Homework 5

Ji Jiabao

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Exer.1

Denote the original r-s-flow and s-t-flow as f_{rs} and f_{st} respectively, we use them to construct a new r-t-flow f_{rt}, thus prove the transitivity of flow.

Let

$$f_{rt} = V \times V \to \mathbb{R} = \begin{cases} f_{rs}(u,v) & f_{rs}(u,v) \neq 0 \\ f_{st}(u,v) & f_{st}(u,v) \neq 0 \\ 0 & otherwise \end{cases}$$

Here V is the set of all vertices.

We just need to prove f_{rt} is a flow.

- Capacity constraint:
  Case1: $f_{rs}(u,v) \neq 0$, then $f_{rt}(u,v) = f_{rs}(u,v) \leq c(u,v)$
  Case2: $f_{st}(u,v) \neq 0$, then $f_{rt}(u,v) = f_{st}(u,v) \leq c(u,v)$
  Case3: Otherwise, $f_{rt}(u,v) = 0 \leq c(u,v)$, since $c(u,v) \geq 0$

- Skew symmetry:
  Case1: $f_{rs}(u,v) \neq 0$, then $f_{rt}(u,v) = f_{rs}(u,v) = -f_{rs}(v,u) = -f_{rt}(v,u)$
  Case2: $f_{st}(u,v) \neq 0$, then $f_{rt}(u,v) = f_{st}(u,v) = -f_{st}(v,u) = -f_{rt}(v,u)$
  Case3: Otherwise, $f_{rs}(u,v) = 0, f_{st}(u,v) = 0 \Rightarrow f_{rt}(u,v) = 0$
  $f_{rs}(v,u) = -f_{rs}(u,v) = 0, f_{st}(v,u) = 0 \Rightarrow f_{rt}(v,u) = 0$
  $f_{rt}(u,v) = 0 = -f_{rt}(v,u))$

- Flow conservation:
  For all $u \in V/\{r,t\}$,

$$\begin{aligned}\sum_{v\in V} f_{rt}(u,v) &= \sum_{v\in V, f_{rs}(u,v)>0} f_{rt}(u,v) + \sum_{v\in V, f_{rs}(u,v)>0} f_{rt}(u,v) + \sum 0 \\ &= \sum f_{rs}(u,v) + \sum f_{st}(u,v) + 0 \\ &= 0+0+0 \\ &= 0\end{aligned}$$

Exer.2

First we prove the easy part, these two cases will not occur simultaneously.Prove it by contradiction.

Suppose both cases are true simultaneously, then there are k vertex disjoint paths $p_1, ..., p_k$ from s to t. and there exists $v_1, ..., v_{k-1}$ such that $G - \{v_1, ..., v_{k-1}\}$ contains no s-t-path.

For any $v_1, ..., v_{k-1}$, the can take place in at most $k-1$ paths in $p_1, ..., p_k$ since $p_1, ..., p_k$ are disjoint vertex paths. Then we know there must be at least one path from $p_1, .., p_k$ left, which connects s and t, leading to a contradiction.

Next we prove that one of them must occur.

The second case is obvious since we can just remove all vertices except s, t from the V, then obviously there're no s-t-path now.

For the first case, we construct such network-graph. Edges connecting with s, t are assigned with ∞ capacity so that it will not affect the inner flow. All other edges are assigned with capacity 1.

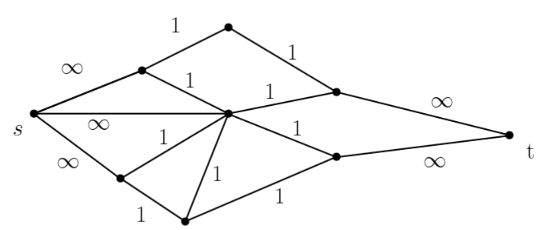

图 1: network

In such a network, the value of a flow is the number of vertex disjoint paths in such a flow. Since the capacity of each edge is 1, we can know for sure that no two s-t-path cross with each other, otherwise there must be a vertex with units bigger than 1. Based on that, we see k, the value of a flow is the number of paths in it. Thus leading to k disjoint paths.