5 Affine Combinations, Convex Combinations (Section 2.4)

5.1 Affine combinations

Last time, we had that a set is affinely dependent if $\sum c_i \vec{v}_i = \vec{0}$, $\sum c_i = 0$.

Theorem 5.1
Let $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$. The following are equivalent (they are either all true or all false):

1. The set of vectors is affinely dependent.
2. The set $\{\vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_1, \cdots, \vec{v}_m - \vec{v}_1\}$ is linearly dependent.
3. The homogeneous forms $\{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_m\}$ (in $\mathbb{R}^{n+1}$) form a linearly dependent set.

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m & \mid & 0 \\ \downarrow & \downarrow & & \downarrow & \mid & \vdots \\ 1 & 1 & & 1 & \mid & 0 \end{bmatrix}$$

Note that bottom row represents the equation $c_1 + c_2 + \cdots c_m = 0$.

Example 5.2
Is $\{(2,1), (5,4), (-3,-2)\}$ affinely dependent?

$$\vec{v}_2 - \vec{v}_1 = (3,3)$$
$$\vec{v}_3 - \vec{v}_1 = (-5,-3)$$

Because the difference vectors are not multiples of each other, they form a linearly independent set, so by our above theorem we have that $\{(2,1), (5,4), (-3,-2)\}$ is affinely independent.

Alternatively, we could use the third property and row reduce the matrix

$$\begin{bmatrix} 2 & 5 & -3 & \mid & 0 \\ 1 & 4 & -2 & \mid & 0 \\ 1 & 1 & 1 & \mid & 0 \end{bmatrix}$$

Just as every vector in the span of a set of linearly independent vectors can be uniquely expressed as a linear combination of those vectors, we have the following analogue with affinely independent sets of vectors:

Theorem 5.3
Let $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$ be affinely independent in $\mathbb{R}^n$. Then each $\vec{u} \in \text{aff}(S)$ can be uniquely written as an affine combination of $\vec{v}_1, \cdots, \vec{v}_m$.

The (unique) coefficients $c_1, \cdots, c_m$ such that $\vec{u} = \sum_{i=1}^{m} c_i \vec{v}_i$ are the Barycentric coordinates of $\vec{u}$.

Note
Note that we are simply looking for how to write $\vec{u}$ as an affine combination of $\vec{v}_1, \cdots, \vec{v}_m$.

We consider coloring a triangle by using RGB values (r, g, b), where $0 \le r, g, b \le 1$ (so $(0,1,0)$ is green).

The RGB values of the vertices of a triangle will be used to interpolate the color inside the triangle. The contribution of each RGB value will be depending on the Barycentric coordinates of the point.

Consider we have a triangle with 3 vertices, the top vertex R having RGB values $(1,0,0)$, the left vertex G having values $(0,1,0)$, and the right vertex B having values $(0,0,1)$.
How much will each vertex contribute to the color of some point P within the triangle?

The contribution of vertex R to the color of point P will be the ratio

$$\frac{\text{area of triangle GPB}}{\text{area of entire triangle RGB}} = c_1$$

Which sort of represents how "close" point P is to vertex R.

We can generalize this to define c_2, c_3 similarly, and we find that $c_1 + c_2 + c_3 = 1$.

Moreover, these are the coefficients that form the affine combination of P using the 3 vertices.

Example 5.4
Given points and RGB values

Point	RGB
(2, 0)	(1, 0.1, 0.2)
(1, 2)	(0, 1, 1)
(3, 2)	(0.2, 0.3, 0.4)

Find the RGB value of the interpolated point $(1.5, 0.8)$.

We express $(1.5, 0.8)$ as an affine combination of $(2, 0)$, $(1, 1)$, $(3, 2)$ by row reducing

$$\begin{bmatrix} 2 & 1 & 3 & | & 1.5 \\ 0 & 1 & 2 & | & 0.8 \\ 1 & 1 & 1 & | & 1 \end{bmatrix}$$

And we get

$$\begin{bmatrix} 1 & 0 & 0 & | & 0.3 \\ 0 & 1 & 0 & | & 0.6 \\ 0 & 0 & 1 & | & 0.1 \end{bmatrix}$$

Which means that the RGB value at the point (1.5, 0.8) is

$$0.3 \cdot (1, 0.1, 0.2) + 0.6 \cdot (0, 1, 1) + 0.1 \cdot (0.2, 0.3, 0.4) = (0.32, 0.66, 0.7)$$

5.2 Convex Combinations (Section 2.4)

Definition 5.5
A **convex combination** of $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$ is a linear combination $\sum_{i=1}^m c_i \vec{v}_i$ such that $\sum_{i=1}^m c_i = 1$ and $c_i \geq 0$, $1 \leq i \leq m$.

The set of all convex combinations of a set S is the **convex hull**, denoted $\text{conv}(S)$.

Example 5.6
Consider $S = \{\vec{v}_1, \vec{v}_2\}$ (where $\vec{v}_1, \vec{v}_2$ are not multiples).
Then, $\text{conv}(S)$ contains points $\vec{y} = (1-t)\vec{v}_1 + \vec{v}_2$, $0 \leq t \leq 1$.

$$\begin{aligned} \vec{y} &= (1-t)\vec{v}_1 + t\vec{v}_2 && 0 \leq t \leq 1 \\ &= \vec{v}_1 + t(\vec{v}_2 - \vec{v}_1) && \text{equivalent to } \vec{p}_0 + t\vec{v} \end{aligned}$$

Which is a line segment.

The convex hull of 3 points (not on a line) exactly create a triangle (the ratios in the last section make sense, because they are all nonnegative).