CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2020

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First submission and questions due on Thursday, 2020-06-12
You will receive feedback from the TA.
Final submission due on Thursday, 2020-06-19

7 Farkas Lemma and LP Duality

7.1 Different Versions of Farkas Lemma

In the following, let $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, and let $\mathbf{x} = (x_1, \dots, x_n)^T$ be a column vector of n variables and $\mathbf{y} = (y_1, \dots, y_m)$ be a row vector of m variables.

Exercise 1. Show that the three versions of Farkas Lemma presented in class are all equivalent (I actually did not present the third version in class):

$$(\neg\exists \mathbf{x} : A\mathbf{x} \le \mathbf{b}) \iff (\exists \mathbf{y} \ge \mathbf{0} : \mathbf{y}^T A = \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) . \tag{1}$$

$$(\neg\exists \mathbf{x} \ge \mathbf{0} : A\mathbf{x} \le \mathbf{b}) \iff (\exists \mathbf{y} \ge \mathbf{0} : \mathbf{y}^T A \ge \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) . \tag{2}$$

$$(\neg\exists \mathbf{x} \ge \mathbf{0} : A\mathbf{x} = \mathbf{b}) \iff (\exists \mathbf{y} \quad : \mathbf{y}^T A \ge \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) . \tag{3}$$

Note that the direction "$\Longleftarrow$" is easy in each case. We will show the "$\Longrightarrow$" of (1) in class using a technique called *Fourier-Motzkin Elimination*. This exercise is actually not that hard. The hardest part is keeping track of what you want to prove and what you can assume.

7.2 A Linear Program for, well, for what?

Let $G = (V, E)$ be a directed graph, $s, t \in V$, and $c : E \to \mathbf{R}^+$ be a cost function. We want to find an $s-t$-flow f of value 1. Every edge e generates cost $f(e) \cdot c(e)$, and we want to minimize the overall cost. There are no capacity constraints. We can easily write this as a linear program MCF (Minimum Cost Flow):

$$
\text{MCF}(G, s, t, c) : \quad
\begin{array}{ll}
\text{minimize} & \sum_{e \in E} c(e) f(e) \\
\\
\text{subject to} & \sum_{v \in V} f(v, t) = 1 \\
& \sum_{u \in V} f(u, v) - \sum_{w \in V} f(v, w) = 0 \quad \forall\, v \in V \setminus \{s, t\} \\
\\
& f(e) \geq 0 \ \forall\ e \in E
\end{array}
$$

Note that we have m variables, one variable $f(e)$ for each edge e. The first constraint says that the value of the flow should be 1. The other constraints say that the inflow at v should equal the outflow.

Exercise 2. Let d be the shortest path distance from s to t in the directed graph G, where distance means sum of the $c(e)$ along the path. Show that $\text{opt}(MCF) = d$. **Hint.** Make sure you show both $\leq$ and $\geq$.

Exercise 3. Write down the dual of MCF. This will be a maximization problem. Don't use any matrix notation.

Exercise 4. Interpret the dual. Show that it is the LP formulation of a "natural" maximization problem on G.

Exercise 5. Describe an optimal solution of the dual program.