Heat Conduction

Notes based on practice, tutorial and past year examination papers

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Typical Assumptions

For examination style problems, we typically assume the following:

1. Steady heat transfer.
2. Constant thermal properties (check if thermal conductivity is constant)
3. Negligible radiation exchange.
4. No heat generation (do check)
5. 1D/2D/3D heat transfer
6. No mass transfer (do check)
7. Idealised geometry (plane, wall, semi-infinite solid)
8. Thermal properties are linearly interpolated.

Integration

BASIC FORMS

(1) $\int x^n dx = \frac{1}{n+1} x^{n+1}$

(2) $\int \frac{1}{x} dx = \ln x$

(3) $\int u dv = uv - \int v du$

(4) $\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$

RATIONAL FUNCTIONS

(5) $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln(ax+b)$

(6) $\int \frac{1}{(x+a)^2} dx = \frac{-1}{x+a}$

(7) $\int (x+a)^n dx = (x+a)^n \left(\frac{a}{1+n} + \frac{x}{1+n} \right), \ n \neq -1$

(8) $\int x(x+a)^n dx = \frac{(x+a)^{1+n}(nx+x-a)}{(n+2)(n+1)}$

(9) $\int \frac{dx}{1+x^2} = \tan^{-1} x$

(10) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}(x/a)$

(11) $\int \frac{xdx}{a^2+x^2} = \frac{1}{2} \ln(a^2+x^2)$

(12) $\int \frac{x^2 dx}{a^2+x^2} = x - a \tan^{-1}(x/a)$

(13) $\int \frac{x^3 dx}{a^2+x^2} = \frac{1}{2} x^2 - \frac{1}{2} a^2 \ln(a^2+x^2)$

(14) $\int (ax^2+bx+c)^{-1} dx = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}} \right)$

(15) $\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \left[\ln(a+x) - \ln(b+x) \right], \ a \neq b$

(16) $\int \frac{x}{(x+a)^2} dx = \frac{a}{a+x} + \ln(a+x)$

(17) $$\int \frac{x}{ax^2+bx+c} dx = \frac{\ln(ax^2+bx+c)}{2a} - \frac{b}{a\sqrt{4ac-b^2}} \tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}} \right)$$

INTEGRALS WITH ROOTS

(18) $\int \sqrt{x-a}\, dx = \frac{2}{3}(x-a)^{3/2}$

(19) $\int \frac{1}{\sqrt{x \pm a}} dx = 2\sqrt{x \pm a}$

(20) $\int \frac{1}{\sqrt{a-x}} dx = 2\sqrt{a-x}$

(21) $\int x\sqrt{x-a}\, dx = \frac{2}{3} a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2}$

(22) $\int \sqrt{ax+b}\, dx = \left(\frac{2b}{3a} + \frac{2x}{3} \right) \sqrt{b+ax}$

(23) $\int (ax+b)^{3/2} dx = \sqrt{b+ax} \left(\frac{2b^2}{5a} + \frac{4bx}{5} + \frac{2ax^2}{5} \right)$

(24) $\int \frac{x}{\sqrt{x \pm a}} dx = \frac{2}{3}(x \pm 2a)\sqrt{x \pm a}$

(25) $\int \sqrt{\frac{x}{a-x}} dx = -\sqrt{x}\sqrt{a-x} - a \tan^{-1}\left(\frac{\sqrt{x}\sqrt{a-x}}{x-a} \right)$

(26) $\int \sqrt{\frac{x}{x+a}} dx = \sqrt{x}\sqrt{x+a} - a \ln\left[\sqrt{x} + \sqrt{x+a} \right]$

(27) $\int x\sqrt{ax+b}\, dx = \left(-\frac{4b^2}{15a^2} + \frac{2bx}{15a} + \frac{2x^2}{5} \right) \sqrt{b+ax}$

(28) $$\int \sqrt{x}\sqrt{ax+b}\, dx = \left(\frac{b\sqrt{x}}{4a} + \frac{x^{3/2}}{2} \right) \sqrt{b+ax} - \frac{b^2 \ln\left(2\sqrt{a}\sqrt{x} + 2\sqrt{b+ax} \right)}{4a^{3/2}}$$

(29) $$\int x^{3/2} \sqrt{ax+b}\, dx = \left(-\frac{b^2\sqrt{x}}{8a^2} + \frac{bx^{3/2}}{12a} + \frac{x^{5/2}}{3} \right) \sqrt{b+ax} - \frac{b^3 \ln\left(2\sqrt{a}\sqrt{x} + 2\sqrt{b+ax} \right)}{8a^{5/2}}$$

(30) $\int \sqrt{x^2 \pm a^2}\, dx = \frac{1}{2} x\sqrt{x^2 \pm a^2} \pm \frac{1}{2} a^2 \ln\left(x + \sqrt{x^2 \pm a^2} \right)$

(31) $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} - \frac{1}{2} a^2 \tan^{-1}\left(\frac{x\sqrt{a^2-x^2}}{x^2-a^2} \right)$

(32) $\int x\sqrt{x^2 \pm a^2} = \frac{1}{3}(x^2 \pm a^2)^{3/2}$

(33) $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln\left(x + \sqrt{x^2 \pm a^2} \right)$

(34) $\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

(35) $\int \frac{x}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$

(36) $\int \frac{x}{\sqrt{a^2-x^2}}dx = -\sqrt{a^2-x^2}$

(37) $\int \frac{x^2}{\sqrt{x^2 \pm a^2}}dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \mp \frac{1}{2}\ln\left(x+\sqrt{x^2 \pm a^2}\right)$

(38) $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = -\frac{1}{2}x\sqrt{a-x^2} - \frac{1}{2}a^2\tan^{-1}\left(\frac{x\sqrt{a^2-x^2}}{x^2-a^2}\right)$

(39)
$$\int \sqrt{ax^2+bx+c}\ dx = \left(\frac{b}{4a}+\frac{x}{2}\right)\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a^{3/2}}\ln\left(\frac{2ax+b}{\sqrt{a}}+2\sqrt{ax^2+bc+c}\right)$$

(40)
$$\int x\sqrt{ax^2+bx+c}\ dx = \left(\frac{x^3}{3}+\frac{bx}{12a}+\frac{8ac-3b^2}{24a^2}\right)\sqrt{ax^2+bx+c} - \frac{b(4ac-b^2)}{16a^{5/2}}\ln\left(\frac{2ax+b}{\sqrt{a}}+2\sqrt{ax^2+bc+c}\right)$$

(41) $\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{a}}\ln\left[\frac{2ax+b}{\sqrt{a}}+2\sqrt{ax^2+bx+c}\right]$

(42)
$$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}}\ln\left[\frac{2ax+b}{\sqrt{a}}+2\sqrt{ax^2+bx+c}\right]$$

LOGARITHMS

(43) $\int \ln x dx = x\ln x - x$

(44) $\int \frac{\ln(ax)}{x}dx = \frac{1}{2}(\ln(ax))^2$

(45) $\int \ln(ax+b)dx = \frac{ax+b}{a}\ln(ax+b) - x$

(46) $\int \ln(a^2x^2 \pm b^2)dx = x\ln(a^2x^2 \pm b^2) + \frac{2b}{a}\tan^{-1}\left(\frac{ax}{b}\right) - 2x$

(47) $\int \ln(a^2-b^2x^2)dx = x\ln(a^2-b^2x^2) + \frac{2a}{b}\tan^{-1}\left(\frac{bx}{a}\right) - 2x$

(48)
$$\int \ln(ax^2+bx+c)dx = \frac{1}{a}\sqrt{4ac-b^2}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) - 2x + \left(\frac{b}{2a}+x\right)\ln\left(ax^2+bx+c\right)$$

(49) $\int x\ln(ax+b)dx = \frac{b}{2a}x - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2-\frac{b^2}{a^2}\right)\ln(ax+b)$

(50) $\int x\ln(a^2-b^2x^2)dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2-\frac{a^2}{b^2}\right)\ln(a^2-bx^2)$

EXPONENTIALS

(51) $\int e^{ax}dx = \frac{1}{a}e^{ax}$

(52) $\int \sqrt{x}e^{ax}dx = \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\mathrm{erf}\left(i\sqrt{ax}\right)$ where

$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$

(53) $\int xe^x dx = (x-1)e^x$

(54) $\int xe^{ax}dx = \left(\frac{x}{a}-\frac{1}{a^2}\right)e^{ax}$

(55) $\int x^2e^x dx = e^x(x^2-2x+2)$

(56) $\int x^2e^{ax}dx = e^{ax}\left(\frac{x^2}{a}-\frac{2x}{a^2}+\frac{2}{a^3}\right)$

(57) $\int x^3e^x dx = e^x(x^3-3x^2+6x-6)$

(58) $\int x^n e^{ax}dx = (-1)^n\frac{1}{a}\Gamma[1+n,-ax]$ where

$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$

(59) $\int e^{ax^2}dx = -i\frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(ix\sqrt{a}\right)$

TRIGONOMETRIC FUNCTIONS

(60) $\int \sin x dx = -\cos x$

(61) $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4}\sin 2x$

(62) $\int \sin^3 x dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x$

(63) $\int \cos x dx = \sin x$

(64) $\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4}\sin 2x$

(65) $\int \cos^3 x dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x$

(66) $\int \sin x\cos x dx = -\frac{1}{2}\cos^2 x$

(67) $\int \sin^2 x \cos x dx = \frac{1}{4}\sin x - \frac{1}{12}\sin 3x$

(68) $\int \sin x \cos^2 x dx = -\frac{1}{4}\cos x - \frac{1}{12}\cos 3x$

(69) $\int \sin^2 x \cos^2 x dx = \frac{x}{8} - \frac{1}{32}\sin 4x$

(70) $\int \tan x dx = -\ln \cos x$

(71) $\int \tan^2 x dx = -x + \tan x$

(72) $\int \tan^3 x dx = \ln[\cos x] + \frac{1}{2}\sec^2 x$

(73) $\int \sec x dx = \ln|\sec x + \tan x|$

(74) $\int \sec^2 x dx = \tan x$

(75) $\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x \tan x|$

(76) $\int \sec x \tan x dx = \sec x$

(77) $\int \sec^2 x \tan x dx = \frac{1}{2}\sec^2 x$

(78) $\int \sec^n x \tan x dx = \frac{1}{n}\sec^n x, \quad n \neq 0$

(79) $\int \csc x dx = \ln|\csc x - \cot x|$

(80) $\int \csc^2 x dx = -\cot x$

(81) $\int \csc^3 x dx = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln|\csc x - \cot x|$

(82) $\int \csc^n x \cot x dx = -\frac{1}{n}\csc^n x, \quad n \neq 0$

(83) $\int \sec x \csc x dx = \ln \tan x$

TRIGONOMETRIC FUNCTIONS WITH x^n

(84) $\int x \cos x dx = \cos x + x \sin x$

(85) $\int x \cos(ax) dx = \frac{1}{a^2}\cos ax + \frac{1}{a} x \sin ax$

(86) $\int x^2 \cos x dx = 2x \cos x + (x^2 - 2)\sin x$

(87) $\int x^2 \cos ax dx = \frac{2}{a^2} x \cos ax + \frac{a^2 x^2 - 2}{a^3}\sin ax$

(88) $\int x^n \cos x dx = -\frac{1}{2}(i)^{1+n}\left[\Gamma(1+n,-ix) + (-1)^n \Gamma(1+n,ix)\right]$

(89) $\int x^n \cos ax dx = \frac{1}{2}(ia)^{1-n}\left[(-1)^n \Gamma(1+n,-iax) - \Gamma(1+n,iax)\right]$

(90) $\int x \sin x dx = -x \cos x + \sin x$

(91) $\int x \sin(ax) dx = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax$

(92) $\int x^2 \sin x dx = (2 - x^2)\cos x + 2x \sin x$

(93) $\int x^3 \sin ax dx = \frac{2 - a^2 x^2}{a^3}\cos ax + \frac{2}{a^3} x \sin ax$

(94) $\int x^n \sin x dx = -\frac{1}{2}(i)^n\left[\Gamma(n+1,-ix) - (-1)^n \Gamma(n+1,-ix)\right]$

TRIGONOMETRIC FUNCTIONS WITH e^{ax}

(95) $\int e^x \sin x dx = \frac{1}{2}e^x[\sin x - \cos x]$

(96) $\int e^{bx} \sin(ax) dx = \frac{1}{b^2 + a^2}e^{bx}[b \sin ax - a \cos ax]$

(97) $\int e^x \cos x dx = \frac{1}{2}e^x[\sin x + \cos x]$

(98) $\int e^{bx} \cos(ax) dx = \frac{1}{b^2 + a^2}e^{bx}[a \sin ax + b \cos ax]$

TRIGONOMETRIC FUNCTIONS WITH x^n AND e^{ax}

(99) $\int x e^x \sin x dx = \frac{1}{2}e^x[\cos x - x \cos x + x \sin x]$

(100) $\int x e^x \cos x dx = \frac{1}{2}e^x[x \cos x - \sin x + x \sin x]$

HYPERBOLIC FUNCTIONS

(101) $\int \cosh x dx = \sinh x$

(102) $\int e^{ax} \cosh bx dx = \frac{e^{ax}}{a^2 - b^2}[a \cosh bx - b \sinh bx]$

(103) $\int \sinh x dx = \cosh x$

(104) $\int e^{ax} \sinh bx dx = \frac{e^{ax}}{a^2 - b^2}[-b \cosh bx + a \sinh bx]$

(105) $\int e^x \tanh x dx = e^x - 2\tan^{-1}(e^x)$

(106) $\int \tanh ax dx = \frac{1}{a}\ln \cosh ax$

(107) $\int \cos ax \cosh bx dx = \frac{1}{a^2 + b^2}[a \sin ax \cosh bx + b \cos ax \sinh bx]$

(108) $\int \cos ax \sinh bx dx = \frac{1}{a^2+b^2}[b\cos ax \cosh bx + a\sin ax \sinh bx]$

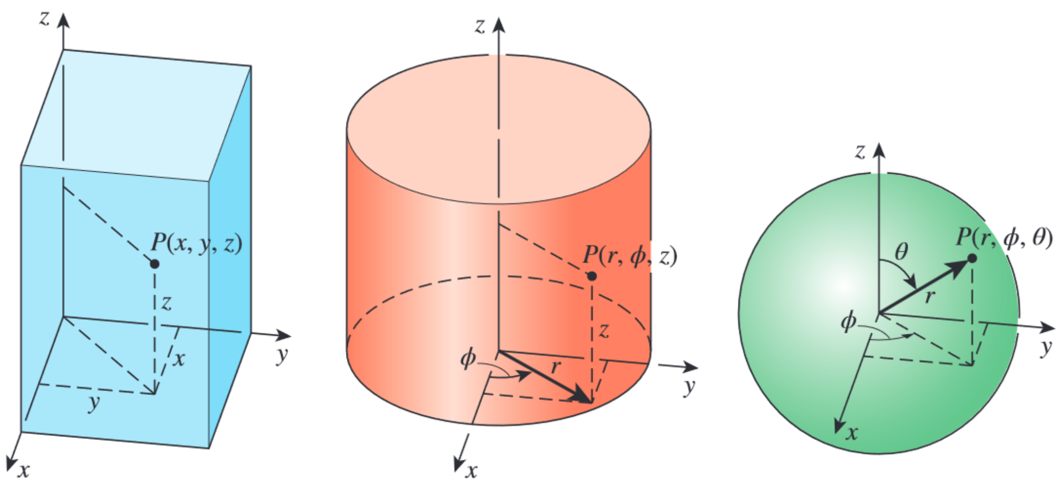
(109) $\int \sin ax \cosh bx dx = \frac{1}{a^2+b^2}[-a\cos ax \cosh bx + b\sin ax \sinh bx]$

(110) $\int \sin ax \sinh bx dx = \frac{1}{a^2+b^2}[b\cosh bx \sin ax - a\cos ax \sinh bx]$

(111) $\int \sinh ax \cosh ax dx = \frac{1}{4a}[-2ax + \sinh(2ax)]$

(112) $\int \sinh ax \cosh bx dx = \frac{1}{b^2-a^2}[b\cosh bx \sinh ax - a\cosh ax \sinh bx]$

Coordinate Systems



(a) Rectangular coordinates (b) Cylindrical coordinates (c) Spherical coordinates

Heat Generation

$$T_{s,\text{plane wall}} = T_\infty + \frac{\dot{e}_{\text{gen}} L}{h}$$

$$T_{s,\text{cylinder}} = T_\infty + \frac{\dot{e}_{\text{gen}} r_o}{2h}$$

$$T_{s,\text{sphere}} = T_\infty + \frac{\dot{e}_{\text{gen}} r_o}{3h}$$

$$\Delta T_{\text{max, plane wall}} = \frac{\dot{e}_{\text{gen}} L^2}{2k}$$

$$\Delta T_{\text{max, cylinder}} = \frac{\dot{e}_{\text{gen}} r_o^2}{4k}$$

$$\Delta T_{\text{max, sphere}} = \frac{\dot{e}_{\text{gen}} r_o^2}{6k}$$

Simplified Equations

General Equations

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \textbf{(2–39)}$$

(1) *Steady-state:*
(called the **Poisson equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{e}_{\text{gen}}}{k} = 0 \quad \textbf{(2–40)}$$

(2) *Transient, no heat generation:*
(called the **diffusion equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \textbf{(2–41)}$$

(3) *Steady-state, no heat generation:*
(called the **Laplace equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad \textbf{(2–42)}$$

1D Heat Transfer Equations

In general:

Variable conductivity:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{e}_{\text{gen}} = \rho c\frac{\partial T}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{e}_{\text{gen}} = \rho c\frac{\partial T}{\partial t} \qquad \textbf{(2–43)}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{e}_{\text{gen}} = \rho c\frac{\partial T}{\partial t} \qquad \textbf{(2–44)}$$

Constant conductivity:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{e}_{\text{gen}}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

(1) *Steady-state:* $(\partial/\partial t = 0)$

$$\frac{d^2T}{dx^2} + \frac{\dot{e}_{\text{gen}}}{k} = 0 \qquad \textbf{(2–15)}$$

(2) *Transient, no heat generation:* $(\dot{e}_{\text{gen}} = 0)$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \textbf{(2–16)}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0 \text{ and } \dot{e}_{\text{gen}} = 0)$

$$\frac{d^2T}{dx^2} = 0 \qquad \textbf{(2–17)}$$

For a cylinder:

Variable conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t} \qquad \text{(2–25)}$$

For the case of constant thermal conductivity, the previous equation reduces to

Constant conductivity: $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(2–26)}$$

where again the property $\alpha = k/\rho c$ is the thermal diffusivity of the material. Eq. 2–26 reduces to the following forms under specified conditions (Fig. 2–15):

(1) *Steady-state:* $(\partial/\partial t = 0)$ $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0 \qquad \text{(2–27)}$$

(2) *Transient, no heat generation:* $(\dot{e}_{gen} = 0)$ $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(2–28)}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0$ and $\dot{e}_{gen} = 0)$ $$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \qquad \text{(2–29)}$$

For a
sphere:

Variable conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2k\frac{\partial T}{\partial r}\right) + \dot{e}_{gen} = \rho c \frac{\partial T}{\partial t} \qquad \text{(2–30)}$$

which, in the case of constant thermal conductivity, reduces to

Constant conductivity: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{\dot{e}_{gen}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(2–31)}$$

(1) *Steady-state:* $(\partial/\partial t = 0)$ $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0 \qquad \text{(2–32)}$$

(2) *Transient, no heat generation:* $(\dot{e}_{gen} = 0)$ $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(2–33)}$$

(3) *Steady-state, no heat generation:* $(\partial/\partial t = 0$ and $\dot{e}_{gen} = 0)$ $$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0 \quad \text{or} \quad r\frac{d^2T}{dr^2} + 2\frac{dT}{dr} = 0 \qquad \text{(2–34)}$$

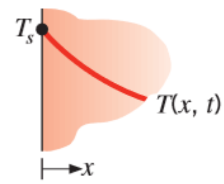
Plane Wall Conduction

Boundary Conditions

TABLE 2.2 Boundary conditions for the heat diffusion equation at the surface ($x = 0$)

1. Constant surface temperature

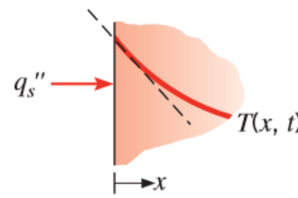
$$T(0, t) = T_s \qquad (2.31)$$



2. Constant surface heat flux

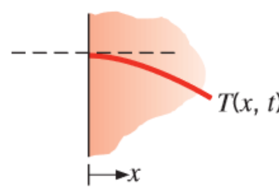
(a) Finite heat flux

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = q_s'' \qquad (2.32)$$



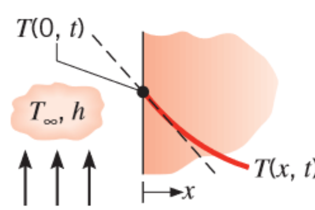
(b) Adiabatic or insulated surface

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad (2.33)$$



3. Convection surface condition

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0, t)] \qquad (2.34)$$



Directions for specified heat flux boundary conditions are :

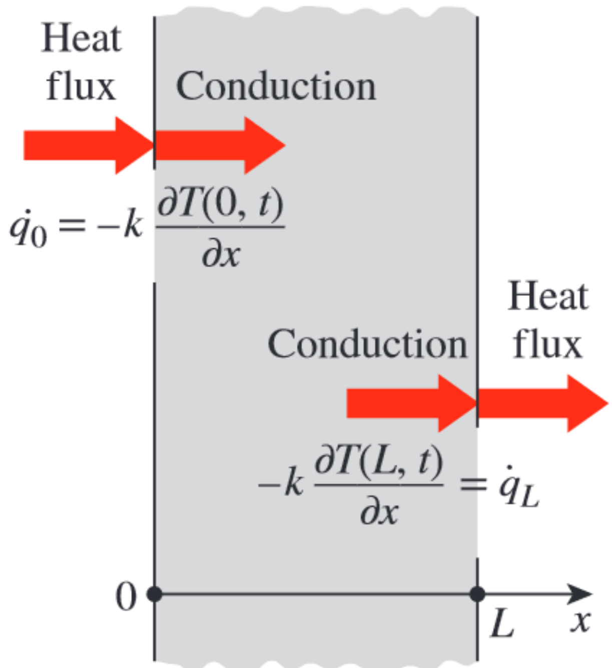
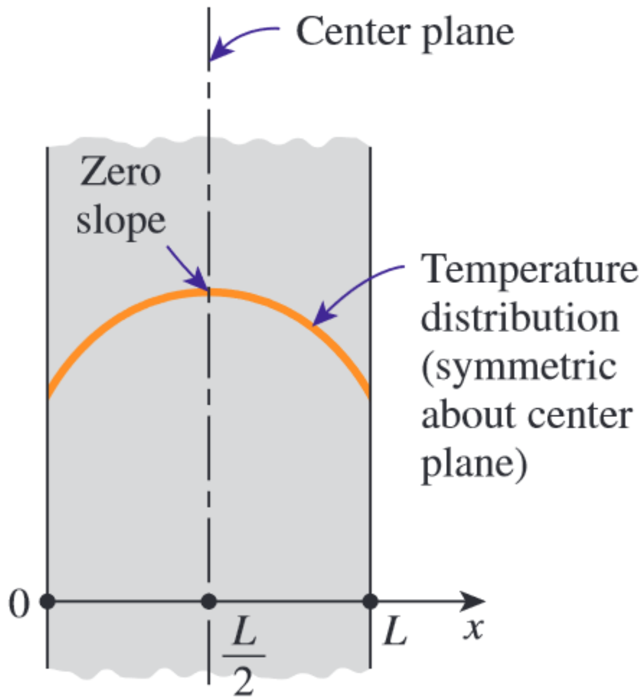


FIGURE 2–28
Specified heat flux boundary conditions on both surfaces of a plane wall.



$$\frac{\partial T(L/2,\, t)}{\partial x} = 0$$

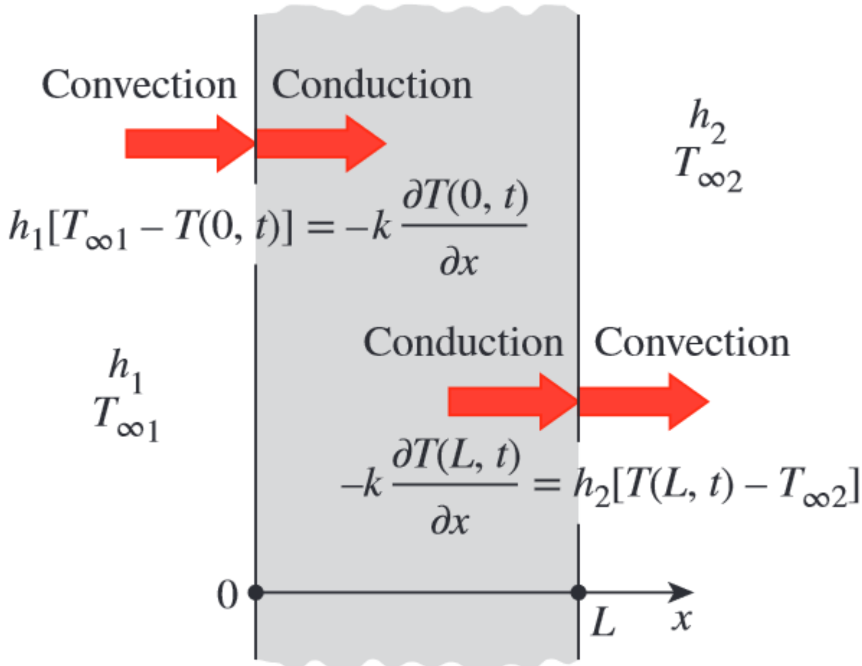


FIGURE 2–32
Convection boundary conditions on the two surfaces of a plane wall.

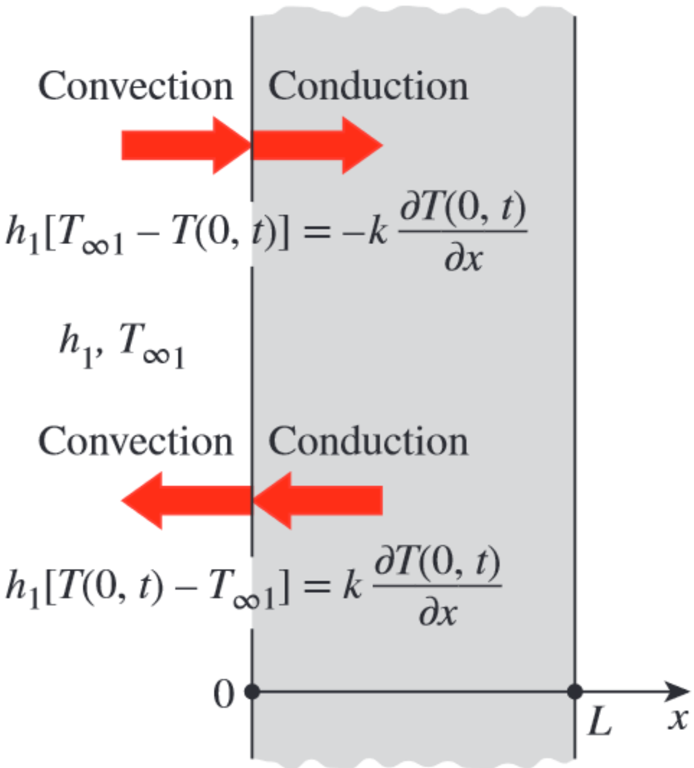


FIGURE 2–33

The assumed direction of heat transfer at a boundary has no effect on the boundary condition expression.

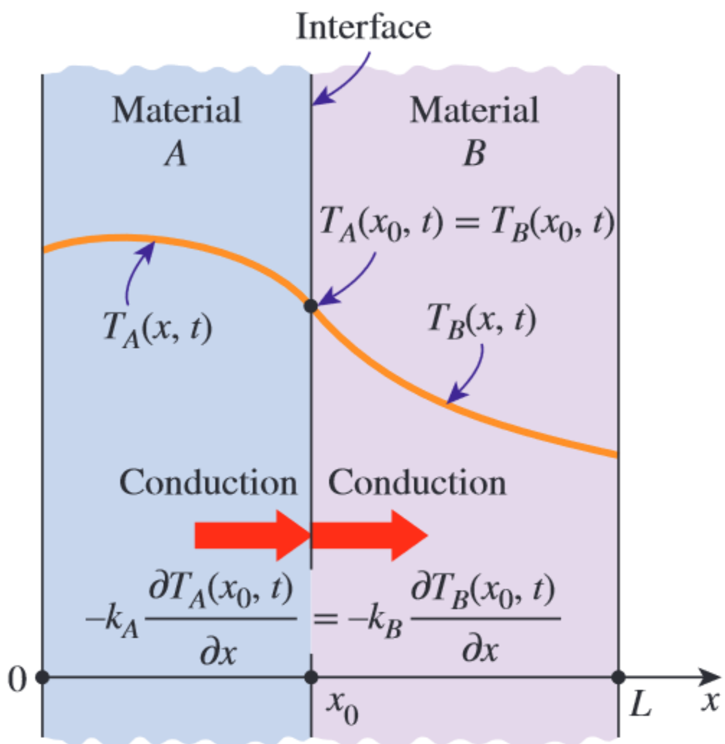


FIGURE 2–36
Boundary conditions at the interface of two bodies in perfect contact.

ODE Solutions

- 1-D steady-state heat conduction equation with constant heat generation—rectangular coordinates (Eq. 2–15)

$$\frac{d^2y}{dx^2} + S = 0$$

Solution: $y(x) = C_1x + C_2 - \frac{1}{2}Sx^2$

1-D steady-state heat conduction equation without heat generation—rectangular coordinates (Eq. 2–17)

$$\frac{d^2y}{dx^2} = 0$$

Solution: $y(x) = C_1x + C_2$

1-D steady-state heat conduction equation with constant heat generation—cylindrical coordinates (Eq. 2–27)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right) + S = 0$$

Solution: $y(r) = C_1 \ln r + C_2 - \frac{1}{4}Sr^2$

1-D steady-state heat conduction equation without heat generation—cylindrical coordinates (Eq. 2–29)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right) = 0$$

Solution: $y(r) = C_1 \ln r + C_2$

1-D steady-state heat conduction equation with constant heat generation—spherical coordinates (Eq. 2–32)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dy}{dr}\right) + S = 0$$

Solution: $y(r) = -\frac{C_1}{r} + C_2 - \frac{1}{6}Sr^2$

1-D steady-state heat conduction equation without heat generation—spherical coordinates (Eq. 2–34)

$$\frac{d}{dr}\left(r^2\frac{dy}{dr}\right) = 0$$

Solution: $y(r) = -\dfrac{C_1}{r} + C_2$

1-D steady-state fin or bioheat transfer equations for uniform cross section with constant coefficients—rectangular coordinates (Eq. 3–56 or Eq. 3–88)

$$\frac{d^2y}{dx^2} - \lambda^2 y = 0$$

Solution: $y(x) = C_1e^{+\lambda x} + C_2e^{-\lambda x}$

Variation of Thermal Conductivity

$$k_{\text{avg}} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1}$$

$$\dot{Q}_{\text{plane wall}} = k_{\text{avg}}A\frac{T_1 - T_2}{L} = \frac{A}{L}\int_{T_2}^{T_1} k(T)dT \qquad \textbf{(2–76)}$$

$$\dot{Q}_{\text{cylinder}} = 2\pi k_{\text{avg}}L\frac{T_1 - T_2}{\ln(r_2/r_1)} = \frac{2\pi L}{\ln(r_2/r_1)}\int_{T_2}^{T_1} k(T)dT \qquad \textbf{(2–77)}$$

$$\dot{Q}_{\text{sphere}} = 4\pi k_{\text{avg}}r_1r_2\frac{T_1 - T_2}{r_2 - r_1} = \frac{4\pi r_1r_2}{r_2 - r_1}\int_{T_2}^{T_1} k(T)\,dT \qquad \textbf{(2–78)}$$

$$k_{\text{avg}} = \frac{\int_{T_1}^{T_2} k_0(1 + \beta T)dT}{T_2 - T_1} = k_0\left(1 + \beta\frac{T_2 + T_1}{2}\right) = k(T_{\text{avg}})$$