CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2019

Handed out on Monday, 2019-10-21
First submission and questions due on Monday, 2019-10-28
You will receive feedback from the TA.
Final submission due on Monday, 2019-11-04

5 More on Network Flows

Exercise 1. Let $G = (V, c)$ be a flow network. Prove that flow is "transitive" in the following sense: if r, s, t are vertices, and there is an r–s-flow of value k and an s–t-flow of value k, then there is an r–t-flow of value k.

Proof. Denote the original r-s-flow and s-t-flow as f_{rs} and f_{st} respectively, we use them to construct a new r-t-flow f_{rt}, thus prove the transitivity of flow.

Let

$$f_{rt} = V \times V \to \mathbb{R} = \begin{cases} f_{rs}(u, v) & f_{rs}(u, v) \neq 0 \\ f_{st}(u, v) & f_{st}(u, v) \neq 0 \\ 0 & otherwise \end{cases}$$

Here V is the set of all vertices.

We just need to prove f_{rt} is a flow.

- Capacity constraint:
  Case1: $f_{rs}(u, v) \neq 0$, then $f_{rt}(u, v) = f_{rs}(u, v) \leq c(u, v)$
  Case2: $f_{st}(u, v) \neq 0$, then $f_{rt}(u, v) = f_{st}(u, v) \leq c(u, v)$
  Case3: Otherwise, $f_{rt}(u, v) = 0 \leq c(u, v)$, since $c(u, v) \geq 0$

- Skew symmetry:
  Case1: $f_{rs}(u,v) \neq 0$, then $f_{rt}(u,v) = f_{rs}(u,v) = -f_{rs}(v,u) = -f_{rt}(v,u)$
  Case2: $f_{st}(u,v) \neq 0$, then $f_{rt}(u,v) = f_{st}(u,v) = -f_{st}(v,u) = -f_{rt}(v,u)$
  Case3: Otherwise, $f_{rs}(u,v) = 0, f_{st}(u,v) = 0 \Rightarrow f_{rt}(u,v) = 0$
  $f_{rs}(v,u) = -f_{rs}(u,v) = 0, f_{st}(v,u) = 0 \Rightarrow f_{rt}(v,u) = 0$
  $f_{rt}(u,v) = 0 = -f_{rt}(v,u))$

- Flow conservation:
  For all $u \in V/\{r,t\}$,

$$\begin{aligned}
\sum_{v\in V} f_{rt}(u,v) &= \sum_{v\in V, f_{rs}(u,v)>0} f_{rt}(u,v) + \sum_{v\in V, f_{rs}(u,v)>0} f_{rt}(u,v) + \sum 0 \\
&= \sum f_{rs}(u,v) + \sum f_{st}(u,v) + 0 \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}$$

□

5.1 Vertex Disjoint Paths

Let G be a directed graph. Two paths p_1, p_2 from s to t are called *vertex disjoint* if they don't share any vertices except s and t.

Theorem 2 (Menger's Theorem)**.** *Let G be a graph and $s \neq t$ two vertices therein. Let $k \in \mathbf{N}_0$. Then exactly one of the following is true:*

1. *There are k vertex disjoint paths $p_1, \ldots, p_k$ from s to t; that is, no two p_i, p_j share any vertex besides s and t.*

2. *There are vertices $v_1, \ldots, v_k \in V \setminus \{s,t\}$ such that $G - \{v_1, \ldots, v_k\}$ contains no s–t-path.*

Exercise 3. Prove Menger's Theorem. You have to prove two things: first, not both cases above can occur (this is rather easy); second, one of them must occur (this requires a tool from the lecture).

Proof. First we prove the easy part, these two cases will not occur simultaneously.Prove it by contradiction.

Suppose both cases are true simultaneously, then there are k vertex disjoint paths $p_1, ..., p_k$ from s to t. and there exists $v_1, ..., v_{k-1}$ such that $G - \{v_1, ..., v_{k-1}\}$ contains no s-t-path.

For any $v_1, ..., v_{k-1}$, the can take place in at most $k-1$ paths in $p_1, ..., p_k$ since $p_1, ..., p_k$ are disjoint vertex paths. Then we know there must be at least one path from $p_1, .., p_k$ left, which connects s and t, leading to a contradiction.

Next we prove that one of them must occur.

The second case is obvious since we can just remove all vertices except s, t from the V, then obviously there're no s-t-path now.

For the first case, we construct such network-graph. Edges connecting with s, t are assigned with ∞ capacity so that it will not affect the inner flow. All other edges are assigned with capacity 1.

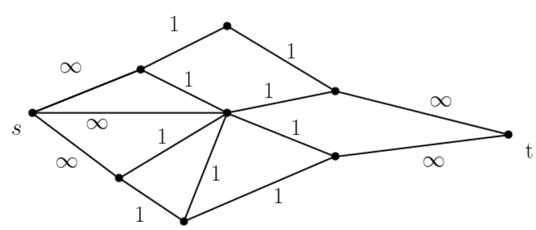


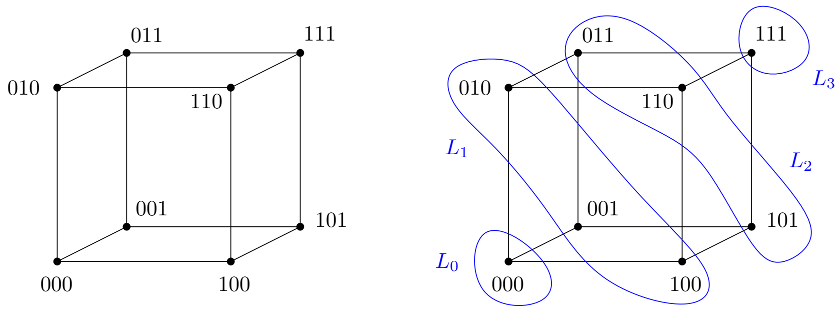
Figure 1: network

In such a network, the value of a flow is the number of vertex disjoint paths in such a flow. Since the capacity of each edge is 1, we can know for sure that no two s-t-path cross with each other, otherwise there must be a vertex with units bigger than 1. Based on that, we see k, the value of a flow is the number of paths in it. Thus leading to k disjoint paths. $\square$

Let $V = \{0, 1\}^n$. The n-dimensional Hamming cube H_n is the graph (V, E) where $\{u, v\} \in E$ if u, v differ in exactly one coordinate. Define the i^{th} level of H_n as

$$L_i := \{u \in V \mid \|u\|_1 = i\} \ ,$$

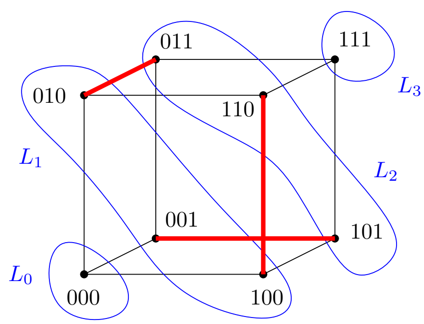
i.e., those vertices u having exactly i coordinates which are 1.



The 3-dimensional Hamming cube and the four sets L_0, L_1, L_2, L_3.

Exercise 4. [Matchings in H_n] Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$. This is the graph on vertex set $L_i \cup L_{i+1}$ where two edges are connected by an edge if and only if they are connected in H_n.

Show that for $i \leq n/2$ the graph $H_n[L_i \cup L_{i+1}]$ has a matching of size $|L_i| = \binom{n}{i}$.



A matching of size 3 between L_1 and L_2.

Proof. Consider $v \in L_i$, we know there are i bits to be 1. Let $a_1, a_2, \ldots, a_i$ be that set of integers or indexes such that $\forall j \leq i, v_{a_j} = 1$. That is, we use the indexes in which v has value 1 to represent the vertex.

Our goal is to construct a mapping $g : \{\mathbb{N} \times \ldots \times \mathbb{N}\} \to \mathbb{N}$, such that $\{v_1, \ldots, v_i, g(v)\} = \{\mathbf{v}, g(v)\} \in L_{i+1}$, plus $\forall u \neq v \in L_i$, we have

$$\{u, g(u)\} \neq \{v, g(v)\}.$$

Let $n = \prod_{i=1}^{n} p_i^{\alpha_i}$, consider the smallest prime $p \notin \{p_1, \ldots, p_n\}$, we construct the mapping g as following:

$$g(u) = (p-1)\left(\sum_{j=1}^{i} u_j\right) \bmod n.$$

We assume u maps to n if $g(u) = 0$. Now we prove this mapping is valid. Otherwise, there exists u, v they have at least 1 different elements, i.e.

$$i - |u \cap v| \geq 1.$$

Plus $\{u, g(u)\} = \{v, g(v)\}$. We must have

$$|u \cap v| = i - 2.$$

Assume $u = \{a_1, \dots, a_{i-1}, g(v)\}, v = \{a_1, \dots a_{i-1}, g(u)\}$. Hence

$$g(u) = (p-1)\left(\sum_{j=1}^{i-1} a_i + g(v)\right) \bmod n$$

$$g(v) = (p-1)\left(\sum_{j=1}^{i-1} a_i + g(u)\right) \bmod n.$$

Which leads that

$$p(g(u) - g(v)) = 0 \pmod{n}.$$

Hence

$$u = \{a_1, \dots, a_{i-1}, g(v)\} = \{a_1, \dots, a_{i-1}, g(u)\} = v.$$

It contradicts with the assumption that $u \neq v$, thus the mapping g is a valid one. And $H_n[L_i \cap L_{i+1}]$ has a matching! $\square$

Exercise 5. Let H_n be the n-dimensional Hamming cube. For $i < n/2$ consider L_i and L_{n-i}. Note that $|L_i| = \binom{n}{i} = \binom{n}{n-i} = L_{n-i}$, so the L_i and L_{n-i} have the same size. Show that there are $\binom{n}{i}$ paths $p_1, p_2, \dots, p_{\binom{n}{i}}$ in H_n such that (i) each p_i starts in L_i and ends in L_{n-i}; (ii) two different paths p_i, p_j do not share any vertices. **Hint 1.** Model this problem as a network flow with vertex capacities. What would the maximum flow be in this network? **Hint 2.** It's not *that* easy. If you try to work from both sides towards the middle by combining matchings between levels, you will certainly run into problems as how to glue things together in the middle. I have never seen any "meet in the middle" proof that works. **Hint 3.** There is a "direct" proof by induction that does not require anything about network flows.

Proof. It is well known that $\binom{i}{n} \leq \binom{i+1}{n} \leq \dots \leq \binom{\frac{n}{2}}{n}$. We construct a graph with $s, t, L_i, \dots, L_{n-i}$. While s connects every vertex in L_i, t connects every vertex in L_{n-i}.

And each edge connects $L_k, L_{k+1}, i \leq k \leq n-i$ has capacity 1. It's clear that the maxflow is less than $|L_i|$.

Plus if maxflow equals $|L_i|$ we are done. $\square$

5.2 Matchings and Vertex Covers

The following exercise was on the final exam of CS 499 (mathematical foundations of computer science) in spring 2019.

Exercise 6. Let $\nu(G)$ denote the size of a maximum matching of G. Show that a bipartite graph G has at most $2^{\nu(G)}$ minimum vertex covers.

Proof. Suppose the number of minumum vertex covers of grapg G is $m(G)$. We can easily Now we prove that there could be no more than $2^{\nu(G)}$ minimum

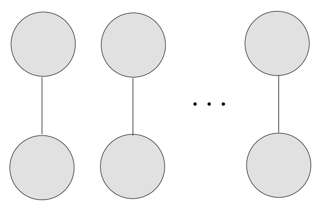


Figure 2: example

vertex cover. Let $G = A \cup B$, since G is a bipartite graph. Assume that the maximum matching is $e_1, e_2, \dots, e_t$, in which $t = \nu(G)$. By **Konig's Theorem** we know that the number of vertices in a minimum vertex cover is exactly t.

Plus we can prove that every vertex in K, the vertex cover, is an endpoint of a matched edge. Hence the result is obvious. $\square$

Obviously, this is not true for general (non-bipartite) graphs: the triangle K_3 has $\nu(K_3) = 1$ but it has three minimum vertex covers. The five-cycle C_5 has $\nu(C_5) = 2$ but has five minimum vertex covers.

Exercise 7. Is there a function $f : \mathbf{N}_0 \to \mathbf{N}_0$ such that every graph with $\nu(G) = k$ has at most $f(k)$ minimum vertex covers? How small a function f can you obtain?

Proof. Since this is for every graph. First consider K_{2r+1}, we have $\frac{k}{r}$ K_{2r+1}. Hence $\nu(G) = \frac{k}{r} \cdot r = k$

$$f(k) \geq (2r+1)^{\frac{k}{r}}.$$

For example, if we choose $r = 1$, there are k K_3, we have $f(k) \geq 3^k$. And let $r \to \infty$ we have

$$f(k) \geq e^{2k}.$$

Next we prove that the number of minimum vertex cover is at most e^{2k}. Consider $\forall$ minimum vertex cover K. We prove

$$|K| \leq e^{2k}.$$

Let $\forall u, v \in K$. If u, v are connected, then at least one of them is on a edge of maximum matching. $\square$