

CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2019

Handed out on Monday, 2019-10-21

First submission and questions due on Monday, 2019-10-28

You will receive feedback from the TA.

Final submission due on Monday, 2019-11-04

5 More on Network Flows

Exercise 1. Let $G = (V, c)$ be a flow network. Prove that flow is “transitive” in the following sense: if r, s, t are vertices, and there is an r – s -flow of value k and an s – t -flow of value k , then there is an r – t -flow of value k .

Proof. Denote the original r - s -flow and s - t -flow as f_{rs} and f_{st} respectively. We prove $f_{rt} = k$ by contradiction.

Suppose $f_{rt} < k$, denote $f_{rt} = p$. By Max-Flow-Min-Cut THM, there is a cut C with $\text{cap}(C) = p < k$. If $s \in C$, we know that C is also a r - s -cut. But by Max-Flow-Min-Cut THM, $\min \text{cap}(r - s - \text{cut}) = k > \text{cap}(C) = p$, which leads to a contradiction. If $s \notin C$, the proof is similar. So $f_{rt} \geq k$, so there is a flow of value k in between r, t .

□

5.1 Vertex Disjoint Paths

Let G be a directed graph. Two paths p_1, p_2 from s to t are called *vertex disjoint* if they don't share any vertices except s and t .

Theorem 2 (Menger's Theorem). *Let G be a graph and $s \neq t$ two vertices therein. Let $k \in \mathbb{N}_0$. Then exactly one of the following is true:*

1. There are k vertex disjoint paths p_1, \dots, p_k from s to t ; that is, no two p_i, p_j share any vertex besides s and t .
2. There are vertices $v_1, \dots, v_{k-1} \in V \setminus \{s, t\}$ such that $G - \{v_1, \dots, v_{k-1}\}$ contains no s - t -path.

Exercise 3. Prove Menger's Theorem. You have to prove two things: first, not both cases above can occur (this is rather easy); second, one of them must occur (this requires a tool from the lecture).

Proof. First we prove the easy part, these two cases will not occur simultaneously. Prove it by contradiction.

Suppose both cases are true simultaneously, then there are k vertex disjoint paths p_1, \dots, p_k from s to t . and there exists v_1, \dots, v_{k-1} such that $G - \{v_1, \dots, v_{k-1}\}$ contains no s - t -path.

For any v_1, \dots, v_{k-1} , the can take place in at most $k - 1$ paths in p_1, \dots, p_k since p_1, \dots, p_k are disjoint vertex paths. Then we know there must be at least one path from p_1, \dots, p_k left, which connects s and t , leading to a contradiction.

Next we prove that one of them must occur.

For each vertex b , other than s or t , replace b by the 2-vertex graph $H(b)$ with vertex set $\{b^+, b\}$ and arc bb^+ with cap 1. Replace s by s^+ and t by t^- . Replace each edge $e = ab$, $\{a, b\}$ by the arcs a^+b with cap \inf .

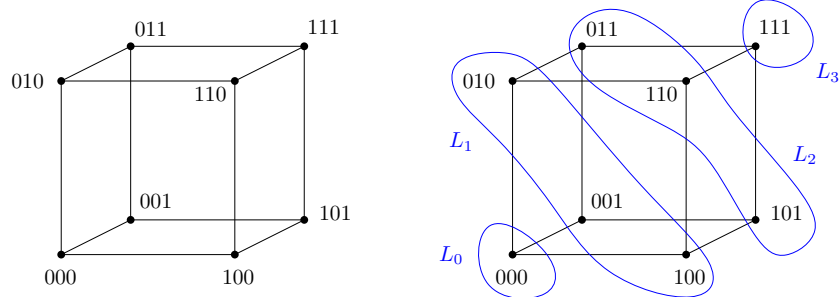
If the maximum flow of the new graph G' is less than k . According to MaxFlow-MinCut Theorem and the min-cut must choose the edge b^-b^+ , we can choose one of the min-cut that makes the graph has no s - t path and the number of points is less than k . That satisfies the situation 1.

Otherwise, the maximum flow is at least k , that means there are at least k augmenting path in new graph G' . And we can correspond one augmenting path in new graph to one vertex disjoint paths in origin graph. That satisfies the situation 2. \square

Let $V = \{0, 1\}^n$. The n -dimensional Hamming cube H_n is the graph (V, E) where $\{u, v\} \in E$ if u, v differ in exactly one coordinate. Define the i^{th} level of H_n as

$$L_i := \{u \in V \mid \|u\|_1 = i\},$$

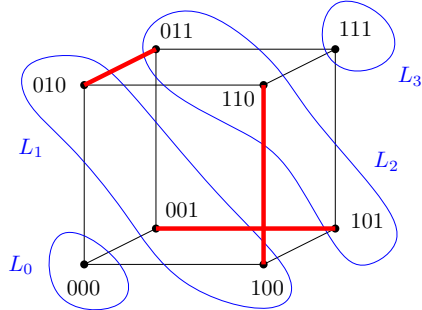
i.e., those vertices u having exactly i coordinates which are 1.



The 3-dimensional Hamming cube and the four sets L_0, L_1, L_2, L_3 .

Exercise 4. [Matchings in H_n] Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$. This is the graph on vertex set $L_i \cup L_{i+1}$ where two edges are connected by an edge if and only if they are connected in H_n .

Show that for $i \leq n/2$ the graph $H_n[L_i \cup L_{i+1}]$ has a matching of size $|L_i| = \binom{n}{i}$.



A matching of size 3 between L_1 and L_2 .

Proof. On the left, sets of size i . On the right, sets of size $i + 1$, and an edge if the left vertex is included in the right vertex. Note that every vertex on the left has $n - i$ neighbors, and every vertex on the right has $i + 1$ neighbors.

Let S be any family of L_i . The number of edges coming out of S is $|S|(n - i)$, therefore, the number of neighbors of S is at least $|S|(n - i)/(i + 1) \geq |S|$. Hall's theorem then completes the proof. \square

Exercise 5. Let H_n be the n -dimensional Hamming cube. For $i < n/2$ consider L_i and L_{n-i} . Note that $|L_i| = \binom{n}{i} = \binom{n}{n-i} = |L_{n-i}|$, so the L_i and L_{n-i} have the same size. Show that there are $\binom{n}{i}$ paths $p_1, p_2, \dots, p_{\binom{n}{i}}$ in H_n such that (i) each p_i starts in L_i and ends in L_{n-i} ; (ii) two different paths p_i, p_j

do not share any vertices. **Hint 1.** Model this problem as a network flow with vertex capacities. What would the maximum flow be in this network? **Hint 2.** It's not *that* easy. If you try to work from both sides towards the middle by combining matchings between levels, you will certainly run into problems as how to glue things together in the middle. I have never seen any “meet in the middle” proof that works. **Hint 3.** There is a “direct” proof by induction that does not require anything about network flows.

Proof. Assume while $n = k$, there are $\binom{n}{i}$ vertex disjoint paths from L_i to L_{n-i} . We want to proof while $k = n + 1$ the conclusion is also right. We can divide the points between L_i and L_{n+1-i} into two set: $A = \{x | x[n + 1] = 0\}$ and $B = \{x | x[n + 1] = 1\}$. And according to induction assumption, there are $\binom{n}{i}$ vertex disjoint paths from $H_n[L_i]$ to $H_n[L_{n-i}]$ \square

5.2 Matchings and Vertex Covers

The following exercise was on the final exam of CS 499 (mathematical foundations of computer science) in spring 2019.

Exercise 6. Let $\nu(G)$ denote the size of a maximum matching of G . Show that a bipartite graph G has at most $2^{\nu(G)}$ minimum vertex covers.

Proof. Suppose the number of minimum vertex covers of graph G is $m(G)$. We can easily Now we prove that there could be no more than $2^{\nu(G)}$ minimum

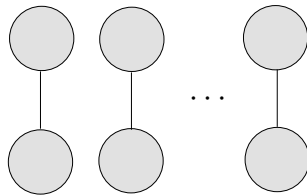


Figure 1: example

vertex cover. Let $G = A \cup B$, since G is a bipartite graph. Assume that the maximum matching is e_1, e_2, \dots, e_t , in which $t = \nu(G)$. By **König's Theorem** we know that the number of vertices in a minimum vertex cover is exactly t .

Plus we can prove that every vertex in K , the vertex cover, is an endpoint of a matched edge. Hence the result is obvious. \square

Obviously, this is not true for general (non-bipartite) graphs: the triangle K_3 has $\nu(K_3) = 1$ but it has three minimum vertex covers. The five-cycle C_5 has $\nu(C_5) = 2$ but has five minimum vertex covers.

Exercise 7. Is there a function $f : \mathbf{N}_0 \rightarrow \mathbf{N}_0$ such that every graph with $\nu(G) = k$ has at most $f(k)$ minimum vertex covers? How small a function f can you obtain?

Proof. Since this is for every graph. First consider K_{2r+1} , we have $\frac{k}{r} K_{2r+1}$. Hence $\nu(G) = \frac{k}{r} \cdot r = k$

$$f(k) \geq (2r+1)^{\frac{k}{r}}.$$

For example, if we choose $r = 1$, there are k K_3 , we have $f(k) \geq 3^k$. And let $r \rightarrow \infty$ we have

$$f(k) \geq e^{2k}.$$

Next we prove that the number of minimum vertex cover is at most e^{2k} . Consider \forall minimum vertex cover K . We prove

$$|K| \leq e^{2k}.$$

Let $\forall u, v \in K$. If u, v are connected, then at least one of them is on a edge of maximum matching. \square