CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2019

Handed out on Thursday, 2019-09-26
First submission and questions due on Monday, 2019-09-30
You will receive feedback from the TA.
Final submission due on Thursday, 2019-10-10

3 Minimum Spanning Trees

Throughout this assignment, let G be a weighted graph, i.e., $G = (V, E, w)$ with $w : E \to \mathbb{R}^+$. For $c \in \mathbb{R}$ and a weighted graph $G = (V, E, w)$, let $G_c := (V, \{e \in E \mid w(e) \leq c\})$. That is, G_c is the subgraph of G consisting of all edges of weight at most c.

Exercise 1 Let T be a minimum spanning tree of G, and let $c \in \mathbb{R}$. Show that T_c and G_c have exactly the same connected components. (That is, two vertices $u, v \in V$ are connected in T_c if and only if they are connected in G_c). You are encouraged to draw pictures to illustrate your proof!

Proof: In other words, we need to prove the following statements are equivalent , given $v_1, v_2 \in V$

$$\exists \text{ path } e_1, \ldots, e_n \in E, \text{ the path connects } v_1, v_2, \text{and } w(e_i) \leq c \quad (*).$$

and

$$\exists \text{ path } e_1, \ldots, e_n \in E(T), \text{ the path connects } v_1, v_2, \text{and } w(e_i) \leq c \quad (**).$$

Notice that $(**) \implies (*)$ is obvious. We only need to prove $(*) \implies (**)$. The path connects v_1, v_2 in T is unique, otherwise there are cycles.

Let the unique path connects $v_1 = u_1, \ldots, u_t = v_2$. And assume $\exists j, 1 \leq j < t$ such that

$$w\left(e(u_j, u_{j+1})\right) > c.$$

we remove the edge $e\left(u_j, u_{j+1}\right)$ from T, and get 2 sets of vertices A, B. A, B are connected respectively. We prove that $\exists e \in E$ that connects A, B plus $w\left(e\right) \leq c$. This is obvious from $(*)$. Hence it leads that T is not a MST, a contradiction. $\square$

Exercise 2 For a weighted graph G, let $m_c(G) := |\{e \in E(G) \mid w(e) \leq c\}|$, i.e., the number of edges of weight at most c (so G_c has $m_c(G)$ edges). Let T, T' be two minimum spanning trees of G. Show that $m_c(T) = m_c(T')$.

Proof: Let the edge set of T_c be

$$E\left(T_c\right) = \{e_1, e_2, \ldots, e_r\}.$$

We know that $E\left(T_c\right)$ forms several connected componets $A_1, A_2, \ldots, A_t \subset V$. And from the last exercise we know that $A_1, A_2, \ldots, A_t$ are also connected componets in T'. And we can assert the connected components in T' are exactly these A_i, otherwise apply the conclusion from last exercise, we can derive more components for T. And specifically, those components must be trees. Hence

$$m_c\left(T\right) = \sum_{i=1}^{t} \left(|A_i| - 1\right) = m_c\left(T'\right).$$

$\square$

Exercise 3 Suppose G is connected, and no two edges of G have the same weight. Show that G has exactly one minimum spanning tree!

Proof: Otherwise consider 2 minimum spanning tree T_1, T_2, let

$$E\left(T_1\right) = \{a_1, a_2, \ldots, a_r\}.$$

in which $a_i < a_{i+1}$, and similarly

$$E\left(T_2\right) = \{b_1, b_2, \ldots, b_s\}.$$

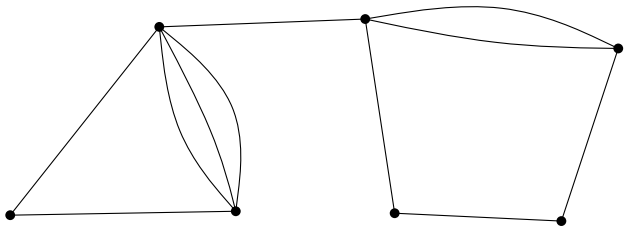
Let j be the minimum such that $a_j \neq b_j$. We can assert j exists since T_1, T_2 are different.

W.L.O.G let $a_j < b_j$, it leads that

$$m_{a_j}(T_1) = j \neq j - 1 = m_{a_j}(T_2).$$

Which contradicts with **Exercise 2**. □

A *multigraph* is a graph that can have multiple edges, called "parallel edges". Without defining it formally, we illustrate it:



A multigraph.

All other definitions, like connected components and spanning trees are the same as for normal (simple) graphs. However, when two spanning trees use different parallel edges, we consider them different:

The same multigraph with two different spanning trees.

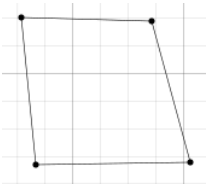
Exercise 4 How many spanning trees does the above multigraph on 7 vertices have? Justify your answer!

Proof: Obviously the edge connecting the left part of the graph and the right part must be included in a spanning tree. To construct a spanning tree, we now pay attention on these two subgraphs.

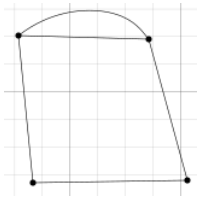
We first consider such a simple graph with 4 edges. Without any effort, it has 4 spanning trees, and this result can be applied to a graph with n edges.

Then we consider a simple multi-graph case. If we remove one of the parallel edges in the multi-graph, it's exactly a simple graph discussed above. So obviously, we know it has $4 \times 2 = 8$ spanning trees.

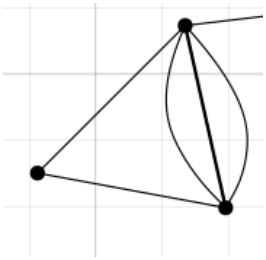
Based on these observations, we consider the subgraphs in *Exer.*4. Left part has 3 parallel edges, which leads to $3 \times 3 = 9$ spanning trees, and right



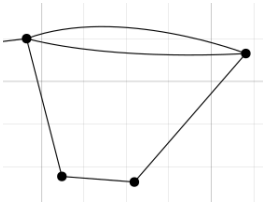
(a) simple graph



(b) simple multi graph

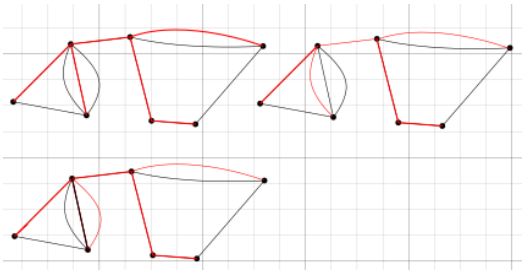


(a) left part

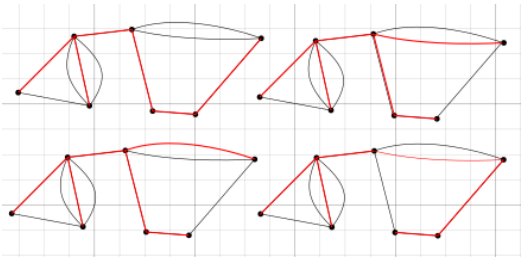


(b) right part

part has $2 \times 4 = 8$ spanning trees. Due to the simple counting method, there're $9 \times 8 = 72$ spanning trees for the graph.Below is some of them. $\square$



(a) left part examples



(b) right part examples

Exercise 5 Suppose you have a polynomial-time algorithm that, given a multigraph H, computes the number of spanning trees of H. Using this algorithm as a subroutine, design a polynomial-time algorithm that, given a weighted graph G, computes the number of minimum spanning trees of G.

Proof: We already know $Prim's$ and $Kruskal's$ algorithm to construct one of the MSTs of a graph, and only those edges having the same weight may lead to different spanning trees and we have a magic polynomial-time

algorithm for cumputing the number of spanning trees. Based on these tools, we just need to find do some midifications for $Kruskal's$ algorithm.

We know that during the process of $Kruskal's$ algorithm, it gets the edges with different weights of an MST. Having these weights, we can get all other *candidate* edges for another MST as long as its weight is the same as one of the known MST's edge's. We can solve all edges with the same weight together. Every time, for a paticular weight, construct an abstract graph G^* with the represented point and the edge connected different connponent in that weight. Then we use that polynomial-time magic algorithm(since i know it's according to Matrix-Tree Theorem) to calculate the answer. And then merge different conponent. Finally we multiplicate the answer for every type of edges.

Data: graph G
Result: number of MSTs of G
sort(E);
$Ans = 1$;
for $e \in E$ **do**
  $EdgeLayer = \{(Find(e.a), Find(e.b))\}$;
  $PointLayer = \{Find(e.a), Find(e.b)\}$;
  $nowe = e.weight$;
  $e = next(e)$;
  while $e.weight == nowe$ **do**
    $EdgeLayer = EdgeLayer \cup (Find(e.a), Find(e.b))$;
    $PointLayer = PointLayer \cup \{Find(e.a), Find(e.b)\}$;
    $e = next(e)$;
  end
  Ans *= ComputeSpanningTreeNumber(PointLayer, EdgeLayer);
  for $e2 \in EdgeLayer$ **do**
    Merge(e2.a, e2.b);
  end
end
if $NotConnected(G)$ **then**
  $Ans = 0$;
end
return Ans

Algorithm 1: Compute number of MST

The running time for this algorithm is really simple, since we know that $ComputeSpanningTreeNumber$ is polynomial-time. And we divide the

edges at most $|E|$ layers, so in all the running time is polynomial. $\square$