Capillary Hypersurfaces and Variational Methods

in Positively Curved Manifolds with Boundary

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Minimal Surfaces in Real Life

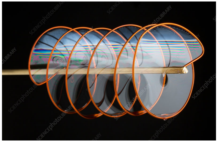 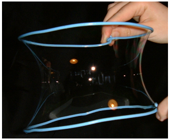 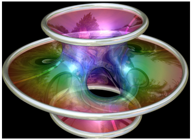

Image credit: Ted Kinsman, Blinking Spirit, Paul Nylander

Minimizing area while fixing the boundary (the wire): existence and regularity. This is the Plateau's problem.

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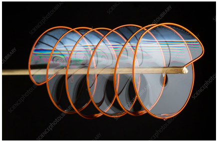 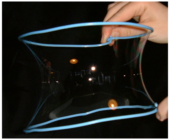 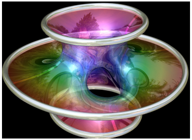

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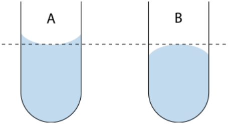
Image credit: Malte Sörensen, Kate Fraser, Joaquim Alves Gaspar

As one blows air into a soap bubble, the surface tension increase while enclosing a fixed "volume" inside the bubble.
Does the sphere minimize area given a fixed volume inside?
This is the isoperimetric problem.

Minimal Surfaces in Real Life

When we put liquid into a tube, the surface tension balances with adhesion between the tube and the liquid.

Capillary Action



A: Capillary attraction.
B: Capillary repulsion.

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No gravity: surfaces have constant mean curvature and constant angle along the container.

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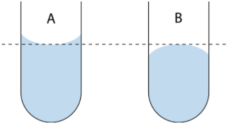

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Applications of capillary action can be seen in many aspects of life.

Image Credit: Pat Hastings, Content Pixie

Minimal Surfaces for Mathematicians

- In 1762, Lagrange found the Euler-Lagrange equation for Plateau's problem of a graph $z = z(x, y)$ in $\mathbb{R}^3$,

$$\operatorname{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}}\right) = 0.$$

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- In 18 and 19th century, more minimal surfaces are discovered, including the catenoid and helicoid (1744 Euler, 1776 Meusnier).
- The Plateau's problem for surfaces was completely solved in 1930 independently by Douglas and Radó.

Minimal Surfaces in Modern Days

- Extending the existence and smoothness of minimizers of Plateau's problem to higher dimensions turn out to be difficult.
- Singularities could occur for hypersurfaces in dimension 8 or higher, or for codimension 2 or more.
- Efforts in these directions contributed massively to the development of geometric measure theory.

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Schoen-Yau 1979, Gromov-Lawson 1983, Geroch Conjecture

Consider X^n a closed manifold and $\mathbb{T}^n$ the n-torus ($3 \leq n \leq 7$), then $\mathbb{T}^n \# X$ has no PSC metric.

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Schoen-Yau 1979, Positive Mass Theorem

Let (M^n, g) be an asymptotically flat manifold with $R_g \geq 0$, $3 \leq n \leq 7$, then its ADM mass $m_g \geq 0$, and $m_g = 0$ if and only if M is isometric to the Euclidean space.

Geroch Conjecture ⇒ Positive Mass Theorem

Idea of Proof.

We show how the Geroch conjecture implies $m_g \geq 0$ in this setting.

- Lokhamp: if $m_g < 0$, then M has a metric $\hat{g}$ with $R_{\hat{g}} \geq 0$, $(M \setminus K, \hat{g})$ is isometric to $\mathbb{R}^n \setminus B_R(0)$, and $R_{\hat{g}}(x_0) > 0$.

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- Kazdan-Warner, Kazdan: for a closed manifold $N^n, n \geq 3$ with $R_N \geq 0, \mathrm{Ric} \not\equiv 0$, then N has a PSC metric.
- Then we obtain a contradiction.

$\square$

Proof of Geroch Conjecture

Geometric Idea

Stable minimal hypersurfaces of a PSC manifolds also admit a metric of PSC (after a conformal change of metric). This method is called Schoen-Yau's conformal descent method.

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- A torus $\mathbb{T}^n$ has enough topology so we can minimize in a non-trivial homology class inductively, to obtain a chain of nested stable minimal hypersurfaces, $\mathbb{T}^n \supset \Sigma_{n-1} \supset \Sigma_{n-2}, ..., \supset \Sigma_2$.

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- For $n \leq 7$, these minimizers must be smooth.
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Roughly speaking, a μ-bubble in a manifold (M^n, g) is a smooth open set Ω that minimizes the following functional, given $h \in C^\infty(M)$,

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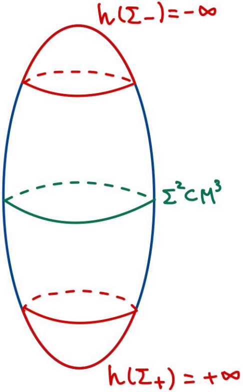
The first variation: $H_\Sigma = h|_\Sigma$. The $\mathcal{A}(\cdot)$ is also called the "prescribed mean curvature" (PMC) functional.

The Method of μ-Bubble: A Model Case for 3-Manifolds

We choose h on (M^3, g) so that the sets $\Sigma_\pm := \{x \in M, h(x) = \pm\infty\}$ serves as "barriers" to constrain and make sure a minimizer must exist.



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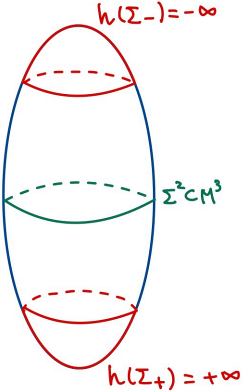

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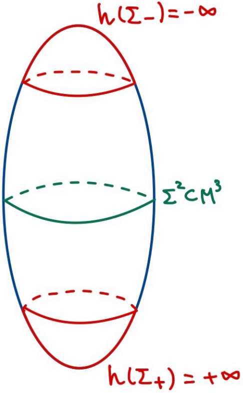

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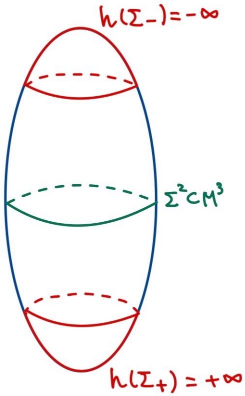

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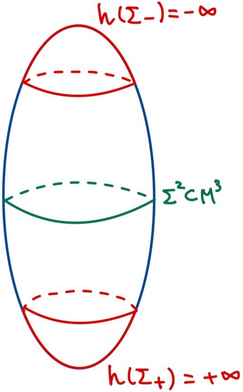
- Localization if M is non-compact.



Topology and Geometry of PSC manifolds

The μ-bubble method allows us to obtain new geometric estimates for PSC 3-manifolds.

- Topologically, a 3-manifold with uniform PSC must be connected sum of $\mathbb{S}^2 \times \mathbb{S}^1$ and space forms (quotients of $\mathbb{S}^3$).
  - Uses Gromov-Lawson 1983, Geometrization proved by Perelman in 2003, and combined works of Chang-Weinberger-Yu 2010, Besseeres-Besson-Maillot 2011, Wang (using μ-bubbles) 2023.

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- More Quantitatively: a complete 3-manifold (M, g) with $R_g \geq R_0 > 0$ is close to being "one-dimensional".
  - Liokumovich-Maximo 2020, Liokumovich-Wang 2023
  - There is a continuous map $f : M \to \mathbb{R}$ such that every component of a fiber must have bounded diameter and area.

Urysohn Width and the μ-Bubble Method

Definition

A metric space (X, d) has k-Urysohn width bounded by $d_0 > 0$, if there is a continuous map to a k-dimensional space $f : X \to G^k$, such that $\mathrm{diam}_d f^{-1}(g) \leq d_0$ for all $g \in G$.

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Any compact n-manifold has bounded 0-Urysohn width.
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Conjecture (Gromov, 2017)

If (X^n, g) with $n \geq 2$ is a closed Riemannian manifold with $R_g \geq 1$, then its $(n-2)$-Urysohn width is bounded by $c(n) > 0$.

Urysohn Width and the μ-Bubble Method

The μ-bubble method can be used to give a short proof of the simply connected case of Gromov's conjecture.

Theorem (Chodosh-Li, 2024)

If (M, g) is a simply connected 3-manifold with $R_g \geq 2$, then the 1-Urysohn width of M is bounded from above by 10π.

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Proof.

Fix a point $x \in M$, and consider the bands $M_k := B^M_{2(k+1)\pi}(x) \setminus B^M_{2k\pi}(x)$, since $R_g \geq 2$ over each band of length at least 2π, we can

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- put a μ-bubble called Σ^2 inside with diameter no more than 2π;
- using simply connectedness we know that Σ is separating in M_k;
- using triangle inequality we get that the diameter of each M_k is no more than 10π.

$\square$

Applications of 1-Urysohn Width Bound

Theorem (Chodosh-Li, 2024)

The following two generalized Geroch Conjecture holds,

- *Closed aspherical 4 or 5 manifolds has no PSC.*
- $\mathbb{T}^n \# X (2 \leq n \leq 7)$ *for any manifold* X *has no complete PSC metric.*

Remark

A torus is aspherical. The extensions follows the idea of the proof of Schoen-Yau but generalized in the sense that here we need to find (generalized) minimal surfaces in a space with little topology.

Rigidity of Stable Minimal Hypersurfaces

Earlier rigidity results using stability of $M^n \subset X^{n+1}$:

$$\int_M |\nabla \phi|^2 \geq \int_M (\mathrm{Ric}_X(\nu_M, \nu_M) + |\mathrm{II}_M|^2)\phi^2.$$

- If $\mathrm{Ric}_X \geq 0$, then any compact stable minimal hypersurface is totally geodesic, and $\mathrm{Ric}_X(\nu_M, \nu_M) = 0$ along M^n (Simons 1968).
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- If $R_X \geq 1$ then M admit a metric of PSC (Scheon and Yau, 1979).
- If $n = 2$ and M is complete non-compact, then $R_X \geq 0$ implies M must be conformal to a plane or a cylinder. In the latter case, M must be totally geodesic, intrinsically flat, (Fischer-Colbrie and Schoen 1980).

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- There exists a stable totally geodesic $\mathbb{R}^3$ embedded in $(\mathbb{R}^4, g)$ with $\sec > 0$. So $\sec > 0$ does not imply non-existence.
- $\mathrm{Ric}_X \geq 1$ also cannot rule out existence using the method of second variation.

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Curvature hierarchy of a manifold X:

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Theorem (Chodosh-Li-Stryker, 2022)

Consider (X^4, g) has weakly bounded geometry and

$$Ric_2^X \geq 0, \quad R_X \geq R_0 > 0.$$

Then any complete two-sided stable minimal hypersurface $M^3 \hookrightarrow X^4$ must have

$$|\mathrm{I\!I}_M| = 0, \quad Ric(\nu_M, \nu_M) = 0,$$

for ν_M a choice of unit normal along M.

In particular, $\mathbb{S}^4$ has no complete two-sided stable minimal hypersurfaces.

Ingredients of the Proof.

We may assume M is simply connected. Recall stability:

$$\int_M |\nabla\phi|^2 \geq \int_M (\mathrm{Ric}_X(\nu_M, \nu_M) + |\mathrm{II}_M|^2)\phi^2.$$

- Goal: show M has almost linear volume growth.

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- If not, $\mathrm{Ric}_2^X \geq 0$ implies a Liouville theorem: harmonic function on M with finite energy must be constant.
- This allows us to show M has at most one non-parabolic end.

$\square$

Free Boundary Minimal Hypersurfaces

Definition

An free boundary minimal hypersurface $(M^n, \partial M) \hookrightarrow (X^{n+1}, \partial X)$ is a critical point to the area functional among all variations that send ∂X to ∂X, we call M a FBMH.

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Consider $(X^4, \partial X, g)$ *has weakly bounded geometry*

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- Hierarchy of convexity: $\mathrm{II}_{\partial X} \geq 0 \implies \mathrm{II}_2^{\partial X} \geq 0 \implies H_{\partial X} \geq 0$.
- Rearranged stability inequality, $\mathrm{Ric}_2^X \geq 0$ and $\mathrm{II}_2^{\partial X} \geq 0$ (2-convexity of the boundary) $\implies$ the same Liouville theorem holds for M.
- Using free boundary μ-bubbles and $H_{\partial X} \geq 0$ we can show the 1-Urysohn width bound also holds for M.

Trading Uniform PSC with Uniform Mean Convexity

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The assumption of PSC allows us to use μ-bubbles, a key tool to obtain geometric control. What should we do now?

Capillary Hypersurfaces

Capillary surfaces help us study manifolds with non-negative scalar curvature (NNSC) and uniformly mean convex boundary.

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A capillary hypersurface $\Sigma^n = \partial\Omega$ in M^{n+1} is a critical point to,

$$E_c(\Omega) := \mathrm{Area}(\partial\Omega) - \cos\theta \mathrm{Area}(\overline{\Omega} \cap \partial M),$$

among variations that fix the volume ratio $\lambda_0 := \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(M)}$.

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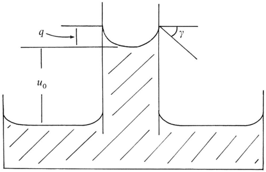

Figure: Robert Finn

Equivalently, Σ is a capillary hypersurface if it has constant mean curvature and intersect with ∂M at a constant angle.

Generalized Capillary Hypersurfaces: θ-Bubbles

The idea: having NNSC and Mean Convex Boundary can be also inherited by (generalized) capillary hypersurfaces.

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Definition

Consider a manifold with boundary $(M^{n+1}, \partial M)$, given a smooth function $\theta \in C^\infty(\partial M)$, a θ-bubble $\Sigma = \partial\Omega$ is a minimizer to,

$$E_\theta(\Omega) := \mathrm{Area}(\partial\Omega) - \int_{\overline{\Omega}\cap\partial M} \cos\theta.$$

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First and Second Variation

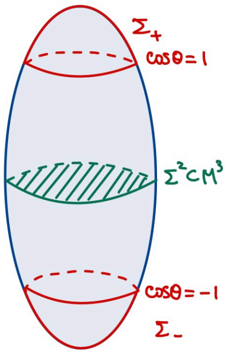
$$H_\Sigma = 0, \quad \langle \nu, \bar{\nu} \rangle = \cos\theta(x)$$

We may call a θ-bubble, a "prescribed contact angle" surface.

The Method of θ-Bubble: A Model Case for 3-Manifolds

We choose θ on ∂M so that the sets $\Sigma_\pm := \{x \in \partial M, \cos\theta = \pm 1\}$ serves as "barriers" to constrain and make sure a minimizer Σ must exist.

- Solomon-White: if $H_{\partial M} \geq 0$ and $\cos\theta \equiv 1$, then Σ must be minimizing across ∂M, either disjoint to ∂M, or equal to a connected component of ∂M.
- Using a similar argument, here $H_{\partial M} > 0$ gives a minimizer always exists and $\partial\Sigma \subset \{|\cos\theta| < 1\}$, Σ is smooth if $\dim(\Sigma) \leq 4$.



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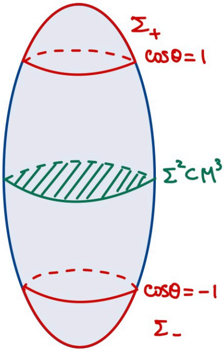

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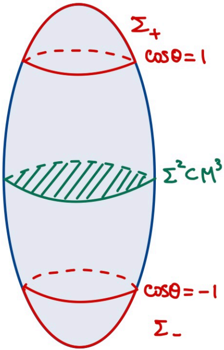
- Stability inequality: $R_M \geq 0, H_{\partial M} \geq 2$ and $d_0 := \operatorname{diam}(\partial M) > \pi$, $\implies R_\Sigma \geq 0, H_{\partial\Sigma} > 0$.
- This leads to localization of $\partial\Sigma$ when M is non-compact. Further estimates $d_\Sigma(x, \partial\Sigma) \leq \frac{2}{a_0}$ localizes Σ totally.



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Using θ-bubbles, we can obtain the following geometric estimates.

Theorem (W., 2024)

If $(S^2, \partial S)$ is a complete connected manifold with $R_S \geq 0$ and $k_{\partial S} \geq 1$, then S is a compact topological disk with $|\partial S| \leq 2\pi$ and $d(x, \partial S) \leq 1$ for any $x \in S$. Furthermore, if $|\partial S| = 2\pi$ then S is isometric to the unit disk in $\mathbb{R}^2$.

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Theorem (W., 2024, Obstruction to Gromov's Fill-In Question)

If $(M^3, \partial M)$ *is a complete simply connected Riemannian manifold with* $R_M \geq 0, H_{\partial M} \geq 3$*, then the* 1*-Urysohn width of* ∂M *with respect to the induced metric is at most* 3π*.*

The Method of θ-Bubble

Theorem (Gromov 2020, Bandwidth Estimate)

Let $2 \leq n \leq 6$*, consider* $M = (\mathbb{T}^n \times [-1,1], g)$ *such that* $R_g \geq n(n+1)$*, then* $d_g(\mathbb{T}^n \times \{+1\}, \mathbb{T}^n \times \{-1\}) \leq \frac{2\pi}{n+1}$*. And the bound is sharp.*

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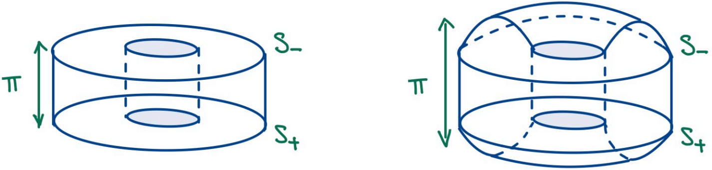
Let $(M^3, \partial M, g) = \Sigma_0 \times [-1,1]$ with $(\Sigma_0, \partial\Sigma_0)$ an orientable surface with $\chi(\Sigma_0) \leq 0$. If $R_M \geq 0, H_{\partial_0 M} \geq 2$ and $H_{\partial M} > 0$, then $d_{\partial M}(\partial S_+, \partial S_-) \leq \pi$, in particular $d_M(S_+, S_-) \leq \pi$.



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The idea is that we can chop ∂M into chunks of bounded diameter.

Corollary (W., 2024, Linear Growth of ∂M)

If $(M^3, \partial M)$ be a complete simply connected NNSC Riemannian manifold. If ∂M is uniformly mean convex and has weakly bounded geometry, then each end of ∂M has linear volume growth. In particular, if ∂M has finitely many ends, then ∂M has linear volume growth.

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Remark

Linear volume growth in the interior can not be obtained for NNSC manifolds.

Back to Rigidity of FBMH in $\mathbb{B}^4$: Trading Mean Convexity for PSC

So far we are using that $(M^3, \partial M)$ inherits the NNSC and mean convexity through stability, and are only able to obtain control of ∂M.
Note M may have compact or disconnnected ∂M.
We need to further exploit stability to control the interior of M.

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- Now using simply-connectedness, we can exhaust the non-parabolic end of M using θ-bubbles and obtain this end has linear growth.

Thank You For Listening!

Jean Siméon Chardin, 1733-34

Marie Gale, 2012