CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2019

Handed out on Monday, 2019-10-21
First submission and questions due on Monday, 2019-10-28
You will receive feedback from the TA.
Final submission due on Monday, 2019-11-04

5 More on Network Flows

Exercise 1. Let $G = (V, c)$ be a flow network. Prove that flow is "transitive" in the following sense: if r, s, t are vertices, and there is an r–s-flow of value k and an s–t-flow of value k, then there is an r–t-flow of value k.

Proof. Denote the original r-s-flow and s-t-flow as f_{rs} and f_{st} respectively. We prove $f_{rt} = k$ by contradiction.

Suppose $f_{rt} < k$, denote $f_{rt} = p$. By Max-Flow-Min-Cut THM, there is a cut C with $cap(C) = p < k$. If $s \in C$, we know that C is also a r-s-cut. But by Max-Flow-Min-Cut THM, min $cap(r - s - cut) = k > cap(C) = p$, which leads to a contradiction. If $s \notin C$, the proof is similar. So $f_{rt} \geq k$, so there is a flow of value k in between r, t.

□

5.1 Vertex Disjoint Paths

Let G be a directed graph. Two paths p_1, p_2 from s to t are called *vertex disjoint* if they don't share any vertices except s and t.

Theorem 2 (Menger's Theorem)**.** *Let G be a graph and $s \neq t$ two vertices therein. Let $k \in \mathbf{N}_0$. Then exactly one of the following is true:*

1. *There are k vertex disjoint paths $p_1, \ldots, p_k$ from s to t; that is, no two p_i, p_j share any vertex besides s and t.*

2. *There are vertices $v_1, \ldots, v_{k-1} \in V \setminus \{s,t\}$ such that $G - \{v_1, \ldots, v_{k-1}\}$ contains no s–t-path.*

Exercise 3. Prove Menger's Theorem. You have to prove two things: first, not both cases above can occur (this is rather easy); second, one of them must occur (this requires a tool from the lecture).

Proof. First we prove the easy part, these two cases will not occur simultaneously.Prove it by contradiction.

Suppose both cases are true simultaneously, then there are k vertex disjoint paths $p_1, ..., p_k$ from s to t. and there exists $v_1, ..., v_{k-1}$ such that $G - \{v_1, ..., v_{k-1}\}$ contains no s-t-path.

For any $v_1, ..., v_{k-1}$, the can take place in at most $k-1$ paths in $p_1, ..., p_k$ since $p_1, ..., p_k$ are disjoint vertex paths. Then we know there must be at least one path from $p_1, .., p_k$ left, which connects s and t, leading to a contradiction.

Next we prove that one of them must occur.

The second case is obvious since we can just remove all vertices except s, t from the V, then obviously there're no s-t-path now.

For the first case, we construct such a network-graph with all edge in the original graph assigned capacity 1. In such a network, the value of a flow is

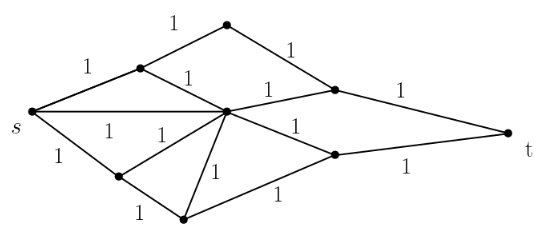


Figure 1: network

the number of vertex disjoint paths in such a flow. Since the capacity of each edge is 1, we can know for sure that no two s-t-path cross with each other, otherwise there must be a vertex with units bigger than 1. Based on that, we see k, the value of a flow is the number of paths in it. Thus leading to k disjoint paths.

In fact, we can prove that the maximum number of disjoint paths is equal to the minimum number of vertex set which separates s, t.

We construct a new network from original graph. For a vertex v other than s, t, we split it to v^+, v^-, and draw a new directed edge v^-v^+. For an edge (s, v), replace it with sv^-. For an edge (v, t), replace it with v^+t. For other edges (u, v), replace it with (u^+v^-, v^+u^-). All new edges are assigned capacity 1. Here's an example.

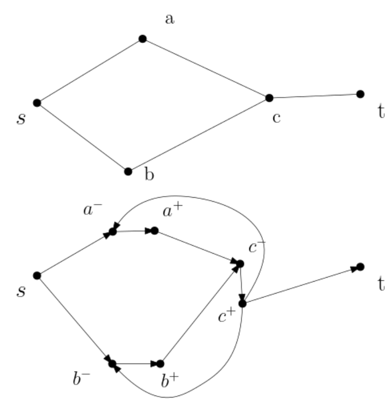


Figure 2: network

In such a network G', we split the nodes and put directed edge to make sure no two flow cross each other at some nodes. Also, any flow f in this network can produce a set P of disjoint (s, t) paths, $|P| = val(f)$. We add an edge (u, v) to a path if (u^+, v^-) is in the flow. Any cut C has a sub set C' with all arcs like (v^-, v^+), and C' still a cut in G'. Then we can produce a set of vertices V in G with all nodes $v((v^-, v^+) \in C')$.

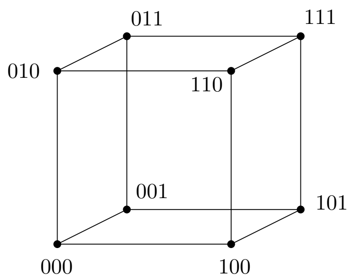
Then we know that for a maximum set $P'$of disjoint paths in G, $|P'| \geq |P| = val(f) = |C| \geq |C'| \geq |V| \geq |P'|$ So all of these values are equal, which finishes the proof.

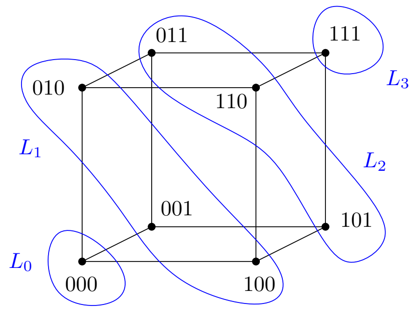
$\square$

Let $V = \{0, 1\}^n$. The n-dimensional Hamming cube H_n is the graph (V, E) where $\{u, v\} \in E$ if u, v differ in exactly one coordinate. Define the i^{th} level of H_n as

$$L_i := \{u \in V \mid \|u\|_1 = i\} ,$$

i.e., those vertices u having exactly i coordinates which are 1.

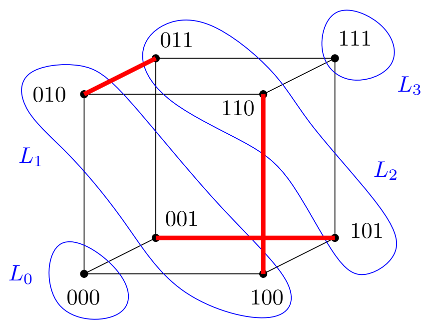




The 3-dimensional Hamming cube and the four sets L_0, L_1, L_2, L_3.

Exercise 4. [Matchings in H_n] Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$. This is the graph on vertex set $L_i \cup L_{i+1}$ where two edges are connected by an edge if and only if they are connected in H_n.

Show that for $i \leq n/2$ the graph $H_n[L_i \cup L_{i+1}]$ has a matching of size $|L_i| = \binom{n}{i}$.



A matching of size 3 between L_1 and L_2.

Proof. □

Exercise 5. Let H_n be the n-dimensional Hamming cube. For $i < n/2$ consider L_i and L_{n-i}. Note that $|L_i| = \binom{n}{i} = \binom{n}{n-i} = L_{n-i}$, so the L_i and L_{n-i} have the same size. Show that there are $\binom{n}{i}$ paths $p_1, p_2, \ldots, p_{\binom{n}{i}}$ in H_n such that (i) each p_i starts in L_i and ends in L_{n-i}; (ii) two different paths p_i, p_j do not share any vertices. **Hint 1.** Model this problem as a network flow with vertex capacities. What would the maximum flow be in this network? **Hint 2.** It's not *that* easy. If you try to work from both sides towards the middle by combining matchings between levels, you will certainly run into problems as how to glue things together in the middle. I have never seen any “meet in the middle” proof that works. **Hint 3.** There is a “direct” proof by induction that does not require anything about network flows.

Proof. We can build a new network like this, create a source vertex s and a terminal vertex t. Connect s with all nodes in L_i and v with all nodes in L_{n-i}, leave all edges between L_i, L_{i+1} as they were. Assign all edges with capacity 1.

The maximum flow in this network is no larger than $|L_i| = \binom{n}{i}$, since the number of edges connecting to s is $|L_i|$ and they all have capacity 1. Then we give a flow with value $|L_i|$, this should tell us the maximum flow is $\binom{n}{i}$

Here is an example.

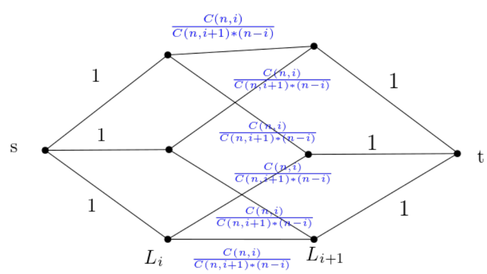


Figure 3: network

For all edges between L_j, L_{j+1}, we assign it with flow $\frac{C_n i}{C_n j+1(n-j)}$ as the edges between L_j, L_{j+1} is $C_n j + 1(n-j)$. Notice that L_{j+1} has $C_n j + 1$ nodes and each node in L_{j+1} is connected with $(n-j)$ nodes in L_j. Since the graph is symmetrical, such flow can be built.

Based on the proof of Menger’s THM, we know that the maximum number of disjoint paths equals the maximum flow in this network. So there’re $\binom{n}{i}$ disjoint paths from L_i to L_{n-i}.

$\square$

5.2 Matchings and Vertex Covers

The following exercise was on the final exam of CS 499 (mathematical foundations of computer science) in spring 2019.

Exercise 6. Let $\nu(G)$ denote the size of a maximum matching of G. Show that a bipartite graph G has at most $2^{\nu(G)}$ minimum vertex covers.

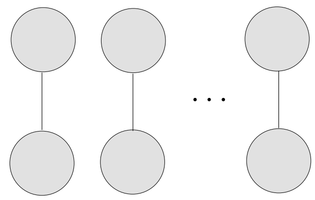


Figure 4: example

Proof. Suppose the number of minimum vertex covers of graph G is $m\,(G)$. We can easily Now we prove that there could be no more than $2^{\nu(G)}$ minimum vertex cover. Let $G = A \cup B$, since G is a bipartite graph. Assume that the maximum matching is $e_1, e_2, \ldots, e_t$, in which $t = \nu\,(G)$. By **Konig's Theorem** we know that the number of vertices in a minimum vertex cover is exactly t.

Plus we can prove that every vertex in K, the vertex cover, is an endpoint of a matched edge. Hence the result is obvious. □

Obviously, this is not true for general (non-bipartite) graphs: the triangle K_3 has $\nu(K_3) = 1$ but it has three minimum vertex covers. The five-cycle C_5 has $\nu(C_5) = 2$ but has five minimum vertex covers.

Exercise 7. Is there a function $f : \mathbf{N}_0 \to \mathbf{N}_0$ such that every graph with $\nu(G) = k$ has at most $f(k)$ minimum vertex covers? How small a function f can you obtain?

Proof. Since this is for every graph. First consider K_{2r+1}, we have $\frac{k}{r}$ K_{2r+1}. Hence $\nu\,(G) = \frac{k}{r} \cdot r = k$

$$f\,(k) \geq (2r+1)^{\frac{k}{r}}\,.$$

For example, if we choose $r = 1$, there are k K_3, we have $f\,(k) \geq 3^k$. And let $r \to \infty$ we have

$$f\,(k) \geq e^{2k}.$$

Next we prove that the number of minimum vertex cover is at most e^{2k}. Consider $\forall$ minimum vertex cover K. We prove

$$|K| \leq e^{2k}.$$

Let $\forall u, v \in K$. If u, v are connected, then at least one of them is on a edge of maximum matching. □