Parallel Computations & Applications

National Tsing-Hua University

2017, Summer Semester

Outline

- Embarrassingly Computations
- Divide-And-Conquer Computations
- Pipelined Computations
- Synchronous Computations

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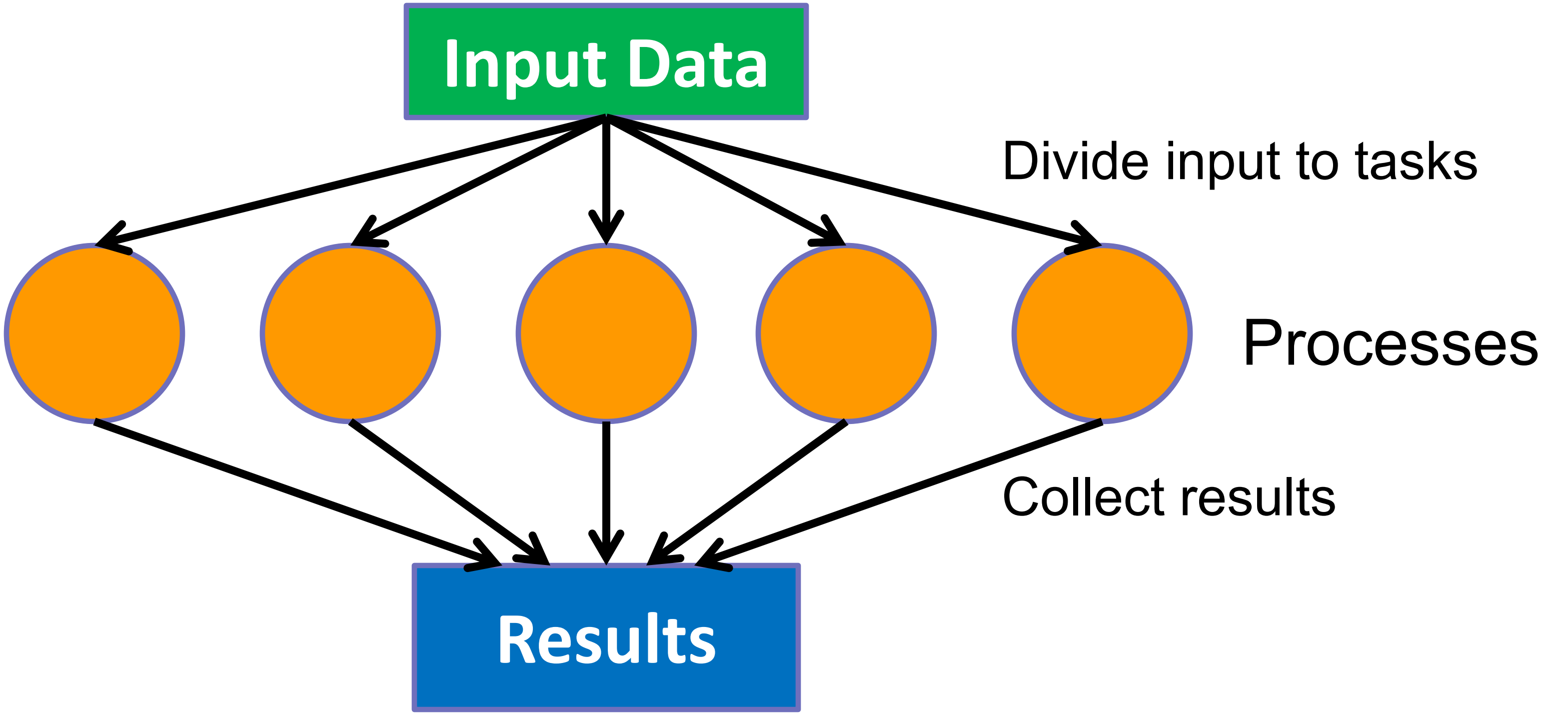
- Embarrassingly Computations
  - Image Transformations
  - Mandelbrot Set
  - Monte Carlo Methods
- Divide-And-Conquer Computations
- Pipelined Computations
- Synchronous Computations

What is Embarrassingly Parallel

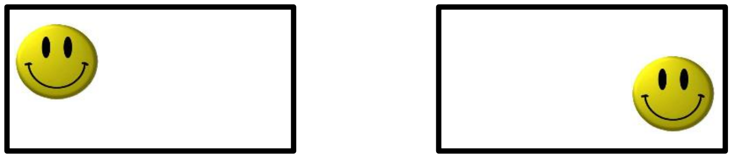
- A computation that can be divided into a number of completely **independent** tasks



Example 1: Image Transformations

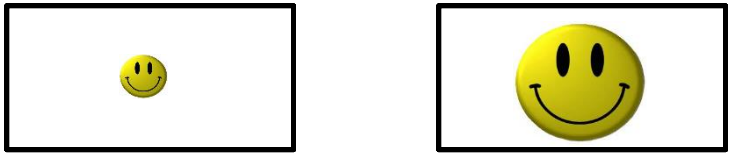
- Low-level image operations:
  - Shifting: object shifted by Δx in the x-dimension and Δy in the y-dimension:

    $$x' = x + \Delta x, \quad y' = y + \Delta y$$

  - Scaling: object scaled by a factor of Sx in the x-direction and Sy in the y-direction;

    $$x' = xS_x, \quad y' = yS_y$$

  - Rotation: object rotated through the angle θ about the origin of the coordinate system:

    $$x' = x\cos\theta + y\sin\theta$$
    $$y' = -x\sin\theta + y\cos\theta$$

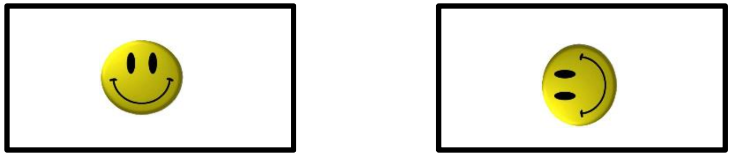
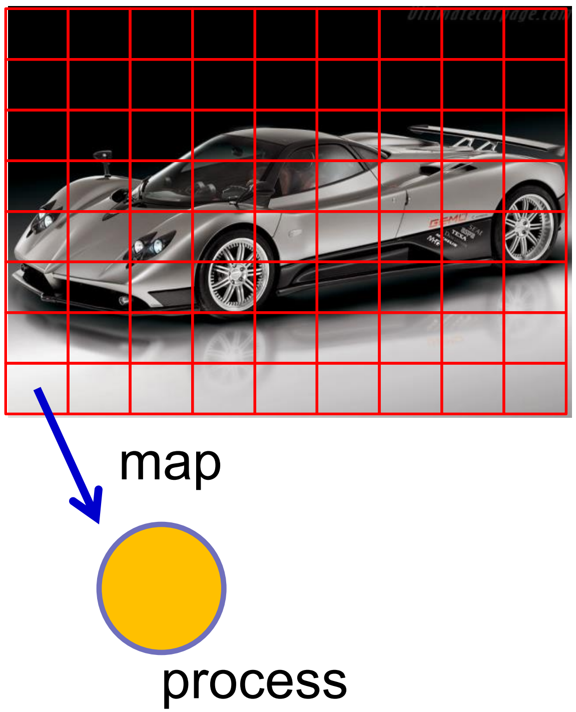
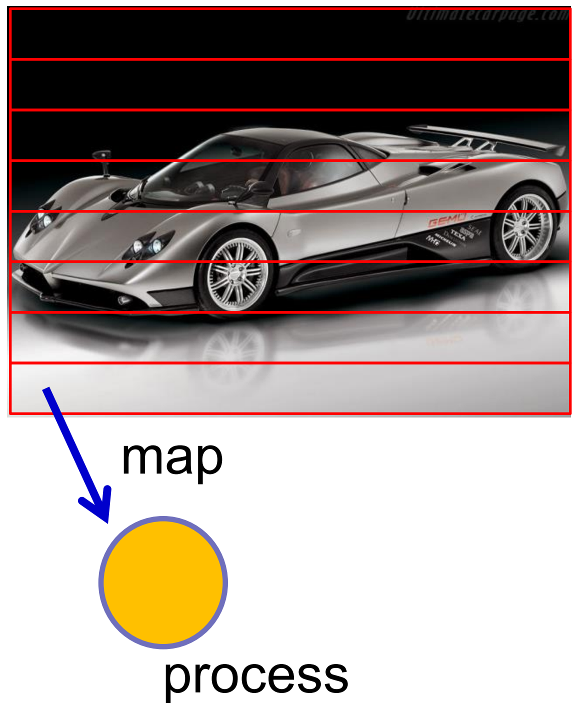
    

Image Region Partitioning

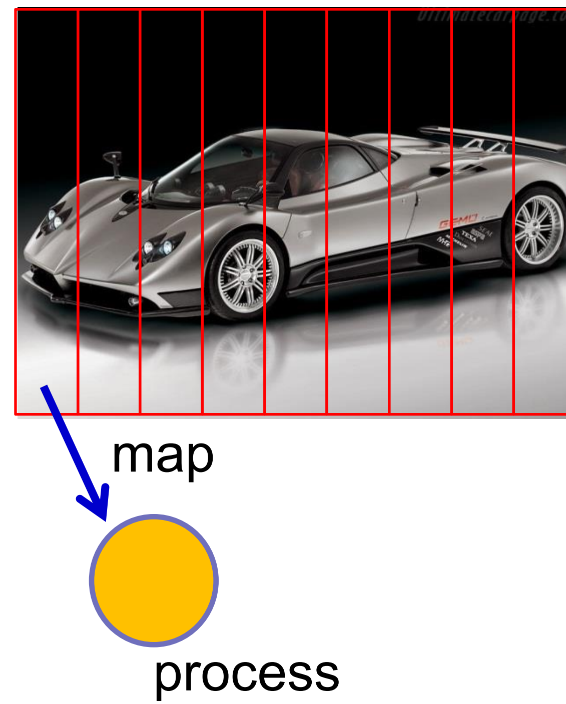
- Square region partition



- Row region partition

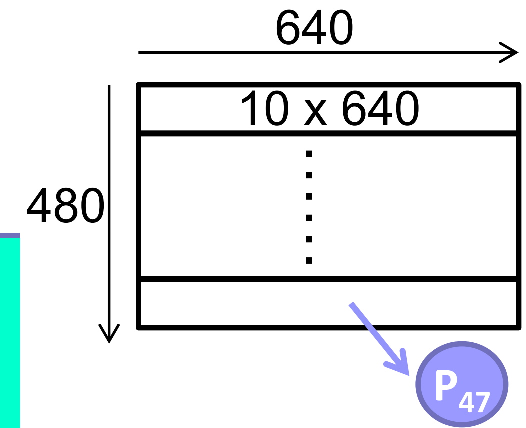


- Column region partition



Pseudo-code for Image Shift

- Partition region by ROW with width 10




```
//master process
for(i=0, row=0; i<48; i++, row+=10)   // for each of 48 processes
      send(row, P_i);                   // send row no.

for(i=0; i<480; i++)  for(j=0; j<640; j++)  temp_map[i][j] = 0; // initialize temp

for(i=0; i<(480*640); i++) {                                  // for each pixel
      recv(oldrow, oldcol, newrow, newcol, P_ANY);            // accept new coordinates
      if !((newrow<0)||((newrow>=480)||(newcol<0)||((newcol>=640))
            temp_map[newrow][newcol] = map[oldrow][oldcol];
}

for(i=0; i<480; i++)  for(j=0; j<640; j++)  map[i][j] = temp_map[i][j];   // update map
```

```
// slave process
recv (row, Pmaster);
for (oldrow = row; oldrow < (oldrow+10); oldrow++)        // for each row in the partition
      for (oldcol = 0; oldcol < 640; oldcol++) {          // for each column in the row
            newrow = oldrow + delta_x;                    // shift along x-dimension
            newcol = oldcol + delta_y;                    // shift along y-dimension
            send(oldrow, oldcol, newrow, newcol, Pmaster); // send out new coordinates
      }
```

Example 2: Mandelbrot Set

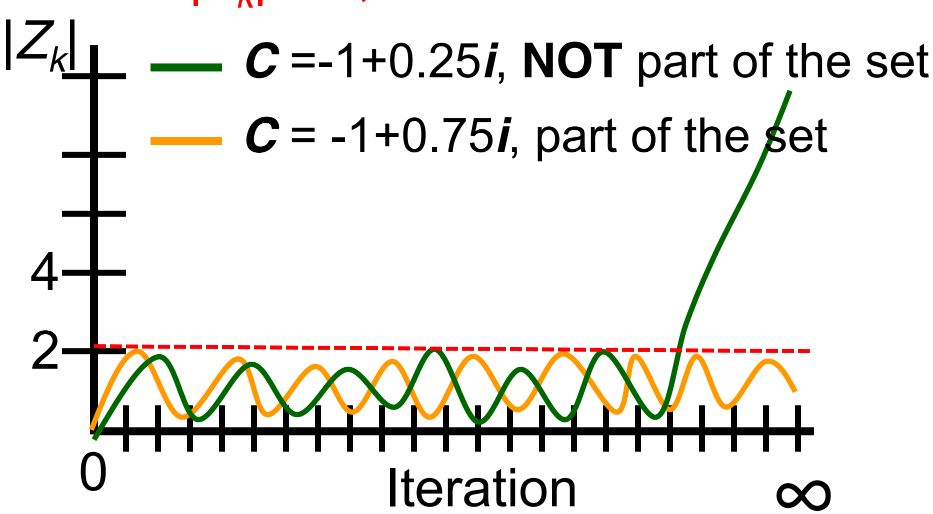
- The Mandelbrot Set is a set of **complex numbers** that are quasi-stable when computed by iterating the function:

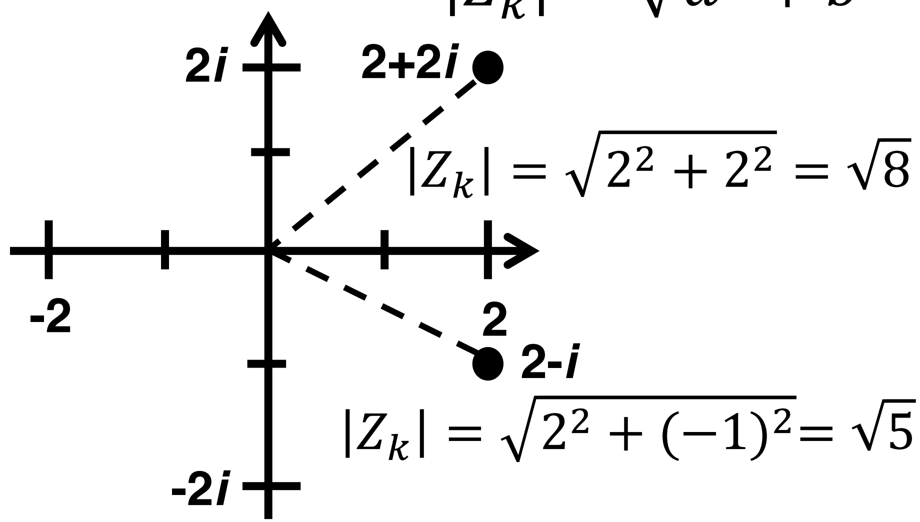
$$Z_0 = C, \qquad Z_{k+1} = {Z_k}^2 + C$$

  - ***C*** is some complex number: $\boldsymbol{C} = a + b\boldsymbol{i}$
  - $\boldsymbol{Z_{k+1}}$ is the (**k+1**)th iteration of the complex number
  - If $|Z_k| \leq 2$ for **ANY** ***k*** ➔ ***C*** belongs to Mandelbrot Set

$$|Z_k| = \sqrt{a^2 + b^2}$$

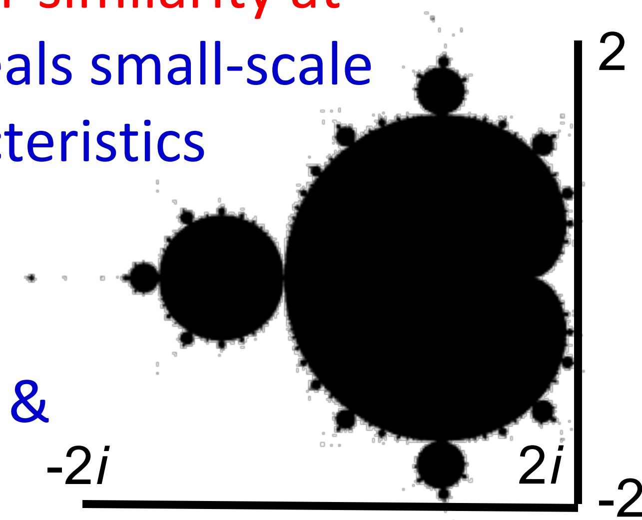
Once $|Z_k| > 2$, it will increase forever!



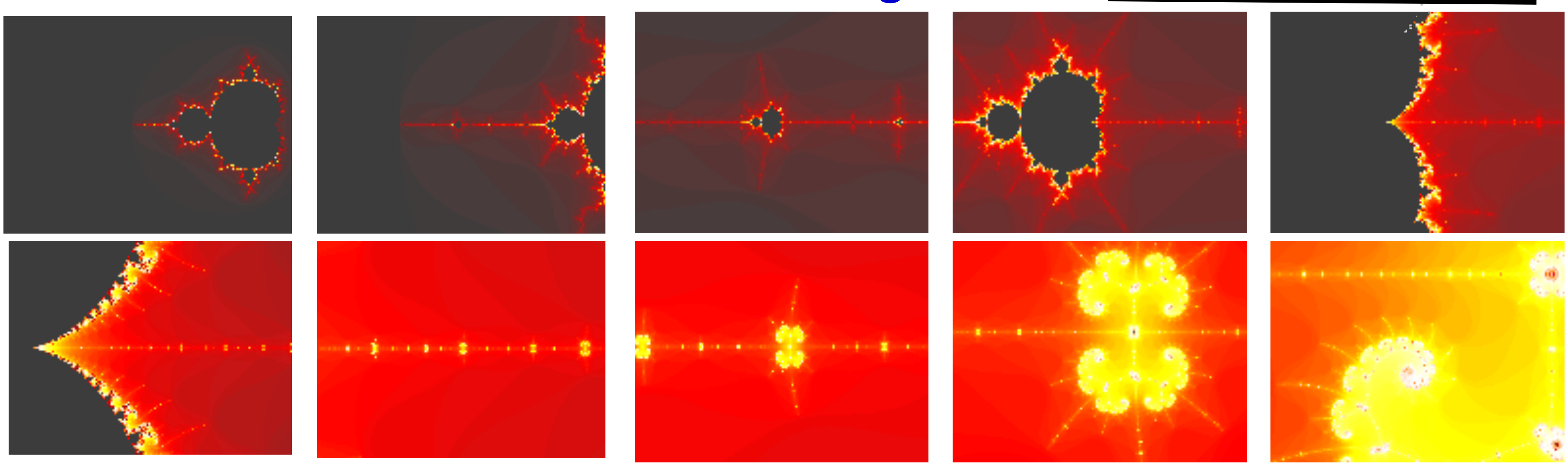


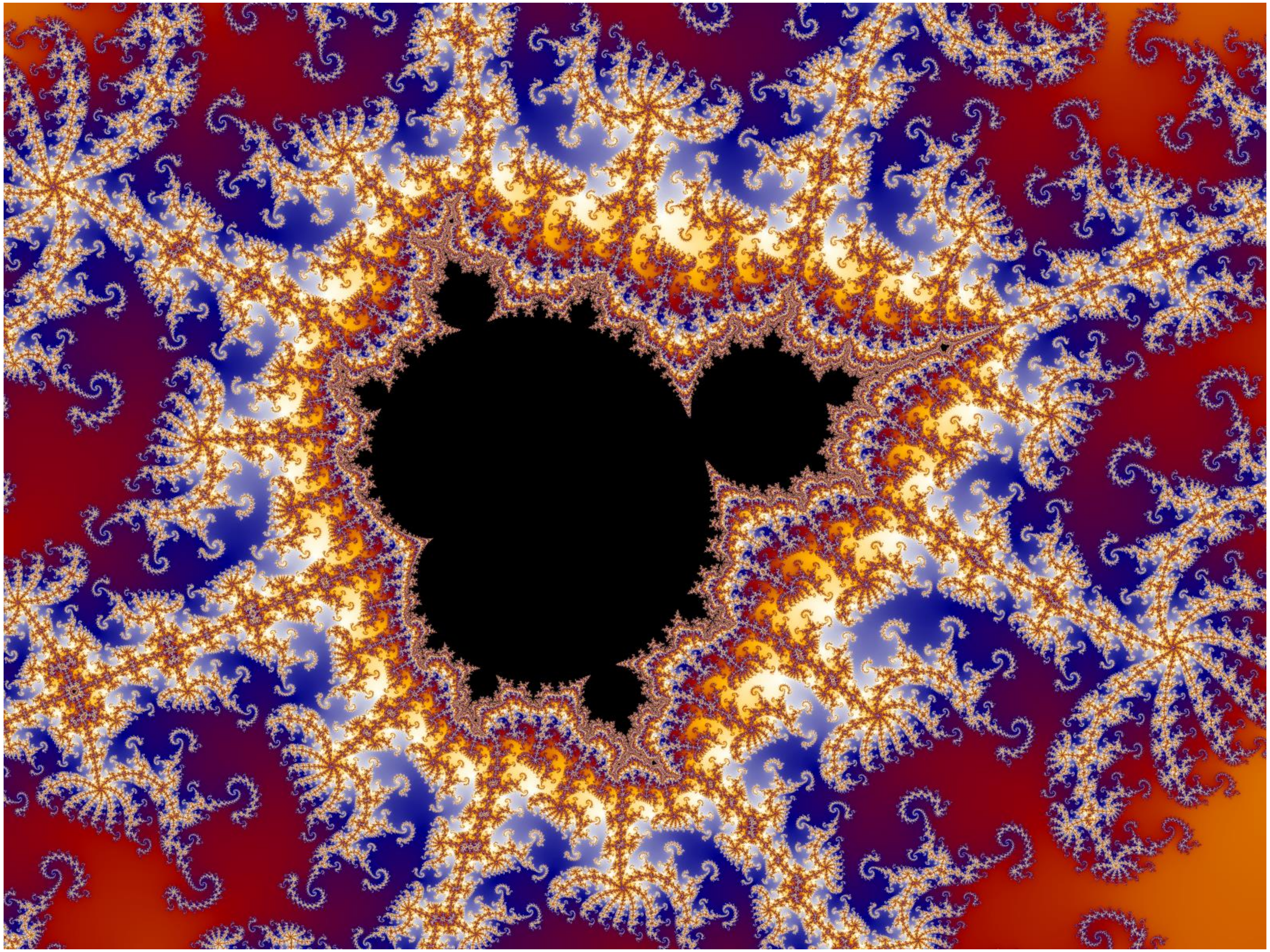
Fractal

- What exact is Mandelbrot Set?
  - It is a **fractal**: An object that display self-similarity at various scale; Magnifying a fractal reveals small-scale details similar to the large-scale characteristics
  - After plotting the Mandelbrot Set determined by thousands of iteration:



  - Add color to the points outside the set & zoom in at the center of the image:





Mandelbrot Set Program

- Compute $Z_{k+1} = {Z_k}^2 + C$
  - Let $C = C_{real} + C_{imag}i$, $Z_k = Z_{real} + Z_{imag}i$
  - $Z_{k+1} = ({Z_{real}}^2 - {Z_{imag}}^2 + 2Z_{real}Z_{imag}i) + (C_{real} + C_{imag}i)$
  - ➔ $Z_{real\_next} = {Z_{real}}^2 - {Z_{imag}}^2 + C_{real}$
  - ➔ $Z_{imag\_next} = 2Z_{real}Z_{imag} + C_{imag}$
- Represent image number in program
  - *C* = 2 + 4***i*** ➔ C.real = 2, C.imag = 4

```
Struct complex {
    float real;
    float imag;
};
```

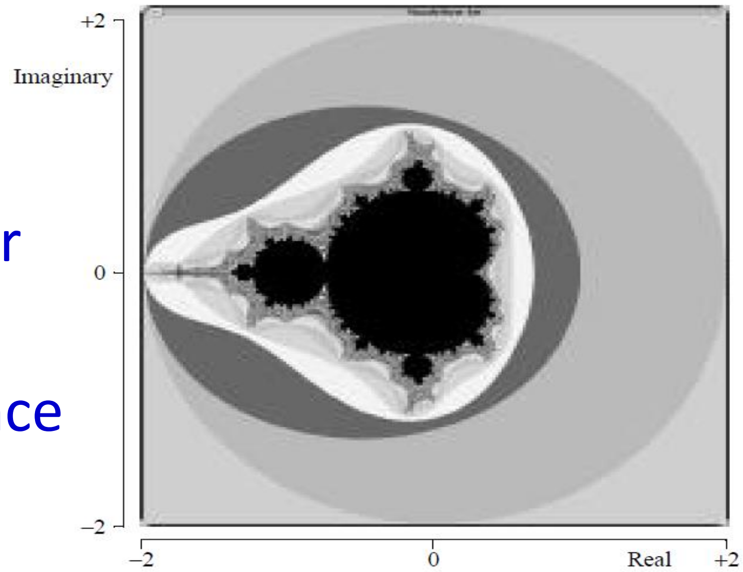
Sequential Mandelbrot Set Program

- Testing program:
  - Giving a complex number
  - Return the iteration number when $|Z_k| > 2$
  - Let the maximum iteration is 256

```
int cal_pixel (complex c) {
   int count = 0;                          // number of iterations
   int max= 256;                           // maximum iteration is 256
   float temp, lengthsq;
   complex z;                              // initialize complex number z
   z.real = 0; z.imag = 0;
   do {
      temp = (z.real * z.real) – (z.imag * z.imag) + c.real;   // compute next z.real
      z.imag = (2 * z.real * z.imag) + c.imag;                  // compute next z.imag
      z.real = temp;
      lengthsq = (z.real * z.real) + (z.imag * z.imag);
      count++;                                                  // update iteration counter
   } while ((lengthsq < 4.0) && (count < max));
   return count;
}
```

Sequential Mandelbrot Set Program

- Scaling Coordinate Display Program:
  - Plot the Mandelbrot Set from the coordinate system
  - Color indicate the iteration number black=256, white=0
  - Points are apart with a fixed distance read_disk, imag_dist




```
for (x=real_min; x < real_max; x += real_dist) {
   for (y=imag_min; y < imag_max; x += imag_dist) {
          c.real = x; c.img = y;
          color = cal_pixel (c);
          display(x, y, color);
   }
}
```

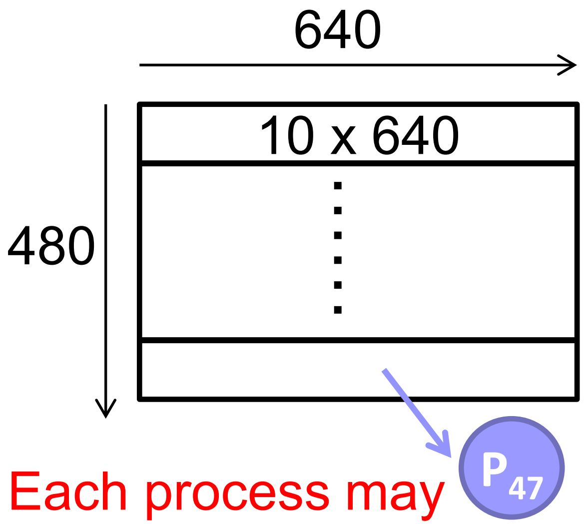
Parallelizing Mandelbrot Set Program

- Partition screen 640*480 by row using 48 processes

```
//master process
for(i=0, row=0; i<48; i++, row+=10)          // for each process
      send(row, P_i);                         // send row no.

for(i=0; i<(480*640); i++) {                  // for each pixel point
      recv(&x, &y, &color, P_ANY);            // receive coordinate/colors
      display(x, y, color);                   // display pixel
}
```

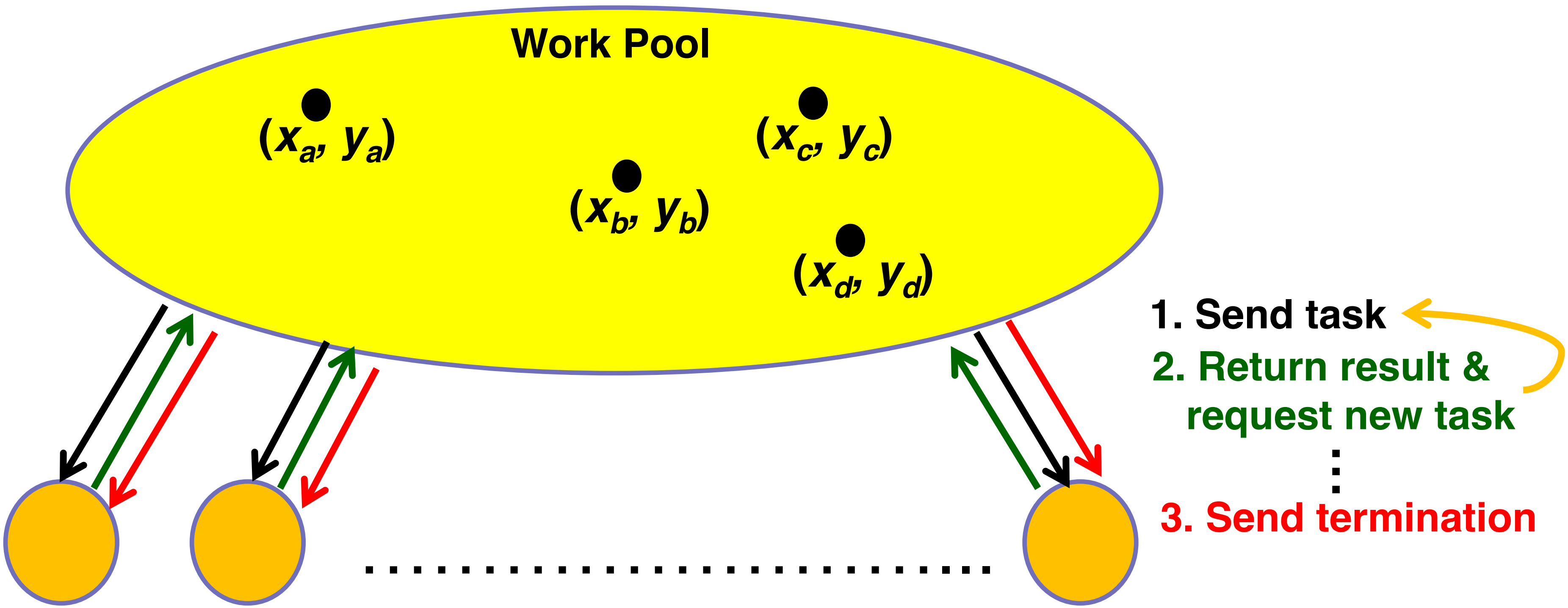
```
//slave process
recv (&row, P_master);
for (x=0; x < 640; x++) {
   for (y=row; y < (row+10); y++) {
         c.real = min_real + (x * scale_real);
         c.imag = min_imag + (y * scale_image);
         color = cal_pixel (c);
         send(x, y, &color, P_master);
   }
}
```



Each process may have different load!

Dynamic Task Assignment

- Work pool / Processor Farm
  - Useful when tasks require different execution time
  - Dynamic load balancing



Coding for Work Pool Approach

```
//master process
count = 0;                              // # of active processes
row = 0;                                // row being sent
for (k=0; k<num_proc; k++) {            // send initial row to each processes
     send(row, P_i , data_tag);
     count++;
     row++;
}
do {
     recv(&slave, &r, color, P_ANY , result_tag);
     count--;
     if (row < num_row) {                       // keep sending until no new task
          send(row, P_slave , data_tag);        // send next row
          count++;
          row++;
     } else {
          send(row, P_slave , terminate_tag);   // terminate
     }
     display(r, color);                         // display row
} while(count > 0);
```

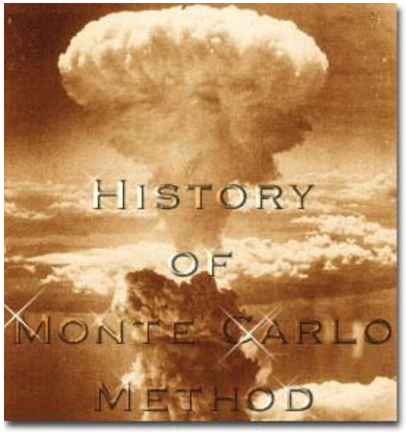
Tag is needed to distinguish between **data** and **termination** msg

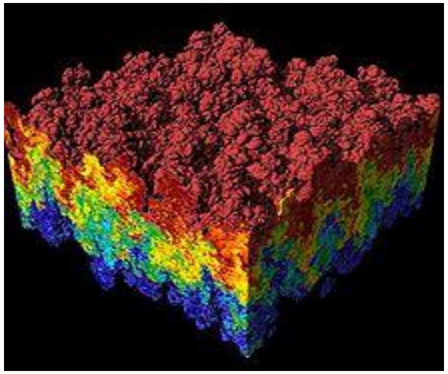
Coding for Work Pool Approach

```
//slave process P ( i )
recv(&row, P_master , source_tag);
while (source_tag == data_tag) {        // keep receiving new task
    c.imag = min_imag + (row * scale_image);
    for (x=0; x<640; x++) {
        c.real = min_real + (x * scale_real);
        color[x] = cal_pixel (c);       // compute color of a single row
    }
    send(i, row, color, P_master , result_tag); // send process id and results
    recv(&row, P_master , source_tag);
}
```

Example 3: Monte Carlo Methods

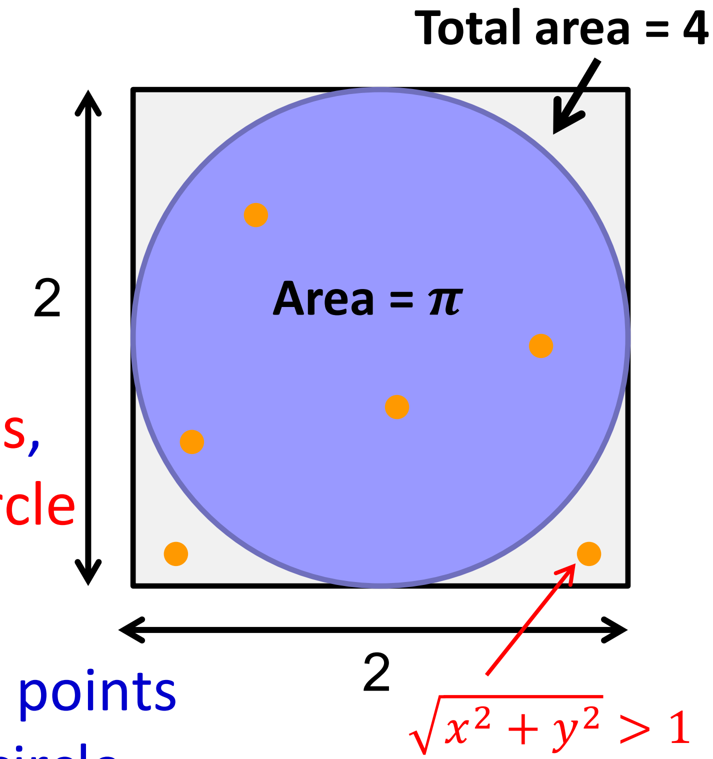
- **Monte Carlo methods**: a class of computational algorithms that rely on **repeated random sampling** to compute their results
  - Invented in 1940s by *John von Neumann*, *Stanislaw Ulam* and *Nicholas Metropolis*, while they were working on nuclear weapon (Manhattan Project)
  - Especially useful for **simulating systems** with many coupled degrees of freedom, such as fluids, disordered material





Monte Carlo Methods --- π calculation

- How to compute π ???
  - Definition of π: the area of a circle with unit radius
  - We know: $\frac{\textbf{Area of circle}}{\textbf{Area of square}} = \frac{\pi}{4}$
  - Randomly choose points from the square
  - Giving sufficient number of samples, the fraction of points **within** the circle will be $\pi/4$!!!
  - E.g.: With 10,000 randomly sample points we expect 7854 points within the circle
    ➔7854/10000 =$\pi/4$ ➔π = 7854/10000*4 = 3.1416

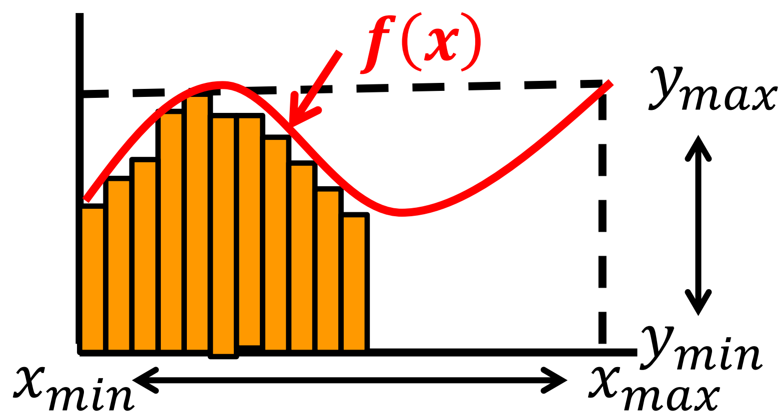


Monte Carlo Methods --- Integral

- Monte Carlo Method can compute ANY **definite** integral!
  - max and min values of the integral must be known
  - Very inefficient….

$$Area = \int_{x_{min}}^{x_{max}} f(x)dx$$

- Method:
  - Randomly choose point (x, y):
    - $x_{max} \leq x \leq x_{min}$
    - $y_{max} \leq y \leq y_{min}$
  - Compute the area (integral) according to the ratio of points inside and outside the area
    ➔ just like the computation of π



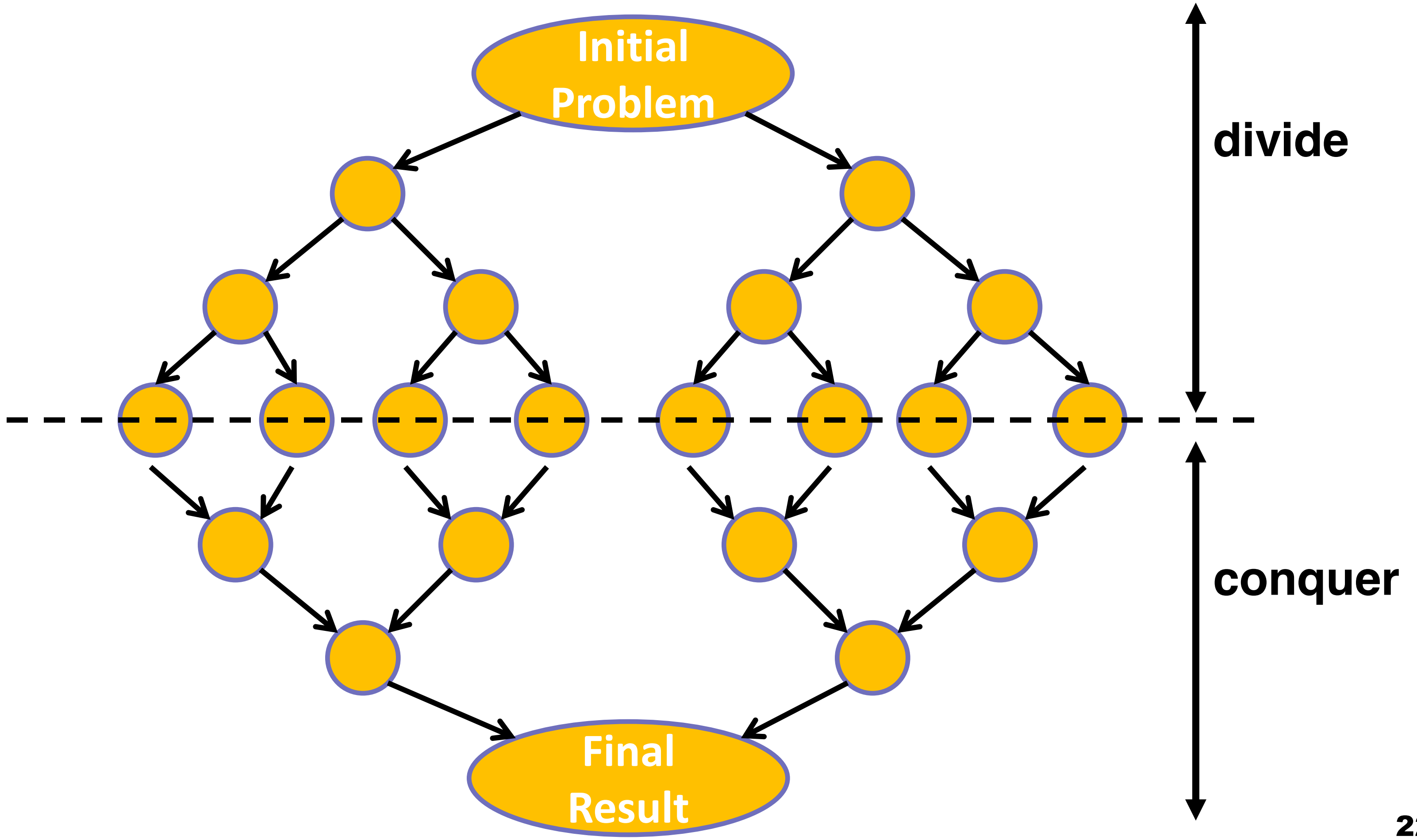
  - Given any point (x, y), outside means : $y > f(x)$

Outline

- Embarrassingly Computations
- Divide-And-Conquer Computations
  - Adding Numbers
  - Bucket Sort
  - N-Body Simulation
- Pipelined Computations
- Synchronous Computations

What is Divide & Conquer

- **Recursively** divide a problem into sub-problems that are of the same form as the larger problem

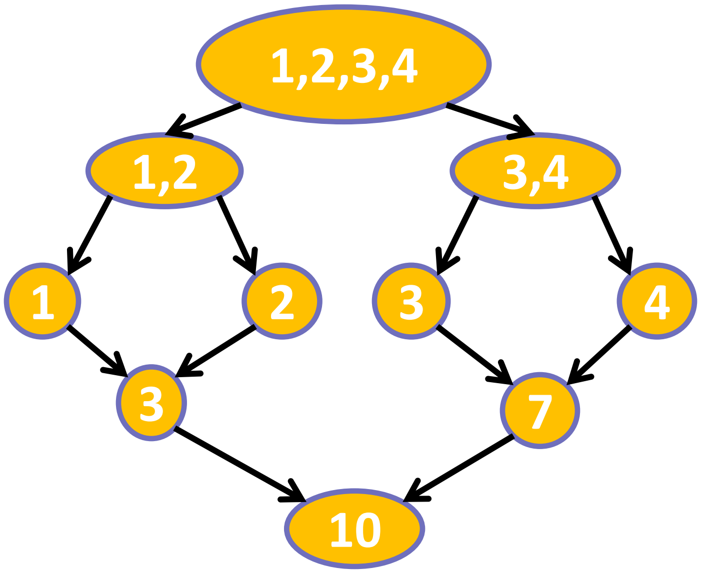


Example 1: Adding Numbers

- Add a sequence of numbers
  - Sequential Recursive Code:

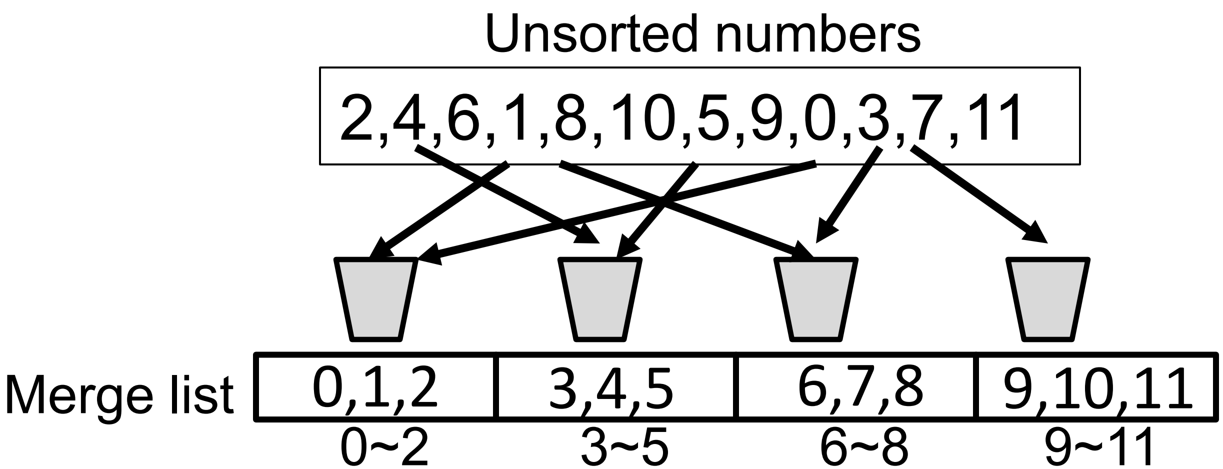
```
int add (int* numbers) {
  if (len(numbers) <=2) {
       return numbers[1]+numbers[2];
   } else {
      divide (numbers, sub_num1, sub_num2);
      part_sum1 = add(sub_num1);
      part_sum2 = add(sub_num2);
  }
  return (part_sum1+part_sum2);
}
```

  - Parallel Code:
    - Scatter the numbers then reduce results



Example 2: Bucket Sort

- Algorithm
  1. Range of numbers is divided into *m* **equal regions**
  2. One bucket is assigned for each region
  3. Place numbers to buckets based on the region
  4. Use sequential sort for each bucket



- Only effective if number of items per bucket is similar!!
  - Numbers should have a known interval ([max, min])
  - Numbers better to be uniformly distributed

Complexity Analysis

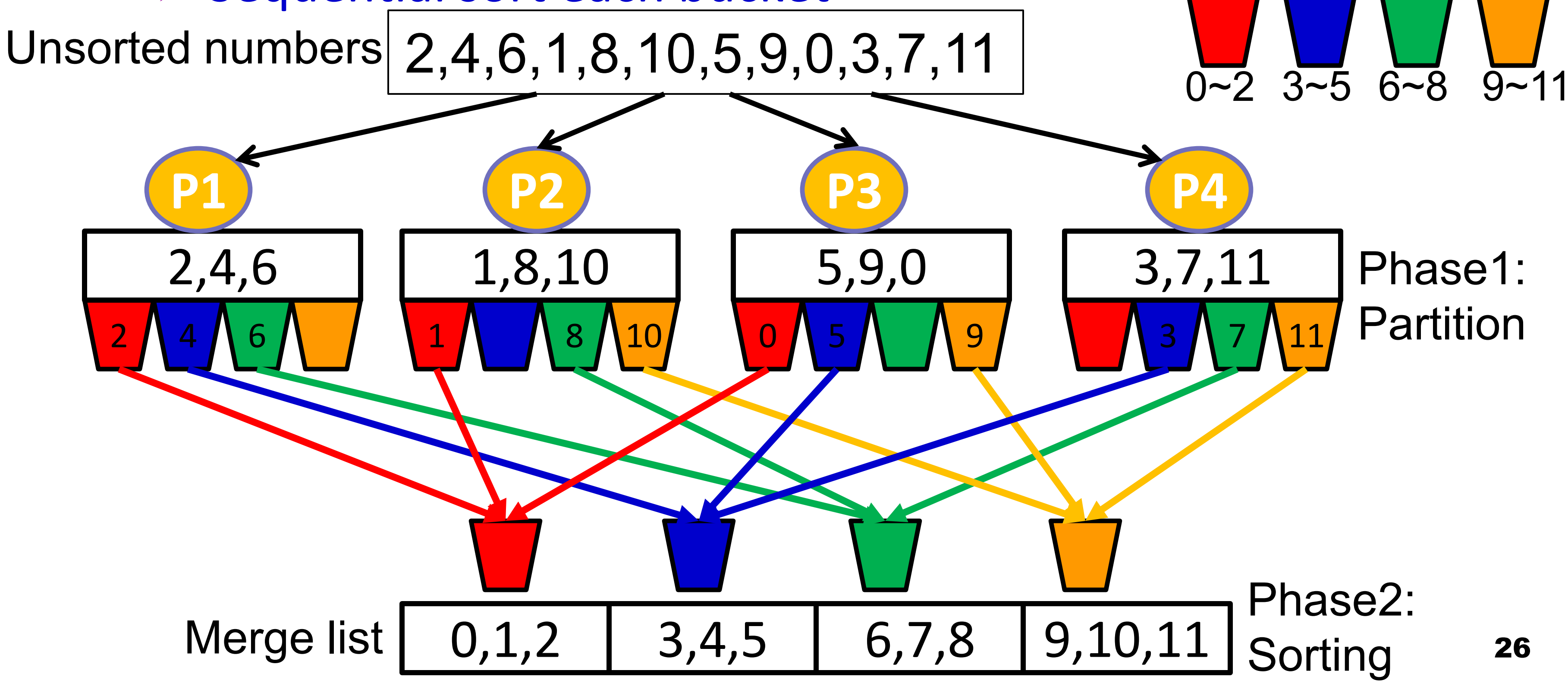
- Sequential:
  1. Distribute numbers to bucket: $O(n)$
  2. Sequential sort each bucket: $(n/m)\log(n/m) \times m$
  - ➢ Overall: $O(n \log(n/m))$

- Parallelize sorting: one process per bucket
  1. Distribute numbers to bucket: $O(n)$
  2. Sequential sort each bucket: $(n/m)\log(n/m)$
  - ➢ Overall: $O(n + n/m \log(n/m))$

  - ➢ A single process must scan through all numbers in step1

Further Parallelized Bucket Sort

- Parallelize partitioning and sorting:
  - Partition numbers to ***m*** parts/processes
  - Each process divides its numbers to small buckets
  - Merge small buckets to large bucket
  - Sequential sort each bucket



Example 3: N-Body Problem

- Newtonian laws of physics
  - The gravitational force between two bodies of masses m_a & m_b :

$$F = \frac{G m_a m_b}{r^2}$$

  - Subject to the force, acceleration occurs

$$F = m \times a$$

- Let the time interval be Δt & current velocity v^t, position x^t
  - New velocity v^{t+1} :

$$F = m\frac{v^{t+1} - v^t}{\Delta t} \Rightarrow v^{t+1} = v^t + \frac{F\Delta t}{m}$$

  - New position x^{t+1}:

$$x^{t+1} = x^t + v^{t+1}\Delta t$$

Three-Dimensional Space

- Considering 2 bodies at (x_a, y_a, z_a)& (x_b, y_b, z_b)

$$r = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2}$$

- The forces, velocities and positions can be resolved in the three direction independently

$$F_x = \frac{Gm_a m_b}{r^2}\left(\frac{x_b - x_a}{r}\right)$$

$$F_y = \frac{Gm_a m_b}{r^2}\left(\frac{y_b - y_a}{r}\right)$$

$$F_z = \frac{Gm_a m_b}{r^2}\left(\frac{z_b - z_a}{r}\right)$$

N-Body Sequential Code

- Assume all bodies have the same mass ***m***

```
for (t=0; t<T; t++) {
    for (i=0; i<N; i++) {
        F = Compute_Force(i);          // compute force in O(N^2)
        v_new[i] = v[i] + F *dt / m;   // compute new velocity
        x_new[i] = x[i] + v_new[i] * dt;  // compute new position
    }
    for(i=0; i<N; i++){
        x[i] = x_new[i];               // update position
        v[i] = v_new[i];               // update velocity
    }
}
```

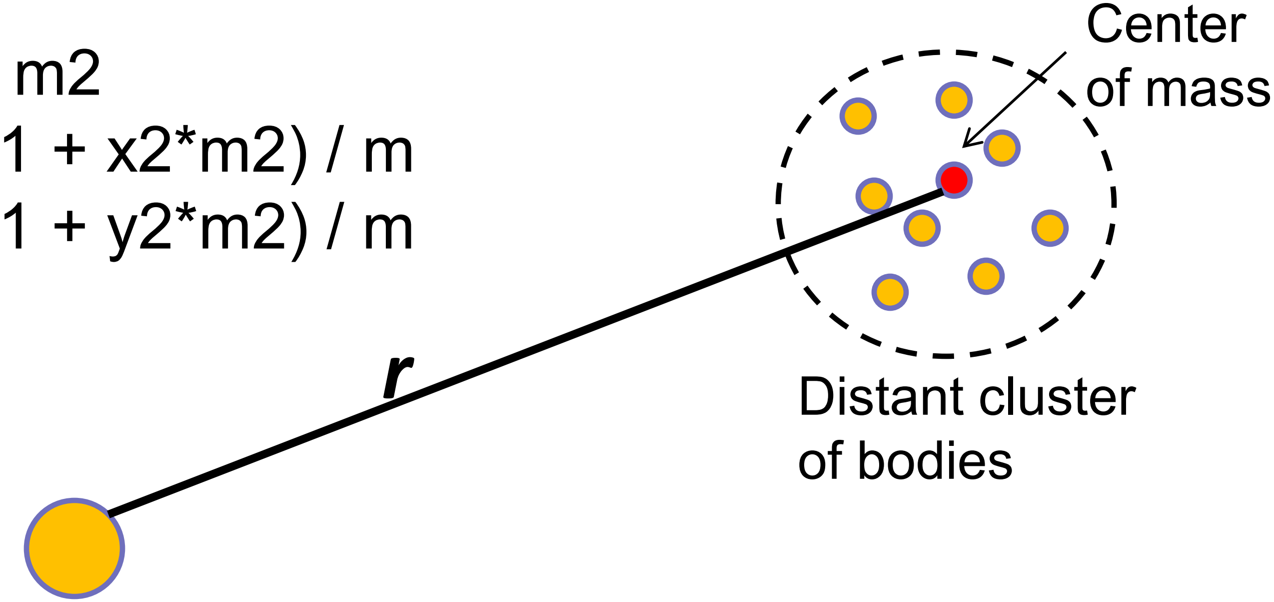
- Non-feasible as *N* increases due to $O(N^2)$complexity

Approximate Algorithms

- Reduce time complexity by approximating a cluster of bodies as a single distant body

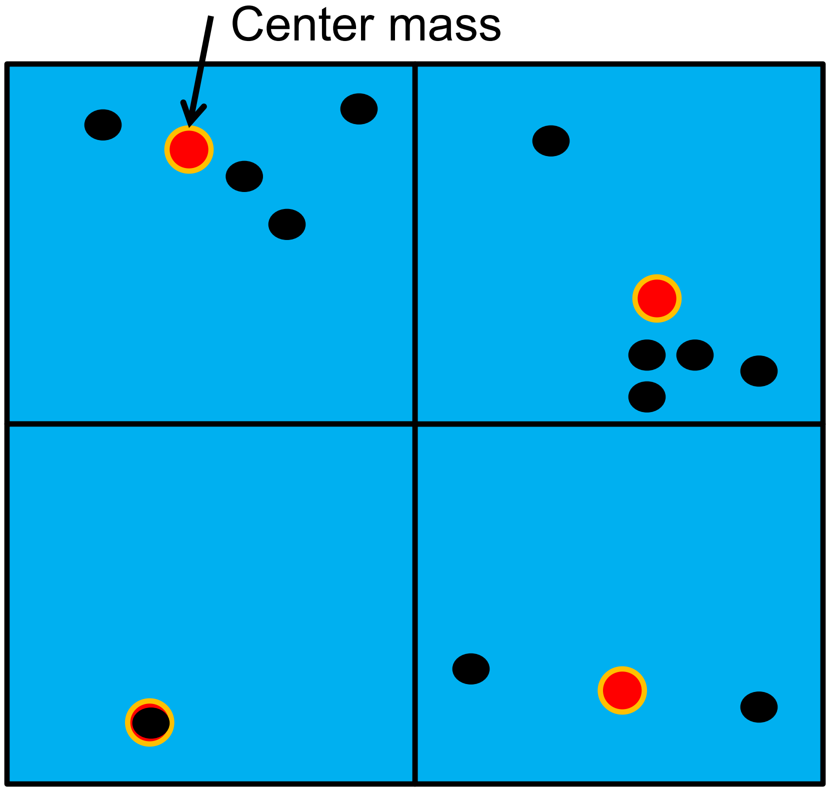
How to find those clusters of bodies?

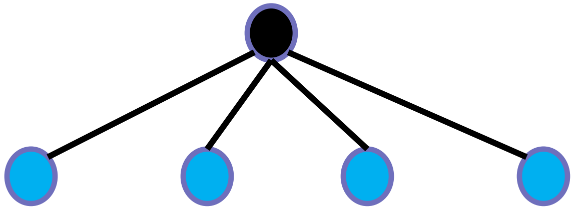
m = m1 + m2
x = (x1*m1 + x2*m2) / m
y = (y1*m1 + y2*m2) / m



Barnes-Hut Algorithm

- Step1: Recursively divide space by two in each dimensions
  - Record the center mass and position of each internal node





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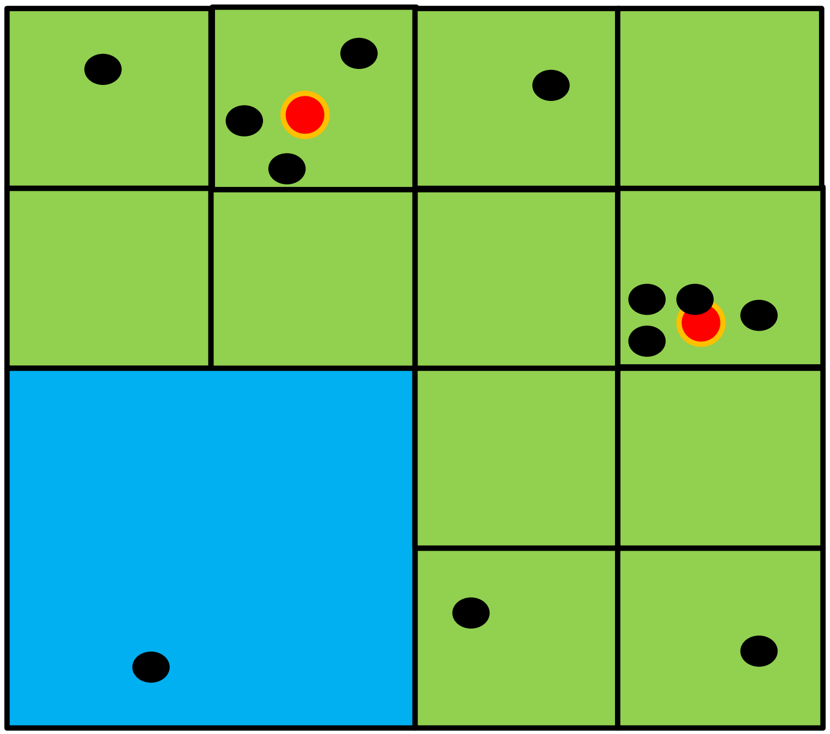

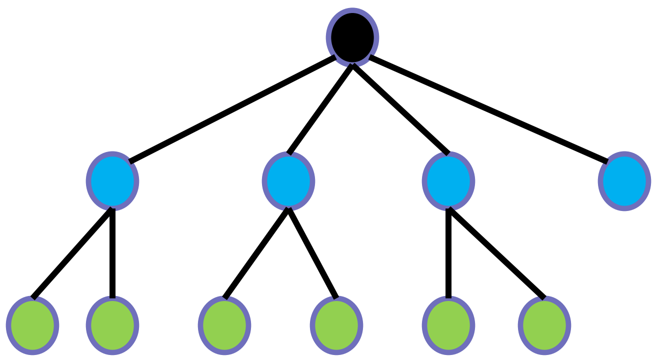

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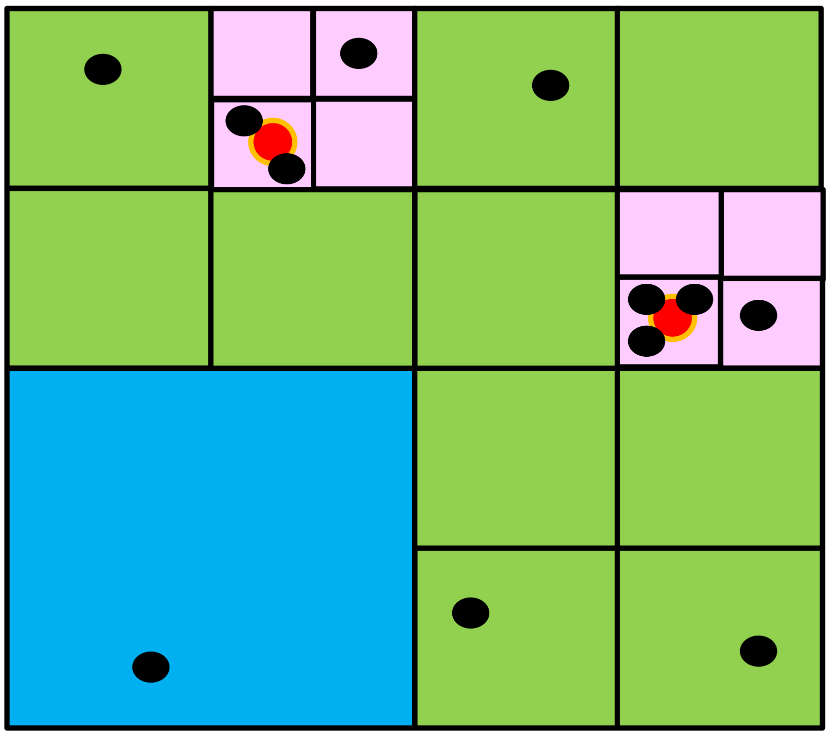

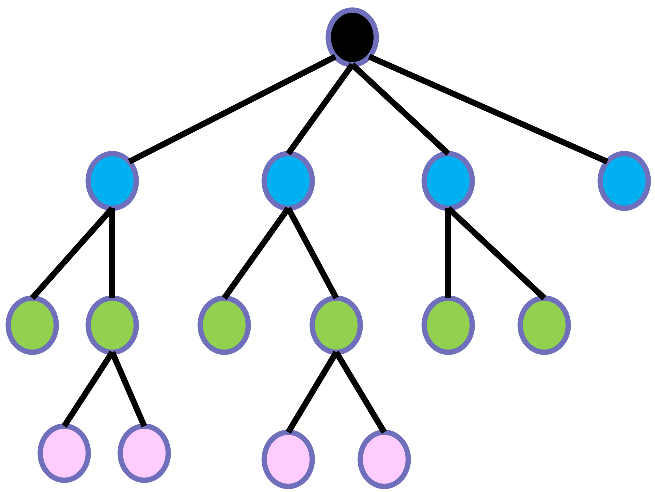

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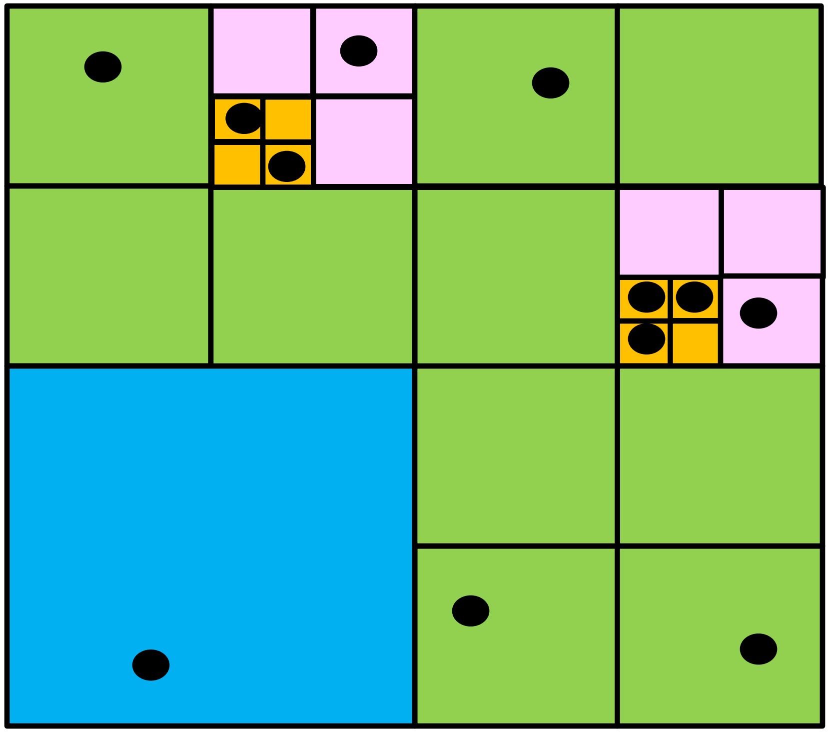
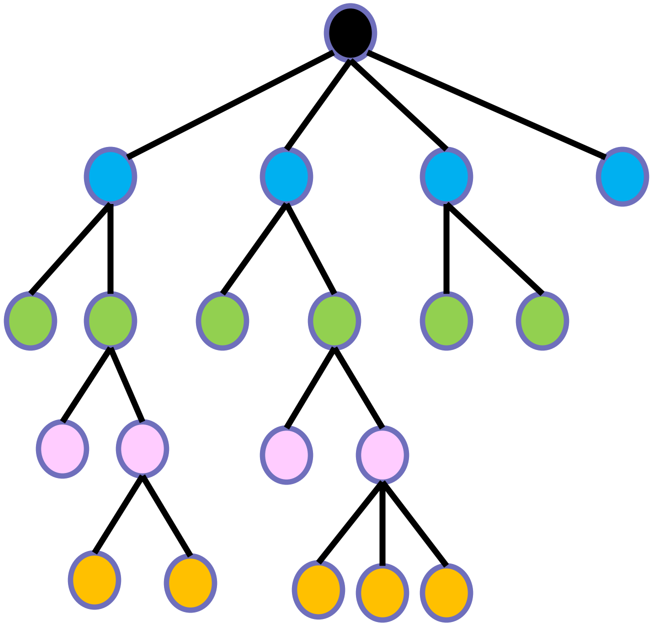
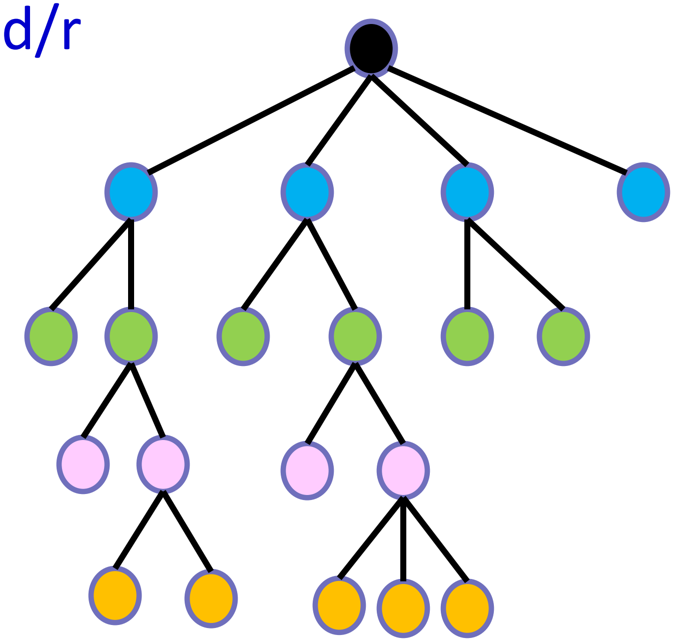


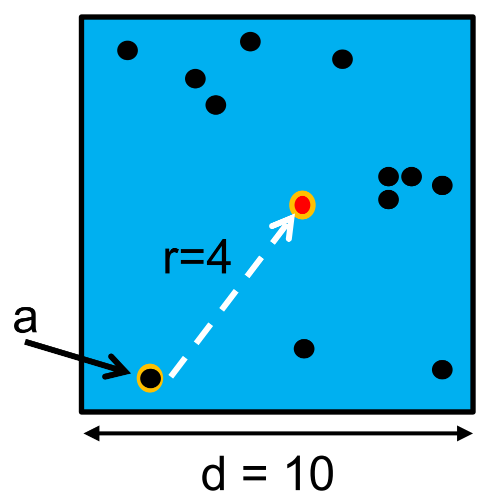


Barnes-Hut Algorithm

- Step2: Compute approximate forces on each object
  1. traverse the nodes of the tree, starting from the root.
  2. If the center-of-mass of an **internal node** is **sufficiently far** from the body, approximate the internal node as a single body
  - Far is determined by a parameter: θ=d/r
    - ***r***: the distance between the body and the node's center-of-mass
    - ***d***: the width of the region



Example: θ=0.5
d/r=2.5 > θ



Barnes-Hut Algorithm

- Step2: Compute approximate forces on each object
  1. Traverse the nodes of the tree, starting from the root.
  2. If the center-of-mass of an **internal node** is **sufficiently far** from the body, approximate the internal node as a **single body**
  - Far means $d/r < \theta$ (e.t. $0 < \theta < 1$)
    - ***r***: the distance between the body and the node's center-of-mass
    - ***d***: the width of the region

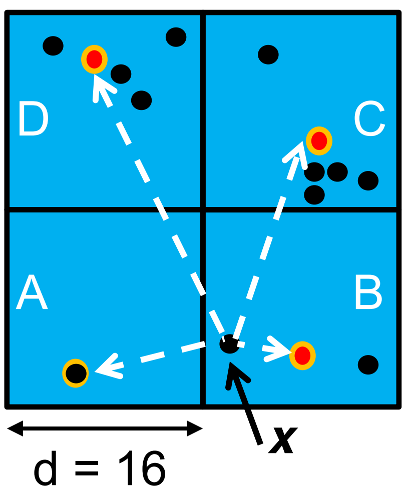
Example: $\theta=1$

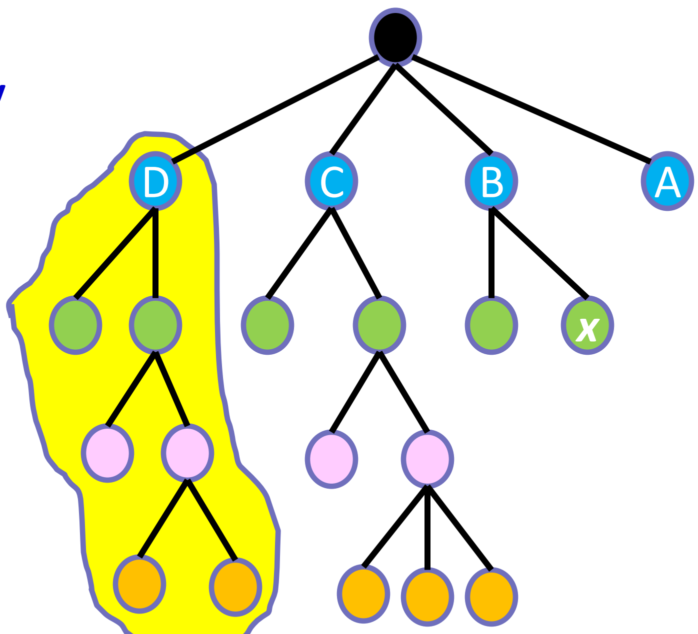
$d/r_A=16/10 > \theta$

$d/r_B=16/2 > \theta$

$d/r_C=16/15 > \theta$

$d/r_D=16/20 < \theta$





Barnes-Hut Algorithm

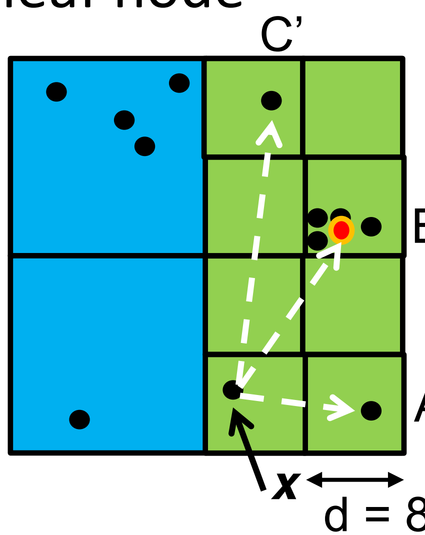
- Step2: Compute approximate forces on each object
  3. If it is a leaf node, calculate the force and add to the object.
  4. Otherwise, recursively compute the force from children of the internal node.

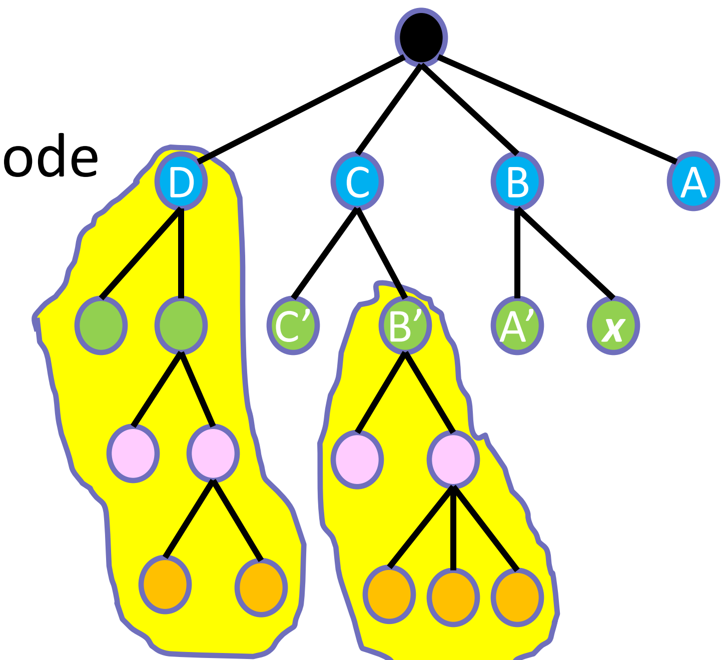
Example: $\theta=1$

$d/r_{A'}=8/7 \ > \theta$ ➔ A' is a leaf node

$d/r_{B'}=8/15 < \theta$ ➔ B' treated like a single node

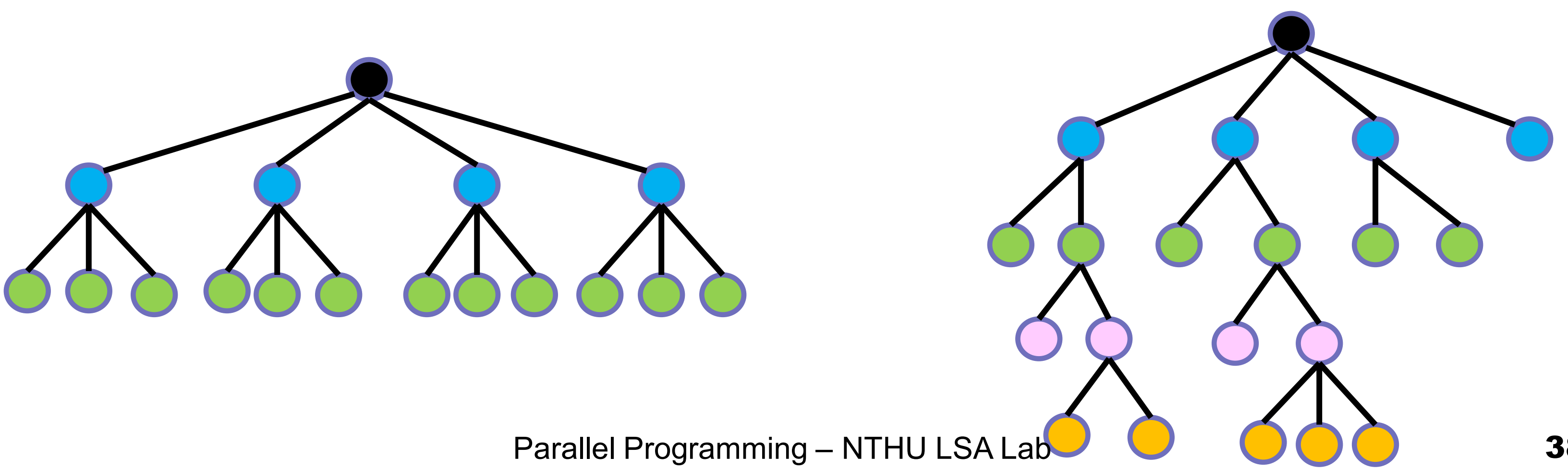
$d/r_{C'}=8/20 < \theta$ ➔ C' is a leaf node





Barnes-Hut Algorithm

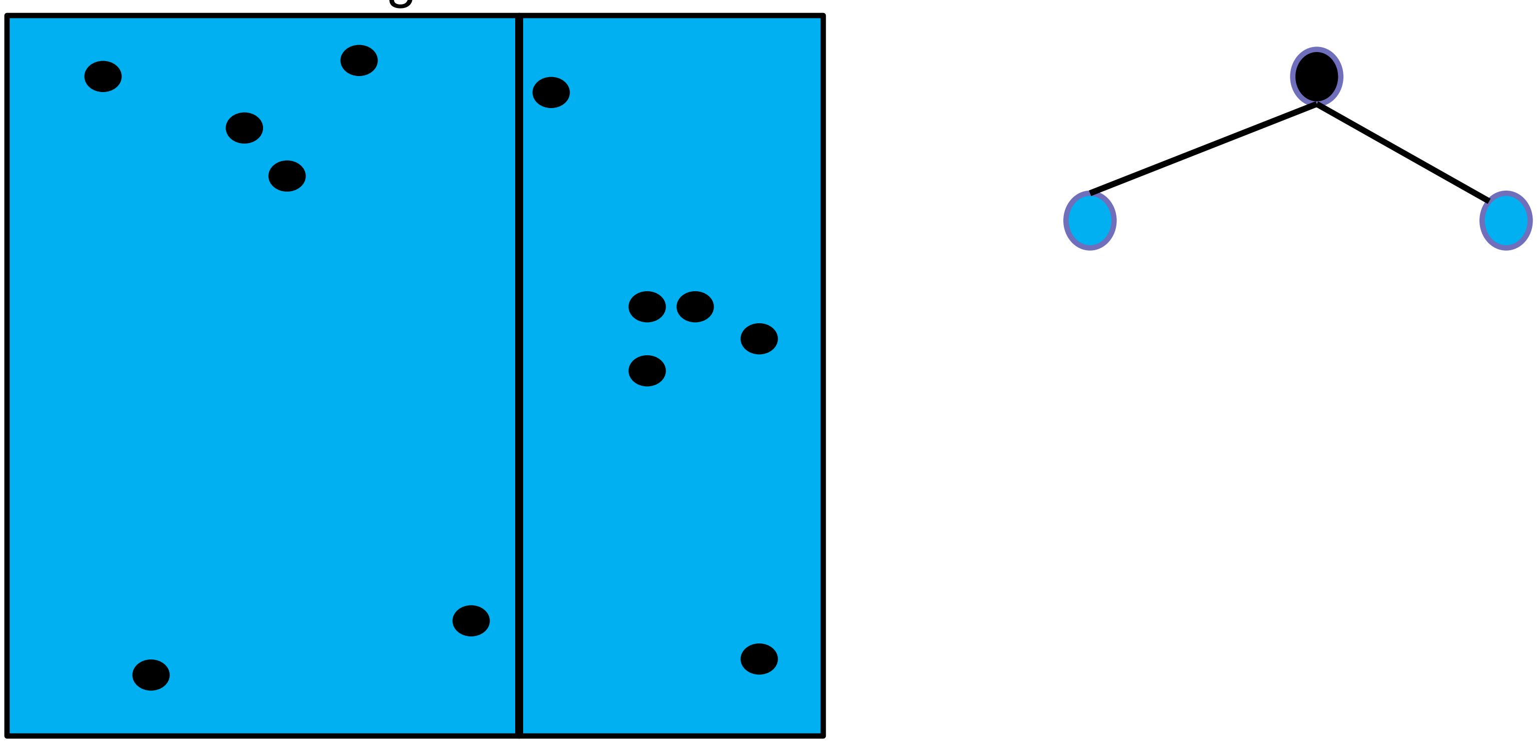
- θ controls the accuracy and approximation error of the algorithm
  - θ = 0 ➔ ***d/r* ALWAYS** larger than θ ➔ same as brute force
  - θ = 1 ➔ most likely only need to consider the object within the same cluster/region
- If the tree is balanced, the complexity is O(***n***log***n***)
  - But in general , the tree could be very unbalanced ....
- The tree must be re-built for each time interval



Orthogonal Recursive Bisection Method

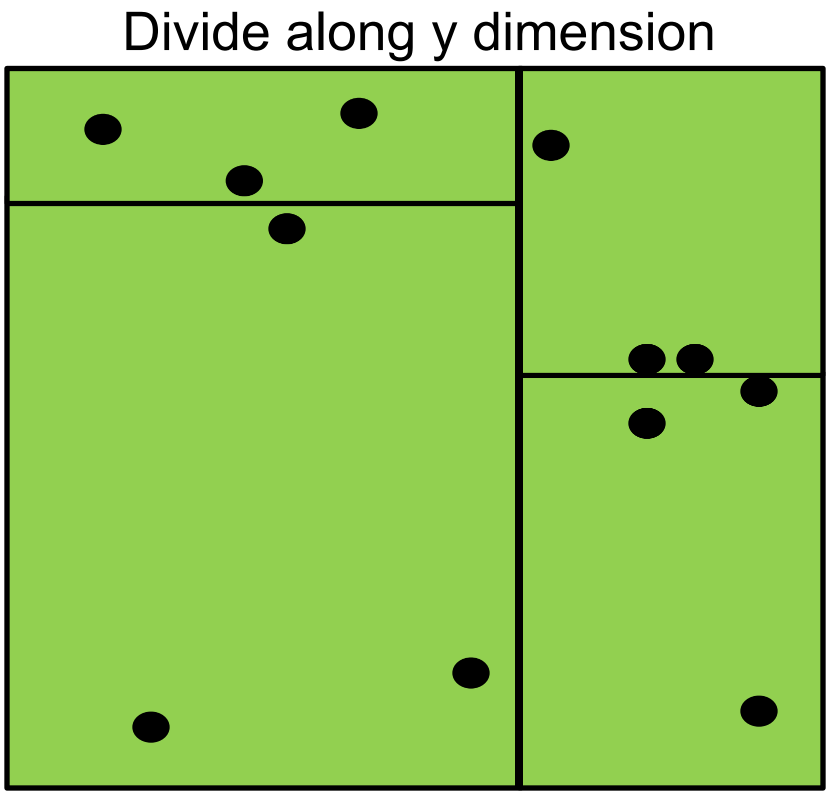
- Recursively evenly divide space with the same number of bodies in each of the dimensions

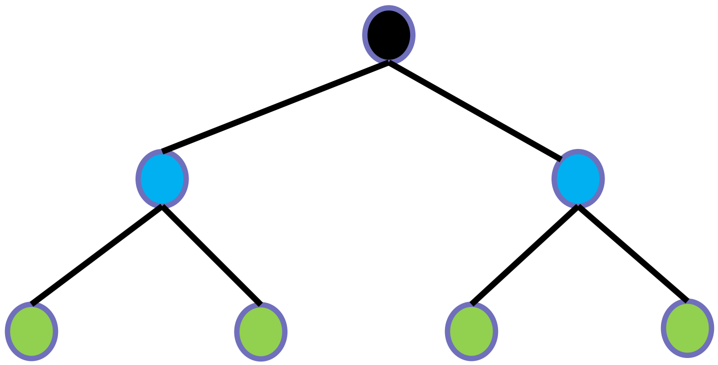
Divide along x dimension



Orthogonal Recursive Bisection Method

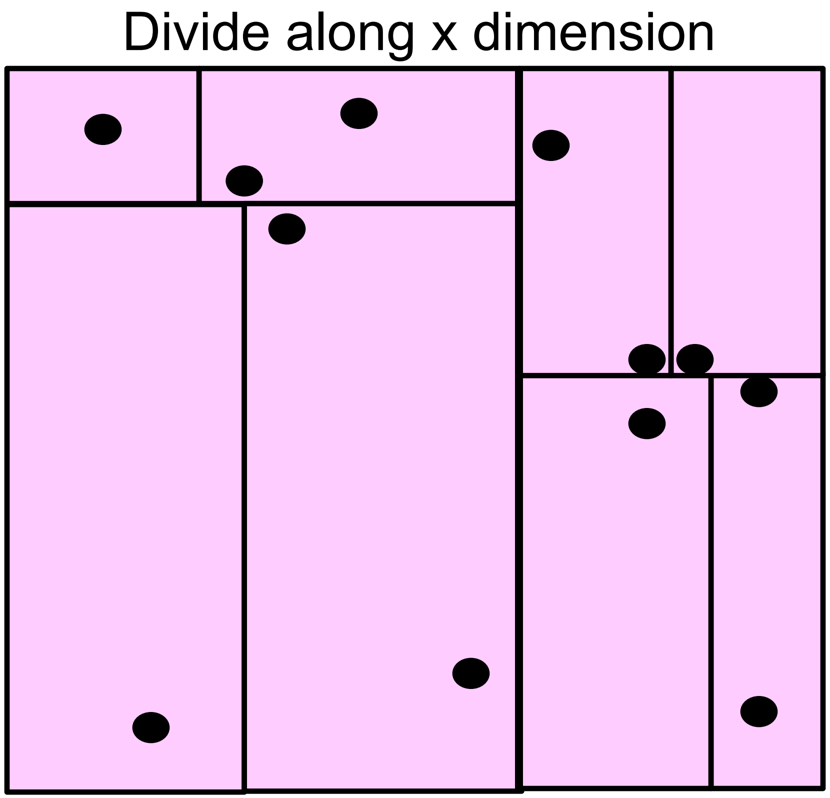
- Recursively evenly divide space with the same number of bodies in each of the dimensions

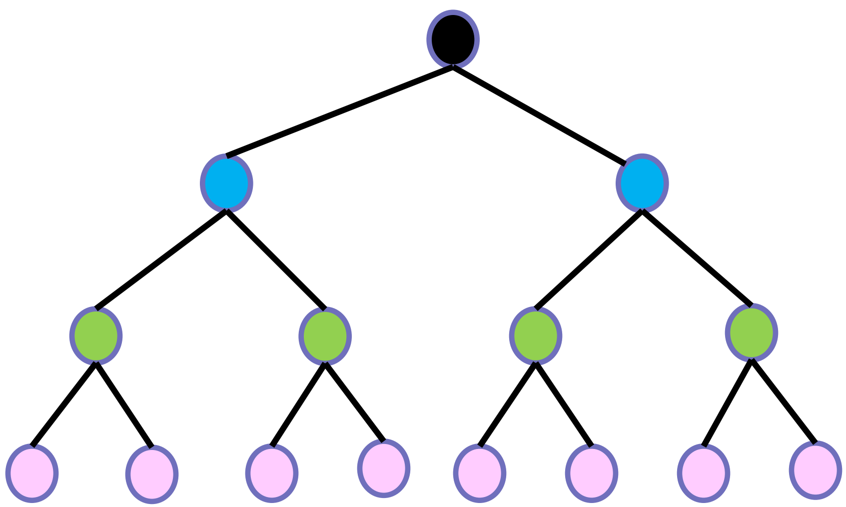




Orthogonal Recursive Bisection Method

- Recursively evenly divide space with the same number of bodies in each of the dimensions

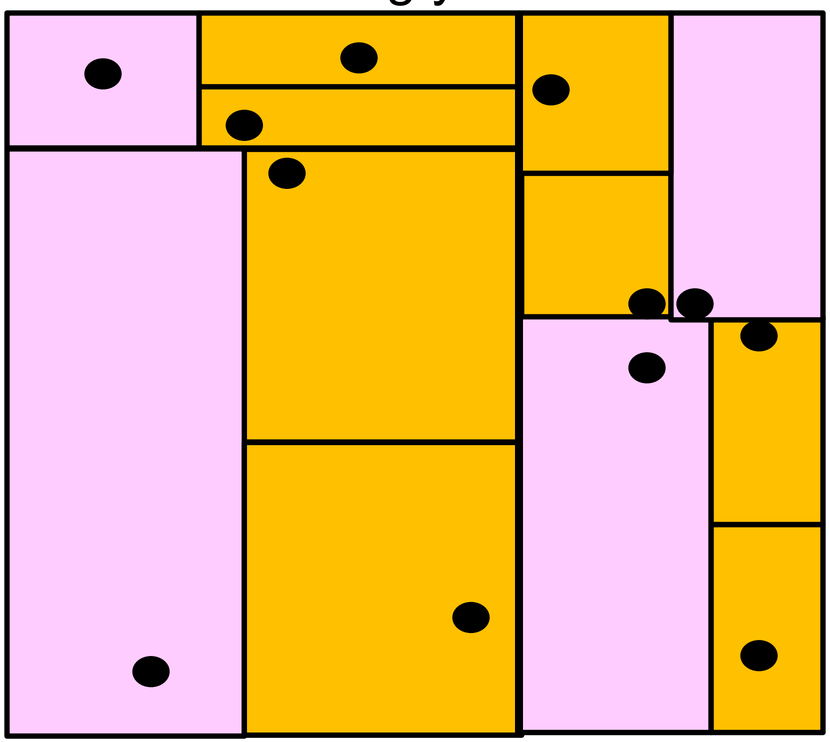




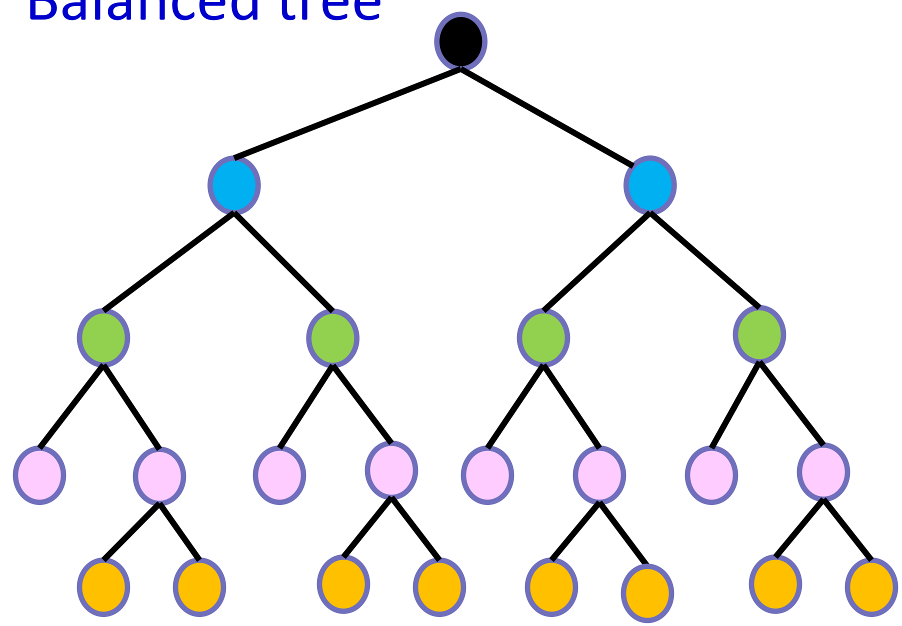
Orthogonal Recursive Bisection Method

- Recursively evenly divide space with the same number of bodies in each of the dimensions

Divide along y dimension



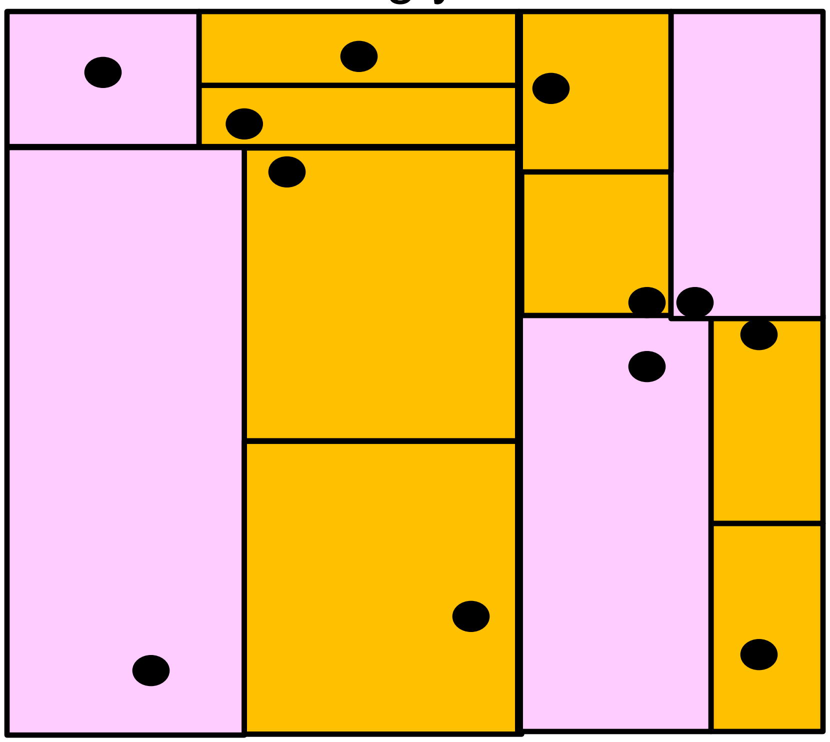
Balanced tree



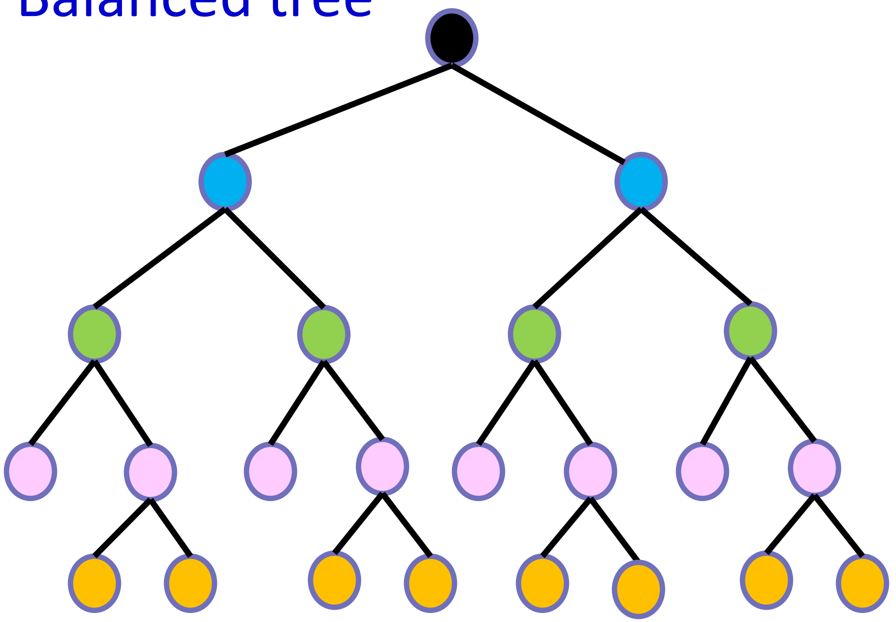
Orthogonal Recursive Bisection Method

- It is more balanced, but less accurate
  - Objects close to each other may not in the same cluster

Divide along y dimension



Balanced tree

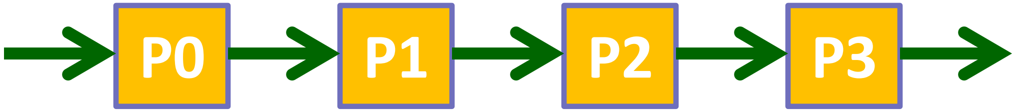


Outline

- Embarrassingly Computations
- Divide-And-Conquer Computations
- Pipelined Computations
  - Adding Numbers
  - Sorting Numbers
  - Linear Equation Solver
- Synchronous Computations

What is Pipelined Computations

- A problem is divided into a series of tasks
- Tasks have to be completed one after the other
- Each task will be executed by a separate process or processor



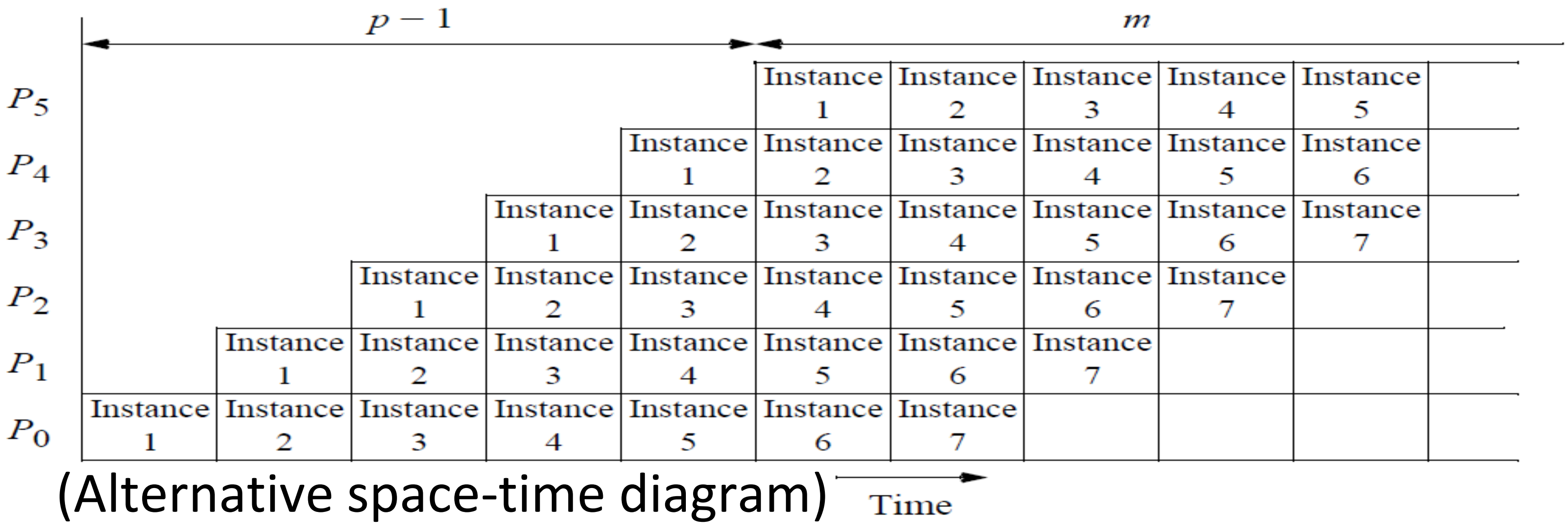
Types of Pipelined Computations

- Pipelined approach can provide increased speed under three types of computations:

1. If more than one instance of the complete problem is to be executed
2. If a single instance has a series of data items must be processed, each requiring multiple operations
3. If information to start the next process can be passed forward before the process has completed all its internal operations

Type 1 Pipelined Computations

1. If more than one **instance** of the complete problem is to be executed

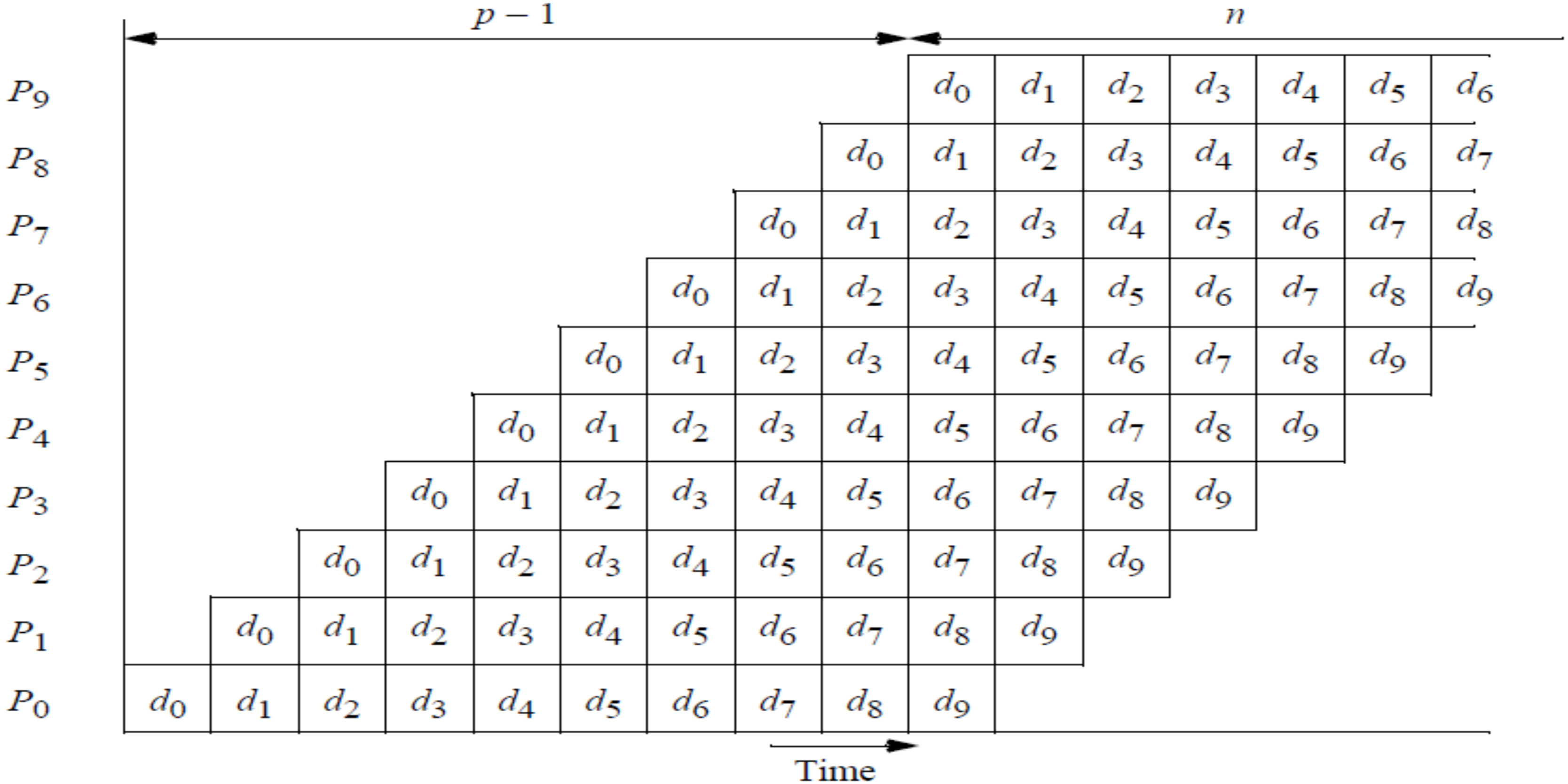


(Alternative space-time diagram)

- After the first (p-1) cycles, one problem instance is completed in each pipeline cycle
- The number of instance should be >> the number of processes

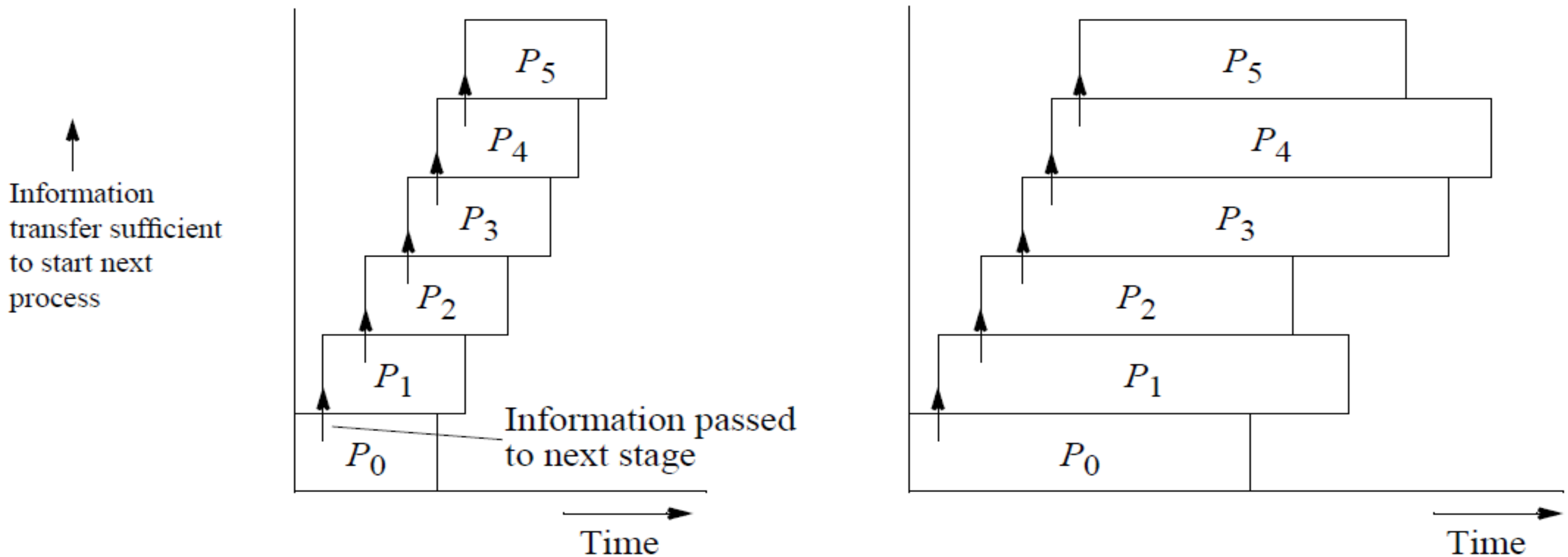
Type 2 Pipelined Computations

2. If a series of **data** items must be processed, each requiring multiple operations



Types 3 Pipelined Computations

3. Only one problem instance, but each process can pass on information to the next process, **before it has completed**

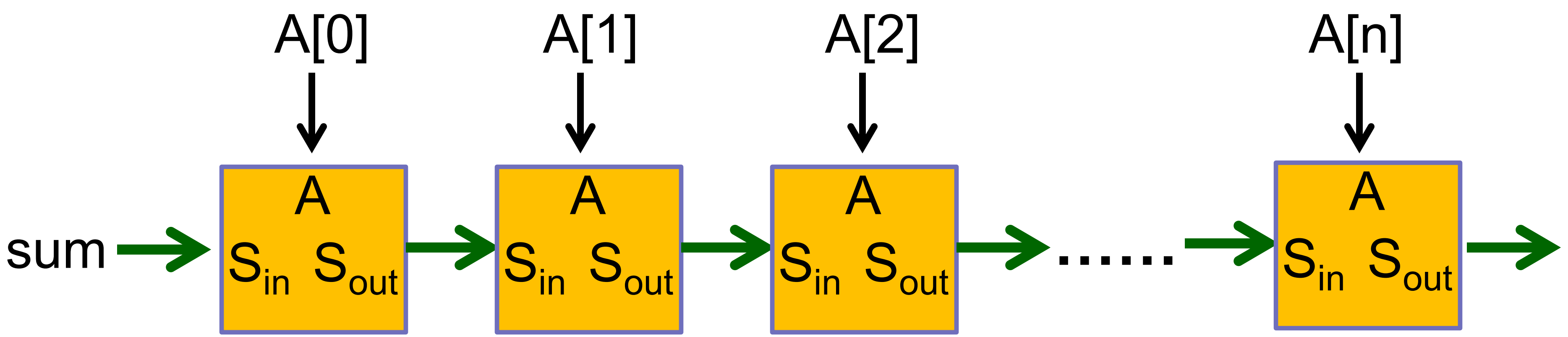


(a) Processes with the same execution time

(b) Processes not with the same execution time

Example1: Adding Numbers

- Compute sum of an array:
  - for(i=0; i<n; i++) sum += A[i]
- Pipeline for an unfolded loop:
  - sum += A[0], sum += A[1], sum += A[2], ……



Example1: Adding Numbers

- The basic code for Pi:

```
recv(&sum, P_{i-1});
sum += number;
send(&sum, P_{i+1});
```

- For the first process, P0:

```
send(&sum, P_{i+1});
```

- For the last process, Pn-1:

```
recv(&sum, P_{i-1});
sum += number;
```

- SPMD Program:

```
// code for process Pi
if (Pi != P0) {
    recv(&sum, P_{i-1});
    sum += number;
}
if (Pi != Pn) {
    send(&sum, P_{i+1});
}
```

Example2: Sorting Numbers

- Insertion Sort:

5 2 1 3 4

5

5 2

5 2 1

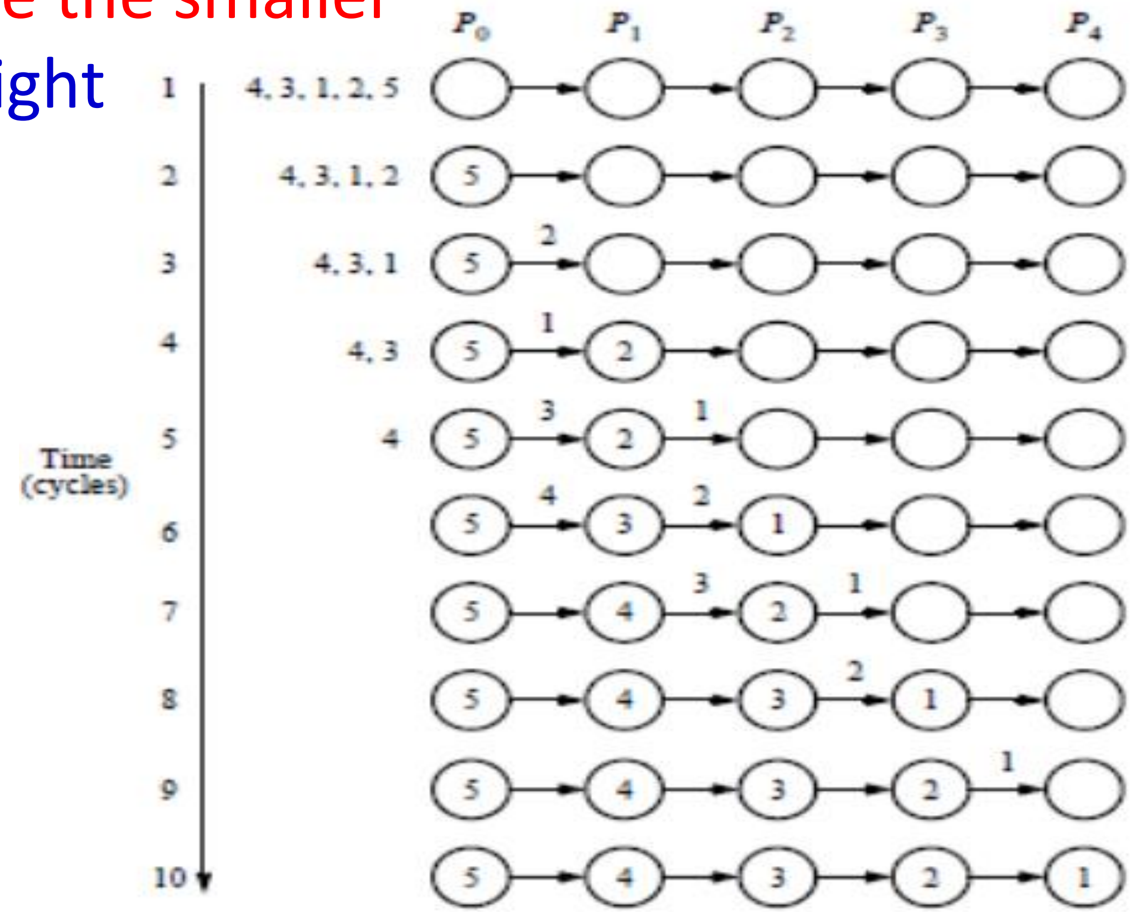
5 3 2 1

5 4 3 2 1

Example2: Sorting Numbers

- Insertion Sort:
  - Each process holds one number
  - Compare & move the smaller number to the right

```
recv(&number, P_{i-1});
if (number > x) {
   send(&x, P_{i+1});
   x= number;
} else {
   send(&number, P_{i+1});
}
```



Example 3: Linear Equation Solver

- Special linear equations of "upper-triangular" form
  - ***a***'s and ***b***'s are constants, ***x***'s are unknown to be found

$$
\begin{pmatrix}
a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots\cdots & a_{n-1,n-1} \\
a_{n-2,0} & a_{n-2,1} & \cdots\cdots & a_{n-2,n-2} & 0 \\
\vdots & \vdots & \vdots & 0 & 0 \\
a_{1,0} & a_{1,1} & 0 & 0 & 0 \\
a_{0,0} & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1}
\end{pmatrix}
=
\begin{pmatrix}
b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0
\end{pmatrix}
$$

$$a_{n-1,0}x_0 + a_{n-1,1}x_1 + a_{n-1,2}x_2 + \cdots + a_{n-1,n-1}x_{n-1} = \mathrm{b}_{n-1}$$

$$\vdots$$

$$a_{2,0}x_0 + a_{2,1}x_1 + a_{2,2}x_2 = \mathrm{b}_2$$

$$a_{1,0}x_0 + a_{1,1}x_1 = \mathrm{b}_1$$

$$a_{0,0}x_0 = \mathrm{b}_0$$

Example 3: Linear Equation Solver

- Back Substitution
  - x_0 is found from the last equation

$$x_0 = \frac{b_0}{a_{0,0}}$$

  - Value for x_0 is substituted into the next equation

$$x_1 = \frac{b_1 - a_{1,0}x_0}{a_{1,1}}$$

  - Values for x_0, x_1 are substituted into the next equation

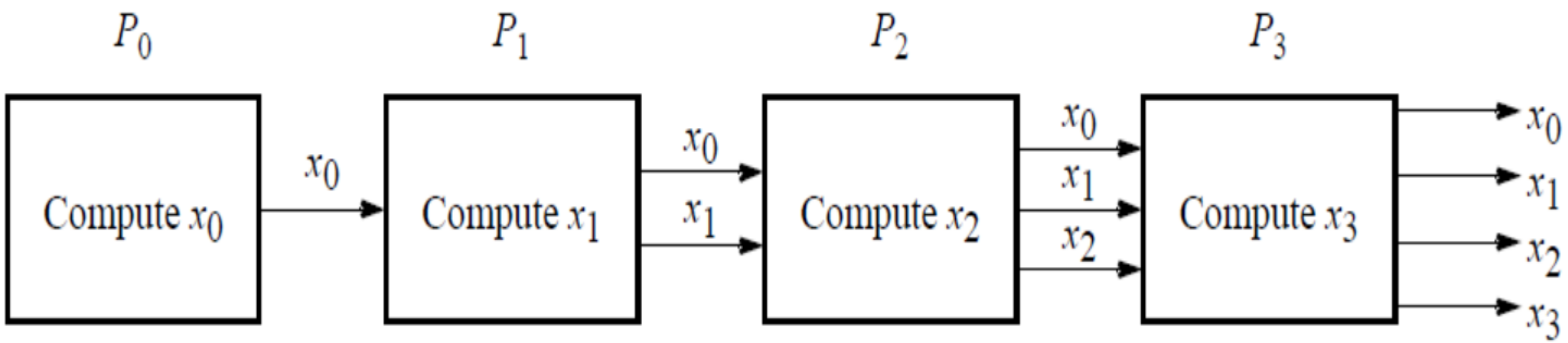
$$x_2 = \frac{b_2 - a_{2,0}x_0 - a_{2,1}x_1}{a_{2,2}}$$

  - So on until all unknowns are found …

$$x_i = \frac{b_i - \sum_{j=0}^{i-1} a_{i,j}x_j}{a_{i,i}}$$

Example 3: Linear Equation Solver

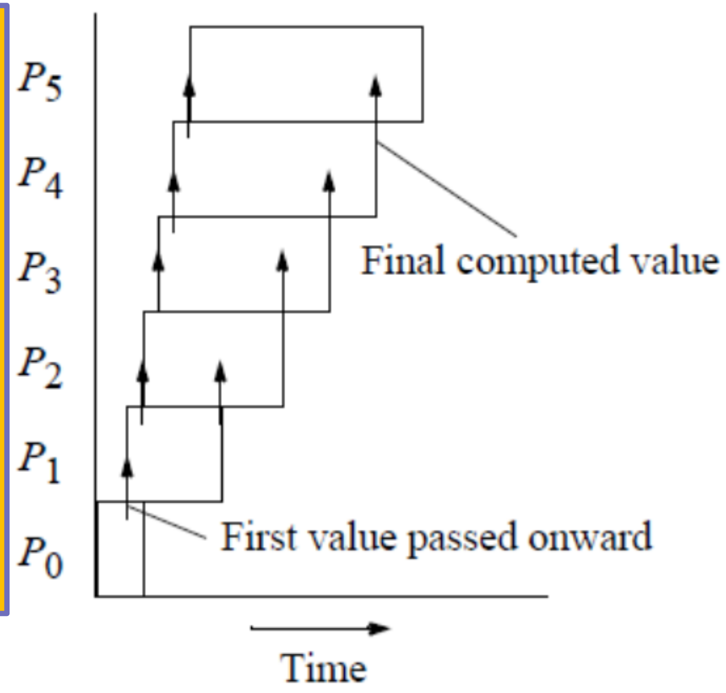
- First pipeline stage computes x_0 and passes x_0 onto the second stage, which computes x_1 from x_0 and passes both x_0 and x_1 onto the next stage, which computes x_2 from x_0 and x_1, and so on



Example 3: Linear Equation Solver

- Parallel Code

```
// code for Pi
sum = 0;
for (j=0; j<i; j++) {           // compute partial result
    recv(&x[j], P_{i-1});       // once data is available
    send(&x[j], P_{i+1});
    sum += a[i][j]*x[j];
}
x[i] = (b[i] - sum) / a[i][j];  // send out final result to
send(&x[j], P_{i+1});           // next process
```



- Time complexity: $O(n^2)$
  - Although later processes have more work for both communications and computations

Outline

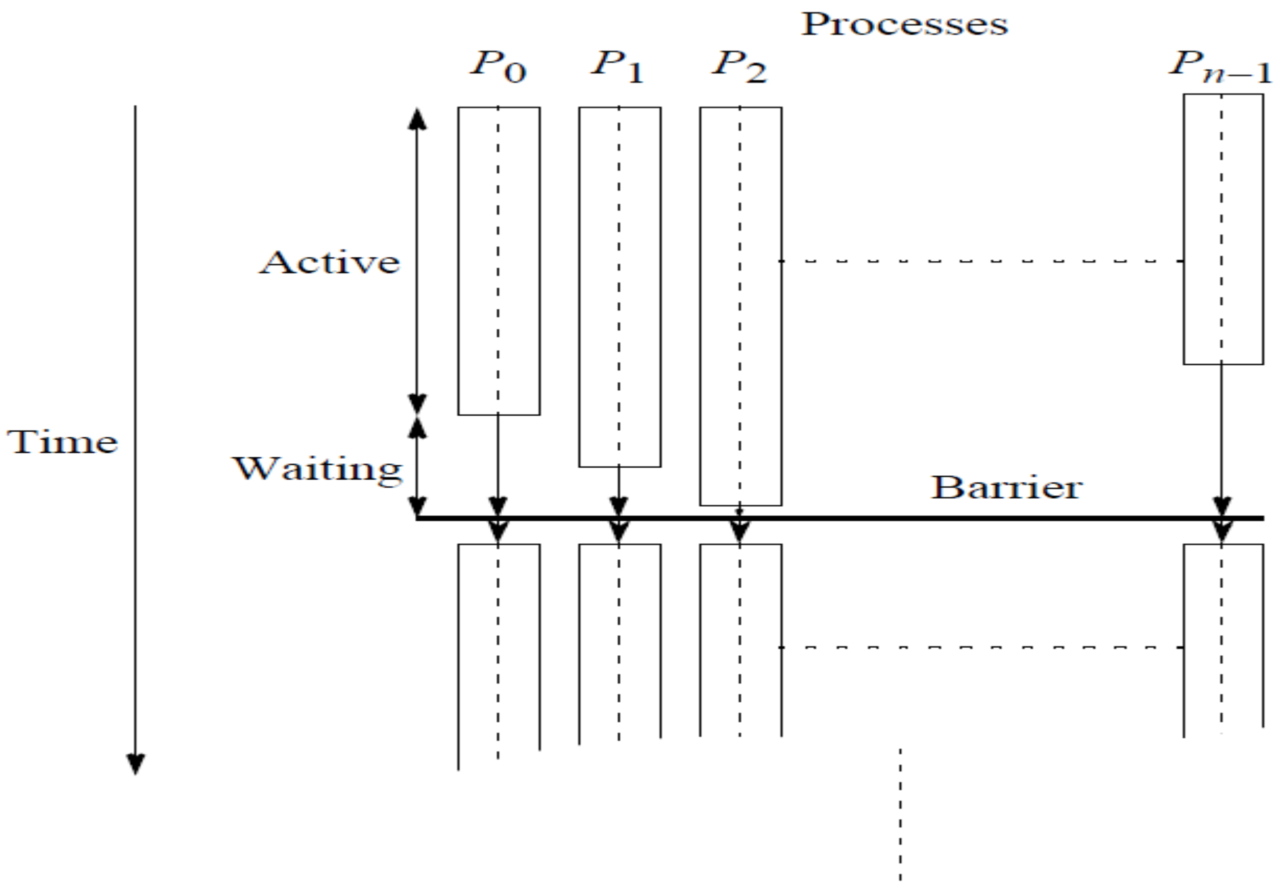
- Embarrassingly Computations
- Divide-And-Conquer Computations
- Pipelined Computations
- Synchronous Computations
  - Prefix Sum
  - System of Linear Equations

Synchronous Computations

- **Definition:** all the processes ***synchronized*** at regular points
- **Barrier**: Basic mechanism for synchronizing processes
  - Inserted at the point in each process where it must wait
  - Message (token) is passed among processes for synchronization
- **Deadlock**: Common problem occurs from synchronization
  - Two or multiple processes waiting for each other

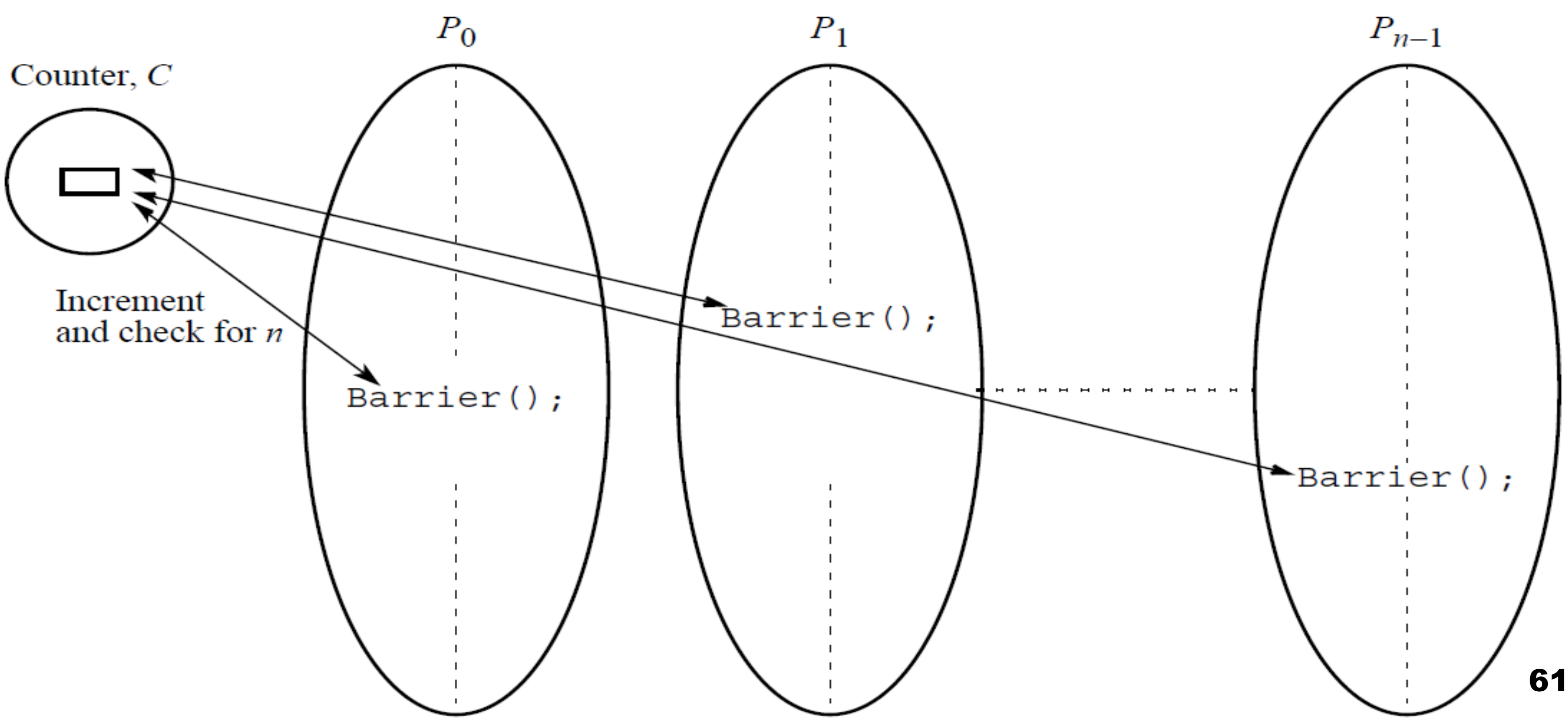
Barrier

- **All** processes can only continue from this *POINT* when **all** the processes have reached it



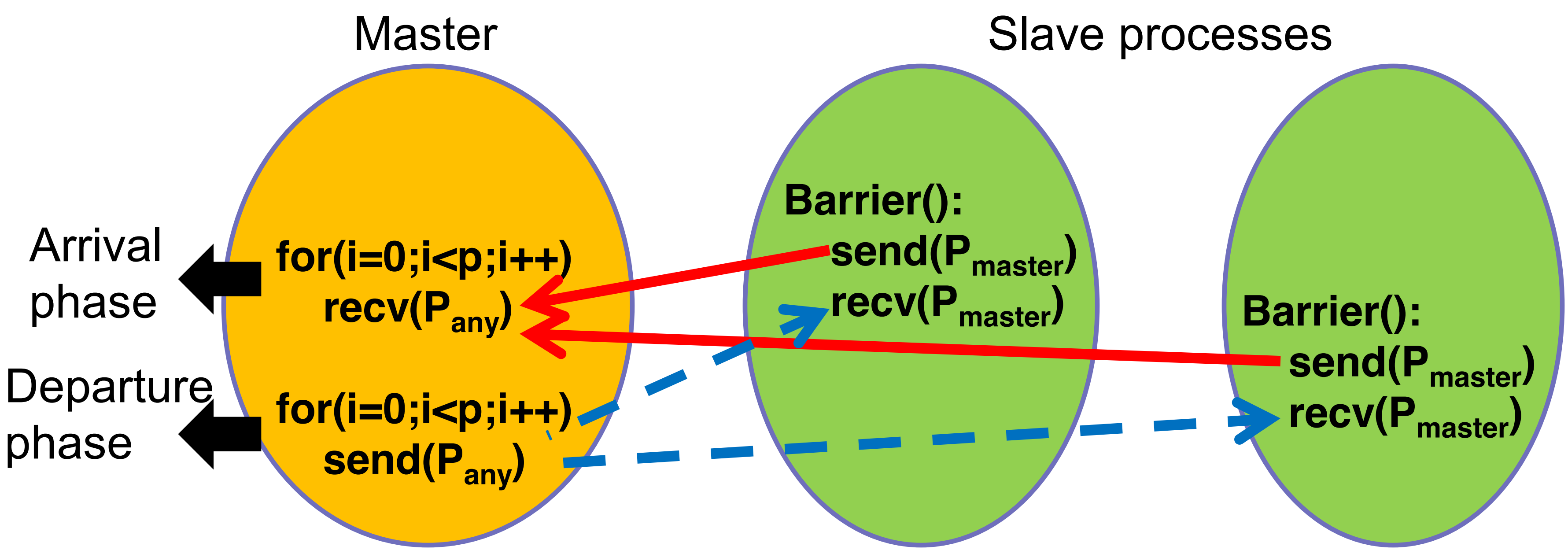
Counter Barrier Implementation

- A.k.a: Linear Barrier
  - Centralized counter: count # of processes reaching the barrier
  - Increase & check the counter for each barrier call
  - Processes is locked by the barrier call until counter == # processes



Counter Barrier Implementation

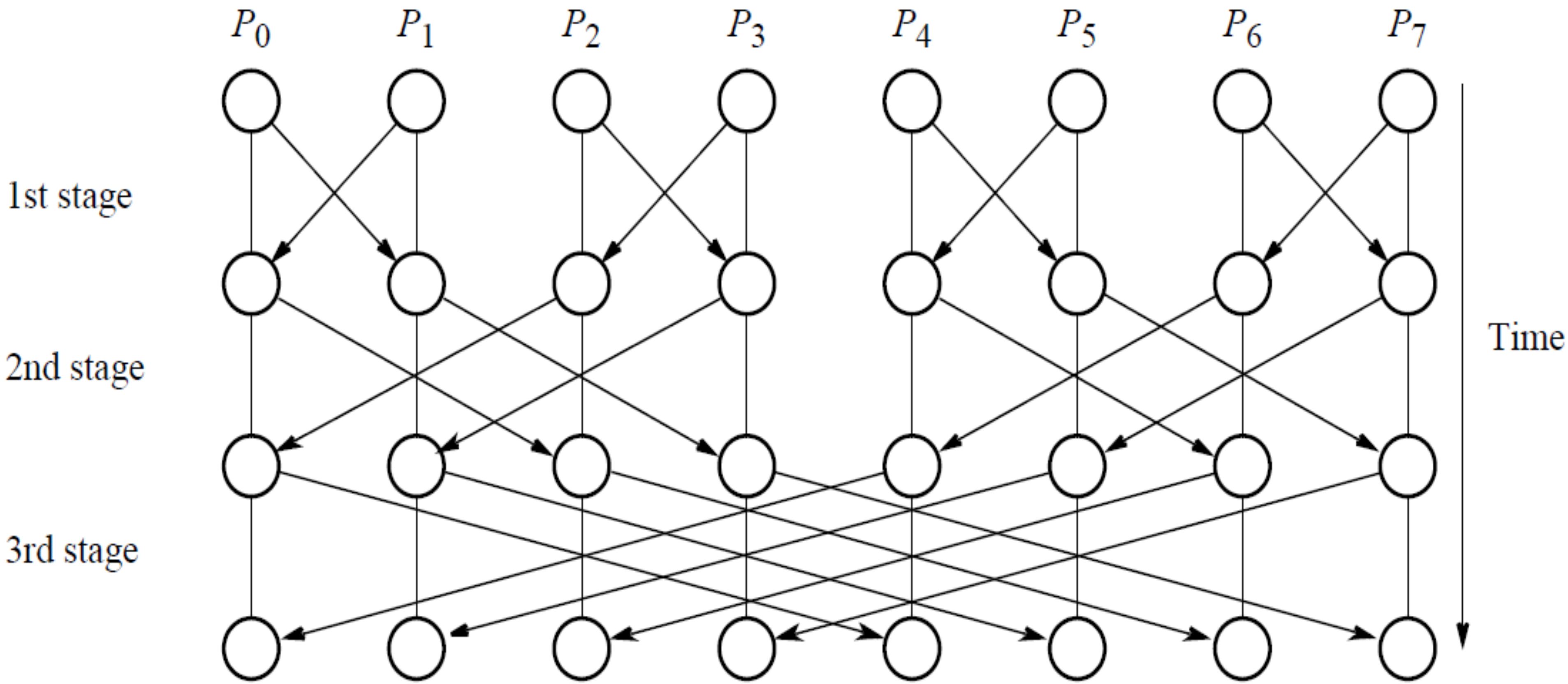
- Counter-based barrier often have two phases
  - **Arrival phase**: a process enters *arrival phase* and does not leave this phase until all processes have arrived in this phase
  - **Departure phase**: Processes are released after moving to the *departure phase*



- Slave processes is blocked by **recv()**
- Master could be a **bottleneck**

Butterfly Barrier Implementation

- At stage ***i***, each process passes a token to the process with 2^i distance away



Butterfly Barrier Implementation

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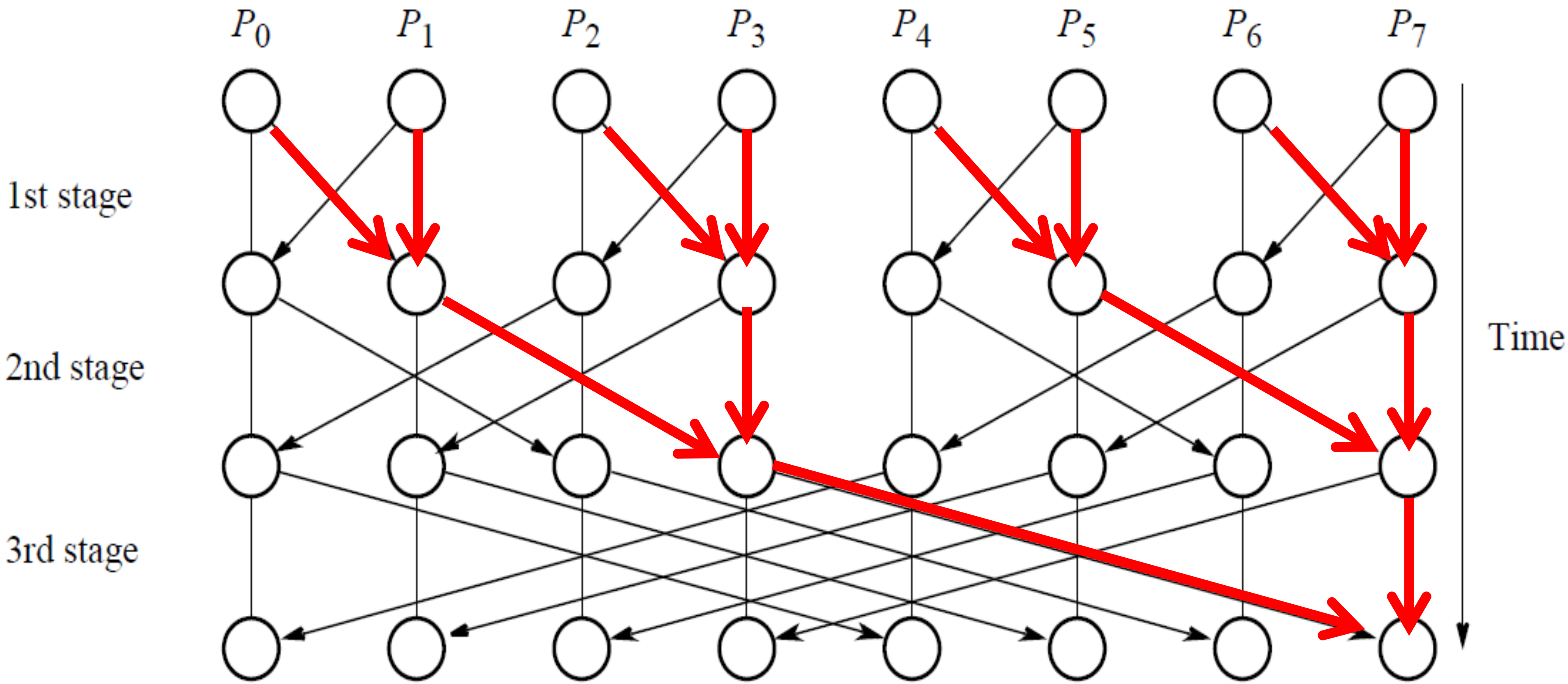

Butterfly Barrier Implementation

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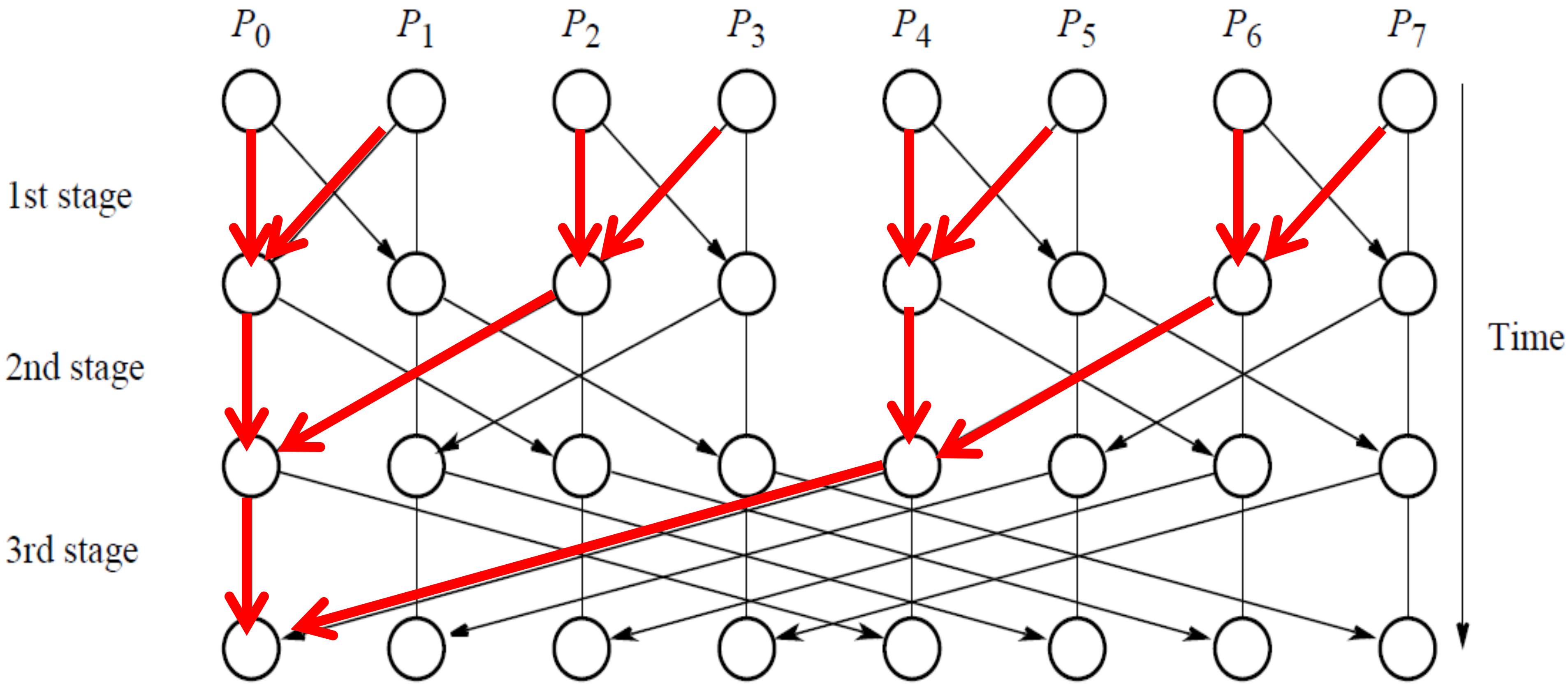
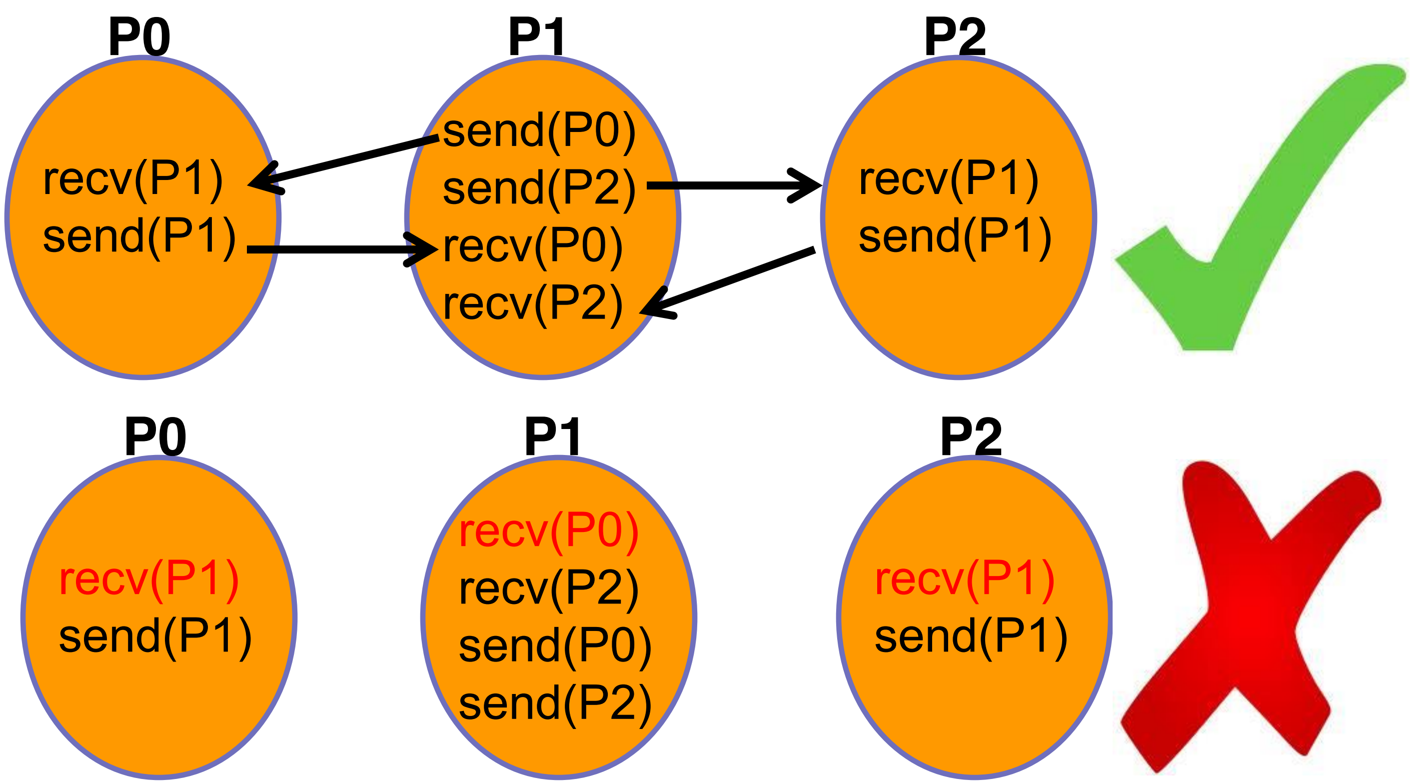


Deadlock Problem

- A set of blocked processes each holding some resources and waiting to acquire a resource held by another process in the set
- Example:

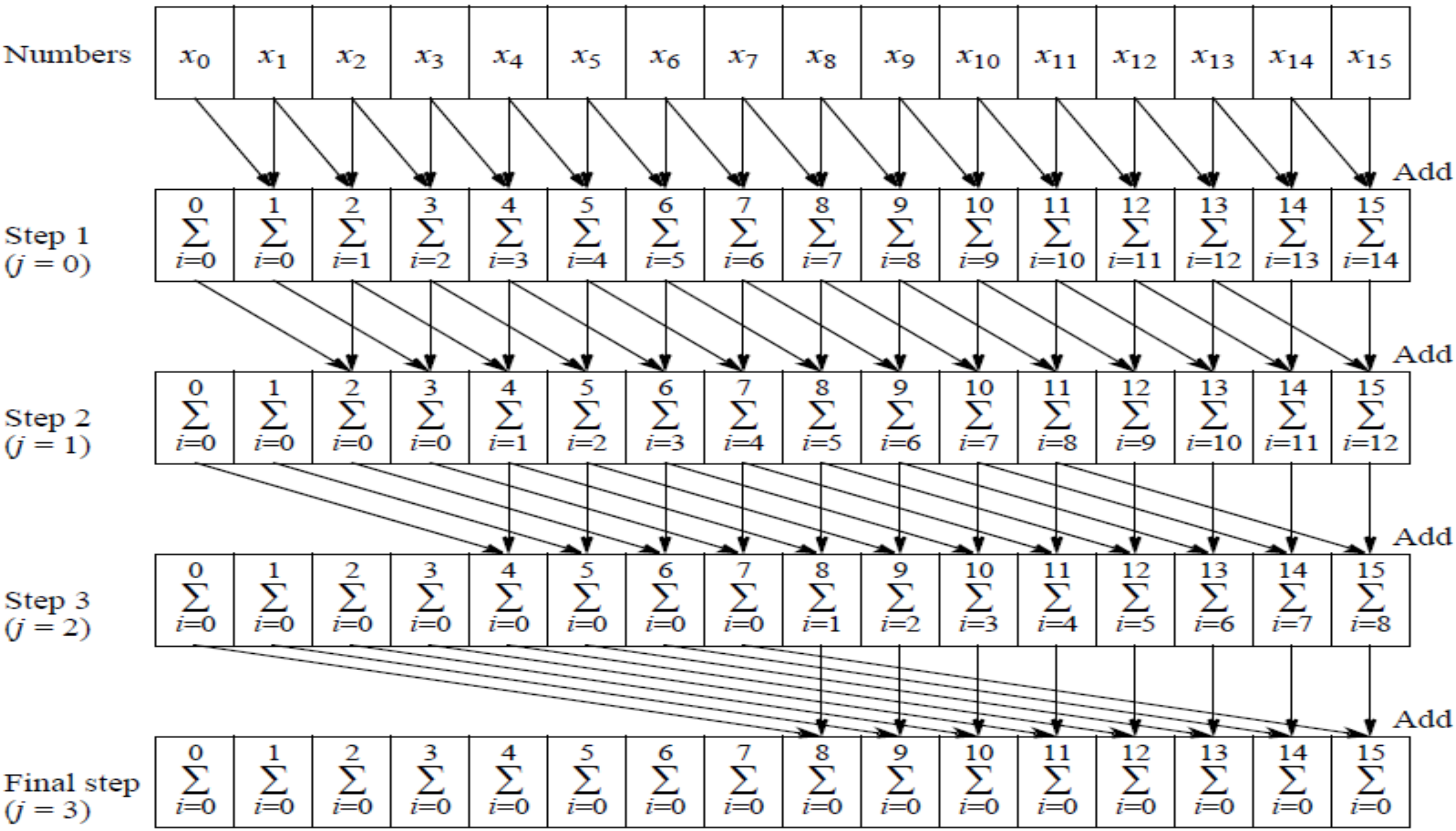


Example 1: Prefix Sum

- Given a list of numbers $x_0, x_1, \ldots. x_{n-1}$, compute all *partial* summations
  - $x_0; x_0 + x_1; x_0 + x_1 + x_2; \ldots \ldots..$
  - Could also replace operator + with AND, OR, *, etc.
- Example:
  - $x = 1,2,3,4,5$
  - Sum = 1,3,6,10,15
- Sequential code: O(n^2)

```
//sequential code
for(i = 0; i < n; i++) {
    sum[i] = 0;
    for (j = 0; j <= i; j++)
        sum[i] = sum[i] + x[j];
}
```

Data Parallelism Solution



Data Parallelism Code

- Sequential Code: $O(n^2)$, optimal: $O(n)$

```
for (j = 0; j < log(n); j++)          /* at each step */
    for (i = 2^j; i < n; i++)         /* add to accumulating sum */
        x[i] = x[i] + x[i - 2^j]
```

- Parallel Code: $O(\log n)$

```
for (j = 0; j < log(n); j++)          /* at each step */
    forall (i = 0; i < n; i++)        /* add to accumulating sum */
        if (i >= 2^j) x[i] = x[i] + x[i - 2^j];
```

Synchronous Parallelism

- Each iteration composed of several processes that start together at beginning of iteration and next iteration cannot begin until all processes have finished previous iteration

- openMP

```
for (j=0; j<n; j++) {     // each iteration
    forall (i=0; i<N; i++) { // each process
        body(i);
    }
}
```

- MPI

```
for (j=0; j<n; j++) {     // each iteration
    i = myrank;
    body(i);
    barrier(mygroup);
}
```

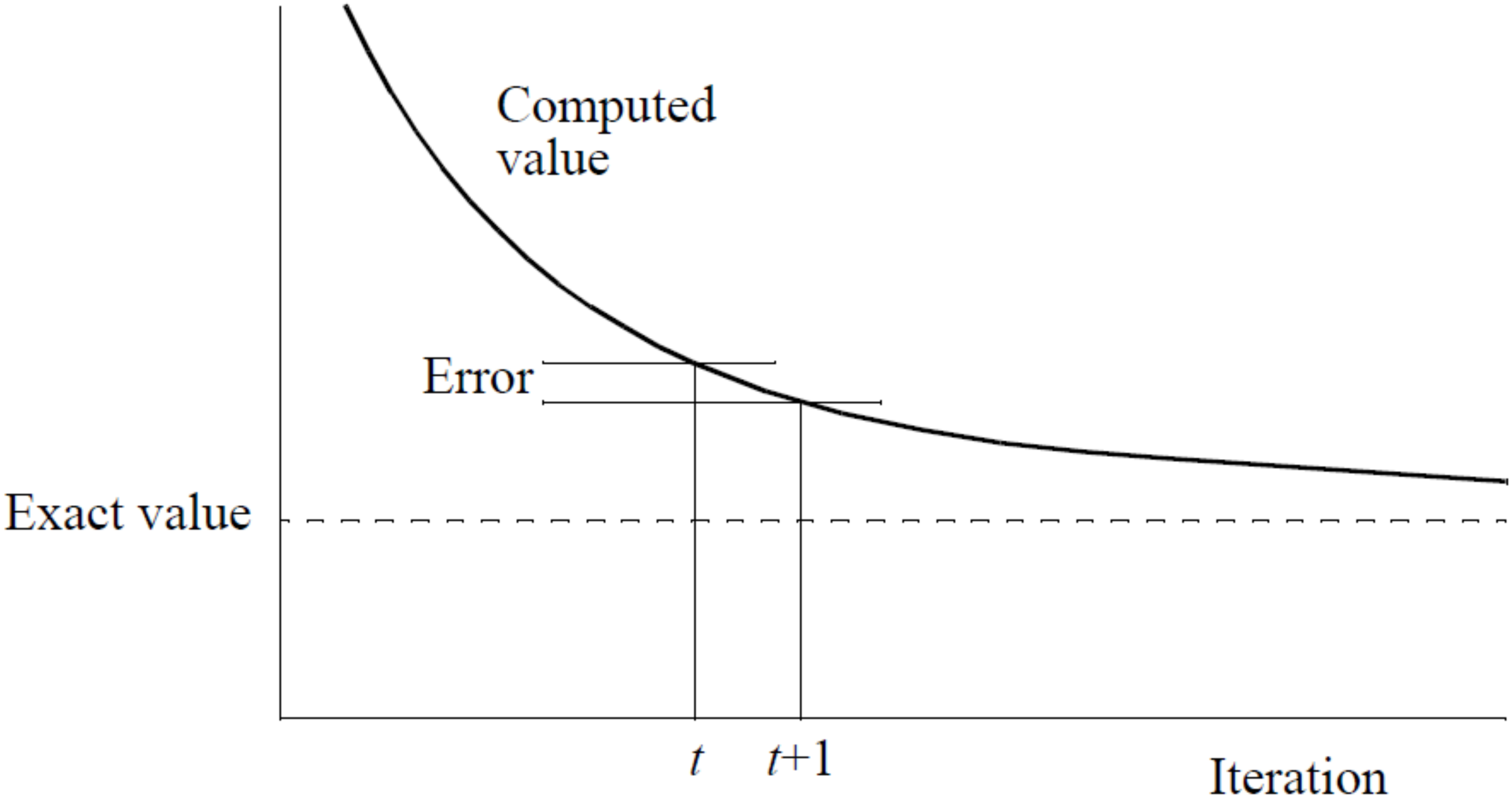
Example 2: System of Linear Equations

- System of linear equations

$$\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & \cdots\cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots\cdots & a_{1,n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-2,0} & a_{n-2,1} & a_{n-2,2} & \cdots\cdots & a_{n-2,n-1} \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots\cdots & a_{n-1,n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{pmatrix}$$

- Jacobi iteration algorithm:
  - Convert *i*th iteration to $x_i = \frac{1}{a_{i,i}}[b_i - \sum_{i \neq j} a_{i,j}\, x_j]$
  - Initial guess with $x_i = b_i$, and calculate new x_i values
  - Repeat until $|{x_i}^t - {x_i}^{t-1}| <$ error tolerance

Jacobi iteration algorithm



Jacobi iteration algorithm example

$$\begin{cases} -x_0 + 2x_1 - x_2 = 2 \\ 2x_0 + x_1 - 2x_2 = 2 \\ 2x_0 - x_1 + 2x_2 = 2 \end{cases}$$

$$x_i = \frac{1}{a_{i,i}}[b_i - \sum_{i \neq j} a_{i,j}\, x_j]$$

$$\begin{cases} x_0 = 2 - \dfrac{(2x_1 - x_2)}{-1} \\ x_1 = 2 - \dfrac{(2x_0 - 2x_2)}{1} \\ x_2 = 2 - \dfrac{(2x_0 - x_1)}{2} \end{cases}$$

- Iter1: $\mathrm{x}_0^1 = 2,\ \ x_1^1 = 2,\ \ x_2^1 = 2$

  ➔ $\mathrm{x}_0^2 = 2 - \frac{2\mathrm{x}_1^1 - x_2^1}{-1} = 4,\ \ x_1^2 = 2,\ \ x_2^2 = 1$

  ➔ $e_0 = |2 - 4| = 2,\ \ e_1 = 0,\ \ e_2 = 1$

- iIter2: $\mathrm{x}_0^2 = 2 - \frac{2\mathrm{x}_1^2 - x_2^2}{-1} = 5,\ \ x_1^3 = -2,\ \ x_2^3 = -1$

  ➔ $e_0 = |4 - 5| = 1,\ \ e_1 = 4,\ \ e_2 = 2$

Jacobi iteration algorithm

- Sequential Code
  - **a[][]** and **b[]** holding constants in the equations
  - **x[]** holding unknowns
  - fixed number of iterations

$$x_i = \frac{1}{a_{i,i}}[b_i - \sum_{i \neq j} a_{i,j}\, x_j]$$

```
for (i = 0; i < n; i++)  x[i] = b[i];          /*initialize unknowns*/
for (iteration = 0; iteration < limit; iteration++) {
    for (i = 0; i < n; i++) {                  /* for each unknown */
        sum = -a[i][i] * x[i];
        for (j = 0; j < n; j++)                /* compute summation */
            sum = sum + a[i][j] * x[j];
        new_x[i] = (b[i] - sum) / a[i][i];  /*compute unknown*/
    }
    for (i = 0; i < n; i++) x[i] = new_x[i];   /*update to new values*/
}
```

Jacobi iteration algorithm

- Parallel Code
  - Process ***i*** handles unknown ***x***[***i***]

```
x[i] = b[i]; /*initialize unknown*/
for (iteration = 0; iteration < limit; iteration++) {
    sum = -a[i][i] * x[i];
    for (j = 0; j < n; j++) /* compute summation */
        sum = sum + a[i][j] * x[j];
    new_x[i] = (b[i] - sum) / a[i][i];  /* compute unknown */
    allGather(&new_x[i]); /* gather & broadcast new value */
    barrier();                           /* wait for all processes */
}
```