Dualities, adjunctions, and representing objects

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Category theory is the mathematics of metaphor. It is unique in the sense that unlike other disciplines of analogy, dualities in category theory allow for perfect metaphors in the form of (natural) isomorphisms.

Exercise 0.1. Work out the adjunction between the floor and ceiling functions on the real line. Related this to the duality between open sets and closed sets. How is this duality by a suitable isomorphism? (Hint: double complementation)

Example 0.1. The Isbell adjunction or the Isbell duality give a precise duality between space and quantity. Alternatively, it is the duality between geometry and algebra.

Consider a smooth space and its colimit (think of the colimit naively as some sort of sum).

The left adjunction taking smooth spaces to smooth ring is also known as a *copresheafification*.

The right adjunction where we have a smooth algebra and then we consider its spectrum or *presheafification* which gives something like a smooth space. Note that you will need a full sheafification to get the smooth space.

Another basic example would be the homogenous nullstellensatz, where there is a duality between the projective varieties in complex projectiv space of order n to the homogenous ideals of a coordinate ring except for the irrelevant ideal.

Dimension theory also has this duality in definitions. The Krull dimension is dual to the depth of a module.

Exercise 0.2. What is the initial object of commutative rings? Argue by duality, from the previous question to determine the terminal object of affine schemes.

Isbell duality is interesting because it highlights an approach one can take to mathematical thinking. One can first build a duality that is obvious, in this case it is the duality between classical affine varieties and nilpotent free finitely generated algebras over a field. Then, slowly relax all of these assumptions (nilpotents exist, finitely generated, field axioms) to get schemes. The idea is to start with some duality, drop as many assumptions as you can, and see how much duality will still hold. This was Grothendieck's approach to algebraic geometry.

Example 0.2. Verdier duality is one part of the formalism of six functors.

There is the adjunction between the direct image and the inverse image for morphisms.

There is also the adjunction between the direct image with compact support with exceptional inverse image for separated morphisms. This is the Verdier duality.

Lastly, there is the adjunction between the symmetric monoidal tensor product and the internal hom.

This also motivates the philosophy of invertible sheaves. Varieties can be looked as classically as topological spaces instead of varieties as functions of them. Similarly, we can study line bundles instead using the sheaf of sections.

Exercise 0.3. Describe Verdier duality as a form of generalised Poincare duality through the existence of an adjoint to the derived pushforward.

Example 0.3. Pontrjagin duality is exemplified with this easy example.

Consider the group of integers under addition and compare it to the circle group. The product of roots of unity if one consider the circle group gives rise to Fourier theory.

Since the circle group is compact and connected, by Pontrjagin duality, the integers must therefore be discrete and torsion free.

The exponential map also gives the self duality of the additive rules as the sum of powers of exponentials. This gives a duality between abelian discrete groups like additivity on the integers to compact commutative topological groups like the circle groups. One can make the construction of the Haar measure make sense, and associate it with this group.

Van-Kampen's theorem is a statement on how the amalgamated free product of the fundamental group is the wedge sum of two path connected topological spaces.

Example 0.4. The Eckmann-Hinton duality refers to how diaagrams for some concept cane be reversed.

This is similar to how one can define the opposite category for category theory.

A similar argument is used for colimits to turn the Eilenberg-Steenrod axioms for homology to give axioms for cohomology.

This also gives the adjunction functor between the reduced suspension which is left adjoint to the loop space, which is the right adjoint.

Homotopy groups can be related to homotopy classes of maps from the n-sphere to our space, we have $\pi_n(X, p) \cong \langle S^n, X \rangle$. The sphere has a single nonzero (reduced) cohomology group.

Cohomology groups are homotopy classes of maps to spaces with a single nonzero homotopy group. This is given by the Eilenberg-Maclane spaces $K(G, n)$ and the relation $H^n(X; G) \cong \langle X, K(G, n) \rangle$. This is an example of Fuks duality. One can apply this to have a homotopy to be dual to cohomology, mapping cylinder to mapping cocylinder, fibrations to cofibrations.

Similarly, I think Lagrange duality can be classified under this. The constraint problem is dual to the abundance problem, the minimisation problem is dual to the maximisation problem. This is related to the formal duality of vector spaces in optimisation problems. Variables in the primal problem is complemnetary to constraints in the dual problem. There cannot be a slack on both the constraint and the corresponding dual variable.

In homological algebra, there is the tensor-hom adjunction and its derived counterpart, the Tor-Ext adjunction. The standard counter example for hom-sets over the module $\mathbf{Z}/2$. Ext must be a left exact functor, Tor must be a right exact functor.

Example 0.5. Subgroups are dual to quotient groups. In a quotient group, similar elements are now made equivalent (if they are in the same subgroup) by an equivalence relation.

Example 0.6. The Baire category theorem is a result that relates the qualitative theory of continuous linear operators and the quantitative theory of estimates.

The Riesz representation are related to representable functors as such. Let F be a linear functional in complex Hilbert space H. If F is continuous, then there exists an unique a in complex Hilbert space H where it is equal to the inner product function $\langle a, - \rangle$ where $F(v) = \langle a, v \rangle$.

For example, continuous linear functions on square integrable functions are representable by the integral inner product $\int a(x)v(x)$. Generally, general continuous functions are representable by test functions. This is distribution theory. The analogy is much deeper, this is a sheaf! It is easier to spot sheaves when you want sections of something to exist. We have test functions as representing objects. The exists in a "sheaf" in some sense, as global sections that represent general continuous functionals.

Example 0.7. A monoid and its module category are dual. This is called Tannaka duality. Consider automorphisms / endomorphisms of some forgetful functor called the fibre functor. This fibre functor gives the monoid structure. Apply Yoneda's lemma, using the fact that the fibre functor is an endofunctor. This is an example of a representable functor. This corresponds to the fact that modules are representations ove ring.

Representation theory is a basic example of this duality. For example, one can consider maps between an object to its algebra. For example, the representation of discrete groups can be thought of as modules over the group ring. Let G be the discrete group or the group ring, and $GL(V)$. Representations or simply maps from $GtoGL(V)$ are endomorphisms of a vector space.

A proof sketch can be done for G-sets as an exercise in a similar fashion to Cayley's theorem.

Example 0.8. The syntax of first order logic (pretopoi) corresponds to the semantics of first order logic (ultracategories). This is known as Makkai duality.

Example 0.9. One can generalise the fact that $dr = 0$ that encodes a commutative law as well as $d^2\omega = 0$. For some long sequences, one can compare the conversion of a free graded commutative algebra on the special linear group SL into a differential grade algebra to be analogous to making it into a Lie algebra by setting the Jacobi identity to vanish. This duality means that the free graded Lie algebra on the special linear group into a differential graded algebra is also a commutative algebra, when you have $d^2 = 0$ to encode a commutative law. This duality is also known as Koszul duality.

Example 0.10. Stone duality refers to the duality between a totally disconnected compact Hausdorff space (Stone spaces) to the Boolean algebras by considering the clopen sets of the Stone space. The set of prime filters for a bounded distributed lattice (a poset admitting all finite meets and joins, or a thin category with all finite limits and finite colimits) is a Stone space. There exists a map from the Boolean algebras to the clopen sets of the prime filters of a Boolean algebras which can be made into an isomorphism of Boolean algebras. This is Stone's representability theorem. Now, weaken complementation to turn a Boolean algebra into a Heyting algebra. Define Heyting spaces appropriately, and establish the isomorphisms to complete Heyting duality.

This is an example of how a suitable isomorphism can be used to find a duality. Further, the poset lattice correspond to topology. The poset lattice interpreted as a form of logic. This corresponds to topology being defined as a semidecidable logic, where arbitrary unions correspond decidable disjunctions, finite intersections correspond to decidable conjunctions in finite time.

Further, one can relate profinite sets as pro objects in the category of finite sets, to be equivalent to compact Hausdorff totally disconnected spaces. From Stone duality, compactness can be thought of as the case where every limit point is a point, or any decidable proposition is computable within finite time. Countability becomes important in a topology, where countably many experiments can be used to determine a result, corresponding to how every limit point being a limit point with some countable subset. This motivates separability. For me, I use the typical example of countable dense balls of rational radii that cover the real line, then replace balls with neighbourhoods for the general topological notion of countability.

Example 0.11. The decomposition of the linear representation of the direct product group of the general linear group on a field k with the symmetric group of size n can be decomposed as a direct sum of tensor product of irreducible representation for either group. This is Schur-Weyl duality.

Consider the field of complex numbers. An example would be to suppose the number of factors to be 2, then the space of two tensors decomposes into symmetric and antisymmetric part, each is a irreducible module for the general linear group of size n. This is because the symmetric group on two elements consists of two elements, the trivial representation and the sign representation. The trivial representation give rise to symmetric tensors that do not change under factor permutation, the sign representation corresponds to skew symmetric tensors that flip the sign.

$$\mathbf{C}^n \otimes \mathbf{C}^n = S^2\mathbf{C}^n \oplus \Lambda^2\mathbf{C}^n \tag{1}$$

A map from V to $S_\lambda V$ for a given tableau λ of n can be upgraded to a covariant functor with maps from V to W inducing covariantly maps from $S_\lambda V$ to $S_\lambda W$.

Some general tips for using dualities. Firstly, one can establish an isomorphism under two successive operations to find an adjunction. Secondly, if there is an estimation, see if there is a dual notion that is also an estimation and imagine the problem to be similar to a thin category or posets. This can help in establishing another adjunction. Thirdly, logic can be related to space by Heyting duality, representing objects to objects or Yoneda nonsense by Tannaka duality, arrow reversal by the Eckmann-Hinton duality, space and quantities by Isbell duality.