Monthly Meeting on July

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matroid

definition

Let $M = (S, \mathcal{I})$ where S is ground set, and $\mathcal{I}$ is a family satisfying $\mathcal{I} \subseteqq 2^S$.
M is called a **matroid** when $\mathcal{I}$ satisfies:

$$\emptyset \in \mathcal{I} \tag{1}$$

$$I_1 \subset I_2, I_2 \in \mathcal{I} \Rightarrow I_1 \in \mathcal{I} \tag{2}$$

$$I_1, I_2 \in \mathcal{I}, |I_1| < |I_2| \Rightarrow \exists i_2 \in I_2 \,;\; I_1 \cup \{i_2\} \in \mathcal{I} \tag{3}$$

$\mathcal{I}$ is called an **independent set family**.
A maximal element in $\mathcal{I}$ with respect to order “$\subseteqq$” is called a **base**.
Evary base is the same size. This size is called **rank**.

matroid examples

vector matroid

Linearly independent sets construct matroid:

$$\boldsymbol{a_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{a_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{a_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{a_4} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

ground set S is $\{a_1, a_2, a_3, a_4\}$, and $\mathcal{I}$ consists of linearly independent combinations, that:

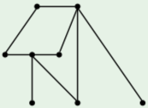
$$\begin{aligned} \mathcal{I} = \{&\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \\ &\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}, \\ &\{a_2, a_4\}, \{a_3, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}, \{a_2, a_3, a_4\}\} \end{aligned}$$

Then $\mathcal{I}$ satisfies (1), (2), (3).

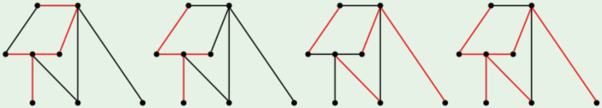
matroid examples

graphic matroid

In graph theory, forests construct matroid:
Let $G = (V, E)$ be a graph where V is a set of verteces and E is a set of edges:



When $\mathcal{I}$ is a family of forests, such as:



etc...

Then $\mathcal{I}$ satisfies (1), (2), (3).

On Disjoint Common Bases in Two Matroids

problem 1

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids on the ground set S, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are the respective families of independent sets. A set $B \subsetneqq S$ that is both a base of M_1 and M_2 is called a common base. The problem is to decide whether S can be partitioned into common bases.

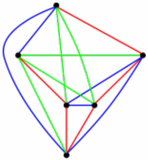


Figure: partitioned into bases

On Disjoint Common Bases in Two Matroids

conjecture 2 (Rota's conjecture)

Let $M = (T, \mathcal{I})$ be a matroid of rank n. Let $A_1, ..., A_n$ be a partition of T into bases of M. Then there are disjoint bases $B_1, ..., B_n$ such that $|A_i \cap B_j| = 1$ for every $i = 1, ..., n$ and $j = 1, ..., n$.

Next conjecture is the generalization of Rota's conjecture:

conjecture 3 (Chow's conjecture)

Let $M = (T, \mathcal{I})$ be a matroid of rank n with the property that T can be partitioned into b bases where $3 \leq b \leq n$. Let $I_1, ..., I_n \in \mathcal{I}$ be disjoint independent sets, each size at most b. Then there exists a partition of T into sets $A_1, ..., A_n$ such that $I_i \subseteqq A_i$ and $|A_i| = b$ for every $i = 1, ..., n$, and there exist disjoint bases $B_1, ..., B_b$ such that $|A_i \cap B_j| = 1$ for every $i = 1, ..., n$ and $j = 1, ..., b$.

On Disjoint Common Bases in Two Matroids

theorem 4

Problem 1 can be solved in polynomial time if and only if this is under the additional assumption that one of the matroids is a direct sum of uniform matroids.

definition

Let $M_1 = (E_1, \mathcal{I}_1),\ M_2 = (E_2, \mathcal{I}_2)$ be matroids where $E_1, E_2 \neq \emptyset, E_1 \cap E_2 = \emptyset$.
M is called a **direct sum** of 2 matroids, M_1 and M_2 when:

$$M = (E_1 \cup E_2, \mathcal{I}_1 \oplus \mathcal{I}_2)$$
$$\textit{where } \mathcal{I}_1 \oplus \mathcal{I}_2 = \{X_1 \cup X_2 \mid X_1 \in \mathcal{I}_1, X_2 \in \mathcal{I}_2\}$$

On Disjoint Common Bases in Two Matroids

uniform matroid

Let $U = (S, \mathcal{I})$ be a matroid where $\mathcal{I} \in 2^S$ satisfies:

$$\mathcal{I} = \{I \mid |I| \leq k\}$$

Then U is a matroid of rank k. This U is called a **uniform matroid**.

On Disjoint Common Bases in Two Matroids

claim 5

Conjecture 3 is false for every b such that $2 \leq b \leq \frac{n}{3}$.

corollary 6

Problem 1 can be solved in polynomial time if and only if this is true under the additional assumption that M_2 is a direct sum of uniform matroids whose blocks are each independent in M_1.

next month

Yet these methods show that Chow's conjecture is false when $3 \leq b \leq \frac{n}{3}$, Rota's conjecture is still alive since it's Chow's conjecture in the case of $b = n$. There's still room for research about Chow's conjecture.
Also, computational complexity of problem 1 is still open. This paper gives neccesary and sufficient condition to solve problem 1 in polynomial time only when problem 1 can be solved. It's yet to be discovered exactly when problem 1 can be solved.
What I'm going to do next month is to think about these problems.

References

Nicholas J. A. Harvey, Tamas Kiraly, and Lap Chi Lau: “On Disjoint Common Bases in Two Matroids”, SIDMA. Vol. 25, No. 4. pp. 1792-1803

Thank you for your attention.