Problem 3 of Assignment 7

1. First we see that: $S_T \sim \mathcal{N}(S_t, \sigma^2 \cdot (T - t))$. To have an explicit expression for $V(t, S_t, W, I)$, we have the following:

$$
\begin{aligned}
V(t, S_t, W, I) &= \mathbb{E}[-e^{-\gamma \cdot (W + I \cdot S_T)}|(t, S_t)] \\
&= -e^{-\gamma \cdot W}\mathbb{E}[e^{-\gamma I \cdot S_T}|(t, S_t)] \\
&= -e^{-\gamma \cdot W}e^{-\gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2}} \\
&= -e^{-\gamma \cdot W - \gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2}}
\end{aligned}
\tag{1}
$$

To calculate the Indifference Bid Price $Q^{(b)}(t, S_t, I)$, we have the following:

$$V(t, S_t, W - Q^{(b)}(t, S_t, I), I + 1) = V(t, S_t, W, I)$$

$$-e^{-\gamma \cdot (W - Q^{(b)}(t, S_t, I)) - \gamma(I+1)S_t + \frac{\sigma^2 \gamma^2 (I+1)^2(T-t)}{2}} = -e^{-\gamma \cdot W - \gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2}}$$

$$e^{-\gamma \cdot (W - Q^{(b)}(t, S_t, I)) - \gamma(I+1)S_t + \frac{\sigma^2 \gamma^2 (I+1)^2(T-t)}{2}} = e^{-\gamma \cdot W - \gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2}}$$

$$-\gamma \cdot (W - Q^{(b)}(t, S_t, I)) - \gamma(I+1)S_t + \frac{\sigma^2 \gamma^2 (I+1)^2(T-t)}{2} = -\gamma \cdot W - \gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2}$$

$$\tag{2}$$

As a result, we have the following:

$$
\begin{aligned}
\gamma Q^{(b)}(t, S_t, I) &= \gamma(I+1)S_t - \frac{\sigma^2 \gamma^2 (I+1)^2(T-t)}{2} - \gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2} \\
Q^{(b)}(t, S_t, I) &= (I+1)S_t - \frac{\sigma^2 \gamma (I+1)^2(T-t)}{2} - I S_t + \frac{\sigma^2 \gamma I^2(T-t)}{2} \\
&= S_t - \frac{\sigma^2 \gamma (T-t)(2I+1)}{2}
\end{aligned}
\tag{3}
$$

To calculate the Indifference Ask Price $Q^{(a)}(t, S_t, I)$, we have the following:

$$V(t, S_t, W + Q^{(a)}(t, S_t, I), I - 1) = V(t, S_t, W, I)$$

$$-e^{-\gamma \cdot (W + Q^{(a)}(t, S_t, I)) - \gamma(I-1)S_t + \frac{\sigma^2 \gamma^2 (I-1)^2(T-t)}{2}} = -e^{-\gamma \cdot W - \gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2}}$$

$$e^{-\gamma \cdot (W + Q^{(a)}(t, S_t, I)) - \gamma(I-1)S_t + \frac{\sigma^2 \gamma^2 (I-1)^2(T-t)}{2}} = e^{-\gamma \cdot W - \gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2}}$$

$$-\gamma \cdot (W + Q^{(a)}(t, S_t, I)) - \gamma(I-1)S_t + \frac{\sigma^2 \gamma^2 (I-1)^2(T-t)}{2} = -\gamma \cdot W - \gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2}$$

$$\tag{4}$$

As a result, we have the following:

$$
\begin{aligned}
-\gamma Q^{(a)}(t, S_t, I) &= \gamma(I-1)S_t - \frac{\sigma^2 \gamma^2 (I-1)^2(T-t)}{2} - \gamma I S_t + \frac{\sigma^2 \gamma^2 I^2(T-t)}{2} \\
Q^{(a)}(t, S_t, I) &= -(I-1)S_t + \frac{\sigma^2 \gamma (I-1)^2(T-t)}{2} + I S_t - \frac{\sigma^2 \gamma I^2(T-t)}{2} \\
&= S_t - \frac{\sigma^2 \gamma (T-t)(-2I+1)}{2}
\end{aligned}
\tag{5}
$$