frac{d^2 v}{d\eta^2}+\left(\frac{\partial^2 V(\phi_0)}{\partial \phi_0{}^2}a^2+k^2-\frac{1}{a}\frac{d^2 a}{d\eta^2}\right)v$$
$$=-2a^2\frac{\partial V(\phi_0)}{\partial\phi_0}u-4a^2 H\frac{d\phi_0}{dt}u+4a\frac{d\phi_0}{dt}\frac{du}{d\eta}, \qquad (4.38)$$

into (4.12)

$$\frac{d^2 v}{d\eta^2}+k^2 v=0\,. \qquad (4.39)$$

Divided both sides by a^2, (4.38) becomes

$$\frac{1}{a^2}\frac{d^2 v}{d\eta^2}+\left(\frac{\partial^2 V(\phi_0)}{\partial \phi_0{}^2}+\frac{k^2}{a^2}-\frac{1}{a^3}\frac{d^2 a}{d\eta^2}\right)v$$
$$=-2\frac{\partial V(\phi_0)}{\partial\phi_0}u-4H\frac{d\phi_0}{dt}u+4\frac{d\phi_0}{dt}\frac{1}{a}\frac{du}{d\eta}\,. \qquad (4.40)$$

We rearrange the equation into this form so that $\frac{k}{a}$ represents the physical wavenumber. We see that for (4.39) to be a valid approximation, the following conditions need to be satisfied,

$$\frac{k^2}{a^2}>>\frac{\partial^2 V(\phi_0)}{\partial\phi_0{}^2}, \qquad (4.41)$$

$$\frac{k^2}{a^2}>>\left|\frac{1}{a^3}\frac{d^2 a}{d\eta^2}\right|, \qquad (4.42)$$

$$\frac{k^2}{a^2}|v|>>2\frac{\partial V(\phi_0)}{\partial\phi_0}|u|, \qquad (4.43)$$

$$\frac{k^2}{a^2}|v|>>4\left|\frac{d\phi_0}{dt}\left(-Hu+\frac{1}{a}\frac{du}{d\eta}\right)\right|. \qquad (4.44)$$

Inequalities (4.42)-(4.44) can be put into the form

$$\frac{k^2}{a^2} >> \left| H^2 + \frac{1}{a}\frac{d^2 a}{dt^2} \right|, \tag{4.45}$$

$$\frac{k^2}{a^2}\left|\delta\phi\right| >> 2\frac{\partial V(\phi_0)}{\partial \phi_0}\left|\Psi\right|, \tag{4.46}$$

$$\frac{k^2}{a^2}\left|\delta\phi\right| >> 4\left|\frac{d\phi_0}{dt}\frac{d\Psi}{dt}\right| \tag{4.47}$$

for later numerical convenience.

To approximate the quantization of the field by that in an *unperturbed* Robertson-Walker metric, we need to make sure that Ψ is negligible compared to that $\delta\phi$. This bases on that the conditions (4.41) and (4.45)-(4.47) being satisfied so that we can relate Ψ to $\delta\phi$ by (4.15). For Ψ to be negligible (4.15) provides the condition

$$\left|\frac{\Psi}{\delta\phi}\right| = \left| 4\pi G \dot{\phi}_0 \frac{1}{\left(\frac{k}{a}\right)} \right| << 1, \tag{4.48}$$

which can be rearranged into

$$\frac{k}{a} >> 4\pi G\left|\dot{\phi}_0\right| \tag{4.49}$$

In (4.31) we have also neglect the Hubble parameter in time derivative of $\delta\phi_k$. For this to be valid, the condition

$$\frac{k}{a} >> H \tag{4.50}$$

needs to be satisfied.

In conclusion, when applying the initial conditions (4.36) and (4.37) in numerical calculations, we need to check that whether conditions (4.41), (4.45)-(4.47), and (4.49)-(4.50) are all satisfied.

4.2 The pre-inflation matter era

4.2.1 Quantization of the inflaton field at early times

In the previous section, the quantization is performed in a pure inflation case, and approximations are made to simplify the analysis. However, the validity conditions for the approximations are no longer satisfied for modes whose wavelengths are comparable to the size of the Hubble radius. The most critical restriction comes from the effect of the expansion of the universe in such a large scale (while approaching the scale of the Hubble radius, conditions (4.45) and (4.50) are the first criterions that break down). To investigate into this regime, we keep the expansion term in (4.6), working with approximate equation

$$\frac{d^2\delta\phi}{dt^2}+3H\frac{d\delta\phi}{dt}+\frac{k^2}{a^2}\delta\phi=0. \tag{4.51}$$

In the case of pre-inflation matter era, as proposed in the works of Scardigli *et al.* [16] and Gruber [17], the matter density at the early times ($10^3 t_p$ in their case) is about 10^6 greater than the energy density of the scalar field, so at the early times, the universe may be taken as in a matter dominated epoch. Neglecting the energy density of the scalar field, the Friedmann equation in the matter dominated epoch is

$$H^2=\frac{8\pi}{3}\frac{\rho_0 a_0{}^3}{a^3}. \tag{4.52}$$

Here we have set $G=1$ as it is in the Planck unit system. Using

$$H=\frac{1}{a^2}\frac{da}{d\eta}, \tag{4.53}$$

one finds the scale factor

$$a(\eta)=\left[\sqrt{\frac{2\pi\rho_0 a_0{}^3}{3}}(\eta-\eta_0)+\sqrt{a_0}\right]^2 . \tag{4.54}$$

Setting $a_0=1$ and $\eta_0=0$ at $t=10^3$ (in Planck units), feeding back into (4.51), one gets

$$\delta\phi=\frac{1}{\overline{\eta}^2}\left[1-\frac{i}{k\overline{\eta}}\right]e^{-ik\overline{\eta}} , \tag{4.55}$$

where

$$\overline{\eta}=\eta+\sqrt{\frac{3}{2\pi\rho_0}} . \tag{4.56}$$

Imposing the commutation relation (4.29), one gets the normalized solution at the early times:

$$\delta\phi=\frac{3}{8\pi^{5/2}\sqrt{k}\rho_0}\frac{1}{\overline{\eta}^2}\left[1-\frac{i}{k\overline{\eta}}\right]e^{-ik\overline{\eta}} . \quad \text{(Early times)} \tag{4.57}$$

Feeding back into (4.5) or (4.8), assuming that $\frac{d\phi_0}{dt}$ is almost constant (which is reasonable for the inflation model we adopt and is also a general characteristic for the slow-roll approximation), we obtain the corresponding solution to the metric perturbation:

$$\Psi=\frac{1}{\sqrt{\pi k}}\frac{d\phi_0}{dt}\left[\frac{i}{k}+\frac{3}{k^2\overline{\eta}}-\frac{3i}{k^3\overline{\eta}^2}\right]e^{-ik\overline{\eta}} . \quad \text{(Early times)} \tag{4.58}$$

4.2.2 Validity conditions of the approximation

As stated in the previous subsection, the treatment we use for the pre-inflation matter era is free from the constraint that the wavelengths of the modes are much smaller than the Hubble radius, while keeping all other approximations, such as neglecting the mass of the inflaton and the metric perturbation. The conditions remained are listed below:

$$\frac{k^2}{a^2} >> \frac{\partial^2 V(\phi_0)}{\partial \phi_0{}^2}, \tag{4.59}$$

$$\frac{k^2}{a^2}|\delta\phi| >> 2\frac{\partial V(\phi_0)}{\partial \phi_0}|\Psi|, \tag{4.60}$$

$$\frac{k^2}{a^2}|\delta\phi| >> 4\left|\frac{d\phi_0}{dt}\frac{d\Psi}{dt}\right|, \tag{4.61}$$

$$\frac{k}{a} >> 4\pi G|\dot{\phi}_0|. \tag{4.62}$$

4.3 The power spectrum of the quantum fluctuations

To obtain the primordial power spectrum, we assume that during inflation the universe is in its vacuum state $|0\rangle$, which satisfies

$$\hat{\alpha}_{\vec{k}}|0\rangle = 0 \text{, and } \langle 0|0\rangle = 1 \tag{4.63}$$

for any $\vec{k}$. This vacuum state is known as the Bunch-Davies vacuum [36]. The expectation value of $|\delta\phi|^2$ on this state is taken to be

$$\left\langle \delta\phi^2 \right\rangle = \langle 0|\delta\hat{\phi}_{\vec{k}'}(t)^{+}\delta\hat{\phi}_{\vec{k}}(t)|0\rangle, \tag{4.64}$$

where

$$\delta\hat{\phi}_{\vec{k}}(t) = \delta\phi_k(t)\hat{\alpha}_{\vec{k}} + \delta\phi_k{}^*(t)\hat{\alpha}_{\vec{k}}{}^{+} \tag{4.65}$$

and

$$\delta\hat{\phi}_{\vec{k}'}(t)^{+} = \delta\phi_{k'}{}^*(t)\hat{\alpha}_{\vec{k}'}{}^{+} + \delta\phi_{k'}(t)\hat{\alpha}_{\vec{k}'}. \tag{4.66}$$

Expanding (4.64) by inserting (4.65) and (4.66), we get

$$\left\langle \delta\phi^2 \right\rangle = \left\langle 0 \middle| \hat{\alpha}_{\vec{k}'}{}^{+} \delta\phi_{k'}{}^{*}(t) \delta\phi_k(t) \hat{\alpha}_{\vec{k}} \middle| 0 \right\rangle + \left\langle 0 \middle| \hat{\alpha}_{\vec{k}'}{}^{+} \delta\phi_{k'}{}^{*}(t) \delta\phi_k{}^{*}(t) \hat{\alpha}_{\vec{k}}{}^{+} \middle| 0 \right\rangle$$
$$+ \left\langle 0 \middle| \hat{\alpha}_{\vec{k}'} \delta\phi_{k'}(t) \delta\phi_k(t) \hat{\alpha}_{\vec{k}} \middle| 0 \right\rangle + \left\langle 0 \middle| \hat{\alpha}_{\vec{k}'} \delta\phi_{k'}(t) \delta\phi_k{}^{*}(t) \hat{\alpha}_{\vec{k}}{}^{+} \middle| 0 \right\rangle.$$

The first three terms in the right hand side vanish since both $\hat{\alpha}_{\vec{k}} \left| 0 \right\rangle$ and $\left\langle 0 \right| \hat{\alpha}_{\vec{k}'}{}^{+}$ – the dual of $\hat{\alpha}_{\vec{k}'} \left| 0 \right\rangle$ – produce a coefficient of 0. The only term left is the last term:

$$\left\langle \delta\phi^2 \right\rangle = \left\langle 0 \middle| \hat{\alpha}_{\vec{k}'} \hat{\alpha}_{\vec{k}}{}^{+} \middle| 0 \right\rangle \delta\phi_{k'}(t) \delta\phi_k{}^{*}(t). \qquad (4.67)$$

Invoking the commutation relation (4.24),

$$\left[\hat{\alpha}_{\vec{k}}, \hat{\alpha}_{\vec{k}'}{}^{+} \right] = \hat{\alpha}_{\vec{k}} \hat{\alpha}_{\vec{k}'}{}^{+} - \hat{\alpha}_{\vec{k}'}{}^{+} \hat{\alpha}_{\vec{k}} = \delta^3\left(\vec{k} - \vec{k}'\right),$$

(4.67) can be put into

$$\left\langle \delta\phi^2 \right\rangle = \left\langle 0 \middle| \hat{\alpha}_{\vec{k}}{}^{+} \hat{\alpha}_{\vec{k}'} \middle| 0 \right\rangle \delta\phi_{k'}(t) \delta\phi_k{}^{*}(t) + \left\langle 0 \middle| 0 \right\rangle \delta^3\left(\vec{k} - \vec{k}'\right) \delta\phi_{k'}(t) \delta\phi_k{}^{*}(t). \qquad (4.68)$$

The first term of (4.68) vanishes again and the second term, after using (4.63) and noting that only at $\vec{k} = \vec{k}'$ it has nonzero value, reduces to $\delta^3\left(\vec{k} - \vec{k}'\right) \left|\delta\phi_k(t)\right|^2$. Thus we arrive at the final result,

$$\left\langle \delta\phi^2 \right\rangle = \delta^3\left(\vec{k} - \vec{k}'\right) \left|\delta\phi_k(t)\right|^2. \qquad (4.69)$$

It is customary to define the power spectrum as[1]

$$P_{\delta\phi} = k^3 \left|\delta\phi_k(t)\right|^2. \qquad (4.70)$$

It is a function of both k and t, but when evaluated at the time of horizon crossing, $\left|\delta\phi_k(t)\right|^2$ has a roughly $\frac{1}{k^3}$ dependence, therefore the pre-factor k^3 in (4.70).

[1]The definition of "power spectrum" diverges in the literature. In the context of structure formation, it is

also customary to define the density power spectrum as $\left|\delta_{\vec{k}}\right|^2$ (see, for example, [37]), where

$$\delta(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\,\delta_{\vec{k}}\,\exp\left(-i\vec{k}\cdot\vec{x}\right),$$

and $\delta(\vec{x})$ is the density contrast

$$\delta(\vec{x}) = \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}}$$

with $\bar{\rho}$ denoting the average density. Our definition here follows [7].

Chapter 5

The primordial power spectra

In this chapter we are going to calculate the primordial power spectra by solving the differential equations derived in chapter 2 and 3. The initial conditions are given in chapter 4 by quantization of the scalar field. In section 5.1 the case of pure inflation is treated. We will first examine the evolution of the modes in order to know what range of k, the comoving wave number of the modes, is interesting to us. We then solve the differential equations and the time-evolutions of the inflaton field and metric perturbation are given. At last, we plot the primordial power spectra.

Section 5.2 provides the corresponding results of the case with pre-inflation matter era. The parameters of the scalar field are also chosen so that the mode whose physical wavelength equal to the physical Hubble radius today leaves the Hubble radius early enough during inflation, making it possible that the pre-inflation matter era may have significant effects on modes that just re-enter the radius today.

5.1 The pure inflation scenario

5.1.1 The expansion of modes

The size of a Fourier mode is denoted by its comoving wave number k, subject to $a=1$ today, so the comoving wave number k is also the *physical* wave number of the mode *today*. The physical wavelength of the mode increases as the scale factor expands, while the comoving wavelength holds still. The relation between the comoving and the physical wavelength is

$$\lambda_{phy}(t) = a(t)\lambda_{com}, \tag{5.1}$$

and the comoving wavelength is

$$\lambda_{com} = \frac{2\pi}{k}. \tag{5.2}$$

Notice that k always denotes the *comoving* wave number.

The physical Hubble radius is

$$r_H(t) = \frac{1}{H(t)}, \tag{5.3}$$

with $c=1$. In the standard scenario, modes are expelled out of the Hubble radius during inflation, freeze out there, and regain their evolution when they re-enter the radius at some moments after the end of inflation. The evolution of the physical size of a mode can be easily understood by plotting the physical wavelength and the physical Hubble radius together against time, as shown in figure 5.1.

In figure 5.1, plots of modes with several different sizes are given. The comoving wave number corresponding to the Hubble radius today is $k = 1.496\times10^{-3}Mpc^{-1}$. The mass of the inflaton is $2.04\times10^{-6}m_{Pl}$, the initial value of ϕ_0 at $t=10^3 t_{Pl}$ is

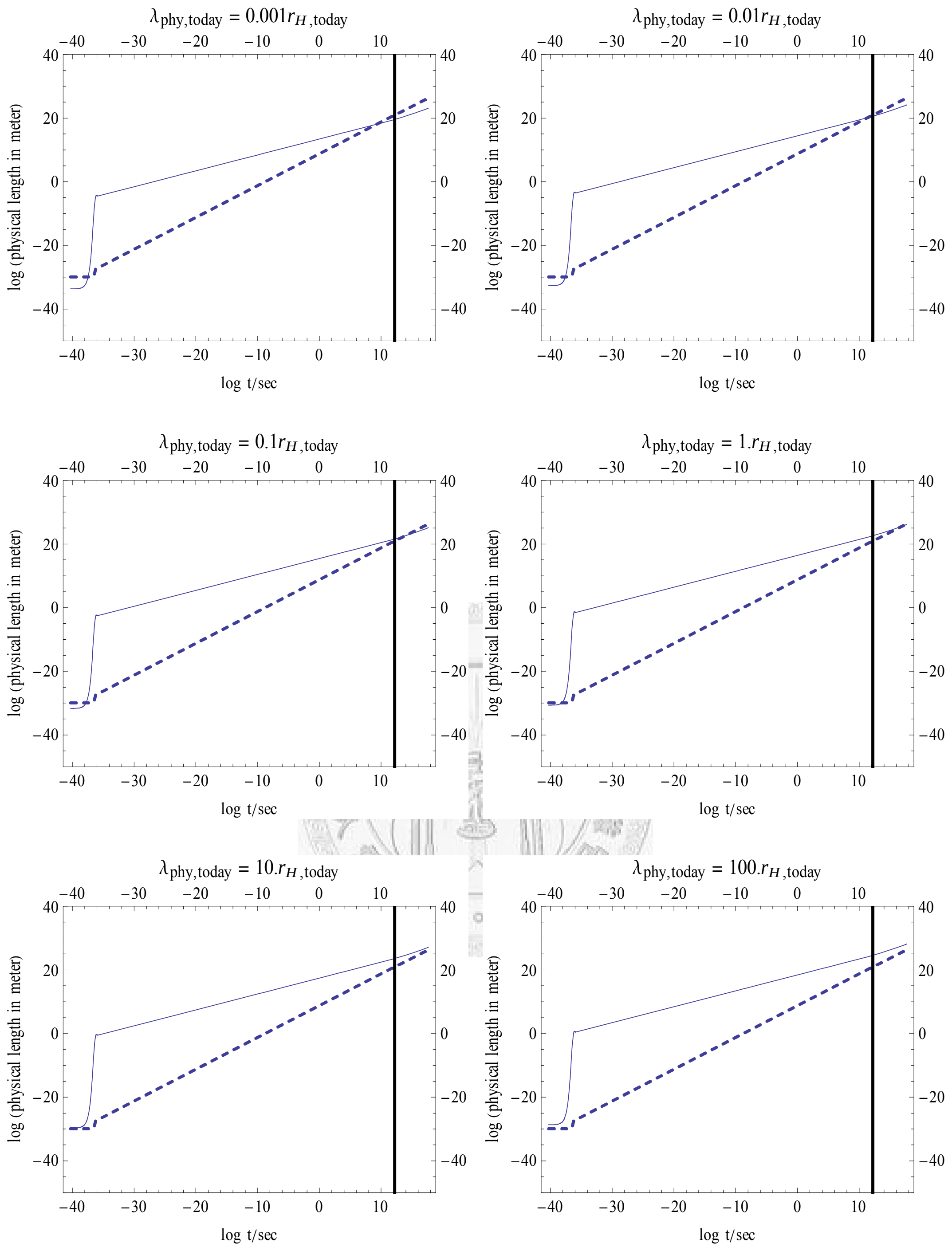


Fig 5.1 The expansion of modes. The solid curve is the physical wavelength of the mode with its physical wavelength today labeled on the top of the plot. The dashed curve is the physical Hubble radius. The horizontal line is the time at which the radiation-matter equality happens.

$3.23 m_{Pl}$, and the time at which we connect the inflation era and the radiation era is $10^7 t_{Pl}$. Inflation sustains for about 67 e-folds. These parameters are chosen to make the connection between the scale factors of the inflation era and radiation era smooth and the inflation provide enough expansion. The parameters of the ΛCDM model is given by the WMAP 7-year data: $\mathrm{h} = 0.714, \Omega_{M0} = 0.262, \Omega_{\Lambda 0} = 0.738, T_{\gamma 0} = 2.725K$, flat universe.

One can see that for modes of small scales, the physical wavelength is smaller than the physical Hubble radius at the early times, and re-enter the Hubble radius at late times. The smallest two modes re-enter before the time of radiation-matter equality. For large modes, they may have been out of the Hubble radius since the beginning, and they may also have not re-entered yet till today.

The modes we can deal with are the ones whose wavelengths are smaller than the Hubble radius at the early times during inflation. This is the basic requirement of the validity condition (4.50),

$$\frac{k}{a} >> H .$$

One thus needs to be careful about the applicable range of k. For k less than about $5\times10^{-4} Mpc^{-1}$ the approximations may fail.

5.1.2 The time-evolution of the inflaton field and the metric perturbation

We have found the differential equations, (4.5) and (4.6),

$$\frac{d\Psi}{dt}+H\Psi=4\pi G\frac{d\phi_0}{dt}\delta\phi \tag{5.4}$$

$$\frac{d^2\delta\phi}{dt^2}+3H\frac{d\delta\phi}{dt}+\left(\frac{\partial^2 V(\phi_0)}{\partial\phi_0{}^2}+\frac{k^2}{a^2}\right)\delta\phi=-2\frac{\partial V(\phi_0)}{\partial\phi_0}\Psi+4\frac{d\phi_0}{dt}\frac{d\Psi}{dt}. \tag{5.5}$$

governing the evolution of the inflaton field and the metric perturbation, and we have also had the initial conditions, (4.36) and (4.37),

$$\delta\phi_k=\frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{2k}}\frac{1}{a}e^{-ik\eta}. \quad \text{(Early times)} \tag{5.6}$$

$$\Psi_k=\frac{1}{\left(\frac{k}{a}\right)}\cdot\frac{i4\pi G\dot{\phi}_0}{(2\pi)^{\frac{3}{2}}\sqrt{2k}}\frac{1}{a}e^{-ik\eta}, \quad \text{(Early times)} \tag{5.7}$$

to these equations. We then proceed to solve these equations to obtain the primordial power spectra.

As we have seen in the previous section, the modes to which the approximations are applicable always start with their physical wavelengths smaller than the Hubble radius at early times, and exit the radius during inflation. The time-evolution of some of these modes are given in figure 5.2 through 5.4.

From these figures we note some properties:

(1) For large modes, as the one in figure 5.2, the variations of the modes do not come to a steady value when they exit the Hubble radius. They continue to

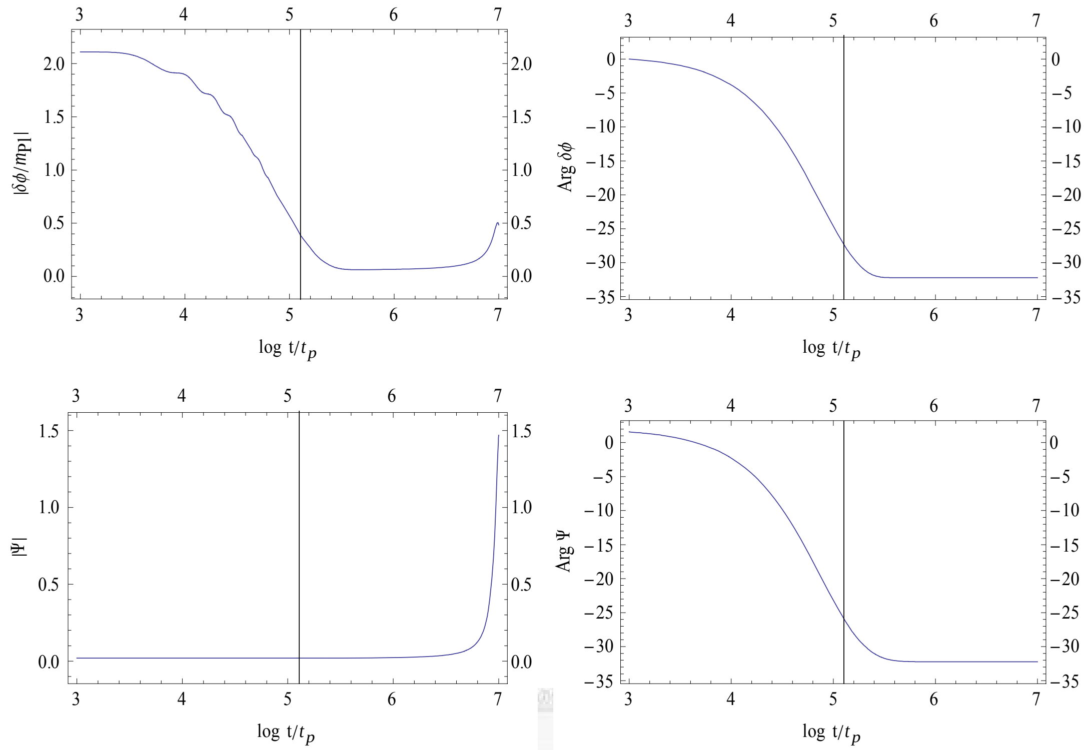


Fig 5.2 The time evolution of the inflaton field and the metric perturbation. The physical wavelength of the mode is equal to the size of the Hubble radius today. The horizontal lines denote the time at which the mode exits the Hubble radius during inflation. The approximations are valid. (Throughout the text the term “valid” means that the condition that requires “$A >> B$” is satisfied with A being greater than B for at least 10 times, which is a little bit loose.)

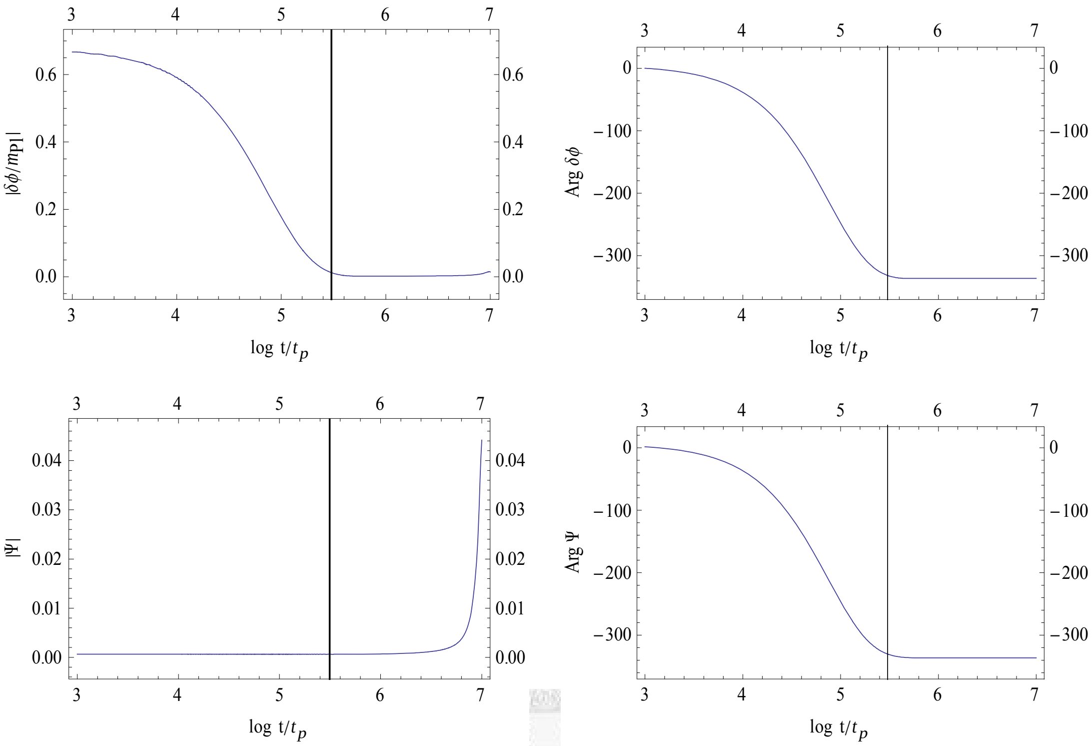


Fig 5.3 The same plots as figure 5.2, for the mode whose physical wavelength is $\frac{1}{10}$ to the size of the Hubble radius today. The approximations are valid.

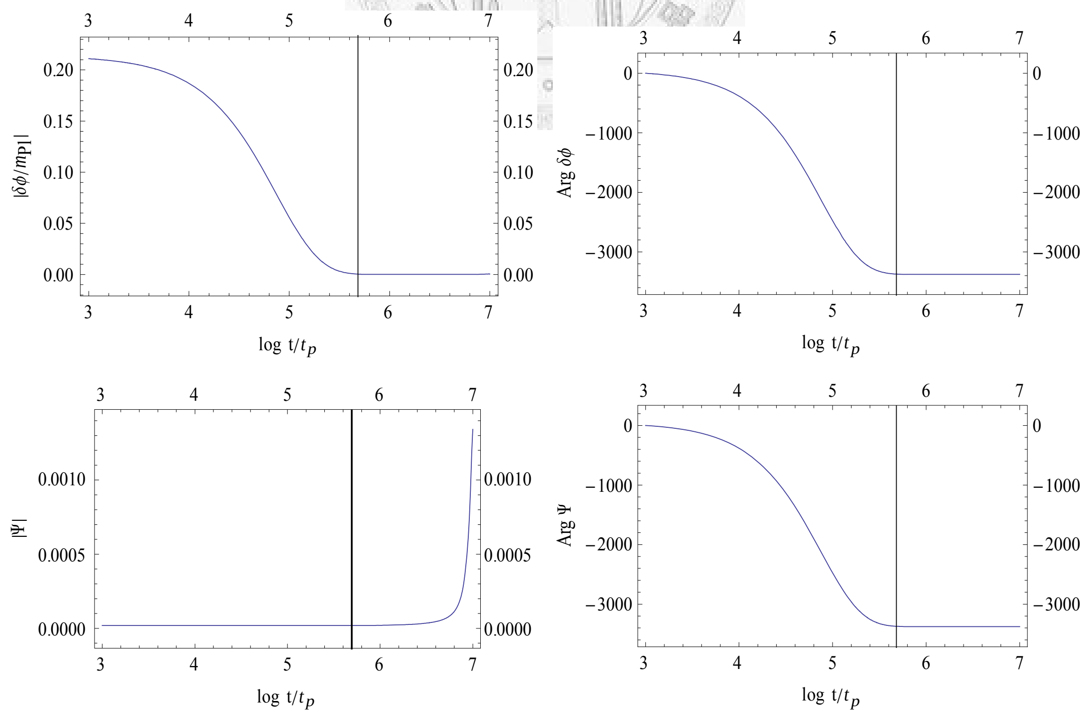


Fig 5.4 The same plots as figure 5.2, for the mode whose physical wavelength is $\frac{1}{100}$ to the size of the Hubble radius today. The approximations are valid.

decay a little bit more after they cross the radius. On the contrary, small modes, as the one in figure 5.4, almost reach their steady values while they exit the radius.

(2) The amplitudes of the fields rise at the end of inflation. This phenomenon is more apparent for large modes.

(3) The ratio of the amplitude of Ψ to the amplitude of $\delta\phi$ gets smaller for smaller modes.

Because currently we do not have a solid understanding to the physics at the end of inflation, such as reheating and the decay process of the particles, we need to find some quantities that are not significantly affected by these details to tell us the initial conditions at the beginning of the radiation era. Now it is obvious that neither the amplitude of $\delta\phi$ nor that of Ψ can provide such an information, since they are all altered when the inflation ends.

However, as Weinberg [36] and others [34, 37] found, there are some quantities that do conserve while the modes are outside the Hubble radius. One of them, represented in the Newtonian gauge, are

$$R = -\Psi - \frac{H}{\dot{\phi}_0}\delta\phi \,. \tag{5.8}$$

We plot the corresponding R's in figure 5.2-5.4 in figure 5.5.

From figure 5.5 we see that R's do conserve while the modes are far outside the

Hubble radius. Therefore it serves as the bridge connecting the inflation and the radiation eras.

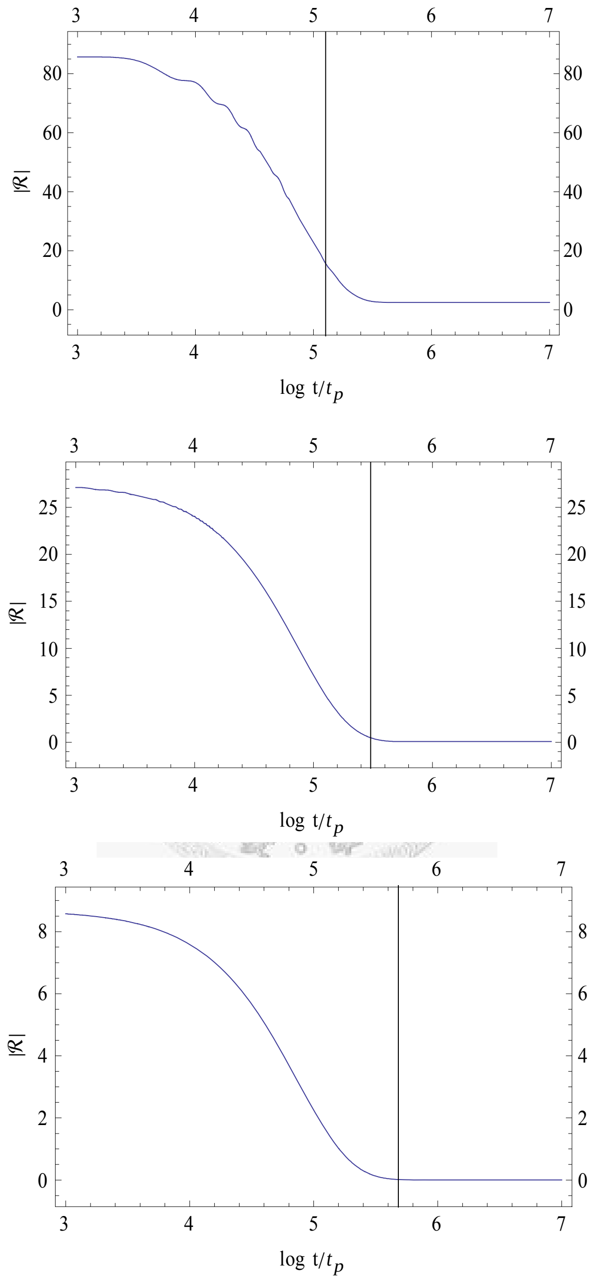
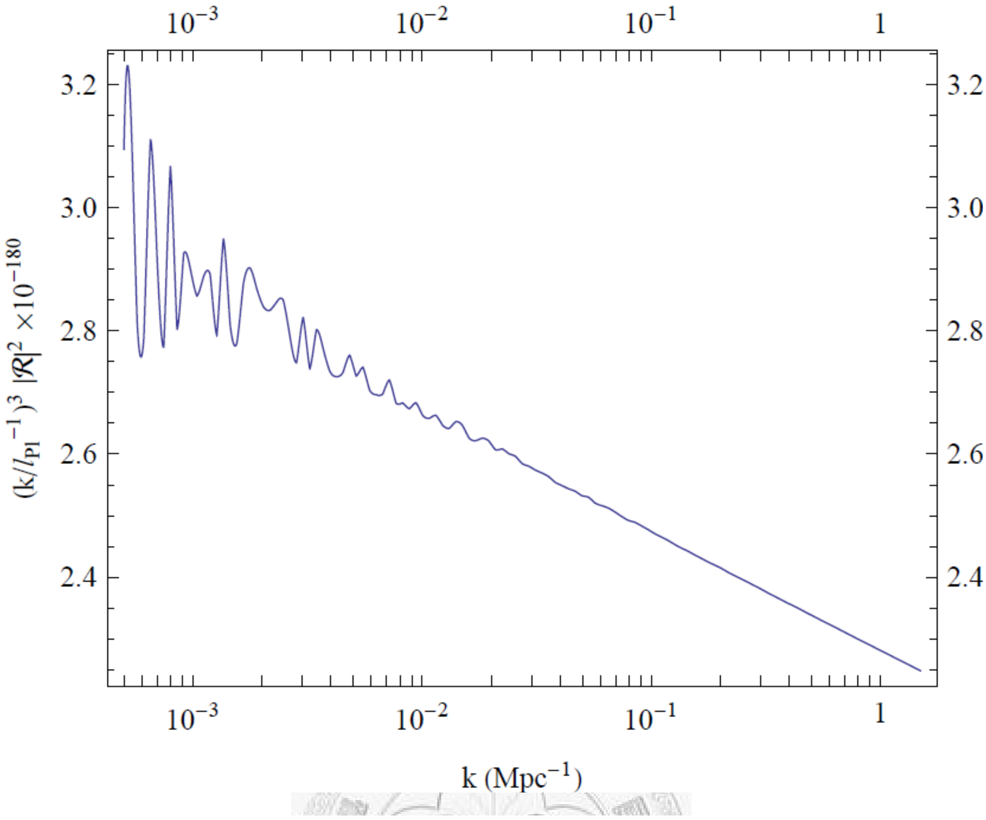


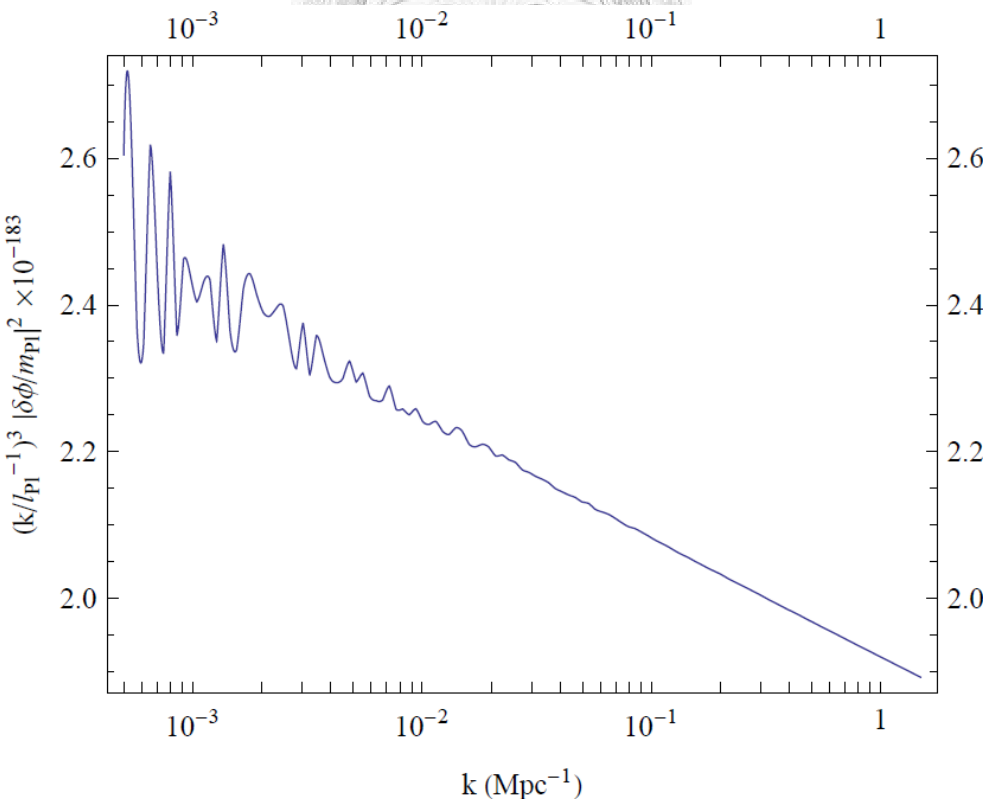
Fig 5.5 The time evolution of R. The physical wavelengths of the modes are equal, $\frac{1}{10}$, and $\frac{1}{100}$ to the size of the Hubble radius today, respectively, from the top to the bottom. The horizontal lines denote the time at which the mode exits the Hubble radius during inflation.

5.1.3 The primordial power spectra

The power spectra change with time. The modes stop evolving roughly after they exit the Hubble radius, as we have seen in the previous section. Thus the amplitude of every mode is commonly recorded at the time at which the mode exits the Hubble radius. However, as we have also noted, the modes do not hold still exactly at the moments they cross the radius, but decay for a little bit longer as shown in figure 5.5. Since the reason we look for the primordial power spectrum is that it conserves outside the radius and serves as the initial conditions in the following radiation era, it is more reasonable to record the value of the amplitude after it settles down. This is exactly what we do when finding the primordial power spectra. For the range of k we are interested in, we record the value of the amplitude at $t = 10^{6.2} t_p$, at which the disturbance ceases. Figure 5.6 shows the primordial power spectra. In addition to the power spectrum of R, the ones of $\delta\phi$ and Ψ are also given for reference.

Among the primordial power spectra of various quantities, the one of R is of the most importance, since it is the one that remain unchanged since the mode exits the Hubble radius till it re-enters. Therefore once we find the primordial power spectrum of R in the inflation era, we can take it as the initial condition at the beginning of the radiation era.


10^{-3}
10^{-2}
10^{-1}
1
3.2
3.0
2.8
2.6
2.4
$(k/l_{Pl}{}^{-1})^3\,|\mathcal{R}|^2 \times 10^{-180}$
k (Mpc^{-1})


10^{-3}
10^{-2}
10^{-1}
1
2.6
2.4
2.2
2.0
$(k/l_{Pl}{}^{-1})^3\,|\delta\phi/m_{Pl}|^2 \times 10^{-183}$
k (Mpc^{-1})

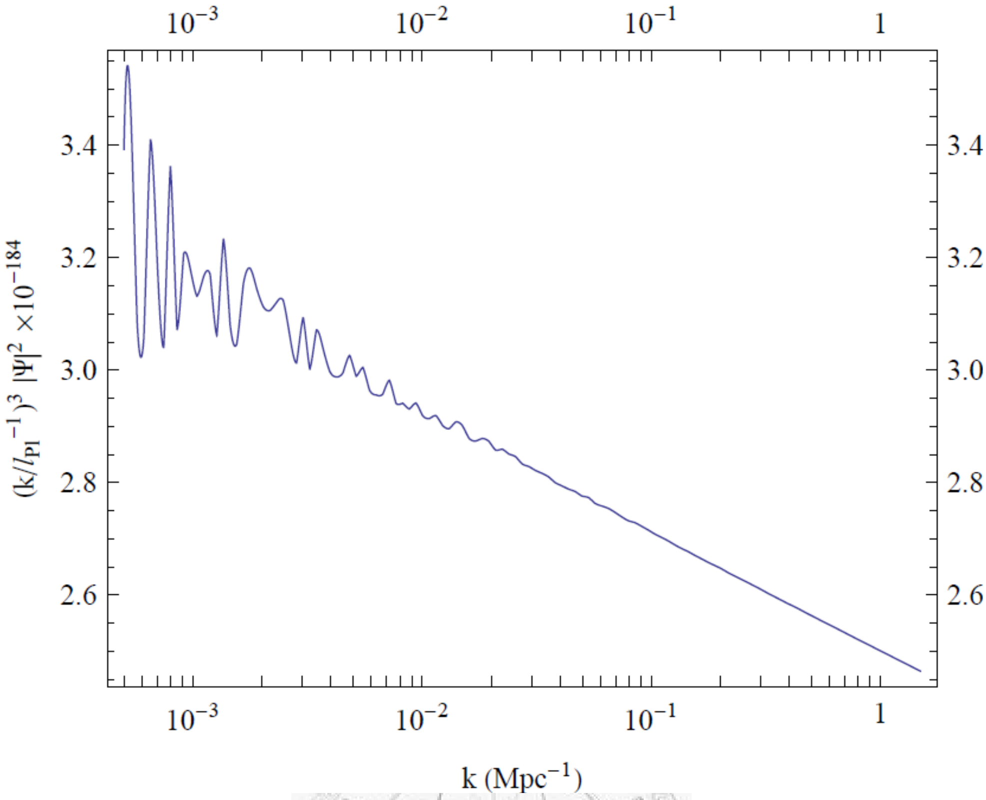


Fig 5.6 The primordial power spectra of R, $\delta\phi$, and Ψ. They look identical in shape except the overall scaling because of the choice of the parameters. Note that the conditions for the approximations to be valid are violated for modes with k smaller than about 10^{-3}; that is, the primordial power spectrum is valid only for $k > 10^{-3}$ roughly.

5.2 The pre-inflation matter era

5.2.1 The expansion of modes

According to the estimation made in the work of Scardigli, Gruber, and Chen [16], the energy density of the primordial black hole remnants at $t = 10^3 t_p$ is about $10^{-5} m_p c^2 / l_p^3$, which is about 10^6 times greater than the energy density of the scalar field then. We thus assume an initial matter density ahead of the beginning of inflation and solve the Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\inf} + \rho_M\right), \tag{5.9}$$

as well as the conservation equations of the scalar field and matter,

$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + m^2\phi_0 = 0\,, \tag{5.10}$$

$$\rho_M = \frac{\rho_{M0}a_0^3}{a^3}\,. \tag{5.11}$$

After combined with the solution of the ΛCDM model, we obtain the mode evolution, as shown in figure 5.7. The parameters of the scalar fields and ΛCDM model are the same as that in the pure inflation case. The overall expansion due to matter and inflation is about 71 e-folds, greater than that in the pure inflation case. In figure 5.8 and 5.9 we enlarge the portion of the inflation era and compare with the pure inflation case.

We can see that the most significant difference is that in the case with pre-inflation matter era, the size of the mode is equal to, or even greater than, the size of the Hubble radius at the beginning, while in the case of pure inflation, the size of the mode is constantly smaller than the size of the Hubble radius at the beginning. This means that the applicability of the approximations we made when performing quantization is more restricted in the case with pre-inflation matter era. In fact, some scales that are valid to apply the approximations in pure inflation case fail to be valid in the case with pre-inflation matter era

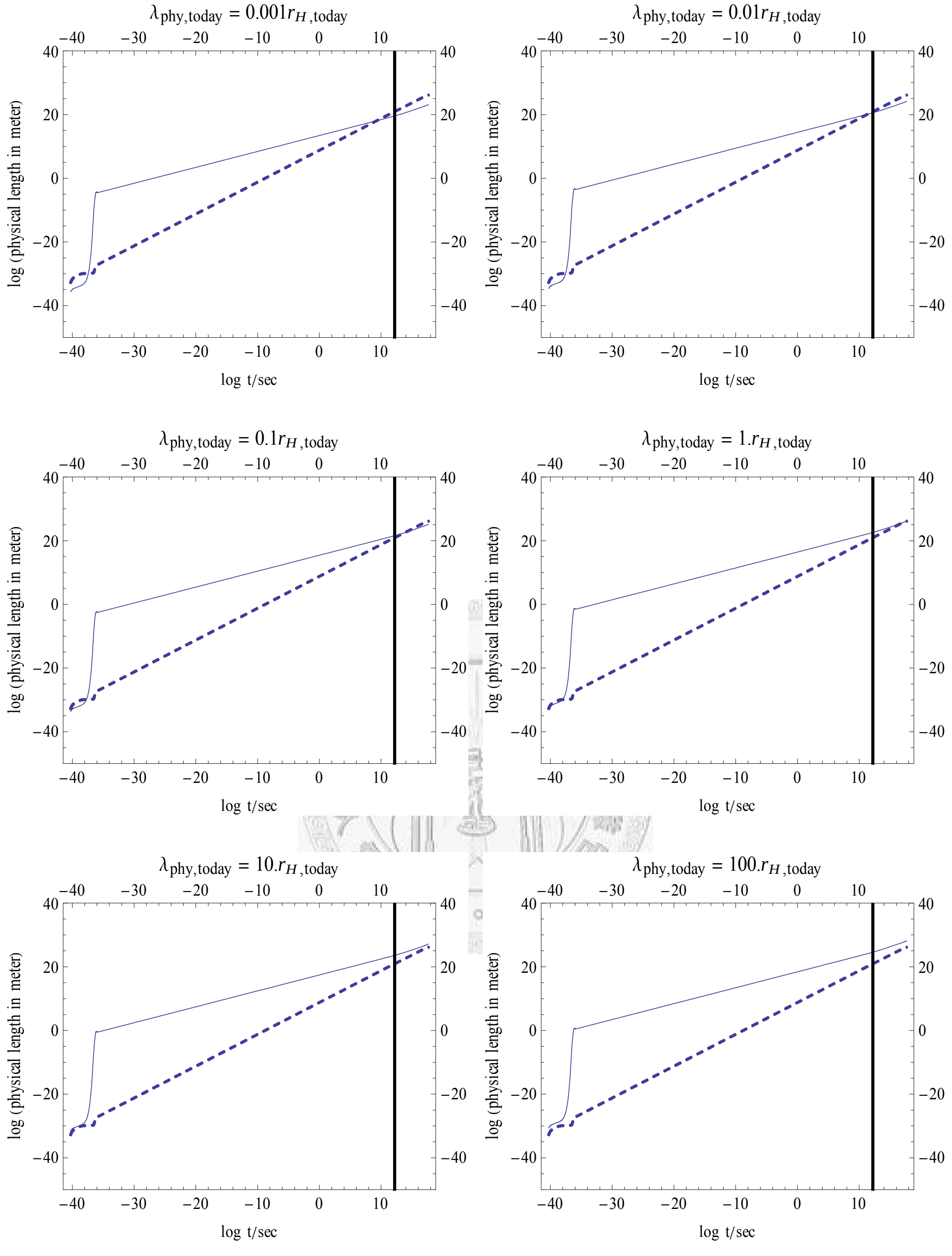
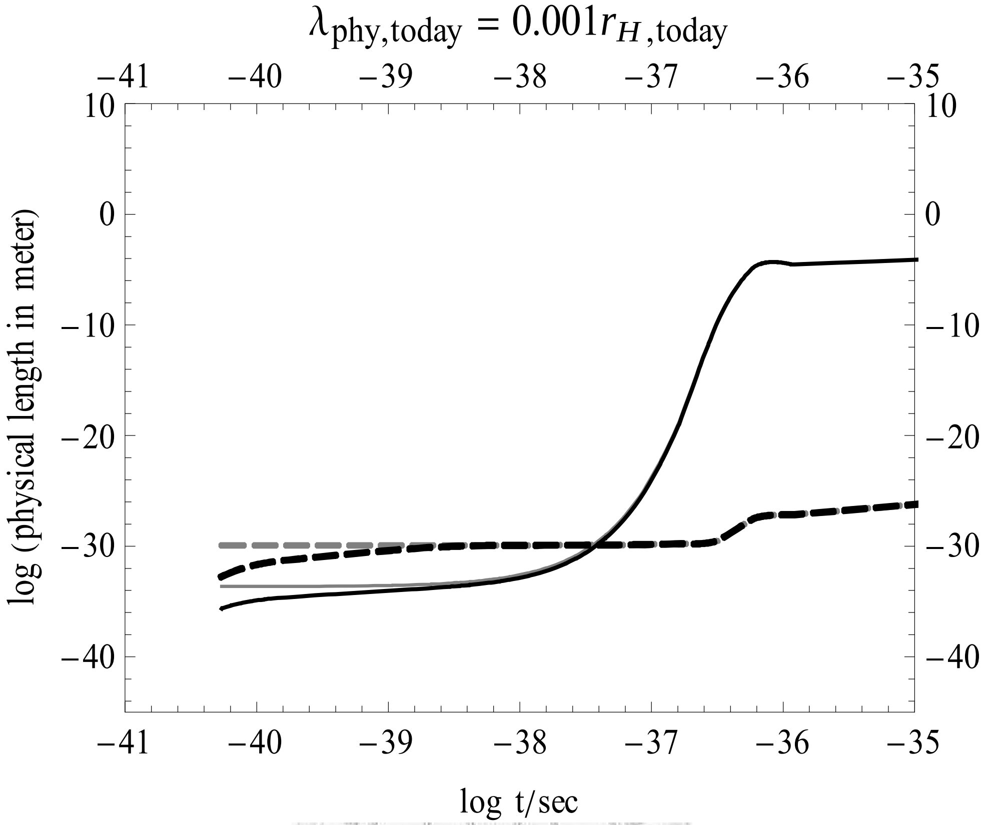
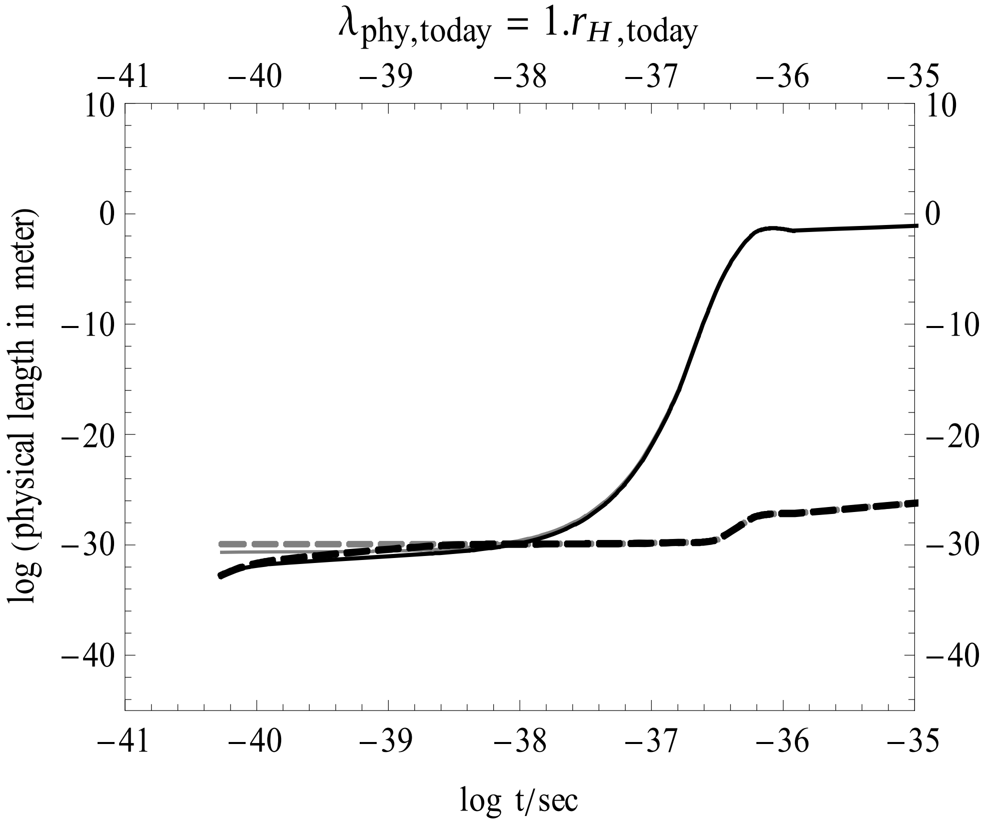


Fig 5.7 The expansion of modes, with pre-inflation matter era. The solid curve is the physical wavelength of the mode with its physical wavelength today labeled on the top of the plot. The dashed curve is the physical Hubble radius. The horizontal line is the time at which the radiation-matter equality happens.


$\lambda_{\text{phy,today}} = 0.001 r_{H,\text{today}}$
log (physical length in meter)
log t/sec


$\lambda_{\text{phy,today}} = 1. r_{H,\text{today}}$
log (physical length in meter)
log t/sec

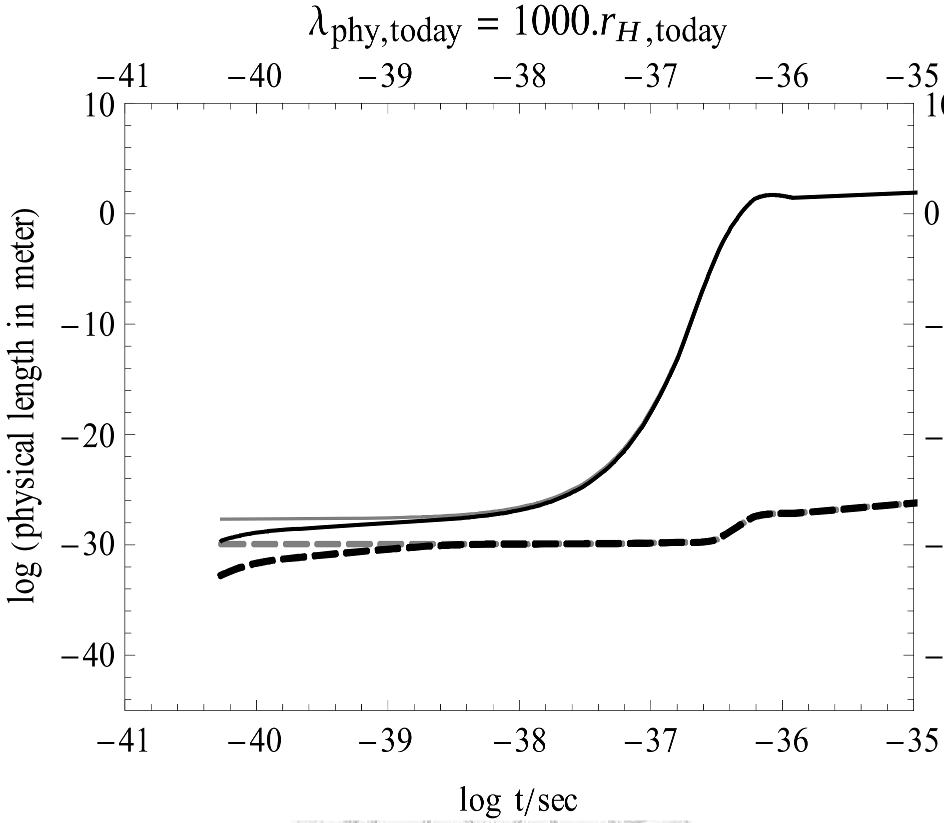
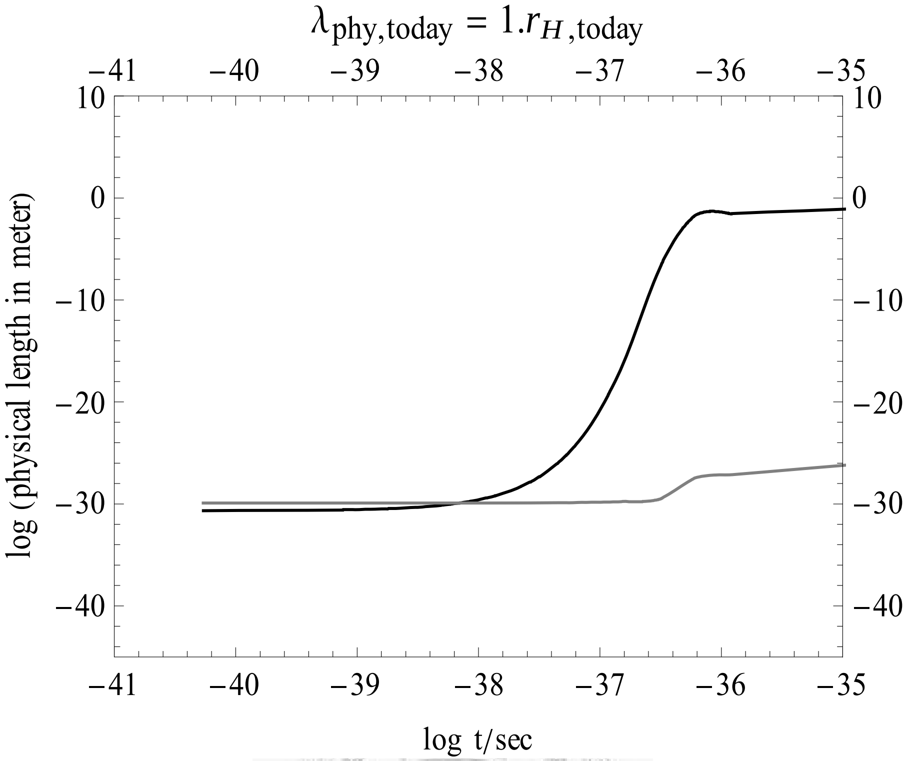


Fig 5.8 The expansion of modes. The darker set of curves, which curve downward at the left end, is the case with pre-inflation matter era. The lighter set of curves, which are flat at the left end, is the pure inflation case. The solid curve is the physical wavelength of the mode, and the dashed curve is the physical Hubble radius. Modes of different wavelengths are shown.



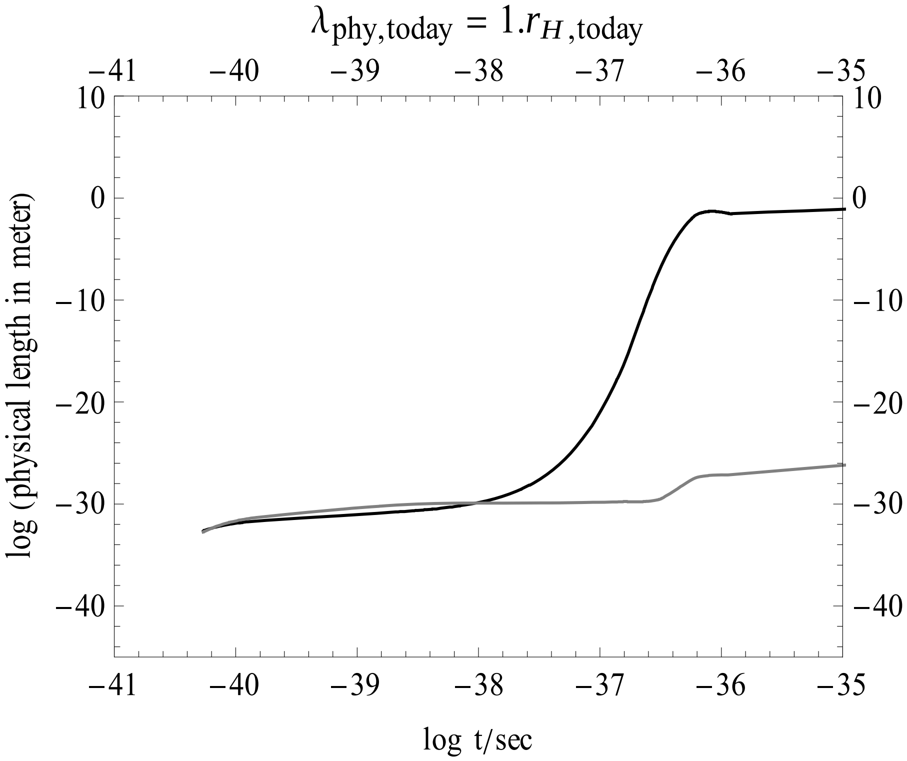


Fig 5.9 The separate enlarged plots of the mode whose physical wavelength today is equal to the Hubble radius today. The top one is the pure inflation case, and the bottom one is the case with pre-inflation matter era.

5.2.2 The time-evolution of the inflaton field and the metric perturbation

In the case with pre-inflation matter era, we solve the equations (5.4) and (5.5) with initial conditions (4.57) and (4.58) in the matter dominated epoch:

$$\delta\phi = \frac{3}{8\pi^{5/2}\sqrt{k}\rho_0}\frac{1}{\overline{\eta}^2}\left[1-\frac{i}{k\overline{\eta}}\right]e^{-ik\overline{\eta}}\,, \quad \text{(Early times)} \qquad (4.57)$$

$$\Psi = \frac{1}{\sqrt{\pi k}}\frac{d\phi_0}{dt}\left[\frac{i}{k}+\frac{3}{k^2\overline{\eta}}-\frac{3i}{k^3\overline{\eta}^2}\right]e^{-ik\overline{\eta}}\,. \quad \text{(Early times)} \qquad (4.58)$$

In figure 5.10-5.12 we give the time-evolutions of the fields in the case with pre-inflation matter era.

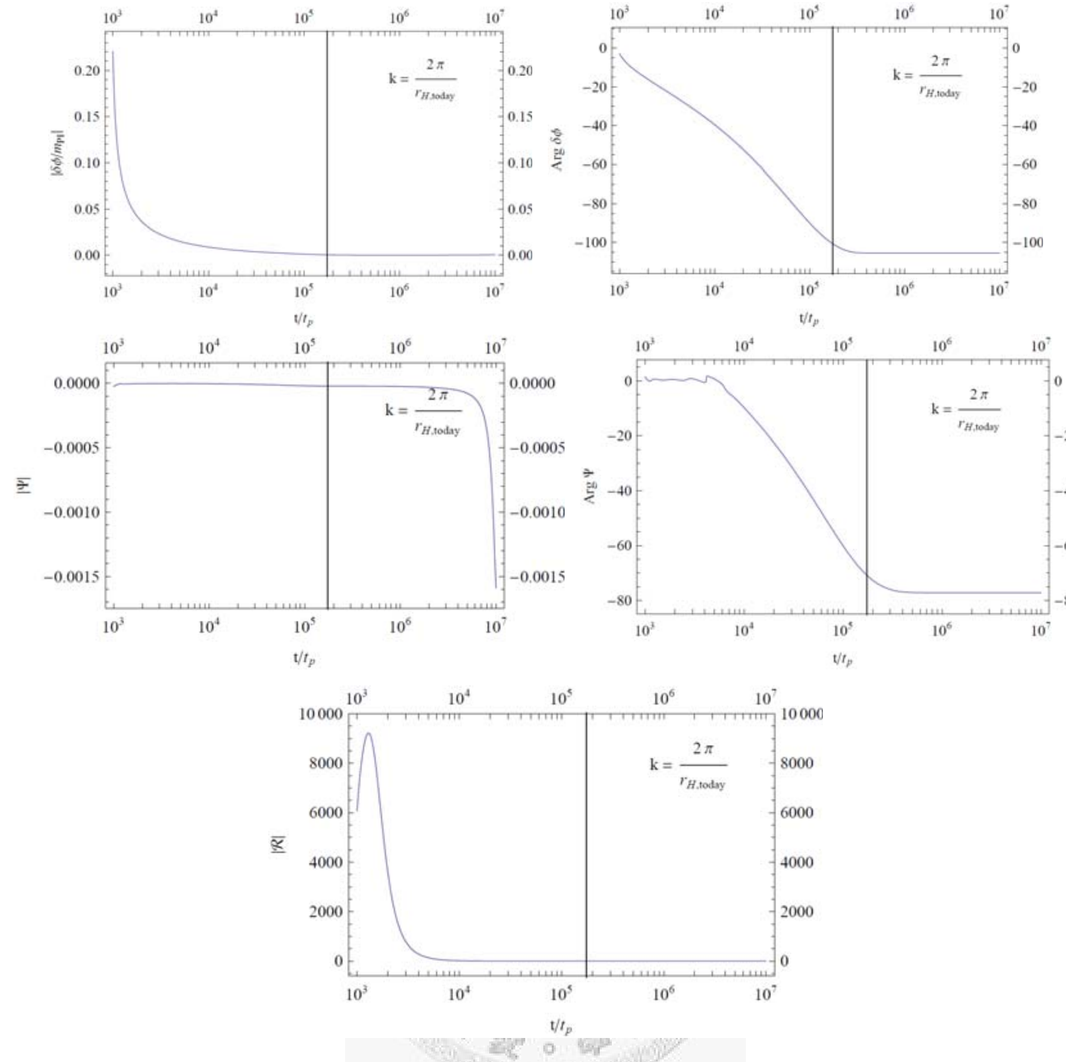


Fig 5.10 The time evolutions of various fields. The physical wavelength of the mode is equal to the size of the Hubble radius today. The horizontal lines denote the time at which the mode exits the Hubble radius during inflation. The approximations are valid.

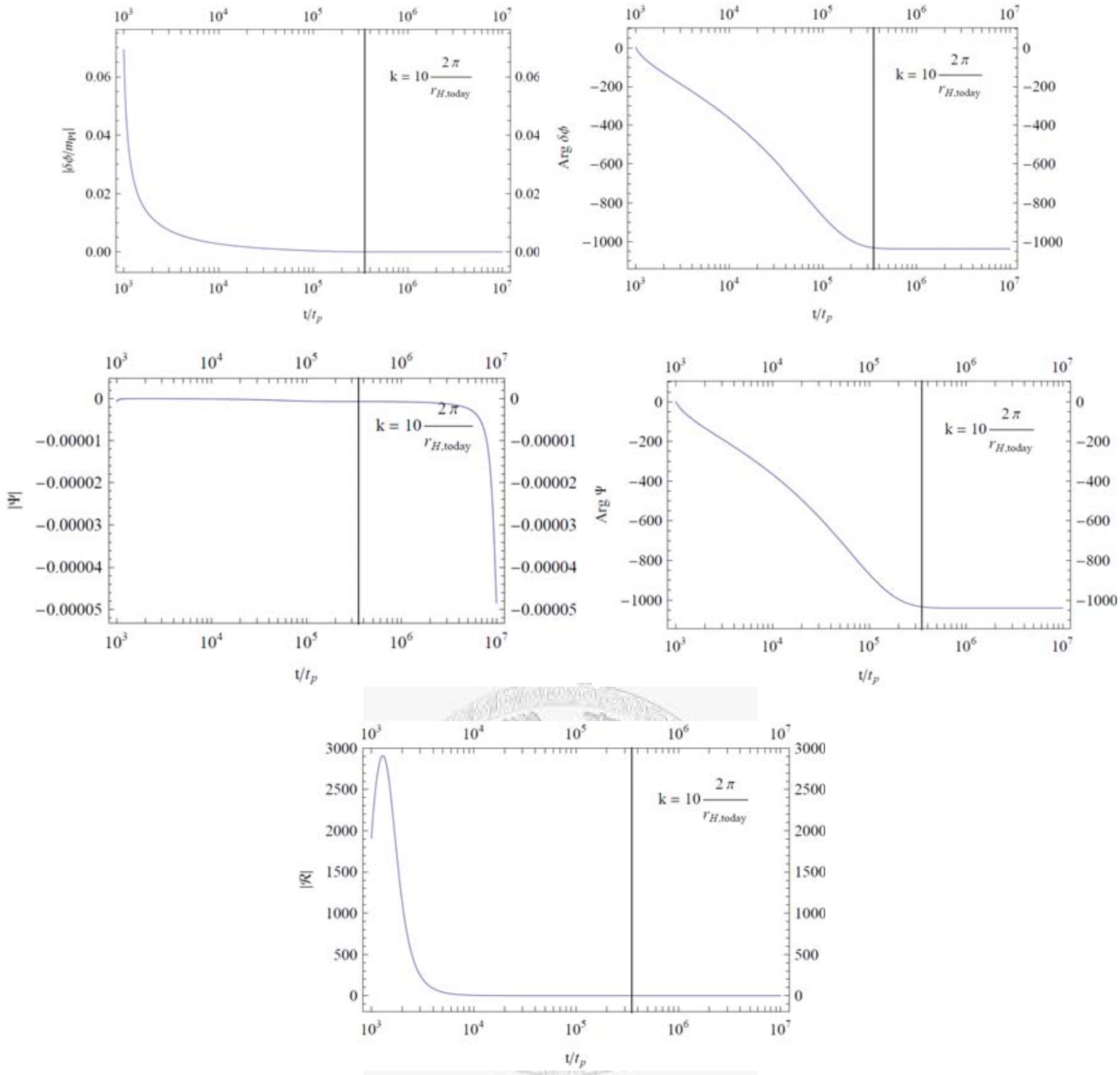


Fig 5.11 The same plots as figure 5.10, for the mode whose physical wavelength is $\frac{1}{10}$ to the size of the Hubble radius today. The approximations are valid.

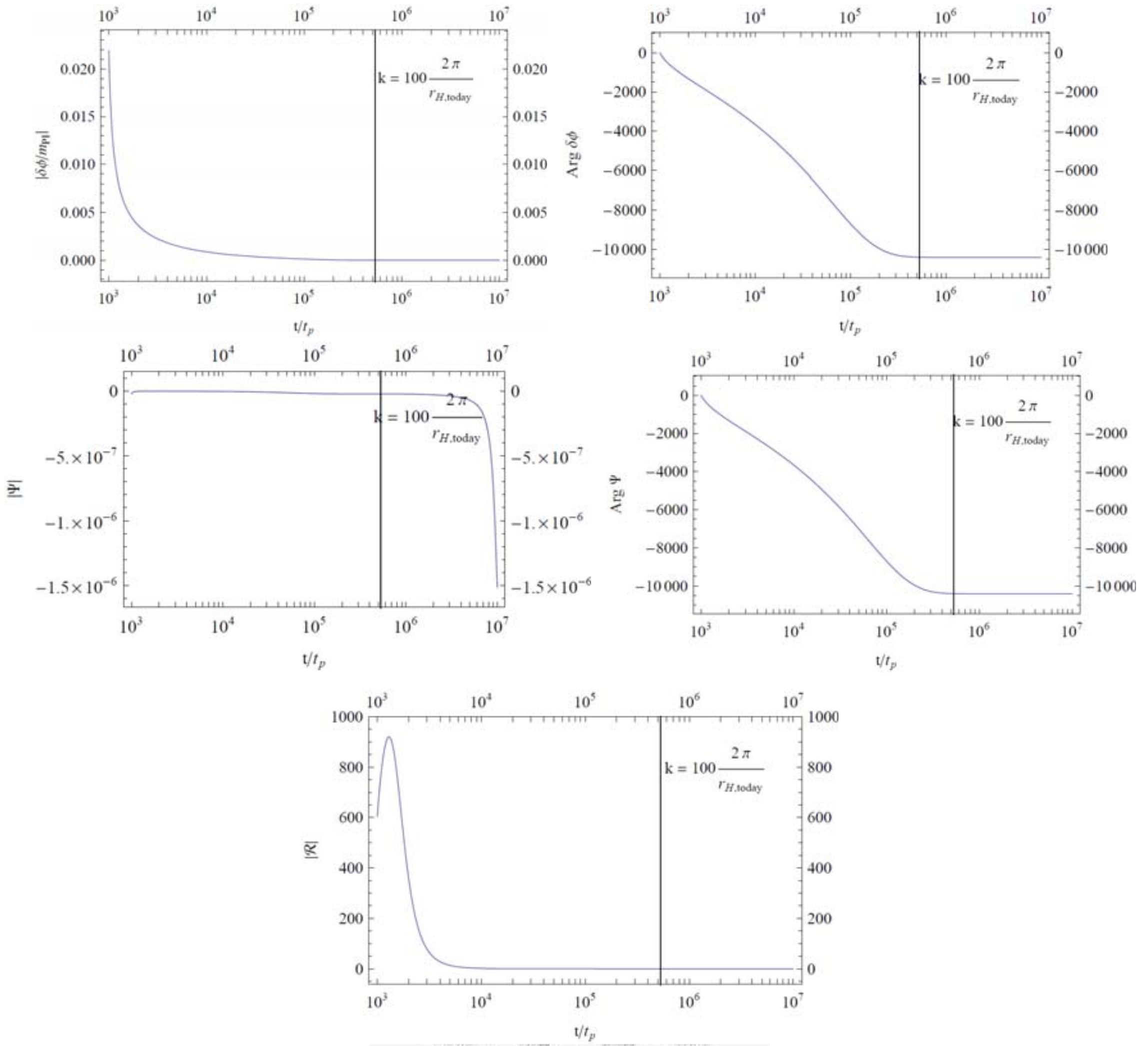
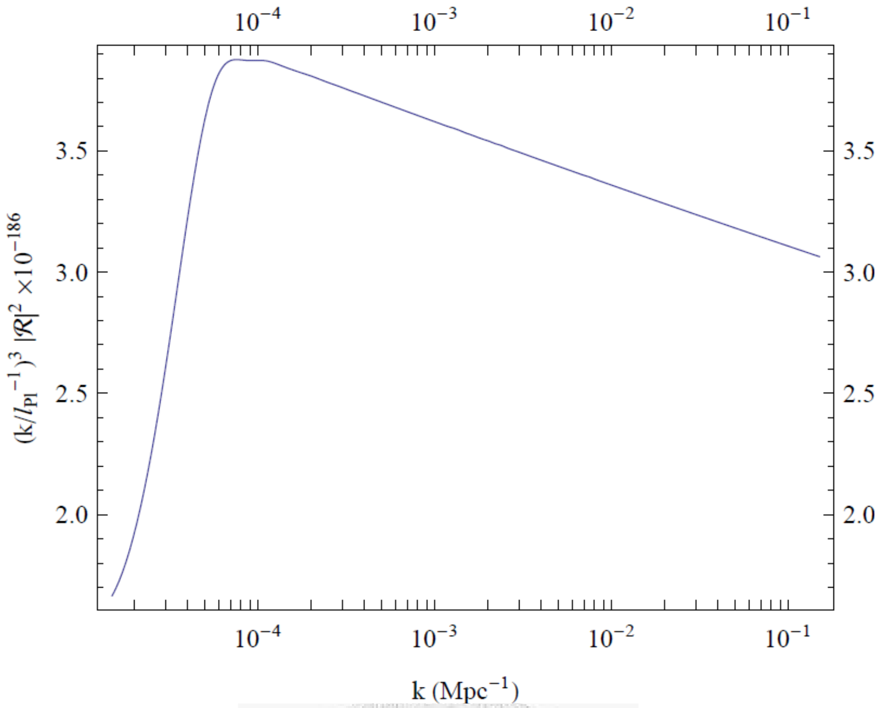
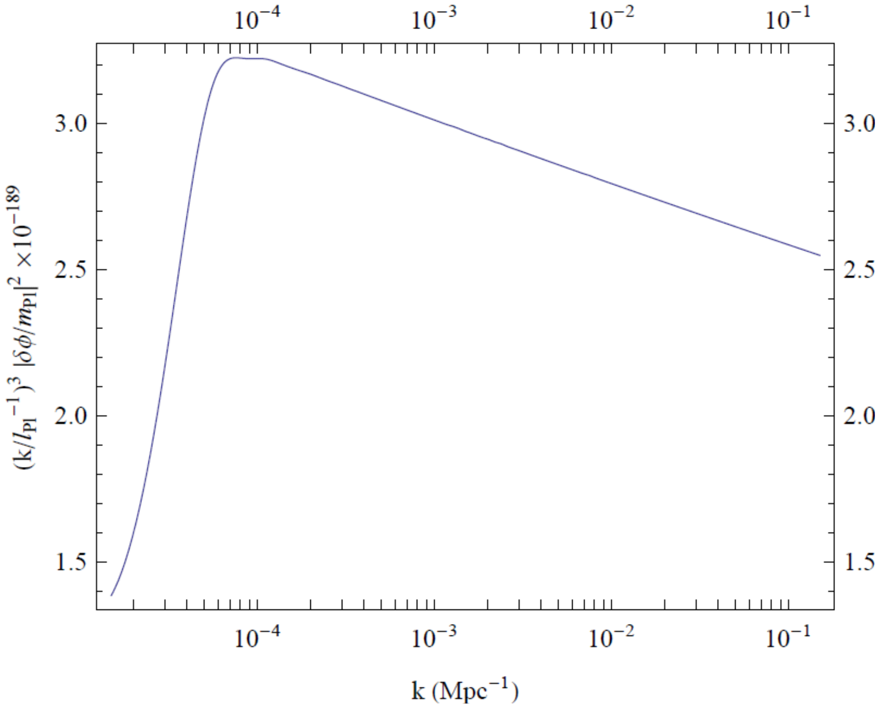


Fig 5.12 The same plots as figure 5.10, for the mode whose physical wavelength is $\frac{1}{100}$ to the size of the Hubble radius today. The approximations are valid.

5.2.3 The primordial power spectra

The primordial power spectra are given in figure 5.13. We can see that there is indeed a cutoff at large scales.


10^{-4} 10^{-3} 10^{-2} 10^{-1}
3.5
3.0
2.5
2.0
$(k/l_{Pl}^{-1})^3\,|\mathcal{R}|^2 \times 10^{-186}$
k (Mpc^{-1})


10^{-4} 10^{-3} 10^{-2} 10^{-1}
3.0
2.5
2.0
1.5
$(k/l_{Pl}^{-1})^3\,|\delta\phi/m_{Pl}|^2 \times 10^{-189}$
k (Mpc^{-1})

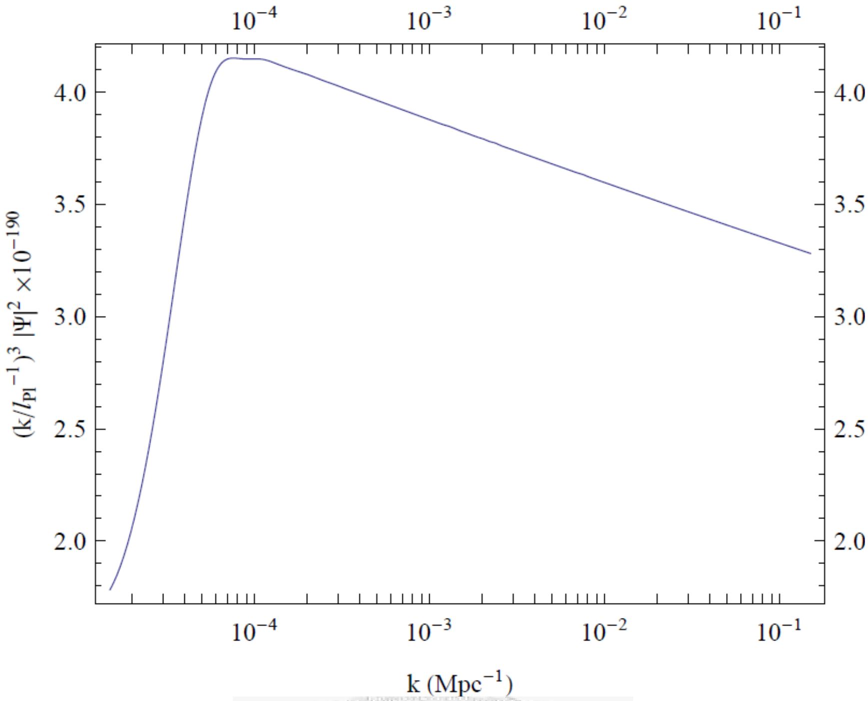


Fig 5.13 The primordial power spectra of R, $\delta\phi$, and Ψ, in the case with pre-inflation matter era. The conditions for the approximations to be valid are satisfied at all ranges in this figure.

Chapter 6

Conclusions

The pre-inflation matter era produces a cutoff at large scales of the primordial power spectrum. The cutoff is due to the fact that one needs to quantize the initial perturbations at the pre-inflation matter dominated era, in which some approximations made in the pure-inflation scenario are no longer valid. Under some suitable choices of parameters, the cutoff occurs at about $10^{-4} Mpc^{-1}$, corresponding to the modes that just re-enter to the Hubble radius today. The lack of power in the primordial power spectrum may account for the low quadrupole moment of the CMB temperature anisotropy spectrum. Therefore, the CMB quadrupole anomaly can be a hint of the existence of the pre-inflation matter era.

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