YU JIE TEO

A MATH BEDTIME STO- RYBOOK

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To my parents.

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Introduction

These are my personal notes in mathematics.

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Conventions

Set Theory

The Zermelo-Fraenkel axioms of set theory with the axiom of choice by default.

Alternative foundations with homotopy type theory and the univalence axiom may be considered when appropriate in the appropriate sections.

Universes may also be used.

Logic

Conjunction always refer to inclusive conjunction. So the word "or" means, x or y or both.

Category theory

We follow these conventions [1] by default.

[1] J. De Jong. *Stacks Project*. Columbia University, 2025

Definition 2.0.1. *A (small) category,* **C** *consists of a set of objects. For each pair of objects there exists a set of morphisms.*

Note that all italicised words can be changed to define enrichment, operads when necessarily.

Algebra

In algebra, a ring is a commutative unitary ring.

In representation theory, we may drop commutativity.

In analysis, we may drop the condition of unitary.

Notation

The natural integers refers to the positive integers, however this will be avoided.

3
Set Theory

Definitions

Zermelo-Fraenkel Axioms

This follows Jech [1].

[1] Thomas Jech. *Set Theory*. Springer-Verlag Berlin Heidelberg New York, 4th edition, 2006

Definition 3.0.1 (Pairing axiom). *For any two sets X and Y, then there exists a set, denote $\{X, Y\}$, where the set contains exactly $\{X, Y\}$.*

Definition 3.0.2 (Extensionality axiom). *If sets X and Y have the same elements, we define equality where the set X is equal to the set Y.*

Definition 3.0.3 (Union axiom). *The union over elements of a set exists.*

Definition 3.0.4 (Infinity axiom). *An infinite set exists.*

Definition 3.0.5 (Regularity axiom). *All nonempty sets have a membership minimal element.*

Definition 3.0.6 (Separation axiom schema). *If P is a property parameterised by p, then for any set X and parmaeter p, then there exists a set Y that has elements y in X that contains all elements y in X that has property P.*

Definition 3.0.7 (Powerset axiom). *For any set X, there exists the set of all subsets of X called the power set of X, and is denoted by $P(X)$.*

Definition 3.0.8 (Replacement axiom schema). *If a class F is a function, there for any set X, there exists a set called the function set with elements of the form $F(x)$ for an element x in set X, this set is denoted $F(X)$.*

Definition 3.0.9 (Strong choice axiom). *All families of nonempty sets have a choice function.*

Naive sets

This section is purely metamathematical and it is not rigorous.

Definition 3.0.10 (Naive set). *A naive set is a list of unique elements.*

Definition 3.0.11 (Naive class). *Suppose we have a formula $p(x, p_1, ...p_n)$.*

A naive class is a set C containing elements x such that x satisfies the formula $p(x, p_1, ...p_n)$.

Definition 3.0.12 (Naive membership). *If an element x in in a naive set X, we say x is a member in X, and it is denoted by $x \in X$.*

Definition 3.0.13 (Union of sets). *Let X and Y be naive sets. Then, there exists a set, called the union of X and Y, denoted $X \cup Y$, which contains the list of unique elements in either X or Y, or both.*

Lemma 3.0.14 (Union exists). *The union of naive sets exists.*

Proof. Use Axiom 3.0.3 on all the elements in a pair of naive sets X and Y. Use Axiom 3.0.4 for infinite sets. □

Definition 3.0.15 (Intersection of sets). *Let X and Y be naive sets. Then, there exists a set, called the intersection of X and Y, denoted $X \cap Y$, which contains the list of unique elements in X and in Y.*

Lemma 3.0.16 (Intersection exists). *The intersection of naive sets exists.*

Proof. Use Axiom 3.0.3 on all the elements in a pair of naive sets X and Y in common. Use Axiom 3.0.4 for infinite sets. □

Definition 3.0.17 (Complementation of sets). *Let X and Y be naive sets. Then, there exists a set, called the complementation of X and Y, denoted $X - Y$, which contains the list of unique elements in X but not in Y.*

Lemma 3.0.18 (Complementation exists). *The complementation of naive sets exists.*

Proof. Use Axiom 3.0.3 on all the elements in a pair of naive sets X and Y in common only considering elements in X but exclude elements that are in both X and Y. Use Axiom 3.0.4 for infinite sets.

□

Lemmas

We shall implicitly use without reference the axioms of set theory.

Lemma 3.0.19 (De Morgan's first law for pairs of naive sets)**.** *Let X and Y be a naive sets.*

Then the set $X^c \cup Y^c$ is equal to $(X \cap Y)^c$.

The rule applies for arbitrary unions of complements to the complement of arbitrary intersections.

Proof. We read the set $X^c \cup Y^c$ as consisting of the elements c such that it is either not in X or it is the elements not in Y, or both.

By case analysis, we take the elements that are not in both set X and set Y. These elements form precisely the set $(X \cap Y)^c$.

This proves the base case. Now consider the arbitrary union of the precedent case with the successive set complement. The law for the pairs applies again, and the successive case is shown. The proof follows by induction. □

Lemma 3.0.20 (De Morgan's second law for pairs of naive sets)**.** *Let X and Y be a naive sets.*

Then the set $X^c \cap Y^c$ is equal to $(X \cup Y)^c$.

The rule applies for arbitrary intersection of complements to the complement of arbitrary unions.

Proof. We read the set $X^c \cap Y^c$ as consisting of the elements c such that they are not in X and they are not elements not in Y.

By case analysis, we take the elements that are not in either set X or set Y or both. These elements form precisely the set $(X \cup Y)^c$.

This proves the base case. Now consider the arbitrary intersection of the precedent case with the successive set complement. The law for the pairs applies again, and the successive case is shown. The proof follows by induction. □

4
Categories

Definitions

Definition 4.0.1 (Set of morphisms). *A set of morphisms between objects X, Y are maps from X to Y. These are denoted* $\mathrm{Hom}(X, Y)$ *and are also called hom-sets.*

Definition 4.0.2 (Composition maps). *A composition map for objects X, Y, Z where is a map from a cartesian product of hom-sets to a hom-set* $\cdot : \mathrm{Hom}(Z, Y) \times \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Z, X)$.

Definition 4.0.3 (Category). *A category* $\mathbf{C}$ *has a set of objects, denoted* $\mathrm{Ob}(X)$ *or with objects X.*

It has a set of morphisms between objects X, Y denoted $\mathrm{Hom}(X, Y)$.

It has a composition map for objects X, Y, Z where $\cdot : \mathrm{Hom}(Z, Y) \times \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Z, X)$ *such that for morphism p in* $\mathrm{Hom}(Y, X)$ *and morphism q in* $\mathrm{Hom}(Z, Y)$ *we have a morphism $q \cdot p$ in the set of morphisms* $\mathrm{Hom}(Z, X)$.

These satisfy these rules:

1. *For every object X in the set of objects* $\mathrm{Ob}(X)$, *there exists an identity morphism $i \in Hom_{\mathbf{C}}(X, X)$ such that it composes with morphisms p and q where $p = i \cdot p$ and $q \cdot i = q$.*
2. *The composition of morphism is associative where $p \cdot (q \cdot r) = (p \cdot q) \cdot r$.*

Definition 4.0.4 (Covariant functor). *A functor category* $\mathbf{FC}$ *has a set of objects, denoted* $\mathrm{Ob}(\mathbf{F}X)$ *or with objects FX.*

It has a set of morphisms between objects $\mathbf{F}X, \mathbf{F}Y$ denoted $\mathrm{Hom}(\mathbf{F}X, \mathbf{F}Y)$.

It has a composition map for objects $\mathbf{F}X$, $\mathbf{F}Y$, $\mathbf{F}Z$.

This map is such that $\cdot : \mathrm{Hom}(\mathbf{F}Z, \mathbf{F}Y) \times \mathrm{Hom}(\mathbf{F}Y, \mathbf{F}X) \to \mathrm{Hom}(\mathbf{F}Z, \mathbf{F}X)$.

This is such that each morphism Fp in $\mathrm{Hom}(\mathbf{F}Y, \mathbf{F}X)$ *and morphism* $\mathbf{F}q$ *in* $\mathrm{Hom}(\mathbf{F}Z, \mathbf{F}Y)$ *we have a morphism $\mathbf{F}q \cdot \mathbf{F}p$ in the set of morphisms* $\mathrm{Hom}(\mathbf{F}Z, \mathbf{F}X)$.

These satisfy these rules:

1. *For every object* $\mathbf{F}X$ *in the set of objects* $\mathrm{Ob}(\mathbf{F}X)$*, there exists an identity morphism* $i \in Hom_{\mathbf{FC}}(\mathbf{F}X, \mathbf{F}X)$ *such that it composes with morphisms* $\mathbf{F}p$ *and* $\mathbf{F}q$ *where* $\mathbf{F}p = \mathbf{F}i \cdot \mathbf{F}p$ *and* $Fq \cdot \mathbf{F}i = \mathbf{F}q$*.*

2. *The composition of morphisms is associative where* $\mathbf{F}p \cdot (\mathbf{F}q \cdot \mathbf{F}r) = (\mathbf{F}p \cdot \mathbf{F}q) \cdot \mathbf{F}r$*.*

3. *The composition keeps arrows so* $\mathbf{F}(p \cdot q) = \mathbf{F}p \cdot \mathbf{F}q$*. We abused notation for composition here.*

A covariant functor $\mathbf{F}$ *takes a category* $\mathbf{C}$ *to the functor category* $\mathbf{FC}$ *satisfying the above rules.*

Definition 4.0.5 (Contravariant functor)**.** *A functor category* $\mathbf{FC}$ *has a set of objects, denoted* $\mathrm{Ob}(\mathbf{F}X)$ *or with objects* FX*.*

It has a set of morphisms between objects.

This is $\mathbf{F}X, \mathbf{F}Y$ *denoted* $\mathrm{Hom}(\mathbf{F}X, \mathbf{F}Y)$*.*

It has a composition map for objects $\mathbf{F}X$*,* $\mathbf{F}Y$*,* $\mathbf{F}Z$*.*

This is $\cdot : \mathrm{Hom}(\mathbf{F}Z, \mathbf{F}Y) \times \mathrm{Hom}(\mathbf{F}Y, \mathbf{F}X) \to \mathrm{Hom}(\mathbf{F}Z, \mathbf{F}X)$*.*

This is such that each morphism $\mathbf{F}p$ *in* $\mathrm{Hom}(\mathbf{F}Y, \mathbf{F}X)$ *and morphism* $\mathbf{F}q$ *in* $\mathrm{Hom}(\mathbf{F}Z, \mathbf{F}Y)$ *we have a morphism* $\mathbf{F}q \cdot \mathbf{F}p$ *in the set of morphisms* $\mathrm{Hom}(\mathbf{F}Z, \mathbf{F}X)$*.*

These satisfy these rules:

1. *For every object* $\mathbf{F}X$ *in the set of objects* $\mathrm{Ob}(\mathbf{F}X)$*, there exists an identity morphism* $i \in Hom_{\mathbf{FC}}(\mathbf{F}X, \mathbf{F}X)$ *such that it composes with morphisms* $\mathbf{F}p$ *and* $\mathbf{F}q$ *where* $\mathbf{F}p = \mathbf{F}i \cdot \mathbf{F}p$ *and* $\mathbf{F}q \cdot \mathbf{F}i = \mathbf{F}q$*.*

2. *The composition of morphisms is associative where* $\mathbf{F}p \cdot (\mathbf{F}q \cdot \mathbf{F}r) = (\mathbf{F}p \cdot \mathbf{F}q) \cdot \mathbf{F}r$*.*

3. *The composition reverses arrows so* $\mathbf{F}(p \cdot q) = \mathbf{F}q \cdot \mathbf{F}p$*. We abused notation for composition here.*

A functor $\mathbf{F}$ *takes a category* $\mathbf{C}$ *to the functor category* $\mathbf{FC}$ *satisfying the above rules.*

Definition 4.0.6 (Opposite category)**.** *A category* $\mathbf{C}$ *has a set of objects, denoted* $\mathrm{Ob}(X)$ *or with objects* X*.*

It has a set of morphisms between objects X, Y *denoted* $\mathrm{Hom}(X, Y)$*.*

The opposite category, denoted $\mathbf{C}^{\mathrm{op}}$ *is a category with the hom-sets of* $\mathrm{Hom}(Y, X)$ *satisfying the definition of a category in Definition 4.0.3.*

The yoga of objects

We need some definitions to build up to this first.

Definition 4.0.7 (Span). *A span is a diagram from a limiting object with projection morphisms to separate objects.* [1]

[1] Spans are known as pushout diagrams.

Definition 4.0.8 (Cospan). *A cospan is a diagram to a colimiting object with morphisms from separate objects to the colimiting object.* [2]

[2] Cospans are known as pullback diagrams.

Definition 4.0.9 (Pullback). *A pullback is a limit of a cospan.*

Definition 4.0.10 (Pushout). *A pushout is a colimit of a span.*

Definition 4.0.11 (Monad). *A monad is a monoid of the monoidal category of endofunctors* $\mathbf{C}^{\mathbf{C}}$ *on the category* $\mathbf{C}$*.*

The key definition is that of internalisation. We use the definition where it has all pullbacks.

Definition 4.0.12 (Internal category). *Let* $\mathbf{C}$ *be a category. It is a category with all pullbacks.*[3]

[3] One can weaken this to pullbacks existing, at the cost of some very messy diagrams. See Exercise 4.0.104.

Recall that a span is a diagram from a limiting object with projection morphisms to separate objects.

A category of spans $\mathrm{Span}(\mathbf{C})$ *is a category with objects and morphism as as spans. These form a category with all pullbacks.*

Form the bicategory $\mathrm{Span}(\mathbf{C})$ *of spans in category* $\mathbf{C}$*.*

A category internalised in a category $\mathbf{C}$ *is precisely a monad in the category of spans* $\mathrm{Span}(\mathbf{C})$*.*

Internal objects internal to a category can alter be externalised to functors with the category as the domain, or as fibrations over the category.

Definition 4.0.13 (Pseudodefinition of Grothendieck fibration). *A Grothendieck fibration is the externalisation of an internal groupoid in a finitely complete category.*

Now, we have all the motivation for the yoga of objects.

Definition 4.0.14 (Initial object). *Let* $\mathbf{C}$ *be a category.*

An initial object is an object x *with all morphisms mapping from it in the category* $\mathbf{C}$*.*

Definition 4.0.15 (Final object). *Let* $\mathbf{C}$ *be a category.*

An final object is an object x *with all morphisms mapping to it in the category* $\mathbf{C}$*.*

Typically, definitions will have morphisms mapping from the terminal object, and not to the final object. This is a source of confusion.

This is best rectified by knowing this definition:

Definition 4.0.16 (Global element, final object). *Let* **C** *be a category with a final object.*

A global element of a object x is a morphism from the final object 1*.*

Definition 4.0.17 (Global element, represented presheaf). *Let* **C** *be a category without a final object.*

It is the global element of the represented presheaf of the object.[4]

[4] This definition works if the category has no terminal object since the Yoneda embedding is fully faithful and preserves all limits.

Conditions on categories for the yoga of objects

Certain size theory concerns are needed to define types of objects using the yoga of objects.

Definition 4.0.18 (Locally small). *Let* **C** *be a category.*

A category is locally small if the hom-sets are small sets.

Definition 4.0.19 (Fibre complete category). *Let* **C** *be a category.*

A category is fibre complete that the category of all topologies on a given set called the fibre of U on a set X form a complete lattice ordered by inclusion.

Definition 4.0.20 (Concrete category). *Let* **C** *be a category.*

A category is concrete if it is equipped with a faithful functor to a category of sets.[5]

[5] This functor makes it possible to think of the objects of the category as sets with additional structure.

Definition 4.0.21 (Finitely complete category). *A finitely complete category is a category that admits all finite limits. It is also called a lex category. Lex is shorthand for left exact.*

Definition 4.0.22 (Finitely cocomplete category). *A finitely cocomplete category is a category that admits all finite colimits.*

Definition 4.0.23 (Complete category). *A complete category has all small limits.*

Definition 4.0.24 (Cocomplete category). *A cocomplete category has all small colimits.*

One can make a category cocomplete with the Yoneda embedding. Then, one can define an object on the free cocompletion of the category instead.

Definition 4.0.25 (Free cocompletion of a category). *The free cocompletion of a category is the presheaf category formed by freely adjoining colimits through the Yoneda embedding.*

Definition 4.0.26 (Regular category). *A regular category is a finitely complete category whose kernel pair on any morphism as a pullback admits a coequaliser on projections, and the pullback of epimorphisms along any morphism is again a regular morphism. It is defined so that the kernel pair is always a congruence on the kernel pair components. The resulting coequaliser is the object of equivalence classes.*

Definition 4.0.27 (Coherent category). *A coherent category is a regular category whose subobject posets all have finite unions preserved under base change functors.*

Definition 4.0.28 (Monoidal category). *A monoidal category has a canonical tensor product as a functor and the terminal object as the tensor unit.*

It has suitable conditions on associators, left unitor, and right unitor so that the triangle identity and the pentagonal identity commute to allow for the bilinearity of maps.

The canonical example for a monoidal category should be rings.

Definition 4.0.29 (Closed category). *A closed category is a category that has an internal hom object.*

Morphisms from source objects to target objects are objects of a closed category defined as the internal hom objects if they are objects of a closed category.

The word closed should remind you of the Yoneda lemma, and Cayley's theorem, groups are closed under group elements as hom objects. Internal homs ensure this closure.

Definition 4.0.30 (Internal hom functor). *An internal hom is a functor admitting the tensor-hom adjunction for every object in the category.*

Definition 4.0.31 (Closed monoidal category). *A category with a canonical internal hom with an appropriate tensor-hom adjunction is a closed monoidal category.*

Definition 4.0.32 (Semicartesian monoidal category). *A semicartesian monoidal category has the tensor unit as a terminal object.*[6]

[6] This is weaker than saying the tensor product is the categorical cartesian product.

Definition 4.0.33 (Semicocartesian monoidal category). *A semicocartesian monoidal category has the tensor unit as a initial object.*

Definition 4.0.34 (Graded category). *The graded category is the functor category from the discrete (or monoidal) category* $\mathbf{S}$ *to the current category* $\mathbf{C}$ *denoted by* $\mathbf{C}^{\mathbf{S}}$ *as an exponential functor category.*

Definition 4.0.35 (Graded object)**.** *A graded object is a object in a graded category.*

Definition 4.0.36 (Cartesian monoidal category)**.** *A cartesian monoidal category is a category with finite products with respect to its cartesian monoidal structure. The internal hom (which exists, since it is closed) of a cartesian closed category is called exponentiation (which can be thought of as a product, since it is cartesian). The tensor unit is the terminal object, it has all finite products, and the tensor product is a product.*

Definition 4.0.37 (Cocartesian monoidal category)**.** *A cocartesian monoidal category is a category with finite coproducts with respect to its cartesian monoidal structure. The tensor unit is the initial object, it has all finite coproducts, and the tensor product is a coproduct.*

We can have functors that preserve the niceness of cartesian closed categories.

Definition 4.0.38 (Cartesian closed functor)**.** *A cartesian closed functor is a functor that preserves products and exponentials.*

Definition 4.0.39 (Bicartesian closed functor)**.** *A bicartesian closed category is a category that is cartesian (admits all finite products) and co-cartesian (admits all finite coproducts) that has an internal hom.*

We would like an intuition for density argumnent that may be useful

Definition 4.0.40 (Dense in category)**.** *An subcategory is dense in a category if every object is a colimit of a diagram of objects in the subcategory in a canonical way. This is defined to be a dense subcategory.*

Lastly we have some remarks on the completion of an object. There are several ontologies worth considering here. Nobody has a good definition to capture the various forms of completion.

Remark 4.0.41. *A completion of an object is an object with the original object as a subobject. The word "free" is used for adjunction of forgetful functor. Typically, this is also used for faithful reflector.*

Examples include Cauchy completions of metric spaces, Dedekind completions of linear order, ring completion, Stone-Cech compactification of a Tychnoff space, profinite completion, Grothendieck group formation, group completion, field of fraction of integral domains, free cocompletion to presheaf categories, ind-completion under filtered colimits, pro-completion under cofiltered limits.

List of objects

We have now come to the meat of what we want to do. We want to use internalisation to define objects in the categories we want.

Definition 4.0.42 (Subobject)**.** *A subobject is equivalently:*

1. *isomorphism classes of monomorphisms. Two monomorphisms are isomorphic if they are both monomorphisms into a object and there is a isomorphism between them such that when the isomorphism composed to a monomorphism, this gives equality to the other monomorphism.* [7]

2. *objects of the full subcategory of the over category of an object in monomorphisms. The product in this over category as a subcategory is an intersection or meet of subobjects. Their coproduct is the union or the join of subobjects. An over category or a slice category over a (base) object is a category whose objects are all arrows with codomain as that object and morphisms all satisfy commutative diagrams that has that object as the cocone.*

[7] If we have monomorphisms not on the level of isomorphisms but on equality and essential uniqueness, this condition can only happen on the level of posets, then this corresponds to subsets. This motivates the definition of subobjects.

Definition 4.0.43 (Complemented object)**.** *A complemented object is a subobject given by a monomorphism in a coherent category and is defined when it has a complement or another subobject such that its intersection is the initial object and the union is the full object of the subobject.*

Definition 4.0.44 (Exponential object)**.** *An exponential object is an internal hom object in a cartesian closed categories.*[8] [9]

[8] The category is carteisan closed, therefor products and multiplications make sense. The closure of the category ensures that the internal hom exists, so products can be defined as internal homs that form exponentials.

[9] Note that the subcategory of compactly generatory Hausdorff spaces is cartesian closed.

Definition 4.0.45 (Differential object)**.** *A differential object in a category with translation is an object equipped if the translation called the differential. This is a special case of suspensions.*

Definition 4.0.46 (Suspension object)**.** *A suspension object is an object in an $(\infty, 1)$ category admitting a terminal object as the suspension object as the homotopy pushout.*

Definition 4.0.47 (Connected object)**.** *A connected object is an object whose hom functor out of the object to a fixed object is preserves coproducts.* [10]

[10] The colimit of connected objects is a connected object.

Definition 4.0.48 (Filtered object)**.** *A filtered object is an object equipped with either an ascending or descending filtration.*[11]

[11] For example, a descending filtration has a sequence of morphisms as a graded object.

Definition 4.0.49 (Interval object)**.** *A interval object I is the colimit of a cospan diagram with equal feet in the category with such that for any two pointing pointing to the interval, 0 and 1 are morphisms.*

Definition 4.0.50 (Cartesian interval object)**.** *A cartesian interval object I is the colimit of a cospan diagram with equal feet in the category with such that for any two termianl objects pointing to the interval,* 0 *and* 1 *are morphisms.*

Definition 4.0.51 (Pointed object)**.** *A pointed object is an object equipped with a global element.*

Recall from Definition 4.0.16 that a global element is a morphism from the terminal object to that object.

If we cannot make it work, use 4.0.17 where a global element is a morphism from the terminal object to the representable presheaf.

Definition 4.0.52 (Integer object)**.** *An integer object in a cartesian closed category with a terminal object is equipped with a morphism from the terminal object to it and an isomorphism called the successor with the universal property such that there is a unique isomorphisms that satisfies conditions with commutative diagrams akin to the Peano axioms. This can be generalised to symmetric monoidal categories using the tensor unit instead of the terminal object.*

Definition 4.0.53 (Braided object)**.** *A braided object is an object* B *in a monoidal category equipped with an invertible morphism* a *on a tensor product satisfying the Yang-Baxter equation* $(aB)(Ba)(aB) = (Ba)(aB)(Ba)$ *where the silent product by parentheses is morphism composition, and the silent product within the parentheses is the tensor product.* [12]

[12] I know this is terrible notation for the Yang-Baxter equation, but it shows the braiding so nicely I think it is worth showing.

Definition 4.0.54 (Choice object)**.** *A choice object is an object such that the axiom of choice holds when making choices from the object. A projective object is an object such that the axiom of choice holds when making choices indexed by the projective object.*

Definition 4.0.55 (Descent object)**.** *A descent object is an hom object that induces a contravariant descent hom object that is an equivalence.*

Definition 4.0.56 (Compact object)**.** *A compact object is a corepresentable functor (hom object) from a locally small category that admits filtered colimits such that homs out of it to a fixed object preserve filtered colimits.*

Pro means projective, ind means inductive.

Definition 4.0.57 (Pro object)**.** *A pro object is an object in the full subcategory inclusion via the opposite of the Yoneda embedding. Recall that the Yoneda embedding is from the category of presheaves to the free cocompletion. Taking the dual of the Yoneda embedding means we start from the category and end up with its free completion.*

Definition 4.0.58 (Ind object). *An ind object is an object in the full subcategory inclusion via the Yoneda embedding from the category of presheaves to the free cocompletion.*

Definition 4.0.59 (Strict pro object). *A strict pro object is representable as a limit of a small cofiltered diagram.*

Definition 4.0.60 (Strict ind object). *A strict ind object is representable as a colimit of a small filtered diagram.*

Definition 4.0.61 (Sifted ind object). *A sind object is a formal sifted colimit taken in the category of presheaves or free completion.*

Remark 4.0.62. *Warning, pro objects are not projective objects.*

Warning, ind objects are not inductive objects.

Example 4.0.63. *A formal scheme is an ind-object in schemes.*

Finitely indexed sets are ind-objects in sets.

Finitely generated groups are ind-objects in groups.

Profinite groups are pro objects of finite groups.

Definition 4.0.64 (Normal subobject). *A monomorphism (between a subobject to a full object) that is normal or conjugate to some internal equivalence relation or it factors through that internal equivalence relation is equipped and defines a normal subobject from a subobject.*

Definition 4.0.65 (Noetherian object). *A Noetherian object is such that only finitely many inclusions in the ascending chain subobjects of the Noetherian object are not isomorphisms in the category.*

Definition 4.0.66 (Artinian object). *An Artinian object is such that only finitely many inclusions in the descending chain of subobjects of the Artinian object are not isomorphisms in the category, there is a terminal subobject in some sense.*

Definition 4.0.67 (Locally small object, using slice categories). *A locally small object is an object in the full subcategory of the slice category on the monomorphisms is essentially small.*

Easier definition: isomorphism classes of monomorphisms with the locally small object as target or the subobjects of the object form a set.

Definition 4.0.68 (Colocally small object, using slice categories). *A colocally small object is an object in the full subcategory of the coslice category on the epimorphisms is essentially small.*

Easier definition: isomorphism classes of epimorphisms with the locally small object as source or the quotient of the object form a set.

Definition 4.0.69 (Projective object). *A projective object is an object whose hom functor out of the object preserves epimorphisms. A morphism out of the projective object factors through epimorphisms to be a projective morphism on the coimage object, this is defined to be the left lifting property.*

Definition 4.0.70 (Injective object). *An injective object is an object whose hom functor into the object preserve monomorphisms.*

Definition 4.0.71 (Tiny object). *A tiny object is a projective object whose hom functor out of the object preserves coequalisers (or all colimits). These are the projective connected objects.*

Definition 4.0.72 (Enough projectives). *A category has enough projectives if all objects in the category admits epimorphisms by a projective object in the category. We say that every object admits a projective presentation.*

Definition 4.0.73 (Enough injectives). *A category has enough injectives if all objects in the category admits monomorphisms into a injective object in the category.* [13]

[13] In a regular category, projectives are also regular projectives.

Definition 4.0.74 (Simple object). *A simple object is an object with precisely the terminal object and the full object as the quotient object.*

Definition 4.0.75 (Semisimple object). *A semisimple object is a coproduct of simple objects.*

Definition 4.0.76 (Group object). *A group object is an object with diagrams that allow an unital associative magma object to commute in a cartesian category.*

Definition 4.0.77 (Ring object). *A ring object is an object with diagram with diagrams that expressive addition, zero, multiplicative identity and additive inverses in a cartesian monoidal category.*

Definition 4.0.78 (Lie algebra object). *A Lie algebra object is an object in a symmetric monoidal k-linear category with braiding such that it is an object and a morphism called the Lie bracket formed from the tensor product, with equivalence classes formed using the Jacobi identity and skew symmetry. Braiding is needed here to define the Jacobi identity.*

Definition 4.0.79 (Simplicial object). *A simplicial object is a presheaf object of the presheaf functor category from the simplicial indexing category.*

Definition 4.0.80 (Graded object, as representable object of a representable functor)**.** *A graded object is an representing object of the representable functor category from the discrete monoidal category.*

Definition 4.0.81 (Associated graded object, as representable object of a representable functor)**.** *An associated graded object is a gaded object whose n-th degree is the cokernel of the n-th inclusion.*

Definition 4.0.82 (Continuous object)**.** *A continuous object is an object in the functor category, or a functor that preserves all small limits.*

Definition 4.0.83 (Connected object, functor category)**.** *A connected object is an object in the functor category, or a functor that preserves all small colimits.*

Definition 4.0.84 (Power object)**.** *A power object is a object with a monomorphism such that there exist a unique monomorphism for each other object into their cartesian object a unique morphism such that the monomorphism is a pullback.*

Definition 4.0.85 (Subobject classifier)**.** *The power object of a terminal object is a subobject classifier.*

Example 4.0.86. *A power object in set is a power set.*

A category with finite limits and power objects for all objects is precisely a topos.

Definition 4.0.87 (Comma object)**.** *A comma object of a pair of morphisms in a cospan in a two category is an object equipped with two projections to the feet of comma object as the apex that also has a 2-morphism that fills this commutative diagram such that the 2-morphism is universal as a 2-limit.*

Ordered objects

We give a definition of several ordered objects here.

Definition 4.0.88 (Internal preorder)**.** *An internal preorder is a subobject of the cartesian product object in equipped with internal reflexibity that is a section of both the source and target subobject of the cartesian product object, and internal transitivity which is an object that factors the left and right projection map from the product of the internal preorder to the product of the preordered objects on both the source and targets. An internal preorder can be the representing object of the representable subpresheaf of the hom functor into the cartesian product of an object so that for each object Y the composite $R(Y)$ into $\hom(Y, X \times X)$ is canonically isomorphic to $\hom(Y, X) \times \hom(Y, X)$ that exhibits $R(Y)$ as a preorder on* $\hom(\mathrm{Y}, \mathrm{X})$.

Definition 4.0.89 (Internal congruence). *A internal congruence is an internal equivalence relation or an internal groupoid and hence an internal category with all morphisms being isomorphisms with no non identity automorphisms. It consists of a subobject of the Cartesian product of an object with itself, with internal reflexivity as sections of projections, internal symmetry which interchanges projections using them as sections of each other, and internal transivity where a suitable fibre product of the subobject with itself factors the projection map through the subobject fibre product to the full cartesian product via a suitable pullback diagram with the fibre product of the subobject with itself as the pullback.*

Definition 4.0.90 (Preorder object). *A preorder object is an object in a category with pullbacks and suobjects with an internal preorder on the object X that is injective of pullbacks of the product $X \times X$.*

Definition 4.0.91 (Quotient object). *A quotient object is the coequaliser of a congruence.*

Definition 4.0.92 (Cartesian monoidal preordered object). *A cartesian monoidal preordered object is a preordered object with an internal preorder as a representable subpresheaf of homs out of the target with monoidal objects with monoidal multiplication and a global unit from the terminal object to any object such that there exists a function τ for all global elements a, it is identified by the source as a section and becomes the global unit under target as a section. It also has suitable left and right unitors λ_l and λ_r as internal hom objects such that for all global elements, composition of the left component subobject with the left unitor gives the internal left projection, right component subobject with the right unitor give the internal right projection, and internal composition by left on the projection on the opposing unitor gives the meet of subobjects of the preorder. Dualising using means that the global unit is the join, and is careful still a morphism from the terminal object.*

Definition 4.0.93 (Semicartesian monoidal preordered object). *A semicartesian monoidal object is an object in a category whose tensor unit is the terminal object.*

Definition 4.0.94 (Semicocartesian monoidal preordered object). *A semicocartesian monoidal object is an object in a category whose tensor unit is the initial object.*

Definition 4.0.95 (Join relative object). *A join relative object is a cocartesian monoidal preordered object that is a partial order object.*

Definition 4.0.96 (Meet relative object). *A meet relative object is a cartesian monoidal preordered object that is a partial order object.*

Definition 4.0.97 (Partial order object). *A partial order object is a preordered object whose internal preorder has an internal antisymmetric relation.*

Definition 4.0.98 (Bicartesian preordered object). *A bicartesian preordered object or a prelattice object is a object that is both a cartesian monoidal preordered object and a cocartesian monoidal preordered object.*

Definition 4.0.99 (Lattice object). *A lattice object is a prelattice object that is also a partially ordered object.*

Definition 4.0.100 (Bicartesian closed preordered object). *A bicartesian closed preordered object is a Heyting prealgebra object exhibits closure with a suitable logical function that is a bicartesian preordered object.*

Definition 4.0.101 (Boolean prealgebra object). *A Boolean prealgebra object is a bicartesian closed preorder object for all element pairs of source and target (s, t) such that composition by source s gives implications of objects $a \implies b$ as an object, and composition by target t gives the join of the statements a being false (vacously true) or b is true. You need bicartesian to admit all small products and coproducts, closure to make this internal hom.*

Definition 4.0.102 (Boolean algebra object). *A Boolean algebra object is a Boolean prealgebra object that is also a partially ordered object that makes statements either true or false but not both.*

Exercises

Exercise 4.0.103 (Opposite of a opposite category). *Show that the opposite category of a opposite category is naturally isomorphic to the original category of the opposite category.*

Proof. The functor taking hom-sets from $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(Y, X) \to \mathrm{Hom}(X, Y)$.

This is in the definition of a opposite category as per Definition 4.0.6 is a canonical identity natural isomorphism. [14] □

[14] The double dual functor yield a natural isomorphism similar to the double dual of a vector space is the motivating example for category theory.

Exercise 4.0.104 (Commutative diagrams for internal categories). *Draw the commutative diagrams to define an internal category.*

Exercise 4.0.105 (Commutative diagrams for monoidal categories). *Draw the commutative diagrams to define an monoidal category.*

5 Topology

Definitions

We follow Steen and Seebach's presentation of topology [1].

[1] Lynn Arthur Steen and Jr. J. Arthur Seebach. *Counterexamples in Topology*. Dover, paperback edition, 1995

Definition 5.0.1 (Topology of open sets)**.** *For a set X, a collection of open subsets of the set X is called a topology, denoted by τ if arbitrary unions and finite intersections of each subset is in τ.*

Definition 5.0.2 (Topology of closed sets)**.** *For a set X, a collection of closed subsets of the set X is called a topology, denoted by τ if arbitrary intersections and finite unions of each subset is in τ.*

Definition 5.0.3 (Open set)**.** *An open set is a set U in a topology τ of a set X.*

Definition 5.0.4 (Open map)**.** *An open map is a map from the topological space $(X, \tau_X) \to (Y, \tau_Y)$ taking sets U_X in a topology τ_X of a set X to ets U_Y in a topology τ_Y of a set Y.*

Definition 5.0.5 (Closed set)**.** *An closed set S is the complement of an open set U of a topology τ with respect to the main set X.*

Lemma 5.0.6 (Topology of closed sets)**.** *The closed sets $(X - U)^c$ of a topological space (X, τ) form a topology τ given the open sets U in the topology τ.*

Proof. The complement of the entire space X relative to finite intersection of open sets U is a arbitrary union of closed sets, this follows by De Morgan's laws or Lemma 3.0.19 and Lemma 3.0.20. The complement of the entire space X relative to arbitrary union of open sets U is a finite intersection of closed sets, this follows by De Morgan's laws. $\square$

Definition 5.0.7 (Topological space). *For a set X, a collection of subsets of the set X is called a topology, denoted by τ if arbitrary unions and finite intersections of each subset is in τ. A pair (X, τ) is a topological space. By abuse of notation, we call X a topological space.*

Definition 5.0.8 (Open covers). *For a set X, a collection of subsets of the set X is called a topology, denoted by τ if arbitrary unions and finite intersections of each subset is in τ. A pair (X, τ) is a topological space.*

An open cover of U on a open subset V of a a topological space (X, τ) is a union of open sets $U = \cup U_i$ in the topology τ such that the open subset V is contained in the open cover U.

Definition 5.0.9 (Finite open covers). *For a set X, a collection of subsets of the set X is called a topology, denoted by τ if arbitrary unions and finite intersections of each subset is in τ. A pair (X, τ) is a topological space.*

A finite open cover of U on a open subset V of a a topological space (X, τ) is a finite union of open sets $U = \cup U_i$ in the topology τ such that the open subset V is contained in the open cover U.

Definition 5.0.10 (Compact topological space). *For a set X, a collection of subsets of the set X is called a topology, denoted by τ if arbitrary unions and finite intersections of each subset is in τ. A pair (X, τ) is a topological space.*

A compact topological space (X, τ) is a topological space where every open cover, that is a union of a collection open sets $U = \cup U_i$ containing X admits a finite subcover, that is a finite union of a subcollection of open sets $U = \cup U_i$.

We now give the definition in terms of category theory.

Remark 5.0.11. *The category of topological spaces is both complete and cocomplete. From Definition 4.0.24 and 4.0.23, all small limits and colimits exist in the category of topological spaces.*

Definition 5.0.12 (Compact topological space, compact object). *A compact topological space (X, τ) is a compact object in the category of topological spaces.*[2]

A compact object is a corepresentable functor (hom object, in this case it is a topological space representing a set of open covers) from a locally small category[3] *that admits filtered colimits such that homs out of it (open covers as homs) to a fixed object (the underlying topological space) preserve filtered colimits (finite subcovers).*

Colimits in the category of topological spaces are precisely the union of open covers. Filtered colimits in the category of topological spaces are precisely the union of finite covers. Admission means that there are finite subcovers.

[2] See Definition 4.0.56 for a general definition

[3] We used the fact that the category of topological spaces is complete and cocomplete.

Definition 5.0.13 (Initial object in the category of topological spaces). *The unique initial object in the category of topological spaces is the empty set.*

Definition 5.0.14 (Final object in the category of topological spaces). *The final object in the category of topological spaces is the singleton space.*

Definition 5.0.15 (Coarser). *Suppose τ_1 and τ_2 are topologies for a set X. Recall that these are collections of subsets.*[4]

If the set τ_1 is contained in the set τ_2, we say the topology τ_1 is coarser than τ_2.

[4] Topologies may not be comparable.

Definition 5.0.16 (Finer). *Suppose τ_1 and τ_2 are topologies for a set X. Recall that these are collections of subsets.*[5]

If the set τ_1 is contained in the set τ_2, we say the topology τ_2 is coarser than τ_1.

[5] Topologies may not be comparable.

Definition 5.0.17 (Neighbourhood). *Suppose τ is a topology for a set X. Let p be a point in the set X. A neighbourhood of a point p in the set X is a subset of an open set U in the topology τ containing the point p.*

Definition 5.0.18 (Open neighbourhood). *Suppose τ is a topology for a set X. Let p be a point in the set X. A open neighbourhood of a point p in the set X is a open subset of an open set U in the topology τ containing the point p.*

Definition 5.0.19 (Limit point). *Suppose τ is a topology for a set X. Let p be a point in the set X. A limit point p in the set X is a point such that it is in every open set contains p and one distinct point that is not p.* [6]

[6] The motivation is that of a limit point of a sequence.

Definition 5.0.20 (Adherent point). *Suppose τ is a topology for a set X. Let p be a point in the set X. A adherent point p in the set X is a point such that it is in every open set contains p and one other point that may be equal to the point p.*

Definition 5.0.21 (ω-accumulation point). *Suppose τ is a topology for a set X. Let p be a point in the set X. A ω-accumulation point p in the set X is a point such that it is in every open set contains p and infinitely many points that is not the point p.*

Definition 5.0.22 (Condensation point). *Suppose τ is a topology for a set X. Let p be a point in the set X. A condensation point p in the set X is a point such that it is in every open set contains p and uncountably infinitely many points that is not the point p.*

Definition 5.0.23 (Derived set). *Suppose τ is a topology for a set X. A derived set $D(A)$ of a set A which is a subset of the set X under the topology τ is the collection of all the limit points of the subset A.*

Definition 5.0.24 (Isolated point). *Suppose τ is a topology for a set X. A derived set $D(A)$ of a set A which is a subset of the set X under the topology τ is the collection of all the limit points of the subset A.*

An isolated point is a point in the subset A that is not in the derived set $D(A)$.

Definition 5.0.25 (Dense in itself). *Suppose τ is a topology for a set X. A derived set $D(A)$ of a set A which is a subset of the set X under the topology τ is the collection of all the limit points of the subset A.*

An isolated point is a point in the subset A that is not in the derived set $D(A)$.

A set without any isolated point is a set that is dense in itself.

Definition 5.0.26 (Perfect set). *Suppose τ is a topology for a set X. A derived set $D(A)$ of a set A which is a subset of the set X under the topology τ is the collection of all the limit points of the subset A.*

An isolated point is a point in the subset A that is not in the derived set $D(A)$.

A closed set without any isolated point is a set that is dense in itself. This is defined to be a perfect set.

Definition 5.0.27 (Closure set). *Suppose τ is a topology for a set X. The closure of a set is a set together with its limit points.*

Definition 5.0.28 (Pointed topological space). *A pointed object is an object equipped with a global element.*[7]

Recall from Definition 4.0.16 that a global element is a morphism from the terminal object to that object.

Therefore, a pointed topological space is a pointed object in the category of set. Explicitly, it is a topological set, equipped with a continuous map from a singleton in that topological space to the pointed topological space.

[7] See Definition 4.0.51.

Definition 5.0.29 (Connected topological space, disjoint union definition). *A topological space (X, τ) is connected if the collection of subsets in the topology τ cannot be written as a disjoint union of two open sets.*

Definition 5.0.30 (Connected topological space, connected objects). *A connected object is an object whose hom functor out of the object to a fixed object is preserves coproducts.* [8]

[8] The colimit of connected objects is a connected object.

A connected topological space is a connected object in the category of topological spaces. This means that the map:

$$\mathrm{Hom}(X, Y) + \mathrm{Hom}(X, Z) \to \mathrm{Hom}(X, Y + Z) \tag{5.1}$$

is a bijection on the level of sets.

Definition 5.0.31 (Filtered topological space, filtered object). *A filtered object is an object equipped with either an ascending or descending filtration.*[9]

A filtered topological space is a topological space (X, τ) where all open sets have a chain of opens under inclusion $U_1 \in U_2 \in ...X$ for every U_i in the topology τ.

[9] For example, a descending filtration has a sequence of morphisms as a graded object.

Definition 5.0.32 (Subspaces in topology, subobject definition). *A subobject can be defined as isomorphism classes of monomorphisms. Two monomorphisms are isomorphic if they are both monomorphisms into a object and there is a isomorphism between them such that when the isomorphism composed to a monomorphism, this gives equality to the other monomorphism.*

A topological subspace can be defined in the same way. Monomorphisms are injective continuous maps. So a subspace topology (S, τ_S) of a topological space (X, τ) for a subset S of X can be defined as an injection from $i : (S, \tau_S) \to (X, \tau)$ made into "isomorphism classes" by picking the coarsest topology of all possible topologies, and this injection is continuous.

Definition 5.0.33 (Topological embedding). *A subspace topology (S, τ_S) of a topological space (X, τ) for a subset S of X can be defined as an injection from $i : (S, \tau_S) \to (X, \tau)$ made into "isomorphism classes" by picking the coarsest topology of all possible topologies, and this injection is continuous. We have that the topological space (S, τ_S) is homeomorphic to its image in the topological space (X, τ).*

This injection is called a topological embedding.

Definition 5.0.34 (Open map). *An open map is a map that maps open sets to open sets.*

Definition 5.0.35 (Closed map). *An closed map is a map that maps closed sets to closed sets.*

Definition 5.0.36 (Open subspace). *An open subspace is a subspace formed by an injective open map.*

Definition 5.0.37 (Closed subspace). *An closed subspace is a subspace formed by an injective closed map.*

Examples

Example 5.0.38 (Euclidean metric space)**.** *Work with one dimension. Suppose X is the real line $\mathbb{R}^1$. One can put the Euclidean metric μ on it. For two points x_1 and x_2 in X, and without loss of generality we use the fact that the reals are well ordered where $x_1 > x_2$:*

$$\mu(x_1, x_2) = \sqrt{x_1^2 - x_2^2} \tag{5.2}$$

The use of the ordering is not ideal. We can actually take the square of the metric μ instead.

Example 5.0.39 (Indiscrete topology)**.** *Let X be a set. The topology consisting of the empty set and the set X is the indiscrete topology.*

Proof. The arbitrary union of all open subsets is the open set which is the space X. The finite intersection of disjoint open subsets is the empty set. □

Remark 5.0.40 (Indiscrete topology remarks)**.** *1. This topology is comparable to every other topology.*

Proof. Every topology τ must contain the space X and the empty set. This is because of the following reasons:

The arbitrary union of all open subsets is the open set which is the space X. The finite intersection of disjoint open subsets is the empty set.

Therefore, the indiscrete topology is comparable to every other topology. □

Example 5.0.41 (Discrete topology)**.** *Let X be a set. The topology consisting of all subsets of the set X is the discrete topology.*

Proof. Arbitrary unions of subsets of a set X is a subset of the set X (recall that the full set X is a subset of the set X).

Finite intersections of subsets of a set X is a subset of the set X.

□

Example 5.0.42 (Finite particular point topology)**.** *Let X be a finite set. Let p be a point in the finite set X. Consider the topology τ consisting of arbitrary unions and finite intersection of subsets of the finite set containing the point p.*

Proof. This is a topology by definition. We make such a topology τ coarser by including only the open sets in the collection τ that contain the point p. □

Example 5.0.43 (Countably infinite particular point topology). *Let X be a countably inffinite set. Let p be a point in the countably infinite set X. Consider the topology τ consisting of arbitrary unions and finite intersection of subsets of the finite set containing the point p.*

Proof. This is a topology by definition. We make such a topology τ coarser by including only the open sets in the collection τ that contain the point p. □

Example 5.0.44 (Uncountably infinite particular point topology). *Let X be a uncountably infinite set. Let p be a point in the uncountably infinite set X. Consider the topology τ consisting of arbitrary unions and finite intersection of subsets of the finite set containing the point p.*

Proof. This is a topology by definition. We make such a topology τ coarser by including only the open sets in the collection τ that contain the point p. □

Example 5.0.45 (Zariski topology). *Let X be the collection of algebraic sets. An algebraic set is the locus of zeros of polynomials. The Zariski topology is the topology formed by the collection of algebraic sets of a space as the closed sets.*[10]

[10] This is not fully defined since we did not say what an algebraic set really is. And we need to prove this.

Proof. Omitted. To be filled in. □

Lemmas

Lemma 5.0.46 (Pigeonhole principle for measure spaces). *Let E_1, E_2 be an at most countably infinite sequence of measurable subsets of a measurable space $(X, \mathcal{X}, \mu)$. If the union up to E_n or $\cup_n E_n$ has a positive measure, then at least one of the E_i in this union has a positive measure.*

Proof. We define a null set to be a set contained in a set of measure zero. By the definition of a measure, a measure is countably additive. The countable union of countably additive null sets of measure zero, is therefore measure zero. By taking contrapositives, if the countable union of countably additive sets are not measure zero, then at least one of these sets have a non-null measure. Lastly, this lemma is a corollary of the contrapositive. □

Exercises

Exercise 5.0.47. *Show that not all topologies are comparable.*

Proof. Consider a set of two elements, with the finite particular point topology on each point as per Example 5.0.42. The topologies generated are not comparable. □

Exercise 5.0.48. *Show that a subset in a topology can be open and closed.*

Proof. Consider an two point set with the indiscrete topology as per Example 5.0.39. □

Exercise 5.0.49. *Show that a topology can be comparable to every other topology.*

Proof. Consider a set with the indiscrete topology as per Example 5.0.39. It is coarser than every other topology.

By definition, a topology must contain the indiscrete topology which consists of arbitrary unions of all subsets which is the full set, as well as finite intersections of all subsets which is the empty set for disjoint sets. □

Exercise 5.0.50 (Motivation for isolated point)**.** *Show that an isolated point is a point contained in an open set with no other point of a subset A in a topological space (X, τ).*

Proof. By Definition 5.0.24, an open set containing an isolated point does not contain a limit point. Since it does not contain a limit point, there is no point whose open neighbourhood in the subset containing the isolated point as a distinct point. Therefore, the open set containing the isolated point must no other point in open set since by case analysis either the isolated point is a limit point in the open set which is not true, or there is a limit point in the open set which is not true. □

Exercise 5.0.51 (Closed set contains all of its limit points)**.** *Show that a closed set contains all of its limit points.*

Proof. The complement of the full set and its limit point is a open set by the definition of a closed set or Defintion 5.0.5, and every point in a closed set satisfies this condition.

Therefore, a closed set contains all of its limit points. □

Exercise 5.0.52 (Perfect set equal to derived set)**.** *Show that a set is perfect if and only if it is equal to its derived set.*

Proof. Suppose a set is perfect. A closed set contains all of its limit points. This is because the complement of the full set and its limit point is a open set by the definition of a closed set or Defintion 5.0.5, and every point in a closed set satisfies this condition. This complement, denoted by $A - A$ for closed set A and full set X is open, containing limit point x and no points of A. It is therefore a derived set.

A set containing all its limit points is closed since $X - A$ contains a neighbourhood of each of its limit points, and hence it is open. This is a perfect set, as this is a closed set without any isolated points.

Therefore, a set is perfect if and only if it is equal to its derived set. $\square$

6
Representation theory

Definitions

Definition 6.0.1 (Skew field)**.** *A skew field k is a possibly noncommutative ring with an identity element* 1*, with* $1 \neq 0$*, where every nonzero element x in k has a multiplicative inverse.*

7
Algebraic geometry

Examples

Example 7.0.1 (Blowing up to get gradients)**.** *The problem is to describe the gradients m of all line $y = mx + c$ in the real plane $\mathbb{R}^2$.*

For vertical lines, add the point at infinity. Therefore, the gradients m can be distinguished by a projective line $\mathbb{P}^1$.

A more general example would be to turn this into Grassmannians.

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