Monthly Meeting on July

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matroid

definition

Let $M = (S, \mathcal{I})$ where S is ground set, and $\mathcal{I}$ is a family satisfying $\mathcal{I} \subseteqq 2^S$.
M is called a **matroid** when $\mathcal{I}$ satisfies:

$$\emptyset \in \mathcal{I} \tag{1}$$

$$I_1 \subset I_2, I_2 \in \mathcal{I} \Rightarrow I_1 \in \mathcal{I} \tag{2}$$

$$I_1, I_2 \in \mathcal{I}, |I_1| < |I_2| \Rightarrow \exists i_2 \in I_2 \setminus I_1 \ ; \ I_1 \cup \{i_2\} \in \mathcal{I} \tag{3}$$

$\mathcal{I}$ is called an **independent set family**.
A maximal element in $\mathcal{I}$ with respect to order "$\subseteqq$" is called a **base**.
Evary base is the same size. This size is called **rank**.

matroid examples

vector matroid

Linearly independent sets construct matroid:

$$\boldsymbol{a_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{a_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{a_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{a_4} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

ground set S is $\{a_1, a_2, a_3, a_4\}$, and $\mathcal{I}$ consists of linearly independent combinations, that:

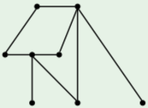
$$\begin{aligned} \mathcal{I} = \{&\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \\ &\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}, \\ &\{a_2, a_4\}, \{a_3, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_3, a_4\}, \{a_2, a_3, a_4\}\} \end{aligned}$$

Then $\mathcal{I}$ satisfies (1), (2), (3).

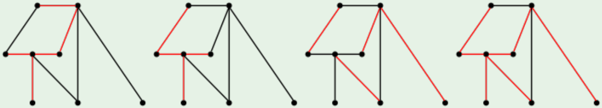
matroid examples

graphic matroid

In graph theory, forests construct matroid:
Let $G = (V, E)$ be a graph where V is a set of verteces and E is a set of edges:



When $\mathcal{I}$ is a family of forests, such as:



etc...

Then $\mathcal{I}$ satisfies (1), (2), (3).

matroid examples

uniform matroid

Let $U = (S, \mathcal{I})$ be a matroid where $\mathcal{I} \in 2^S$ satisfies:

$$\mathcal{I} = \{I \mid |I| \leq k\}$$

Then U is a matroid of rank k. This U is called a **uniform matroid**.

1. Introduction
2. Progress
3. Next step

On Disjoint Common Bases in Two Matroids

definition

When $\mathcal{A} = \{A_1, ..., A_n\}$ is a **partition** of S,

$$S = A_1 \oplus, ..., \oplus A_n (\forall i, j;\ A_i \cap A_j = \emptyset)$$

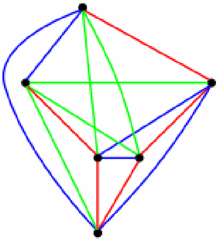


Figure: partitioned into bases

On Disjoint Common Bases in Two Matroids

problem 1

Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids on the ground set S, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are the respective families of independent sets. A set $B \subseteq S$ that is both a base of M_1 and M_2 is called a common base. The problem is to decide whether S can be partitioned into common bases.

input: common ground set S and matroids M_1 and M_2 on it
output: partition of S into common bases of M_1 and M_2

On Disjoint Common Bases in Two Matroids

theorem 4

Problem 1 can be solved in polynomial time if and only if problem 1 can be solved in polynomial time under the additional assumption that one of the matroids is a direct sum of uniform matroids.

definition

Let $M_1 = (E_1, \mathcal{I}_1),\ M_2 = (E_2, \mathcal{I}_2)$ be matroids where $E_1, E_2 \neq \emptyset, E_1 \cap E_2 = \emptyset$.
M is called a **direct sum** of two matroids, M_1 and M_2 when:

$$M = (E_1 \cup E_2, \mathcal{I}_1 \oplus \mathcal{I}_2)$$
$$where\ \mathcal{I}_1 \oplus \mathcal{I}_2 = \{X_1 \cup X_2 \mid X_1 \in \mathcal{I}_1, X_2 \in \mathcal{I}_2\}$$

also we denote this $M = M_1 \oplus M_2$.

On Disjoint Common Bases in Two Matroids

Preliminaries

S : ground set whose size is $|S| = r(k+1)$
$M_1,\ M_2$: two different matroids of rank r on S

$S^{[k]} := \{\{(s,1),(s,2),...,(s,k)\} \mid s \in S\}$
$\hat{\mathcal{S}} := \{\{s,(s,1),(s,2),...,(s,k)\} \mid s \in S\}$ $(note :$ $\hat{\mathcal{S}}$ is a partition of $S \cup S^{[k]})$
X^* := dual matroid of X $(X : matroid)$

then, define two new matroid:
$M := M_1 \oplus M_2^{*[k]}$
$U_1(\hat{\mathcal{S}}) := (S \cup S^{[k]}, \mathcal{I}_U)$ where $\mathcal{I}_U = \{I \mid I \cup s^{[k]} \leq 1\}$
also note that they are both rank $r(k+1)$

On Disjoint Common Bases in Two Matroids

claim 7

The common bases of M and $U_1(\hat{\mathcal{S}})$ are precisely the subsets $B \subseteq S \cup S^{[k]}$ satisfying

$$|B \cap \hat{s}| = 1 \ \forall s \in S \text{ and } B \cap S \text{ is a common base of } M_1 \text{ and } M_2$$

corollary 8

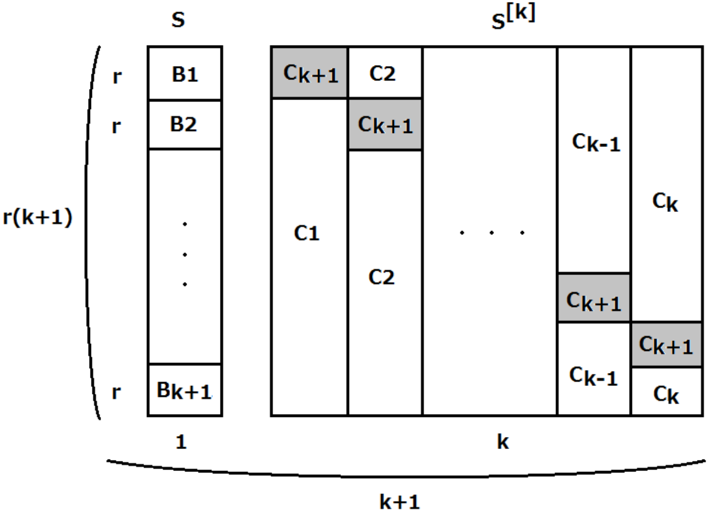
if $B_1, ..., B_{k+1}$ is a partition of $S \cup S^{[k]}$ into common bases of M and $U_1(\hat{\mathcal{S}})$, then $B_1 \cap S, ..., B_{k+1} \cap S$ is a partition of S into common bases ofM_1 and M_2.

On Disjoint Common Bases in Two Matroids

claim 9

Given a partition $B_1, ..., B_{k+1}$ of S into common bases of M_1 and M_2, we can constract a partition $B'_1, ..., B'_{k+1}$ of $S \cup S^{[k]}$ into common bases of M and $U_1(\hat{\mathcal{S}})$.

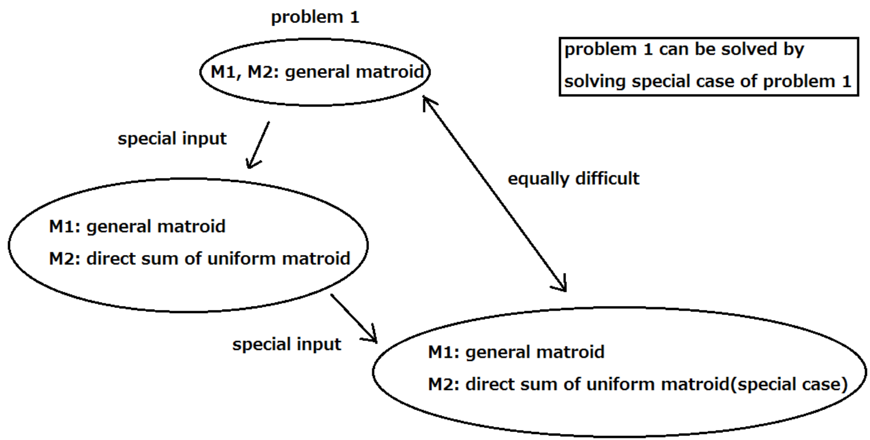
On Disjoint Common Bases in Two Matroids



then let $B'_j := B_j \cup C_j$, B'_j will be a partition of $S \cup S^{[k]}$.

On Disjoint Common Bases in Two Matroids

by corollary 8 and claim 9...



next month

TODO:

1. examinate what ploblem 1 will be in simple matroid such as an uniform matroid and a graphic matroid
2. or try other things like Rota's conjecture

References

Nicholas J. A. Harvey, Tamas Kiraly, and Lap Chi Lau: “On Disjoint Common Bases in Two Matroids”, SIDMA. Vol. 25, No. 4. pp. 1792-1803

Thank you for your attention.