3

LIMITING DISTRIBUTION AND CONVERGENCE THEOREM

3.1 Converge in Distribution

Consider a sequence of random variables

$$Y_1, Y_2, \ldots, Y_n, \ldots$$

with a corresponding sequence of CDF's

$$G_1(y) = \mathbb{P}\{Y_1 \le y\}, G_2(y) = \mathbb{P}\{Y_2 \le y\}, \ldots, G_n(y) = \mathbb{P}\{Y_n \le y\}, \ldots$$

and let a random variable

$$Y$$

has CDF

$$G(y) = \mathbb{P}\{Y \le y\}$$

Definition 3.1.1. **(Converge in distribution)**. If

$$\lim_{n \to \infty} G_n(y) = G(y)$$

for all y at which $G(y)$ is continuous, we say Y_n is converge in distribution to Y, denoted by

$$Y_n \xrightarrow{d} Y \quad \text{as } n \to \infty$$

and $G(y)$ is called the limiting distribution of Y_n.

Theorem 3.1.1. **(Central limit theorem (CLT)).** Let $X_1, X_2, \ldots, X_n, \ldots$ be i.i.d. with mean $\mathbb{E}(X_i) = \mu$ and variance $\mathbb{V}ar(X_i) = \sigma^2$. Then

$$\frac{\sqrt{n}\left(\overline{X} - \mu\right)}{\sigma} \xrightarrow{d} Z \sim \mathcal{N}(0, 1)$$

as $n \to \infty$.

Lemma 3.1.1. **(General lemma).** If a_n is a real sequence such that

$$\lim_{n\to\infty} a_n = a$$

Then

$$\lim_{n\to\infty} (1 + \frac{a_n}{n})^n = e^a$$

Proof. Sufficient to show

$$\lim_{n\to\infty} n \ln(1 + \frac{a_n}{n}) = a$$

Because $n \ln(1 + \frac{a_n}{n}) = a_n \frac{\ln(1+\frac{a_n}{n})}{a_n/n}$

$$\lim_{n\to\infty} n \ln(1 + \frac{a_n}{n}) = \lim_{n\to\infty} a_n \lim_{n\to\infty} \frac{\ln(1 + \frac{a_n}{n})}{a_n/n} = a \lim_{n\to\infty} \frac{\ln(1 + \frac{a_n}{n})}{a_n/n}$$

So sufficient to show

$$\lim_{n\to\infty} \frac{\ln(1 + \frac{a_n}{n})}{a_n/n} = 1$$

Let sequence $x_n = a_n/n \to 0$, by L'Hopital's rule

$$\lim_{n\to\infty} \frac{\ln(1 + \frac{a_n}{n})}{a_n/n} = \lim_{n\to\infty} \frac{\ln(1 + x_n)}{x_n} = \lim_{n\to\infty} \frac{\frac{d}{dx_n} \ln(1 + x_n)}{\frac{d}{dx_n} x_n} = \lim_{n\to\infty} \frac{1}{1 + x_n} = 1$$

□

Example 3.1.1. **(Exponential distribution).** Suppose we have a complex system which break down into n parts. The failure of any of the parts will make the whole system fail. Let $T_i, i = 1, 2, \ldots, n$ be the time to failure of each of the parts ans suppose $T_i \sim \mathcal{U}nif(0, n\theta), i = 1, 2, \ldots, n$[1] and are independent. let Y_n be the time to failure of the whole system. What is the limiting distribution of Y_n?

The time to failure of the whole system can be expressed as

$$Y_n = \min_{1 \le i \le n} T_i$$

Then

$$\mathbb{P}\{Y_n \le t\} = \mathbb{P}\{\min_{1 \le i \le n} T_i \le t\} = 1 - \mathbb{P}\{\min_{1 \le i \le n} T_i > t\}$$

$$= 1 - \mathbb{P}\{T_1 > t, T_2 > t, \ldots, T_n > t\} \stackrel{i.i.d.}{=} 1 - (\mathbb{P}\{T_1 > t\})^n$$

$$= 1 - \left(1 - \frac{t}{n\theta}\right)^n$$

which requires $0 \le t \le n\theta$. Now take limit on both sides and apply the general lemma 3.1.1

$$\lim_{n \to \infty} \mathbb{P}\{Y_n \le t\} = 1 - \lim_{n \to \infty} \left(1 - \frac{t}{n\theta}\right)^n = 1 - e^{-t/\theta}$$

with $0 \le t \le \infty$, which is an exponential distribution[2]. Therefore,

$$Y_n \xrightarrow{d} Y \sim \mathcal{E}xp(\theta) \quad \text{as } n \to \infty$$

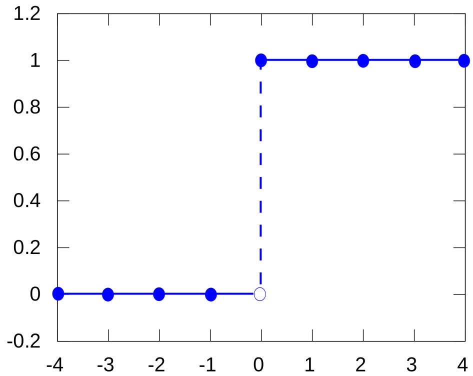
3.2 Converge Stochastically

$\boxed{\text{Definition 3.2.1.}}$ **(Degenerate distribution).** The function $G(y)$ is the CDF of a degenerate distribution at the value $y = c$ if

[1] Recall the CDF of uniform distribution $\mathcal{U}nif(0, n\theta)$: $\mathbb{P}\{T_i \le t\} = \begin{cases} 0, & t < 0 \\ t/(n\theta), & 0 \le t \le n\theta \\ 1, & t > n\theta \end{cases}$

[2] Recall the CDF of exponential distribution $\mathcal{E}xp(\theta)$ is $1 - e^{-x/\theta}$

$$G(y) = \begin{cases} 0, & y < c \\ 1, & y \geq c \end{cases}$$



$\boxed{\text{Definition 3.2.2.}}$ **(Converge stochastically).** A sequence of random variables $Y_1, Y_2, \ldots$ is said to converge stochastically to a constant c if $Y_n \xrightarrow{d} Y$ as $n \to \infty$ where Y has CDF G which is degenerate at c.

3.3 Converge in Probability

$\boxed{\text{Definition 3.3.1.}}$ **(Converge in Probability).** The sequence of random variables $Y, Y_1, Y_2, \ldots$ is said to converge in probability to Y, written $Y_n \xrightarrow{P} Y$ as $n \to \infty$, if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\{|Y_n - Y| > \varepsilon\} = 0$$

or equivalently,

$$\lim_{n\to\infty} \mathbb{P}\{|Y_n - Y| \leq \varepsilon\} = 1$$

$\boxed{\text{Theorem 3.3.1.}}$ **(Converge stochastically and converge in Probability).** Y_n converges stochastically to c if and only if $Y_n \xrightarrow{P} c$ as $n \to \infty$.

Proof. Suppose Y_n converge stochastically to c. Then

$$\mathbb{P}\{Y_n \leq y\} = G_n(y) \to \begin{cases} 0, & y < c \\ 1, & y \geq c \end{cases}$$

as $n \to \infty$. For any $\varepsilon > 0$

$$\mathbb{P}\left\{|Y_n - c| > \varepsilon\right\} = \mathbb{P}\left\{Y_n > c + \varepsilon, Y_n < c - \varepsilon\right\}$$

$$\overset{1}{\leq} \mathbb{P}\left\{Y_n > c + \varepsilon\right\} + \mathbb{P}\left\{Y_n < c - \varepsilon\right\}$$

$$= 1 - \underbrace{G_n(c + \varepsilon)}_{\to 1} + \underbrace{G_n(c - \varepsilon)}_{\to 0} \to {}^{2}0$$

as $n \to \infty$, which implies $Y_n \xrightarrow{P} c$ as $n \to \infty$.

Now suppose $Y_n \xrightarrow{P} c$ as $n \to \infty$, or for every $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\{|Y_n - c| > \varepsilon\} = 0 \quad \text{or} \quad \lim_{n\to\infty} \mathbb{P}\{|Y_n - c| \leq \varepsilon\} = 1$$

Let $y > c$

$$G_n(y) = \mathbb{P}\left\{Y_n \leq y\right\} = \mathbb{P}\left\{Y_n - c \leq y - c\right\} \overset{3}{\geq} \mathbb{P}\left\{-(y - c) \leq Y_n - c \leq y - c\right\}$$

$$= \mathbb{P}\left\{|Y_n - c| \leq \underbrace{y - c}_{\text{some } \varepsilon > 0}\right\} \to 1$$

as $n \to \infty$. Next let $y < c$

$$G_n(y) = \mathbb{P}\left\{Y_n \leq y\right\} = \mathbb{P}\left\{c - Y_n \geq c - y\right\}$$

$$\overset{4}{\leq} \mathbb{P}\left\{|c - Y_n| \geq \underbrace{c - y}_{\text{some } \varepsilon > 0}\right\}$$

[1] Recall $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$, which implies $\mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B)$

[2] Note that $\lim_{n\to\infty} x_n = x$ is equivalent to $x_n \to x$ as $n \to \infty$

[3] Recall $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B|A) \leq \mathbb{P}(A)$ or $\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A|B) \leq \mathbb{P}(B)$

[4] Recall $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \geq \mathbb{P}(A)$, since $\mathbb{P}(A \cap B) \leq \mathbb{P}(B)$. Or $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \geq \mathbb{P}(B)$, since $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)$

$$\stackrel{1}{\leq} \mathbb{P}\left\{|Y_n - c| > \underbrace{\frac{c-y}{2}}_{\text{some } \varepsilon>0}\right\} \to 0$$

as $n \to \infty$. Hence,

$$G_n(y) \to \begin{cases} 0, & y < c \\ 1, & y \geq c \end{cases}$$

as $n \to \infty$. So Y_n converge stochastically to c. □

$\boxed{\text{Theorem 3.3.2.}}$ **(Law of large numbers)**. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}X_i = \mu$, $\mathbb{V}arX_i = \sigma^2$, and $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Then $\overline{X}_n$ converges stochastically to μ. I.e., $\forall \varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\{|\overline{X}_n - \mu| > \varepsilon\} = 0$$

or $\overline{X}_n \xrightarrow{P} \mu$ as $n \to \infty$.

Proof.

$$\mathbb{V}ar\overline{X}_n = \mathbb{V}ar\frac{1}{n}\sum_{i=1}^n X_i \stackrel{i.i.d.}{=} \frac{1}{n^2}\sum_{i=1}^n \mathbb{V}arX_i = \frac{1}{n^2}\sum_{i=1}^n \sigma^2 = \frac{\sigma^2}{n}$$

$$\mathbb{E}\overline{X}_n = \mathbb{E}\frac{1}{n}\sum_{i=1}^n X_i = \frac{1}{n}\sum_{i=1}^n \mathbb{E}X_i = \mu$$

Apply Chebychev's inequality

$$\mathbb{P}\{|\overline{X}_n - \mu| > \varepsilon\} \stackrel{2}{\leq} \frac{1}{\varepsilon^2}\frac{\sigma^2}{n} \to 0$$

as $n \to \infty$. □

[1]Recall if $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$, since A and $A^c \cap B$ are disjoint, and $\mathbb{P}(B) = \mathbb{P}(A \cup (A^c \cap B)) = \mathbb{P}(A) + \mathbb{P}(A^c \cap B) \geq \mathbb{P}(A)$

[2]Recall Chebychev's inequality $\mathbb{P}(|X - \mathbb{E}X| \geq a) \leq \frac{\mathbb{V}arX}{a^2}$, $a > 0$

Theorem 3.3.3. **(Property of converge in probability).** Let $g : \mathbb{R}^2 \to \mathbb{R}$, continuous at the point (c, d), and suppose $X_n \xrightarrow{P} c$ and $Y_n \xrightarrow{P} d$. Then

$$g(X_n, Y_n) \xrightarrow{P} g(c, d)$$

as $n \to \infty$.

Proof. g continuous at (c, d) means that given $\varepsilon > 0$, $\exists \delta > 0$ such that $|x - c| \leq \delta$ and $|y - d| \leq \delta$ imply $|g(x, y) - g(c, d)| \leq \varepsilon$. So

$$\mathbb{P}\{|g(X_n, Y_n) - g(c, d)| \leq \varepsilon\} \overset{1}{\geq} \mathbb{P}\{|X_n - c| \leq \delta, |Y_n - d| \leq \delta\}$$

$$\overset{2}{=} 1 - \mathbb{P}\{\{|X_n - c| > \delta\} \cup \{|Y_n - d| > \delta\}\}$$

$$\overset{3}{\geq} 1 - \underbrace{\mathbb{P}\{|X_n - c| > \delta\}}_{\to 0} - \underbrace{\mathbb{P}\{|Y_n - d| > \delta\}}_{\to 0}$$

as $n \to \infty$. Therefore

$$\lim_{n \to \infty} \mathbb{P}\{|g(X_n, Y_n) - g(c, d)| \leq \varepsilon\} \geq 1$$

which implies

$$\lim_{n \to \infty} \mathbb{P}\{|g(X_n, Y_n) - g(c, d)| \leq \varepsilon\} = 1$$

□

Corollary 3.3.1. **(Properties of converge in probability).** Suppose $X_n \xrightarrow{P} c$ and $Y_n \xrightarrow{P} d$

(i) $aX_n + bY_n \xrightarrow{P} ac + bd$

[1] Recall if $A \implies B$, then $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$
[2] Recall $\overline{A \cap B} = \overline{A} \cup \overline{B}$
[3] Recall $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq \mathbb{P}(A) + \mathbb{P}(B)$

(ii) $X_nY_n \xrightarrow{P} cd$

(iii) $X_n/c \xrightarrow{P} 1$, if $c \neq 0$

(iv) $1/X_n \xrightarrow{P} 1/c$, if $c \neq 0$, $\mathbb{P}\{X_n \neq 0\} = 1$

(v) $\sqrt{X_n} \xrightarrow{P} \sqrt{c}$, if $c > 0$, $\mathbb{P}\{X_n \geq 0\} = 1$

Proof. Apply Theorem 3.3.3 by $g(x, y) = ax + by$, $g(x, y) = xy$, $g(x, y) = x/c$, $g(x, y) = 1/x$, $g(x, y) = \sqrt{x}$. □

3.4 Advanced Probability: Converge in Distribution and Converge in Probability

$\boxed{\text{Theorem 3.4.1.}}$ **(Converge in probability implies converge in distribution).** If $Y_n \xrightarrow{P} Y$, as $n \to \infty$ then $Y_n \xrightarrow{d} Y$ as $n \to \infty$.

Proof. $\forall \varepsilon > 0$

$$F_{Y_n}(y) = \mathbb{P}\{Y_n \leq y\} = \mathbb{P}\{Y_n \leq y, Y \leq y+\varepsilon\} + \mathbb{P}\{Y_n \leq y, Y > y+\varepsilon\}$$

$$= \mathbb{P}\{Y_n \leq y | Y \leq y+\varepsilon\}\mathbb{P}\{Y \leq y+\varepsilon\} + \mathbb{P}\{Y_n \leq y, Y > y+\varepsilon\}$$

$$\overset{1}{\leq} \mathbb{P}\{Y \leq y+\varepsilon\} + \mathbb{P}\{Y_n < Y-\varepsilon\}$$

$$\overset{2}{\leq} F_Y(y+\varepsilon) + \mathbb{P}\{|Y_n - Y| > \varepsilon\}$$

Similarly

$$F_Y(y-\varepsilon) = \mathbb{P}\{Y \leq y-\varepsilon, Y_n \leq y\} + \mathbb{P}\{Y \leq y-\varepsilon, Y_n > y\}$$

$$= \mathbb{P}\{Y \leq y-\varepsilon | Y_n \leq y\}\mathbb{P}\{Y_n \leq y\} + \mathbb{P}\{Y \leq y-\varepsilon, Y_n > y\}$$

$$\leq \mathbb{P}\{Y_n \leq y\} + \mathbb{P}\{Y < Y_n - \varepsilon\} \leq F_{Y_n}(y) + \mathbb{P}\{|Y_n - Y| > \varepsilon\}$$

[1] Recall if $A \implies B$, then $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$ and recall $\mathbb{P}(A)\mathbb{P}(B) \leq \mathbb{P}(A)$
[2] Recall $\mathbb{P}(A \cup B) \geq \mathbb{P}(A)$

Thus

$$F_Y(y-\varepsilon) - \underbrace{\mathbb{P}\{|Y_n - Y| > \varepsilon\}}_{\to 0} \leq F_{Y_n}(y) \leq F_Y(y+\varepsilon) + \underbrace{\mathbb{P}\{|Y_n - Y| > \varepsilon\}}_{\to 0}$$

Taking now $n \to \infty$

$$F_Y(y-\varepsilon) \leq \liminf_{n\to\infty} F_{Y_n}(y) \leq \limsup_{n\to\infty} F_{Y_n}(y) \leq F_Y(y+\varepsilon)$$

Since this holds for any $\varepsilon > 0$, $\lim_{\varepsilon\to 0} F_Y(y-\varepsilon) = \lim_{\varepsilon\to 0} F_Y(y+\varepsilon) = F_Y(y)$

$$\lim_{n\to\infty} F_{Y_n}(y) = F_Y(y)$$

□

Theorem 3.4.2. **(Continuous mapping theorem)**. Suppose $Y_n \xrightarrow{d} Y$ and $X_n \xrightarrow{P} c$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function everywhere except at possibly a countable set of points in $\mathbb{R}^2$. Then if $\mathbb{P}\{g \text{ is continuous at } (c, Y)\} = 1$,

$$g(X_n, Y_n) \xrightarrow{d} g(c, Y)$$

as $n \to \infty$.

Corollary 3.4.1.. Suppose $Y_n \xrightarrow{d} Y$ and $X_n \xrightarrow{P} c$

(i) $X_n + Y_n \xrightarrow{d} c + Y$

(ii) $X_n Y_n \xrightarrow{d} cY$

(iii) $Y_n / X_n \xrightarrow{d} Y/c$ if $c \neq 0$

Proof. Apply the continuous mapping theorem 3.4.2 and let $g(x,y) = x + y$ $g(x,y) = xy$, $g(x,y) = y/x$ (continuous at $(x,y) \neq (0,y)$). □

Remark 3.4.1. **(Approaches to find limiting distributions)**.

(i) Use CDF: $G_n(y) \to G(y)$

(ii) Use MGF: $\mathbb{M}_{Y_n} \to \mathbb{M}_Y$

(iii) Use combination of (i) (ii), and Theorem 3.3.3 and the continuous mapping theorem 3.4.2 (and their corollaries)

$\boxed{\textit{Example 3.4.1.}}$. Let $Z \sim \mathcal{N}(0,1)$, $V_n \sim \chi^2(n)$ (not necessarily independent!). Find the limiting distribution of $T_n = \frac{Z}{\sqrt{V_n/n}}$.

V_n has MGF

$$\mathbb{M}_{V_n}(t) = (1-2t)^{-n/2}$$

for $t < \frac{1}{2}$. Then

$$\mathbb{M}'_{V_n}(t) = n\,(1-2t)^{-n/2-1}, \quad \mathbb{E}V_n = \mathbb{M}'_{V_n}(0) = n$$

$$\mathbb{M}''_{V_n}(t) = n(n+2)\,(1-2t)^{-n/2-2}, \quad \mathbb{E}V_n^2 = \mathbb{M}''_{V_n}(0) = n(n+2)$$

So

$$\mathbb{V}ar V_n = n(n+2) - n^2 = 2n$$

Then $\forall \varepsilon > 0$, apply Chebychev's inequality

$$\mathbb{P}\{|V_n/n - 1| > \varepsilon\} \le \frac{1}{\varepsilon^2}\mathbb{V}ar\frac{V_n}{n} = \frac{1}{\varepsilon^2}\frac{2n}{n^2} = \frac{2}{n\varepsilon^2} \to 0$$

as $n \to \infty$. So $\frac{V_n}{n} \xrightarrow{P} 1$ as $n \to \infty$. Apply Corollary 3.3.1 (v) and (iv) (or Theorem 3.3.3)

$$\frac{1}{\sqrt{V_n/n}} \xrightarrow{P} 1 \quad \text{as } n \to \infty$$

Then apply corollary 3.4.1 (ii) of the continuous mapping theorem with $Y_n = Z$ and $X_n = \frac{1}{\sqrt{V_n/n}}$ in (ii)

$$T_n = \frac{Z}{\sqrt{V_n/n}} \xrightarrow{d} Z \sim \mathcal{N}(0,1) \quad \text{as } n \to \infty$$

$\boxed{\textit{Example 3.4.2.}}$. Let $X_1, X_2, \ldots$ be i.i.d. $\mathcal{U}nif(0, \theta)$ and $X_{n:n} = \max\{X_1, X_2, \ldots, X_n\}$, $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the limiting distribution of $W_n = \frac{n(\theta - X_{n:n})}{2\overline{X}_n}$.

Use the result in Theorem 3.3.2, $\overline{X}_n \xrightarrow{P} \mathbb{E}X_i = \frac{\theta}{2}$ as $n \to \infty$. Apply Corollary 3.3.1 (i), (iv), $\frac{1}{2\overline{X}_n} \xrightarrow{P} \frac{1}{\theta}$ as $n \to \infty$.

$$\mathbb{P}\{n(\theta - X_{n:n}) \leq y\} = \mathbb{P}\{X_{n:n} \geq \theta - y/n\}$$

$$= 1 - \mathbb{P}\{X_1 \leq \theta - y/n, X_2 \leq \theta - y/n, \ldots, X_n \leq \theta - y/n\}$$

$$\overset{i.i.d.}{=} 1 - \left(\frac{\theta - y/n}{\theta}\right)^n = 1 - \left(1 - \frac{y}{\theta n}\right)^n$$

Apply Lemma 3.1.1

$$1 - \left(1 - \frac{y}{\theta n}\right)^n \to 1 - e^{-y/\theta}$$

as $n \to \infty$, which is the CDF of an exponential variable. So $n(\theta - X_{n:n}) \xrightarrow{d} Y \sim \mathcal{E}xp(\theta)$. Then apply corollary 3.4.1 of the continuous mapping theorem,

$$W_n = \frac{n(\theta - X_{n:n})}{2\overline{X}_n} \xrightarrow{d} \frac{Y}{\theta}$$

as $n \to \infty$. But

$$\mathbb{P}\{\frac{Y}{\theta} \leq y\} = \mathbb{P}\{Y \leq \theta y\} = 1 - e^{-\theta y/\theta} = 1 - e^{-y}$$

Hence,

$$W_n = \frac{n(\theta - X_{n:n})}{2\overline{X}_n} \xrightarrow{d} \frac{Y}{\theta} \sim \mathcal{E}xp(1) \quad \text{as } n \to \infty$$

3.5 Delta Rule

$\boxed{\text{Theorem 3.5.1.}}$ **(Delta rule: for asymptotic normality).** If

$$\sqrt{n}\,(Y_n - m) \xrightarrow{d} Y \sim \mathcal{N}\left(0, c^2\right)$$

as $n \to \infty$ and if $g(y)$ is differentiable at $y = m$ with $g'(m) \neq 0$. Then

$$\frac{\sqrt{n}\left(g\left(Y_n\right) - g(m)\right)}{cg'(m)} \xrightarrow{d} Z \sim \mathcal{N}(0,1) \quad \text{as } n \to \infty$$

or

$$\sqrt{n}\left(g\left(Y_n\right) - g(m)\right) \xrightarrow{d} g'(m)Y \quad \text{where } Y \sim \mathcal{N}\left(0, c^2\right) \quad \text{as } n \to \infty$$

or

$$\sqrt{n}\left(g\left(Y_n\right) - g(m)\right) \xrightarrow{d} W \sim \mathcal{N}\left(0, \left(g'(m)c\right)^2\right) \quad \text{as } n \to \infty$$

or

$$g\left(Y_n\right) \overset{d}{\approx} V \sim \mathcal{N}\left(g(m), \frac{\left(g'(m)c\right)^2}{n}\right) \quad \text{for n very large}$$

Proof. Suppose $\sqrt{n}\left(Y_n - m\right) \xrightarrow{d} Y \sim \mathcal{N}\left(0, c^2\right)$, then by corollary 3.4.1 (ii) of the continuous mapping theorem

$$Y_n - m = \underbrace{\frac{1}{\sqrt{n}}}_{\xrightarrow{P} 0^{1}} \underbrace{\sqrt{n}\left(Y_n - m\right)}_{\xrightarrow{d} Y} \xrightarrow{d} 0$$

as $n \to \infty$. By Theorem 3.3.1

$$Y_n - m \xrightarrow{P} 0$$

as $n \to \infty$. Expand g in Taylor series for y near m

$$g(y) = g(m) + g'(m)(y - m) + R(y)$$

where $\frac{R(y)}{y-m} \to 0$ as $y \to m$. Put $y = Y_n$ and multiply by $\sqrt{n}$

$$\sqrt{n}\left(g\left(Y_n\right) - g\left(m\right)\right) = g'(m)\sqrt{n}(Y_n - m) + \sqrt{n}R(Y_n)$$

[1] Recall the definition of converge in probability $\forall \varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\{|\frac{1}{\sqrt{n}}| \leq \varepsilon\} = \mathbb{P}\{0 \leq \varepsilon\} = 1$

and by corollary 3.4.1

$$\sqrt{n}R(Y_n) = \underbrace{\sqrt{n}\,(Y_n - m)}_{\xrightarrow{d} Y} \underbrace{\frac{R\,(Y_n)}{(Y_n - m)}}_{\xrightarrow{P} 0^{1}} \xrightarrow{d} 0$$

as $n \to \infty$. By Theorem 3.3.1

$$\sqrt{n}R(Y_n) \xrightarrow{P} 0$$

as $n \to \infty$. So

$$\sqrt{n}\,(g\,(Y_n) - g\,(m)) = \underbrace{g'(m)\sqrt{n}(Y_n - m)}_{\xrightarrow{d} g'(m)Y} + \underbrace{\sqrt{n}R(Y_n)}_{\xrightarrow{P} 0}$$

and by corollary 3.4.1 again

$$\sqrt{n}\,(g\,(Y_n) - g\,(m)) \xrightarrow{d} g'(m)Y$$

as $n \to \infty$. □

$\boxed{\textit{Example 3.5.1.}}$. Let $X_1, X_2, \ldots$ be i.i.d. $\mathcal{P}ois(\mu)$ with PMF $\mathbb{P}\{X_i = k\} = \frac{\mu^k e^{-\mu}}{k!}$ for $k = 0, 1, 2, 3, \ldots$.

$$\mathbb{E}(X_i) = \mu \qquad \mathbb{V}ar(X_i) = \mu$$

By CLT

$$\frac{\sqrt{n}\,(\overline{X}_n - \mu)}{\sqrt{\mu}} \xrightarrow{d} Z \sim \mathcal{N}(0, 1)$$

as $n \to \infty$, i.e.

$$\sqrt{n}\,(\overline{X}_n - \mu) \xrightarrow{d} W \sim \mathcal{N}(0, \mu)$$

as $n \to \infty$. Consider $g(x) = 2\sqrt{x}$, $g'(x) = 1/\sqrt{x}$ and $g'(\mu) = 1/\sqrt{\mu}$. By the Delta rule

$$\sqrt{n}\,(g\,(\overline{X}_n) - g\,(\mu)) = \sqrt{n}\left(2\sqrt{\overline{X}_n} - 2\sqrt{\mu}\right) \xrightarrow{d} Z \sim \mathcal{N}\left(0, \mu\,(1/\sqrt{\mu})^2\right) = \mathcal{N}\,(0, 1)$$

as $n \to \infty$.

[1] Apply Theorem 3.3.3, since $Y_n \xrightarrow{P} m$, $\frac{R(Y_n)}{(Y_n - m)} \xrightarrow{P} \frac{R(m)}{(m-m)} = \lim_{y \to m} \frac{R(y)}{(y-m)} = 0$