Homework 5

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2020 年 6 月 1 日

Exer.1

We just need to prove that $int-val(MLP(G)) = val(MLP(G))$, since $val(MLP(G)) \geq \nu(G)$, we do this following the proof for $VCLP(G)$ in class, the idea is to get int value solution through modifying a non-int value solution a bit.

Suppose we already have a solution $\mathbf{x}$, and denote $\sum_{e\in E, u\in e} x_e = y_u, \forall u \in V$, we want to change $\mathbf{x}$ to int value, and meanwhile $\mathbf{y}$ becomes int value.

At each step, those edges with $x_e = 0$ could just be ignored, while those edges with $x_e = 1$ and its both ends could be ignored. In the following situations, we just don't take these edges and vertices into account.

1. For two vertices u, v with non integral y_u, y_v, suppose there exits a path $e_1, e_2, ...e_m$, where m is odd. Let $d = min(1 - y_u, 1 - y_v, 1 - x_{e_1}, x_{e_2}, ..., 1 - x_{e_k})$. Next we modify edges as following. For those $e_i, iodd$, let $e_i = e_i + d$, and for those $e_i, ieven$, let $e_i = e_i - d$. By doing this, all constraints remains, and at least one of $y_u, y_v, x_{e_1}, ...x_{e_k}$ becomes integral. In all, $val(MLP(G))$ increases by d.

2. Similar to the situation above, except that the path has an even size. Let $d = min(1 - y_u, y_v, 1 - x_{e_1}, x_{e_2}, ..., 1 - x_{e_{k-1}}, x_{e_k})$. Next we modify edges as the above situation. By doing this, all constraints still remains, and at least one of $y_u, y_v, x_{e_1}, ...x_{e_k}$ becomes integral. In all, $val(MLP(G))$ remains unchanged.

3. If y_u is non integral but y_v integral, we just find all vertices connected with u, v. Denote the left side as L and the right side as R, We shall have $\sum_{u \in L} y_u = \sum_{u \in R} y_u$. But the left side is non integral while the right side is integral, leading to a contradiction.

4. If all y_u are integral, but some x_e may still remain non integral. To solve this, we find one of the non integral edge e_1, with u as one of its end. Since y_u integral, there exists another e_2 non integral connected to u. Continue this, we will find a cycle $e_1, ..., e_k$ with even size since G is bipartite. Let $d = min(1 - x_{e_1}, x_{e_2}, ...1 - x_{e_{k-1}}, x_{e_k}$, do the updates as the above updates. We can see that $\mathbf{y}$ remains, while at least one x_e becomes integral, and $val(MLP(G))$ remains.

In the above situations, at least one y_u or one x_e becomes integral. Since both $\mathbf{x}, \mathbf{y}$ are finite, this procedure will end. And since x_e becomes all integral, the original solution becomes an int value solution while not making $\sum_{e \in E} x_e$ worse.