# Simple Linear Price Impact Model Proof:

$$P_{t+1} = P_t - \alpha N_t + \varepsilon_t.$$

$$g_t(P_t, N_t) = \beta N_t$$

$$V_t^{\pi}((P_t, R_t)) = E\Big(\sum_{i=t}^{T-1} N_i (P_i - \beta N_i) \,\Big|\, (t, P_t, R_t)\Big).$$

$$V_t^{*}((P_t, R_t)) = \max_{\pi} V_t^{\pi}((P_t, R_t))$$

$$= \max_{N_t} \{ N_t P_t - \beta N_t^2 + E(V_{t+1}^{*}((P_{t+1}, R_{t+1}))) \}.$$

$$V_{T-1}^{*}((P_{T-1}, R_{T-1})) = R_{T-1} P_{T-1} - \beta R_{T-1}^2$$

$$V_{T-2}^{*}((P_{T-2}, R_{T-2})) = \max_{N_{T-2}} \{ N_{T-2} P_{T-2} - \beta N_{T-2}^2 + R_{T-1}(P_{T-2} - \alpha N_{T-2}) - \beta R_{T-1}^2 \}.$$

$$= \max_{N_{T-2}} \{ N_{T-2} P_{T-2} - \beta N_{T-2}^2 + (R_{T-2} - N_{T-2})(P_{T-2} - \alpha N_{T-2}) - \beta (R_{T-2} - N_{T-2})^2 \}.$$

$$= \max_{N_{T-2}} \{ N_{T-2} P_{T-2} - \beta N_{T-2}^2 + R_{T-2} P_{T-2} - \alpha R_{T-2} N_{T-2} + P_{T-2} N_{T-2} + \alpha N_{T-2}^2 - \beta (R_{T-2} - N_{T-2})^2 \}.$$

$$= \max_{N_{T-2}} \{ \underline{N_{T-2} P_{T-2}} \underline{- \beta N_{T-2}^2} + R_{T-2} P_{T-2} - \underline{\alpha R_{T-2} N_{T-2}} + \underline{P_{T-2} N_{T-2}} + \underline{\alpha N_{T-2}^2} - \beta R_{T-2}^2 + \underline{2\beta R_{T-2} N_{T-2}}$$
$$\underline{- \beta N_{T-2}^2} \}$$

$$= \max_{N_{T-2}} \{ (\alpha - 2\beta) N_{T-2}^2 - (\alpha - 2\beta) R_{T-2} N_{T-2} + R_{T-2} P_{T-2} - \beta R_{T-2}^2 \}$$

$$= \max_{N_{T-2}} \{ R_{T-2} P_{T-2} - \beta R_{T-2}^2 + (\alpha - 2\beta)(N_{T-2}^2 - R_{T-2} N_{T-2}) \}$$

$N_{T-2}^2 - R_{T-2} N_{T-2} \le 0$ since $N_{T-2} \le R_{T-2}$. Thus, if $\alpha - 2\beta \ge 0$ or $\alpha \ge 2\beta$, $N_{T-2}^{*} = 0$ or $R_{T-2}$.

If $\alpha - 2\beta < 0$, $N_{T-2}^{*} = \frac{R_{T-2}}{2}$. Thus, $V_{T-2}^{*}((P_{T-2}, R_{T-2})) = R_{T-2} P_{T-2} - \beta R_{T-2}^2 + (\alpha - 2\beta)\left(-\frac{R_{T-2}^2}{4}\right)$.

$$V_{T-2}^{*}((P_{T-2}, R_{T-2})) = R_{T-2} P_{T-2} - \underbrace{\frac{\alpha + 2\beta}{4}}_{//\ \beta'} R_{T-2}^2.$$

$$V_{T-3}^{*}((R_{T-3}, P_{T-3})) = \max_{N_{T-3}} \{ N_{T-3} P_{T-3} - \beta N_{T-3}^2 + R_{T-2}(P_{T-3} - \alpha N_{T-3}) - \frac{\alpha + 2\beta}{4} R_{T-2}^2 \}.$$

$$= \max_{N_{T-3}} \{ \underline{- \beta N_{T-3}^2} + R_{T-3} P_{T-3} \underline{- \alpha R_{T-3} N_{T-3}} + \underline{\alpha N_{T-3}^2} - \beta' R_{T-3}^2 \underline{- \beta' N_{T-3}^2} \underline{+ 2\beta' R_{T-3} N_{T-3}} \}$$

$$= \max_{N_{T-3}} \{ (\alpha - \beta - \beta') N_{T-3}^2 - (\alpha - 2\beta') R_{T-3} N_{T-3} + R_{T-3} P_{T-3} - \beta' R_{T-3}^2 \}.$$

$N_{T-3}^{*} = \frac{(\alpha - 2\beta') R_{T-3}}{2(\alpha - \beta - \beta')}$. $\alpha - 2\beta' = \alpha - \frac{\alpha + 2\beta}{2} = \frac{\alpha - 2\beta}{2}$. $\alpha - \beta - \beta' = \alpha - \beta - \frac{\alpha + 2\beta}{4} = \frac{3\alpha - 2\beta}{4}$

$N_{T-3}^{*} = \frac{\alpha - 2\beta}{3\alpha - 2\beta} R_{T-3} = \frac{R_{T-3}}{3}$ ? $\Rightarrow$ Office Hour!

$$V_{T-3}^{*}((R_{T-3}, P_{T-3})) = \frac{(\alpha - 2\beta')^2 R_{T-3}^2}{4(\alpha - \beta - \beta')} - \frac{(\alpha - 2\beta')^2 R_{T-3}^2}{2(\alpha - \beta - \beta')} + R_{T-3} P_{T-3} - \beta' R_{T-3}^2$$

$$= R_{T-3} P_{T-3} - R_{T-3}^2 \left( \beta' + \frac{(\alpha - 2\beta')^2}{4(\alpha - \beta - \beta')} \right)$$

$$\beta' + \frac{\alpha - 4\alpha\beta' + 4\beta'^2}{4(\alpha-\beta-\beta')} = \frac{4\alpha\beta' - 4\beta\beta' - 4\beta'^2 + \alpha - 4\alpha\beta' + 4\beta'^2}{4(\alpha-\beta-\beta')} = \frac{\alpha - 4\beta\beta'}{4(\alpha-\beta-\beta')} = \frac{\alpha - \beta(\alpha+2\beta)}{3\alpha - 2\beta} = \frac{\alpha - \alpha\beta - 2\beta^2}{3\alpha - 2\beta}.$$

Thus, $V^*_{T-3}((R_{T-3}, P_{T-3})) = R_{T-3}P_{T-3} - R^2_{T-3}\left(\frac{\alpha - \alpha\beta - 2\beta^2}{3\alpha - 2\beta}\right)$. $\Rightarrow$ Office Hour!