Introduction to finiteness conditions

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Consider a unital associative commutative ring Mod_A as A-modules.

A module M in Mod_A is finitely generated if and only if there exists an A-linear surjection $A^{\oplus n} \to M$ for nonzero n. The intuition for this is that there is a finite generating set (not a basis, since need not be linearly independent).

A module M in Mod_A is finitely presented if and only if there exists an exact sequence $A^{\oplus m} \to A^{\oplus n} \to M \to 0$ for nonzero m and n. The motivation is that categorically, these are the compact objects in the category of A-modules. An example would be that finite sets are the compact objects in the category of sets. If the commutative ring A is a field, then the compact objects are precisely the finite dimensional vector spaces.

Lemma 0.1. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence.*

If M' and M'' are finitely generated, then so is M.

If M' and M'' are finitely presented, then so is M.

If M' is finitely generated and M is finitely presented, then M'' is finitely presented.

If M is finitely generated then M'' is finitely generated.

If M is finitely generated, and M'' is finitely presented, then M' is finitely generated.

Lemma 0.2. *If A is noetherian, then M is a finitely generated A-module if and only if M is also a finitely presented A-module.*

Proof. Suppose A-module M is finitely generated. Consider $A^{\oplus n} \to M$. Since the A-module M is finitely generated, we can choose K to be such that $0 \to K \to A^{\oplus n} \to M \to 0$. This is the key step: since A is noetherian, therefore K is finitely generated. Since K is also finitely generated, by definition one can construct $A^{\oplus m} \to K \to A^{\oplus n}$. This gives the construction of an exact sequence for M to be finitely presented where $A^{\oplus m} \to A^{\oplus n} \to M \to 0$. Therefore, the A-module M must be finitely presented since this construction satisfies the definition. $\square$

Example 0.1. An example would be an ideal. Let I be in an ideal contained in module A. Consider the surjection from A to the quotient of A over the ideal A/I. This quotient A/I must be finitely generated.