Problem 1 of Assignment 6

1. The Expected Utility $\mathbb{E}[U(x)]$ is given as:

$$\begin{aligned}
\mathbb{E}[U(x)] &= \mathbb{E}[x - \frac{\alpha x^2}{2}] \\
&= \mu - \frac{\alpha}{2}\mathbb{E}[x^2] \\
&= \mu - \frac{\alpha}{2}\mathbb{E}[x^2 \cdot e^{0\cdot x}] \\
&= \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)
\end{aligned} \tag{1}$$

The Certainty-Equivalent Value x_{CE} is given by solving:

$$\begin{aligned}
\mu - \frac{\alpha}{2}(\mu^2 + \sigma^2) &= x_{\text{CE}} - \frac{\alpha x_{\text{CE}}^2}{2} \\
\frac{\alpha x_{\text{CE}}^2}{2} - x_{\text{CE}} &= -\mu + \frac{\alpha}{2}(\mu^2 + \sigma^2) \\
\alpha x_{\text{CE}}^2 - 2x_{\text{CE}} &= -2\mu + \alpha(\mu^2 + \sigma^2) \\
\alpha x_{\text{CE}}^2 - 2x_{\text{CE}} + \frac{1}{\alpha} &= -2\mu + \alpha(\mu^2 + \sigma^2) + \frac{1}{\alpha} \\
(\sqrt{\alpha}x_{\text{CE}} - \frac{1}{\sqrt{\alpha}})^2 &= -2\mu + \alpha(\mu^2 + \sigma^2) + \frac{1}{\alpha} \\
\sqrt{\alpha}x_{\text{CE}} - \frac{1}{\sqrt{\alpha}} &= \pm\sqrt{-2\mu + \alpha(\mu^2 + \sigma^2) + \frac{1}{\alpha}} \\
\sqrt{\alpha}x_{\text{CE}} &= \pm\sqrt{-2\mu + \alpha(\mu^2 + \sigma^2) + \frac{1}{\alpha}} + \frac{1}{\sqrt{\alpha}} \\
x_{\text{CE}} &= \pm\frac{1}{\sqrt{\alpha}}\sqrt{-2\mu + \alpha(\mu^2 + \sigma^2) + \frac{1}{\alpha}} + \frac{1}{\alpha}
\end{aligned} \tag{2}$$

The Absolute Risk-Premium π_{A} is thus given by:

$$\begin{aligned}
\pi_{\text{A}} &= \mu - x_{\text{CE}} \\
&= \mu \pm \frac{1}{\sqrt{\alpha}}\sqrt{-2\mu + \alpha(\mu^2 + \sigma^2) + \frac{1}{\alpha}} - \frac{1}{\alpha}
\end{aligned} \tag{3}$$

If we have 1000000 and want to invest z in the risky asset, we will have the following wealth W in one year: $W \sim \mathcal{N}((1000000 - z)(1 + r) + z\mu, z^2\sigma^2)$. Thus, to find the optimal z^*, we need to minimize the following:

$$\mathbb{E}[U(x)] = (1000000 - z)(1 + r) + z\mu - \frac{\alpha}{2}(((1000000 - z)(1 + r) + z\mu)^2 + z^2\sigma^2) \tag{4}$$

We achieve this by take the derivative of $\mathbb{E}[U(x)]$ w.r.t. z. Therefore:

$$\begin{aligned}
-(1+r)+\mu-\alpha(((1000000-z^*)(1+r)+z^*\mu)(-(1+r)+\mu)+z^*\sigma^2)=0\\
\alpha(((1000000-z^*)(1+r)+z^*\mu)(-(1+r)+\mu)+z^*\sigma^2)=-(1+r)+\mu\\
((1000000-z^*)(1+r)+z^*\mu)(-(1+r)+\mu)+z^*\sigma^2=\frac{-(1+r)+\mu}{\alpha}\\
(1000000(1+r)+z^*(-(1+r)+\mu))(-(1+r)+\mu)+z^*\sigma^2=\frac{-(1+r)+\mu}{\alpha}\\
1000000(1+r)(-(1+r)+\mu)+z^*(-(1+r)+\mu)^2+z^*\sigma^2=\frac{-(1+r)+\mu}{\alpha}\\
z^*((-(1+r)+\mu)^2+\sigma^2)=1000000(1+r)((1+r)-\mu)+\frac{-(1+r)+\mu}{\alpha}\\
z^*=\frac{1000000(1+r)((1+r)-\mu)+\frac{-(1+r)+\mu}{\alpha}}{(-(1+r)+\mu)^2+\sigma^2}
\end{aligned} \tag{5}$$

We get the following plot for $r=0.02$, $\sigma=0.04$, $\mu=0.04$ and $\alpha\in[0.1,1]$: