

Introduction to ADaaa

Kaiwen Jin

香港科技大学

2024-11-25

Spectral Density

It has two kinds of forms.

$$1. \quad A(x) = \sum_{k=1}^{N} \delta(x - x_i)$$

2.
$$A(x) \in \mathcal{S}, \quad A(z) \in \mathbb{H}(\mathbb{C})$$

Now we nonly consider the second kind.

Green Function

Define 1. Green function

Kaiwen Jin 2024-11-25 Introduction to ADaaa 2 / 14

$$\begin{split} G(z) &= \int_{\mathbb{R}} \frac{A(x)}{z-x} dx \quad Imz > 0, \\ G(w) &= \lim_{\eta \to 0^+} \int \frac{A(x)}{w+i\eta-x} dx = P.V. \int \frac{A(x)}{w-x} dx - i\pi A(w) \\ A(w) &= -\frac{1}{\pi} Im(G(w)) \end{split}$$

Theorem 1.

$$\lim_{z \to w} \int \frac{A(x)}{z - x} dx = P.V. \int \frac{A(x)}{w - x} dx - i\pi * sgn(\eta) A(w)$$

Kaiwen Jin 2024-11-25 3 / 14 Introduction to ADaaa



Proposition 1.1. The green function G(z) on Imz>0 can be analytically continued to the whole complex plane $\mathbb C.$

It means that G(z) has no pole on \mathbb{C} .

Kaiwen Jin 2024-11-25 Introduction to ADaaa 4 / 14

2. Complex Differentiation

Definition 2. For a function

$$f: \mathbb{C} \to \mathbb{C}, \quad f(x,y) = u(x,y) + iv(x,y), \quad u,v \in C^{\infty}$$

We define that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial z^*}dz^*$$

In above formula,

$$dz = dx + idy, \quad dz^* = dx - idy$$

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$



2. Complex Differentiation

Proposition 2.1

$$\left(\frac{\partial f}{\partial z}\right)^* = \left(\frac{\partial f^*}{\partial z^*}\right)$$

Proposition 2.2

$$dg(f) = \left(\frac{\partial g}{\partial f}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial f^*}\frac{\partial f^*}{\partial z}\right)dz + \left(\frac{\partial g}{\partial f}\frac{\partial f}{\partial z^*} + \frac{\partial g}{\partial f^*}\frac{\partial f^*}{\partial z^*}\right)dz^*$$



It's a interesting algorithm because it's most important idea is not barycentric but deviding a set of points into 2 parts. One of them is insert points set and the second is checking points set.

The reason is you can't decide the weight just with all ponits as insert points.

Denote chosen points as A, and set of waiting points as B. Assume that

$$A = \{z_1, ..., z_n\}, \quad B = \{z_{\{n+1\}}, ..., z_m\}$$

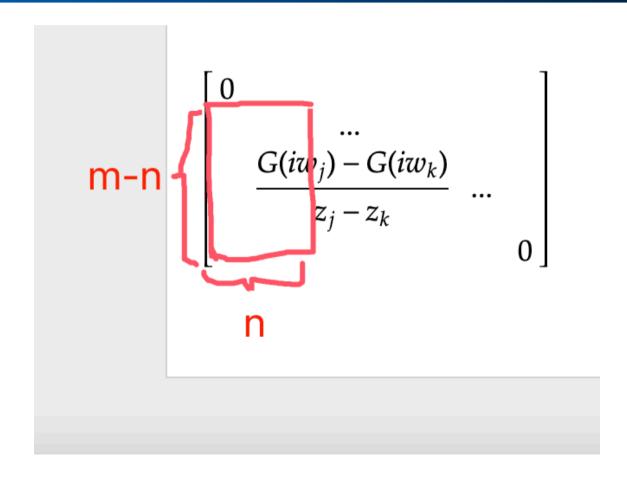
Now consider:

$$L = \left(\frac{G\!\left(z_{j}\right) - G\!\left(z_{k}\right)}{z_{j} - z_{k}}\right)_{jk}$$

Then we get a sub matrix L_n from it by getting the first n columns and n+1,...,m rows.

Kaiwen Jin 2024-11-25 7 / 14 Introduction to ADaaa





Kaiwen Jin 2024-11-25 Introduction to ADaaa 8 / 14

Now for

$$G(z) \approx \frac{N_{n(z)}}{D_{n(z)}}$$

$$N_{n(z)} = \sum_{j=1}^{n} rac{w_{j} G(z_{j})}{z - z_{j}}, \quad D_{n(z)} = \sum_{j=1}^{n} rac{w_{j}}{z - z_{j}}$$

We have

$$(GD_n-N_n)(B)=L_nw$$

$$\Rightarrow \min_{w} \lVert (GD_n-L_n)(B)\rVert_{L^2} = \min_{w} \lVert L_nw\rVert = \min \sigma(L_n)$$

Then we can use svd to find such min $\sigma(L_n)$ and related w.

Then we chose

$$z_{new} = argmax_{z \in B} \quad \|G(z) - \frac{N_{n(z)}}{D_{n(z)}}\|$$

Add z_{new} into A and delete it from B and continue iteration.

Get G(z) and then we can reconstruct A(w)

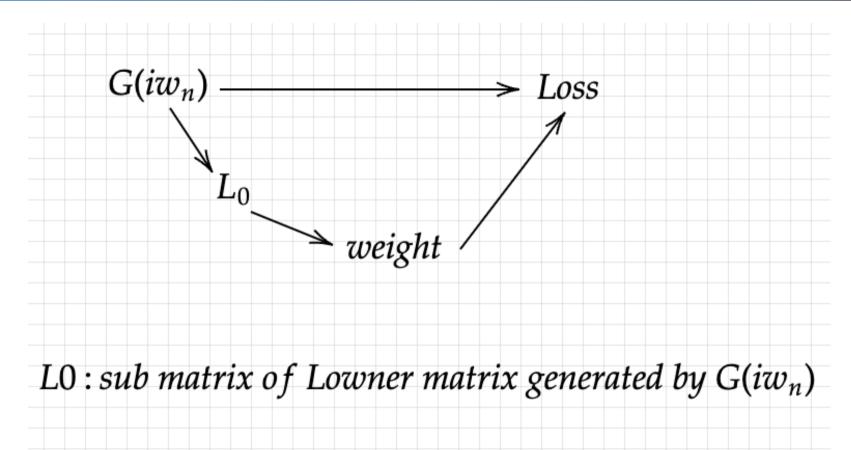
Kaiwen Jin 2024-11-25 Introduction to ADaaa 10 / 14

Given $G_0(iwn)$ and wn, the way we calcuate Loss function is as following chart and the Loss function is defined as

$$Loss(G(iwn), weight) = \|A(x) - A_0(x)\|_2$$

Kaiwen Jin 2024-11-25 Introduction to ADaaa 11 / 14





Kaiwen Jin Introduction to ADaaa 12 / 14 2024-11-25

Finite difference (FD) works awful for calculating derivative of

So we use AD to calculate $\nabla Loss$

But because L_0 is an ill-condition number matrix and the log of its condition number is approximately proportional to the number of G(iwn), formulas of calculating compelx SVD performs disastrously. So we directly use FD and infact

$$\frac{weight(G_0 + \varepsilon) - weight(G_0)}{\varepsilon}$$

performs very stably as ε changes.

Then with proposition 2.2 we have

Kaiwen Jin 2024-11-25 Introduction to ADaaa 13 / 14

Theorem 4.

$$\nabla_G Loss(G, w(G)) = 2 \left(\frac{\partial L}{\partial G}\right)^* = \nabla_1 L + 2 \left(\frac{\partial L}{\partial w}\right)^\dagger * \left(\frac{Jw}{JG}\right)^* + 2 \left(\frac{\partial L}{\partial w}\right)^T * \frac{Jw}{JG^*}$$

Here $\frac{Jy}{Ix}$ means the jacobian matrix.