Yujie Xu
yujiex@andrew.cmu.edu

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Linear Algebra Review

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Lecture 1: Intro

Instructor: Notes taken: Yujie Xu
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1.1 Concepts

Definition 1.1. *Column Space: In linear algebra, the column space C(A) of a matrix A (sometimes called the range of a matrix) is the set of all possible linear combinations of its column vectors.*

A matrix with n columns and m rows corresponds to n vectors in m-dimensional space.

Linear equations lead to geometry of planes. Intersection of planes leads to lines: in three dimensions a line requires two equations; in a n dimensional space it will require $n - 1$.

There are two ways of looking at a system of linear equation: from rows or from columns.

- Row picture: Intersection of planes

  If looking at rows, each row is a plain and their intersection is the solution. For an n-equation-n-variable system of equations, each equation is a $n - 1$ dimensional plain in a n dimensional space. For example, $2u + v + w = 10$ is a 2D plain in a 3D space.

  The first equation produces an $(n - 1)$-dimensional plane in n dimensions, The second plane intersects it (we hope) in a smaller set of "dimension $n - 2$." Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. It is a point, it lies on all the planes, and its coordinates satisfy all n equations. It is the solution!

- Col picture: Combination of columns

  If looking at columns, the problem of solving the system of linear equations becomes to find the combination of the column vectors on the left side that produces the vector on the right side. The co-efficient of each variable is considered as a point in n-dimensional space, which is also a vector. The multipliers of vectors that does the job is also the point where in the row picture all plane intersects.

  Definition 1.2. *Linear combination of vectors:*
  vectors are multiplied by numbers and then added. The result is called a linear combination.

Definition 1.3. *Singular cases: no solution or infinite solution*
Non-singular case: there is one solution to the system of equations.

- No solution: Inconsistent equations results in no solution.

  For row picture, this happens when two or planes are parallel, or the intersection line of two planes is parallel to other planes.

  For column picture, this happens when all vectors are in the same plane but the target vector is not in the plane.

- Infinite many solution

  For row picture, when all planes intersects in one line, or even, when all planes are the same, there are infinitely many solutions.

  For column picture, this happens when all vectors are in the same plane and the target vector is also in the same plane.

If the n planes have no point in common, or infinitely many points, then the n columns lie in the same plane

1.2 Gaussian Elimination

The algorithm:

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Subtracting multiples of equation 1 from equation 2 to n to eliminate
the first variable

Recurse on step 1 on the remaining
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The (forward) elimination produces a triangular system, the values on the hypotenuse is called pivots. By definition, pivots cannot be zero.

Definition 1.4. *In the following example, there are 9 coefficients, 3 unknowns, 3 right-hand sides*

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

Definition 1.5. *Square Matrix:*
For a coefficient matrix, when the number of unknowns equals the number of equations.

Definition 1.6. *inner product:*
The number produced from the multiplication of a row vector and a column vector.

There are two ways of thinking about the multiplication of Ax:

- n inner product, one for each row.

- a combination of columns in A: Ax is a combination of the columns of A. The coefficients are the components of x

Remark: The notation $x = (2, 5, 0)$ is equivalent to $x = [2\ 5\ 0]^T$

Definition 1.7. *The identity matrix I has 1s on the diagonal and 0s everywhere else*

Definition 1.8. *The elementary matrix E_{ij} subtracts l times row j from row i.*

Theorem 1.9. *Matrix multiplication is associative: $(AB)C = A(BC)$*
Matrix multiplication is distributive: $A(B + C) = AB + AC$. Matrix multiplication is NOT commutative

Theorem 1.10. *The i, j entry of AB is the inner product of the ith row of A and the jth column of B.*

Every column of AB is a combination of the columns of A. Column i is the combination columns in A with the ith column in B as multipliers: column j of $AB = A$ times (column j of B)

Each row of AB is a combination of the rows of B:

row i of $AB =$ (row i of A) times B.

Definition 1.11. *Row exchange matrix:*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix}$$

Remark: the product of lower triangular matrices is again lower triangular

Definition 1.12. *upper triangle:*

All entries below the diagonal are zero.

1.3 Summations

- $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
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