Homework 5

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Exer.1

1. $\Leftarrow$: Suppose $\exists \mathbf{x} : A\mathbf{x} \leq \mathbf{b}$, Then $0 = 0 \cdot \mathbf{x} = (\mathbf{y}^T A)\mathbf{x} \leq \mathbf{y}^T \mathbf{b} < 0$, $0 < 0$, contradiction, so $\neg\exists \mathbf{x} : A\mathbf{x} \leq \mathbf{b}$
2. $\Leftarrow$: Similar to the proof in case (1), $0 \leq 0 \cdot \mathbf{x} \leq (\mathbf{y}^T A)\mathbf{x} \leq \mathbf{y}^T \mathbf{b} < 0$ $0 < 0$, contradiction
3. $\Leftarrow$: Similar to the proof above, $0 \leq 0 \cdot \mathbf{x} \leq (\mathbf{y}^T A)\mathbf{x} = \mathbf{y}^T \mathbf{b} < 0$, $0 < 0$, contradiction

Exer.2

To prove $opt(MCF) \leq d$, we just need to prove that the shortest path is a solution to MCF. We set $f(e) = 1$ along all edges in the shortest path, since there is only one path with flow 1, The constraints are obviously satisfied. So it is a solution of MCF, and its value is $1 * d = d$, so $opt(MCF) \leq d$

To prove $opt(MCF) \geq d$, we need to prove that all solutions of MCF is not better than d. We try to improve the value of all possible solutions to d.

Suppose we have a solution with x different $s - t$ path. Define $b(path)$ be the smallest flow in all edges of $path$, $d(path)$ be the length of $path$. Let sp be the shortest $s - t$ path. We do as follows, choose any path p besides sp, put $b(p)$ units of flow on p to sp. Repeat it until there is only 1 unit flowing through sp.

We need to show in each turn, $val(MCF)$ is not worse than previous and no constraints are broke. We fist prove $val(MCF)$ is not worse. In each turn, $val(MCF)' = val(MCF) + d * b(p) - d(p) * b(p),\ d \leq d(p)$, so $val(MCF)' \leq val(MCF)$. As for the constraints, inflow of t remains to be 1 since we just move $b(p)$ units between two different paths. Flow constraints remains since we modify the flow in one path, which means we move inflow and outflow of a single vertex at the same time. Since we only have x different $s - t$ path, and the flow value on each path is finite, the process terminates. So $val(MCF) \geq d$

In all $val(MCF) = d$

Exer.3

We introduce a dual coefficient $g_v, v \in V$. The dual program is:

- Maximize g_t, subject to:
- $g_v - g_u \leq c(u,v), \forall (u,v) \in E$
- $g_v \in \mathbb{R}, v \in V$

Exer.4

If we set $g_s = 0$, then the g_v can be thought as the cost of some $s - t$-path. Since each edge (u,v) must satisfy $g_v - g_u \leq c(u,v))$, we can not just choose the maximal $s - t$-path as solution. Under this constraint, we can see that the solution must at first be a **safe** path, so the program is actually the shortest path problem.

Exer.5

The optimal solution is the shortest $s - t$-path sp, and for each vertex along the path, we must set g_v for $(u,v), u \in sp, v \notin sp$ accordingly to $g_v - g_u$ constraints.