Problem 1 of Assignment 10

1. First we see that: $S_T \sim \mathcal{N}(S_t, \sigma^2 \cdot (T-t))$. To have an explicit expression for $V(t, S_t, W, I)$, we have the following:

$$\begin{aligned} V(t,S_t,W,I) &= \mathbb{E}[-e^{-\gamma\cdot(W+I\cdot S_T)}|(t,S_t)] \\ &= -e^{-\gamma\cdot W}\mathbb{E}[e^{-\gamma I\cdot S_T}|(t,S_t)] \\ &= -e^{-\gamma\cdot W}e^{-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2}} \\ &= -e^{-\gamma\cdot W-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2}} \end{aligned} \tag{1}$$

To calculate the Indifference Bid Price $Q^{(b)}(t, S_t, I)$, we have the following:

$$\begin{aligned} V(t,S_t,W-Q^{(b)}(t,S_t,I),I+1) &= V(t,S_t,W,I) \\ -e^{-\gamma\cdot(W-Q^{(b)}(t,S_t,I))-\gamma(I+1)S_t+\frac{\sigma^2\gamma^2(I+1)^2(T-t)}{2}} &= -e^{-\gamma\cdot W-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2}} \\ e^{-\gamma\cdot(W-Q^{(b)}(t,S_t,I))-\gamma(I+1)S_t+\frac{\sigma^2\gamma^2(I+1)^2(T-t)}{2}} &= e^{-\gamma\cdot W-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2}} \\ -\gamma\cdot(W-Q^{(b)}(t,S_t,I))-\gamma(I+1)S_t+\frac{\sigma^2\gamma^2(I+1)^2(T-t)}{2} &= -\gamma\cdot W-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2} \end{aligned} \tag{2}$$

As a result, we have the following:

$$\begin{aligned} \gamma Q^{(b)}(t,S_t,I) &= \gamma(I+1)S_t-\frac{\sigma^2\gamma^2(I+1)^2(T-t)}{2}-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2} \\ Q^{(b)}(t,S_t,I) &= (I+1)S_t-\frac{\sigma^2\gamma(I+1)^2(T-t)}{2}-IS_t+\frac{\sigma^2\gamma I^2(T-t)}{2} \\ &= S_t-\frac{\sigma^2\gamma(T-t)(2I+1)}{2} \end{aligned} \tag{3}$$

To calculate the Indifference Ask Price $Q^{(a)}(t, S_t, I)$, we have the following:

$$\begin{aligned} V(t,S_t,W+Q^{(a)}(t,S_t,I),I-1) &= V(t,S_t,W,I) \\ -e^{-\gamma\cdot(W+Q^{(a)}(t,S_t,I))-\gamma(I-1)S_t+\frac{\sigma^2\gamma^2(I-1)^2(T-t)}{2}} &= -e^{-\gamma\cdot W-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2}} \\ e^{-\gamma\cdot(W+Q^{(a)}(t,S_t,I))-\gamma(I-1)S_t+\frac{\sigma^2\gamma^2(I-1)^2(T-t)}{2}} &= e^{-\gamma\cdot W-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2}} \\ -\gamma\cdot(W+Q^{(a)}(t,S_t,I))-\gamma(I-1)S_t+\frac{\sigma^2\gamma^2(I-1)^2(T-t)}{2} &= -\gamma\cdot W-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2} \end{aligned} \tag{4}$$

As a result, we have the following:

$$\begin{aligned} -\gamma Q^{(a)}(t,S_t,I) &= \gamma(I-1)S_t-\frac{\sigma^2\gamma^2(I-1)^2(T-t)}{2}-\gamma I S_t+\frac{\sigma^2\gamma^2 I^2(T-t)}{2} \\ Q^{(a)}(t,S_t,I) &= -(I-1)S_t+\frac{\sigma^2\gamma(I-1)^2(T-t)}{2}+IS_t-\frac{\sigma^2\gamma I^2(T-t)}{2} \\ &= S_t-\frac{\sigma^2\gamma(T-t)(-2I+1)}{2} \end{aligned} \tag{5}$$