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A MATH BEDTIME STORYBOOK

TEMPLATE AND FORMAT MADE BY

TUFTE-LATEX.GITHUB.IO/TUFTE-LATEX/

First printing, May 2025

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To my parents.

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Introduction

These are my personal notes in mathematics.

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Conventions

Set Theory

The Zermelo-Fraenkel axioms of set theory with the axiom of choice by default.

Alternative foundations with homotopy type theory and the univalence axiom may be considered when appropriate in the appropriate sections.

Universes may also be used.

Logic

Conjunction always refer to inclusive conjunction. So the word "or" means, x or y or both.

Category theory

We follow these conventions [1] by default.

[1] J. De Jong. *Stacks Project*. Columbia University, 2025

Definition 2.0.1. *A (small) category,* $\mathbf{C}$ *consists of a set of objects. For each pair of objects there exists a set of morphisms.*

Note that all italicised words can be changed to define enrichment, operads when necessarily.

Algebra

In algebra, a ring is a commutative unitary ring.

In representation theory, we may drop commutativity.

In analysis, we may drop the condition of unitary.

Notation

The natural integers refers to the positive integers, however this will be avoided.

3
Set Theory

Definitions

Zermelo-Fraenkel Axioms

This follows Jech [1].

[1] Thomas Jech. *Set Theory*. Springer-Verlag Berlin Heidelberg New York, 4th edition, 2006

Definition 3.0.1 (Pairing axiom). *For any two sets X and Y, then there exists a set, denote $\{X, Y\}$, where the set contains exactly $\{X, Y\}$.*

Definition 3.0.2 (Extensionality axiom). *If sets X and Y have the same elements, we define equality where the set X is equal to the set Y.*

Definition 3.0.3 (Union axiom). *The union over elements of a set exists.*

Definition 3.0.4 (Infinity axiom). *An infinite set exists.*

Definition 3.0.5 (Regularity axiom). *All nonempty sets have a membership minimal element.*

Definition 3.0.6 (Separation axiom schema). *If P is a property parameterised by p, then for any set X and parmaeter p, then there exists a set Y that has elements y in X that contains all elements y in X that has property P.*

Definition 3.0.7 (Powerset axiom). *For any set X, there exists the set of all subsets of X called the power set of X, and is denoted by $P(X)$.*

Definition 3.0.8 (Replacement axiom schema). *If a class F is a function, there for any set X, there exists a set called the function set with elements of the form $F(x)$ for an element x in set X, this set is denoted $F(X)$.*

Definition 3.0.9 (Strong choice axiom). *All families of nonempty sets have a choice function.*

Naive sets

This section is purely metamathematical and it is not rigorous.

Definition 3.0.10 (Naive set). *A naive set is a list of unique elements.*

Definition 3.0.11 (Naive class). *Suppose we have a formula $p(x, p_1, ...p_n)$.*

A naive class is a set C containing elements x such that x satisfies the formula $p(x, p_1, ...p_n)$.

Definition 3.0.12 (Naive membership). *If an element x in in a naive set X, we say x is a member in X, and it is denoted by $x \in X$.*

Definition 3.0.13 (Union of sets). *Let X and Y be naive sets. Then, there exists a set, called the union of X and Y, denoted $X \cup Y$, which contains the list of unique elements in either X or Y, or both.*

Lemma 3.0.14 (Union exists). *The union of naive sets exists.*

Proof. Use Axiom 3.0.3 on all the elements in a pair of naive sets X and Y. Use Axiom 3.0.4 for infinite sets. □

Definition 3.0.15 (Intersection of sets). *Let X and Y be naive sets. Then, there exists a set, called the intersection of X and Y, denoted $X \cap Y$, which contains the list of unique elements in X and in Y.*

Lemma 3.0.16 (Intersection exists). *The intersection of naive sets exists.*

Proof. Use Axiom 3.0.3 on all the elements in a pair of naive sets X and Y in common. Use Axiom 3.0.4 for infinite sets. □

Definition 3.0.17 (Complementation of sets). *Let X and Y be naive sets. Then, there exists a set, called the complementation of X and Y, denoted $X - Y$, which contains the list of unique elements in X but not in Y.*

Lemma 3.0.18 (Complementation exists). *The complementation of naive sets exists.*

Proof. Use Axiom 3.0.3 on all the elements in a pair of naive sets X and Y in common only considering elements in X but exclude elements that are in both X and Y. Use Axiom 3.0.4 for infinite sets.

□

Lemmas

We shall implicitly use without reference the axioms of set theory.

Lemma 3.0.19 (De Morgan's first law for pairs of naive sets)**.** *Let X and Y be a naive sets.*

Then the set $X^c \cup Y^c$ is equal to $(X \cap Y)^c$.

The rule applies for arbitrary unions of complements to the complement of arbitrary intersections.

Proof. We read the set $X^c \cup Y^c$ as consisting of the elements c such that it is either not in X or it is the elements not in Y, or both.

By case analysis, we take the elements that are not in both set X and set Y. These elements form precisely the set $(X \cap Y)^c$.

This proves the base case. Now consider the arbitrary union of the precedent case with the successive set complement. The law for the pairs applies again, and the successive case is shown. The proof follows by induction. □

Lemma 3.0.20 (De Morgan's second law for pairs of naive sets)**.** *Let X and Y be a naive sets.*

Then the set $X^c \cap Y^c$ is equal to $(X \cup Y)^c$.

The rule applies for arbitrary intersection of complements to the complement of arbitrary unions.

Proof. We read the set $X^c \cap Y^c$ as consisting of the elements c such that they are not in X and they are not elements not in Y.

By case analysis, we take the elements that are not in either set X or set Y or both. These elements form precisely the set $(X \cup Y)^c$.

This proves the base case. Now consider the arbitrary intersection of the precedent case with the successive set complement. The law for the pairs applies again, and the successive case is shown. The proof follows by induction. □

4
Categories

Definitions

Definition 4.0.1 (Set of morphisms)**.** *A set of morphisms between objects X, Y are maps from X to Y. These are denoted* $\mathrm{Hom}(X, Y)$ *and are also called hom-sets.*

Definition 4.0.2 (Composition maps)**.** *A composition map for objects X, Y, Z where is a map from a cartesian product of hom-sets to a hom-set* $\cdot : \mathrm{Hom}(Z, Y) \times \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Z, X)$.

Definition 4.0.3 (Category)**.** *A category* $\mathbf{C}$ *has a set of objects, denoted* $\mathrm{Ob}(X)$ *or with objects X.*

It has a set of morphisms between objects X, Y denoted $\mathrm{Hom}(X, Y)$.

It has a composition map for objects X, Y, Z where $\cdot : \mathrm{Hom}(Z, Y) \times \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Z, X)$ *such that for morphism p in* $\mathrm{Hom}(Y, X)$ *and morphism q in* $\mathrm{Hom}(Z, Y)$ *we have a morphism $q \cdot p$ in the set of morphisms* $\mathrm{Hom}(Z, X)$.

These satisfy these rules:

1. *For every object X in the set of objects* $\mathrm{Ob}(X)$*, there exists an identity morphism $i \in Hom_{\mathbf{C}}(X, X)$ such that it composes with morphisms p and q where $p = i \cdot p$ and $q \cdot i = q$.*

2. *The composition of morphism is associative where $p \cdot (q \cdot r) = (p \cdot q) \cdot r$.*

Definition 4.0.4 (Covariant functor)**.** *A functor category* $\mathbf{FC}$ *has a set of objects, denoted* $\mathrm{Ob}(\mathbf{F}X)$ *or with objects FX.*

It has a set of morphisms between objects $\mathbf{F}X, \mathbf{F}Y$ denoted $\mathrm{Hom}(\mathbf{F}X, \mathbf{F}Y)$.

It has a composition map for objects $\mathbf{F}X$, $\mathbf{F}Y$, $\mathbf{F}Z$.

This map is such that $\cdot : \mathrm{Hom}(\mathbf{F}Z, \mathbf{F}Y) \times \mathrm{Hom}(\mathbf{F}Y, \mathbf{F}X) \to \mathrm{Hom}(\mathbf{F}Z, \mathbf{F}X)$.

This is such that each morphism Fp in $\mathrm{Hom}(\mathbf{F}Y, \mathbf{F}X)$ *and morphism $\mathbf{F}q$ in* $\mathrm{Hom}(\mathbf{F}Z, \mathbf{F}Y)$ *we have a morphism $\mathbf{F}q \cdot \mathbf{F}p$ in the set of morphisms* $\mathrm{Hom}(\mathbf{F}Z, \mathbf{F}X)$.

These satisfy these rules:

1. *For every object* $\mathbf{F}X$ *in the set of objects* $\mathrm{Ob}(\mathbf{F}X)$*, there exists an identity morphism* $i \in Hom_{\mathbf{FC}}(\mathbf{F}X, \mathbf{F}X)$ *such that it composes with morphisms* $\mathbf{F}p$ *and* $\mathbf{F}q$ *where* $\mathbf{F}p = \mathbf{F}i \cdot \mathbf{F}p$ *and* $Fq \cdot \mathbf{F}i = \mathbf{F}q$*.*

2. *The composition of morphisms is associative where* $\mathbf{F}p \cdot (\mathbf{F}q \cdot \mathbf{F}r) = (\mathbf{F}p \cdot \mathbf{F}q) \cdot \mathbf{F}r$*.*

3. *The composition keeps arrows so* $\mathbf{F}(p \cdot q) = \mathbf{F}p \cdot \mathbf{F}q$*. We abused notation for composition here.*

A covariant functor $\mathbf{F}$ *takes a category* $\mathbf{C}$ *to the functor category* $\mathbf{FC}$ *satisfying the above rules.*

Definition 4.0.5 (Contravariant functor)**.** *A functor category* $\mathbf{FC}$ *has a set of objects, denoted* $\mathrm{Ob}(\mathbf{F}X)$ *or with objects* FX*.*

It has a set of morphisms between objects $\mathbf{F}X, \mathbf{F}Y$ *denoted* $\mathrm{Hom}(\mathbf{F}X, \mathbf{F}Y)$*.*

It has a composition map for objects $\mathbf{F}X$*,* $\mathbf{F}Y$*,* $\mathbf{F}Z$*.*

This map is such that $\cdot : \mathrm{Hom}(\mathbf{F}Z, \mathbf{F}Y) \times \mathrm{Hom}(\mathbf{F}Y, \mathbf{F}X) \to \mathrm{Hom}(\mathbf{F}Z, \mathbf{F}X)$*.*

This is such that each morphism $\mathbf{F}p$ *in* $\mathrm{Hom}(\mathbf{F}Y, \mathbf{F}X)$ *and morphism* $\mathbf{F}q$ *in* $\mathrm{Hom}(\mathbf{F}Z, \mathbf{F}Y)$ *we have a morphism* $\mathbf{F}q \cdot \mathbf{F}p$ *in the set of morphisms* $\mathrm{Hom}(\mathbf{F}Z, \mathbf{F}X)$*.*

These satisfy these rules:

1. *For every object* $\mathbf{F}X$ *in the set of objects* $\mathrm{Ob}(\mathbf{F}X)$*, there exists an identity morphism* $i \in Hom_{\mathbf{FC}}(\mathbf{F}X, \mathbf{F}X)$ *such that it composes with morphisms* $\mathbf{F}p$ *and* $\mathbf{F}q$ *where* $\mathbf{F}p = \mathbf{F}i \cdot \mathbf{F}p$ *and* $\mathbf{F}q \cdot \mathbf{F}i = \mathbf{F}q$*.*

2. *The composition of morphisms is associative where* $\mathbf{F}p \cdot (\mathbf{F}q \cdot \mathbf{F}r) = (\mathbf{F}p \cdot \mathbf{F}q) \cdot \mathbf{F}r$*.*

3. *The composition reverses arrows so* $\mathbf{F}(p \cdot q) = \mathbf{F}q \cdot \mathbf{F}p$*. We abused notation for composition here.*

A functor $\mathbf{F}$ *takes a category* $\mathbf{C}$ *to the functor category* $\mathbf{FC}$ *satisfying the above rules.*

Definition 4.0.6 (Opposite category)**.** *A category* $\mathbf{C}$ *has a set of objects, denoted* $\mathrm{Ob}(X)$ *or with objects* X*.*

It has a set of morphisms between objects X, Y *denoted* $\mathrm{Hom}(X, Y)$*.*

The opposite category, denoted $\mathbf{C}^{\mathrm{op}}$ *is a category with the hom-sets of* $\mathrm{Hom}(Y, X)$ *satisfying the definition of a category in Definition 4.0.3.*

Exercises

Exercise 4.0.7 (Opposite of a opposite category)**.** *Show that the opposite category of a opposite category is naturally isomorphic to the original category of the opposite category.*

Proof. The functor taking hom-sets from $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(Y, X) \to \mathrm{Hom}(X, Y)$.

This is in the definition of a opposite category as per Definition 4.0.6 is a canonical identity natural isomorphism. [1] $\square$

[1] The double dual functor yield a natural isomorphism similar to the double dual of a vector space is the motivating example for category theory.

5
Topology

Definitions

We follow Steen and Seebach's presentation of topology [1].

Definition 5.0.1 (Topology of open sets)**.** *For a set X, a collection of open subsets of the set X is called a topology, denoted by τ if arbitrary unions and finite intersections of each subset is in τ.*

Definition 5.0.2 (Topology of closed sets)**.** *For a set X, a collection of closed subsets of the set X is called a topology, denoted by τ if arbitrary intersections and finite unions of each subset is in τ.*

Definition 5.0.3 (Open set)**.** *An open set is a set U in a topology τ of a set X.*

Definition 5.0.4 (Closed set)**.** *An closed set S is the complement of an open set U of a topology τ with respect to the main set X.*

Lemma 5.0.5 (Topology of closed sets)**.** *The closed sets $(X - U)^c$ of a topological space (X, τ) form a topology τ given the open sets U in the topology τ.*

Proof. The complement of the entire space X relative to finite intersection of open sets U is a arbitrary union of closed sets, this follows by De Morgan's laws or Lemma 3.0.19 and Lemma 3.0.20. The complement of the entire space X relative to arbitrary union of open sets U is a finite intersection of closed sets, this follows by De Morgan's laws. □

Definition 5.0.6 (Topological space)**.** *For a set X, a collection of subsets of the set X is called a topology, denoted by τ if arbitrary unions and finite intersections of each subset is in τ. A pair (X, τ) is a topological space. By abuse of notation, we call X a topological space.*

[1] Lynn Arthur Steen and Jr. J. Arthur Seebach. *Counterexamples in Topology*. Dover, paperback edition, 1995

Definition 5.0.7 (Coarser). *Suppose τ_1 and τ_2 are topologies for a set X. Recall that these are collections of subsets.*[2]

If the set τ_1 is contained in the set τ_2, we say the topology τ_1 is coarser than τ_2.

[2] Topologies may not be comparable.

Definition 5.0.8 (Finer). *Suppose τ_1 and τ_2 are topologies for a set X. Recall that these are collections of subsets.*[3]

If the set τ_1 is contained in the set τ_2, we say the topology τ_2 is coarser than τ_1.

[3] Topologies may not be comparable.

Definition 5.0.9 (Neighbourhood). *Suppose τ is a topology for a set X. Let p be a point in the set X. A neighbourhood of a point p in the set X is a subset of an open set U in the topology τ containing the point p.*

Definition 5.0.10 (Open neighbourhood). *Suppose τ is a topology for a set X. Let p be a point in the set X. A open neighbourhood of a point p in the set X is a open subset of an open set U in the topology τ containing the point p.*

Definition 5.0.11 (Limit point). *Suppose τ is a topology for a set X. Let p be a point in the set X. A limit point p in the set X is a point such that it is in every open set contains p and one distinct point that is not p.* [4]

[4] The motivation is that of a limit point of a sequence.

Definition 5.0.12 (Adherent point). *Suppose τ is a topology for a set X. Let p be a point in the set X. A adherent point p in the set X is a point such that it is in every open set contains p and one other point that may be equal to the point p.*

Definition 5.0.13 (ω-accumulation point). *Suppose τ is a topology for a set X. Let p be a point in the set X. A ω-accumulation point p in the set X is a point such that it is in every open set contains p and infinitely many points that is not the point p.*

Definition 5.0.14 (Condensation point). *Suppose τ is a topology for a set X. Let p be a point in the set X. A condensation point p in the set X is a point such that it is in every open set contains p and uncountably infinitely many points that is not the point p.*

Definition 5.0.15 (Derived set). *Suppose τ is a topology for a set X. A derived set $D(A)$ of a set A which is a subset of the set X under the topology τ is the collection of all the limit points of the subset A.*

Definition 5.0.16 (Isolated point). *Suppose τ is a topology for a set X. A derived set $D(A)$ of a set A which is a subset of the set X under the topology τ is the collection of all the limit points of the subset A.*

An isolated point is a point in the subset A that is not in the derived set $D(A)$.

Definition 5.0.17 (Dense in itself). *Suppose τ is a topology for a set X. A derived set $D(A)$ of a set A which is a subset of the set X under the topology τ is the collection of all the limit points of the subset A.*

An isolated point is a point in the subset A that is not in the derived set $D(A)$.

A set without any isolated point is a set that is dense in itself.

Definition 5.0.18 (Perfect set). *Suppose τ is a topology for a set X. A derived set $D(A)$ of a set A which is a subset of the set X under the topology τ is the collection of all the limit points of the subset A.*

An isolated point is a point in the subset A that is not in the derived set $D(A)$.

A closed set without any isolated point is a set that is dense in itself. This is defined to be a perfect set.

Definition 5.0.19 (Closure set). *Suppose τ is a topology for a set X. The closure of a set is a set together with its limit points.*

Examples

Example 5.0.20 (Indiscrete topology). *Let X be a set. The topology consisting of the empty set and the set X is the indiscrete topology.*

Remark 5.0.21 (Indiscrete topology remarks). *1. This topology is comparable to every other topology.*

Example 5.0.22 (Discrete topology). *Let X be a set. The topology consisting of all subsets of the set X is the discrete topology.*

Example 5.0.23 (Finite particular point topology). *Let X be a finite set. Let p be a point in the finite set X. Consider the topology τ consisting of arbitrary unions and finite intersection of subsets of the finite set containing the point p.*

Example 5.0.24 (Countably infinite particular point topology). *Let X be a countably inffinite set. Let p be a point in the countably infinite set X. Consider the topology τ consisting of arbitrary unions and finite intersection of subsets of the finite set containing the point p.*

Example 5.0.25 (Uncountably infinite particular point topology). *Let X be a uncountably infinite set. Let p be a point in the uncountably infinite set X. Consider the topology τ consisting of arbitrary unions and finite intersection of subsets of the finite set containing the point p.*

Example 5.0.26 (Zariski topology). *Let X be the collection of algebraic sets. An algebraic set is the locus of zeros of polynomials. The Zariski topology is the topology formed by the collection of algebraic sets of a space as the closed sets.*[5]

[5] This is not fully defined since we did not say what an algebraic set really is. And we need to prove this.

Exercises

Exercise 5.0.27. *Show that not all topologies are comparable.*

Proof. Consider a set of two elements, with the finite particular point topology on each point as per Example 5.0.22. The topologies generated are not comparable. □

Exercise 5.0.28. *Show that a subset in a topology can be open and closed.*

Proof. Consider an two point set with the indiscrete topology as per Example 5.0.19. □

Exercise 5.0.29. *Show that a topology can be comparable to every other topology.*

Proof. Consider a set with the indiscrete topology as per Example 5.0.19. It is coarser than every other topology.

By definition, a topology must contain the indiscrete topology which consists of arbitrary unions of all subsets which is the full set, as well as finite intersections of all subsets which is the empty set for disjoint sets. □

Exercise 5.0.30 (Motivation for isolated point). *Show that an isolated point is a point contained in an open set with no other point of a subset A in a topological space (X, τ).*

Proof. By Definition 5.0.16, an open set containing an isolated point does not contain a limit point. Since it does not contain a limit point, there is no point whose open neighbourhood in the subset containing the isolated point as a distinct point. Therefore, the open set containing the isolated point must no other point in open set since by case analysis either the isolated point is a limit point in the open set which is not true, or there is a limit point in the open set which is not true. □

Exercise 5.0.31 (Closed set contains all of its limit points). *Show that a closed set contains all of its limit points.*

Proof. The complement of the full set and its limit point is a open set by the definition of a closed set or Defintion 5.0.4, and every point in a closed set satisfies this condition.

Therefore, a closed set contains all of its limit points. □

Exercise 5.0.32 (Perfect set equal to derived set)**.** *Show that a set is perfect if and only if it is equal to its derived set.*

Proof. Suppose a set is perfect. A closed set contains all of its limit points. This is because the complement of the full set and its limit point is a open set by the definition of a closed set or Defintion 5.0.4, and every point in a closed set satisfies this condition. This complement, denoted by $A - A$ for closed set A and full set X is open, containing limit point x and no points of A. It is therefore a derived set.

A set containing all its limit points is closed since $X - A$ contains a neighbourhood of each of its limit points, and hence it is open. This is a perfect set, as this is a closed set without any isolated points.

Therefore, a set is perfect if and only if it is equal to its derived set. □

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Thomas Jech. *Set Theory*. Springer-Verlag Berlin Heidelberg New York, 4th edition, 2006.

J. De Jong. *Stacks Project*. Columbia University, 2025.

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