# CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2019

Handed out on Monday, 2019-10-21
First submission and questions due on Monday, 2019-10-28
You will receive feedback from the TA.
Final submission due on Monday, 2019-11-04

## 5 More on Network Flows

**Exercise 1.** Let $G = (V, c)$ be a flow network. Prove that flow is "transitive" in the following sense: if $r, s, t$ are vertices, and there is an $r$–$s$-flow of value $k$ and an $s$–$t$-flow of value $k$, then there is an $r$–$t$-flow of value $k$.

### 5.1 Vertex Disjoint Paths

Let $G$ be a directed graph. Two paths $p_1, p_2$ from $s$ to $t$ are called *vertex disjoint* if they don't share any vertices except $s$ and $t$.

**Theorem 2** (Menger's Theorem)**.** *Let $G$ be a graph and $s \neq t$ two vertices therein. Let $k \in \mathbf{N}_0$. Then exactly one of the following is true:*

1. *There are $k$ vertex disjoint paths $p_1, \ldots, p_k$ from $s$ to $t$; that is, no two $p_i$, $p_j$ share any vertex besides $s$ and $t$.*
2. *There are vertices $v_1, \ldots, v_k \in V \setminus \{s, t\}$ such that $G - \{v_1, \ldots, v_k\}$ contains no $s$–$t$-path.*

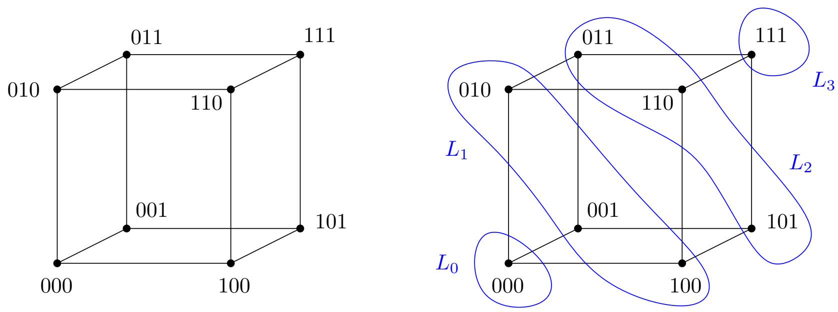
**Exercise 3.** Prove Menger's Theorem. You have to prove two things: first, not both cases above can occur (this is rather easy); second, one of them must occur (this requires a tool from the lecture).

*Proof.* $k$ is the value of the flow? □

Let $V = \{0,1\}^n$. The $n$-dimensional Hamming cube $H_n$ is the graph $(V, E)$ where $\{u, v\} \in E$ if $u, v$ differ in exactly one coordinate. Define the $i^{\text{th}}$ level of $H_n$ as

$$L_i := \{u \in V \mid \|u\|_1 = i\} ,$$

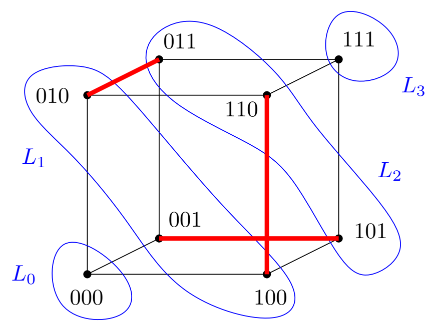
i.e., those vertices $u$ having exactly $i$ coordinates which are 1.




The 3-dimensional Hamming cube and the four sets $L_0$, $L_1$, $L_2$, $L_3$.

**Exercise 4.** [Matchings in $H_n$] Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$. This is the graph on vertex set $L_i \cup L_{i+1}$ where two edges are connected by an edge if and only if they are connected in $H_n$.

Show that for $i \leq n/2$ the graph $H_n[L_i \cup L_{i+1}]$ has a matching of size $|L_i| = \binom{n}{i}$.




A matching of size 3 between $L_1$ and $L_2$.

*Proof.* Consider $v \in L_i$, we know there are $i$ bits to be 1. Let $a_1, a_2, \ldots, a_i$ be that set of integers or indexes such that $\forall j \leq i, v_{a_j} = 1$. That is, we use the indexes in which $v$ has value 1 to represent the vertex.

Our goal is to construct a mapping $g : \{\mathbb{N} \times \ldots \times \mathbb{N}\} \to \mathbb{N}$, such that $\{v_1, \ldots, v_i, g(v)\} = \{\mathbf{v}, g(v)\} \in L_{i+1}$ , plus $\forall u \neq v \in L_i$, we have

$$\{u, g(u)\} \neq \{v, g(v)\}.$$

Let $n = \prod_{i=1}^{n} p_i^{\alpha_i}$, consider the smallest prime $p \notin \{p_1, \ldots, p_n\}$, we construct the mapping $g$ as following:

$$g(u) = (p-1)\left(\sum_{j=1}^{i} u_j\right) \bmod n.$$

We assume $u$ maps to $n$ if $g(u) = 0$. Now we prove this mapping is valid. Otherwise, there exists $u, v$ they have at least 1 different elements, i.e.

$$i - |u \cap v| \geq 1.$$

Plus $\{u, g(u)\} = \{v, g(v)\}$. We must have

$$|u \cap v| = i - 2.$$

Assume $u = \{a_1, \ldots, a_{i-1}, g(v)\}, v = \{a_1, \ldots a_{i-1}, g(u)\}$. Hence

$$g(u) = (p-1)\left(\sum_{j=1}^{i-1} a_i + g(v)\right) \bmod n$$
$$g(v) = (p-1)\left(\sum_{j=1}^{i-1} a_i + g(u)\right) \bmod n.$$

Which leads that

$$p(g(u) - g(v)) = 0 \pmod{n}.$$

Hence

$$u = \{a_1, \ldots, a_{i-1}, g(v)\} = \{a_1, \ldots, a_{i-1}, g(u)\} = v.$$

It contradicts with the assumption that $u \neq v$, thus the mapping $g$ is a valid one. And $H_n[L_i \cap L_{i+1}]$ has a matching! $\square$

**Exercise 5.** Let $H_n$ be the $n$-dimensional Hamming cube. For $i < n/2$ consider $L_i$ and $L_{n-i}$. Note that $|L_i| = \binom{n}{i} = \binom{n}{n-i} = L_{n-i}$, so the $L_i$ and $L_{n-i}$ have the same size. Show that there are $\binom{n}{i}$ paths $p_1, p_2, \ldots, p_{\binom{n}{i}}$ in $H_n$

such that (i) each $p_i$ starts in $L_i$ and ends in $L_{n-i}$; (ii) two different paths $p_i, p_j$ do not share any vertices. **Hint 1.** Model this problem as a network flow with vertex capacities. What would the maximum flow be in this network? **Hint 2.** It's not *that* easy. If you try to work from both sides towards the middle by combining matchings between levels, you will certainly run into problems as how to glue things together in the middle. I have never seen any "meet in the middle" proof that works. **Hint 3.** There is a "direct" proof by induction that does not require anything about network flows.

*Proof.* It is well known that $\binom{i}{n} \leq \binom{i+1}{n} \leq \ldots \leq \binom{\frac{n}{2}}{n}$. We construct a graph with $s, t, L_i, \ldots, L_{n-i}$. While $s$ connects every vertex in $L_i$, $t$ connects every vertex in $L_{n-i}$.

And each edge connects $L_k, L_{k+1}, i \leq k \leq n-i$ has capacity 1. It's clear that the maxflow is less than $|L_i|$.

Plus if maxflow equals $|L_i|$ we are done. □

## 5.2 Matchings and Vertex Covers

The following exercise was on the final exam of CS 499 (mathematical foundations of computer science) in spring 2019.

**Exercise 6.** Let $\nu(G)$ denote the size of a maximum matching of $G$. Show that a bipartite graph $G$ has at most $2^{\nu(G)}$ minimum vertex covers.

*Proof.* Suppose the number of minumum vertex covers of grapg $G$ is $m(G)$. We can easily Now we prove that there could be no more than $2^{\nu(G)}$ minimum

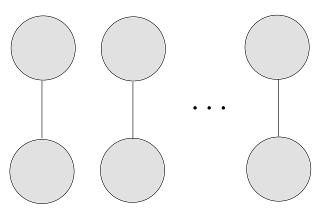


Figure 1: example

vertex cover. Let $G = A \cup B$, since $G$ is a bipartite graph. Assume that the maximum matching is $e_1, e_2, \ldots, e_t$, in which $t = \nu(G)$. By **Konig's**

**Theorem** we know that the number of vertices in a minimum vertex cover is exactly $t$.

Plus we can prove that every vertex in $K$, the vertex cover, is an endpoint of a matched edge. Hence the result is obvious. $\square$

Obviously, this is not true for general (non-bipartite) graphs: the triangle $K_3$ has $\nu(K_3) = 1$ but it has three minimum vertex covers. The five-cycle $C_5$ has $\nu(C_5) = 2$ but has five minimum vertex covers.

**Exercise 7.** Is there a function $f : \mathbf{N}_0 \to \mathbf{N}_0$ such that every graph with $\nu(G) = k$ has at most $f(k)$ minimum vertex covers? How small a function $f$ can you obtain?

*Proof.* Since this is for every graph. First consider $K_{2r+1}$, we have $\frac{k}{r}$ $K_{2r+1}$. Hence $\nu\left(G\right) = \frac{k}{r} \cdot r = k$

$$f\left(k\right) \geq \left(2r+1\right)^{\frac{k}{r}}.$$

For example, if we choose $r = 1$, there are $k$ $K_3$, we have $f\left(k\right) \geq 3^k$. And let $r \to \infty$ we have

$$f\left(k\right) \geq e^{2k}.$$

Next we prove that the number of minimum vertex cover is at most $e^{2k}$. Consider $\forall$ minimum vertex cover $K$. We prove

$$|K| \leq e^{2k}.$$

Let $\forall u, v \in K$. If $u, v$ are connected, then at least one of them is on a edge of maximum matching. $\square$