CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2020

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First submission and questions due on Monday, 2020-6-1
You will receive feedback from the TA.
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6 Matching LP and Vertex Cover LP

Let $G = (V, E)$ be a graph and consider the Vertex Cover Linear Program VCLP(G):

$$\text{VCLP}(G): \qquad \begin{array}{lrll} \text{minimize} & \sum_{u\in V} y_u & & \\ \text{subject to} & y_u + y_v & \geq 1 & \forall \text{ edges } \{u,v\} \in E \\ & \mathbf{y} & \geq \mathbf{0} & \end{array}$$

Every vertex cover of G corresponds to a feasible solution $\mathbf{y} \in \text{sol}(\text{VCLP}(G))$, but not vice versa. However, every $\mathbf{y} \in \text{sol}(\text{VCLP}(G)) \cap \{0,1\}^V$ does. Let $\tau(G)$ denote the size of a minimum vertex cover of G. In class, we showed that $\tau(G) = \text{val}(\text{VCLP}(G))$ for all *bipartite* graphs G. We achieved this by taking an arbitrary feasible solution $\mathbf{y}$ and "shaking" it until it becomes integral, while making sure its value does not go up.

Next, recall the Matching Linear Program MLP(G):

$$\text{MLP}(G): \qquad \begin{array}{lrll} \text{maximize} & \sum_{e\in E} x_e & & \\ \text{subject to} & \sum_{e\in E: u \in e} x_e & \leq 1 & \forall\, u \in V \\ & \mathbf{x} & \geq \mathbf{0} & \end{array}$$

Every matching of G corresponds to a feasible solution $\mathbf{x} \in \text{sol}(\text{MLP}(G))$, but not vice versa. However, every $\mathbf{x} \in \text{sol}(\text{MLP}(G)) \cap \{0,1\}^E$ does.

Exercise 1. Let $\nu(G)$ denote the size of a maximum matching of G. Obviously, val(MLP(G)) $\geq \nu(G)$ for all graphs. Show that $\nu(G) =$ val(MLP(G)) for all *bipartite* graphs G.

Proof. We just need to prove that $int - val(MLP(G)) = val(MLP(G))$, since $val(MLP(G)) \geq \nu(G)$, we do this following the proof for $VCLP(G)$ in class, the idea is to get int value solution through modifying a non-int value solution a bit.

Suppose we already have a solution $\mathbf{x}$, and denote $\sum_{e \in E, u \in e} x_e = y_u, \forall u \in V$, we want to change $\mathbf{x}$ to int value, and meanwhile $\mathbf{y}$ becomes int value.

At each step, those edges with $x_e = 0$ could just be ignored, while those edges with $x_e = 1$ and its both ends could be ignored. In the following situations, we just don't take these edges and vertices into account.

1. For two vertices u, v with non integral y_u, y_v, suppose there exits a path $e_1, e_2, ...e_m$, where m is odd. Let $d = min(1 - y_u, 1 - y_v, 1 - x_{e_1}, x_{e_2}, ..., 1 - x_{e_k})$. Next we modify edges as following. For those $e_i, iodd$, let $e_i = e_i + d$, and for those $e_i, ieven$, let $e_i = e_i - d$. By doing this, all constraints remains, and at least one of $y_u, y_v, x_{e_1}, ...x_{e_k}$ becomes integral. In all, $val(MLP(G))$ increases by d.

2. Similar to the situation above, except that the path has an even size. Let $d = min(1 - y_u, y_v, 1 - x_{e_1}, x_{e_2}, ..., 1 - x_{e_{k-1}}, x_{e_k})$. Next we modify edges as the above situation. By doing this, all constraints still remains, and at least one of $y_u, y_v, x_{e_1}, ...x_{e_k}$ becomes integral. In all, $val(MLP(G))$ remains unchanged.

3. If y_u is non integral but y_v integral, we just find all vertices connected with u, v. Denote the left side as L and the right side as R, We shall have $\sum_{u \in L} y_u = \sum_{u \in R} y_u$. But the left side is non integral while the right side is integral, leading to a contradiction.

4. If all y_u are integral, but some x_e may still remain non integral. To solve this, we find one of the non integral edge e_1, with u as one of its end. Since y_u integral, there exists another e_2 non integral connected to u. Continue this, we will find a cycle $e_1, ..., e_k$ with even size since G is bipartite. Let $d = min(1 - x_{e_1}, x_{e_2}, ...1 - x_{e_{k-1}}, x_{e_k}$, do the updates as the above updates. We can see that $\mathbf{y}$ remains, while at least one x_e becomes integral, and $val(MLP(G))$ remains.

In the above situations, at least one y_u or one x_e becomes integral. Since both $\mathbf{x}, \mathbf{y}$ are finite, this procedure will end. And since x_e becomes all integral, the original solution becomes an int value solution while not making $\sum_{e \in E} x_e$ worse. Thus we know $\nu(G) = val(MLP(G))$ □

Exercise 2. We know that $\nu(G) = \tau(G)$ for all bipartite graphs (Kőnig's Theorem) and $\nu(G) \leq \tau(G)$ for all graphs (since every matched edge must be covered by a distinct vertex). Show that $\tau(G) \leq 2\,\nu(G)$ for all graphs G.

Proof. Assume that α be the size of maximum independent set, and β be the size of a minimum vertex cover of G. Let U be the set of vertex cover, and V/U must be an independent set, because if there is any edge between S/U, U is not a cover, contradiction. So we have $\alpha + \beta = |V|(1)$.

Suppose $(a_1, b_1), (a_2, b_2), \cdots, (a_{\nu(G)}, b_{\nu(G)})$ be a maximum match. Let W be the vertex set that is not in the maximum match. Obviously, W is an independent set. By definition, $2\nu(G) + |W| = |V|$. Because W is an independent set, $|W| \leq \alpha = |V| - \tau(G)$. So we get $\tau(G) \leq 2\nu(G)$. □

Exercise 3. Show that $\tau(G) \leq 2\,\mathrm{opt}(\mathrm{VCLP}(G))$ for all graphs G (including non-bipartite graphs).

Proof. By the result of last exercise

$$\tau(G) \leq 2\nu(G).$$

And $\nu(G) \leq \mathrm{opt}(VCLP(G))$, the result is obvious. □

Exercise 4.

Proof. (1) Consider such a graph: $|V| = 4, E = E(K_4)/e_{1,2}$, obviously it's not bipartite. We have

$$\nu(G) = 2 = \nu_f(G) = \tau_f(G) = \tau(G).$$

(2) Let $G = K_4$, then

$$\nu(G) = 2 = \nu_f(G) = \tau_f(G) < \tau(G) = 3.$$

(3) Since G is VCLP exact. A MVC Y also corresponds to a optimal solution in VCLP. If $e = (u_0, v_0) \in Y$. We have

$$y_{u_0} = y_{v_0} = 1 \implies y_{u_0} + y_{v_0} = 2 > 1.$$

Hence we wouldn't use this inequality $y_{u_0} + y_{v_0} \geq 1$ when converting the VCLP to its dual, therefore the respective coefficient $x_e = 0$. More specifically:

$$\sum_{u \in V} y_u \geq \sum_{e \in E} x_e \left(y_u + y_v\right) \geq \sum_{e \in E} x_e.$$

As $\min \sum_{u \in V} y_u = \max \sum_{e \in E} x_e$. These inequalities are all tight, which leads that

$$x_e = 0 \text{ or } y_u + y_v = 1.$$

If $y_u + y_v > 1, x_e \neq 0$, the inequality is not tight, leads to a contradiction.
(4)First we prove there is a matching of size $s = |Y|$. Consider all the vertexes $v_1, v_2, \ldots, v_s \in Y$. If $\exists i \neq j$ s.t. $e = (v_i, v_j) \in E$, then $x_e = 0$. We can assert that $\exists u_i \neq u_j$ s.t.

$$\left(v_i, u_i\right), \left(v_j, u_j\right) \in E.$$

Plus $u_i \neq v_j, u_j \neq v_i$, otherwise we can obtain a smaller vertex cover by removing one of v_i, v_j. And otherwise v_i has no neighbors in Y, we just choose an arbitrary neighbor(since it's MVC, it must have a neighbor), this neighbor cannot be one of u_i mentioned above, otherwise there is smaller vertex cover by removing this v_i. Hence we get a matching of size $|Y|$

$$\max \sum_{e \in E} x_e \geq |Y|.$$

While

$$\max \sum_{e \in E} x_e \leq \tau_f(G) = |Y|.$$

It follows that

$$\nu(G) = \nu_f(G).$$

Hence G is MLP-exact too. $\square$