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Template adapted from CMU's 10725 Fall 2012 Optimization course taught by Geoff Gordon and Ryan Tibshirani.

Note of Computational Geometry

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Lecture 1: Introduction, The Line Intersection Problem using Sweepline

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1.1 Intro

1.1.1 Course description from the syllabus

"How do you sort points in space? What does it even mean? This course takes the ideas of a traditional algorithms course, sorting, searching, selecting, graphs, and optimization, and extends them to problems on geometric inputs. We will cover many classical geometric construc- tions and novel algorithmic methods. Some of the topics to be covered are convex hulls, Delaunay triangulations, graph drawing, point location, geometric medians, polytopes, configuration spaces, computational topology, approximation algorithms, and others. This course is a natural extension to 15-451, for those who want to learn about algorithmic problems in higher dimensions."

Textbook: Computational Geometry: Algorithms and Applications [1].

Traditional algorithm courses mainly discuss 1D problems, such as BST. The topics of this course contains the following main topics:

- Large dimensional problems
- The change of the nature of simple geometry problems when dimension increases

The applications of computational geometry include:

- 2D: graphics
- high dimension: machine learning

The basic issues or standard computational problems that will be discussed in recent lectures include:

- Line segment intersection (sweepline algorithm, random incremental algorithm)
- Convex Hull: given a set of points, compute the convex hull of these points.
- 2D-LP

The standard geometry problems that will be discussed recently include:

- Line side test

- In circle test

First the abstract objects and their representation that will be used in this course are discussed and is listed in the Table 1.1

Table 1.1: Abstract Object and Their Representation

Absract Object	Representation	Issue
Real Number	Float	Rounding Err
	Bignum (with arbitrary precision, normally they use arbitrary length array of digits)	Memory Intense
	Computer Algebra (Symbolic Computation)	
Point	Pair of Real	
Line	Pair of Points	
Line Segment	Pair of Points	
Triangle	Tripple of Points	

1.2 How to use points to generate object

Suppose $P_1, P_2, \ldots P_k \in \omega^d$ where $P_k \in M$ is a d-dimensional vector space with each point being a M-dimensional point, the following combinations of points creates the linear subspace of the vector space and generates geometric objects:

- Linear Combination:

$$Subspace = \sum_i \alpha_i \cdot P_i, \alpha_i \in \mathbb{R}$$

  For the $d = 2$ and $P_i \in \mathbb{R}^3$ case, the linear combination of P_1 and P_2 forms a plane with $\vec{OP_1}, \vec{OP_2}$ being the basis.

- Affine Combination:

$$Plane = \sum_i \alpha_i \cdot P_i, s.t.\ \alpha_i \in \mathbb{R} \wedge \sum_i \alpha_i = 1$$

  For the $d = 2$ and $P_i \in \mathbb{R}^3$ case, the affine combination of P_1 and P_2 forms a line that passes P_1, P_2.

- Convex Combination:

$$Body = \sum_i \alpha_i \cdot P_i, s.t.\ \alpha_i \in \mathbb{R} \wedge \sum_i \alpha_i = 1 \wedge \alpha_i \geq 0$$

$S \subseteq \mathbb{R}^d$ is a convex set iff $\forall p, q \in S$., $[p, q] \subseteq S$ (A set S in a vector space over R is called a convex set if the line segment joining any pair of points of S lies entirely in S [2])

For the $d = 2$ and $P_i \in \mathbb{R}^3$ case, the convex combination of P_1 and P_2 forms a line segment between P_1, P_2.

Convex Closure/Hull: The minimal convex set $S' \supseteq S$

A subset S of the plane is called convex if and only if for any pair of points $p, q \in S$ the line segment $\overline{pq}$ is completely contained in S. The convex hull CH(S) of a set S is the smallest convex set that contains S [1].

Theorem 1.1. $CC(P_1 \ldots P_k) = \{\alpha \in \mathbb{R} \mid \sum_i \alpha_i = 1 \wedge \alpha_i \geq 0\}$

In convex geometry Carathodory's theorem states that if a point x of $\mathbb{R}^d$ lies in the convex hull of a set P, there is a subset P' of P consisting of $d+1$ or fewer points such that x lies in the convex hull of P'.

1.3 Primitives

Problems related to geometry primitives

1) Test equality $p = q$?

2) Line segment intersection in 2D

   Let $L_1 = [P_1, P_2]$, $L_2 = [P_3, P_4]$, let $P_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$.

$$L_1 \cap L_2 \neq 0 \iff (P_1 P_2) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (P_3 P_4) \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix} \text{ and } \alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = 1 \text{ and } \alpha_i \geq 0. \tag{1.1}$$

   If written in matrix form, Equation 1.1 becomes:

$$\begin{pmatrix} x_1 & x_2 & -x_3 & -x_4 \\ y_1 & y_2 & -y_3 & -y_4 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{1.2}$$

   So the general process of solving the line segment intersection problem is

   Step I. Solveeq:lineSeg2 with some solver

   Step II. Check if $\forall i, \alpha_i \geq 0$

   There could be more than one solutions when L_1 and L_2 intersect in more than one point: the four point are collinear and there is an overlay. But in terms of the problem, the solution is still unique, since the solution is either True of False.

   Remark: The good practice is make this line segment intersection test be a sum-routine and call an existing solver to solve the matrix. Don't try to inline the code

   Some cases when the algorithm output False:

- L_1 and L_2 parallel but not collinear
- the intersection is on the extension of the two line segment
- the intersection is on one line segment and on the extension of the other.

3) Line side test

- input: three points in 2D: P_1, P_2, P_3
- output: if P_3 is to the left of ray P_1P_2

  One process of solving the problem: Subtract P_1 from both of the other vectors. Let $V_2 = P_2 - P_14$ and $V_3 = P_3 - P_1$. Now the cross product $V_2 \times V_3$ is the signed area of the parallogram formed by V_2 and V_3. This area is > 0 if and only if P_3 is to the left of ray $P_1 \mapsto P_2$.

  Alternatively, suppose $P_1 = O$ (the origin), then the signed area of the parallogram formed by $P_1\vec{P}_2, P_1,\vec{P}_3$ is

$$\det \begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix} \tag{1.3}$$

$$LHS = \det \begin{bmatrix} x_1 & x_2 & x_2 \\ y_1 & y_2 & y_2 \\ 1 & 1 & 1 \end{bmatrix} = det \begin{bmatrix} x_1 & x_2 - x_1 & x_3 - x_1 \\ y_1 & y_2 - y_1 & y_3 - y_1 \\ 1 & 0 & 0 \end{bmatrix} \tag{1.4}$$

  Just report the sign of determinant of LHS of Equation 1.4

Remark: The good property of method 2 is that it can be generalized to higher dimension easily: for a 4D space, the test checks the determinant of a 4 by 4 matrix.

4) In circle test

- input: four points in 2D: P_1, P_2, P_3, P_4
- output: if P_4 is in the circle of (P_1, P_2, P_3)

References

[1] Mark de Berg, Otgried Cheong, Marc van Kreveld, and Mark Overmars. Computational Geometry : Algorithms and Applications. Springer Berlin Heidelberg, 2008.

[2] Wolfram Research, Inc. Convex set. web, August 2015. `http://mathworld.wolfram.com/ConvexSet.html`.