Full, faithful, essential surjective

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Definition 0.1. A functor between small categories is a homomorphism of underlying graphs that respects the compositions of objects, consisting of a component function of classes of objects and sets of morphisms such that for each pair of objects, it has a component function between hom-sets respecting the source and target of morphisms with coincident source and target objects, respects the identity morphisms and respect composition of images.

Definition 0.2. A functor is full if for each pair of objects, the function between hom-sets is surjective.

Definition 0.3. A functor is faithful if for each pair of objects, the function between hom-sets is injective.

Definition 0.4. A functor is essentially surjective if there merely exists an isomorphism between the functor on a object in the source category to an object in the target category.

Why the distinctions?

Example 0.1. An essentially surjective functor is additionally fully faithful precisely when it is an equivalence of categories. This serves a good test for equivalence of categories.

What about infinity categories?

Definition 0.5. A functor of infinity categories (in Lurie's construction, I copied this from Kerodon) is fully faithful if for every pair of objects in the infinity category, the induced map of morphism spaces is a homotopy equivalence of Kan complexes.

Exercise 0.1. Compare the definition of fully faithful to that in normal spaces to better understand what is important in infinity categories.