Theoretical Portion of Assignment 9

1. See code portion.

2.

$$
\begin{aligned}
V_t^\pi((P_t,R_t)) &= \mathbb{E}(\sum_{i=t}^{T-1} r_i|(t,P_t,R_t)) \\
&= \mathbb{E}(\sum_{i=t}^{T-1} N_iQ_i|(t,P_t,R_t)) \\
&= \mathbb{E}(\sum_{i=t}^{T-1} N_iP_i(1-\beta N_i-\theta X_i)|(t,P_t,R_t))
\end{aligned}
$$

$$
\begin{aligned}
V_t^*((P_t,R_t)) &= \max_\pi\{V_t^\pi((P_t,R_t))\} \\
&= \max_{N_t}\{N_tP_t(1-\beta N_t-\theta X_t)+\mathbb{E}(V_{t+1}^*((P_{t+1},R_{t+1})))\}
\end{aligned}
$$

Notice that: $V_{T-1}^*((P_{T-1},R_{T-1})) = R_{T-1}P_{T-1}(1-\beta R_{T-1}-\theta X_{T-1})$. Then, we have:

$$
V_{T-2}^*((P_{T-2},R_{T-2})) = \max_{N_{T-2}}\{N_{T-2}P_{T-2}(1-\beta N_{T-2}-\theta X_{T-2})+\mathbb{E}(V_{T-1}^*((P_{T-1},R_{T-1})))\}
$$

We first calculate $\mathbb{E}(V_{T-1}^*((P_{T-1},R_{T-1})))$:

$$
\begin{aligned}
V_{T-1}^*((P_{T-1},R_{T-1})) &= R_{T-1}P_{T-1}(1-\beta R_{T-1}-\theta X_{T-1}) \\
&= (R_{T-2}-N_{T-2})P_{T-1}(1-\beta(R_{T-2}-N_{T-2})-\theta(\rho X_{T-2}+\eta_{T-2})) \\
&= P_{T-1}((R_{T-2}-N_{T-2})-\beta(R_{T-2}-N_{T-2})^2-\theta(R_{T-2}-N_{T-2})(\rho X_{T-2}+\eta_{T-2}))
\end{aligned}
$$

Since Z_t and η_t are independent for all t, we have:

$$
\begin{aligned}
\mathbb{E}(V_{T-1}^*((P_{T-1},R_{T-1}))) &= P_{T-2}\mathbb{E}(e^{Z_{T-2}})((R_{T-2}-N_{T-2})-\beta(R_{T-2}-N_{T-2})^2-\theta\rho X_{T-2}(R_{T-2}-N_{T-2})) \\
&= P_{T-2}e^{\mu_Z+\frac{1}{2}\sigma_Z^2}((R_{T-2}-N_{T-2})-\beta(R_{T-2}-N_{T-2})^2-\theta\rho X_{T-2}(R_{T-2}-N_{T-2}))
\end{aligned}
$$

Using the moment generating function of normal distribution. Therefore, the optimal N_{T-2}, denoted N_{T-2}^* here, will satisfy:

$$
P_{T-2}(1-2\beta N_{T-2}^*-\theta X_{T-2})+P_{T-2}e^{\mu_Z+\frac{1}{2}\sigma_Z^2}(-1+2\beta(R_{T-2}-N_{T-2}^*)+\theta\rho X_{T-2})=0
$$

$$
(1-2\beta N_{T-2}^*-\theta X_{T-2})+e^{\mu_Z+\frac{1}{2}\sigma_Z^2}(-1+2\beta(R_{T-2}-N_{T-2}^*)+\theta\rho X_{T-2})=0
$$

$$
1-2\beta N_{T-2}^*-\theta X_{T-2}-e^{\mu_Z+\frac{1}{2}\sigma_Z^2}+2\beta e^{\mu_Z+\frac{1}{2}\sigma_Z^2}R_{T-2}-2\beta e^{\mu_Z+\frac{1}{2}\sigma_Z^2}N_{T-2}^*+\theta\rho e^{\mu_Z+\frac{1}{2}\sigma_Z^2}X_{T-2}=0
$$

$$
2\beta N_{T-2}^*+2\beta e^{\mu_Z+\frac{1}{2}\sigma_Z^2}N_{T-2}^*=1-\theta X_{T-2}-e^{\mu_Z+\frac{1}{2}\sigma_Z^2}+2\beta e^{\mu_Z+\frac{1}{2}\sigma_Z^2}R_{T-2}+\theta\rho e^{\mu_Z+\frac{1}{2}\sigma_Z^2}X_{T-2}
$$

$$
N_{T-2}^*=\frac{1-\theta X_{T-2}-e^{\mu_Z+\frac{1}{2}\sigma_Z^2}+2\beta e^{\mu_Z+\frac{1}{2}\sigma_Z^2}R_{T-2}+\theta\rho e^{\mu_Z+\frac{1}{2}\sigma_Z^2}X_{T-2}}{2\beta+2\beta e^{\mu_Z+\frac{1}{2}\sigma_Z^2}}
$$

Let $c=\frac{1-e^{\mu_Z+\frac{1}{2}\sigma_Z^2}}{2\beta+2\beta e^{\mu_Z+\frac{1}{2}\sigma_Z^2}}$, $A=\frac{\theta\rho e^{\mu_Z+\frac{1}{2}\sigma_Z^2}-\theta}{2\beta+2\beta e^{\mu_Z+\frac{1}{2}\sigma_Z^2}}$, $B=\frac{e^{\mu_Z+\frac{1}{2}\sigma_Z^2}}{1+e^{\mu_Z+\frac{1}{2}\sigma_Z^2}}$, we have:

$$
N_{T-2}^*=c+AX_{T-2}+BR_{T-2}
$$

While β cannot be 0.