6.4 Universal Pivot Functions

It may not always be possible to find a pivotal quantity based on MLE's, but for a sample from a continuous distribution with a single unknown parameter, at least one pivotal quantity can be derived by universal Pivot Functions.

Let $X_1, \dots, X_n$ is a random sample from PDF $f(x;\theta)$, CDF $F(x;\theta) = \mathbb{P}\{X_1 \le x\}$, $\theta \in \Theta$.

Lemma 6.4.1. **(Probability integral transformation).**

$$F(X_1;\theta), F(X_2;\theta), \dots, F(X_n;\theta)$$

are i.i.d. $\mathcal{U}nif(0,1)$.

Proof. Independent is clear.

$$\mathbb{P}\{F(X_1;\theta) \le x\} = \mathbb{P}\{X_1 \le F^{-1}(x)\} = F(F^{-1}(x)) = x$$

which is the CDF of $\mathcal{U}nif(0,1)$. □

Lemma 6.4.2.. If $U \sim \mathcal{U}nif(0,1)$ then $-2\ln U \sim \chi^2(2)$ and $-2\ln(1-U) \sim \chi^2(2)$.

Proof. If $U \sim \mathcal{U}nif(0,1)$ then $1-U \sim \mathcal{U}nif(0,1)$, so just need to verify $-2\ln U \sim \chi^2(2)$. Recall $\chi^2(2) \sim \mathcal{E}xp(2)$

$$\mathbb{P}\{-2\ln U \le t\} = \mathbb{P}\{\ln U \ge -t/2\} = \mathbb{P}\{U \ge e^{-t/2}\} = 1 - e^{-t/2}$$

which is the CDF of $\chi^2(2)$ or $\mathcal{E}xp(2)$. □

Lemma 6.4.3.. $-2\sum_{i=1}^n \ln F(X_i;\theta) \sim \chi^2(2n)$, $\theta \in \Theta$ and $-2\sum_{i=1}^n \ln(1-F(X_i;\theta)) \sim \chi^2(2n)$, $\theta \in \Theta$. These are pivotal quantities.

Proof. It follows from Lemmas 6.4.1, 6.4.2 and $\sum_{i=1}^n \chi^2(v_i) \sim \chi^2(\sum_{i=1}^n v_i)$ (theorem 2.3.1) for independent χ^2. □

Theorem 6.4.1. **(Universal pivot functions).** A $(1-\alpha)100\%$ confidence region for θ is given by

1. $\left\{\theta : \chi^2_{\alpha/2}(2n) < -2\sum_{i=1}^n \ln F(x_i;\theta) < \chi^2_{1-\alpha/2}(2n)\right\}$
2. $\left\{\theta : \chi^2_{\alpha/2}(2n) < -2\sum_{i=1}^n \ln\left(1 - F(x_i;\theta)\right) < \chi^2_{1-\alpha/2}(2n)\right\}$

Proof. By Lemma 6.4.3, we have

$$\mathbb{P}\left\{\chi^2_{\alpha/2}(2n) < -2\sum_{i=1}^n \ln F(X_i;\theta) < \chi^2_{1-\alpha/2}(2n)\right\} = 1-\alpha \quad \forall\theta \in Ⓗ$$

Similarly for 2. □

$\boxed{\textit{Example 6.4.1.}}$. $X_1, \ldots, X_n$ is a random sample from $\mathcal{W}ei(\theta, 2)$

$$f(x;\theta) = \frac{2}{\theta^2} x e^{-(x/\theta)^2}, \quad F(x;\theta) = 1 - e^{-(x/\theta)^2}, \quad x > 0$$

Use 2 of the theorem of universal pivot functions 6.4.1, a $(1-\alpha)100\%$ confidence region for θ is

$$\left\{\theta : \chi^2_{\alpha/2}(2n) < -2\sum_{i=1}^n \ln\left(1 - 1 + e^{-(x_i/\theta)^2}\right) < \chi^2_{1-\alpha/2}(2n)\right\}$$

$$= \left\{\theta : \chi^2_{\alpha/2}(2n) < \frac{2\sum_{i=1}^n x_i^2}{\theta^2} < \chi^2_{1-\alpha/2}(2n)\right\}$$

$$= \left\{\theta : \sqrt{\frac{2\sum_{i=1}^n x_i^2}{\chi^2_{1-\alpha/2}(2n)}} < \theta < \sqrt{\frac{2\sum_{i=1}^n x_i^2}{\chi^2_{\alpha/2}(2n)}}\right\}$$

for $x > 0$. So a $(1-\alpha)100\%$ confidence interval for θ is

$$\left(\sqrt{\frac{2\sum_{i=1}^n x_i^2}{\chi^2_{1-\alpha/2}(2n)}}, \sqrt{\frac{2\sum_{i=1}^n x_i^2}{\chi^2_{\alpha/2}(2n)}}\right)$$

6.5 General Method

When we can not find a pivotal quantity based on MLE's and there are one or more than one unknown parameters, the following methods are applied.

6.5.1 Part I: Universal Pivotal Quantity for Continuous Case

Step 1: Find a (minimal) sufficient statistic S for θ

Step 2: Find $\mathbb{P}\{S \leq s\} = G(s;\theta)$

Step 3: Use $G(S;\theta) \sim \mathcal{U}nif(0,1)$ as pivotal quantity

$$\mathbb{P}\{\alpha_1 < G(S;\theta) < 1-\alpha_2\} = 1-\alpha_1-\alpha_2 \quad \forall\theta$$

Step 4: The set $\{\theta : \alpha_1 < G(s;\theta) < 1-\alpha_2\}$ is a $(1-\alpha_1-\alpha_2)100\%$ confidence region for θ, if $S = s$ is observed.

Step 5: If applicable, use the following theorem to obtain confidence interval.

Theorem 6.5.1.. Suppose statistic S has continuous CDF $G(s;\theta)$ and for any fixed observation $S = s$, $G(s;\theta)$ is monotone and continuous in θ.

Case 1: $G(s;\theta)$ decreasing in θ

1. A one-sided $(1-\alpha_2)100\%$ lower confidence limit for θ is θ_L, solution to $G(s;\theta_L) = 1-\alpha_2$.
2. A one-sided $(1-\alpha_1)100\%$ upper confidence limit for θ is θ_U, solution to $G(s;\theta_U) = \alpha_1$.
3. A $(1-\alpha_1-\alpha_2)100\%$ confidence interval for θ is (θ_L, θ_U) defined by 1 and 2, $0 < \alpha_1+\alpha_2 < 1$.

Case 2: $G(s;\theta)$ increasing in θ. Reverse roles of θ_L, θ_U.

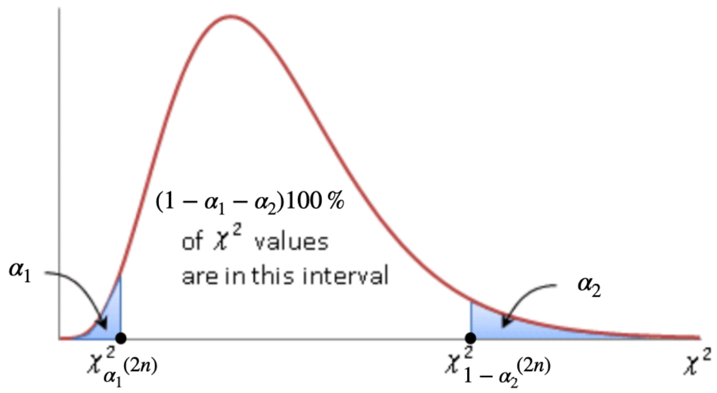
Example 6.5.1.. $X_1, \dots, X_n$ is a random sample from $\mathcal{P}ar(1,\kappa)$

$$f(x;\kappa) = \kappa(1+x)^{-(1+\kappa)}\mathbb{1}_{(0,\infty)}(x)$$

$$F(x;\kappa) = 1-(1+x)^{-\kappa}\mathbb{1}_{(0,\infty)}(x)$$

Use the REC theorem for easily identity complete sufficient statistics 5.4.1

$$f(x;\kappa) = \kappa(1+x)^{-(1+\kappa)}\mathbb{1}_{(0,\infty)}(x) = \kappa e^{-(\kappa+1)\ln(1+x)}\mathbb{1}_{(0,\infty)}(x)$$



So $t_1(x) = \ln(1+x)$ and $q_1(\kappa) = -(\kappa+1)$. By the REC theorem, $S = \sum_{i=1}^n t_1(X_i) = \sum_{i=1}^n \ln(1+X_i)$ is complete sufficient for κ.

Now what is the CDF of S? For $t > 0$

$$\mathbb{P}\{\ln(1+X_i) \leq t\} = \mathbb{P}\{X_i \leq e^t - 1\} = \mathbb{P}\{X_i \leq e^t - 1\} = 1 - e^{-\kappa t}$$

which is CDF of $\mathcal{E}xp(1/\kappa)$.

$$\mathbb{P}\{2\kappa \ln(1+X_i) \leq t\} = \mathbb{P}\{\ln(1+X_i) \leq t/(2\kappa)\} = 1 - e^{-\kappa t/(2\kappa)} = 1 - e^{-t/2}$$

which is the CDF of $\mathcal{E}xp(2)$ or $\mathcal{G}AM(2,1)$. Recall the summation of independent gamma variables is still a gamma (theorem 2.2.1)

$$\sum_{i=1}^n 2\kappa \ln(1+X_i) \sim \mathcal{G}AM(2,n) \sim \chi^2(2n) \implies S \sim \frac{\chi^2(2n)}{2\kappa}$$

So

$$G(s;\kappa) = \mathbb{P}\left\{S \leq s\right\} = \mathbb{P}\left\{\frac{\chi^2(2n)}{2\kappa} \leq s\right\} = \mathbb{P}\{\chi^2(2n) \leq 2\kappa s\}$$

which is increasing in κ for fixed s. So solve

$$\mathbb{P}\{\chi^2(2n) \leq 2\kappa_U s\} = 1 - \alpha_2 \implies 2\kappa_U s = \chi^2_{1-\alpha_2}(2n) \implies \kappa_U = \frac{\chi^2_{1-\alpha_2}(2n)}{2s}$$

Now solve

$$\mathbb{P}\{\chi^2(2n) \le 2\kappa_L s\} = \alpha_1 \implies 2\kappa_L s = \chi^2_{\alpha_1}(2n) \implies \kappa_L = \frac{\chi^2_{\alpha_1}(2n)}{2s}$$

So $(1-\alpha_1-\alpha_2)100\%$ confidence interval for κ is $\left(\frac{\chi^2_{\alpha_1}(2n)}{2\sum_{i=1}^n \ln(1+x_i)}, \frac{\chi^2_{1-\alpha_2}(2n)}{2\sum_{i=1}^n \ln(1+x_i)}\right)$. If let $\alpha_1 = \alpha_2 = \alpha/2$, the corresponding equal tailed confidence interval for κ is $\left(\frac{\chi^2_{\alpha/2}(2n)}{2\sum_{i=1}^n \ln(1+x_i)}, \frac{\chi^2_{1-\alpha/2}(2n)}{2\sum_{i=1}^n \ln(1+x_i)}\right)$.

6.5.2 Part II: Universal Pivotal Quantity for Discrete or Continuous Case

Suppose (minimal) sufficient statistic S is discrete, with CDF $G(s;\theta)$, but $G(s;\theta)$ is continuous function of θ if s is fixed. The probability integral transformation lemma is invalid now, i.e. $G(s;\theta) \not\sim \mathcal{U}nif(0,1)$ since it is discrete, so can not use $G(s;\theta)$ as pivotal. In general, it is not possible to find l, u such that $\mathbb{P}\{l(S) < \theta < u(S)\} = 1-\alpha \ \forall \theta \in \Theta$ due to discreteness. We need the notion of conservative confidence interval.

Definition 6.5.1. **(Conservative and exact confidence interval)**. $(l(S), u(S))$ is a conservative $(1-\alpha)100\%$ confidence interval for θ if $\mathbb{P}\{l(S) < \theta < u(S)\} \ge 1-\alpha \ \forall \theta \in \Theta$. In addition, if $\mathbb{P}\{l(S) < \theta < u(S)\} = 1-\alpha$ for some $\theta \in \Theta$, the confidence interval is called exact.

Theorem 6.5.2. **(Angus, 1990)**. Define $G(s;\theta) = \mathbb{P}\{S < s\}$, $G(s-;\theta) = \lim\limits_{\epsilon \to 0+} G(s-\epsilon;\theta)$. Then

$$\mathbb{P}\{G(s;\theta) \ge \alpha_1, G(s-;\theta) \le 1-\alpha_2\} \ge 1-\alpha_1-\alpha_2 \qquad \forall \theta \in \Theta$$

For details, see *Generalization of the "universal pivot function" and its use in constructing confidence intervals and sets. John E. Angus, Mathematics, Computer Science, SIAM Review, 1990.*

Theorem 6.5.3. **(Generalization of universal pivotal quantity)**. Suppose S, statistic, has CDF $G(s;\theta)$, and $G(s;\theta)$ is monotone and continuous in θ, s fixed, for each possible observation value s of S. Let $G(s-;\theta) = \lim\limits_{\epsilon \to 0+} G(s-\epsilon;\theta)$ and let θ_L, θ_U be solutions to $G(s-;\theta_L) = 1-\alpha_2$, $G(s;\theta_U) = \alpha_1$.

Case 1: $G(s;\theta)$ decreasing in θ, fixed s

- (θ_L, θ_U) is conservative $(1-\alpha_1-\alpha_2)100\%$ confidence interval for θ
- θ_U is conservative $(1-\alpha_1)100\%$ upper confidence limit for θ
- θ_L is conservative $(1-\alpha_2)100\%$ lower confidence limit for θ

Case 2: $G(s;\theta)$ increasing in θ, fixed s. Reverse roles of θ_L, θ_U.

Example 6.5.2.. $X_1, \dots, X_n$ is a random sample from $\mathcal{B}in(1,p)$. The minimal sufficient statistic is $S = \sum_{i=1}^n X_i \sim \mathcal{B}in(n,p)$ (HW3 5) a)). So $G(s;p) = \sum_{i=0}^s \binom{n}{i} p^i (1-p)^{n-i}$ and is decreasing in p, s fixed.

To verify $f(p) = \sum_{i=0}^s \binom{n}{i} p^i (1-p)^{n-i}$ is decreasing

$$\frac{\partial p^i (1-p)^{n-i}}{\partial p} = ip^{i-1}(1-p)^{n-i} - (n-i)p^i(1-p)^{n-i-1}$$

$$= \frac{i}{p} p^i (1-p)^{n-i} - \frac{n-i}{1-p} p^i (1-p)^{n-i}$$

$$= \frac{i}{p(1-p)} p^i (1-p)^{n-i} - \frac{n}{1-p} p^i (1-p)^{n-i}$$

Recall for binomial distribution, $np = E(S)$

$$\frac{\partial f(p)}{\partial p} = \frac{1}{p(1-p)} \sum_{i=0}^s \binom{n}{i} ip^i (1-p)^{n-i} - \frac{n}{1-p} \sum_{i=0}^s \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \frac{f(p)}{p(1-p)} \frac{\sum_{i=0}^s \binom{n}{i} ip^i (1-p)^{n-i}}{f(p)} - \frac{f(p)np}{p(1-p)}$$

$$= \frac{f(p)}{p(1-p)} \left(\frac{\sum_{i=0}^s i\mathbb{P}\{S=i\}}{\mathbb{P}\{S \le s\}} - \mathbb{E}S \right)$$

$$= \frac{f(p)}{p(1-p)} \left(\mathbb{E}S^R - \mathbb{E}S \right) \le 0$$

where S^R is the right truncated random variable[1] of S. So $f(p) = \sum_{i=0}^s \binom{n}{i} p^i (1-p)^{n-i}$ is decreasing.

[1]Let X be a random variable with PDF and the right truncated random variable of X be X^R with right truncation μ. The PDF of X^R is $f_{X^R}(x) = \frac{f(x)}{\mathbb{P}\{X \le \mu\}}$. Then $\mathbb{E}X^R \le \mathbb{E}X$. To verify it,

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)\,dx = \mu + \int_{-\infty}^{\mu} (x-\mu)f(x)\,dx + \int_{\mu}^{\infty} (x-\mu)f(x)\,dx$$

Now by the generalization of universal pivotal quantity theorem, for $0 < \epsilon < 1$

$$G(s-;p_L) = 1-\alpha_2 \implies \lim_{\epsilon\to 0}\sum_{i=0}^{\lfloor s-\epsilon\rfloor}{}^{1}\binom{n}{i}p_L^i(1-p_L)^{n-i} = 1-\alpha_2$$

$$\implies \sum_{i=0}^{s-1}\binom{n}{i}p_L^i(1-p_L)^{n-i} = 1-\alpha_2 \implies \sum_{i=s}^{n}\binom{n}{i}p_L^i(1-p_L)^{n-i} = \alpha_2$$

and

$$G(s;p_U) = \alpha_1 \implies \sum_{i=0}^{s}\binom{n}{i}p_U^i(1-p_U)^{n-i} = \alpha_1$$

In your HW 8 1) c), you will see how to find the value of p_L, p_U and (p_L, p_U) is a conservative $(1-\alpha)100\%$ confidence interval for p.

$\boxed{\textit{Example 6.5.3.}}$. $X_1, \ldots, X_n$ is a random sample from $\mathcal{P}oi(\mu)$. The minimal sufficient statistic is $S = \sum_{i=1}^n X_i \sim \mathcal{P}oi(n\mu)$. So $G(s;\mu) = \sum_{i=0}^{s}\frac{(n\mu)^i e^{-n\mu}}{i!}$ and is decreasing in μ, s fixed.

To verify $G(s;\mu)$ is decreasing in θ, recall for Poisson distribution $\mathbb{E}S = n\mu$ and then one can verify that $f(\mu) = \sum_{i=0}^{s}\frac{(n\mu)^i e^{-n\mu}}{i!}$ is decreasing by

$$\frac{\partial f(\mu)}{\partial\mu} = \sum_{i=0}^{s}\frac{n^i i\mu^{i-1}e^{-n\mu}}{i!} - \sum_{i=0}^{s}\frac{n(n\mu)^i e^{-n\mu}}{i!} = \frac{f(\mu)}{\mu}\frac{\sum_{i=0}^{s} i\frac{n^i\mu^i e^{-n\mu}}{i!}}{f(\mu)} - \frac{f(\mu)}{\mu}n\mu$$

$$= \frac{f(\mu)}{\mu}\left(\mathbb{E}S^R - \mathbb{E}S\right) \leq 0$$

$$= \mu + \frac{\int_{-\infty}^{\mu}(x-\mu)f(x)\,dx}{\mathbb{P}\{X\leq\mu\}}\mathbb{P}\{X\leq\mu\} + \underbrace{\frac{\int_{\mu}^{\infty}(x-\mu)f(x)\,dx}{\mathbb{P}\{X\geq\mu\}}}_{\geq\int_{\mu}^{\infty}(x-\mu)f(x)\,dx\geq\int_{-\infty}^{\infty}(x-\mu)f(x)\,dx}\mathbb{P}\{X\geq\mu\}$$

$$\geq \mu + \mathbb{E}(X^R-\mu)\mathbb{P}\{X\leq\mu\} + \mathbb{E}(X-\mu)\mathbb{P}\{X\geq\mu\} = \mathbb{E}X^R\mathbb{P}\{X\leq\mu\} + \mathbb{E}X\mathbb{P}\{X\geq\mu\}$$

$$\implies \mathbb{E}X \geq \mathbb{E}X^R$$

Similarly result also holds for the discrete case.

[1] $\lfloor x\rfloor$= greatest integer less than or equal to x

So by the generalization of universal pivotal quantity theorem, for $0 < \epsilon < 1$

$$G(s-;\mu_L) = 1-\alpha_2 \implies \lim_{\epsilon\to 0}\sum_{i=0}^{\lfloor s-\epsilon\rfloor}\frac{(n\mu_L)^i e^{-n\mu_L}}{i!} = 1-\alpha_2$$

$$\implies \sum_{i=0}^{s-1}\frac{(n\mu_L)^i e^{-n\mu_L}}{i!} = 1-\alpha_2 \implies \sum_{i=s}^{\infty}\frac{(n\mu_L)^i e^{-n\mu_L}}{i!} = \alpha_2$$

and

$$G(s;\mu_U) = \alpha_1 \implies \sum_{i=0}^{s}\frac{(n\mu_U)^i e^{-n\mu_U}}{i!} = \alpha_1$$

We can simplify this using $\mathbb{P}\{\mathcal{P}oi(\mu)\le s\} = \mathbb{P}\{\chi^2(2s+2)\ge 2\mu\}$[1]

$$\alpha_1 = \mathbb{P}\{\mathcal{P}oi(n\mu_U)\le s\} = \mathbb{P}\{\chi^2(2s+2)\ge 2n\mu_U\}$$

$$\implies 2n\mu_U = \chi^2_{1-\alpha_1}(2s+2) \implies \mu_U = \chi^2_{1-\alpha_1}(2s+2)/2n$$

and

$$1-\alpha_2 = \mathbb{P}\{\mathcal{P}oi(n\mu_L)\le s-1\} = \mathbb{P}\{\chi^2(2s)\ge 2n\mu_L\}$$

[1]By change-of-variables and s is integer

$$\int_{2\mu}^{\infty}\frac{1}{2^{s+1}\Gamma(s+1)}y^s e^{-\frac{y}{2}}\,dy = \frac{1}{s!}\int_{\mu}^{\infty}x^s e^{-x}\,dx$$

By integral-by-parts

$$\frac{1}{s!}\int_{\mu}^{\infty}x^s e^{-x}\,dx = \frac{1}{s!}\left(\mu^s e^{-\mu} + s\int_{\mu}^{\infty}x^{s-1}e^{-x}\,dx\right) = \frac{1}{s!}\mu^s e^{-\mu} + \frac{1}{(s-1)!}\int_{\mu}^{\infty}x^{s-1}e^{-x}\,dx$$

$$= \frac{1}{s!}\mu^s e^{-\mu} + \frac{1}{(s-1)!}\left(\mu^{s-1}e^{-\mu} + (s-1)\int_{\mu}^{\infty}x^{s-2}e^{-x}\,dx\right)$$

$$= \frac{1}{s!}\mu^s e^{-\mu} + \frac{1}{(s-1)!}\mu^{s-1}e^{-\mu} + \frac{1}{(s-2)!}\int_{\mu}^{\infty}x^{s-2}e^{-x}\,dx = \cdots$$

$$= \frac{1}{s!}\mu^s e^{-\mu} + \frac{1}{(s-1)!}\mu^{s-1}e^{-\mu} + \cdots + \underbrace{\int_{\mu}^{\infty}e^{-x}\,dx}_{=e^{-\mu}} = \sum_{i=0}^{s}\frac{\mu^i e^{-\mu}}{i!}$$

$$\implies 2n\mu_L = \chi^2_{\alpha_2}(2s) \implies \mu_L = \chi^2_{\alpha_2}(2s)/2n$$

So conservative $100(1-\alpha_1-\alpha_2)\%$ confidence interval for μ is

$$\left(\frac{\chi^2_{\alpha_2}(2s)}{2n}, \frac{\chi^2_{1-\alpha_1}(2s+2)}{2n}\right)$$

where $s = \sum_{i=1}^n x_i$.

6.6 Approximate Confidence Intervals

Suppose $\hat{\theta}_n$ is an estimator/estimate (MLE, UMVUE, MM) of θ such that

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2(\theta)) \quad \text{as } n \to \infty$$

$\hat{\theta}_n \xrightarrow{P} \theta$ as $n \to \infty$, and $\sigma^2(\theta)$ is continuous. Then

$$\sqrt{n}\frac{(\hat{\theta}_n - \theta)}{\sigma(\hat{\theta}_n)} = \underbrace{\sqrt{n}\frac{(\hat{\theta}_n - \theta)}{\sigma(\theta)}}_{\xrightarrow{d}\mathcal{N}(0,1)} \underbrace{\frac{\sigma(\theta)}{\sigma(\hat{\theta}_n)}}_{\xrightarrow{P}1} \xrightarrow{d} \mathcal{N}(0,1)$$

So for large n

$$\mathbb{P}\left\{z_{\alpha/2} \le \sqrt{n}\frac{(\hat{\theta}_n - \theta)}{\sigma(\hat{\theta}_n)} \le z_{1-\alpha/2}\right\} \approx \Phi(z_{1-\alpha/2}) - \Phi(z_{\alpha/2}) = 1-\alpha \qquad \forall\theta$$

so $\sqrt{n}\frac{(\hat{\theta}_n-\theta)}{\sigma(\hat{\theta}_n)}$ is approximate pivotal and approximate (large n) $(1-\alpha)100\%$ confidence interval for θ is $\left(\hat{\theta}_n + z_{\alpha/2}\frac{\sigma(\hat{\theta}_n)}{\sqrt{n}}, \hat{\theta}_n + z_{1-\alpha/2}\frac{\sigma(\hat{\theta}_n)}{\sqrt{n}}\right)$.

Sometimes it is better to let $g(\theta) = \int \frac{1}{\sigma(\theta)}\, d\theta$. Then for large n

$$\sqrt{n}\left(g\left(\hat{\theta}_n\right) - g(\theta)\right) \overset{d}{\approx} \mathcal{N}(0,1)$$

Hence,

$$\mathbb{P}\left\{z_{\alpha/2} \le \sqrt{n}\left(g\left(\hat{\theta}_n\right) - g(\theta)\right) \le z_{1-\alpha/2}\right\} \approx 1-\alpha$$

So for large n, $\left\{\theta : g\left(\hat{\theta}_n\right) + z_{\alpha/2}/\sqrt{n} < g(\theta) < g\left(\hat{\theta}_n\right) + z_{1-\alpha/2}/\sqrt{n}\right\}$ gives approximate $(1-\alpha)100\%$ confidence interval for $g(\theta)$. If g is monotone, we get interval for θ.

$\boxed{\textit{Example 6.6.1.}}$. $X_1, \ldots, X_n$ is a random sample from $\mathcal{P}oi(\mu)$. $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the MLE. By central limit theory

$$\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} \mathcal{N}(0, \mu) \quad \text{as } n \to \infty$$

$g(\mu) = \int \frac{d\mu}{\sqrt{\mu}} = 2\sqrt{\mu}$. So, by variance stabilization transformation

$$\sqrt{n}(2\sqrt{\overline{X}} - 2\sqrt{\mu}) \overset{d}{\approx} \mathcal{N}(0,1)$$

So if $\overline{x}$ is an observation of $\overline{X}$

$$\left\{\mu : 2\sqrt{\overline{x}} + z_{\alpha/2}/\sqrt{n} < 2\sqrt{\mu} < 2\sqrt{\overline{x}} + z_{1-\alpha/2}/\sqrt{n}\right\}$$

gives an approximate $(1-\alpha)100\%$ confidence interval for $2\sqrt{\mu}$. So

$$\frac{\left(2\sqrt{\overline{x}} + z_{\alpha/2}/\sqrt{n}\right)^2}{4} < \mu < \frac{\left(2\sqrt{\overline{x}} + z_{1-\alpha/2}/\sqrt{n}\right)^2}{4}$$

$$\left(\sqrt{\overline{x}} + z_{\alpha/2}/(2\sqrt{n})\right)^2 < \mu < \left(\sqrt{\overline{x}} + z_{1-\alpha/2}/(2\sqrt{n})\right)^2$$

$$\Longrightarrow \left(\left(\sqrt{\overline{x}} + z_{\alpha/2}/(2\sqrt{n})\right)^2, \left(\sqrt{\overline{x}} + z_{1-\alpha/2}/(2\sqrt{n})\right)^2\right)$$

is approximate $(1-\alpha)100\%$ confidence interval for μ.

$\boxed{\textit{Example 6.6.2.}}$. $X_1, \ldots, X_n$ is a random sample from $\mathcal{B}in(1, p)$. $\overline{X}$ is the MLE of p. By central limit theory

$$\sqrt{n}(\overline{X} - p) \xrightarrow{d} \mathcal{N}(0, p(1-p)) \quad \text{as } n \to \infty$$

For binomial distribution, $\sigma^2(p) = p(1-p)$, then $g(p) = \int \frac{dp}{\sqrt{p(1-p)}} = \int \frac{d(2p-1)}{\sqrt{1-(2p-1)^2}} \overset{1}{=} \arcsin(2p-1)$. So

$$\sqrt{n}\left(\arcsin(2\overline{X} - 1) - \arcsin(2p-1)\right) \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \to \infty$$

[1] $\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C$

So approximate $(1-\alpha)100\%$ confidence interval for $\arcsin(2p-1)$ is

$$\arcsin(2\overline{x}-1) + z_{\alpha/2}/\sqrt{n} < \arcsin(2p-1) < \arcsin(2\overline{x}-1) + z_{1-\alpha/2}/\sqrt{n}$$

So approximate $(1-\alpha)100\%$ confidence interval for p is

$$\left(\frac{1+\sin\left(\arcsin(2\overline{x}-1) + z_{\alpha/2}/\sqrt{n}\right)}{2}, \frac{1+\sin\left(\arcsin(2\overline{x}-1) + z_{1-\alpha/2}/\sqrt{n}\right)}{2}\right)$$