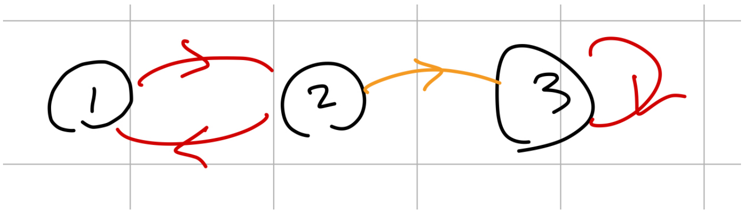
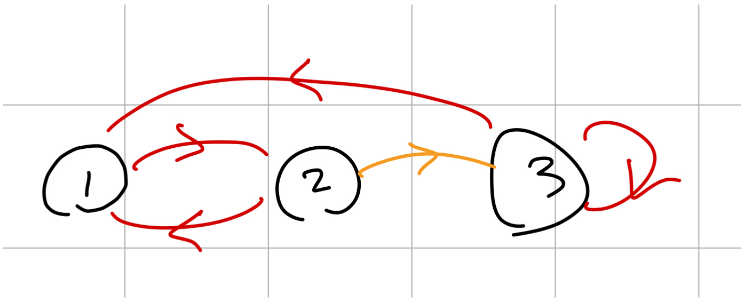
15 Name of Lecture

15.1 Communication Classes

From last time, we had that state i and j communicate if P^n and P^m is strictly positive for entry (i, j) and (j, i) respectively.



In this case, we can see that states 1 and 3 do not communicate.



In this case, states 1 and 3 do communicate.

Definition 15.1
Given a binary relation $\sim$ on a set, an **equivalence relation** satisfies the following:

1. $A \sim A$ (reflexive)
2. If $A \sim B$, then $B \sim A$ (symmetric)
3. If $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive)

Example 15.2
For non-negative integers, $x \sim y \iff$ leave the same remainder when divided by 3

Here, we can divide all of the non-negative integer into 3 classes representing remainders 0, 1, and 2.

Theorem 15.3
The communication relation is an equivalence relation on the set of states of the Markov chain.

Proof. (informally)

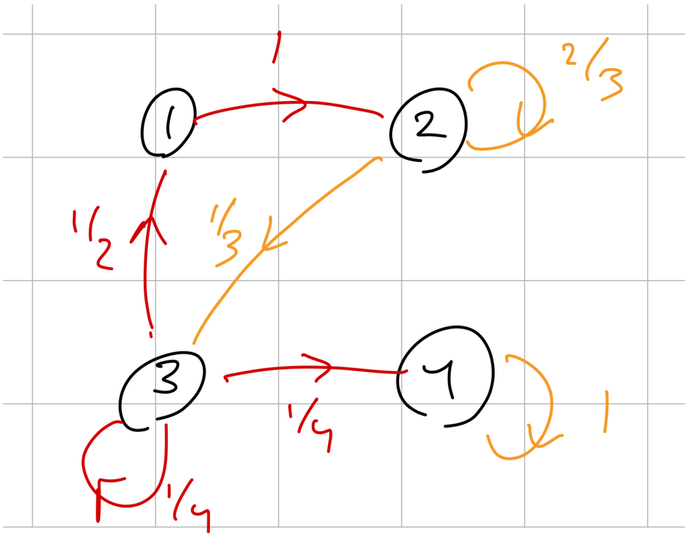
1. If P is the transition matrix, $P^0 = I$, which means all (i, i) entries are strictly positive, so every state communicates with itself.
2. Given that i communicates with j, by definition j communicates with i.
3. Given that i communicates with j and j communicates with k, we know that we can go from i to j and we can go from j to k, and we also know we can go from k to j and then j to i.
   Then, clearly we can go to k starting from i, and we can go to i starting from k in some number of steps.

$\square$

With this theorem, we can partition the state into equivalence classes, i.e. communication classes, where if states are in the same communication class, then they can communicate with each other.

Example 15.4
Classify the communication classes for Markov chain with transition matrix

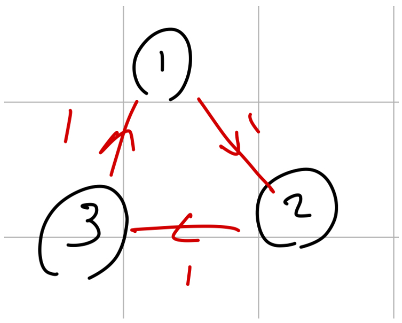
$$P = \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1 & 2/3 & 0 & 0 \\ 0 & 1/3 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1 \end{bmatrix}$$



We can see here that the classes are $\{1, 2, 3\}$ and $\{4\}$.

Definition 15.5
A Markov chain with only 1 communication class is **irreducible**. Otherwise it is called **reducible**.

Note that being irreducible does NOT mean that the Markov chain is regular.



Here, the only communication class is $\{1, 2, 3\}$ so the Markov chain is irreducible, but it is not regular. We will never have a P^k that has all positive values, because we can not get to any state from any other state in any number of transitions.
We can start and end in state 1 only in multiples of 3, so P^k will never have all positive entries for some integer k.

<u>**Irreducibility is the more relaxed condition that guarantees a stable vector.**</u>

Note however that the converse is true: a regular Markov chain is also irreducible.

Definition 15.6
A **random variable** X of an experiment is a function that assigns each element in the sample space to a real number.

Definition 15.7
Given a random variable X taking on values $x_1, x_2, \cdots, x_k$, then the **expected value** of X is

$$E(X) = \sum_{i=1}^{k} x_i P(X = x_i)$$

Consider a Markov chain on n states and the sample space to be all n^2 ordered pairs (i, j). If X is the random variable on the number of transitions to go from state j to i for the first time, we are interested in $E(X)$.