CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2020

Yujie Lu, Tong Chen, Jiabao Ji

Handed out on Friday, 2020-06-05
First submission and questions due on Thursday, 2020-06-12
You will receive feedback from the TA.
Final submission due on Thursday, 2020-06-19

7 Farkas Lemma and LP Duality

7.1 Different Versions of Farkas Lemma

In the following, let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, and let $\mathbf{x} = (x_1, \ldots, x_n)^T$ be a column vector of n variables and $\mathbf{y} = (y_1, \ldots, y_m)$ be a row vector of m variables.

Exercise 1. Show that the three versions of Farkas Lemma presented in class are all equivalent (I actually did not present the third version in class):

$$(\neg \exists \mathbf{x} : A\mathbf{x} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T A = \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) . \tag{1}$$

$$(\neg \exists \mathbf{x} \geq \mathbf{0} : A\mathbf{x} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T A \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) . \tag{2}$$

$$(\neg \exists \mathbf{x} \geq \mathbf{0} : A\mathbf{x} = \mathbf{b}) \iff (\exists \mathbf{y} \quad : \mathbf{y}^T A \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) . \tag{3}$$

Note that the direction "$\Longleftarrow$" is easy in each case. We will show the "$\Longrightarrow$" of (1) in class using a technique called *Fourier-Motzkin Elimination*. This exercise is actually not that hard. The hardest part is keeping track of what you want to prove and what you can assume.

7.2 A Linear Program for, well, for what?

Let $G = (V, E)$ be a directed graph, $s, t \in V$, and $c : E \to \mathbf{R}^+$ be a cost function. We want to find an $s - t$-flow f of value 1. Every edge e generates cost $f(e) \cdot c(e)$, and we want to minimize the overall cost. There are no capacity constraints. We can easily write this as a linear program MCF (Minimum Cost Flow):

$$\text{MCF}(G, s, t, c): \quad \begin{array}{ll} \text{minimize} & \sum_{e \in E} c(e) f(e) \\ \\ \text{subject to} & \sum_{v \in V} f(v, t) = 1 \\ & \sum_{u \in V} f(u, v) - \sum_{w \in V} f(v, w) = 0 \quad \forall\, v \in V \setminus \{s, t\} \\ \\ & f(e) \geq 0 \;\forall\; e \in E \end{array}$$

Note that we have m variables, one variable $f(e)$ for each edge e. The first constraint says that the value of the flow should be 1. The other constraints say that the inflow at v should equal the outflow.

Exercise 2. Let d be the shortest path distance from s to t in the directed graph G, where distance means sum of the $c(e)$ along the path. Show that $\text{opt}(MCF) = d$. **Hint.** Make sure you show both $\leq$ and $\geq$.

Proof. To prove $opt(MCF) \leq d$, we just need to prove that the shortest path is a solution to MCF. We set $f(e) = 1$ along all edges in the shortest path, since there is only one path with flow 1, The constraints are obviously satisfied. So it is a solution of MCF, and its value is $1 * d = d$, so $opt(MCF) \leq d$

To prove $opt(MCF) \geq d$, we need to prove that all solutions of MCF is not better than d. We try to improve the value of all possible solutions to d.

Suppose we have a solution with x different $s - t$ path. Define $b(path)$ be the smallest flow in all edges of $path$, $d(path)$ be the length of $path$. Let sp be the shortest $s - t$ path. We do as follows, choose any path p besides sp, put $b(p)$ units of flow on p to sp. Repeat it until there is only 1 unit flowing through sp.

We need to show in each turn, $val(MCF)$ is not worse than previous and no constraints are broke. We fist prove $val(MCF)$ is not worse. In each turn, $val(MCF)' = val(MCF) + d * b(p) - d(p) * b(p)$, $d \leq d(p)$, so $val(MCF)' \leq val(MCF)$. As for the constraints, inflow of t remains to be 1 since we just move $b(p)$ units between two different paths. Flow constraints remains since we modify the flow in one path, which means we move inflow and outflow of a single vertex at the same time. Since we only have x different $s-t$ path, and the flow value on each path is finite, the process terminates. So $val(MCF) \geq d$

In all $val(MCF) = d$ $\square$

Exercise 3. Write down the dual of MCF. This will be a maximization problem. Don't use any matrix notation.

Proof. We introduce a dual coefficient $g_v, v \in V$. The dual program is:

- Maximize g_t, subject to:
- $g_v - g_u \leq c(u,v), \forall (u,v) \in E$
- $g_v \in \mathbb{R}, v \in V$

$\square$

Exercise 4. Interpret the dual. Show that it is the LP formulation of a "natural" maximization problem on G.

Proof. If we set $g_s = 0$, then the g_v can be thought as the cost of some $s-t$-path. Since each edge (u,v) must satisfy $g_v - g_u \leq c(u,v))$, we can not just choose the maximal $s-t$-path as solution. Under this constraint, we can see that the solution must at first be a **safe** path, so the program is actually the shortest path problem. $\square$

Exercise 5. Describe an optimal solution of the dual program.

Proof. The optimal solution is the shortest $s-t$-path sp, and for each vertex along the path, we must set $g_v = g_u$ for $(u,v), u \in sp, v \notin sp$ accordingly to $g_v - g_u$ constraints. $\square$