CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2020

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First submission and questions due on Monday, 2020-6-1
You will receive feedback from the TA.
Final submission due on Thursday, 2020-11-14

6 Matching LP and Vertex Cover LP

Let $G = (V, E)$ be a graph and consider the Vertex Cover Linear Program VCLP(G):

$$\text{VCLP}(G): \qquad \begin{array}{lll} \text{minimize} & \sum_{u\in V} y_u & \\ \text{subject to} & y_u + y_v \geq 1 & \forall \text{ edges } \{u,v\} \in E \\ & \mathbf{y} \geq \mathbf{0} & \end{array}$$

Every vertex cover of G corresponds to a feasible solution $\mathbf{y} \in \text{sol}(\text{VCLP}(G))$, but not vice versa. However, every $\mathbf{y} \in \text{sol}(\text{VCLP}(G)) \cap \{0,1\}^V$ does. Let $\tau(G)$ denote the size of a minimum vertex cover of G. In class, we showed that $\tau(G) = \text{val}(\text{VCLP}(G))$ for all *bipartite* graphs G. We achieved this by taking an arbitrary feasible solution $\mathbf{y}$ and "shaking" it until it becomes integral, while making sure its value does not go up.

Next, recall the Matching Linear Program MLP(G):

$$\text{MLP}(G): \qquad \begin{array}{lll} \text{maximize} & \sum_{e\in E} x_e & \\ \text{subject to} & \sum_{e\in E: u\in e} x_e \leq 1 & \forall\, u \in V \\ & \mathbf{x} \geq \mathbf{0} & \end{array}$$

Every matching of G corresponds to a feasible solution $\mathbf{x} \in \text{sol}(\text{MLP}(G))$, but not vice versa. However, every $\mathbf{x} \in \text{sol}(\text{MLP}(G)) \cap \{0,1\}^E$ does.

Exercise 1. Let $\nu(G)$ denote the size of a maximum matching of G. Obviously, $\text{val}(\text{MLP}(G)) \geq \nu(G)$ for all graphs. Show that $\nu(G) = \text{val}(\text{MLP}(G))$ for all *bipartite* graphs G.

Proof. $\square$

Exercise 2. We know that $\nu(G) = \tau(G)$ for all bipartite graphs (Kőnig's Theorem) and $\nu(G) \leq \tau(G)$ for all graphs (since every matched edge must be covered by a distinct vertex). Show that $\tau(G) \leq 2\,\nu(G)$ for all graphs G.

Proof. $\square$

Exercise 3. Show that $\tau(G) \leq 2\,\text{opt}(\text{VCLP}(G))$ for all graphs G (including non-bipartite graphs).

Proof. By the result of last exercise

$$\tau\left(G\right) \leq 2\nu\left(G\right).$$

And $\nu\left(G\right) \leq \text{opt}\left(VCLP\left(G\right)\right)$, the result is obvious. $\square$

Basic Solutions. Recall our definition of basic solutions. Let P be the following linear program.

$$P: \qquad \begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \ \leq \mathbf{b} \end{array}$$

where we translated the constraint $\mathbf{x} \geq 0$ into n constraints $-x_i \leq 0$ and integrated them into A, so the n last rows of A form the negative identity matrix $-I_n$. We introduce some notation: $\mathbf{a}_i$ is the i^{th} row of i; for $I \subseteq [m+n]$ let A_I be the matrix consisting of the rows $\mathbf{a}_i$ for $i \in I$.

Definition 4. *For $\mathbf{x} \in \mathbb{R}^n$ let $I(\mathbf{x}) := \{i \in [m+n] \mid \mathbf{a}_i\mathbf{x} = b_i\}$ be the set of indices of the constraints that are "tight", i.e., satisfied with equality (we include non-negativity constraints here). We call $\mathbf{x} \in \mathbb{R}^n$ a* basic point *if $\text{rank}(A_{I(x)}) = n$. If $\mathbf{x}$ is a basic point and feasible, we call it a* basic feasible solution *or simply a* basic solution.

We can define the same concept for minimization programs.

We say a set $C \subseteq V$ is a minimal vertex cover of $G = (V, E)$ if (1) it is a vertex cover and (2) it is minimal, i.e., for every $u \in C$ the set $C \setminus \{u\}$ is not a vertex cover anymore.

Exercise 5.

Proof. (1) Consider such a graph: $|V| = 4, E = E(K_4)/e_{1,2}$, obviously it's not bipartite. We have

$$\nu\left(G\right) = 2 = \nu_f\left(G\right) = \tau_f\left(G\right) = \tau\left(G\right).$$

(2) Let $G = K_4$, then

$$\nu\left(G\right) = 2 = \nu_f\left(G\right) = \tau_f\left(G\right) < \tau\left(G\right) = 3.$$

(3) Since G is VCLP exact. A MVC Y also corresponds to a optimal solution in VCLP. If $e = (u_0, v_0) \in Y$. We have

$$y_{u_0} = y_{v_0} = 1 \implies y_{u_0} + y_{v_0} = 2 > 1.$$

Hence we wouldn't use this inequality $y_{u_0} + y_{v_0} \geq 1$ when converting the VCLP to its dual, therefore the respective coefficient $x_e = 0$. More specifically:

$$\sum_{u\in V} y_u \geq \sum_{e\in E} x_e\left(y_u + y_v\right) \geq \sum_{e\in E} x_e.$$

As $\min \sum_{u\in V} y_u = \max \sum_{e\in E} x_e$. These inequalities are all tight, which leads that

$$x_e = 0 \text{ or } y_u + y_v = 1.$$

If $y_u + y_v > 1, x_e \neq 0$, the inequality is not tight, leads to a contradiction.

(4)First we prove there is a matching of size $s = |Y|$. Consider all the vertexes $v_1, v_2, \ldots, v_s \in Y$. If $\exists i \neq j$ s.t. $e = (v_i, v_j) \in E$, then $x_e = 0$. We can assert that $\exists u_i \neq u_j$ s.t.

$$\left(v_i, u_i\right), \left(v_j, u_j\right) \in E.$$

Plus $u_i \neq v_j, u_j \neq v_i$, otherwise we can obtain a smaller vertex cover by removing one of v_i, v_j. And otherwise v_i has no neighbors in Y, we just choose an arbitrary neighbor(since it's MVC, it must have a neighbor), this neighbor cannot be one of u_i mentioned above, otherwise there is smaller vertex cover by removing this v_i. Hence we get a matching of size $|Y|$

$$\max \sum_{e\in E} x_e \geq |Y|.$$

While

$$\max \sum_{e\in E} x_e \leq \tau_f\left(G\right) = |Y|.$$

It follows that

$$\nu(G) = \nu_f(G).$$

Hence G is MLP-exact too. $\square$