CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Spring 2020

Yujie LU, Jiabao JI, Tong CHEN

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First submission and questions due on Friday, 2020-05-08
You will receive feedback from the TA.
Final submission due on Tuesday, 2020-05-19

5 More on Network Flows

Exercise 1. Let $G = (V, c)$ be a flow network. Prove that flow is “transitive” in the following sense: if r, s, t are vertices, and there is an r–s-flow of value k and an s–t-flow of value k, then there is an r–t-flow of value k.

Proof. Denote the original r-s-flow and s-t-flow as f_{rs} and f_{st} respectively. We prove $f_{rt} = k$ by contradiction.

Suppose $f_{rt} < k$, denote $f_{rt} = p$. By Max-Flow-Min-Cut THM, there is a cut C with $cap(C) = p < k$. If $s \in C$, we know that C is also a r-s-cut. But by Max-Flow-Min-Cut THM, min $cap(r - s - cut) = k > cap(C) = p$, which leads to a contradiction. If $s \notin C$, the proof is similar. So $f_{rt} \geq k$, so there is a flow of value k in between r, t.

□

5.1 Vertex Disjoint Paths

Let G be a directed graph. Two paths p_1, p_2 from s to t are called *vertex disjoint* if they don't share any vertices except s and t.

Theorem 2 (Menger's Theorem)**.** *Let G be a graph and $s \neq t$ two vertices therein. Let $k \in \mathbf{N}_0$. Then exactly one of the following is true:*

1. There are k vertex disjoint paths $p_1, \ldots, p_k$ from s to t; that is, no two p_i, p_j share any vertex besides s and t.

2. There are vertices $v_1, \ldots, v_{k-1} \in V \setminus \{s, t\}$ such that $G - \{v_1, \ldots, v_{k-1}\}$ contains no s–t-path.

Exercise 3. Prove Menger's Theorem. You have to prove two things: first, not both cases above can occur (this is rather easy); second, one of them must occur (this requires a tool from the lecture).

Proof. First we prove the easy part, these two cases will not occur simultaneously.Prove it by contradiction.

Suppose both cases are true simultaneously, then there are k vertex disjoint paths $p_1, ..., p_k$ from s to t. and there exists $v_1, ..., v_{k-1}$ such that $G - \{v_1, ..., v_{k-1}\}$ contains no s-t-path.

For any $v_1, ..., v_{k-1}$, the can take place in at most $k-1$ paths in $p_1, ..., p_k$ since $p_1, ..., p_k$ are disjoint vertex paths. Then we know there must be at least one path from $p_1, .., p_k$ left, which connects s and t, leading to a contradiction.

Next we prove that one of them must occur.

For each vertex b, other than s or t, replace b by the 2-vertex graph $H(b)$ with vertex set $\{b^+, b\}$ and arc bb^+ with cap 1. Replace s by s^+ and t by t. Replace each edge $e = ab$, $\{a, b\}$ by the arcs a^+b with cap inf.

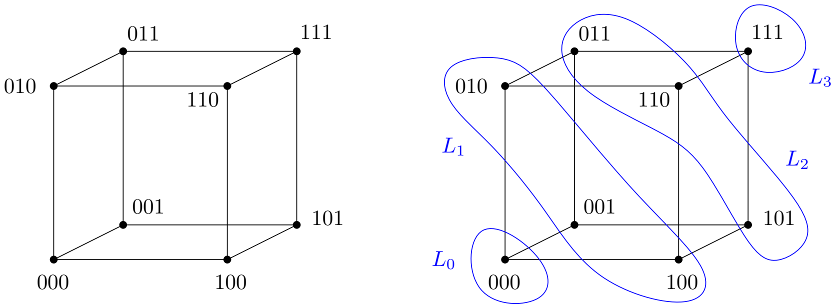
If the maxinum flow of the new graph G' is less than k. According to MaxFlow-MinCut Theroem and the min-cut must choose the edge b^-b^+, we can choose one of the min-cut that makes the graph has no s-t path and the number of points is less than k. That satisfies the situation 1.

Otherwise, the maxinum flow is at least k, that means there are at least k augmenting path in new graph G'. And we can correspond one augmenting path in new graph to one vertex disjoint paths in origin graph. That satisfies the situation 2. □

Let $V = \{0, 1\}^n$. The n-dimensional Hamming cube H_n is the graph (V, E) where $\{u, v\} \in E$ if u, v differ in exactly one coordinate. Define the i^{th} level of H_n as

$$L_i := \{u \in V \mid \|u\|_1 = i\} ,$$

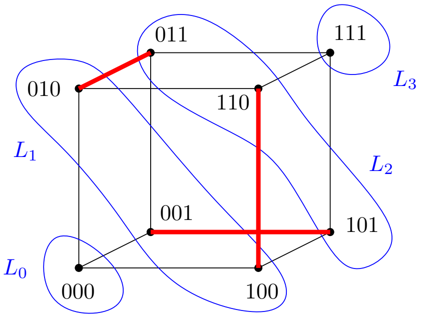
i.e., those vertices u having exactly i coordinates which are 1.



The 3-dimensional Hamming cube and the four sets L_0, L_1, L_2, L_3.

Exercise 4. [Matchings in H_n] Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$. This is the graph on vertex set $L_i \cup L_{i+1}$ where two edges are connected by an edge if and only if they are connected in H_n.

Show that for $i \leq n/2$ the graph $H_n[L_i \cup L_{i+1}]$ has a matching of size $|L_i| = \binom{n}{i}$.



A matching of size 3 between L_1 and L_2.

Proof. On the left, sets of size i. On the right, sets of size $i+1$, and an edge if the left vertex is included in the right vertex. Note that every vertex on the left has $n-i$ neighbors, and every vertex on the right has $i+1$ neighbors.

Let S be any family of L_i. The number of edges coming out of S is $|S|(n-i)$, therefore, the number of neighbors of S is at least $|S|(n-i)/(i+1) \geq |S|$. Hall's theorem then completes the proof. □

Exercise 5. Let H_n be the n-dimensional Hamming cube. For $i < n/2$ consider L_i and L_{n-i}. Note that $|L_i| = \binom{n}{i} = \binom{n}{n-i} = L_{n-i}$, so the L_i and L_{n-i} have the same size. Show that there are $\binom{n}{i}$ paths $p_1, p_2, \ldots, p_{\binom{n}{i}}$ in H_n such that (i) each p_i starts in L_i and ends in L_{n-i}; (ii) two different paths p_i, p_j

do not share any vertices. **Hint 1.** Model this problem as a network flow with vertex capacities. What would the maximum flow be in this network? **Hint 2.** It's not *that* easy. If you try to work from both sides towards the middle by combining matchings between levels, you will certainly run into problems as how to glue things together in the middle. I have never seen any "meet in the middle" proof that works. **Hint 3.** There is a "direct" proof by induction that does not require anything about network flows.

Proof. We can build a new network like this, create a source vertex s and a terminal vertex t. Connect s with all nodes in L_i and v with all nodes in L_{n-i}, leave all edges between L_i, L_{i+1} as they were. Assign all edges with capacity 1.

The maximum flow in this network is no larger than $|L_i| = \binom{n}{i}$, since the number of edges connecting to s is $|L_i|$ and they all have capacity 1. Then we give a flow with value $|L_i|$, this should tell us the maximum flow is $\binom{n}{i}$

Here is an example.

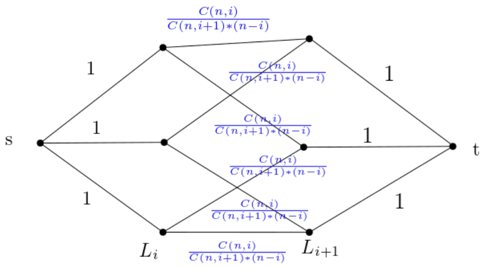

$\frac{C(n,i)}{C(n,i+1)*(n-i)}$
1
$\frac{C(n,i)}{C(n,i+1)*(n-i)}$
1
$\frac{C(n,i)}{C(n,i+1)*(n-i)}$
s
1
1
t
$\frac{C(n,i)}{C(n,i+1)*(n-i)}$
$\frac{C(n,i)}{C(n,i+1)*(n-i)}$
1
1
L_i
$\frac{C(n,i)}{C(n,i+1)*(n-i)}$
L_{i+1}

Figure 1: network

For all edges between L_j, L_{j+1}, we assign it with flow $\frac{C_n i}{C_n j+1(n-j)}$ as the edges between L_j, L_{j+1} is $C_n j + 1(n-j)$. Notice that L_{j+1} has $C_n j + 1$ nodes and each node in L_{j+1} is connected with $(n-j)$ nodes in L_j. Since the graph is symmetrical, such flow can be built.

Based on the proof of Menger's THM, we know that the maximum number of disjoint paths equals the maximum flow in this network. So there're $\binom{n}{i}$ disjoint paths from L_i to L_{n-i}.

□

5.2 Matchings and Vertex Covers

The following exercise was on the final exam of CS 499 (mathematical foundations of computer science) in spring 2019.

Exercise 6. Let $\nu(G)$ denote the size of a maximum matching of G. Show that a bipartite graph G has at most $2^{\nu(G)}$ minimum vertex covers.

Proof. Suppose the number of minimum vertex covers of graph G is $m\left(G\right)$. We can easily Now we prove that there could be no more than $2^{\nu(G)}$ minimum vertex cover. Let $G = A \cup B$, since G is a bipartite graph. Assume that the maximum matching is $e_1, e_2, \ldots, e_t$, in which $t = \nu\left(G\right)$. By **Konig's Theorem** we know that the number of vertices in a minimum vertex cover is exactly t.

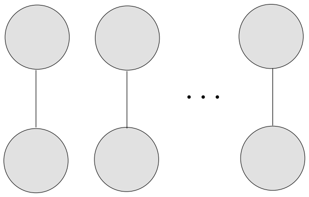


Figure 2: example

Plus we can prove that every vertex in K, the vertex cover, is an endpoint of a matched edge. Hence the result is obvious. □

Obviously, this is not true for general (non-bipartite) graphs: the triangle K_3 has $\nu(K_3) = 1$ but it has three minimum vertex covers. The five-cycle C_5 has $\nu(C_5) = 2$ but has five minimum vertex covers.

Exercise 7. Is there a function $f : \mathbf{N}_0 \to \mathbf{N}_0$ such that every graph with $\nu(G) = k$ has at most $f(k)$ minimum vertex covers? How small a function f can you obtain?

Proof. Assume that there are m connected componets $G_1, \ldots, G_m$, and each have c_i vertexes, hence the number of minimum vertex cover is

$$S = \prod_{i=1}^{m} c_i \leq f\left(k\right).$$

And suppose that connected component G_i has maximum matching k_i. We have $k_i \leq \frac{c_i}{2}$, therefore

$$k = \sum_{i=1}^{m} k_i \leq \frac{1}{2} \sum_{i=1}^{m} c_i.$$

Plus

$$k = \sum_{i=1}^{m} k_i \geq m.$$

Hence we need to find the smallest function f such that

$$\prod_{i=1}^{m} c_i \leq f\left(k\right) \leq f\left(\frac{1}{2} \sum_{i=1}^{m} c_i\right).$$

Notice that

$$\sum_{i=1}^{m} c_i \geq m \left(\prod_{i=1}^{m} c_i\right)^{\frac{1}{m}}.$$

Let $t = \prod_{i=1}^{m} c_i$, we must have

$$t \leq f\left(\frac{m}{2} t^{\frac{1}{m}}\right).$$

Which is equivalent to find the largest $g = f^{-1}$ such that

$$\forall m, m t^{\frac{1}{m}} - 2g\left(t\right) \geq 0.$$

Thus g cannot be polynomial, the maximum of g could only be something like $c \ln\left(x\right)$ i.e.

$$h\left(t\right) = m t^{\frac{1}{m}} - 2c \ln\left(t\right), \forall t \geq 1, m \geq 1.$$

It easy to find the largest c is $\frac{1}{2}$. Hence the smallest f is $f\left(k\right) = e^{2k}$.

□